Magdi S. Mahmoud

Switched Time-Delay Systems

Stability and Control

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Magdi S. Mahmoud
Department of Systems Engineering
King Fahd University of Petroleum and
Minerals (KFUPM)
31261 Dhahran
Saudi Arabia
msmahmoud@kfupm.edu.sa

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In the Name of

Allah

the All-Compassionate, the All-Merciful

*“***My Lord!** *Enhance me in knowledge”*

To the Memory of My Parents
(Source of My Big Dreams)
To My Family: Salwa, Medhat, Monda, Mohamed
(For Their Love and Gratitude)
To My Grandchildren: Malak, Mostafa
(Representative of New Generations)

MsM
Dhahran, Saudi Arabia, 2009

Preface

In many practical applications we deal with a wide class of dynamical systems that are comprised of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between the subsystems. This class of systems is frequently called *switched system*. Switched linear systems provide a framework that bridges the linear systems and the complex and/or uncertain systems. The motivation for investigating this class of systems is twofold: first, it has an inherent multi-modal behavior in the sense that several dynamical subsystems are required to describe their behavior, which might depend on various environmental factors. Second, the methods of intelligent control systems are based on the idea of switching between different controllers. Looked at in this light, switched systems provide an integral framework to deal with complex system behaviors such as chaos and multiple limit cycles and gain more insights into powerful tools such as intelligent control, adaptive control, and robust control. Switched systems have been investigated for a long time in the control and systems literature and have increasingly attracted more attention for the past three decades. The number of journal articles, books, and conference papers have grown exponentially and a number of fundamental concepts and powerful tools have been developed. It has been pointed out that switched systems have been studied from various viewpoints. One viewpoint is that the switching signal is an exogenous variable, and then the problem is to investigate whether there exists a switching signal such that the switched system has the desired performance including stability, certain disturbance attenuation level, and the like. Another viewpoint is that the switching signal is available to system designers and thus it may be used for control purposes. This books aims at integrating the main issues of switched systems in a systematic way.

On the contrary, the existence of transfer phenomena, including material, energy, and information, is an integral part of several physical and man-made systems. In turn, this gives rise to *delay* element and, consequently, the overall system representation would be the delay differential equations (DDEs) as opposed to the conventional ordinary differential equations (ODEs). Over the years, it is recorded that DDEs are used in modeling other phenomena arising in different fields, including biosciences (heredity in population dynamics), chemistry (behaviors in chemical kinetics), economics (dynamics of business cycles), engineering (water quality, hot and cold mills, vibration in cutting machines), to name a few. Time-delay systems

(TDs) have a long-standing history, and early treatment of DDEs dates back to the work of Bernoulli and Condorcet. The development of mathematical theory for TDs however started in the second half of the 20th century by the pioneering work of Myshkis, Krasovakii, Halanay, and Pinney in the frequency domain and Bellman, Cooke, and Hale in the time domain. From a control systems standpoint, delays give rise to stabilizing/destabilizing effects depending on the situation under consideration. By now it is fair to say the fundamental results of the theory of functional differential equations (FDEs), as equivalent to DDEs), are well known and well understood. However, there are increasing number of applications involving large-scale systems that exhibit the *delay* (transport, propagation, communication, decision) as a crucial parameter in the control analysis and design methods. Recent approaches in *robust control* opened interesting perspectives and issues in dealing with delays in dynamical systems, where delays are eventually treated as *uncertainty*.

Since most of the time delays have a crucial impact on the plant performance, the employment of FDEs rather than ODEs in the modeling effort becomes the rule, not the exception. Putting them together, a new class of system configuration readily emerges, which, from now onward, we call *switched time-delay systems (STDS)*. This class possesses the main ingredients of multi-modes of operation, nominally inherent time-delay model and parametric uncertainties and external disturbances. Indeed, this class reflects several important features on the performance analysis and control design and emphasizes the existence of a hybrid system: state-space delay dynamics and switching dynamics.

There are numerous applications that can be cast in the framework of such STDS. Examples include, but not limited to, water quality control, electric power systems, productive manufacturing systems, and cold steel rolling mills. For obvious reasons, STDS can be best represented in the time domain by a hybrid state-space formalism the major part of which is a state-space hereditary model and a switching model forming the remaining part.

Recently, there has been considerable research interest in stability analysis and control design of STDS and satisfactory results have been obtained in the literature. While most of these excellent publications are for specialists and researchers in the field, so far there is no single book in the literature that presents a systematic and structured approach to the modeling, stability, and control of STDS. With this in mind, this book is about stability analysis and control design methodologies for such a new class of systems, STDs. Thus, the primary objective of the book is to present an introductory, yet comprehensive, treatment of STD systems by jointly combining the two fundamental attributes: the system dynamics possesses an inherent time delay and the system operational mode undergoes switching among different modes. Although each attribute has been examined individually in several texts, the integration of both attributes is quite unique and deserves special consideration.

Dhahran, Saudi Arabia Magdi S. Mahmoud

Acknowledgments

The material contained in this book is an outgrowth of my academic research activities over the past several years. The idea of writing the book arose and developed during fall 2006 and it is now revived after joining the KFUPM. In this regard, the overall scientific environment at KFUPM is gratefully recorded.

In writing this volume, I took the approach of referring within the text to papers and/or books which I believed taught me some ideas and methods. I further complement this by adding some observations and notes at the end of each chapter to shed some light on other related results. I apologize in advance in case I committed injustice and assure my colleagues that any mistake was unintentional.

Needless to stress that 'interaction' with people in general and colleagues and friends in particular manifests our life. This is true in my technical career where I benefited from listening, discussing, and collaborating with several colleagues. Foremost, I would like to express my deep gratitude to Prof. Shoukri Selim, who has been a good supporter, but most importantly an example of a true friend. I owe him for his deep insight as well as for his trust and perseverance.

I owe a measure of gratitude to the Deanship of Scientific Research, for providing excellent opportunity of academic research activities through grants and technical support. This book is written based on the research project **IN 090030** and, in this regard, the unfailing assistance of the DSR team is gratefully acknowledged. The continuous encouragements of Dr Omar Al-Turki, Dean of Computer Science and Engineering, and Professor Fouad AL-Sunni, Chairman of the Systems Engineering Department, are wholeheartedly acknowledged. The process of fine tuning and producing the final draft was pursued at the Systems Engineering Department and special thanks must go to my colleagues Professors Moustafa El-Shafie, Hosam E. Emara-Shabaik, and Drs Abdul-Wahed Saif, Smair Al-Amer, and Sami ElFreik for their superb interactions, helpful comments, and assistance throughout my stay at the KFUPM.

This book is intended for graduate students and researchers in robust control theory, serving as both a summary of recent results and a source of new research problems.

For the development of the book, I am immensely pleased for many stimulating and fruitful discussions and helpful suggestions from colleagues, students, and friends throughout my technical career, which have definitely enriched my knowledge and experience. I have had the good fortune to interact and be inspired by conversations with international experts in systems and control theory, including Profs. Basar (Illinois), Khalil (MSU), Mickle and Vogt (Pitt), P. Shi (Glamorgan), Boukas (Polytechnique of Montreal), and Yuanqing Xie (BIT). I am particularly indebted to Professor P. Shi who has helped me focused on the ultimate goals of hybrid systems research. The great support of Alexander Greene, Springer editorial director, Ciara Vincent and the unfailing expert assistance of Vignesh Kumar are gratefully acknowledged.

Most of all, however, I would like to thank all the members of my family and especially my wife Salwa for unfailing encouragements and proofreading parts of the book. Without their constant love, incredible amount of patience, and (mostly) enthusiastic support, this volume would not have been finished.

Dhahran, Saudi Arabia Magdi S. Mahmoud

Notations

Throughout this book, the following terminologies, conventions, and notations have been adopted. All of them are quite standard in the scientific media and only vary in form or character.

I^+	$\triangleq$	the set of positive integers
$\Re$	$\triangleq$	the set of real numbers
$\Re_+$	$\triangleq$	the set of nonnegative real numbers
$\Re^n$	$\triangleq$	the set of all n-dimensional real vectors
$\Re^{n \times m}$	$\triangleq$	the set of $n \times m$-dimensional real matrices
A^t	$\triangleq$	the transpose of matrix A
A^{-1}	$\triangleq$	the inverse of matrix A
I	$\triangleq$	an identity matrix
I_s	$\triangleq$	the identity matrix of dimension $s \times s$
e_j	$\triangleq$	the jth column of matrix I
x^t *or* A^t	$\triangleq$	the transpose of vector x or matrix A
$\lambda(A)$	$\triangleq$	the set of eigenvalues of matrix A (spectrum)
$\varrho(A)$	$\triangleq$	the spectral radius of matrix A
$\lambda_j(A)$	$\triangleq$	the jth eigenvalue of matrix A
$\lambda_m(A)$	$\triangleq$	the minimum eigenvalue of matrix A where $\lambda(A)$ are real
$\lambda_M(A)$	$\triangleq$	the maximum eigenvalue of matrix A where $\lambda(A)$ are real
A^{-1}	$\triangleq$	the inverse of matrix A
$A^{-\dagger}$	$\triangleq$	the Moore–Penrose inverse of matrix A
$P > 0$	$\triangleq$	matrix P is real symmetric and positive definite
$P \geq 0$	$\triangleq$	matrix P is real symmetric and positive semidefinite
$P < 0$	$\triangleq$	matrix P is real symmetric and negative definite
$P \leq 0$	$\triangleq$	matrix P is real symmetric and negative semidefinite

$A(i,j), A_{ij}$	$\triangleq$	the ij-th element of matrix A
$\det(A)$	$\triangleq$	the determinant of matrix A
$\mathrm{trace}(A)$	$\triangleq$	the trace of matrix A
$\mathrm{rank}(A)$	$\triangleq$	the rank of matrix A
$\|a\|$	$\triangleq$	the absolute value of scalar a
$\|\|x\|\|$	$\triangleq$	the Euclidean norm of vector x
$\|\|A\|\|$	$\triangleq$	the induced Euclidean norm of matrix A
$\|\|x\|\|_p$	$\triangleq$	the ℓ_p norm of vector x
$\|\|A\|\|_p$	$\triangleq$	the induced ℓ_p norm of matrix A
$Im(A)$	$\triangleq$	the image of operator/matrix A
$Ker(A)$	$\triangleq$	the kernel of operator/matrix A
$\max \mathbf{D}$	$\triangleq$	the maximum element of set **D**
$\min \mathbf{D}$	$\triangleq$	the minimum element of set **D**
$\sup \mathbf{D}$	$\triangleq$	the smallest number that is larger than or equal to each element of set **D**
$\inf \mathbf{D}$	$\triangleq$	the largest number that is smaller than or equal to each element of set **D**
$\arg\max \mathbf{D}$	$\triangleq$	the index of maximum element of ordered set **S**
$\arg\min \mathbf{D}$	$\triangleq$	the index of minimum element of ordered set **S**
$\mathbf{B}_r$	$\triangleq$	the ball centered at the origin with radius r
$\mathbf{R}_r$	$\triangleq$	the sphere centered at the origin with radius r
$\mathcal{N}$	$\triangleq$	the fixed index set $\{1, 2, ..., N\}$
$[a, b)$	$\triangleq$	the real number set $\{t \in \Re : a \leq t < b\}$
$[a, b]$	$\triangleq$	the real number set $\{t \in \Re : a \leq t \leq b\}$
$\mathbf{S}$	$\triangleq$	the set of modes $\{1, 2, ..., s\}$
iff	$\triangleq$	if and only if
$\mathbf{O}(.)$	$\triangleq$	order of *(.)*
$\mathrm{diag}(\ldots)A$	$\triangleq$	diagonal matrix with given diagonal elements

Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry.

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Part I
Mathematical Tools

Chapter 1
Introduction

Recent years have witnessed an enormous growth of interest in dynamic systems that are characterized by a mixture of both continuous and discrete dynamics. Such systems are commonly found in engineering practice and are referred to as hybrid or switching systems. The widespread application of such systems is motivated by ever-increasing performance requirements, and by the fact that high-performance control systems can be realized by switching between relatively simple LTI systems. However, the potential gain of switched systems is offset by the fact that the switching action introduces behavior in the overall system that is not present in any of the composite subsystems. For example, it can be easily shown that switching between stable subsystems may lead to instability or chaotic behavior of the overall system, or that switching between unstable subsystems may result in a stable overall system. In this book, we closely examine two classes of systems: switched systems (SS) and time-delay systems (TDS), which will eventually pave the way toward studying a new class of systems, switched time-delay systems (STDS).

1.1 Introduction

Motivated by the desire for a high degree of automation and excellent performance capabilities, control system design has been the focal point of extensive research work during the past several decades. Increasingly sophisticated tools from modern control theories have been developed for improved and better tracking performance. Concurrent advances in microprocessor technology have made the implementation of complex nonlinear control algorithms practically feasible. To meet the explosive social demands, contemporary engineering applications and real-life systems are becoming more complex, interconnected, and spatially distributed. By careful consideration of such systems and phenomena, it turns out that they have a distinct property that the future evolution of the systems states is affected by their previous values, this is frequently called the time-delay effect or simply time delay. This effect can be produced from different sources and in some cases it may affect the system behavior and performance and complicate the system analysis. By and large, the delays are perhaps the main causes of instability and poor performance in

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dynamical systems and frequently encountered in various engineering and physical systems [24, 108, 216]. Formally, a system with time delay can be defined as the system in which the future states depend not only on the present but also on the past history of the system [304] and there are many names used in literature for these phenomena, such as system with aftereffect, system with time lag, and hereditary system. In general, such systems are often described by functional differential equation; *a functional equation* is an equation involving an unknown function for different argument values [304]. When this is a differential equation we have a functional differential equation (FDE) or delay differential equation (DDE), where the rate of change of the state in a system model is determined not only by the present state but also by past values. The wide appearance of DDE as a model for several physical and man-made systems is especially important for control systems where actuators, sensors, and transmission lines introduce time delays.

On the contrary, a switched system is a wide class of dynamical systems that are comprised of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between the subsystems. This class of systems has an inherent multi-modal behavior in the sense that several dynamical subsystems are required to describe their behavior that might depend on various environmental factors. Switched systems provide an integral framework to deal with complex system behaviors such as chaos and multiple limit cycles and gain more insights into powerful tools such as intelligent control, adaptive control, and robust control. Switched systems have been investigated for a long time in the control and systems literature and have increasingly attracted more attention for the past three decades.

In the remainder of this chapter, we will review some basic notions of dynamical system representation before providing an organization chart of the book.

1.2 Functional Differential Equations

Let $\mathbf{C}_{n,\tau} = \mathbf{C}([-\tau, 0], \Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the topology of uniform convergence and designate the norm of an element $\phi \in \mathbf{C}_{n,\tau}$ by

$$||\phi||_* \overset{\Delta}{=} \sup_{\theta \in [-\tau, 0]} ||\phi(\theta)||_2 \tag{1.1}$$

If $\alpha \in \Re$, $d \geq 0$ and $x \in \mathbf{C}([\alpha - \tau, \alpha + d], \Re^n)$ then for any $t \in [\alpha, \alpha + d]$, we let $x_t \in \mathbf{C}$ be defined by $x_t(\theta) := x(t + \theta), \ -\tau \leq \theta \leq 0$. If $\mathcal{D} \subset \Re \times \mathbf{C}$, $f : \mathbf{D} \to \Re^n$ is a given function, the relation $\dot{x}(t) = f(t, x_t)$ is a retarded functional differential equation (RFDE) [109] on $\mathbf{D}$ where $x_t(t), t \geq t_o$ denotes the restriction of $x(\cdot)$ to the interval $[t - \tau, t]$ translated to $[-\tau, 0]$. Here, $\tau \geq 0$ is termed the *delay factor*. In the sequel, if $\alpha \in \Re$, $d \geq 0$ and $x \in \mathbf{C}([\alpha - \tau, \alpha + d], \Re^n)$ then for any $t \in [\alpha, \alpha + d]$, we let $x_t \in \mathbf{C}$ be defined by $x_t(\theta) \overset{\Delta}{=} x(t + \theta), \ -\tau \leq \theta \leq 0$. In addition, if $\mathcal{D} \subset \Re \times \mathbf{C}$, $f : \mathbf{D} \to \Re^n$ is given function, then the relation

$$\dot{x}(t) = f(t, x_t) \tag{1.2}$$

is a retarded functional differential equation (RFDE) on $\mathbf{D}$ where $x_t, t \geq t_0$ denotes the restriction of $x(.)$ on the interval $[t-\tau, t]$ translated to $[-\tau, 0]$. A function x is said to be a *solution* of (1.2) on $[\alpha-\tau, \alpha+d]$ if there $\alpha \in \Re$ and $d > 0$ such that

$$x \in \mathbf{C}([\alpha-\tau, \alpha+d], \Re^n), \quad (t, x_t) \in \mathbf{D}, \quad t \in [\alpha, \alpha+d] \tag{1.3}$$

and $x(t)$ satisfies (1.2) for $t \in [\alpha, \alpha+d]$. For a given $\alpha \in \Re,\ \phi \in \mathbf{C},\ x(\alpha, \phi, f)$ is said to be a solution of (1.2) with *initial value* ϕ at α.

In the linear case, the RFDE (1.2) assume the form

$$\dot{x}(t) = A_o x(t) + A_d x(t-\tau), \quad x(\theta) = \phi(\theta),\ -\tau \leq \theta \leq 0 \tag{1.4}$$

We note from [108] that when $\phi(.)$ is continuous then there exists a unique solution $x(\phi)$ defined on $[-\tau, \infty)$ that coincides with ϕ on $[-\tau, 0]$ and satisfies (1.4) for all $t \geq 0$. By the Lagrange's formula, this solution is given by

$$\begin{aligned} x(t) &= \exp^{A_o t}\ x(0) + \int_0^t \exp^{A_o(t-\theta)}\ A_d x(t-\theta)\ \mathrm{d}\theta \\ &= \exp^{A_o t}\ x(0) + \int_{-\tau}^{t-\tau} \exp^{A_o(t-\theta-\tau)}\ A_d x(\theta)\ \mathrm{d}\theta \end{aligned} \tag{1.5}$$

In the case where $\tau \equiv 0$, system (1.4) reduces to

$$\dot{x}(t) = (A_o + A_d) x(t) \tag{1.6}$$

which is asymptotically stable when all the eigenvalues of $(A_o + A_d)$ have negative real parts.

