Chapter 8
Delay-Dependent Switched Filtering

In this chapter, the filtering problem for a class of discrete-time switched systems with state delays is thoroughly investigated. We will focus on discrete-time systems. Attention will be equally focused on the design of stable filters guaranteeing different prescribed performance criteria including the $\mathcal{L}_2$ sense and in the $\mathcal{L}_2 - \mathcal{L}_\infty$ sense. In all cases, switched Lyapunov functionals are employed to derive sufficient conditions for the solvability of the filtering problem and expressed in terms of linear matrix inequalities (LMIs).

8.1 $\mathcal{H}_\infty$ Filter Design

The problem of $\mathcal{H}_\infty$ filtering for a class of discrete-time switched systems with state delays is investigated in this section. Attention is focused on the design of a stable filter guaranteeing a prescribed noise attenuation level in the $\mathcal{H}_\infty$ sense. By using switched Lyapunov functionals, sufficient conditions for the solvability of this problem are obtained in terms of linear matrix inequalities (LMIs), by solving which a desired $\mathcal{H}_\infty$ filter can be constructed.

8.1.1 Introduction

It is well known that state estimation has been widely studied and has found many practical applications during the past decades. When a priori information on the external noise is not precisely known, the celebrated Kalman filtering scheme is no longer applicable. In this case, $\mathcal{H}_\infty$ filter was introduced in [57], where the noise signal was assumed to be energy bounded and the main objective was to minimize the $\mathcal{H}_\infty$ norm of the filtering error system [78, 282, 346, 394, 408]. When time delays are taken into account in a system, linear matrix inequality-based (LMI-based) results on the $\mathcal{H}_\infty$ filtering problem have also been reported in the literature; see, for example, [79, 106, 347, 381, 393] and the references therein.

Recently, the control synthesis of switched systems has been extensively investigated and many methodologies have been used in the study of switched systems

M.S. Mahmoud, *Switched Time-Delay Systems*,
DOI 10.1007/978-1-4419-6394-9_8, © Springer Science+Business Media, LLC 2010

[42, 52, 56, 86, 170, 191, 441]. For example, multiple Lyapunov functions were employed to establish certain general Lyapunov-like results for nonlinear switched systems [56]; dwell-time and average dwell-time approaches were employed to study the stability and disturbance attenuation of switched systems [377, 426]; piecewise Lyapunov function approach was adopted in [156, 388]; and a switched Lyapunov function method has been applied in [42] to study the stability problem of discrete-time switched systems.

On the contrary, time delays are the inherent features of many physical process and the big sources of instability and poor performances. Switched systems with time delays have strong engineering background in network control systems [170] and power systems [291]. More recently, some theoretical studies were conducted for switched systems with time delays [370, 395, 425]. Till date, to the best of the authors' knowledge, the $\mathcal{H}_\infty$ filtering problem has not been addressed for time-delayed switched systems. In this paper, an $\mathcal{H}_\infty$ filtering design is developed using switched Lyapunov functional approach for discrete-time switched systems with time delay. The filtering design solution is facilitated by introducing some additional instrumental matrix variables. These additional matrix variables decouple the Lyapunov and the system matrices, which makes the filtering design feasible.

8.1.2 Problem Formulation

Consider the following discrete-time switched system with state delay :

$$\Sigma_0: \qquad x_{k+1} = \sum_{i=1}^{S} \alpha_i(k) A_i x_k + \sum_{i=1}^{S} \alpha_i(k) A_{di} x_{k-d} + \sum_{i=1}^{S} \alpha_i(k) B_i \omega_k \tag{8.1}$$

$$y_k = \sum_{i=1}^{S} \alpha_i(k) C_i x_k + \sum_{i=1}^{S} \alpha_i(k) C_{di} x_{k-d} + \sum_{i=1}^{S} \alpha_i(k) D_i \omega_k \tag{8.2}$$

$$z_k = \sum_{i=1}^{S} \alpha_i(k) G_i x_k \tag{8.3}$$

where $x_k \in R^n$ is the state, $y_k \in R^r$ is the measured output, $z_k \in R^q$ is the signal to be estimated, $\omega_k \in R^p$ is the disturbance input, which is assumed to belong to $l_2[0, \infty)$, and the positive integer d denotes the known state delay. $\alpha_i(k)$ is the switching signal:

$$\alpha_i : Z^+ \longrightarrow \{0, 1\}, \qquad \sum_{i=1}^{S} \alpha_i(k) = 1, \qquad k \in Z^+ = \{0, 1, \cdots\}$$

which specifies which subsystem will be activated at certain discrete time. A_i, A_{di}, B_i, C_i, C_{di}, D_i, and G_i are system matrices with compatible dimensions.

Here we are interested in designing a filter described by

$$\Sigma_f: \quad \hat{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k) A_{fi} \hat{x}_k + \sum_{i=1}^{S} \alpha_i(k) B_{fi} y_k \tag{8.4}$$

$$\hat{z}_k = \sum_{i=1}^{S} \alpha_i(k) C_{fi} \hat{x}_k \tag{8.5}$$

where $\hat{x}_k \in R^n$ and $\hat{z}_k \in R^q$, the matrices A_{fi}, B_{fi}, and C_{fi} are to be determined. Augmenting the model of Σ_0 to include the system Σ_f, we obtain the following system (called filtering error system):

$$\Sigma_c: \quad \tilde{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k) \tilde{A}_i \tilde{x}_k + \sum_{i=1}^{S} \alpha_i(k) \tilde{A}_{di} \tilde{x}_{k-d} + \sum_{i=1}^{S} \alpha_i(k) \tilde{B}_i \omega_k \tag{8.6}$$

$$\tilde{z}_k = \sum_{i=1}^{S} \alpha_i(k) \tilde{C}_i \tilde{x}_k \tag{8.7}$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ B_{fi} C_i & A_{fi} \end{bmatrix}, \ \tilde{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ B_{fi} C_{di} & 0 \end{bmatrix}, \ \tilde{B}_i = \begin{bmatrix} B_i \\ B_{fi} D_i \end{bmatrix},$$
$$\tilde{x}_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T, \quad \tilde{z}_k = z_k - \hat{z}_k, \ \tilde{C}_i = \begin{bmatrix} G_i & -C_{fi} \end{bmatrix} \tag{8.8}$$

Our objective is to develop a filter in the form of (8.4) and (8.5) such that the following specifications are met for the filtering error system Σ_c:

(H1): The filtering error system Σ_c is globally asymptotically stable when $\omega_k = 0$.
(H2): The filtering error system Σ_c guarantees, under zero-initial condition, $\|\tilde{z}_k\|_2 \le \gamma \|\omega_k\|_2$ for all nonzero $\omega_k \in l_2[0, \infty)$ and a given positive constant γ.

In the sequel, we will refer systems satisfying (H1) and (H2) as stable and with $\mathcal{H}_\infty$ norm bound γ.

Remark 8.1 The robust filter design problem for switched systems has been investigated in [86], where the minimax linear filters are developed for discrete-time systems whose dynamics switches are within a finite set of stochastic behaviors. In this paper, our attention is focused on the design of delay-independent robust $\mathcal{H}_\infty$ filters for the system Σ_0 under arbitrary switching signal.

8.1.3 Stability and Performance Analysis

This section gives a new characterization involving switched Lyapunov functional for the filtering error system Σ_c to be stable and with $\mathcal{H}_\infty$ norm bound γ.

Theorem 8.2 *The filtering error system Σ_c is stable and with $\mathcal{H}_\infty$ norm bound γ, if there exist matrices $\{P_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ for all $\{i, j, l\} \in \mathcal{S} = \{1, 2, \cdots, S\}$ such that*

$$\begin{bmatrix} -P_j^{-1} & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 \\ \bullet & -P_i + Q_i & 0 & 0 & \tilde{C}_i^t \\ \bullet & \bullet & -Q_l & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{8.9}$$

where $*$ denotes the corresponding transposed block matrix due to symmetry.

Proof First, we establish the stability of system (6). When $\omega_k = 0$, (8.6) becomes

$$\tilde{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k)\tilde{A}_i\tilde{x}_k + \sum_{i=1}^{S} \alpha_i(k)\tilde{A}_{di}\tilde{x}_{k-d} \tag{8.10}$$

Define

$$V_k = \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)P_i\right)\tilde{x}_k + \sum_{s=k-d}^{k-1} \tilde{x}_s^t \left(\sum_{i=1}^{S} \alpha_i(s)Q_i\right)\tilde{x}_s \tag{8.11}$$

Then

$$\begin{aligned} \Delta V_k|_{(8.10)} &= V_{k+1} - V_k \\ &= \tilde{x}_{k+1}^t \left(\sum_{i=1}^{S} \alpha_i(k+1)P_i\right)\tilde{x}_{k+1} - \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)P_i\right)\tilde{x}_k \\ &\quad + \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)Q_i\right)\tilde{x}_k - \tilde{x}_{k-d^tT} \left(\sum_{i=1}^{S} \alpha_i(k-d)Q_i\right)\tilde{x}_{k-d} \end{aligned}$$

It follows that for any nonzero vector $\tilde{x}_k$ and the particular case $\alpha_i(k) = 1$, $\alpha_{r\neq i}(k) = 0, \alpha_j(k+1) = 1, \alpha_{r\neq j}(k+1) = 0, \alpha_l(k-d) = 1, \alpha_{r\neq l}(k-d) = 0$. Then, we have

$$\Delta V_k|_{(8.10)} = \eta_k^t \left(\begin{bmatrix} \tilde{A}_i^t \\ \tilde{A}_{d^t} \end{bmatrix} P_j \begin{bmatrix} \tilde{A}_i & \tilde{A}_{di} \end{bmatrix} + \begin{bmatrix} -P_i + Q_i & 0 \\ 0 & -Q_l \end{bmatrix} \right) \eta_k$$

where $\eta_k = \left[\, \tilde{x}_k^t \ \tilde{x}_{k-d^t} \,\right]^t$. By the Schur complement formula, it follows from (8.9) that $\Delta V_k|_{(8.10)} < 0$, which establishes the stability of system (8.10).

Let

$$J_K = \sum_{k=0}^{K-1} \left(\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^t \omega_k \right)$$

where K is an arbitrary positive integer. For any nonzero $\omega_k \in l_2[0, \infty)$ and zero initial condition $\tilde{x}_0 = 0$, one has

$$\begin{aligned} J_K &= \sum_{k=0}^{K-1} \left(\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \right) - V_K \\ &\leq \sum_{k=0}^{K-1} \left(\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \right) \end{aligned}$$

where $\Delta V_k|_{(8.6)}$ defines the increment of V_k along the solution of system (8.6). It is noted that

$$\begin{aligned} &\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \\ &\quad = \tilde{\eta}_k^t \begin{bmatrix} \tilde{A}_i^t \\ \tilde{A}_{di}^t \\ \tilde{B}_i^t \end{bmatrix} P_j \left[\, \tilde{A}_i \ \tilde{A}_{di} \ \tilde{B}_i \,\right] \tilde{\eta}_k \\ &\quad\quad + \tilde{\eta}_k^t \begin{bmatrix} -P_i + Q_i + \tilde{C}_i^T \tilde{C}_i & 0 & 0 \\ 0 & -Q_l & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \tilde{\eta}_k \end{aligned} \tag{8.12}$$

where

$$\tilde{\eta}_k = \left[\, \tilde{x}_k^t \ \tilde{x}_{k-d^t} \ \omega_k^t \,\right]^t$$

It follows from (8.9) and Schur complement that

$$\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} < 0$$

which implies, for any K, $J_K < 0$. Then one has that for any nonzero $\omega_k \in l_2[0, \infty)$, $\|\tilde{z}_k\|_2 < \gamma \|\omega_k\|_2$ ■.

Motivated by the idea in [44], we present the following theorem.