1.3 Piecewise Linear Dynamical Systems

Piecewise linear (PL) systems are naturally due to the presence of a range of system nonlinearities, such as dead zones, saturation, relays, and hysteresis. Indeed, stability properties of system components, especially actuators which are piecewise linear, have been studied for decades. However, in recent times engineers have started to employ control laws that are piecewise linear in nature. Important examples are rule-based control, gain scheduling, and programmable logic control [334]. There has also been a recent interest in what has been termed hybrid systems [99]. Indeed, this term has been used for a wide range of systems, from timed finite state automation to complete integrated factory control and scheduling problems to the extent that some definitions used would encompass the piecewise linear systems. In [334], a computational tool for the analysis of PL dynamical systems was developed. An example of such a system is the ABS (anti-skid braking) system in a car, where the controller is rule-based and designed using the engineer's knowledge of the system. The only current viable approach to testing such a system is by using extensive simulation and prototype testing, which must be repeated for each of the different car models on which it is installed. The work reported in [334] provides useful insight into the logic and dynamic interaction of such a system. Similarly, systems with programmable logic controllers and gain schedulers also fall into the class of piecewise linear systems.

By taking ideas and known results from linear systems, convex set theory, and computational geometry, the work of [334] aims to synthesize an analysis tool for studying a class of systems that mix logic and dynamics.

The attractions of piecewise linear (PL) systems in control can be recognized by representing a PL system as a set of convex polytopes $\Pi_j\ \Re^n$, each containing some linear system of the form

$$\dot{x} = A_m x + b_m, \quad x \in \Pi_m \tag{1.7}$$

where the Π_j form a partition of $\Re^n$ such that

$$\cup\ \Pi_j = \Re^n, \quad \Pi_j \cap \Pi_k = \oslash, \quad j \neq k \tag{1.8}$$

In a geometric setting, the problem has a complex picture of *boxes* stacked together in state space with each box containing a different linear dynamic system. Any global analysis must somehow identify the behaviors in each box and then link them together to form a global picture of the dynamics. Loosely speaking, the associated state space will comprise of $n \times p$ linear regions, where n and p represent the number of states and number of PL functions, respectively. Note that the PL functions of the system would eventually result in switching surfaces in the state space. These surfaces act as the boundaries of the convex polytopes that contain each linear dynamic region. The difficulties presented in analyzing this setup are bound up in the need to manipulate high-dimensional convex polytopes and the dynamic systems within them. One analysis technique, using the phase portrait, fulfills many of the analysis aims. In the phase portrait, PL functions can be represented as lines in the plane and trajectories or isoclines plotted to represent the dynamics. The result is a graphical plot of the system dynamics that gives global stability information and shows how the dynamic patterns change due to the switching lines and hence the PL functions. The major drawback is the limitation of the phase portrait to two states.

In [334], the idea of mapping a piecewise linear system into a connected graph was developed, the idea being based on the phase portrait. Each convex polytope or region in the state space will have dynamics entering and exiting that region. If the boundaries of every region were partitioned into sections containing only dynamics entering a region (termed an *Nface*) and only dynamics exiting a region (termed an *Xface*) then the boundaries can be characterized into sections of homogeneous dynamic behavior. Each section thus identified is then represented as a node of a graph. The connections between nodes are then characterized by tracking the set of trajectories (or trajectory bundle) entering via some *Nface* and identifying which (if any) *Xface* the trajectory bundle leaves that region.

Piecing together the nodes and connections for each region results in a directed graph that captures the global dynamic patterns of the system. The nodes of the graph represent the PL functions and the directed connections represent the interaction of the PL functions with the system's dynamics. As will be explained in the subsequent sections, the realization of this apparently simple idea is not easy.

Piecing together the nodes and connections for each region yields a directed graph that captures the global dynamic patterns of the system. The nodes of the graph represent the PL functions and the directed connections represent the interaction of the PL functions with the system's dynamics.

More about piecewise linear (PL) systems with time delays will be provided later in the book.

1.4 Fundamental Stability Theorems

In this section, we present the fundamental stability theorems that can be used in studying the stability behavior of switched systems and time-delay systems. Further details of these theorems can be found in the classical books [96, 108, 109, 171].

1.4.1 Lyapunov–Razumikhin Theorem

Here the idea is based on the following argument: because the future states of the system depend on the current and past states' values, the Lyapunov function should become functional – more details in Lyapunov–Krasovskii method – which may complicate the condition formulation and the analysis. To avoid using functional, Razumakhin made his theorem, which is based on formulating Lyapunov functions, not functionals. First, one should build a Lyapunov function $V(x(t))$, which is zero when $x(t) = 0$ and positive otherwise, then the theorem does not require $\dot{V} < 0$ always but only when the $V(x(t))$ for the current state becomes equal to $\bar{V}$, which is given by

$$\bar{V} = \max_{\theta \in [-\tau, 0]} V(x(t + \theta)) \tag{1.9}$$

The theorem statement is given by [105]:

suppose f is a functional that takes time t and initial values x_t and gives a vector of n states $\dot{x}$ and u, v, w are class $\mathcal{K}$ functions, $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. If there exists a continuously differentiable function $V : \Re \times \Re^n \rightarrow R$ such that

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|) \tag{1.10}$$

and the time derivative of V along the solution $x(t)$ satisfies $\dot{V}(t,x) \le -w(\|x\|)$ whenever $V(t+\theta, x(t+\theta)) \le V(t, x(t))\theta \in [-\tau, 0]$, then the system is uniformly stable.

If, in addition, $w(s) > 0$ for $s > 0$ and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that

$$\dot{V}(t,x) \le -w(\|x\|) \text{ whenever } V(t+\theta, x(t+\theta)) \le p(V(t, x(t)))$$

for $\theta \in [-\tau, 0]$, then the system is uniformly asymptotically stable. If in addition $\lim\ u(s)_{s\to\infty} = \infty$ then the system is globally asymptotically stable.

The argument behind the theorem is like this: $\bar{V}$ is serving as a measure for the V in the interval $t-\tau$ to t then if $V(x(t))$ is less than $\bar{V}$ then it is not necessary that $\dot{V} < 0$, but if $V(x(t))$ becomes equal to $\bar{V}$ then $\dot{V}$ should be < 0 such that V will not grow.

The procedure can be explained by the following discussion: consider a system and a selected Lyapunov function $V(x)$, which is positive semidefinite. By taking the time derivative of this Lyapunov function we get $\dot{V}$. According to the Razumikhin theorem this term does not always need to be negative, but if we add the following term $a(V(x) - V(x_t))\ \ a > 0$ to $\dot{V}$, then the term

$$\dot{V} + a(V(x) - V(x_t)) \tag{1.11}$$

should always be negative. Then by looking at this term we find that this condition is satisfied if $\dot{V} < 0$ and $V(x) \le V(x_t)$, meaning that the system states are not growing in magnitude and it is approaching the origin (stable system) or $a(V(x) < V(x_t))$ and $\dot{V} > 0$ but $\dot{V} < |a(V(x) - V(x_t))|$ then although $\dot{V}$ is positive and the states increasing, the Lyapunov function is limited by an upper bound and it will not grow without limit. The third case is that both of them are negative and it is clear that it is stable. This condition insures uniform stability, meaning that the states may not reach the origin but it is contained in a domain, say ε which obeys the primary definition of stability. To extend this theorem for asymptotic stability, we can consider adding the term $p(V(x(t))) - V(x_t)$, where $p(.)$ is a function that has the following characteristics:

$$p(s) > s$$

and then the condition becomes

$$\dot{V} + a(p(V(x(t))) - V(x_t)) < 0,\ a > 0 \tag{1.12}$$

By this, when the system reaches some value, which makes $p(V(x(t))) = V(x_t)$, requires $\dot{V}$ to be negative but at this instant $V(x(t) < V(x_t)$ then in the coming τ interval the $V(x)$ will never reach $V(x_t)$ and the maximum value in this interval is the new $V(x_t)$, which is less than the previous value, and with time the function keeps decreasing until the states reach the origin.

1.4.2 Lyapunov–Krasovskii Theorem

The Razumikhin theorem attempts to construct the Lyapunov function while the Lyapunov–Krasovskii theorem uses functionals because V, which can be considered as an indicator for the internal power in the system, is function of x_t, then it is logical to consider V, which is a function of function and hence a functional. The terms of $V(x_t)$ should contain terms for the x in the interval $(t-\tau)$ to t and $\dot{V}$ should be < 0 to ensure asymptotic stability. This method will be covered in more detail in the next section.

In many cases, the Lyapunov–Razumikhin theorem can be found as a special case of Lyapunov–Krasovskii, theorem which makes the former more conservative. The Lyapunov–Krasovskii method tries to build a Lyapunov functional, which is function in x_t, and the time derivative of this Lyapunov function should be negative for the system to be stable. Previously there were criticism on the Lyapunov–Krasovskii method that it can be used for systems with the third category of delay mentioned in Section 2.2.2 only when $\dot{\tau} \le \mu \le 1$ [338], but the recent results resolve this problem as we see in the next chapter. Another criticism is that the Krasovskii methods cannot deal with delay in the second category and also the recent results in this method succeed to include this case [153, 155, 168–170]. The remaining advantage of the Razumikhin method is its simplicity, but the Krasovskii method proved to give less conservative results, the object of interest of most of the researchers in the recent years. Before going to the theorem we have to define the following notations

$$\begin{aligned} \phi &= x_t \\ \|\phi\|_c &= \max_{\theta\in[-\tau,0]} x(t+\theta) \end{aligned} \tag{1.13}$$

Lyapunov–Krasovskii theorem statement [105]:

Suppose f is a functional that takes time t and initial values x_t and gives a vector of n states $\dot{x}$ and u, v, w are class $\mathcal{K}$ functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. If there exists a continuously differentiable function $V : R \times R_n \to R$ such that

$$u(\|\phi\|) \le V(t, x) \le v(\|\phi\|_c) \tag{1.14}$$

and the time derivative of V along the solution $x(t)$ satisfies

$$\dot{V}(t, x) \le -w(\|\phi\|) \text{ for } \theta \in [-\tau, 0]$$

then the system is uniformly stable. If in addition $w(s) > 0$ for $s > 0$ then the system is uniformly asymptotically stable. If in addition lim $u(s)_{s\to\infty} = \infty$ then the system is globally asymptotically stable.

It is clear that V is a functional and $\dot{V}$ should always be negative.

When considering a special class of systems that considers the case of linear time invariant system with multiple discrete time delay, which is given by [167]

$$\dot{x}(t) = A_o x(t) + \sum_{j=1}^{m} A_j x(t - h_j) \tag{1.15}$$

where h_j $j = 1, 2, ..., m$ are constants then this case is a simplified case, and in spite of that the Lyapunov–Krasovskii functional that gives a necessary and sufficient condition for the system stability is given by

$$\begin{aligned} V(x_t) = {} & x'(t)U(0)x(t) \\ & + \sum_{k=1}^{m} \sum_{k=1}^{m} x'(t+\theta_2)A_k' \times \int_0^{-h_k} U(\theta_1 + \theta_2 + h_k - h_j) \\ & \times A_j x(t+\theta_1) d\theta_1 d\theta_2 \\ & + \sum_{m}^{k=1} \int_0^{-h_k} x'(t+\theta)[(h_k+\theta)R_k + W_k]x(t+\theta)d\theta \end{aligned} \tag{1.16}$$

where W_0; W_1; ...; W_m; R_1, R_2; ...; R_m are positive definite matrices and U is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) = U(\tau)A_0 + \sum_{k=1}^{m} U(\tau - h_k)A_k \qquad \tau \in [0, \max_k(h_k)] \tag{1.17}$$

This theorem were found by trying to imitate the situation of delay-free systems by finding the state transition matrix and then using it to find P that makes

$$x'(t)(PA + A'P)x(t) = -Q, \quad Q > 0, \quad P > 0$$

This Lyapunov functional gives a necessary and sufficient condition for the system stability, but finding the U for this equation is very difficult “and involves solving algebraic ordinary and partial differential equations with appropriate boundary conditions which is obviously unpromising” [105]. Even if we can find this U, the resulting functional leads to a complicated system of partial differential equations yielding infinite dimension LMI. Thus, many authors considered special forms of it and thus derived simpler but more conservative, sufficient conditions, which can be represented by an appropriate set of LMIs.

This is the case for LTI system with a fixed time delay, then considering time varying delay or a generally nonlinear system makes it more difficult. But looking at these terms one can have an idea about the possible terms that can be used in the simplified functional.

1.4.3 Halanay Theorem

The following fundamental result plays an important role in the stability analysis of time-delay systems. Suppose the constant scalars k_1 and k_2 satisfy $k_1 > k_2 > 0$ and $y(t)$ is a nonnegative continuous function on $[t_o - \tau, t_o]$ satisfying

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \;\leq\; -\,k_1\; y(t) + k_2\; \bar{y}(t) \tag{1.18}$$

for $t \geq t_o,$ where $\tau \geq 0$ and

$$\bar{y}(t) = \sup_{t-\tau \leq s \leq t} \{y(s)\}$$

Then, for $t \geq t_o,$ we have

$$y(t) \;\leq\; \bar{y}(t_o)\; \exp(-\sigma(t - t_o))$$

where $\sigma > 0$ is the unique solution of the following equation

$$\sigma = k_1 - k_2 \exp(\sigma\tau)$$

It must be emphasized that the Lyapunov–Krasovskii theorem, Lyapunov–Razumikhin theorem, and Halanay theorem can be effectively used to derive stability conditions when the time delay is time varying and continuous, but not necessarily differentiable. Experience and the available literature show that the Lyapunov–Krasovskii theorem is more usable particularly for obtaining delay-dependent stability and stabilization conditions.

In this book we are going to adopt the use of a simplified sufficient condition Lyapunov–Krasovskii method for continuous-time as well as discrete-time nominally linear system, with single time-varying delay. Of course, the general case is to consider

- nonlinear system
- distributed delay.

When one looks at the real application, it is found that dealing with a nonlinear system cannot give a general result because every family of nonlinear systems has its own characteristics, so trying to build a method of a nonlinear system is not useful, in addition to the difficulties of dealing with a nonlinear system even in delay-free systems. The general practice is to linearize around some operating point and to use the linearized model and treat the nonlinearities as perturbations. In spite of this, the proposed method in Chapter 5 can be used for some families of nonlinear system, which are and not necessarily coming from a linearized mode. Regarding the distributed delay, again the difficulties in obtaining a good result in this field prevent one from selecting this direction in addition to the fact that many systems not only have discrete delay but also there are techniques to approximate [338] or even transform [140] the distributed delay into discrete delays, and the problem becomes of multiple discrete time-delay types, but as we will see in Chapter 3 if the Lyapunov functional is selected properly then a theorem made for single delay can be easily extended to multiple delays. The reason behind selecting time-varying delay is that it can cover a large class of systems and it can also be modified to cover fixed delay.

1.5 Outline of the Book

Toward our goal, this book has been carefully tailored to

(i) give a comprehensive study of STD modeling and dynamics,
(ii) present theoretical explorations on several fundamental problems for switched time-delay systems, and
(iii) provide systematic approaches for switching design and feedback control by integrating fresh concepts and the state-of-the-art results to the distinct theories on switched systems and time-delay systems.

Essentially, a basic theoretical framework is formed toward a switched time-delay theory, which not only extends the theory of time-delay systems, but also applies to more realistic problems.

In dealing with STDS, we follow a systematic modeling approach in that a convenient representation of the system state would be by observing a finite-dimensional vector at a particular instant of time and then examining the subsequent behavior to arrive at the dynamical relations. Looked at in this light, the primary objective of this book is to present an introductory, yet comprehensive, treatment of STDS by jointly combining the two fundamental attributes: the system dynamics possesses an inherent time delay and the system behavior is managed by a switching signal. Although each attribute has been examined individually in several texts, the integration of both attributes is quite unique and deserves special consideration. Additionally, STDS are nowadays receiving increasing attention by numerous investigators as evidenced by the number of articles appearing in journals and conference proceedings.

1.5.1 Methodology

Throughout the monograph, our methodology in each Chapter/section is composed of five steps:

- *Mathematical Modeling*
  in which we discuss the main ingredients of the state-space model under consideration.

- *Definitions and/or Assumptions*
  here we state the definitions and/or constraints on the model variables to pave the way for subsequent analysis.
- *Analysis and Examples*
  this signifies the core of the respective sections and subsections, which contains some solved examples for illustration.
- *Results*
  which are provided most of the time in the form of theorems, lemmas, and corollaries.
- *Remarks*
  which are given to shed some light of the relevance of the developed results vis-a-vis published work.

In the sequel, theorems (lemmas, corollaries) are keyed to chapters and stated in *italic* font with **bold titles**, for example, **Theorem 3.4** means Theorem 4 in Chapter 3 and so on. For convenience, we have grouped the reference in one major bibliography cited toward the end of the book. Relevant notes and research issues are offered at the end of each chapter for the purpose of stimulating the reader.

We hope that this way of articulating the information will attract the attention of a wide spectrum of readership.

1.5.2 Chapter Organization

Switched linear systems have been investigated for a long time in the control literature and have attracted increasingly more attention for more than two decades. The literature grew progressively and quite a number of fundamental concepts and powerful tools have been developed from various disciplines. Despite the rapid progress made so far, many fundamental problems are still either unexplored or less well understood. In particular, there still lacks a unified framework that can cope with the core issues in a systematic way. This motivated us to write the current monograph. The book presents theoretical explorations on several fundamental problems for switched linear systems. By integrating fresh concepts and the state-of the-art results to form a systematic approach for the switching design and feedback control, a basic theoretical framework is formed toward a switched system theory, which not only extends the theory of linear systems, but also applies to more realistic problems.