Theorem 8.3 *The filtering error system Σ_c is stable and with H_∞ norm bound γ, if there exist matrices $\{R_i\}_{i=1}^N$, $\{\Psi_i\}_{i=1}^N$, and Ω for all $\{i, j, l\} \in \mathcal{S} = \{1, 2, \cdots, S\}$ such that*

$$\begin{bmatrix} -R_j & \tilde{A}_i\Omega & \tilde{A}_{di}\Omega & \tilde{B}_i & 0 & 0 \\ \bullet & R_i-(\Omega+\Omega^T) & 0 & 0 & \Omega^T\tilde{C}_i^T & \Omega^T \\ \bullet & \bullet & \Psi_l-(\Omega+\Omega^T) & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \quad (8.13)$$

Proof Suppose that (8.13) holds, then it is easy to see from (8.13) that

$$(R_i-\Omega)^t R_i^{-1}(R_i-\Omega) \geq 0$$

which implies

$$-\Omega^t R_i^{-1}\Omega \leq R_i-(\Omega+\Omega^t)$$

Similarly, we can get $-\Omega^t\Psi_i^{-1}\Omega \leq \Psi_i-(\Omega+\Omega^T)$. Then, (8.13) is transformed into

$$\begin{bmatrix} -R_j & \tilde{A}_i\Omega & \tilde{A}_{di}\Omega & \tilde{B}_i & 0 & 0 \\ \bullet & -\Omega^T R_i^{-1}\Omega & 0 & 0 & \Omega^T\tilde{C}_i^T & \Omega^T \\ \bullet & \bullet & -\Omega^T\Psi_l^{-1}\Omega & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \quad (8.14)$$

Pre-multiplying (8.14) by

$$\mathrm{diag}\{I, \Omega^{-t}, \Omega^{-t}, I, I, I\}$$

and post-multiplying by

$$\mathrm{diag}\{I, \Omega^{-1}, \Omega^{-1}, I, I, I\}$$

then (8.13) is transformed into

$$\begin{bmatrix} -R_j & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 & 0 \\ \bullet & -R_i^{-1} & 0 & 0 & \tilde{C}_i^T & I \\ \bullet & \bullet & -\Psi_l^{-1} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \quad (8.15)$$

Notice that $R_i = P_i^{-1}, \Psi_i = Q_i^{-1}$. Then, by using the Schur complement formula we can see that (8.15) is equivalent to (8.9). The proof is completed. ■

Remark 8.4 With the introduction of a new additional matrix Ω, we obtain a sufficient condition in which the matrices R_i and Ψ_i are not involved in any product with matrices $\tilde{A}_i$, $\tilde{A}_{di}$, $\tilde{B}_i$, and $\tilde{C}_i$. This makes a filter design feasible.

8.1.4 Filter Design

In this section, we will present a sufficient condition for the existence of H_∞ filter in the form of (8.4) and (8.5), and show how to construct a filter based on **Theorem 8.2**.

Theorem 8.5 *Consider system Σ_0 and given a constant $\gamma > 0$. If there exist matrices $0 < R_{1j} = R_{1j}^t$, $0 < R_{3j} = R_{3j}^t$, $0 < X_{1m} = X_{1m}^t$, $0 < X_{3m} = X_{3m}^t$ and R_{2j}, X_{2m}, Z_i, Y_i, H_i, L_i, M_i, S_i such that the following inequality holds:*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 & 0 \\ \bullet & \widehat{\Theta}_{22} & 0 & 0 & \Theta_{25} & \Theta_{26}^t \\ \bullet & \bullet & \widehat{\Theta}_{33} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Theta_{66} \end{bmatrix} < 0 \tag{8.16}$$

where

$$\Theta_{11} = \begin{bmatrix} R_{1j} & R_{2j} \\ \bullet & R_{3j} \end{bmatrix}, \; \Theta_{12} = \begin{bmatrix} Z_i A_i & Z_i A_i \\ Y_i A_i + H_i C_i + L_i & Y_i A_i + H_i C_i \end{bmatrix}$$

$$\Theta_{13} = \begin{bmatrix} Z_i A_{di} & Z_i A_{di} \\ Y_i A_{di} + H_i C_{di} & Y_i A_{di} + H_i C_{di} \end{bmatrix}, \; \Theta_{14} = \begin{bmatrix} Z_i B_i \\ Y_i B_i + H_i D_i \end{bmatrix}$$

$$\Theta_{22} = \begin{bmatrix} R_{1i} & R_{2i} \\ \bullet & R_{3i} \end{bmatrix}, \; \Theta_{25} = \begin{bmatrix} G_i^t - S_i^t \\ G_i^t \end{bmatrix}, \; \Theta_{26} = \begin{bmatrix} Z_i & Z_i \\ Y_i + M_i & Y_i \end{bmatrix}$$

$$\Theta_{33} = \begin{bmatrix} X_{1m} & X_{2m} \\ \bullet & X_{3m} \end{bmatrix}, \; \Theta_{66} = \begin{bmatrix} X_{1i} & X_{2i} \\ \bullet & X_{3i} \end{bmatrix}$$

$$\widehat{\Theta}_{22} = \Theta_{22} - \Theta_{26} - \Theta_{26}^t, \; \widehat{\Theta}_{33} = \Theta_{33} - \Theta_{26} - \Theta_{26}^t$$

then, there exists a filter in the form of (8.4) and (8.5) such that the filtering error system Σ_c is asymptotically stable with $\mathcal{H}_\infty$ norm bound γ. Moreover, if LMI (8.16) has a feasible solution, then the filter matrix

$$\mathcal{F} := \begin{bmatrix} A_{fi} & B_{fi} \\ C_{fi} & 0 \end{bmatrix} \tag{8.17}$$

can be constructed by

$$\mathcal{F} := \begin{bmatrix} V_i^{-1} L_i M_i^{-1} V_i & V_i^{-1} H_i \\ S_i M_i^{-1} V_i & 0 \end{bmatrix} \tag{8.18}$$

Proof Suppose the inequality (8.16) holds. It can be obtained that

$$\begin{bmatrix} Z_i + Z_i^t & Z_i + Y_i^t + M_i^t \\ \bullet & Y_i + Y_i^t \end{bmatrix} > \begin{bmatrix} R_{1i} & R_{2i}^t \\ \bullet & R_{3i} \end{bmatrix} > 0 \tag{8.19}$$

which implies that matrices Z_i and Y_i are nonsingular. Pre-multiplying (8.19) by $\begin{bmatrix} I & -I \end{bmatrix}$ and post-multiplying the result by $\begin{bmatrix} I & -I \end{bmatrix}^t$, one obtains

$$-M_i - M_i^t > 0 \tag{8.20}$$

which implies that M_i is also nonsingular. Hence there exist nonsingular matrices U_i and V_i satisfying $M_i = V_i U_i$ such that (8.16) holds.

Let

$$\begin{aligned} \Pi_i^t &= \begin{bmatrix} Z_i & 0 \\ Y_i & V_i \end{bmatrix}, \ \Omega \Pi_i = \begin{bmatrix} I & I \\ U_i & 0 \end{bmatrix} \\ H_i &= V_i B_{fi}, \ L_i = V_i A_{fi} U_i, \ S_i = C_{fi} U_i, \ M_i = V_i U_i \\ R_j &= \Pi_i^{-t} \Psi_{11} \Pi_i^{-1}, \ R_i = \Pi_i^{-t} \Psi_{22} \Pi_i^{-1} \\ \Phi_m &= \Pi_i^{-t} \Psi_{33} \Pi_i^{-1}, \ \Phi_i = \Pi_i^{-t} \Psi_{66} \Pi_i^{-1} \end{aligned} \tag{8.21}$$

By (8.8) and (8.21), one has

$$\begin{aligned} \Pi_i^t \tilde{A}_i \Omega \Pi_i &= \Psi_{12}, \ \Pi_i^t \tilde{A}_{di} \Omega \Pi_i = \Psi_{13}, \ \Pi_i^t \tilde{B}_i = \Psi_{14} \\ \tilde{C}_i \Omega \Pi_i &= \Psi_{25}^t, \ \Pi_i^t \tilde{A}_i \Omega \Pi_i = \Psi_{26} \end{aligned} \tag{8.22}$$

Pre-multiplying (8.13) by

$$\text{diag} \begin{bmatrix} \Pi_i^t & \Pi_i^t & \Pi_i^t & I & I & \Pi_i^t \end{bmatrix}$$

and post-multiplying the result by

$$\text{diag} \begin{bmatrix} \Pi_i & \Pi_i & \Pi_i & I & I & \Pi_i \end{bmatrix}$$

and using (8.21) and (8.22), we readily obtain (8.16). Finally, it is not difficult to verify from (8.21) that the filter matrices are given by (8.18), which completes the proof.

Remark 8.6 The filter expressed in the form of (8.4) and (8.5) not only guarantees analytical properties, such as stability and guaranteed $\mathcal{H}_\infty$ performance of the filtering error system Σ_c, but is itself a switched system.

Remark 8.7 By using the techniques in [30] and [444], the result of **Theorem 8.3** can be readily extended to the discrete-time switched systems with state delay, which contain norm-bounded parameter uncertainties or linear fractional form parameter uncertainties.

8.1.5 Illustrative Example A

Consider the system Σ_0 with $N = 2$ and

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.4 & 0.05 \\ 0 & -0.35 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.025 & 0 \\ -0.1 & -0.35 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.34 \\ -0.3 \end{bmatrix} \\
C_{d1} &= \begin{bmatrix} 0.02 \; 0 \end{bmatrix}, \; D_1 = 0.02, \; G_1 = \begin{bmatrix} 0.24 \; 0.23 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.29 \; 0.15 \end{bmatrix} \\
A_2 &= \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.05 & -0.1 \\ 0 & 0.15 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.1 \\ -1 \end{bmatrix} \\
C_{d2} &= \begin{bmatrix} 0 \; 0.017 \end{bmatrix}, \; D_2 = 0.015, \; G_2 = \begin{bmatrix} 0.2 \; 0.1 \end{bmatrix}, \; C_2 = \begin{bmatrix} -0.19 \; 0.17 \end{bmatrix}
\end{aligned}$$

The purpose here is to design a filter such that the filtering error system is stable and with a given $\mathcal{H}_\infty$ norm bound γ. Here the performance level is chosen as $\gamma = 0.6$. By using the Matlab LMI Control Toolbox to solve LMI (8.16), we can get a feasible set of solutions. By **Theorem 8.3**, a filter in the form of (8.4) and (8.5) as follows:

$$\begin{aligned}
A_{f1} &= \begin{bmatrix} 0.3497 & -0.5481 \\ 0.1094 & -0.1653 \end{bmatrix}, \; B_{f1} = \begin{bmatrix} -8.3430 \\ 4.3427 \end{bmatrix}, \; C_{f1} = \begin{bmatrix} -0.0030 \; -0.0758 \end{bmatrix} \\
A_{f2} &= \begin{bmatrix} -0.1385 & -0.0975 \\ 0.0049 & 0.0157 \end{bmatrix}, \; B_{f2} = \begin{bmatrix} -4.9351 \\ -1.4790 \end{bmatrix}, C_{f2} = \begin{bmatrix} -0.0059 \; -0.0282 \end{bmatrix}
\end{aligned}$$

The simulation results of the state responses of the plant and filter are, respectively, given in Figs. 8.1 and 8.2, where the initial conditions $x_0 = \begin{bmatrix} 1.0 \; -0.8 \end{bmatrix}^t$ and $\hat{x}_0 = \begin{bmatrix} 0 \; 0 \end{bmatrix}^t$, respectively, and the noise signal is chosen as $\omega_k = 1/(k+1)$, which belongs to $l_2[0, \infty)$. The simulation results of signal z_k and $\hat{z}_k$ are shown in Figs. 8.3 and 8.4. Figure 8.5 shows the simulation result of the filtering error $\tilde{z}_k = z_k - \hat{z}_k$. It is observed that the designed H_∞ filter meets the specified requirements, and works well.

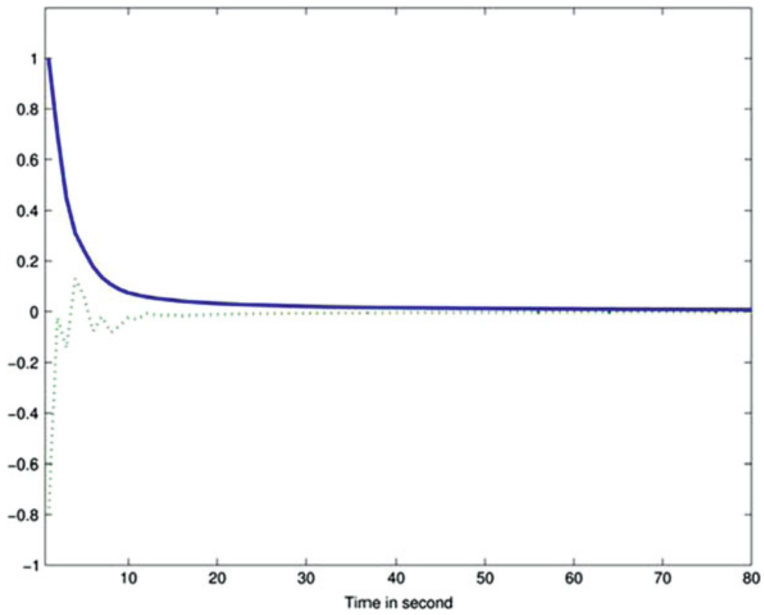

Fig. 8.1 Step response of plant states

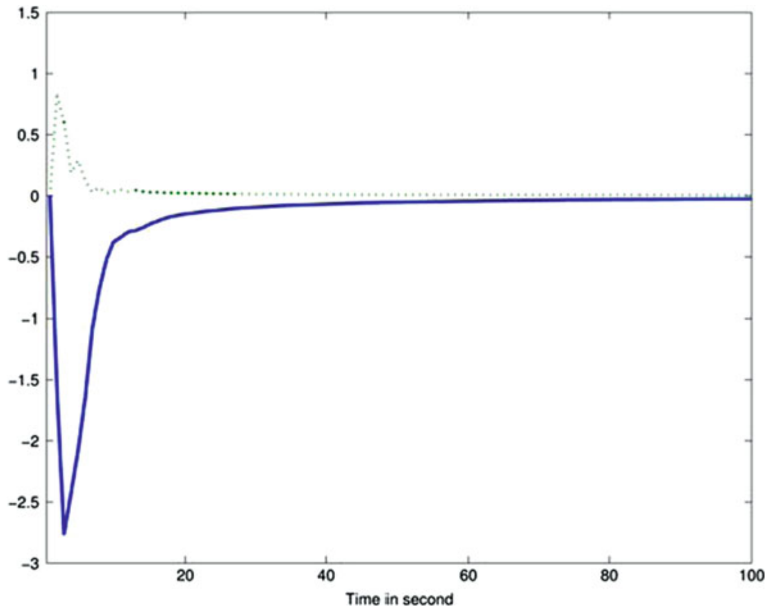

Fig. 8.2 Step response of plant states

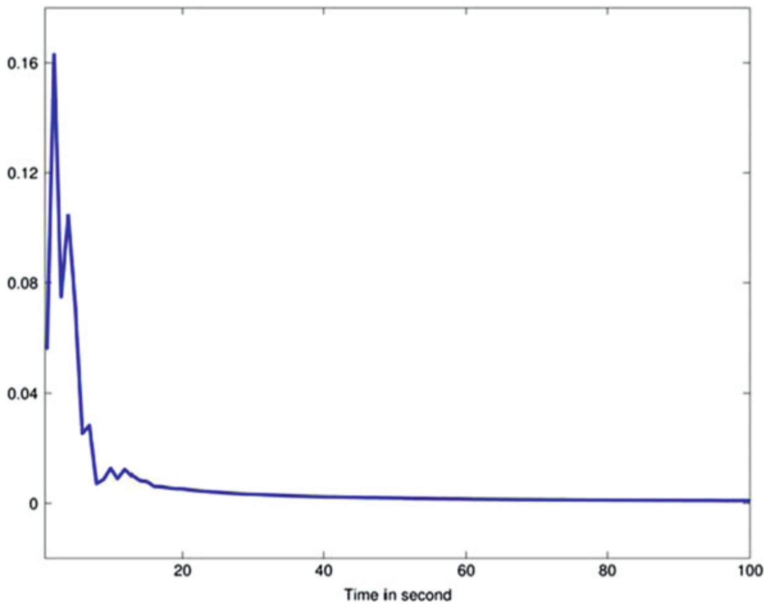

Fig. 8.3 Step response of plant states

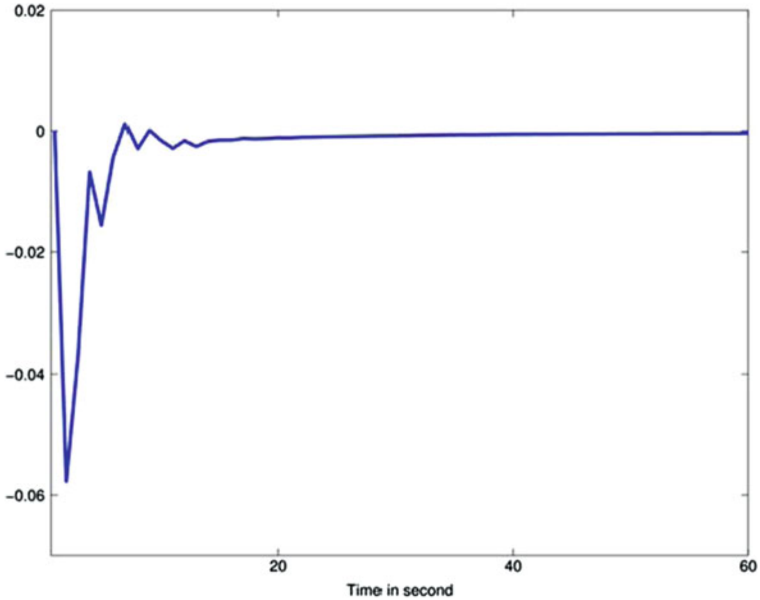

Fig. 8.4 Step response of plant states

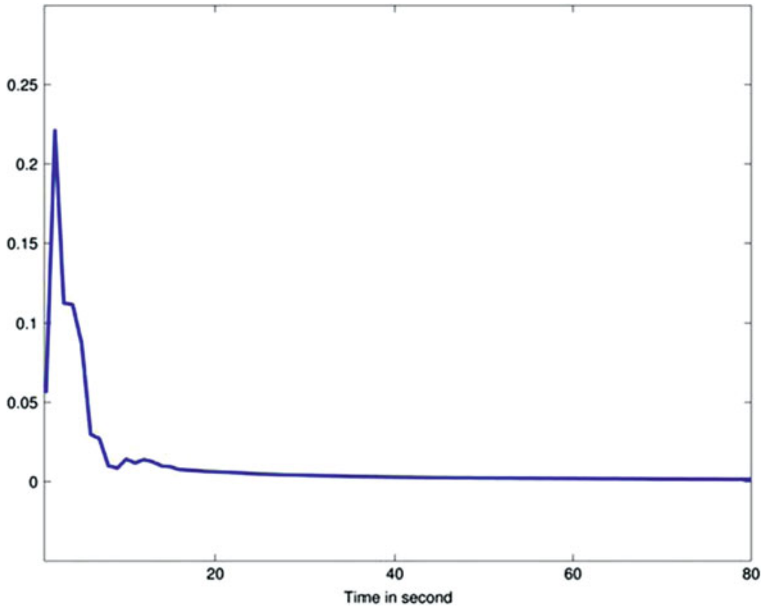

Fig. 8.5 Step response of plant states

8.2 Filter Design for Piecewise Systems

Broadly speaking, hybrid systems have proved to be an effective tool for multi-modeling, analysis, and design of a large number of evolving technological systems, in which digital devices interact with an analog environment. Systems of this type are common in embedded computation, robotics, mechatronics, avionics, and process control. Owing to the rapid advances in computer technology, hybrid systems are becoming increasingly relevant and important and consequently have attracted considerable research interests. A wide class of hybrid systems is *piecewise dynamical systems* for which some of the research results relevant to this study have been reported in [2, 63, 144, 293, 334] and their references. Common to these activities is the development of piecewise Lyapunov function approaches for stability analysis [156, 176, 313] and linear control design [118, 336, 431, 447] of piecewise continuous-time systems. In a parallel development, similar results are obtained for piecewise discrete-time linear systems [184, 293, 334, 376, 428]. For a class of piecewise discrete-time linear systems, the output feedback control problem has been investigated in [61] and the design of $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ filters are performed in [62] using observer-type filters (without parametric uncertainties or time delays). The solution is attained via the solution of a set of LMIs.

On another research front, the filtering problem has been the focal point of numerous research activities in the past four decades due to its central role in systems, control, and signal processing. The celebrated Kalman filter [3, 158, 352, 356] provides a recursive algorithm to minimize the variance of the state estimation error when the power spectral density of the process and measurement noise is known. During the past four decades, Kalman filtering techniques have found widespread applications in aerospace guidance, navigation, and control problems [213–221, 235, 250, 256, 257, 262, 263, 266, 267, 269, 287, 352, 356]. When a priori information on the external noise is not precisely known, Kalman filtering approach is no longer applicable. In such cases, $\mathcal{H}_\infty$ filtering was introduced [87, 305], in which the input signal is assumed to be energy bounded and the main objective is to minimize the energy of the estimation error for the worst possible bounded energy disturbance. The solution to this problem guarantees that the $\mathcal{L}_2$-induced norm from the noise signals to the filtering error will be less than a prescribed performance bound, where the noise are arbitrary energy-bounded signals. In the literature, there have been different approaches to solve $\mathcal{H}_\infty$ filtering problem [16, 67, 69, 71–216, 244, 245, 249, 250, 253, 254, 262–265, 269, 276, 277, 287, 305, 373, 438, 439]. When the systems are subjected to norm-bonded parametric uncertainties, robust $\mathcal{H}_\infty$ filtering were developed in [72] based on a Riccati equation approach and in [189] using a convex optimization approach. For systems with polytopic parameter uncertainties, linear matrix inequalities-based sufficient conditions were derived for robust $\mathcal{H}_\infty$ filters in [87, 317].

By contrast, the objective of $\mathcal{L}_2 - \mathcal{L}_\infty$ filtering problem is to minimize the peak value of the estimation error for all possible bounded energy disturbances. Hence, the $\mathcal{L}_2 - \mathcal{L}_\infty$ (energy-to-peak) filtering can be considered as a deterministic formulation of the Kalman filter [223, 318]. The class of robust filtering arose out of the desire to determine estimates of nonmeasurable state variables for dynamical systems with uncertain parameters. The past decade has witnessed major developments in robust filtering problem using various approaches [16, 305].

In recent years, research investigations into dynamical systems with time delays have been intensified and spread to several domains, including neural networks [35, 37, 194] and nonlinear systems [385, 390, 420]. In addition, the development of $\mathcal{H}_\infty$ filters and robust $\mathcal{H}_\infty$ filters were accomplished, leading to delay-independent and delay-dependent sufficient conditions [69, 217–223, 235–237, 250, 255–258, 266, 267, 278, 282]. By considering the developed conditions of $\mathcal{H}_\infty$ filters, it turns out that the results are generally conservative due to two sources: one introduced after using finite filters for infinite-dimensional systems like time-delay systems and the other source arose from uncertainties. To reduce overdesign conservatism, a new approach to $\mathcal{H}_\infty$ filtering was introduced using a bounded-real lemma (BRL) derived for the corresponding adjoint system. This approach was further refined in [69] using overbounding inequalities. In spite of the considerable advantages of the $\mathcal{H}_\infty$ filtering design results, it still entails some appreciable amount of conservatism due to the majorization procedure in filter design.

The design of robust $\mathcal{H}_\infty$ piecewise filters based on piecewise Lyapunov functional method for a class of piecewise discrete-time linear systems with time-varying delays has not been fully addressed before, which is very challenging. In this paper, we attend to this problem and consider the design of novel filters for a class of linear piecewise discrete-time systems with polytopic parametric uncertainties and time-varying delays. The time delays appear in the state as well as the output and measurement channels. We consider a general full-order filter that guarantees the desired estimation accuracy over the entire uncertainty polytope and accordingly develop two new types of filters by deploying piecewise Lyapunov–Krasovskii functional. The first filter is based on $\mathcal{H}_\infty$ criteria and the design incorporates new parametrization coupled with Finsler's Lemma to establish sufficient conditions for delay-dependent filter feasibility. The other one utilizes the $\mathcal{L}_2 - \mathcal{L}_\infty$ criteria and accomplishes the design via elegant use of Schur complement operations. In both cases, the filter gains are determined by solving linear matrix inequalities (LMIs).

8.2.1 Problem Statement and Definitions

We consider the following class of piecewise discrete-time linear (PDTL) systems:

$$x_{k+1} = A_j x_k + A_{dj} x_{k-d_k} + \Gamma_j \omega_k \tag{8.23}$$

$$y_k = C_j x_k + C_{dj} x_{k-d_k}$$

$$y_k \in \mathbf{\Omega}_j, \quad j = 1, 2, ..., r \tag{8.24}$$

$$z_k = G_j x_k + G_{dj} x_{k-d_k} + \Phi_j \omega_k \tag{8.25}$$

$$x_j = \psi_j, \quad j = -d_M, -d_M + 1, ..., 0 \tag{8.26}$$

where $\{\mathbf{\Omega}_j\}_{j \in S} \subseteq \Re^p$ denotes a partition of the output space into a number of closed polyhedral regions, with $\mathcal{S}$ being the index set of regions, $x_k \in \Re^n$ is the state vector, $\omega_k \in \Re^q$ is the disturbance input, which belongs to $\ell_2[0, \infty)$, $y_k \in \Re^p$ is the measured output and $z_k \in \Re^m$ is the signal to be estimated, $\{\psi_k,\ k = -d_M, -d_M + 1, ..., 0\}$ is a real-valued initial condition and $\{A_j, A_{dj}, \Gamma_j, C_j, C_{dj}, \Psi_j, G_j, G_{dj}, \Phi_j\}$ is the sth local model of the discrete system. In the sequel, we define the set

$$\Pi \stackrel{\Delta}{=} \{j, s | y_k \in \mathbf{\Omega}_j,\ \ y_{k+1} \in \mathbf{\Omega}_s\}$$

to represent all possible transitions from one region to itself or another region. In the sequel, it is assumed that the delay d_k is a time-varying function satisfying

$$d_m \le d_k \le d_M \tag{8.27}$$

where the lower bound $d_m > 0$ and the upper bound $d_M > 0$ are known constant scalars. For well-posedness of the problem and the subsequent results [144, 156, 336], we invoke the following assumptions:

Assumption 8.8 *The solution of the PDTL system (8.23), (8.24), (8.25), and (8.26) starting from any initial condition ψ_k is unique for all $k > 0$.*

Assumption 8.9 *When the state of the PDTL system (8.23), (8.24), (8.25), and (8.26) propagates from region* $\mathbf{\Omega}_j$ *to* $\mathbf{\Omega}_s$ *at time k, then the local model* $\mathbf{\Omega}_j$ *governs the system dynamics at that time.*

Assumption 8.10 *The state variables of the PDTL system (8.23), (8.24), (8.25), and (8.26) are bounded for every initial condition and all admissible disturbances.*

Definition 8.11 The energy-to-peak gain of system (8.23), (8.24), (8.25), and (8.26) is defined as

$$\sup_{0 \neq w \in \ell_2} \{||z_k||_{\ell_\infty} / ||w_k||_{\ell_2}\}$$

Remark 8.12 It should be noted that **Assumption 8.8** and **8.9** give a rule that characterize the piecewise state trajectories of the PDTL system (8.23), (8.24), (8.25), and (8.26). The partition is performed in the output space to ensure measurement consideration. Further details are presented in [144, 156, 336].