The book is primarily intended for researchers and engineers in the system and control community. It can also serve as complementary reading for linear/nonlinear system theory at the postgraduate level.

The book is divided into six parts:

Part I covers the mathematical ingredients needed for switching systems and time-delay systems and comprised of two chapters: Chapter 1 introduces the system description and motivation of the study and presents several analytical tools and stability theories that serve as the main vehicle throughout the book. Chapter 2 reviews some basic elements of mathematical analysis, calculus and algebra of matrices to build up the foundations for the remaining topics of stability, stabilization, control, and filtering of switched time-delay systems.

Part II treats switched stability and consists of three chapters: Chapter 3 establishes an overview of the recent progress of time-delay systems and presents a comprehensive picture about the contemporary results and methods. Chapter 4 gives a general framework of switched systems and addresses the main concepts and ideas. Chapter 5 draws the picture of switched time-delay systems with emphasis on the major properties.

Part III deals with switching stabilization and feedback control and contains two chapters: Chapter 6 includes delay-dependent switched stabilization techniques using different switching strategies and Chapter 7 gives different delay-dependent switched feedback techniques and compares among their merits, features, and computational requirements.

Part IV focuses on switched filtering and summarizes the results in two chapters: Chapter 8 is devoted to switched systems and the corresponding methods for switched time-delay systems are presented in Chapter 9. In both chapters, the design of Kalman, $\mathcal{H}_\infty$, and $\mathcal{H}_2$ filters are presented.

Part V treats switched interconnected systems by concentrating on switching decentralized control in Chapter 10. In this chapter, pertinent materials are selected and presented in a unified way.

Part VI provides applications of switched time-delay systems in terms of water-quality studies and control policies in streams as the subject of Chapter 11. Multirate control is presented in Chapter 13.

An appendix containing some relevant mathematical lemmas and basic algebraic inequalities is provided at the end of the book.

We selected the arrangement of references to be in alphabetical order for the purpose of convenience and easy tracking.

Throughout the book and seeking computational convenience, all the developed results are cast in the format of a family of LMIs. In writing up the different topics, emphasis is primarily placed on the major developments attained thus far and then reference is made to other related work.

In summary, this book covers the analysis and design for switched time-delay systems supplemented with rigorous proofs of closed-loop stability properties and simulation studies. The material contained in this book is not only organized to focus on the new developments in the analysis and control methodologies for such STD systems, but it also integrates the impact of the delay factor on important issues such as delay-dependent stability and control design. After an introductory chapter, it is intended to split the book into self-contained chapters with each chapter being equipped with illustrative examples, problems, and questions. The book will be supplemented by an extended bibliography, appropriate appendices, and indexes. It is planned while organizing the material that this book would be appropriate for use either as a graduate-level textbook in applied mathematics as well as different engineering disciplines (electrical, mechanical, civil, chemical, systems), a good volume for independent study, or a suitable reference for graduate students, practicing engineers, interested readers, and researchers from a wide spectrum of engineering disciplines, science, and mathematics.

Chapter 2
Mathematical Foundations

This chapter contains a collection of useful mathematical concepts and tools, which are useful, directly or indirectly, for the subsequent development to be covered in the main portion of the book. While much of the material is standard and can be found in classical textbooks, we also present a number of useful items that are not commonly found elsewhere. Essentially, this chapter serves as a brief overview and as a convenient reference when necessary.

2.1 Introduction

Hybrid systems are certainly pervasive today. Recently, we have witnessed a resurgence in examining quantization effects and a heightened interest in analog computation. There has also been recent progress in analyzing switched, hierarchical, and discretely controlled continuous-variable systems. It is time to focus on developing formal modeling, analysis, and control methodologies for *hybrid systems*. Therefore, hybrid systems research [357–359] is devoted to modeling, design, and validation of interacting systems of continuous process and computer programs. Therefore, the identifying characteristic of hybrid systems is that they incorporate both continuous components, usually called plants, which are governed by ordinary or functional differential equations, and also digital components such as digital computers, sensors, and actuators controlled by programs. Moreover, the growing demands for control systems that are capable of controlling complex nonlinear continuous plants with discrete intelligent controllers can be addressed by the method of hybrid systems.

Throughout this book, by a switched system we mean a class of hybrid dynamical systems consisting of a family of continuous-time subsystems and a rule that orchestrates the switching between them. An integral part of this book surveys recent developments in three basic problems regarding stability and design of switched systems. These problems are:

- stability for arbitrary switching sequences,
- stability for certain useful classes of switching sequences, and
- construction of stabilizing switching sequences.

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We also provide motivation for studying these problems within the framework of time-delay systems. In practice, many systems encountered exhibit switching between several subsystems (are inherently multimodal) that is dependent on various environmental factors. Another source of motivation for studying switched systems comes from the rapidly developing area of switching control. Control techniques based on switching between different controllers have been applied extensively in recent years, particularly in the adaptive context, where they have been shown to achieve stability and improve transient response. The importance of such control methods also stems in part from the existence of systems that cannot be asymptotically stabilized by a single continuous feedback control law. Additionally, the fact that some of intelligent control methods are based on the idea of switching between different controllers. The existence of systems that cannot be asymptotically stabilized by a single static continuous feedback controller [47] also motivates the study. A survey of basic problems in stability and design of switched systems is given in [193].

In this book, we treat switched systems as a class of hybrid systems consisting of a family of subsystems and a switching law that specifies which subsystem will be activated along the system trajectory at each instant of time. Switched systems deserve investigation for theoretical development as well as for practical applications. To switch between different system structures is an essential feature of many control systems, for example, in power systems and power electronics [47]. There have been many studies for switched systems without uncertainties, primarily on stability analysis and design [358]. But for robust stability analysis of uncertain switched systems, there has been comparatively little work. A notable exception is the study of quadratic stability and stabilization by state-based feedback for both continuous-time and discrete-time switched linear systems composed of polytopic uncertain systems in [357]. For performance analysis of switched systems, authors of [357] investigated the disturbance attenuation properties of time-controlled switched systems consisting of several linear time invariant subsystems by using an average dwell-time approach incorporated with a piecewise Lyapunov function. Reference [133] computed the $\mathcal{L}_2$-induced norm of a switched linear system when the interval between consecutive switching is large. However, uncertainty is not considered in these two papers although it is ubiquitous in the system model due to the complexity of the system itself, exogenous disturbance, measurement errors, and so on. During the past decade, there have also been many papers concerning robust (or quadratic) stability, stabilization, and robust $\mathcal{H}_\infty$ control of uncertain systems without switchings [331, 441].

2.2 Basic Mathematical Concepts

Let $x_j,\ y_j,\ j = 1, 2, \ldots, n \in \Re(or\ \mathbf{C})$. Then the n-dimensional vectors $x,\ y$ are defined by $x = [x_1\ x_2\ \ldots\ x_n]^t,\ y = [y_1\ y_2\ \ldots\ y_n]^t\ \in \Re^n$, respectively.

A nonempty set $\mathcal{X}$ of elements $x, \; y, \ldots$ is called the *real* (*or complex*) *vector space* (*or real* (*complex*) *linear space*) by defining two algebraic operations, *vector additions and scalar multiplication, in* $x = [x_1, \; x_2, \; \ldots, \; x_n]^t$ [46]

2.2.1 Euclidean Space

The n-dimensional Euclidean space, denoted in the sequel by $\Re^n$ is the linear vector space $\Re^n$ equipped by the inner product

$$\langle x, y \rangle = x^t \; y = \sum_{j=1}^{n} x_j y_j$$

Let $\mathcal{X}$ be a linear space over the *field* **F** (typically **F** is the field of real numbers $\Re$ or complex numbers **C**). Then a function

$$||.|| : \mathcal{X} \rightarrow \Re$$

that maps $\mathcal{X}$ into the real numbers $\Re$ is a norm on $\mathcal{X}$ iff

1. $||x|| \geq 0, \;\; \forall x \in \mathcal{X}$ (nonnegativity)
2. $||x|| = 0, \;\; \Longleftrightarrow x = 0$ (positive definiteness)
3. $||\alpha \; x|| = |\alpha| ||x|| \forall x \in \mathcal{X}$ (homogeneity with respect to $|\alpha|$)
4. $||x + y|| \leq ||x|| + ||y||, \quad \forall x, y \in \mathcal{X}$ (triangle inequality)

Given a linear space $\mathcal{X}$, there are many possible norms on it. For a given norm $||.||$ *on* $\mathcal{X}$, the pair $(\mathcal{X}, \; ||.||)$ is used to indicate $\mathcal{X}$ endowed with the norm $||.||$.

2.2.2 Norms of Vectors

The class of L_p-norms is defined by

$$||x||_p = \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p}, \quad f \text{ or } 1 \leq p < \infty$$

$$||x||_\infty = \max 1 \leq j \leq n \; |x_j|$$

The three most commonly used norms are $||x||_1$, $||x||_2$, and $||x||_\infty$. All p-norms are equivalent in the sense that if $||x||_{p1}$ and $||x||_{p2}$ are two different p-norms, then there exist positive constants c_1 and c_s such that

$$c_1 \; ||x||_{p1} \;\leq\; ||x||_{p2} \; c_2 \; ||x||_{p1}, \qquad \forall x \in \Re^n$$

2.2.2.1 Induced Norms of Matrices

For a matrix $A \in \Re^{n\times n}$, the *induced p-norm* of A is defined by

$$||A||_p \stackrel{\Delta}{=} \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} = \sup_{||x||_p=1} \; ||Ax||_p$$

Obviously, for matrices $A \in \Re^{m\times n}$ and $A \in \Re^{n\times r}$, we have *the triangle inequality*:

$$||A+B||_p \;\leq\; ||A|||_p + ||B||_p$$

It is easy to show that the *induced norms* are also equivalent in the same sense as for the vector norms, and satisfying

$$||AB||_p \;\leq\; ||Ax||_p \; ||B||_p, \quad \forall A \in \Re^{n\times m}, \; B \in \Re^{m\times r}$$

which is known as the *submultiplicative property*. For $p = 1, 2, \ldots \infty$, we have the corresponding induced norms as follows:

$$\begin{aligned} ||A||_1 &= \max_j \sum_{s=1}^{n} \; |a_{sj}|, \quad \text{(column sum)} \\ ||A||_2 &= \max_j \sqrt{\lambda_j(A^t A)} \\ ||A||_\infty &= \max_s \sum_{j=1}^{n} \; |a_{sj}|, \quad \text{(row sum)} \end{aligned}$$

2.2.3 Convex Sets

A set $\mathbf{S} \subset \Re^n$ is said to be *open* if every vector $x \in \mathbf{S}$, there is an ϵ-neighborhood of x

$$\mathcal{N}(x,\epsilon) = \{z \in \; \Re^n |||z - x|| \; < \; \epsilon\}$$

such that $\mathcal{N}(x,\epsilon) \subset \mathbf{S}$.

A set is *closed* iff its complement in $\Re^n$ is open; *bounded* if there is $r > 0$ such that $||x|| < r, \ \forall x \in \mathbf{S}$; and *compact* if it is closed and bounded; *convex* if for every $x, y \in \mathbf{S}$, and every real number α, $0 < \alpha < 1$, the point $\alpha \ x + (1-\alpha) \ x \in \mathbf{S}$.

A set $\mathbf{K} \subset \Re^n$ is said to be *convex* if for any two vectors x and y in $\mathbf{K}$ any vector of the form $(1-\lambda)x + \lambda y$ is also in $\mathbf{K}$, where $0 \leq \lambda \leq 1$. This simply means that given two points in a convex set, the line segment between them is also in the set. Note, in particular, that subspaces and linear varieties (a linear variety is a translation of linear subspaces) are convex. Also the empty set is considered convex. The following facts provide important properties for convex sets .

1. Let $\mathcal{C}_j, \ j = 1, \ldots, m$ be a family of m convex sets in $\Re^n$. Then the intersection $\mathcal{C}_1 \cap \mathcal{C}_2 \cap ... \cap \mathcal{C}_m$.
2. Let $\mathcal{C}$ be a convex set in $\Re^n$ and $x_o \in \Re^n$. Then the set $\{x_o + x : x \in \mathcal{C}\}$ is convex.
3. A set $\mathbf{K} \subset \Re^n$ is said to be *convex cone* with vertex x_o if $\mathbf{K}$ is convex, and $x \in \mathbf{K}$ implies that $x_o + \lambda x \in \mathbf{K}$ for any $\lambda \geq 0$.

An important class of convex cones is the one defined by the positive semidefinite ordering of matrices, that is, $A_1 \geq A_2 \geq A_3$. Let $P \in \Re^{n \times n}$ be a positive semidefinite matrix. The set of matrices $X \in \Re^{n \times n}$, such that $X \geq P$ is a convex cone in $\Re^{n \times n}$.

2.2.4 Continuous Functions

A function $f : \Re^n \longrightarrow \Re^m$ is said to be *continuous* at a point x if $f(x + \delta x) \longrightarrow f(x)$ whenever $\delta x \longrightarrow 0$. Equivalently, f is continuous at x if, given $\epsilon > 0$, there is $\delta > 0$ such that

$$||x - y|| < \epsilon \implies ||f(x) - f(y)|| < \epsilon$$

A function f is continuous on a set of $\mathbf{S}$ if it is continuous at every point of $\mathbf{S}$, and it is uniformly continuous on $\mathbf{S}$ if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ (dependent only on ϵ), such that the inequality holds for all $x, y \in \mathbf{S}$

A function $f : \Re \longrightarrow \Re$ is said to be *differentiable* at a point x if the limit

$$\dot{f}(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

exists. A function $f : \Re^n \longrightarrow \Re^m$ is *continuously differentiable* at a point x (a set $\mathbf{S}$) if the partial derivatives $\partial f_j / \partial x_s$ exist and continuous at x (at every point of $\mathbf{S}$) for $1 \leq j \leq m, \ 1 \leq s \leq n$ and the *Jacobian matrix* is defined as

$$\mathbf{J} = \left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{bmatrix} \in \Re^{m \times n}$$

2.2.5 Function Norms

Let $f(t) : \Re_+ \longrightarrow \Re$ be a continuous function or piecewise continuous function. The p-norm of f is defined by

$$||f||_p = \left(\int_0^\infty |f(t)|^p \, \mathrm{d}t \right)^{1/p}, \quad f \text{ or } p \in [1, \infty)$$
$$||f||_\infty = \sup \; t \in [0, \infty)|f(t)|, \quad f \text{ or } p = \infty$$

By letting $p = 1, 2, \infty,$ the corresponding normed spaces are called $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_\infty}$, respectively. More precisely, let $f(t)$ be a function on $[0, \infty)$ of the signal spaces, they are defined as

$$\mathbf{L_1} \stackrel{\Delta}{=} \left\{ f(t) : \Re_+ \longrightarrow \Re | ||f||_1 = \int_0^\infty |f(t)| \, \mathrm{d}t < \infty, \text{ convolution kernel} \right\}$$
$$\mathbf{L_2} \stackrel{\Delta}{=} \left\{ f(t) : \Re_+ \longrightarrow \Re | ||f||_2 = \int_0^\infty |f(t)|^2 \, \mathrm{d}t < \infty, \text{ finite energy} \right\}$$
$$\mathbf{L_\infty} \stackrel{\Delta}{=} \left\{ f(t) : \Re_+ \longrightarrow \Re | ||f||_\infty = \sup_{t \in [0,\infty)} |f(t)| < \infty, \text{ bounded signal} \right\}$$

From a signal point of view, the 1-*norm*, $||x||_1$ of the signal $x(t)$ is the integral of its absolute value, the square $||x||_2^2$ of the *2-norm* is often called the energy of the signal $x(t)$, and the ∞-*norm* is its absolute maximum amplitude or peak value. It must be emphasized that the definitions of the norms for vector functions are not unique.

In the case of $f(t) : \Re_+ \longrightarrow \Re^n, \; f(t) = [f_1(t) \quad f_2(t) \ldots f_n(t)]^t$ which denote a continuous function or piecewise continuous vector function, the corresponding p-norm spaces are defined as

$$\mathbf{L_p^n} \stackrel{\Delta}{=} \left\{ f(t) : \Re_+ \longrightarrow \Re^n | ||f||_p = \int_0^\infty ||f(t)||^p \, \mathrm{d}t < \infty, \; f \text{ or } p \in [1, \infty) \right\}$$
$$\mathbf{L_\infty^n} \stackrel{\Delta}{=} \left\{ f(t) : \Re_+ \longrightarrow \Re^n | ||f||_\infty = \sup_{t \in [0,\infty)} ||f(t)|| < \infty \right\}$$

2.3 Calculus and Algebra of Matrices

In this section, we solicit some basic facts and useful relations from linear algebra and calculus of matrices. The materials are stated along with some hints whenever needed but without proofs unless we see the benefit of providing a proof. Reference is made to matrix M or matrix function $M(t)$ in the form

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \dots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}, \text{ or } M(t) = \begin{bmatrix} M_{11}(t) & \cdots & M_{1n}(t) \\ \vdots & \ddots & \dots \\ M_{m1}(t) & \cdots & M_{mn}(t) \end{bmatrix}$$

2.3.1 Fundamental Subspaces

A nonempty subset $\mathcal{G} \subset \Re^n$ is called a *linear subspace* of $\Re^n$ if $x+y$ and αx are in $\mathcal{G}$ whenever x and y are in $\mathcal{G}$ for any scalar α. A set of elements $X = \{x_1,\ x_2,\ \dots,\ x_n\}$ is said to be a *spanning set* for a linear subspace $\mathcal{G}$ of $\Re^n$ if every element $g \in \mathcal{G}$ can be written as a linear combination of the $\{x_j\}$. That is, we have

$$\mathcal{G} = \{g \in \Re :\ g = \alpha_1 x_1 + \alpha_2 x_2 + \ \dots\ \alpha_n x_n$$

for some scalars $\alpha_1,\ \alpha_2, \dots,\ \alpha_n$.

A spanning set X is said to be a *basis* for $\mathcal{G}$ if no element x_j of the spanning set X of $\mathcal{G}$ can written as a linear combination of the remaining elements $x_1,\ x_2, \dots,\ x_{j-1},\ x_{j+1}, \dots,\ x_n,$ that is, $x_j,\ 1 \le i \le n$ form a linearly independent set. It is frequent to use $x_j = [0\ 0\ \dots\ 0\ 1\ 0\ \dots\ 0]^t$ the kth unit vector.

The geometric ideas of linear vector spaces had led to the concepts of *spanning a space* and a *basis for a space*. The idea now is to introduce four important subspaces which are useful. The entire linear vector space of a specific problem can be decomposed into the sum of these subspaces.

The *column space* of a matrix $A \in Re^{n\times m}$ is the space spanned by the columns of A, also called the *range space* of A, denoted by $\mathcal{R}[A]$. Similarly, the *row space* of A is the space spanned by the rows of A. Since the column rank of a matrix is the dimension of the space spanned by the columns and the row rank is the dimension of the space spanned by the rows, it is clear that the spaces $\mathcal{R}[A]$ and $\mathcal{R}[A^t]$ have the same dimension $r = \text{rank}(A)$.

The *right null space* of $A \in Re^{n\times m}$ is the space spanned by all vectors x that satisfy $A\ x = 0$, and is denoted by $\mathcal{N}[A]$. The right null space of A is also called the *kernel* of A. The *left null space* of A is the space spanned by all vectors y that satisfy $y^t\ A = 0$. This space is denoted by $\mathcal{N}[A^t]$, since it is also characterized by all vectors y such that $A^t\ y = 0$.

The dimensions of the four spaces $\mathcal{R}[A],\ \mathcal{R}[A^t],\ \mathcal{N}[A]$, and $\mathcal{N}[A^t]$ are to be determined in the sequel. Since $A \in \Re^{n\times m}$, we have the following

$$\begin{aligned} r &\stackrel{\Delta}{=} \text{rank}(A) = \text{dimension of column space } \mathcal{R}[A] \\ \dim \mathcal{N}[A] &\stackrel{\Delta}{=} \text{dimension of right null space } \mathcal{N}[A] \\ n &\stackrel{\Delta}{=} \text{total number of columns of } A \end{aligned}$$

Hence the dimension of the null space $\dim \mathcal{N}[A] = n - r$. Using the fact that $\text{rank}(A) = \text{rank}(A^t)$, we have

$$\begin{aligned} r &\triangleq \text{rank}(A^t) = \text{dimension of row space } \mathcal{R}[A^t] \\ \dim \mathcal{N}[A^t] &\triangleq \text{dimension of left null space } \mathcal{N}[A^t] \\ m &\triangleq \text{total number of rows of } A \end{aligned}$$

Hence the dimension of the null space $\dim \mathcal{N}[A^t] = m - r$. These facts are summarized below.

Note from these facts that the entire n-dimensional space can be decomposed into the sum of the two subspaces $\mathcal{R}[A^t]$ and $\mathcal{N}[A]$. Alternatively, the entire m-dimensional space can be decomposed into the sum of the two subspaces $\mathcal{R}[A]$ and $\mathcal{N}[A^t]$.

An important property is that $\mathcal{N}[A]$ and $\mathcal{R}[A^t]$ are *orthogonal subspaces*, that is, $\mathcal{R}[A^t]^{\perp} = \mathcal{N}[A]$. This has the meaning that every vector in $\mathcal{N}[A]$ is orthogonal to every vector in $\mathcal{R}[A^t]$. In the same manner, $\mathcal{R}[A]$ and $\mathcal{N}[A^t]$ are *orthogonal subspaces*, that is, $\mathcal{R}[A]^{\perp} = \mathcal{N}[A^t]$. The construction of the fundamental subspaces is appropriately attained by the singular value decomposition.

$$\begin{aligned} \mathcal{R}[A^t] &\triangleq \text{row space of } A: \text{dimension } r \\ \mathcal{N}[A] &\triangleq \text{right null space of } A: \text{dimension } n - r \\ \mathcal{R}[A] &\triangleq \text{column space of } A: \text{dimension } r \\ \mathcal{N}[A^t] &\triangleq \text{left null space of } A: \text{dimension } n - r \end{aligned}$$

2.3.2 Calculus of Vector–Matrix Functions of a Scalar

The differentiation and integration of time functions involving vectors and matrices arise in solving state equations, optimal control, and so on. This section summarizes the basic definitions of differentiation and integration on vectors and matrices . A number of formulas for the derivative of vector–matrix products are also included.

The derivative of a matrix function $M(t)$ of a scalar is the matrix of the derivatives of each element in the matrix

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \begin{bmatrix} \frac{\mathrm{d}M_{11}(t)}{\mathrm{d}t} & \cdots & \frac{\mathrm{d}M_{1n}(t)}{\mathrm{d}t} \\ \vdots & \ddots & \cdots \\ \frac{\mathrm{d}M_{m1}(t)}{\mathrm{d}t} & \cdots & \frac{\mathrm{d}M_{mn}(t)}{\mathrm{d}t} \end{bmatrix}$$

The integral of a matrix function $M(t)$ of a scalar is the matrix of the integral of each element in the matrix

$$\int_a^b M(t)\mathrm{d}t = \begin{bmatrix} \int_a^b M_{11}(t)\mathrm{d}t & \cdots & \int_a^b M_{1n}(t)\mathrm{d}t \\ \vdots & \ddots & \cdots \\ \int_a^b M_{m1}(t)\mathrm{d}t & \cdots & \int_a^b M_{mn}(t)\mathrm{d}t \end{bmatrix}$$

The Laplace transform of a matrix function $M(t)$ of a scalar is the matrix of the Laplace transform of each element in the matrix

$$\int_a^b M(t)e^{-st}\mathrm{d}t = \begin{bmatrix} \int_a^b M_{11}(t)e^{-st}\mathrm{d}t & \cdots & \int_a^b M_{1n}(t)e^{-st}\mathrm{d}t \\ \vdots & \ddots & \cdots \\ \int_a^b M_{m1}(t)e^{-st}\mathrm{d}t & \cdots & \int_a^b M_{mn}(t)e^{-st}\mathrm{d}t \end{bmatrix}$$

The scalar derivative of the product of two matrix time functions is

$$\frac{\mathrm{d}(A(t)B(t))}{\mathrm{d}t} = \frac{A(t)}{\mathrm{d}t}B(t) + A(t)\frac{B(t)}{\mathrm{d}t}$$

This result is analogous to the derivative of a product of two scalar functions of a scalar, except caution must be used in reserving the order of the product. An important special case follows:

The scalar derivative of the inverse of a matrix time function is

$$\frac{\mathrm{d}A^{-1}(t)}{\mathrm{d}t} = -A^{-1}\frac{A(t)}{\mathrm{d}t}A(t)$$

2.3.3 Derivatives of Vector–Matrix Products

The derivative of a real scalar-valued function $f(x)$ of a real vector $x = [x_1, \ldots, x_n]^t \in Re^n$ is defined by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

where the partial derivative is defined by

$$\frac{\partial f(x)}{\partial x_j} \stackrel{\Delta}{=} \lim_{\Delta x_j \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x_j}, \quad \Delta x = [0 \ldots \Delta x_j \ldots 0]^t$$

An important application arises in the Taylor's series expansion of $f(x)$ about x_o in terms of $\delta x \stackrel{\Delta}{=} x - x_o$. The first three terms are

$$f(x) = f(x_o) + \left(\frac{\partial f(x)}{\partial x}\right)^t \delta x + \frac{1}{2}\delta x^t \left[\frac{\partial^2 f(x)}{\partial x^2}\right] \delta x$$

where

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)^t = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \cdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

The derivative of a real scalar-valued function $f(A)$ with respect to a matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \in Re^{n \times n}$$

is given by

$$\frac{\partial f(A)}{\partial A} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \vdots & \ddots & \cdots \\ \frac{\partial f(A)}{\partial A_{n1}} & \cdots & \frac{\partial f(A)}{\partial A_{nn}} \end{bmatrix}$$

A vector function of a vector is given by

$$v(u) = \begin{bmatrix} v_1(u) \\ \vdots \\ \vdots \\ v_n(u) \end{bmatrix}$$

where $v_j(u)$ is a function of the vector u. The derivative of a vector function of a vector (the *Jacobian*) is defined as follows:

$$\frac{\partial v(u)}{\partial u} = \begin{bmatrix} \frac{\partial v_1(u)}{\partial u_1} & \cdots & \frac{\partial v_1(u)}{\partial u_m} \\ \vdots & \ddots & \cdots \\ \frac{\partial v_n(u)}{\partial u_1} & \cdots & \frac{\partial v_n(u)}{\partial u_m} \end{bmatrix}$$

Note that the *Jacobian* is sometimes defined as the transpose of the foregoing matrix. A special case is given by

$$\frac{\partial(S\,u)}{\partial u} = S, \quad \frac{\partial(u^t Ru)}{\partial u} = 2\,u^t R$$

for arbitrary matrix S and symmetric matrix R.

The following section includes useful relations and results from linear algebra.

2.3.4 The Dini Theorem

2.3.5 Positive Definite and Positive Semidefinite Matrices

A matrix P is positive definite if P is real, symmetric, and $x^t Px > 0, \quad \forall x \neq 0$. Equivalently, all the eigenvalues of P have positive real parts. A matrix S is positive semidefinite if S is real, symmetric, and $x^t Px \geq 0, \quad \forall x \neq 0$.

Since the definiteness of the scalar $x^t Px$ is a property only of the matrix P, we need a test for determining definiteness of a constant matrix P. Define a *principal submatrix* of a square matrix P as any square submatrix sharing some diagonal elements of P. Thus the constant, real, symmetric matrix $P \in \Re^{n\times n}$ is positive definite ($P > 0$) if either of these equivalent conditions holds:

- All eigenvalues of P are positive
- The determinant of P is positive
- All successive principal submatrices of P (minors of successively increasing size) have positive determinants

2.3.6 Trace Properties

The trace of a square matrix P, trace (P), equals the sum of its diagonal elements or equivalently the sum of its eigenvalues. A basic property of the trace is invariant under cyclic perturbations, that is,

$$\text{trace}(AB) = \text{trace}(BA)$$

where AB is square. Successive applications of the above results yield

$$\text{trace}(ABC) = \text{trace}(BCA) = \text{trace}(CAB)$$

where ABC is square. In general,

$$\text{trace}(AB) = \text{trace}(B^t A^t)$$

Another result is that

$$\text{trace}(A^t BA) = \sum_{k=1}^{p} a_k^t B a_k$$

where $A \in \Re^{n\times p}$, $B \in \Re^{n\times n}$, and $\{a_k\}$ are the columns of A. The following identities on trace derivatives are noted:

$$\begin{aligned}
\frac{\partial(\text{trace}(AB))}{\partial A} &= \frac{\partial(\text{trace}(A^t B^t))}{\partial A} = \frac{\partial(\text{trace}(B^t A^t))}{\partial A} \\
&= \frac{\partial(\text{trace}(BA))}{\partial A} = B^t \\
\frac{\partial(\text{trace}(AB))}{\partial B} &= \frac{\partial(\text{trace}(A^t B^t))}{\partial B} = \frac{\partial(\text{trace}(B^t A^t))}{\partial B} \\
&= \frac{\partial(\text{trace}(BA))}{\partial B} = A^t \\
\frac{\partial(\text{trace}(BAC))}{\partial A} &= \frac{\partial(\text{trace}(B^t C^t A^t))}{\partial A} = \frac{\partial(\text{trace}(C^t A^t B^t))}{\partial A} \\
&= \frac{\partial(\text{trace}(ACB))}{\partial A} = \frac{\partial(\text{trace}(CBA))}{\partial A} \\
&= \frac{\partial(\text{trace}(A^t B^t C^t))}{\partial A} = B^t\ C^t \\
\frac{\partial(\text{trace}(A^t BA))}{\partial A} &= \frac{\partial(\text{trace}(BAA^t))}{\partial A} = \frac{\partial(\text{trace}(AA^t B))}{\partial A} \\
&= (B + B^t)A
\end{aligned}$$

Using these basic ideas, a list of matrix calculus results are given below:

$$\begin{aligned}
&\frac{\partial(\text{trace}(AX^t))}{\partial X} = A, \quad \frac{\partial(\text{trace}(AXB))}{\partial X} = A^t\ B^t \\
&\frac{\partial(\text{trace}(AX^t B))}{\partial X} = B\ A, \quad \frac{\partial(\text{trace}(AX))}{\partial X^t} = A \\
&\frac{\partial(\text{trace}(AX^t))}{\partial X^t} = A^t, \quad \frac{\partial(\text{trace}(AXB))}{\partial X^t} = B\ A \\
&\frac{\partial(\text{trace}(AX^t B))}{\partial X^t} = A^t\ B^t, \quad \frac{\partial(\text{trace}(XX))}{\partial X} = 2\ X^t \\
&\frac{\partial(\text{trace}(XX^t))}{\partial X} = 2\ X \\
&\frac{\partial(\text{trace}(AX^n))}{\partial X} = \left(\sum_{j=0}^{n-1} X^j\ A\ X^{n-j-1}\right)^t
\end{aligned}$$

$$\frac{\partial(\mathrm{trace}(AXBX))}{\partial X} = A^t X^t B^t + B^t X^t A^t$$
$$\frac{\partial(\mathrm{trace}(AXBX^t))}{\partial X} = A^t X B^t + AXB$$
$$\frac{\partial(\mathrm{trace}(X^{-1}))}{\partial X} = -\left(X^{-2}\right)^t$$
$$\frac{\partial(\mathrm{trace}(AX^{-1}B))}{\partial X} = -\left(X^{-1}BAX^{-1}\right)^t$$
$$\frac{\partial(\mathrm{trace}(AB))}{\partial A} = B^t + B - \mathrm{diag}(B)$$

2.3.7 Partitioned Matrices

Given a partitioned matrix (matrix of matrices) of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A,\ B,\ C$, and D are of compatible dimensions. Then

(1) if A^{-1} exists, a Schur complement of M is defined as $D - CA^{-1}B$, and
(2) if D^{-1} exists, a Schur complement of M is defined as $A - BD^{-1}C$.

When $A,\ B,\ C$, and D are all $n \times n$ matrices, then

$$a) \quad \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B),\ \det(A) \neq 0$$
$$b) \quad \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\det(A - BD^{-1}C),\ \det(D) \neq 0$$

In the special case, we have

$$\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \det(A)\det(C)$$

where A and C are square. Since the determinant is invariant under row, it follows

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} A & B \\ C - CA^{-1}A & D - CA^{-1}B \end{bmatrix}$$
$$= \det\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \det(A)\det(D - CA^{-1}B)$$

which justifies the forgoing result.

Given matrices $A \in \Re^{m\times n}$ and $B \in \Re^{n\times m}$, then

$$\det(I_m - AB) = \det(I_n - BA)$$

In case that A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

2.3.8 The Matrix Inversion Lemma

Suppose that $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$, $C \in \Re^{p\times p}$, and $D \in \Re^{p\times n}$. Assume that A^{-1} and C^{-1} both exist, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

In the case of partitioned matrices, we have the following result

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Xi^{-1}CA^{-1} & -A^{-1}B\Xi^{-1} \\ -\Xi^{-1}CA^{-1} & \Xi^{-1} \end{bmatrix}$$
$$\Xi = (D - CA^{-1}B)$$

provided that A^{-1} exists. Alternatively,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \Xi^{-1} & -\Xi^{-1}BD^{-1} \\ -D^{-1}C\Xi^{-1} & D^{-1} + D^{-1}C\Xi^{-1}BD^{-1} \end{bmatrix}$$
$$\Xi = (D - CA^{-1}B)$$

provided that D^{-1} exists.

For a square matrix Y, the matrices Y and $(I + Y)^{-1}$ commute, that is, given that the inverse exists

$$Y\,(I + Y)^{-1} = (I + Y)^{-1}\,Y$$

Two additional inversion formulas are given below:

$$\begin{aligned} Y\ (I + XY)^{-1} &= (I + YX)^{-1}\ Y \\ (I + YX)^{-1} &= I - YX\ (I + YX)^{-1} \end{aligned}$$

The following result provides conditions for the positive definiteness of a partitioned matrix in terms of its submatrices. The following three statements are equivalent:

$$\begin{aligned} &1)\ \begin{bmatrix} A_o & A_a \\ A_a^t & A_c \end{bmatrix} > 0 \\ &2)\ A_c > 0, \quad A_o - A_a A_c^{-1} A_a^t > 0 \\ &3)\ A_a > 0, \quad A_c - A_a^t A_o^{-1} A_a > 0 \end{aligned}$$

2.3.9 The Singular Value Decomposition

The singular value decomposition (SVD) is a matrix factorization that has found a number of applications to engineering problems. The SVD of a matrix $M \in Re^{n\times m}$ is

$$M = U\ S\ V^\dagger = \sum_{j=1}^{p} \sigma_j\ U_j\ V_j^\dagger$$

where $U \in Re^{\alpha\times\alpha}$ and $V \in Re^{\beta\times\beta}$ are unitary matrices ($U^\dagger\ U = U\ U^\dagger = I$ and $V^\dagger\ V = V\ V^\dagger I$); $S \in Re^{\alpha\times\beta}$ is a real, diagonal (but not necessarily square); and $pmin(\alpha, \beta)$*. The singular values* $\{\sigma_1, \sigma_2, \ldots, \sigma_\beta\}$ *of M are defined as the positive square roots of the diagonal elements of* $S^t S$, and are ordered from largest to smallest.