In case the PDTL system undergoes parametric uncertainties, we consider the following class of uncertain piecewise discrete-time linear (UPDTL) systems

$$\begin{aligned} x_{k+1} &= A_{j\Delta} x_k + A_{dj\Delta} x_{k-d_k} + \Gamma_{j\Delta} \omega_k && (8.28) \\ y_k &= C_{j\Delta} x_k + C_{dj\Delta} x_{k-d_k} \\ y_k &\in \boldsymbol{\Omega}_j, \quad j = 1, 2, ..., r && (8.29) \\ z_k &= G_{j\Delta} x_k + G_{dj\Delta} x_{k-d_k} + \Phi_{j\Delta} \omega_k && (8.30) \end{aligned}$$

whose matrices contain uncertainties that belong to a real convex-bounded polytopic model of the type

$$\begin{bmatrix} A_{j\Delta} & A_{dj\Delta} & \Gamma_{j\Delta} \\ C_{j\Delta} & C_{dj\Delta} & \\ G_{j\Delta} & G_{dj\Delta} & \Phi_{j\Delta} \end{bmatrix} \triangleq \left\{ \begin{bmatrix} A_{j\lambda} & A_{dj\lambda} & \Gamma_{j\lambda} \\ C_{j\lambda} & C_{dj\lambda} & \\ G_{j\lambda} & G_{dj\lambda} & \Phi_{j\lambda} \end{bmatrix} = \right.$$
$$\left. \sum_{m=1}^{N} \lambda_m \begin{bmatrix} A_{jm} & A_{djm} & \Gamma_{jm} \\ C_{jm} & C_{djm} & \\ G_{jm} & G_{djm} & \Phi_{jm} \end{bmatrix}, \lambda \in \Lambda \right\} \qquad (8.31)$$

where Λ is the unit simplex

$$\Lambda \triangleq \left\{ (\lambda_1, \cdots, \lambda_N) : \sum_{m=1}^{N} \lambda_m = 1 \ , \ \lambda_m \geq 0 \right\} \qquad (8.32)$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A_j, ..., \Phi_j\}$ to imply generic system matrices and $\{A_{jm}, ..., \Phi_{jm}, \ \ m \in \mathcal{N}\}$ to represent the respective values at the vertices.

The objective of this paper is to develop delay-dependent methods for piecewise filtering of the class of PDTL systems of the type (8.23), (8.24), (8.25), and (8.26) and subsequently generalize them to the UPDTL systems (8.28), (8.29), and (8.30). We investigate this problem by recourse to linear filter structure. Specifically, we seek to design an estimate $\hat{z}_k$ of z_k given by the linear state-space realization:

$$\begin{aligned} \hat{x}_{k+1} &= A_{fj}\hat{x}_k + B_{fj}\ y_k, \quad \hat{x}(0) = 0, \ \ y_k \in \mathbf{\Omega}_j \\ \hat{z}_k &= G_{fj}\hat{x}_k \end{aligned} \tag{8.33}$$

In (8.33), $\hat{x}(t) \in \Re^n$ is the state vector of the filter, $\hat{z}(t) \in \Re^q$ is the estimate of $z(t)$ and $A_{fs} \in \Re^{n\times n},\ B_{fs} \in \Re^{n\times m},\ G_{fs} \in \Re^{q\times n}$ are unknown filter matrices to be determined in the sequel based on prescribed performance criteria.

8.2.2 Error Dynamics

In terms of the filtering error $\widetilde{z}_k := z_k - \hat{z}_k$ and the augmented state $\widetilde{x}_k := \begin{bmatrix} x_k^t & \hat{x}_k^t \end{bmatrix}^t$, we get from the PDTL system (8.23) and the piecewise filter (8.33) the error dynamic model described by

$$\begin{aligned} \widetilde{x}_{k+1} &= \widetilde{A}_j\ \widetilde{x}_k + \widetilde{A}_{dj}\ \widetilde{x}_{k-d_k} + \widetilde{\Gamma}_j\omega_k \\ \widetilde{y}_k &= \widetilde{C}_j\widetilde{x}_k + \widetilde{C}_{dj}\ \widetilde{x}_{k-d_k} \\ \widetilde{z}_k &= \widetilde{G}_j\widetilde{x}_k + \widetilde{G}_{dj}\ \widetilde{x}_{k-d_k} + \widetilde{\Phi}_j\omega_k, \quad y_k \in \mathbf{\Omega}_j \end{aligned} \tag{8.34}$$

where the associated matrices are given by

$$\begin{aligned} \widetilde{A}_j &= \begin{bmatrix} A_j & 0 \\ B_{fj}C_j & A_{fj} \end{bmatrix},\ \widetilde{\Gamma}_j = \begin{bmatrix} \Gamma_j \\ B_{fj}\Phi_j \end{bmatrix} \\ \widetilde{G}_j &= \begin{bmatrix} G_j & -G_{fj} \end{bmatrix},\ \widetilde{A}_{dj} = \begin{bmatrix} A_{dj} & 0 \\ B_{fj}C_{dj} & 0 \end{bmatrix} \\ \widetilde{C}_j &= \begin{bmatrix} C_j & 0 \end{bmatrix},\ \widetilde{C}_{dj} = \begin{bmatrix} C_{dj} & 0 \end{bmatrix} \\ \widetilde{G}_{dj} &= \begin{bmatrix} G_{dj} & 0 \end{bmatrix},\ \widetilde{\Phi}_j = \Phi_j \end{aligned} \tag{8.35}$$

In this regard, the piecewise filtering problem of the PDTL system under consideration can be phrased as follows: *Given the PDTL system (8.23), (8.24), (8.25), and (8.26) and the piecewise filter (8.33), it is desired to determine the unknown piecewise matrices* $\{A_{fj},\ B_{fj},\ G_{fj}\}$ *such that the filtered system (8.34) is asymptotically stable and a prescribed performance criterion is achieved for all admissible uncertainties satisfying (8.31) and (8.32).* Two performance criteria will be considered in the sequel:

(1) $\mathcal{H}_\infty$-performance meaning that for a given prescribed performance bound $\gamma_\infty > 0$, $||\widetilde{z}_k||_2 < \gamma_\infty||\omega_k||_2,\ \forall\omega \in \ell_2[0, \infty)$,

This means that the $\gamma_\infty-$*suboptimal* $\mathcal{H}_\infty$*-piecewise filtering problem is to find a piecewise filter such that energy-to-peak value gain of the filtered system from the disturbance* ω_k *to the filtering error* $\widetilde{z}_k$ *is less than* γ_∞.

(2) $\mathcal{L}_2 - \mathcal{L}_\infty$-performance meaning that for a given prescribed performance bound $\gamma_2 > 0$ $||\widetilde{y}_k||_\infty < \gamma_2||\omega_k||_2,\ \forall\omega \in \ell_2[0, \infty)$, and

This means that the γ_2-suboptimal generalized $\mathcal{H}_2$-piecewise filtering problem is to find a piecewise filter such that energy-to-peak value gain of the filtered system from the disturbance ω_k to the output filtering error $\widetilde{y}_k$ is less than γ_2.

8.2.3 Delay-Dependent Stability

In this section, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and ℓ_2 gain analysis of the singular filtered. The criteria include some parameter matrices aiming at expanding the range of applicability of the developed conditions. The major thrust is based on the fundamental stability theory of Lyapunov, which states that for asymptotic stability, it suffices to find a Lyapunov function candidate $V_\sigma(x_k, k) > 0,\ \forall x_k \neq 0,\ k \in \mathbf{N}$ satisfying $\Delta V_\sigma(x_k, k) = V_\sigma(x_{k+1}, k+1) - V_\sigma(x_k, k) < 0$. We apply this theorem hereafter for arbitrary switching.

8.2.4 Piecewise Lyapunov Functional

For convenience, we define $\hat{d} = (d_M - d_m + 1)$ as the number of delay samples. The following theorem summarizes the main result.

Theorem 8.13 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable if there exist matrices $0 < P_j^t = P_j,\ 0 < P_s^t = P_s,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^3$ satisfying the LMIs for $(j, s) \in \Pi$*

$$\widehat{X}_j\overline{A}_j + \overline{A}_j^t\widehat{X}_j^t + \widetilde{P}_{js} < 0 \tag{8.36}$$

$$\widetilde{P}_{js} = \begin{bmatrix} -\Xi_{1s} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.37}$$

$$\Xi_{1s} = P_s - d_M\mathcal{W},\ \Xi_2 = \mathcal{M}_1 - d_M\mathcal{W}$$

$$\Xi_{3j} = P_j - \hat{d}\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t$$

$$\Xi_4 = -\mathcal{M}_2 - \mathcal{M}_2^t,\ \Xi_5 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q}$$

$$\overline{A}_j = \begin{bmatrix} -I & \widetilde{A}_j & \widetilde{A}_{dj} & 0 \end{bmatrix} \tag{8.38}$$

Proof Let the switching rule $\sigma(.)$ have an activated subsystem $j \in \Pi$ at instant k then an activated subsystem $s \in \Pi$ at instant $k+1$. In the sequel, we use $\xi_m = \widetilde{x}_{m+1} - \widetilde{x}_m$ and consider the following switched Lyapunov–Krasovskii functional

$$
\begin{aligned}
V_\sigma(\widetilde{x}_k, k) &\overset{\Delta}{=} V_{a\sigma}(\widetilde{x}_k, k) + V_{b\sigma}(\widetilde{x}_k, k) + V_{c\sigma}(\widetilde{x}_k, k) + V_{d\sigma}(\widetilde{x}_k, k) \\
V_{a\sigma}(\widetilde{x}_k, k) &= \widetilde{x}_k^t \; P_\sigma \widetilde{x}_k, \quad V_{b\sigma}(\widetilde{x}_k, k) = \sum_{j=k-d_k}^{k-1} \widetilde{x}_j^t \; Q \widetilde{x}_j \\
V_{c\sigma}(\widetilde{x}_k, k) &= \sum_{m=-d_M+2}^{-d_m+1} \sum_{j=k+m-1}^{k-1} \widetilde{x}_j^t \; Q \widetilde{x}_j \\
V_{d\sigma}(\widetilde{x}_k, k) &= \sum_{m=-d_M}^{-1} \sum_{j=k+m}^{k-1} \xi_j^t \; W \xi_j \\
0 < P_\sigma^t = P_\sigma, &\quad 0 < Q^t = Q, \; 0 < W^t = W, \; \sigma \in \mathcal{S}
\end{aligned}
\tag{8.39}
$$

Define $\Delta V_\sigma(\widetilde{x}_k, k) = V_\sigma(\widetilde{x}_{k+1}, k+1) - V_\sigma(\widetilde{x}_k, k)$, along the solution of (8.23) we obtain

$$
\Delta V_{a\sigma}(\widetilde{x}_k, k) = \widetilde{x}_{k+1}^t P_s \widetilde{x}_{k+1} - \widetilde{x}_k^t P_j \widetilde{x}_k \tag{8.40}
$$

$$
\begin{aligned}
\Delta V_{b\sigma}(\widetilde{x}_k, k) &= \sum_{m=k-d_{k+1}+1}^{k} \widetilde{x}_m^t \; Q \widetilde{x}_m - \sum_{j=k-d_k}^{k-1} \widetilde{x}_j^t \; Q \widetilde{x}_j \\
&= \widetilde{x}_k^t \; Q \widetilde{x}_k - \widetilde{x}_{k-d_k}^t \; Q \widetilde{x}_{k-d_k} + \sum_{m=k-d_{k+1}+1}^{k-1} \widetilde{x}_m^t \; Q \widetilde{x}_m \\
&\quad - \sum_{m=k-d_k+1}^{k-1} \widetilde{x}_m^t \; Q \widetilde{x}_m \\
&\le \widetilde{x}_k^t \; Q \widetilde{x}_k - \widetilde{x}_{k-d_k}^t \; Q \widetilde{x}_{k-d_k} + \sum_{m=k-\bar{d}+1}^{k-\underline{d}} \widetilde{x}_m^t \; Q \widetilde{x}_m
\end{aligned}
\tag{8.41}
$$

$$
\Delta V_{c\sigma}(\widetilde{x}_k, k) = (d_M - d_m) \, \widetilde{x}_k^t \; Q \widetilde{x}_k - \sum_{m=k-d_M+1}^{k-d_m} \widetilde{x}_m^t \; Q \widetilde{x}_m \tag{8.42}
$$

$$
\begin{aligned}
\Delta V_{d\sigma}(\widetilde{x}_k, k) &\le \bar{d}(\widetilde{x}_{k+1} - \widetilde{x}_k)^t \; W (\widetilde{x}_{k+1} - \widetilde{x}_k) \\
&\quad - d_M \sum_{m=k-d_M}^{k-1} \xi_m^t \; W \xi_m
\end{aligned}
\tag{8.43}
$$

Since $\widetilde{x}_{k-d_k} = \widetilde{x}_k - \sum_{m=k-d_k}^{k-1} \xi_m$, then for arbitrary parameter matrices (a set of free-weighting matrices) $\mathcal{M}_p, \; p = 1, \ldots, 5,$ we have

$$
\begin{aligned}
\widehat{x}(k, m) &= \left[\widetilde{x}_{k+1}^t \; \widetilde{x}_k^t \; \widetilde{x}_{k-d_k}^t \; \xi_m^t \right]^t \\
\widehat{\mathcal{M}} &= \left[\mathcal{M}_1^t \; \mathcal{M}_2^t \; \mathcal{M}_3^t \; 0 \right]^t \\
\widehat{\mathcal{S}} &= \left[0 \; I \; -I \; -d_k I \right]
\end{aligned}
\tag{8.44}
$$

such that the following equation holds

$$2 \sum_{j=k-d_k}^{k-1} \widehat{x}^t(k,m)\ \widehat{\mathcal{M}}\widehat{\mathcal{S}}\widehat{x}(k,m) = 0 \tag{8.45}$$

On considering (8.40), (8.41), (8.42), and (8.43) in the light of (8.39) for $d_k \leq \bar{d}$, $w_k \equiv 0$, it is not difficult to show that $\Delta V(x_k, k) < 0$ is equivalent to the following set of inequalities:

$$\sum_{m=k-d_k}^{k-1} \widehat{x}^t(k,m)\ \widetilde{\mathcal{P}}_{sj}\widehat{x}(k,m) < 0, \quad (s,j) \in \mathbf{N} \times \mathbf{N} \tag{8.46}$$

More importantly, in view of (10.45) with $u_k \equiv 0$, $w_k \equiv 0$, we have

$$\overline{A}_j\ \widehat{x}(k,m) = 0 \tag{8.47}$$

where $\widetilde{\mathcal{P}}_{sj}$, $\widetilde{A}_j$ are given by (8.37) and (8.38), respectively. Application of Finsler's **Lemma A.12** (from the Appendix) to (8.46) and (8.47) with $\widehat{x}(k,j) \equiv x$, $\widetilde{\mathcal{P}}_{sj} \equiv \mathcal{P}$, $\widetilde{A}_s \equiv \mathcal{Z}^t$, $\widehat{X}_s \equiv \mathcal{B}$, we readily obtain LMI (8.37) as desired, which establishes the asymptotic stability. ■

8.2.5 Robust Stability

Corollary 8.14 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) and vertex representation (8.31) and (8.32) is delay-dependent asymptotically stable if there exist matrices $0 < P_j^t = P_j$, $0 < P_s^t = P_s$, $\widehat{X}_j$, $0 < Q^t = Q$, $0 < W^t = W$, $\{\mathcal{M}\}_1^3$ satisfying the LMIs for $\forall (j,s) \in \Pi$*

$$\widehat{X}_j\overline{A}_{jp} + \overline{A}_{jp}^t\widehat{X}_j^t + \widetilde{\mathcal{P}}_{jps} < 0 \tag{8.48}$$

$$\widetilde{\mathcal{P}}_{jps} = \begin{bmatrix} -\Xi_1 & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_3 & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.49}$$

$$\overline{A}_{jp} = \begin{bmatrix} -I & \widetilde{A}_{jp} & \widetilde{A}_{djp} & 0 \end{bmatrix} \tag{8.50}$$

Proof Obtained from Theorem (8.13) by using the polytopic representation (8.31) and (8.32) to get (8.49) from (8.36).

8.2.6 Common Lyapunov Functional

In the special case of using a common Lyapunov functional, the ensuing delay-dependent stability results are summarized by the following corollaries:

Corollary 8.15 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable if there exist matrices* $0 < P_j^t = P_j,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^3$ *satisfying the LMIs for* $\forall (j, s) \in \Pi$

$$\widehat{X}_j \overline{A}_j + \overline{A}_j^t \widehat{X}_j^t + \widetilde{P}_j < 0 \tag{8.51}$$

$$\widetilde{P}_j = \begin{bmatrix} -\Xi_{1j} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.52}$$

$$\Xi_{1j} = P_j - \bar{d}\mathcal{W},\ \Xi_2 = \mathcal{M}_1 - \bar{d}\mathcal{W}$$
$$\Xi_{3j} = P_j - \hat{d}\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t$$
$$\Xi_4 = -\mathcal{M}_2 - \mathcal{M}_2^t,\ \Xi_5 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q}$$
$$\overline{A}_j = \begin{bmatrix} -I & \widetilde{A}_j & \widetilde{A}_{dj} & 0 \end{bmatrix} \tag{8.53}$$

Corollary 8.16 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) and vertex representation (8.31) and (8.32) are delay-dependent asymptotically stable if there exist matrices* $0 < P_j^t = P_j,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^3,\ \forall (j) \in \Pi$ *satisfying the LMIs for* $\forall (j) \in \Pi$

$$\widehat{X}_j \overline{A}_{jp} + \overline{A}_{jp}^t \widehat{X}_j^t + \widetilde{\mathcal{P}}_{jp} < 0 \tag{8.54}$$

$$\widetilde{P}_{jp} = \begin{bmatrix} -\Xi_{1j} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.55}$$

$$\overline{A}_{jp} = \begin{bmatrix} -I & \widetilde{A}_{jp} & \widetilde{A}_{djp} & 0 \end{bmatrix} \tag{8.56}$$

Remark 8.17 The main stability results are derived from feasibility testing in the enlarged state space in contrast with existing similar techniques [184, 368, 372, 438]. The novelty of our approach relies on the deployment of Finsler's Lemma in conjunction with a set of free-weighting matrices without using bounding techniques to ensure that the system matrices are readily separated from the Lyapunov matrices. This decoupling feature simplifies numerical implementation and, as will be shown in the subsequent sections, paves the way to flexible feedback stabilization synthesis. A simple comparison would support our intuition that the LMI results are less conservative and in the nonswitching case are superior than the existing methods [345, 354].

8.2.7 $\mathcal{H}_\infty$ Performance

Here, we consider the performance measure

$$J_{1K} = \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j \right)$$

The following theorem states the main result

Theorem 8.18 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched system (8.23), (8.24), and (8.25) with $u_k \equiv 0$ is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $0 < P_j^t = P_j$, $0 < P_s^t = P_s$, $\widehat{X}_j$, $0 < Q^t = Q$, $0 < W^t = W$, $\{\mathcal{M}\}_1^5$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall (j, s) \in \Pi$*

$$\widehat{X}_j \mathcal{A}_j + \mathcal{A}_j^t \widehat{X}_j^t + \widehat{P}_{js} < 0 \tag{8.57}$$

$$\widehat{P}_{js} = \begin{bmatrix} -\Xi_1 & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\Xi_3 & \Xi_4 & -\bar{d}\mathcal{M}_2 & \widetilde{G}_j^t \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 & \widetilde{G}_{dj}^t \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \widetilde{\Phi}_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} \tag{8.58}$$

$$\mathcal{A}_j = \left[-I \; \widetilde{A}_j \; \widetilde{A}_{dj} \; 0 \; \widetilde{\Gamma}_j \right] \tag{8.59}$$

where $\Xi_1, \ldots, \Xi_5$ are given in (8.38).

Proof For any $\omega_k \in \ell_2(0, \infty) \neq 0$ and zero initial condition $x_o = 0$, we have

$$J_{1K} \leq \sum_{j=0}^{K} \left(z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j + \Delta V_\sigma(x_j, j) \right)$$

Standard algebraic manipulation using (8.25) leads to

$$\begin{aligned} & z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j + \Delta V_\sigma(x_j, j) = \\ & \widetilde{x}^t(k, m) \, \widehat{\mathcal{P}}_{js} \, \widetilde{x}(k, m), \; \widetilde{x}(k, m) = \left[\widehat{x}^t(k, m) \quad \omega_k^t \right]^t \end{aligned} \tag{8.60}$$

and $\widehat{\mathcal{P}}_{js}$ is given by (8.57). It follows from [279] that for the switched system (8.23), (8.24), and (8.25) to be asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ it suffices that $z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j + \Delta V_\sigma(x_j, j) < 0$, $\forall j \in \{0, K\}$ holds for arbitrary switching, which in turn implies that $J_{1K} < 0$. The desired result is achieved by Finsler's Lemma and LMI (8.37) subject to (8.36). ■

8.2.8 $\ell_2 - \ell_\infty$ Performance

Here, we consider the performance measure

$$J_{2K} = V_\sigma(x_K, K) - \sum_{j=0}^{K-1} \omega_j^t \, \omega_j$$

where K is an arbitrary positive integer. The following theorem states the desired stability result

Theorem 8.19 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable with generalized $\mathcal{H}_2 - gain < \gamma_2$ if there exist matrices $0 < P_j^t = P_j,\ 0 < P_s^t = P_s,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^5$ and scalars $\gamma_2 > 0, \varepsilon_j > 0$ satisfying the LMIs for $\forall (j, s) \in \Pi$*

$$\begin{bmatrix} -P_s + \hat{d}Q & 0 & 0 & \widetilde{A}_j^t P_j & \bar{d}\left(\widetilde{A}_j^t - I\right) W \\ \bullet & -Q & 0 & \widetilde{A}_{dj}^t P_j & \bar{d}\widetilde{A}_{dj}^t W \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t P_j & \bar{d}\widetilde{\Gamma}_j^t W \\ \bullet & \bullet & \bullet & -P_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\bar{d}W \end{bmatrix} < 0 \tag{8.61}$$

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \widetilde{C}_j^t \\ \bullet & -\varepsilon_j I & \widetilde{C}_{dj}^t \\ \bullet & \bullet & -\gamma_2^2 I \end{bmatrix} < 0 \tag{8.62}$$

Proof For any sequence $0 \neq \omega_j \in \ell_2[0, \infty),\ j \in \{1, \ldots, K-1\}$ and zero initial condition $\tilde{x}_o = 0$, one has

$$J_{2K} = \sum_{j=0}^{K-1} \left[\Delta V_K|_{(8.23)} - \omega_j^t \, \omega_j \right] \tag{8.63}$$

Using (8.40) (8.41), (8.42), and (8.43) and manipulating, we get

$$J_{2K} = \begin{bmatrix} x_k \\ x_{k-d_k} \\ \omega_k \end{bmatrix}^t \Xi_{sj} \begin{bmatrix} x_k \\ x_{k-d_k} \\ \omega_k \end{bmatrix} \tag{8.64}$$

$$\Xi_{sj} = \begin{bmatrix} \Xi_{1sj} & \Xi_{2sj} & \Xi_{3sj} \\ \bullet & \Xi_{4sj} & \Xi_{5sj} \\ \bullet & \bullet & \Xi_{6sj} \end{bmatrix}$$

$$\Xi_{1sj} = \widetilde{A}_j^t P_s \widetilde{A}_j - P_j + (\bar{d} - \underline{d} + 1)Q + \bar{d}\left(\widetilde{A}_j^t - I\right) W(\widetilde{A}_j - I)$$

$$\Xi_{2sj} = \widetilde{A}_j^t P_s \widetilde{A}_{dj} + \bar{d}\left(\widetilde{A}_j^t - I\right) W \widetilde{A}_{dj}$$

$$\begin{aligned}
\Xi_{3sj} &= \widetilde{A}_j^t P_s \Gamma_j + \bar{d}\left(\widehat{A}_j^t - I\right) W \widetilde{\Gamma}_j \\
\Xi_{4sj} &= -Q + \widetilde{A}_{dj}^t P_s \widetilde{A}_{dj} + \bar{d} \widetilde{A}_{dj}^t W \widetilde{A}_{dj} \\
\Xi_{5sj} &= \widetilde{A}_{dj}^t P_s \widetilde{A}_{dj} + \bar{d} \widehat{A}_{dj}^t W \widetilde{A}_{dj}, \\
\Xi_{6sj} &= -I + \widetilde{\Gamma}_j^t P_s \widetilde{\Gamma}_j + \bar{d} \widetilde{\Gamma}_j^t W \widetilde{\Gamma}_j
\end{aligned} \tag{8.65}$$

By virtue of (8.61) and Schur complements, it is easy to see that $J_{2K} < 0$ for any K. Subsequently, for any $0 \neq \omega_j \in \ell_2[0, \infty)$, it follows that

$$V_K < \sum_{j=0}^{K-1} \omega_j^t \, \omega_j \tag{8.66}$$

In turn, Schur complements on LMI (8.62) and applying the **S**-procedure, it yields

$$\begin{bmatrix} -\gamma_2^2 \mathcal{P}_j + \widetilde{C}_j^t \widetilde{C}_j & \widetilde{C}_j^t \widetilde{C}_{dj} \\ \bullet & \widetilde{C}_{dj}^t \widetilde{C}_{dj} \end{bmatrix} < 0 \tag{8.67}$$

from which it is readily evident that

$$\widetilde{y}_K^t \widetilde{y}_K - \gamma_2^2 \, V_K < 0 \tag{8.68}$$

Finally, by LMIs (8.66) and (8.68), it follows that switched filtered system (8.34) has a generalized $\mathcal{H}_2$ norm bound γ_2. ■

Remark 8.20 We note from that the $\mathcal{L}_2-$ gain under arbitrary switching can be looked as the worst-case energy amplitude gain for the switched system (8.23, 8.24, 8.25, and 8.26) over all possible inputs, switching signals, and all admissible uncertainties. The functional (8.39) is called a switched Lyapunov function (SLF) since it has the same switching signals as system (8.23), (8.24), and (8.25), which is known to yield less conservative results than the constant Lyapunov functional. A novel feature of the developed approach is the arbitrary selection of the matrix $\widetilde{X}_j$, which helps much in the feedback stabilization later on as well as in the numerical simulation.