To proceed further, we recall a result on unitary matrices. If U is a unitary matrix ($U^\dagger\ U = I$), then the transformation U preserves length, that is,

$$\begin{aligned} \|U\ x\| &= \sqrt{(Ux)^\dagger (Ux)} = \sqrt{x^\dagger\ U^\dagger\ U\ x} \\ &= \sqrt{x^\dagger\ x} = \|x\| \end{aligned}$$

As a consequence, we have

$$\begin{aligned} \|M\ x\| &= \sqrt{x^\dagger\ M^\dagger\ M\ x} = \sqrt{x^\dagger\ V S^t U^\dagger\ U S V^\dagger\ x} \\ &= \sqrt{x^\dagger\ V S^t S V^\dagger\ x} \end{aligned}$$

To evaluate the maximum gain of matrix M, we calculate the maximum norm of the above equation to yield

$$\max_{||x||=1} ||M\ x|| = \max_{||x||=1} \sqrt{x^{\dagger}\ V S^t S V^{\dagger}\ x} = \max_{||\tilde{x}||=1} \sqrt{\tilde{x}^{\dagger}\ V S^t S\ \tilde{x}}$$

Note that maximization over $\tilde{x} = Vx$ is equivalent to maximizing over x since V is invertible and preserves the norm (equals 1 in this case). Expanding the norm yields

$$\begin{aligned} \max_{||x||=1} ||M\ x|| &= \max_{||\tilde{x}||=1} \sqrt{\tilde{x}^{\dagger}\ V S^t S\ \tilde{x}} \\ &= \max_{||\tilde{x}||=1} \sqrt{\sigma_1^2|\tilde{x}_1|^2 + \sigma_2^2|\tilde{x}_2|^2 + \cdots + \sigma_{\beta}^2|\tilde{x}_{\beta}|^2} \end{aligned}$$

The foregoing expression is maximized, given the constraint $||\tilde{x}|| = 1$, when $\tilde{x}$ is concentrated at the largest singular value; that is, $|\tilde{x}| = [1\ 0\ \ldots\ 0]^t$. The maximum gain is then

$$\max_{||x||=1} ||M\ x|| = \sqrt{\sigma_1^2|1|^2 + \sigma_2^2|0|^2 + \cdots + \sigma_{\beta}^2|0|^2} = \sigma_1 = \sigma_M$$

In words, this reads *the maximum gain of a matrix is given by the maximum singular value σ_M*. Following similar lines of development, it is easy to show that

$$\begin{aligned} \min_{||x||=1} ||M\ x|| &= \sigma_{\beta} = \sigma_m \\ &= \begin{cases} \sigma_p & \alpha \geq \beta \\ 0 & \alpha < \beta \end{cases} \end{aligned}$$

A property of the singular values is expressed by

$$\sigma_M(M^{-1}) = \frac{1}{\sigma_m(M)}$$

2.4 Notes and References

The topics covered in this chapter is meant to provide the reader with a general platform containing the basic mathematical information needed for further examination of switched time-delay systems. These topics are properly selected from standard books and monographs on mathematical analysis. For further details, the reader is referred to the standard texts [29, 46, 157, 160, 443] where fundamentals are provided.

Part II
System Stability

Chapter 3
Time-Delay Systems: Recent Progress

In preparation for the several chapters on stability and stabilization methods for time-delay systems, it is considered beneficial to provide in this chapter a precise and concise appraisal of the existing results. The appraisal will be conducted in a quantitative manner in addition to numerical simulation on a representative example.

3.1 Time Delays: Overview

Time delay occurs for different reasons and from different sources; one of these sources is the nature of the system or the way it works, for example, in a internal combustion engine a period of time is required to mix the air and the fuel and a time delay appears in the system dynamics. Another source for time delay is the transport delay for some material to travel through the system in heat or mass transfer. Delay also might occur due to the communication among the system parts, for example, time is needed for the signals to travel among the controllers, the sensors, and the actuators in any typical closed-loop system. Some controllers may contribute in producing time delay, for example, consider the standard *PID* controller, by closing the loop, some time delay may be introduced in the system dynamics due to the I part in the *PID* controller since this part accumulates the error from past values which is a function of delayed states. Finally in some cases the delay is deliberately introduced in the system to attain some goals like quenching the overshoot.

In the sequel, the types of delay sources will be discussed in more detail with an example

1. *Nature of the process:*
   This arises in chemical reactors (finite reaction time), diesel engines (ignition delay), and recycled processes (recycle delay) where in all of these cases time for build up or decay down occurs due to the internal functioning of the system.
2. *Transport delay:*
   This occurs in systems containing materials transfer like in rolling mills in which the controller takes finite time to affect the process and in a heating system where the delay appears because of transport of the heated air.

M.S. Mahmoud, *Switched Time-Delay Systems*,

DOI 10.1007/978-1-4419-6394-9_3,

3. *Communication delay*
   Communication delay can generally occur due to:

   1) *propagation time delay* of signals among the actuators, controllers, and sensors, particularly in networked control systems and fault-tolerant systems. This is crucial in remote control systems like in teleportation over the Internet and in guided rocket operations, and
   2) *access time delay* arising in finite time required to gain access to a shared media. One example can be found again in the networked control systems where many nodes are sharing the same communication media and there is access time delay, which can be considerably large, since the sensor, actuator, and the controllers are all connected through the network. The data at the controller are a delayed version of the current state and when the controller sends the control action (e.g., state feedback) it again suffers time delay.

There is a typical example including network congestion control where the amount of this traffic depends on the previous load in the buffer for preselected protocols. Another interesting example occurs in biology of the evolution of a single species consuming a common self-renewing food where time delay takes place due to finite production time for the food.

One should realize that in some cases the delay may be intentionally introduced into the system with a hope to improve the cost function. This delay should be introduced carefully in order to obtain the required target. This delay can be used also to reduce the overshoot and yield a smooth and fast transient response. For further details, the reader is referred to [304].

3.2 Literature Survey

An integral ingredient of research investigations into systems engineering is that of ‘Mathematical Modeling’ or ‘Modeling’ in short. Simply stated, the process of exploring any aspect or examining any problem needs a ‘Mathematical Model’, which would provide a reasonably accurate representation of the system behavior. In standard books, it is sometimes said that a mathematical model is an abstraction of reality to the extent that a ‘good choice’ of a mathematical model would reflect on the quality of the results. Thus it has been our firm belief that mathematical modeling is the corner stone of systems engineering disciplines. With focus on lumped-parameter systems, it has been recently recognized that the best mathematical model would be developed by deploying functional differential equations (FDEs) [109, 171] as the main vehicle of system representation in the time domain. Thus state-space formulation with delay patterns (time-delay systems) has been considered [54, 96] as the backbone in the analysis, synthesis, and design of problems in systems engineering areas. In this regard, we look at control problems of time-delay systems with the objective of developing improved stabilization and control design methods. Broadly speaking, there are three directions of research:

(1) Development of new bounding techniques for the Lyapunov–Krasovskii functionals (LKFs),
(2) Transformation to an appropriate system with distributed delay, and
(3) Construction of new LKFs with a proper distribution of the time delay.
In this chapter, we address equally on all of these directions, although we focus on the third direction when presenting contemporary results.

3.2.1 Stability Methods

System and control problems associated with time-delay systems have been the subject matter of numerous publications, the most relevant of which are [26, 30–448]. Stability analysis and control design of time-delay systems have attracted the attention of numerous investigators, see [221, 338] for a modest coverage. In this regard, stability criteria for linear state-delay systems can be broadly classified into two categories:

- Delay-independent, which are applicable to delays of arbitrary size, and
- Delay-dependent, which include information on the size of the delay.

3.2.2 Delay-Independent Stability Tests

When considering delay-independent stability (DIS) tests, one wants to check for a given system whether it can preserve its stability in spite of the presence of a delay of any size. It is hoped that the magnitude of the delay term is very small relative to the current state and the value of the delayed state can take any value. The DIS test tries to check if the delayed term's value is significant/insignificant to change the original system stability. No information about the delay is needed and only the values of the matrices of the current state and the delayed states are considered. Clearly, this direction does not require any information about the nature of the delay and when it yields positive results, meaning that a system is found to be stable independent of the delay value, then it can be used regardless of what is the magnitude of the delay or how fast it changes.

From the published results in this area, it was found that generally this type of test is relatively easier to be derived and some system can satisfy its condition. On the other hand, it was concluded that it suffers from some degree of conservatism due to the following:

- Not all systems have delayed states with small magnitude and, in these cases, systems will not satisfy the test conditions,
- In many cases, the delay is fixed and the system is time-invariant, and applying delay-independent stability test yields unnecessary conditions on the system,
- When the delay is not fixed but bounded by some relatively small values, then the delay-independent test is unsuitable, and

- It is based on the assumption that the system is stable and it therefore cannot be used for unstable systems. It can however be used in feedback stabilization. In addition, it cannot work well in this case, the system can be suffering from a delay in the input.

For these reasons many researchers shift their interest to the delay dependent stability tests.

3.2.3 Delay-Dependent Stability Tests

In contrast to the DIS test, in the delay-dependent stability (DDS) tests some *a priori* information about the delay is needed to check the system stability. Depending on the delay pattern and related information required, appropriate stability results can be readily obtained. This information can be used in either one of the following scenarios:

Given a dynamical system with some delay information, check whether the system is stable or not, or

Given a dynamical system, check for at what delay limit the system still remains stable

Generally, the second scenario is used in qualifying the stability theorems, because as we will see later, in some cases, we have to make a test of sufficient (not necessary) conditions type to check for the system stability. This means that it is for a stable system to satisfy these conditions. If a system succeed in satisfying them, then the system is stable. All the methods attempt to reduce the conservatism as much as possible and produce measures or criteria to judge with how much delay the system remains stable.

3.2.4 Stability Results

When dealing with time-varying delays, a fundamental problem arises when estimating the upper bound of cross-product terms. Algebraic inequalities [301, 322] and majorization procedures [24] have been used. This introduces a source of overdesign conservatism. There have been different approaches to reduce the level of conservatism, including full-size quadratic functionals [167], discretized LKF [100], and free-weighting matrices techniques [121, 123, 124, 127, 128] and [391]. In particular, in [124] it was pointed out that the significance of bypassing extra conservatism introduced after enlarged integration time span in some LKF terms. From the published results, it appears that further reduction of design conservatism can be achieved with

- Appropriate LKF with moderate number of terms,
- Avoiding bounding methods, and
- Effective use of parametrized relations and variables to avoid redundancy.

Initial results on deriving delay-dependent stability and stabilization criteria have been reported in [188, 265] based on the Leibniz–Newton formula and cast into the Riccati-inequality format. Some recent views and improved methods pertaining to the problems of determining robust stability criteria and robust control design of uncertain time-delay systems have been reported, see, for example, [124, 181, 188, 238, 324] and their references. With the availability of efficient interior-point minimization methods, all the recent results have been cast in linear-matrix inequalities (LMIs) format [27].

Two distinct features of the contemporary research activities are identified, the first feature of which concerns the choice of an appropriate Lyapunov–Krasovskii functional (LKF) for stability and performance analysis within the framework of LMIs [27]. General LKF forms might lead to a complicated system of inequalities [124] and the selection of new and effective LKF forms is becoming crucial for deriving less-conservative stability criteria. The second feature is the introduction of additional parameters for developing improved sufficient stability conditions by importing some basic system identities [65–181]. Parallel to this effort is that several fixed-model transformation methods and parameterization schemes have been derived in the literature to derive delay-dependent stability conditions, see [65–100, 198–325, 338–448] and their references.

3.2.5 Stabilization Results

Increasing attention is being paid to the delay-dependent stability, stabilization, and $\mathcal{H}_\infty$ control of linear systems with state delays (see for example [66, 80, 105, 114, 127, 128, 152, 155, 156, 198, 338, 392, 440]. *For continuous-time systems with time delay*, the main methods so far reported are based on four fixed-model transformation models (see [66]). Among them, the descriptor systems approach combined with Park's [322] and Moon's inequalities [301] are the most effective way to deal with delay-dependent problems, see [66, 301, 322]. In [127] however, it is pointed out that [1, 11, 358] do not consider the relationships between the terms of the Leibniz–Newton formula in the derivative of the Lyapunov functional. In order to overcome the conservativeness of methods based on a fixed-model transformation between those terms (see [114, 124, 127, 440]). Jiang and Han [152, 155] applied this method to systems with an interval time-varying delay. However, as mentioned in [124], they ignored some useful terms in the derivative of the Lyapunov functional, which may lead to conservativeness.

3.3 Stability Approaches: Continuous Time

In the following section, a review and evaluation will be made on the time-delay research in different directions along with few comments on each one. These comments are the points that are to be considered in further development.

3.3.1 Basic Models

In the sequel, we closely treat the stability problems for the single-delay case and aim at deriving LMI-based stability conditions. Extension to the multiple-delay case is a straightforward job and is therefore omitted. We look at two distinct classes of time-delay systems:

$$\Sigma_c : \quad \dot{x}(t) = A_o x(t) + A_d x(t-\tau), \quad x(t) = \phi(t), \quad \forall\, t \in [-\tau, 0] \tag{3.1}$$

and

$$\Sigma_v : \quad \dot{x}(t) = A_o x(t) + A_d x(t-\tau(t)), \quad x(t) = \phi(t), \quad \forall\, t \in [-\varrho, 0] \tag{3.2}$$

where $x(t) \in \Re^n$ is the state; $\phi(t)$ is the continuous initial condition. In (3.1), the scalar τ is the constant delay for system Σ_c and in (3.2), $\tau(t)$ is the time-varying delay of system Σ_v, which is assumed to be continuous, and satisfies

$$0 \; < \; \tau(t) \; \leq \; \varrho \tag{3.3}$$

In both models of time-delay systems Σ_c and Σ_v, $A_o \in \Re^{n \times n}$ and $A_d \in \Re^{n \times n}$ are known real constant matrices.

3.3.2 LMI Stability Conditions

For system Σ_c, by selecting the Lyapunov–Krasovskii functional

$$V(t, x_t) = x^t(t)\mathcal{P}x(t) + \int_{t-\tau}^{t} x^t(s)\mathcal{Q}x(s)\, ds \tag{3.4}$$

and invoking the Lyapunov–Krasovskii theorem, the following stability condition can be derived [206]:

Theorem 3.1 *The time-delay system Σ_c is asymptotically stable if there exist matrices $\mathcal{P} > 0$ and $\mathcal{Q} > 0$ such that*

$$\begin{bmatrix} \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} & \mathcal{P}A_d \\ \bullet & -\mathcal{Q} \end{bmatrix} \; < \; 0 \tag{3.5}$$

On the other hand, for system Σ_v, by selecting the Lyapunov functional

$$V(x(t)) = x^t(t)\mathcal{P}x(t) \tag{3.6}$$

and invoking the Lyapunov–Razumikhin theorem and setting $p(s) = \delta s,\ w(s) = \epsilon s^2$ such that $\delta > 1,\ \epsilon > 0$, the following stability condition can be obtained [206]:

Theorem 3.2 (Mahmoud [206]) *The time-delay system Σ_c is asymptotically stable if there exists a matrix $\mathcal{P} > 0$ such that*

$$\begin{bmatrix} \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{P} & \mathcal{P}A_d \\ \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{3.7}$$

Remark 3.3 Based on the foregoing theorems, we are now in a position to make three key observations.

1. The LMI (3.7) is a special case of the LMI (3.5). Therefore, **Theorem 3.1** is less conservative than **Theorem 3.2**.
2. Both LMIs are delay-independent since they are satisfied regardless of the size of delay τ.
3. **Theorem 3.2** can be applied to the case when the delay τ is time varying and continuous, which may not be differentiable. Alternatively, **Theorem 3.1** usually requires the time-varying delay τ to be differentiable.

These simple observations have motivated numerous researchers to adopt the Lyapunov–Krasovskii theorem in conducting research seeking improved delay-dependent stability and stabilization conditions. We follow this trend throughout the book unless otherwise considered beneficial to apply the Lyapunov–Razumikhin theorem.

3.3.3 Newton–Leibniz Formula

Initial efforts made to get a delay-dependent criteria were using model transformation for the time-delay systems Σ_c or Σ_v. On applying the fundamental Newton–Leibniz formula

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s \\ &= x(t) - \int_{t-\tau}^{t} [A_o x(s) + A_d x(s-\tau)]\mathrm{d}s \end{aligned}$$

to get around the delayed state, then system (3.1) becomes

$$\begin{aligned} \Sigma_w: \quad \dot{x}(t) &= A_o x(t) + A_d \left[x(t) - \int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s \right] \\ &= (A_o + A_d)x(t) - A_d \int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s \\ &= (A_o + A_d)x(t) - A_d \int_{t-\tau}^{t} [A_o x(s) + A_d x(s-\tau)]\mathrm{d}s \end{aligned} \tag{3.8}$$

It should be observed that the asymptotic stability of the time-delay system (3.8) implies that of the system (3.1) or (3.2). For this reason, we focus on studying the stability of (3.8). To this end, we choose a Lyapunov–Krasovskii functional of the form

$$
\begin{aligned}
V &= V_o + V_a + V_c \\
V_o &= x^t(t)\mathcal{P}x(t), \ \ V_a = \int_{t-\tau}^{t} x^t(s)\mathcal{Q}x(s)\mathrm{d}s \\
V_c &= \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^t(s)A_d^t\mathcal{Z}A_d\dot{x}(s)\mathrm{d}s\mathrm{d}\theta
\end{aligned}
\tag{3.9}
$$

where $\mathcal{P} > 0, \ \mathcal{Q} > 0, \ \mathcal{Z} > 0$.

Remark 3.4 In terms of dynamic models with generalized coordinates, one can interpret the first component V_o of the LKF V as a measure of the internal energy of system (3.1), the second term V_a is intuitively seen to provide a measure of the signal energy during the delay period $[\tau, \ 0]$, and the third term V_c gives a measure of the energy corresponding to the difference between the state $x(t)$, the signal sought for feedback stabilization, and the delayed state $x(t-\tau)$, the one that might be available for feedback processing, that is given by the Newton–Leibniz formula $x(t) - x(t-\tau) = \int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s$.