Remark 8.21 The optimal $\mathcal{L}_2 - gain$ of switched system (8.23, 8.24, and 8.25) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned}
& Minimize \;\; \gamma \\
& s.t. \quad LMIs \; (8.36) - (8.37), \quad \forall \, (j, s) \in \mathbf{N} \times \mathbf{N} \\
& P_j, \; P_s, \; \widehat{X}_j, \; Q, \; W, \; \{\mathcal{M}\}_1^5, \; \forall (j, s), \quad \gamma > 0, \; \sigma > 0, \; \kappa > 0
\end{aligned}$$

which can be conveniently solved by the existing LMI software.

8.2.9 $\mathcal{H}_\infty$ Filter Design

To facilitate further development, define

$$\widehat{\mathcal{X}}_j = \left[\, \widehat{\Upsilon}^t \; 0\; 0\; 0\; 0\; 0\; 0 \,\right]^t, \;\; \widehat{\Upsilon} \in \Re^{2n\times 2n}$$

Next, we express $\widehat{\Upsilon}$ and $\widehat{\Lambda} = \widehat{\Upsilon}^{-1}$ and other relevant matrices into the convenient form

$$\begin{aligned}
\widehat{\Upsilon} &= \begin{bmatrix} \Upsilon_s & 0 \\ \Upsilon_o & \Upsilon_c \end{bmatrix}, \;\; \widehat{\mathcal{R}} = \begin{bmatrix} \mathcal{R}_1 & 0 \\ \mathcal{R}_2 & \mathcal{R}_3 \end{bmatrix}, \;\; \widehat{\Lambda} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_1 & \Lambda_2 \end{bmatrix} \\
\widehat{\mathcal{S}} &= \begin{bmatrix} \mathcal{S}_1 & 0 \\ \mathcal{S}_2 & \mathcal{S}_3 \end{bmatrix}, \;\; \Psi_k = \begin{bmatrix} \Psi_{1k} & 0 \\ \Psi_{2k} & \Psi_{3k} \end{bmatrix} \\
\mathcal{P}_j &= \begin{bmatrix} \mathcal{P}_{1j} & 0 \\ \mathcal{P}_{2j} & \mathcal{P}_{3j} \end{bmatrix}, \;\; \mathcal{X}_j = \mathcal{P}_j^{-1} = \begin{bmatrix} \mathcal{X}_{1j} & 0 \\ \mathcal{X}_{2j} & \mathcal{X}_{3j} \end{bmatrix}
\end{aligned} \tag{8.69}$$

The following design result is established:

Theorem 8.22 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.69). Switched filtered system (8.34) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $\{\mathcal{X}_{kj}\}_{k=1}^3$, $\{\mathcal{X}_{ks}\}_{k=1}^3$, $\{\mathcal{S}_k\}_1^3$, $\{\mathcal{R}_k\}_1^3$, B_{fj}, $\{\Psi_k\}_1^5$, $\{\mathcal{Y}_{kj}\}_{k=1}^3$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall(j,s) \in \Pi$*

$$\begin{bmatrix} -\Sigma_{1s} & \Sigma_{2j} & \Sigma_{3j} & -\bar{d}\Psi_1 & \widetilde{\Gamma}_j \\ \bullet & -\Sigma_4 & -\Sigma_5 & -\bar{d}\Psi_2 & \Sigma_{7j} \\ \bullet & \bullet & -\Sigma_6 & -\bar{d}\Psi_3 & \Sigma_{8j} \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} < 0, \tag{8.70}$$

$$\begin{aligned}
\Sigma_{1s} &= \begin{bmatrix} \Lambda_1 + \Lambda_1^t + \mathcal{X}_{1s} + \bar{d}\mathcal{S}_1 & \Lambda_1^t \\ \Lambda_1 + \mathcal{X}_{2s} + \bar{d}\mathcal{S}_2 & \Lambda_2 + \Lambda_2^t + \mathcal{X}_{3s} - \bar{d}\mathcal{S}_1 \end{bmatrix} \\
\Sigma_{2j} &= \begin{bmatrix} \Psi_{11} + A_j\Lambda_1^t - \bar{d}\mathcal{S}_1 & A_j\Lambda_1^t \\ \Psi_{21} - \bar{d}\mathcal{S}_2 + \mathcal{Y}_1 & \mathcal{Y}_1 + \mathcal{Y}_2^t + \Psi_{31} - \bar{d}\mathcal{S}_3 \end{bmatrix} \\
\Sigma_{7j} &= \begin{bmatrix} \Lambda_1 G_j^t \\ \Lambda_1 G_j^t - \mathcal{Y}_{4j} \end{bmatrix}, \;\; \Sigma_{8j} = \begin{bmatrix} \Lambda_1 G_{dj}^t \\ \Lambda_1 G_{dj}^t \end{bmatrix} \\
\Sigma_{3j} &= \begin{bmatrix} -\Psi_{11} + A_{dj}\Lambda_1^t & A_{dj}\Lambda_1^t \\ -\Psi_{12} + \mathcal{Y}_3 & -\Psi_{13} + \mathcal{Y}_3 \end{bmatrix} \\
\Sigma_5 &= \begin{bmatrix} \Psi_{21} + \Psi_{21}^t & 0 \\ \Psi_{22} + \Psi_{22}^t & \Psi_{23} + \Psi_{23}^t \end{bmatrix}, \;\; \Sigma_4 = \begin{bmatrix} \Sigma_{41} & 0 \\ \Sigma_{42} & \Sigma_{43} \end{bmatrix} \\
\Sigma_6 &= \begin{bmatrix} \Psi_{31} + \Psi_{31}^t + \mathcal{R}_1 & 0 \\ \Psi_{32} + \Psi_{32}^t + \mathcal{R}_2 & \Psi_{33} + \Psi_{33}^t + \mathcal{R}_3 \end{bmatrix} \\
\Sigma_{41} &= \mathcal{P}_{1s} - \widehat{d}\mathcal{S}_1 - \mathcal{R}_1 - \Psi_{21} - \Psi_{21}^t,
\end{aligned}$$

$$\Sigma_{42} = \mathcal{P}_{2s} - \widehat{d}\mathcal{S}_2 - \mathcal{R}_2 - \Psi_{22} - \Psi_{22}^t,$$
$$\Sigma_{43} = \mathcal{P}_{3s} - \widehat{d}\mathcal{S}_3 - \mathcal{R}_3 - \Psi_{23} - \Psi_{23}^t \quad (8.71)$$

Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \quad (8.72)$$

Proof Applying the congruence transformation

$$\text{diag}[\widehat{\Lambda}, \ \widehat{\Lambda}, \ \widehat{\Lambda}, \ \widehat{\Lambda}, \ I]$$

to inequality (8.58) using (8.57), and (8.59), and the linearizations

$$X_s = \Upsilon^{-t}P_s\Upsilon^{-1}, \ \mathcal{S} = \Upsilon^{-t}\mathcal{W}\Upsilon^{-1}, \ \mathcal{Y}_{j3} = B_{fj}C_{dj}\Lambda_1^t$$
$$X_j = \Upsilon^{-t}P_j\Upsilon^{-1}, \ \mathcal{Y}_{j1} = B_{fj}C_j\Lambda_1^t, \ \mathcal{Y}_{j2} = \Lambda_2 A_{fj}$$
$$\mathcal{Y}_{4j} = \Lambda_2 G_{fj}^t, \ \{\Psi\}_1^5 = \Upsilon^{-t}\{\mathcal{M}\}_1^5\Upsilon^{-1}$$

we immediately obtain LMI (8.70) subject to (8.71). ■

A special design procedure based on the common Lyapunov functional is given below:

Corollary 8.23 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.69). Switched filtered system (8.34) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $\{\mathcal{X}_{kj}\}_{k=1}^3, \ \{\mathcal{S}_k\}_1^3, \ \{\mathcal{R}_k\}_1^3, \ B_{fj}$, $\{\Psi_k\}_1^5, \ \{\mathcal{Y}_{kj}\}_{k=1}^3, \ \forall (j,s) \in \boldsymbol{N}$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall (j) \in \Pi$*

$$\begin{bmatrix} -\Sigma_{1j} & \Sigma_{2j} & \Sigma_{3j} & -\bar{d}\Psi_1 & \widetilde{\Gamma}_j \\ \bullet & -\Sigma_4 & -\Sigma_5 & -\bar{d}\Psi_2 & \Sigma_{7j} \\ \bullet & \bullet & -\Sigma_6 & -\bar{d}\Psi_3 & \Sigma_{8j} \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} < 0 \quad (8.73)$$

$$\Sigma_{1j} = \begin{bmatrix} \Lambda_1 + \Lambda_1^t + \mathcal{X}_{1j} + \bar{d}\mathcal{S}_1 & \Lambda_1^t \\ \Lambda_1 + \mathcal{X}_{2j} + \bar{d}\mathcal{S}_2 & \Lambda_2 + \Lambda_2^t + \mathcal{X}_{3j} - \bar{d}\mathcal{S}_1 \end{bmatrix} \quad (8.74)$$

where $\Sigma_{2j}, ..., \Sigma_{43}$ are given by (8.71). Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \quad (8.75)$$

8.2.10 $\ell_2 - \ell_\infty$ Filter Design

Initially, we recall the following result:

Lemma 8.24 *The matrix inequality*

$$-\mathcal{M} + \mathcal{N}\,\Omega^{-1}\,\mathcal{N}^t \;<\; 0 \tag{8.76}$$

holds for some $0 < \Omega = \Omega^t \in \Re^{n\times n}$, *if and only if*

$$\begin{bmatrix} -\mathcal{M} & \mathcal{N}\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} \;<\; 0 \tag{8.77}$$

holds for some matrices $\mathcal{X} \in \Re^{n\times n}$ *and* $\mathcal{Z} \in \Re^{n\times n}$.

Proof ($\Longrightarrow$) By Schur complements, inequality (8.76) is equivalent to

$$\begin{bmatrix} -\mathcal{M} & \mathcal{N}\Omega^{-1} \\ \bullet & -\Omega^{-1} \end{bmatrix} \;<\; 0 \tag{8.78}$$

Setting $\mathcal{X} = \mathcal{X}^t = \mathcal{Z} = \Omega^{-1}$, we readily obtain inequality (8.77).
($\Longleftarrow$) Since the matrix $[I \quad \mathcal{N}]$ is of full rank, we obtain

$$\begin{aligned} &\begin{bmatrix} I \\ \mathcal{N}^t \end{bmatrix}^t \begin{bmatrix} -\mathcal{M} & \mathcal{N}\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} \begin{bmatrix} I \\ \mathcal{N}^t \end{bmatrix} \;<\; 0 \Longleftrightarrow \\ &-\mathcal{M} + \mathcal{N}\,\mathcal{Z}\,\mathcal{N}^t \;<\; 0 \Longleftrightarrow, \\ &-\mathcal{M} + \mathcal{N}\,\Omega^{-1}\,\mathcal{N}^t \;<\; 0, \quad \mathcal{Z} = \Omega^{-1} \end{aligned} \tag{8.79}$$

which completes the proof. ■

In preparation for the filter design, we use **Lemma 8.24** to introduce relaxation variables and establish the theorem below:

Theorem 8.25 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable with* $\ell_2 - \ell_\infty \;< \gamma_2$ *if there exist matrices* $\{\mathcal{X}\}_{i=1}^N$, $\mathcal{Y}$, $\mathcal{G}$, $\mathcal{F}$ $\forall (i, j, s) \in \Pi$ *and scalars* $\gamma_2 > 0, \varepsilon_j > 0$ *such that the LMIs*

$$\begin{bmatrix} -\mathcal{X}_s & 0 & 0 & \mathcal{G}^t\widetilde{A}_j^t & \bar{d}\mathcal{G}^t\left(\widetilde{A}_j^t - I\right) & \bar{d}\mathcal{F} \\ \bullet & -\mathcal{F} - \mathcal{F}^t + \mathcal{Y} & 0 & \mathcal{G}^t\widetilde{A}_{dj}^t & \bar{d}\mathcal{G}^t\widetilde{A}_{dj}^t & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_i^t & \bar{d}\widetilde{\Gamma}_i^t & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{G} - \mathcal{G}^t + \bar{d}\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{F} - \mathcal{F}^t + \hat{d}\mathcal{Y} \end{bmatrix} < 0 \tag{8.80}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \widetilde{C}_j\mathcal{G} \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\mathcal{G} - \mathcal{G}^t + \mathcal{X}_j \end{bmatrix} < 0 \tag{8.81}$$

have a feasible solution.