On the other hand, by selecting a Lyapunov–Krasovskii functional of the form

$$
\begin{aligned}
\hat{V} &= V_o + V_a + V_c \\
V_o &= x^t(t)\mathcal{P}^{-1}x(t), \ \ V_a = \int_{-h}^{0}\int_{t+\theta}^{t} x^t(s)A_d^t\mathcal{Q}^{-1}A_dx(s)\mathrm{d}\theta \\
V_c &= \int_{-\tau}^{0}\int_{t-\tau+\theta}^{t} \dot{x}^t s A_d^t\mathcal{Z}^{-1}A_d\dot{x}(s)\mathrm{d}s
\end{aligned}
$$

the following stability condition can be derived:

Theorem 3.5 *The time-delay system Σ_w is asymptotically stable for any constant delay τ satisfying*

$$0 < \tau \le \bar{\tau}$$

if there exist matrices $\mathcal{P} > 0$ and $\mathcal{Q} > 0$ such that

$$
\begin{bmatrix} \Pi & \bar{\tau}\mathcal{P}A_o^t & \bar{\tau}\mathcal{P}A_d^t \\ \bullet & -\mathcal{Q} & 0 \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix} < 0 \tag{3.10}
$$

where

$$\Pi = \mathcal{P}(A_o + A_d) + (A_o + A_d)^t\mathcal{P} + A_d(\mathcal{Q} + \mathcal{Z})A_d^t$$

Had we considered the case of continuous delay $\tau(t)$ in the system

$$\begin{aligned}\Sigma_s :\ \dot{x}(t) &= A_o x(t) + A_d \left[x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right] \\ &= (A_o + A_d)x(t) - A_d \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s \\ &= (A_o + A_d)x(t) - A_d \int_{t-\tau(t)}^{t} [A_o x(s) + A_d x(s-\tau(t))]\mathrm{d}s \quad (3.11)\end{aligned}$$

we would have arrived at the following result:

Theorem 3.6 *The time-delay system Σ_s is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \varrho$$

if there exist matrices $\mathcal{R} > 0,\ \mathcal{M} > 0$, and $\mathcal{N} > 0$ such that

$$\begin{aligned}(A_o + A_d)\mathcal{R} + (A_o + A_d)^t\mathcal{R} + \varrho A_d(\mathcal{M} + \mathcal{N})A_d^t + 2\varrho\mathcal{R} &< 0 \\ \begin{bmatrix} \mathcal{R} & \mathcal{R}A_o^t \\ \bullet & -\mathcal{M} \end{bmatrix} &\geq 0 \\ \begin{bmatrix} \mathcal{R} & \mathcal{R}A_d^t \\ \bullet & -\mathcal{N} \end{bmatrix} &\geq 0 \quad (3.12)\end{aligned}$$

Remark 3.7 It should be emphasized that the LMIs (3.12) are not strict and, in general, suffer from computational difficulties. The Newton–Leibniz formula has been adopted by many researchers to change the time-delay systems Σ_c and Σ_v to (3.8) and (3.11), respectively, in studying various types of time-delay systems to derive delay-dependent stability conditions. The results are reported in [146, 164, 173, 200, 312, 375, 442], and the references therein.

An alternative way to utilize the Newton–Leibniz formula is to change system Σ_c into

$$\Sigma_x :\quad \dot{x}(t) = (A_o + A_d)x(t) - A_d \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s \qquad (3.13)$$

or

$$\Sigma_z :\quad \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) + A_d \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right] = (A_o + A_d)x(t) \qquad (3.14)$$

Study of systems (3.13) and (3.14) has commanded the attention of various groups through appropriate construction of the Lyapunov–Krasovskii functionals to arrive at delay-dependent stability conditions.

Remark 3.8 We note that all the time-delay systems in (3.8), (3.13), and (3.14) are transformed from the time-delay system Σ_c using the Newton–Leibniz formula. A crucial point to address here is that the transformations carried out on system Σ_c to yield either system Σ_w, Σ_s, Σ_x or system Σ_z are not unique and, more importantly, these systems are not equivalent to system Σ_c. Compared with system Σ_c, additional dynamics are introduced in systems $\Sigma_w - \Sigma_z$, which might cause conservatisms as the delay-dependent conditions are derived based on them [103, 104, 162–165], and the references therein.

3.3.4 Cross-Product Terms

One of the main purposes in the study of delay-dependent stability for time-delay systems is to develop methods to reduce conservatism of existing delay-dependent stability conditions. On taking the time derivative of V or $\hat{V}$, the following cross-product term

$$-2x^t(t)\mathcal{P}A_d\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s \tag{3.15}$$

appears, and since it is neither positive nor negative definite, it may lead to a complication in establishing the negative definiteness of $\dot{V}$ or $\dot{\hat{V}}$. It is known that the finding of better bounds on some weighted cross products arising in the analysis of the delay-dependent stability problem plays a key role in reducing conservatism. There was a common practice to resolve this complication by majorizing the term and replacing it by upper-bound terms which are of either positive or negative definite nature. In some effort, they used the following algebraic inequality

$$-2a^tb < a^tXa + b^tX^{-1}b \quad X > 0 \tag{3.16}$$

for some vectors a, b, and matrix X. This solves the problem; however, it makes the result more conservative because we are adding positive terms in $\dot{V}$ and it has smaller chance to become negative. To get the smallest possible upper bound, matrix X should be selected such that the term $-2a^tb$ is replaced by M, given by

$$M = \inf_{X>0}\,(a^tXa + b^tX^{-1}b) \tag{3.17}$$

which means that select a $X > 0$ that gives the minimum M.

3.3.5 Bounding Inequalities

By focusing on the case of constant but unknown delay and seeking an alternative way, it is suggested to employ the following inequality

$$-2a^t b < (a + Mb)^t X(a + Mb) + b^t X^{-1} b + 2b^t Mb \quad X > 0 \tag{3.18}$$

Here M can take any value and if one puts $M = 0$, equation (3.16) is obtained. Therefore, (3.16) is a special case of (3.18) and at the worst case one is able to find the same result as in (3.16) by setting $M = 0$. The following result stands out:

Theorem 3.9 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{R} > 0$, $\mathcal{S} > 0$, $\mathcal{M} > 0$ and $\mathcal{N} > 0$ such that

$$\begin{bmatrix} \widehat{\Pi} & -\mathcal{N} A_d^t & A_o^t A_d^t \mathcal{M} & \hat{\tau}(\mathcal{S} + \mathcal{N}^t) \\ \bullet & -\mathcal{S} & A_d^t A_d^t \mathcal{M} & 0 \\ \bullet & \bullet & -\mathcal{M} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{M} \end{bmatrix} < 0 \tag{3.19}$$

where

$$\widehat{\Pi} = (A_o + A_d)\mathcal{R} + (A_o + A_d)^t \mathcal{R} + \mathcal{N}^t A_d + A_d^t \mathcal{N} + \mathcal{S}$$

A summary of the features of the method developed as follows:

- Employs first-order transformation given in (3.8),
- Incorporates the bounding technique (3.18),
- Deals with unknown fixed delay pattern,
- Introduces LKF with three terms,
- Manipulates three Lyapunov matrices and two free-weighting matrices, and
- Considers nominal time-delay models only.

Further improvement over the inequality in (3.16) to reduce the conservatism can be attained by replacing the cross-product term mentioned in (3.15) with its upper bound by using the following inequality

$$-2\int a^t(s) N b(s) \mathrm{d}s \leq \int \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^t \begin{bmatrix} X & Y - N^t \\ \bullet & Z \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} \mathrm{d}s$$

$$\text{such that} \quad \begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} \geq 0 \tag{3.20}$$

It is not difficult to show that with suitable substitution for matrices Z and Y, one can show that inequalities (3.16) and (3.18) are special cases of (3.20). Therefore, so it is expected to give less conservative stability results. The corresponding stability condition is established belowbased on the LKF (3.9):

Theorem 3.10 *The time-delay system* Σ_c *is asymptotically stable for any constant delay* $\tau(t)$ *satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ X,\ Y$ *and* Z *such that*

$$\begin{bmatrix} \widetilde{\Pi} & \mathcal{P}A_d - Y & \hat{\tau}A_o^t Z \\ \bullet & -\mathcal{Q} & \hat{\tau}A_d^t Z \\ \bullet & \bullet & -\hat{\tau}Z \end{bmatrix} < 0 \tag{3.21}$$

$$\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} \geq 0 \tag{3.22}$$

where

$$\widetilde{\Pi} = A_o\mathcal{P} + A_o^t\mathcal{P} + \hat{\tau}X + Y + Y^t + \mathcal{Q}$$

When dealing with state feedback stabilization, it turns out that the foregoing method determines the feedback gain matrices using iterative computational procedure. By and large, the method is capable of accommodating norm-bounded uncertainties. It is important to note that the iterative method should start with stable system for some delay factor $\tau > 0$, which means this method is not applicable for unstable systems. In addition, for relatively large systems the iterative method takes quite a long time to yield the desired results.

Remark 3.11 The use of inequality (3.20) has been extensively used in dealing with various issues related to time-delay systems to derive delay-dependent results; see [190, 301, 319, 320, 323, 373, 419]. However, it has been analytically established in [402, 407] that the results of **Theorem 3.10** is more conservative, and less conservative results could be obtained by introducing some slack matrices.

Along another direction by using the Jensen's integral inequality (see the Appendix) and choosing the LKF

$$\begin{aligned} V &= V_o + V_a + V_c \\ V_o &= x^t(t)\mathcal{P}x(t),\ V_a = \int_{t-\tau}^{t} x^t(s)\mathcal{Q}x(s)\mathrm{d}s \\ V_c &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^t(s)\mathcal{Z}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{3.23}$$

the following results can be obtained.

Theorem 3.12 *The time-delay system* Σ_c *is asymptotically stable for any constant delay* $\tau(t)$ *satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0$, *and* $\mathcal{Z} > 0$ *such that*

$$\begin{bmatrix} \widetilde{\Xi} & \mathcal{P}A_d - \mathcal{Z} & \hat{\tau}A_o^t\mathcal{Z} \\ \bullet & -\mathcal{Q} - \mathcal{Z} & \hat{\tau}A_d^t\mathcal{Z} \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix} < 0 \tag{3.24}$$

where

$$\widetilde{\Xi} = A_o\mathcal{P} + A_o^t\mathcal{P} + \mathcal{Q} - \mathcal{Z}$$

Alternatively, on using the Jensen's integral inequality and choosing the LKF

$$\begin{aligned} \tilde{V} &= V_o + V_e \\ V_o &= x^t(t)\mathcal{P}x(t),\ \ V_e = \varrho \int_{-\varrho}^{0}\int_{t+\theta}^{t} \dot{x}^t(s)\mathcal{Z}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{3.25}$$

the following results can be obtained.

Theorem 3.13 *The time-delay system* Σ_v *is asymptotically stable for any constant delay* $\tau(t)$ *satisfying*

$$0 < \tau(t) \leq \varrho$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0$ *and* $\mathcal{Z} > 0$ *such that*

$$\begin{bmatrix} A_o\mathcal{P} + A_o^t\mathcal{P} - \mathcal{Z} & \mathcal{P}A_d + \mathcal{Z} & \varrho A_o^t\mathcal{Z} \\ \bullet & -\mathcal{Z} & \varrho A_d^t\mathcal{Z} \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix} < 0 \tag{3.26}$$

We note that **Theorem 3.12** establishes that the time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying $0 < \tau(t) \leq \hat{\tau}$ when the LMI (3.24) has a feasible solution, which implies that for $\tau(t)$ satisfying $0 < \tau(t) \leq \hat{\tau}/2$, the time-delay system Σ_c is asymptotically stable too. A way to reduce the conservatism is by introducing the half-delay into system Σ_c to gain more information. Consequently, we consider the augmented system

$$\begin{aligned} \Sigma_y: \quad \dot{x}(t) &= A_o x(t) + A_d x(t-\tau) \\ \dot{x}(t+\tau/2) &= A_o x(t+\tau/2) + A_d x(t-\tau/2) \end{aligned} \tag{3.27}$$

In terms of

$$y(t) = \begin{bmatrix} x^t(t+\tau/2) & x^t(t) \end{bmatrix}^t$$

system Σ_y can be cast into the form

$$\begin{aligned}\dot{y}(t) &= \begin{bmatrix} A_o & 0 \\ 0 & A_o \end{bmatrix} y(t) + \begin{bmatrix} A_d & 0 \\ 0 & A_d \end{bmatrix} y(t-\tau) \\ &= \mathsf{A}_o y(t) + \mathsf{A}_d y(t-\tau) \end{aligned} \tag{3.28}$$

Choosing the LKF

$$\begin{aligned}\bar{V}(y_t) &= \bar{V}_o + \bar{V}_a + \bar{V}_c \\ \bar{V}_o &= y^t(t)\mathcal{P}y(t),\ \ \bar{V}_a = \sum_{j=1}^{2} \int_{t-(j\tau/2)}^{t} y^t(s)\mathcal{Q}_j y(s)\mathrm{d}s \\ \bar{V}_c &= \tau \ \sum_{j=1}^{2} \int_{-(j\tau/2)}^{0} \int_{t+\theta}^{t} \dot{y}^t(s)\mathcal{Z}_j \dot{y}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{3.29}$$

the following result can be obtained

Theorem 3.14 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 \ < \ \tau \ \le \ \hat{\tau}$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q}_1 > 0,\ \mathcal{Q}_2 > 0,\ \mathcal{Z}_1 > 0$, and $\mathcal{Z}_2 > 0$ such that

$$\mathcal{B}^{\perp t}\mathcal{W}\mathcal{B}^t \ < \ 0$$

where $\mathcal{B}^{\perp}$ is an orthogonal complement of $\mathcal{B}$ given by

$$\begin{aligned}\mathcal{B} &= \begin{bmatrix} I & \mathsf{A}_o & 0 & \mathsf{A}_d & 0 & 0 \\ 0 & -I & I & 0 & I & 0 \\ 0 & -I & 0 & I & 0 & I \\ 0 & I_r & I_f & I_s & 0 & 0 \end{bmatrix} \\ \mathcal{W} &= \begin{bmatrix} \tau^2/2\mathcal{Z}_1 + \tau^2\mathcal{Z}_2 & \mathcal{P} & 0 & 0 \\ \mathcal{P} & \mathcal{Q}_1 + \mathcal{Q}_2 & 0 & 0 \\ 0 & 0 & -\mathcal{Q} & 0 \\ 0 & 0 & 0 & -\mathcal{Z} \end{bmatrix} \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} I_r &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},\ I_f = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},\ I_s = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix} \\ \mathcal{Q} &= \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix},\ \mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 & 0 \\ 0 & \mathcal{Z}_2 \end{bmatrix} \end{aligned}$$

Remark 3.15 Theorem 3.14 is attributed to [97, 98]. It can be shown that this theorem can be further improved by discretizing $r > 2$ times of the interval $[-\tau, 0]$. It is fair to state that Jensen's integral inequality HAS been used to deal with different kinds of time-delay systems in order to derive delay-dependent results [105, 112, 115, 152, 393].

3.3.6 Descriptor System Approach

We have observed in the foregoing sections that delay-dependent stability results obtained via a transformed model, which is not equivalent to the original time-delay system, may lead to conservatism. In an effort to reduce such potential conservatism, a method based on descriptor system model has been introduced in the literature to derive delay-dependent stability conditions, which is equivalent to the original time-delay system.

To introduce the descriptor system approach, we first consider the time-delay system Σ_c and represent (3.1) in the following form:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ 0 &= -y(t) + (A_o + A_d)x(t) - A_d \int_{t-\tau}^{t} y(s)\mathrm{d}s \end{aligned} \tag{3.31}$$

which can be rewritten in the compact form

$$\begin{aligned} E\,\dot{\tilde{x}}(t) &= \bar{A}_o\tilde{x}(t) + \bar{A}_d\tilde{x}(t-\tau) \\ E &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\ \bar{A}_d = \begin{bmatrix} 0 \\ A_d \end{bmatrix} \\ \bar{A}_o &= \begin{bmatrix} 0 & I \\ A_o + A_d & -I \end{bmatrix} \end{aligned} \tag{3.32}$$

where the time delay has the pattern

$$0 \le \tau \le h, \quad \dot{\tau} \le \mu < 1 \tag{3.33}$$

and the system is subjected to either norm bounded or polytopic uncertainty. Given that the descriptor model (3.31) is equivalent to system (3.1), it was concluded that improved stability results can be obtained by choosing the LKF

$$\begin{aligned} \widehat{V}(\tilde{x}_t) &= \widehat{V}_o + \widehat{V}_a + \widehat{V}_c \\ \widehat{V}_o &= \tilde{x}^t(t)E\,\mathcal{P}\tilde{x}(t),\ \widehat{V}_a = \int_{t-\tau}^{t} x^t(s)\mathcal{Q}x(s)\mathrm{d}s \\ \widehat{V}_c &= \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^t(s)\mathcal{Z}\dot{y}(s)\mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{3.34}$$

and deploying the bounding inequality (3.20), the following result can be obtained:

Theorem 3.16 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P}_1 > 0,\ \mathcal{P}_2,\ \mathcal{P}_3,\ \mathcal{Q} > 0,\ \mathcal{Z} > 0,\ Y_{11},\ Y_{12},\ S_{11},\ S_{12}$, and S_{13}, ; such that

$$\begin{bmatrix} \widetilde{\Pi} + \hat{\tau} S_1 & \mathcal{P}^t \bar{A}_d - Y_1^t \\ \bullet & -Q \end{bmatrix} < 0 \tag{3.35}$$

$$\begin{bmatrix} \mathcal{Z} & Y_1 \\ \bullet & S_1 \end{bmatrix} \geq 0 \tag{3.36}$$

where

$$\widetilde{\Pi} = \mathcal{P}^t \begin{bmatrix} 0 & I \\ A_o & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_o & -I \end{bmatrix}^t \mathcal{P} + \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & \hat{\tau}\mathcal{Z} \end{bmatrix} + \begin{bmatrix} Y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} Y_1 \\ 0 \end{bmatrix}^t \tag{3.37}$$

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & 0 \\ \mathcal{P}_2 & \mathcal{P}_3 \end{bmatrix},\ Y_1 = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix},\ S_1 = \begin{bmatrix} S_{11} & S_{12} \\ \bullet & S_{13} \end{bmatrix} \tag{3.38}$$

An equivalent form of **Theorem 3.16** is stated below

Theorem 3.17 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P}_1 > 0,\ \mathcal{P}_2,\ \mathcal{P}_3,\ \mathcal{Q} > 0,\ \mathcal{Z} > 0,\ Y_{11},\ Y_{12}$ such that

$$\begin{bmatrix} \widetilde{\Pi} & \mathcal{P}^t \bar{A}_d - Y_1^t & -\hat{\tau} Y_1^t \\ \bullet & -Q & 0 \\ \bullet & \bullet & -\hat{\tau}\mathcal{Z} \end{bmatrix} < 0 \tag{3.39}$$

where $\widetilde{\Pi}$, $\mathcal{P}$, and Y_1 are given in (3.37) and (3.38).