Proof Applying the congruent transformations

$$[\mathcal{X}_s,\ I,\ I,\ \mathcal{X}_j,\ I,\ I]$$

to LMI (8.61) and

$$[\mathcal{X}_j,\ I,\ I]$$

to LMI (8.62), respectively, with $\mathcal{X}_i = \mathcal{P}_i^{-1},\ i = j, s$ and Schur complements, it yields

$$\begin{bmatrix} -\mathcal{X}_s & 0 & 0 & \mathcal{X}_s \widetilde{A}_j^t & \bar{d}\left(\widetilde{A}_j^t - I\right) & \bar{d}\mathcal{X}_s \mathcal{Q} \\ \bullet & -\mathcal{Q} & 0 & \widetilde{A}_{dj}^t & \bar{d}\widetilde{A}_{dj}^t & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\bar{d}W^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{Q} \end{bmatrix} < 0 \tag{8.82}$$

$$\begin{bmatrix} -\mathcal{X}_j & 0 & \mathcal{X}_j \widetilde{C}_j^t \\ \bullet & -\varepsilon_j I & \widetilde{C}_{dj}^t \\ \bullet & \bullet & -\gamma_2^2 I \end{bmatrix} < 0 \tag{8.83}$$

When (8.80) and (8.81) hold, it is not difficult to infer that $0 < \mathcal{X}_j < \mathcal{G} + \mathcal{G}^t$. The inequality $(\mathcal{X}_j - \mathcal{G})^t \mathcal{X}_j^{-1} (\mathcal{X}_j - \mathcal{G}) \geq 0$ implies that $-\mathcal{G}^t \mathcal{X}_j^{-1} \mathcal{G} \leq \mathcal{X}_j - (\mathcal{G} + \mathcal{G}^t)$ and in the same way, the inequality $(\mathcal{Y} - \mathcal{F})^t \mathcal{Y}^{-1} (\mathcal{Y} - \mathcal{F}) \geq 0$ implies that $-\mathcal{F}^t \mathcal{Y}^{-1} \mathcal{F} \leq \mathcal{Y} - (\mathcal{F} + \mathcal{F}^t)$. Alternatively, it follows from **Lemma A.2** that there exist matrices $\mathcal{G},\ \mathcal{F},\ \mathcal{Y}_{i=1}^N$ such that LMIs (8.80) and (8.81) are readily obtained. ■

Next, to determine the unknown matrices of the piecewise filter we proceed and define the following matrices

$$\mathcal{X}_k = \begin{bmatrix} \mathcal{X}_{1k} & 0 \\ \mathcal{X}_{2k} & \mathcal{X}_{3k} \end{bmatrix},\ k = s, j,\ \mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 \\ \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix}$$
$$\Psi_k = \begin{bmatrix} \Psi_{1k} & 0 \\ \Psi_{2k} & \Psi_{3k} \end{bmatrix},\ \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 & 0 \\ \mathcal{F}_1 & \mathcal{F}_2 \end{bmatrix} \tag{8.84}$$

and the linearizations

$$\mathcal{D}_{1j} = \mathcal{G}_2^t A_{fj}^t,\ \mathcal{D}_{2j} = \mathcal{G}_1^t C_j^t B_{fj}^t + \mathcal{G}_1^t A_{fj}^t$$

The following design results are established.

Theorem 8.26 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.84). Switched filtered system (8.34) is delay-dependent asymptotically stable with* *an* $\mathcal{L}_2 - \mathcal{L}_\infty < \gamma_2$ *if there exist matrices* $\{\mathcal{X}_{kj}\}_{k=1}^3,\ \{\mathcal{X}_{ks}\}_{k=1}^3,\ \{\mathcal{S}_k\}_1^3,\ \{\mathcal{R}_k\}_1^3,\ B_{fj},$ $\{\Psi_k\}_1^5,\ \{\mathcal{Y}_{kj}\}_{k=1}^3,\ \forall(j,s) \in \boldsymbol{N}$ *and scalars* $\gamma_2 > 0, \varepsilon_j > 0$ *satisfying the LMIs for* $\forall(j,s) \in \Pi$

$$\begin{bmatrix} -\Pi_{1s} & 0 & 0 & \Pi_{2j} & \bar{d}\Pi_{3j} & \bar{d}\mathcal{F} \\ \bullet & -\Pi_4 & 0 & -\Pi_4 & -\bar{d}\Pi_4 & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t & -\bar{d}\widetilde{\Gamma}_j^t & 0 \\ \bullet & \bullet & \bullet & -\Pi_{1j} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Pi_6 \end{bmatrix} < 0 \tag{8.85}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \Pi_7 \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\Pi_8 \end{bmatrix} < 0 \tag{8.86}$$

$$\begin{aligned} \Pi_{1s} &= \begin{bmatrix} \mathcal{X}_{1s} & 0 \\ \mathcal{X}_{2s} & \mathcal{X}_{3s} \end{bmatrix},\ \Pi_{2j} = \begin{bmatrix} \mathcal{G}_1^t A_j^t & \mathcal{D}_{2j} \\ 0 & \mathcal{D}_{1j} \end{bmatrix} \\ \Pi_{3j} &= \begin{bmatrix} \mathcal{G}_1^t \left(A_j^t - I\right) & \mathcal{D}_{2j} \\ 0 & \mathcal{D}_{1j} - \mathcal{G}_2^t \end{bmatrix},\ \Pi_7 = \begin{bmatrix} C_j \mathcal{G}_1 \\ 0 \end{bmatrix} \\ \Pi_4 &= \begin{bmatrix} \mathcal{F}_1 + \mathcal{F}_1^t - \mathcal{Y}_1 & 0 \\ \mathcal{F}_2 + \mathcal{F}_2^t - \mathcal{Y}_2 & \mathcal{F}_3 + \mathcal{F}_3^t - \mathcal{Y}_3 \end{bmatrix} \\ \Pi_5 &= \begin{bmatrix} \mathcal{G}_1 + \mathcal{G}_1^t - \bar{d}\mathcal{Z}_1 & 0 \\ \mathcal{G}_2 + \mathcal{G}_2^t - \bar{d}\mathcal{Z}_2 & \mathcal{G}_3 + \mathcal{G}_3^t - \bar{d}\mathcal{Z}_3 \end{bmatrix} \\ \Pi_6 &= \begin{bmatrix} \mathcal{F}_1 + \mathcal{F}_1^t - \bar{d}\mathcal{Y}_1 & 0 \\ \mathcal{F}_2 + \mathcal{F}_2^t - \bar{d}\mathcal{Y}_2 & \mathcal{F}_3 + \mathcal{F}_3^t - \bar{d}\mathcal{Y}_3 \end{bmatrix} \\ \Pi_8 &= \begin{bmatrix} \mathcal{G}_1 + \mathcal{G}_1^t - \mathcal{X}_{1j} & 0 \\ \mathcal{G}_2 + \mathcal{G}_2^t - \mathcal{X}_{2j} & \mathcal{G}_3 + \mathcal{G}_3^t - \mathcal{G}_{3j} \end{bmatrix} \end{aligned} \tag{8.87}$$

Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2},\quad B_{fj},\quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \tag{8.88}$$

A special design procedure based on a common Lyapunov functional is given below:

Corollary 8.27 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27) and the matrices in (8.84). Switched filtered system (8.34) is delay-dependent asymptotically stable with an* $\ell_2 - \ell_\infty < \gamma_2$ *if there exist matrices* $\{\mathcal{X}_{kj}\}_{k=1}^3,\ \{\mathcal{S}_k\}_1^3,\ \{\mathcal{R}_k\}_1^3,\ B_{fj},$ $\{\Psi_k\}_1^5,\ \{\mathcal{Y}_{kj}\}_{k=1}^3,\ \forall(j,s) \in \boldsymbol{N}$ *and scalars* $\gamma_2 > 0, \varepsilon_j > 0$ *satisfying the LMIs for* $\forall(j,s) \in \Pi$

$$\begin{bmatrix} -\Pi_{1j} & 0 & 0 & \Pi_{2j} & \bar{d}\Pi_{3j} & \bar{d}\mathcal{F} \\ \bullet & -\Pi_4 & 0 & -\Pi_4 & -\bar{d}\Pi_4 & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t & -\bar{d}\widetilde{\Gamma}_j^t & 0 \\ \bullet & \bullet & \bullet & -\Pi_{1j} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Pi_6 \end{bmatrix} < 0 \tag{8.89}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \Pi_7 \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\Pi_8 \end{bmatrix} < 0 \tag{8.90}$$

$$\Pi_{1j} = \begin{bmatrix} \mathcal{X}_{1j} & 0 \\ \mathcal{X}_{2j} & \mathcal{X}_{3j} \end{bmatrix} \tag{8.91}$$

where $\Pi_{2j}, \ldots, \Pi_8$ are given by (8.87). Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \tag{8.92}$$

8.2.11 Illustrative Example B

Consider the following system of the type (8.23), (8.24), and (8.25) where the switching occurs between four modes described by the following coefficients:

$$A_1 = \begin{bmatrix} 0.7 & 0.09 \\ 0 & 0.35 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; \Gamma_1 = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} 0.25 \\ 0.15 \end{bmatrix}, \; G_{d1} = \begin{bmatrix} -0.1 \\ -0.01 \end{bmatrix}, \; \Phi_1 = 0.01$$
$$C_1 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \; C_{d1} = \begin{bmatrix} -0.1 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.41 & 0.11 \\ 0 & 0.97 \end{bmatrix}, \; A_{d3} = \begin{bmatrix} 0 & 0.05 \\ 0 & -0.15 \end{bmatrix}, \; \Phi_2 = 0.02$$
$$G_2 = \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix}, \; G_{d2} = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.2 \\ -0.02 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, \; C_{d2} = \begin{bmatrix} 0 & -0.1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0.6 & 0.02 \\ 0 & 0.49 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} -0.1 & 0.01 \\ -0.1 & -0.1 \end{bmatrix}, \; \Phi_3 = 0.02$$
$$G_3 = \begin{bmatrix} 0.17 \\ 0.19 \end{bmatrix}, \; G_{d3} = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \; \Gamma_3 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \; C_{d3} = \begin{bmatrix} -0.1 & 0 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} -0.33 & 0.22 \\ 0 & -0.45 \end{bmatrix}, \; A_{d4} = \begin{bmatrix} 0 & 0.25 \\ 0 & -0.05 \end{bmatrix}, \; \Phi_4 = 0.02$$

$$G_4 = \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix}, \; G_{d4} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \; \Gamma_4 = \begin{bmatrix} 0.1 \\ -0.02 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 0.3 \; 0.7 \end{bmatrix}, \; C_{d4} = \begin{bmatrix} -0.1 \; 0.1 \end{bmatrix}$$

and the corresponding two sets $\{j = 1 \; if \; y_k < 1\}$ and $\{j = 2 \; if \; y_k \geq 1\}$ respectively.

A computational summary of applying **Theorem 8.22** and **Corollary 8.23** and using the tools of [17], such that the above piecewise system is asymptotically stable is depicted in Table 8.1. The piecewise filter matrices are given by

$$A_{f1} = \begin{bmatrix} -0.8118 & -0.2795 \\ 0.2105 & -0.7467 \end{bmatrix}, \; B_{f1} = \begin{bmatrix} -0.7833 \\ -1.2554 \end{bmatrix}$$
$$G_{f1} = \begin{bmatrix} -1.3024 & -0.1185 \end{bmatrix}$$
$$A_{f2} = \begin{bmatrix} 0.7767 & -0.2665 \\ 0.1905 & -0.6885 \end{bmatrix}, \; B_{f2} = \begin{bmatrix} -0.8452 \\ -1.3725 \end{bmatrix}$$
$$G_{f2} = \begin{bmatrix} -1.4513 & -0.1335 \end{bmatrix}$$
$$A_{f3} = \begin{bmatrix} -0.7467 & -0.2835 \\ 0.2019 & 0.7645 \end{bmatrix}, \; B_{f3} = \begin{bmatrix} -1.3675 \\ -0.9008 \end{bmatrix}$$
$$G_{f3} = \begin{bmatrix} -0.2025 & -1.4366 \end{bmatrix}$$
$$A_{f4} = \begin{bmatrix} 0.8258 & -0.2193 \\ 0.2005 & -0.7534 \end{bmatrix}, \; B_{f4} = \begin{bmatrix} -1.5364 \\ -0.8111 \end{bmatrix}$$
$$G_{f4} = \begin{bmatrix} -1.4448 & -0.2167 \end{bmatrix}$$

The state x and filtered state $\hat{x}$ trajectories using $\mathcal{H}_\infty$-performance are plotted in Figs. 8.6 and 8.7.