Theorem 3.18 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P}_1 > 0,\ \mathcal{P}_2,\ \mathcal{P}_3,\ \mathcal{Q} > 0,\ \mathcal{Z} > 0,\ Y_{11},\ Y_{12}$ *such that*

$$\begin{bmatrix} \widetilde{\Pi} & \mathcal{P}^t \bar{A}_d - Y_1^t & -\hat{\tau} Y_1^t \\ \bullet & -\mathcal{Q} & 0 \\ \bullet & \bullet & -\hat{\tau}\mathcal{Z} \end{bmatrix} < 0 \tag{3.40}$$

where $\widetilde{\Pi}$, $\mathcal{P}$, *and* Y_1 *are given in (3.37) and (3.38).*

The following is a summary of the features of the descriptor model transformation method

- Incorporates the bounding inequality (3.20),
- Deals with unknown fixed delay,
- Introduces an LKF with three terms,
- Manipulates five free-weighting matrices, and
- Considers norm-bounded and polytopic uncertainties.

Remark 3.19 Since its introduction to the control literature through [65, 68], the descriptor system approach has been widely used to deal with various problems of time-delay systems in order to provide delay-dependent results; see [66, 80, 114, 145, 368], and the references therein.

At the end of this chapter, we provide an overview of the research efforts and identify some of the merits and demerits of the developed methods.

3.3.7 Free-Weighting Matrices Method

Subsequent research studies focused on further reduction of conservatism. It becomes clear that new methods should be developed which do not arise from model transformation nor upper-bounding. A new approach was developed through the introduction free-weighting matrices (slack matrix variables) method, which is based on adding zero-valued equations to the linear matrix inequality (LMI) under consideration plus incorporating the Newton–Leibniz formula. Candidate examples of this type are

$$2x^t(t)\ Y\left[x(t) - x(t-\tau) - \int_{t-\tau}^{t} x(s)\mathrm{d}s\right] \tag{3.41}$$

$$2x^t(t-\tau)\ W\left[x(t) - x(t-\tau) - \int_{t-\tau}^{t} x(s)\mathrm{d}s\right] \tag{3.42}$$

Here Y and T are free matrices to be manipulated to reach a feasible solution. Furthermore, it can be seen that

$$\tau \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^t \begin{bmatrix} X_{11} & X_{12} \\ \bullet & X_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} - \int_{t-\tau}^{t} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^t \begin{bmatrix} X_{11} & X_{12} \\ \bullet & X_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \mathrm{d}s = 0 \tag{3.43}$$

Now by using the LKF (3.23) in addition to (3.41), (3.42), and (3.43), the following theorem summarizes the main delay-dependent stability result:

Theorem 3.20 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{Z} > 0,\ X_{11},\ X_{12},\ X_{22},\ Y$ and W such that

$$\begin{bmatrix} \Psi_a & \Psi_c & \hat{\tau} A_o^t \mathcal{Z} \\ \bullet & -\Psi_e & \hat{\tau} A_d^t \mathcal{Z} \\ \bullet & \bullet & -\hat{\tau}\mathcal{Z} \end{bmatrix} < 0 \tag{3.44}$$

$$\begin{bmatrix} X_{11} & X_{12} & Y \\ \bullet & X_{22} & W \\ \bullet & \bullet & \mathcal{Z} \end{bmatrix} \geq 0 \tag{3.45}$$

where

$$\begin{aligned} \Psi_a &= \mathcal{P}A_o + A_o^t\mathcal{P} + Y + Y^t + \mathcal{Q} + \tau X_{11}, \\ \Psi_c &= \mathcal{P}A_d - Y + W^t + \tau X_{12}, \\ \Psi_e &= \mathcal{Q} + W + W^t - \tau X_{22}. \end{aligned} \tag{3.46}$$

In the case that equation (3.43) was not utilized, then simple manipulations can show that **Theorem 3.20** reduces to

Theorem 3.21 *The time-delay system Σ_c is asymptotically stable for any constant delay $\tau(t)$ satisfying*

$$0 < \tau \leq \hat{\tau}$$

if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{Z} > 0,\ Y$ and W such that

$$\begin{bmatrix} \Phi_a & \Phi_c & -\hat{\tau}Y & \hat{\tau} A_o^t \mathcal{Z} \\ \bullet & -\Psi_e & -\hat{\tau}W & \hat{\tau} A_d^t \mathcal{Z} \\ \bullet & \bullet & -\hat{\tau}\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & -\hat{\tau}\mathcal{Z} \end{bmatrix} < 0 \tag{3.47}$$

where

$$\begin{aligned} \Phi_a &= \mathcal{P}A_o + A_o^t\mathcal{P} + Y + Y^t + \mathcal{Q} \\ \Phi_c &= \mathcal{P}A_d - Y + W^t \\ \Phi_e &= \mathcal{Q} + W + W^t \end{aligned} \tag{3.48}$$

Admittedly, the method based on introducing slack variables has been extensively used in the derivation of delay-dependent results for time-delay systems, which is also effective in reducing conservatism in the existing delay-dependent results.

Till date, there are no quantitative methods that yield analytical comparisons among different existing techniques. Rather, a common numerical example is usually implemented and an evaluation is made with respect to the ensuing numerical results. The following is a summary of the features of the slack variables method:

- It does not employ any model transformation;
- It does not incorporate any bounding method;
- It deals with unknown differentiable time-varying delay.

When the differentiable time-varying delay pattern

$$0 \le \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu$$

is considered, a slack variables method can be developed by selecting a three-term LKF in the type (3.9) and achieve the delay-dependent stability conditions by using additional six free-weighting matrices.

Subsequent research activities examined the following distinct types of delay

(A1) $0 \le \tau(t) \le \varrho, \quad \dot{\tau}(t) < \mu$
(A2) $0 \le \tau \le \varrho$

The reason for differentiating between the two types of the delay stems from the application under consideration with regard to the delay rate of change. The argument is that type (A1) puts some upper bound on both the delay and its derivative while type (A2) does not. Then a stability method that adopts type (A2) becomes applicable for any system regardless of its delay rate of change. The supporters of this research direction considers it as a method for fast dynamics in comparison to a stability method uses type (A1), was considered with $\mu < 1$ to be only applicable for slow dynamics. Even with this limitation, a time-delay system has slow dynamics can more easily satisfy the conditions based on (A1). The elimination of the condition $\mu < 1$ is a great contribution in its own.

3.3.8 Interval Time Delays

Generalizing both delay types (A1) and (A2) lead to consideration of interval time-varying delay of the type

(A3) $\varphi \le \tau(t) \le \varrho, \quad \dot{\tau}(t) < \mu$
(A4) $\varphi \le \tau \le \varrho$

where $0 < \varphi < \varrho$ are known constants. Obviously, types (A3) and (A4) generalize (A1) and (A2), respectively.

Basically, the combined use of the free-weighting matrices method for systems with interval time delays opens a new systematic approach to look at the stability-stabilization problem and develops a delay-dependent stability result that is applicable to wider classes of time delay systems. Note that the condition $\mu < 1$ is no longer applicable, and the criticism of fast dynamic is resolved by this because μ may take any value. It was shown that this system formulation serves as a general setup to the extent that previously published methods can be considered as special cases.

To provide a numerical evaluation, the following example is implemented by several methods

$$A_o = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix} \quad A_d = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix} \tag{3.49}$$

For every method, the bound μ was fixed at some value and from the LMI feasibility testing, the largest τ is recorded. Typical values of μ were selected as $0,\ 0.5,\ 0.9$, and 3. In Table 3.1 the values of the maximum allowable value of τ are presented.

Table 3.1 Comparison between different methods: largest τ

Method	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$	$\mu = 3$
[65]	4.47	2.0	1.180	X
[392]	4.472	2.008	1.180	X
[128]	4.472	2.008	1.180	0.999
[155]	4.472	2.008	1.180	0.999
[198]	4.472	2.008	1.180	X
[124]	4.472	2.0430	1.3780	1.3450

We first notice that there is a negligible difference between selected methods since they are using the same Lyapunov functional. The effect of free-weighting matrices is small except for the method of [124] because the Lyapunov function there has an additional term, which gives the cited method some advantage over the others.

From the table it is clear that the method of [124] is computationally superior and, therefore in the subsequent work, we will take it as a reference to compare our results with. Another point to notice is that the recent method of [124] was not extended to deal with stabilization through state or observer feedback and investigate the introduction of polytopic type or norm-bounded uncertainties.

3.3.9 Improved Stability Method

Consider the class of linear time-delay systems

$$\dot{x}(t) = A_o x(t) + A_{do} x(t - \tau(t)) \tag{3.50}$$

where $x(t) \in \Re^n$ is the state vector, $A_o \in \Re^{n\times n}$ and $A_{do} \in \Re^{n\times n}$ are real and known constant matrices. The delay $\tau(t)$ is a differentiable time-varying function satisfying

$$0 < \tau(t) \leq \varrho, \quad \dot{\tau}(t) \leq \mu \tag{3.51}$$

where the bounds ϱ and μ are known constant scalars. The following theorem summarizes an improved stability method based on the free-weighting matrices approach.

Theorem 3.22 *Given $\varrho > 0$ and $\mu > 0$. System (3.50) is delay-dependent asymptotically stable if there exist weighting matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{R} > 0,\ \mathcal{W} > 0$ and parameter matrices N_a N_c satisfying the following LMI*

$$\Xi = \begin{bmatrix} \Xi_o & \varrho\mathcal{N} & \Xi_c \\ \bullet & -\varrho\mathcal{W} & 0 \\ \bullet & \bullet & -\varrho\mathcal{W} \end{bmatrix} < 0 \tag{3.52}$$

where

$$\Xi_o = \begin{bmatrix} \Xi_{o1} & \Xi_{o2} & N_a \\ \bullet & \Xi_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix},\ \mathcal{N} = \begin{bmatrix} N_a \\ N_c \\ 0 \end{bmatrix},\ \Xi_c = \begin{bmatrix} \varrho A_o^t\mathcal{W} \\ \varrho A_d^t\mathcal{W} \\ 0 \end{bmatrix} \tag{3.53}$$

$$\begin{aligned} \Xi_{o1} &= \mathcal{P}A_o + A_o^t\mathcal{P}^t + \mathcal{Q} + \mathcal{R} + N_a + N_a^t \\ \Xi_{o2} &= \mathcal{P}A_{do} - 2N_a + N_c^t \\ \Xi_{o3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t \end{aligned} \tag{3.54}$$

Proof Consider the Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned} V(t) &= V_o(t) + V_m(t) + V_c(t) + V_a(t) \\ V_o(t) &= x^t(t)\mathcal{P}x(t),\ V_a(t) = \int_{t-\tau(t)}^{t} x^t(s)\mathcal{Q}x(s)\,\mathrm{d}s \\ V_c(t) &= \int_{t-\varrho}^{t} x^t(s)\mathcal{R}x(s)\mathrm{d}s \\ V_m(t) &= \int_{-\varrho}^{0}\int_{t+s}^{t} \dot{x}^t(\alpha)\mathcal{W}\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}s \end{aligned} \tag{3.55}$$

where $0 < \mathcal{P},\ 0 < \mathcal{W},\ 0 < \mathcal{Q},\ 0 < \mathcal{R}$ are weighting matrices of appropriate dimensions. The first term in (3.55) is standard to the delay-less nominal system while the second and fourth terms correspond to the delay-dependent conditions and the third term is introduced to compensate for the enlarged time interval from $t-\varrho \to t$ to $t-\tau \to t$. A straightforward computation gives the time derivative of $V(x)$ along the solutions of (3.50) as

$$\dot{V}_o(t) = 2x^t \mathcal{P}[A_o x(t) + A_{do} x(t-\tau)] \tag{3.56}$$

$$\begin{aligned} \dot{V}_a(t) &= x^t(t)\mathcal{Q}x(t) - (1-\dot{\tau})\, x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\ &\le x^t(t)\mathcal{Q}x(t) - (1-\mu)\, x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \end{aligned} \tag{3.57}$$

$$\dot{V}_c(t) = x^t(t)\mathcal{R}x(t) - x^t(t-\varrho)\mathcal{R}x(t-\varrho) \tag{3.58}$$

$$\dot{V}_m(t) = \varrho\, \dot{x}^t(t)\mathcal{W}\dot{x}(t) - \int_{t-\varrho}^{0} \dot{x}^t(s)\mathcal{W}\dot{x}(s)\mathrm{d}s \tag{3.59}$$

In terms of

$$\xi(t) = \left[x^t(t)\ x^t(t-\tau(t))\ x^t(t-\varrho) \right]^t$$

and using the classical Leibniz rule $x(t-\theta) = x(t) - \int_{t-\theta}^{t} \dot{x}(s)\mathrm{d}s$ for any matrices $N_a,\ N_c$ of appropriate dimensions and using $\mathcal{N}$ from (3.53), the following equations hold:

$$\begin{aligned} &2\,\xi^t(t)(2\mathcal{N})\left[- \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s + x(t) - x(t-\tau) \right] = 0 \\ &2\,\xi^t(t)(-\mathcal{N})\left[- \int_{t-\varrho}^{t} \dot{x}(s)\mathrm{d}s + x(t) - x(t-\varrho) \right] = 0 \end{aligned} \tag{3.60}$$

From (3.55), (3.56), (3.57), (3.58), and (3.59) and using (3.60), we have

$$\begin{aligned} \dot{V}(t)|_{(3.50)} \le\ & x^t(t)\left[\mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t \right] x(t) \\ & - x^t(t-\varrho)\mathcal{R}x(t-\varrho) \\ & + 2x^t(t)\left[\mathcal{P}A_{do} - 2N_a + N_c^t \right] x(t-\tau) \\ & + 2x^t(t)N_a x(t-\varrho) + 2x^t(t-\tau)N_c x(t-\varrho) \\ & - x^t(t-\tau)\left[(1-\mu)\mathcal{Q} + 2N_c + 2N_c^t \right] x(t-\tau(t)) \\ & - 2\,\xi^t(t)(2\mathcal{N}) \int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s - \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)\mathrm{d}s \\ & + 2\,\xi^t(t)\mathcal{N} \int_{t-\varrho}^{t} \dot{x}(s)\mathrm{d}s \\ & + \varrho\, \xi^t(t)\left[A_o\ A_{do}\ 0 \right]^t \mathcal{W} \left[A_o\ A_{do}\ 0 \right] \end{aligned} \tag{3.61}$$

where $\dot{V}(x)|_{(3.50)}$ defines the Lyapunov derivative along the solutions of system (3.50). Regrouping the terms of (3.61) leads to

$$\begin{aligned}
\dot{V}(t)|_{(3.50)} &= \xi^t(t)\Xi_o\xi(t) \\
&\quad - \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)\mathrm{d}s \\
&\quad + \xi^t(t)\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}\mathcal{W}\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}^t \xi(t) \\
&\quad - 2\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t}\dot{x}(s)\mathrm{d}s - 2\xi^t(t)(-\mathcal{N})\int_{t-\varrho}^{t-\tau(t)}\dot{x}(s)\mathrm{d}s \\
&\le \xi^t(t)\Xi_o\xi(t) + \xi^t(t)\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}\mathcal{W}\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}^t \xi(t) \qquad (3.62)
\end{aligned}$$

where matrices Ξ_o, $\mathcal{N}$ are given in (3.53). From (3.52) and Schur complements, it follows from (3.62) that $\dot{V}(t)|_{(3.50)} < 0$, which establishes the desired asymptotic stability.

Remark 3.23 It is significant to recognize that the foregoing method, based on the implementation requirements of the stability conditions, provides a substantial improvement over the recently developed free-weighting matrices method of [124]. Hence it is expected to yield less-conservative delay-dependent stability results in terms of two aspects. One aspect would be due to reduced computational load as evidenced by a simple comparison with less number of manipulated variables and faster processing. Another aspect arises by noting that LMIs (3.52), (3.53), and (3.54) theoretically cover the results of [155, 181, 188] as special cases. Furthermore, in the absence of delay ($A_d \equiv 0$, $\mathcal{Q} \equiv 0$, $\mathcal{W} \equiv 0$), it is easy to infer that LMIs (3.52) and (3.54) will eventually reduce to a parametrized delay-independent criteria.