It is quite evident the developed piecewise $\mathcal{H}_\infty$ filter gives improved performance.

Turning to the implementation of **Theorem 8.26** and **Corollary 8.27** such that the piecewise discrete-time system under consideration is asymptotically stable, comparison of the feasible results is presented in Table 8.2 and the corresponding state x and filtered state $\hat{x}$ trajectories using $\mathcal{L}_2 - \mathcal{L}_\infty$-performance are plotted in Figs. 8.8 and 8.9.

The foregoing results come in support with the effectiveness of our filtering approach.

Table 8.1 A summary of $\mathcal{H}_\infty$-performance bound: illustrative example B

$\underline{d}$	$\bar{d}$	$The.8.22$	$Coro.8.23$
2	6	2.145	2.335
3	9	2.774	3.021
4	11	3.182	3.664
5	13	3.534	4.875
6	13	3.732	6.438

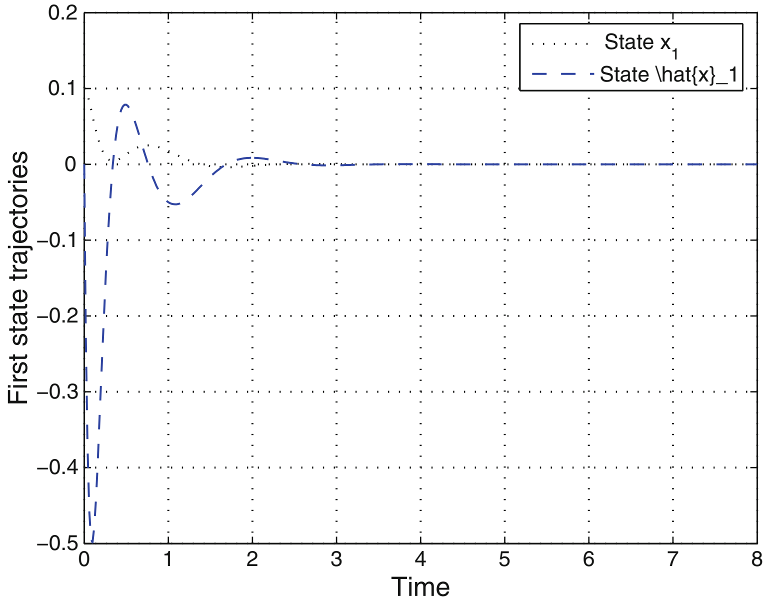

Fig. 8.6 Plot of x_1 and $\hat{x}_1$ versus time: $\mathcal{H}_\infty$ filter

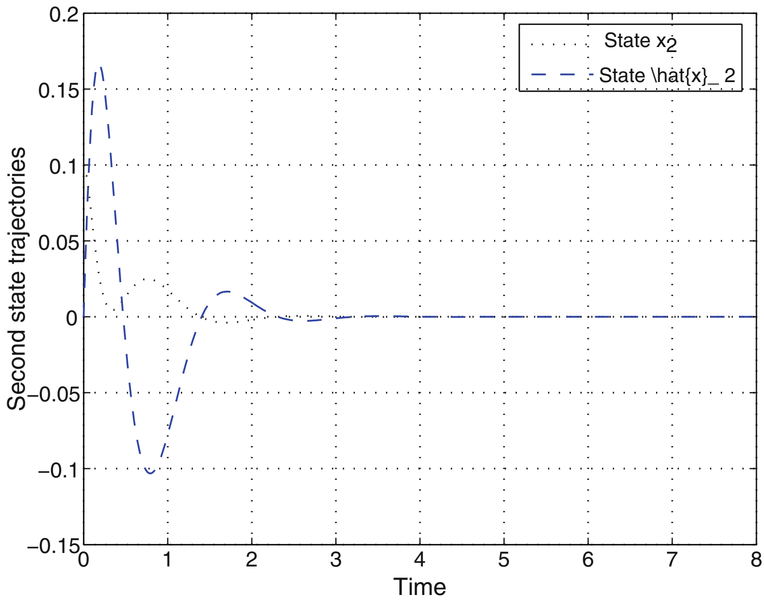

Fig. 8.7 Plot of x_2 and $\hat{x}_2$ versus time: $\mathcal{H}_\infty$ filter

Table 8.2 A summary of $\ell_2 - \ell_\infty$-performance bound: illustrative example B

$\underline{d}$	$\bar{d}$	$The.8.26$	$Coro.8.27$
2	6	3.015	3.532
3	9	3.684	4.021
4	11	5.182	6.224
5	13	6.534	7.694
6	13	6.732	9.015

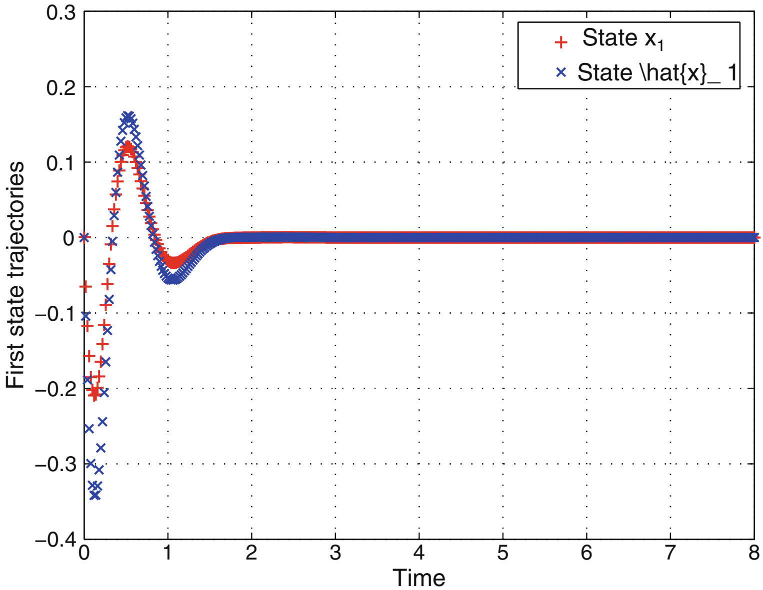

Fig. 8.8 Plot of x_1 and $\hat{x}_1$ versus time: $\mathcal{L}_2 - \mathcal{L}_\infty$ filter

8.2.12 Illustrative Example C

Consider a third-order system of the type (8.23), (8.24), and (8.25) where the switching occurs between two modes described by the following coefficients:

$$Mode\ 1 = y_k \geq 0$$

$$A_1 = \begin{bmatrix} 0 & 0.2 & 0.3 \\ -0.3 & 0 & 0.2 \\ -0.1 & 0.4 & 0 \end{bmatrix}, \quad G_{d1} = \begin{bmatrix} -0.1 \\ 0 \\ -0.01 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -0.3 \\ 0 \\ 0.7 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & -0.2 & 0.4 \\ 0 & 0.2 & -0.3 \\ 0.5 & 0.1 & 0 \end{bmatrix}$$

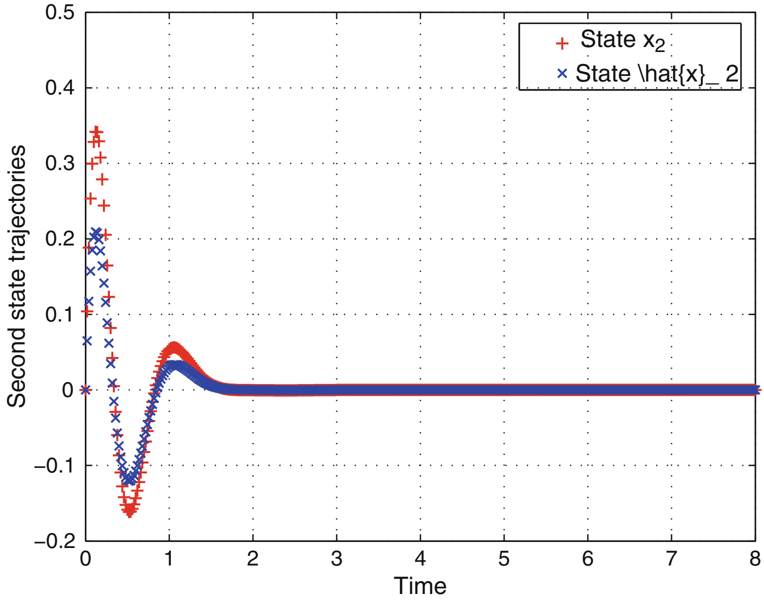

Fig. 8.9 Plot of x_2 and $\hat{x}_2$ versus time: $\mathcal{L}_2 - \mathcal{L}_\infty$ filter

$$C_1 = \begin{bmatrix} 0.8 \\ 0.2 \\ 0.2 \end{bmatrix}, \; C_{d1} = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \; \Gamma_1 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0 \end{bmatrix}$$
$$\Phi_1 = 0.1$$

$$Mode\ 2 = y_k \;\leq\; 0$$
$$A_2 = \begin{bmatrix} 0.3 & 0.2 & 0 \\ 0.3 & 0 & 0.5 \\ 0 & 0.4 & -0.1 \end{bmatrix}, \; G_{d2} = \begin{bmatrix} -0.1 \\ 0.1 \\ 0 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} 0.8 \\ -0.2 \\ 0.3 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.1 & 0.2 & -0.4 \\ 0 & 0.2 & -0.5 \\ 0 & -0.1 & 0.3 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0.7 \\ 0.1 \\ 0.4 \end{bmatrix}, \; C_{d2} = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.1 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0.4 \end{bmatrix}$$
$$\Phi_2 = 0.3$$

and the corresponding two sets $\{j = 1 \; if \;\; y_k < 0\}$ and $\{j = 2 \; if \;\; y_k \geq 0\}$, respectively.

The piecewise filter matrices such that the above piecewise system is asymptotically stable are given by

$$A_{f1} = \begin{bmatrix} -0.6110 & 1.5279 & -1.4688 \\ -1.6789 & 3.6755 & -3.1187 \\ -1.6333 & 4.1765 & -3.8337 \end{bmatrix}$$
$$B_{f1} = \begin{bmatrix} -0.3753 \\ -0.4955 \\ -0.4675 \end{bmatrix}$$
$$G_{f1} = \begin{bmatrix} -2.8874 & -0.8225 & 2.8795 \end{bmatrix}$$
$$A_{f2} = \begin{bmatrix} -1.6022 & 2.7952 & -2.4268 \\ -3.9458 & 5.3355 & -5.1167 \\ -3.6443 & 6.1385 & -34.9837 \end{bmatrix}$$
$$B_{f2} = \begin{bmatrix} -0.8883 \\ -3.0495 \\ -3.0465 \end{bmatrix}$$
$$G_{f2} = \begin{bmatrix} 1.2098 & 0.0224 & -0.2615 \end{bmatrix}$$

In Tables 8.3 and 8.4, computational summaries of applying **Theorem 8.22–Corollary 8.23** for $\mathcal{H}_\infty$-filter and **Theorem 8.26–Corollary 8.27** for $\ell_2 - \ell_\infty$-filter are depicted.

Once again, the foregoing results come in support with the effectiveness of our filtering approach.

Table 8.3 A summary of $\mathcal{H}_\infty$-performance bound: illustrative example C

$\underline{d}$	$\bar{d}$	*The.8.22*	*Coro.8.23*
2	6	0.889	1.035
3	8	0.924	1.044
4	10	0.965	1.067
5	12	0.977	1.095
6	14	0.989	1.105

Table 8.4 A summary of $\mathcal{L}_2 - \mathcal{L}_\infty$-performance bound: illustrative example C

$\underline{d}$	$\bar{d}$	*The.8.26*	*Coro.8.27*
2	6	0.975	1.045
3	8	0.986	1.076
4	10	1.015	1.088
5	12	1.117	1.096
6	14	1.229	1.107

8.3 Notes and References

In this chapter, novel delay-dependent filtering design approaches have been developed for a class of linear piecewise discrete-time systems with convex-bounded parametric uncertainties and time-varying delays appearing in the state as well as the output and measurement channels. The filters have linear full-order structure and guarantee the desired estimation accuracy over the entire uncertainty polytope. We have used switched Lyapunov functionals and introduced some additional instrumental matrix variables to pave the way toward deriving sufficient conditions for the asymptotic stability of the filtering error system.

The desired accuracy has been assessed in terms of either $\mathcal{H}_\infty$-performance or $\ell_2 - \ell_\infty$ criteria. A new parametrization procedure based on a combined Finsler's Lemma and piecewise Lyapunov–Krasovskii functional has been established to yield sufficient conditions for delay-dependent filter feasibility. The filter gains have been subsequently determined by solving a convex optimization problem over LMIs. In comparison to the existing design methods, the developed methodology has been shown to yield the least conservative measures since all previous overdesign limitations are almost eliminated. By means of simulation examples, the advantages of the developed technique have been readily demonstrated.