3.3.10 Delay-Partitioning Projection Method

To shed light on the delay-partitioning projection approach, we consider the time-delay model

$$\dot{x}(t) = A_o x(t) + A_{do} x\left(t - \sum_{j=1}^{p} \tau_j\right) \qquad (3.63)$$

$$x(t) = \phi, \qquad \forall\, t \in [-\hat{\tau}, 0] \qquad (3.64)$$

where the scalars $\tau_j > 0$, $j = 1, ..., p$ and $\sum_{j=1}^{p} \tau_j \le \hat{\tau}$. Thus, the factors τ_j, $j = 1, ..., p$ represents a partition of the lumped delay $\hat{\tau}$. We proceed by letting

$$\alpha_o = 0, \quad \alpha_k = \sum_{j=1}^{k} \tau_j$$

By choosing the Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned}
V(t) &= V_o(t) + V_m(t) + V_c(t) + V_a(t) \\
V_o(t) &= x^t(t)\mathcal{P}x(t), \ V_a(t) = \int_{t-\tau(t)}^{t} x^t(s)\mathcal{R}x(s)\,\mathrm{d}s \\
V_c(t) &= \sum_{j=1}^{p} \int_{t-\alpha_j}^{t-\alpha_{j-1}} x^t(s)\mathcal{Q}_j x(s)\mathrm{d}s \\
V_m(t) &= \int_{-\varrho}^{0} \int_{t+s}^{t} \dot{x}^t(\alpha)\mathcal{W}\dot{x}(\alpha)\mathrm{d}\alpha\ \mathrm{d}s
\end{aligned} \tag{3.65}$$

where $0 < \mathcal{P},\ 0 < \mathcal{W},\ 0 < \mathcal{Q}_j,\ 0 < \mathcal{R}$. The following result can be obtained

Theorem 3.24 *System (3.63) and (3.64) is delay-dependent asymptotically stable if there exist weighting matrices* $\mathcal{P} > 0,\ \mathcal{Q}_j > 0,\ \mathcal{R} > 0,\ \mathcal{W} > 0,\ j = 1, ..., p$ *satisfying the following LMI*

$$\mathcal{B}^{\perp t} \begin{bmatrix} \Xi_v + \Xi_w & 0 \\ 0 & \Xi_s \end{bmatrix} \mathcal{B}^{\perp} < 0 \tag{3.66}$$

where $\mathcal{B}^{\perp} \in \Re^{(2p+1)n \times (p+1)n}$ *is the orthogonal complement of*

$$\mathcal{B} = \begin{bmatrix} I & -I & 0 & \dots & 0 & -I & 0 \dots & 0 \\ 0 & I & -I & \dots & 0 & -I & 0 \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & \dots & I & -I & 0 & 0 \dots & -I \end{bmatrix} \tag{3.67}$$

and

$$\begin{aligned}
\Xi_v &= \begin{bmatrix} \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q}_1 & 0 & \dots 0 & \mathcal{P}A_d \\ \bullet & \mathcal{Q}_2 - \mathcal{Q}_1 & \dots 0 & 0 \\ \vdots & \vdots & \ddots \vdots & \vdots \\ \bullet & \bullet & \dots \bullet & -\mathcal{Q}_p \end{bmatrix} \\
\Xi_w &= \left(\sum_{j=1}^{p} \hat{\tau}_j\right)\left[A_o\ 0\ \dots\ 0\ A_d\right]^t \mathcal{W}\left[A_o\ 0\ \dots\ 0\ A_d\right] \\
\Xi_s &= \mathrm{diag}\left[-\hat{\tau}_1^{-1}\mathcal{W},\ \dots,\ -\hat{\tau}_p^{-1}\mathcal{W}\right].
\end{aligned} \tag{3.68}$$

3.3.11 Numerical Examples

To complete the picture, the following examples provide numerical evaluations

Illustrative Example A

$$A_o = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.5 & -1 \\ 0 & -0.6 \end{bmatrix}$$

In terms of the number of system variables N_v, the number of LMI iterations to reach a feasible solution N_i, the total elapsed time T_e to reach at a desirable ϱ and the maximum ϱ, a sequence of numerical experiments is performed on a standard computing facility.[1] Table 3.2 contains a summary of the computational results of our methods as compared to the other existing method.

Table 3.2 Computational summary with $\mu = 2$: example 1

Method	N_v	N_i	T_e	ϱ
[124]	54	100	14.27 s	0.9
Theorem 3.22	20	100	3.77 s	1.1

Illustrative Example B

An open-loop stable time-delay system for chemical reactor is considered here as a state-feedback design [177]. In the reactor, raw materials A and B take part in three chemical reactions that produce a product P along with some other by products. By linearization and time scaling, the state variables are the deviations from their nominal values in the weight composition of reactant A, in the weight composition of reactant B, in the weight composition of intermediate product C and in the weight composition of reactant P. The control variables are relative deviations in the feed rates. Using typical values [177], the model matrices are

$$A_o = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.2 & -5.3 & -12.8 & 0 \\ 6.4 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \; A_d = \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix}$$

The open-loop system response is plotted in Fig. 3.1. In Table 3.3, a summary of the computational results of our method as compared to the other existing method.

[1] This is comprised of Intel Core Duo- 2.66 GHz both processors with 980MB RAM employing Matlab 7.

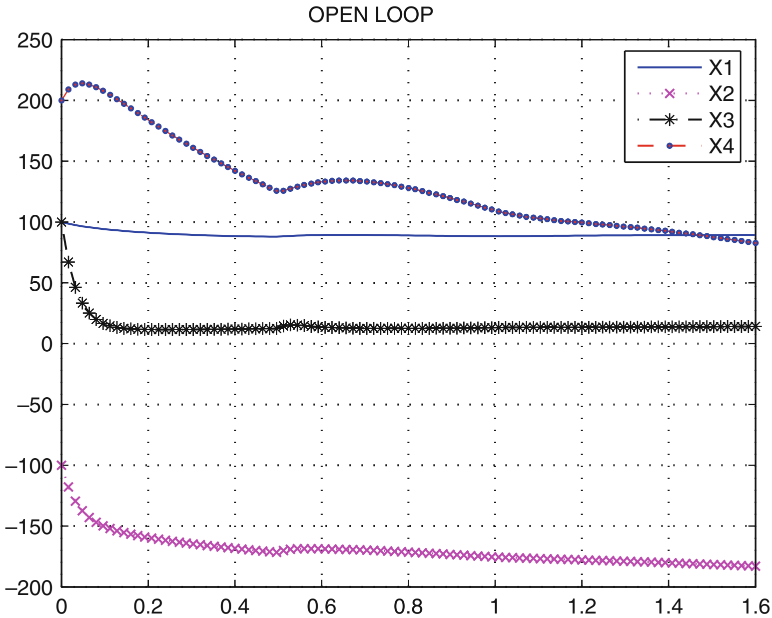

Fig. 3.1 Open-loop state trajectories: example 2

Table 3.3 Computational summary with $\mu = 2$: example 3.2

Method	N_v	N_i	T_e	ϱ
[121, 123, 124, 128]	204	10	14.66 s	0.652
Theorem 3.22	72	10	2.295 s	0.874

It is evidently clear that the improved method is quite superior to [121, 123, 124, 128] since the computational time is much less and, in addition, their storage requirement is almost three times that of our method which is quite excessive. More importantly

Illustrative Example C

The example is used [418] and has the following matrices

$$A_o = \begin{bmatrix} -2 & 0 \\ 0 & -.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

In this example every time a set of values is put for μ and h_l. Then the maximum h_u above which the system fails to satisfy the theorem condition is used in Table 3.3 to compare their results, and the result obtained by the method developed in this book (Table 3.4):

Table 3.4 Comparison between different methods

φ	Method	$\mu = 0.5$	$\mu = 0.9$
$\varphi = 0$	[152]	1.01	1.01
$\varphi = 0$	[418]	2.04	1.37
$\varphi = 0$	Proposed	2.33	1.87
$\varphi = 2$	[152]	2.39	2.39
$\varphi = 2$	[418]	2.43	2.43
$\varphi = 2$	Proposed	4.472	2.6
$\varphi = 4$	[152]	4.06	4.06
$\varphi = 4$	[418]	4.07	4.07
$\varphi = 4$	Proposed	4.09	4.09

3.4 Stability Approaches: Discrete Time

Less attention has been paid to discrete-time systems with a time-delay because a linear discrete-time system with a constant time-delay can be transformed into a delay-free system by means of a state-augmentation approach [207]. However this approach is not suitable for systems with either unknown or time-varying delays. Over the past decade, several articles have appeared on this topic. There are two types of time delays discussed in the literature. For small time-varying delays [34], the descriptor model transformation approach was employed to study the delay-dependent guaranteed-cost control of uncertain discrete-time delay systems, and in [12, 34, 168, 354]. In later chapters, we present switched time-delay models of continuous-time and discrete-time dynamical systems. The purpose is to lay down the mathematical formulations needed in the subsequent study on stability and feedback stabilization of linear time delay (LTD) and nonlinear time-delay (NTD) systems. Specifically, we will seek to generalize the formulations in order to encompass the widespread analytical results.

3.4.1 A Discrete-Time Model

A class of discrete-time systems with state delay is represented by

$$x(k+1) = A_o x(k) + D_o x(k - d(k)) \tag{3.69}$$

where for $k \in \mathsf{Z}_+ \overset{\Delta}{=} \{0, 1, ...\}$ and $x(k) \in \Re^n$ is the state, control, and $A_o \in \Re^{n\times n}$, $D_o \in \Re^{n\times n}$ are constant matrices. The delay factor $d(k)$ is unknown but bounded in the form

$$0 < d_m \leq d(k) \leq d_M, \quad d_s = d_M - d_m + 1 \tag{3.70}$$

where the scalars d_m and d_M represent the lower and upper bounds, respectively, and d_s denotes the number of samples within the delay interval. By setting $d(k) = \equiv 0$

in (3.69), it is readily seen that $|\lambda(A_o + A_d)| < 1$ is a necessary condition for stability of system (3.69). From now onwards, we assume that this is the case.

Remark 3.25 The class of systems (3.69) represents a nominally linear model emerges in many areas dealing with the applications functional difference equations or delay-difference equations [216]. These applications include cold rolling mills, decision-making processes, and manufacturing systems. Related results for a class of discrete-time systems with time-varying delays can be found in [24] where delay-dependent stability and stabilization conditions are derived. It should be stressed that although we consider only the case of single time delay, extension to multiple time-delay systems can be easily attained using an augmentation procedure.

3.4.2 Lyapunov Theorem

Intuitively if we associate with system (3.69) a positive-definite Lyapunov–Krasovskii functional $V(k, x(k)) > 0$ and we find its first difference $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative definite along the solutions of (3.69), then the origin of system (3.69) is globally asymptotically stable. Formally, we present the following theorem for discrete-time systems of the type (3.69):

Theorem 3.26 *The equilibrium* **0** *of the discrete-time system*

$$x(k+1) = h(x(k)) \tag{3.71}$$

is globally asymptotically stable if there is a function $V : \{0, 1, 2, ...\} \times \Re^n \rightarrow \Re$ *such that*

- $V(k, x(k))$ *is a positive-definite function, decrescent, and radially unbounded,*
- $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ *is negative definite along the solutions of system (3.69)*

For arbitrary value of $d(k)$, denote

$$z(k) = \begin{bmatrix} x(k) \\ \vdots \\ x(k - d(k)) \end{bmatrix}$$

We have

$$z(k+1) = \begin{bmatrix} A_o & 0 & \dots & 0 & D_o \\ I & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} z(k) \tag{3.72}$$

It is obvious that system (3.69) is globally asymptotically stable if and only if system (3.71) is globally asymptotically stable. For system (3.71), we define

$$\hat{V}(k, z(k)) = z^t(k) \begin{bmatrix} \mathcal{P} & 0 & \dots & 0 \\ 0 & \mathcal{Q} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{Q} \end{bmatrix} z(k) \tag{3.73}$$

where $\mathcal{P} > 0,\ \mathcal{Q} > 0$. It is easy to see that $\hat{V}(k, z(k)) > 0,$ decrescent, and radially unbounded and hence system (3.71) thereby system (3.69) is globally asymptotically stable.

There are two main classes of stability analysis that have been investigated in the literature, namely delay-dependent and delay-independent conditions. For a discrete-time delay system whose stability does not depend on the time-delay value, the analysis performed through delay-dependent conditions can be very conservative. Also, delay-independent conditions cannot be obtained as a limit case of delay-dependent ones just by imposing the maximum delay value $d_M \to \infty$, leading to a gap between these two types of delay-stability conditions.

3.4.3 Delay-Independent Stability

Given weighting matrices $0 < \mathcal{P}^t = \mathcal{P},\ 0 < \mathcal{Q}^t = \mathcal{Q}$ of appropriate dimensions. By selecting the Lyapunov–Krasovskii functional

$$V(k) = x^t(k)\mathcal{P}x_j(k) + \sum_{m=k-d(k)}^{k-1} x^t(m)\mathcal{Q}x_j(m) \tag{3.74}$$

and invoking the Lyapunov–Krasovskii theorem, the following stability condition can be derived [206]:

Theorem 3.27 *The discrete-delay system (3.69) is asymptotically stable if there exist matrices $\mathcal{P} > 0$ and $\mathcal{Q} > 0$ such that*

$$\begin{bmatrix} -(\mathcal{P} - \mathcal{Q}) & 0 & A_o^t\mathcal{P} \\ \bullet & -\mathcal{Q} & D_o^t\mathcal{P} \\ \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{3.75}$$

We stress that LMI (3.75) is virtually delay independent since it is satisfied, regardless of the size of delay $d(k)$.

3.4.4 Delay-Dependent Stability

In the sequel, sufficient delay-dependent LMI-based stability conditions are given. The approach used here does not introduce any dynamics and leads to a product separation between the matrices of the system and those from the

Lyapunov–Krasovskii functional. The following theorem provides some LMI conditions depending on the values d_m and d_M.

Theorem 3.28 *Given the delay sample number d_s. System (3.69) subject to (3.70) is delay-dependent asymptotically stable if one of the following equivalent conditions is satisfied*

(A) *there exist matrices* $0 < \mathcal{P} \in \Re^{n\times n},\ 0 < \mathcal{Q} \in \Re^{n\times n}$ *such that*

$$\Xi_a = \begin{bmatrix} A_o^t \mathcal{P} A_o + d_s \mathcal{Q} - \mathcal{P} & A_o^t \mathcal{P} D_o^t \\ \bullet & D_o^t \mathcal{P} D_o - \mathcal{Q} \end{bmatrix} < 0 \tag{3.76}$$

(B) *there exist matrices* $0 < \mathcal{P} \in \Re^{n\times n},\ 0 < \mathcal{Q} \in \Re^{n\times n},\ \mathcal{X} \in \Re^{n\times n},\ \mathcal{Y} \in \Re^{n\times n}$, *and* $\mathcal{Z} \in \Re^{n\times n}$ *such that*

$$\Xi_c = \begin{bmatrix} \mathcal{P} + \mathcal{X} + \mathcal{X}^t & \mathcal{Y} - \mathcal{X} A_o & \mathcal{Z} - \mathcal{X} D_o \\ \bullet & \Gamma_v & -A_o^t \mathcal{Z}^t - \mathcal{Y} D_o \\ \bullet & \bullet & \Gamma_w \end{bmatrix} < 0 \tag{3.77}$$

where

$$\begin{aligned} \Gamma_v &= -A_o^t \mathcal{Z}^t - \mathcal{Y} A_o + d_s \mathcal{Q} - \mathcal{P} \\ \Gamma_v &= -\mathcal{Q} - \mathcal{Z} D_o - D_o^t \mathcal{Z}^t \end{aligned} \tag{3.78}$$

In this case, the Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned} \tilde{V}(k) &= V_o(k) + V_a(k) + V_c(k) > 0 \\ V_o &= x^t(k) \mathcal{P} x(k), \quad V_a(k) = \sum_{m=k-d(k)}^{k-1} x^t(m) \mathcal{Q} x(m) \\ V_c &= \sum_{s=2-d_M}^{1-d_m} \sum_{m=k+s-1}^{k-1} x^t(m) \mathcal{Q} x(m) \end{aligned} \tag{3.79}$$

is such that

$$\Delta \tilde{V}(k) < 0, \quad \forall \left[x^t(k)\ x^t(k-d(k)) \right]^t \neq \mathbf{0} \tag{3.80}$$

Proof The positivity of the LKF (3.79) is guaranteed by the requirement that $0 < \mathcal{P} \in \Re^{n\times n},\ 0 < \mathcal{Q} \in \Re^{n\times n}$. Next, it is necessary to verify (3.80). A straightforward computation gives the first-difference of $\Delta \tilde{V}(k) = \tilde{V}(k+1) - \tilde{V}(k)$ along the solutions of (3.69) as

$$\begin{aligned} \Delta V_o(k) &= [A_o x(k) + D_o x(k-d(k))]^t \mathcal{P} [A_o x(k) + D_o x(k-d(k))] \\ &\quad - x^t(k) \mathcal{P} x(k) \end{aligned} \tag{3.81}$$

$$\Delta V_a(k) = x^t(k)\mathcal{Q}x(k) - x^t(k-d_j(k))\mathcal{Q}x(k-d_j(k)) + \sum_{m=k+1-d(k+1)}^{k-1} x^t(m)\mathcal{Q}x(m) - \sum_{m=k+1-d(k)}^{k-1} x^t(m)\mathcal{Q}x(m) \quad (3.82)$$

$$\Delta V_c(k) = (d_M - d_m)x^t(k)\mathcal{Q}x(k) - \sum_{m=k+1-d_M)}^{k-d_m} x^t(m)\mathcal{Q}x(m) \quad (3.83)$$

Observe from (3.82) that

$$\begin{aligned}\sum_{m=k+1-d(k+1)}^{k-1} x^t(m)\mathcal{Q}x(m) &= \sum_{m=k+1-d_m}^{k-1} x^t(m)\mathcal{Q}x(m) + \sum_{m=k+1-d(k+1)}^{k-d_m} x^t(m)\mathcal{Q}x(m) \\ &\leq \sum_{m=k+1-d(k)}^{k-1} x^t(m)\mathcal{Q}x(m) + \sum_{m=k+1-d_M}^{k-d_m} x^t(m)\mathcal{Q}x(m)\end{aligned} \quad (3.84)$$

Then using (3.84) into (3.82) and manipulating, we reach

$$\Delta V_a(k) \leq x^t(k)\mathcal{Q}x(k) - x^t(k-d(k))\mathcal{Q}x(k-d(k)) + \sum_{m=k+1-d_M}^{k-d_m} x^t(m)\mathcal{Q}x(m) \quad (3.85)$$

Taking into consideration (3.81), (3.83), and (3.85), the following upper bound for $\Delta\tilde{V}(k)$ can be obtained:

$$\Delta\tilde{V}(k) \leq [A_o x(k) + D_o x(k-d(k))]^t \mathcal{P}[A_o x(k) + D_o x(k-d(k))] + x^t(k)[d_s\mathcal{Q} - \mathcal{P}]x(k) - x^t(k-d(k))\mathcal{Q}x(k-d(k)) < 0 \quad (3.86)$$

By Schur complement, one gets LMI (3.76). Next, the equivalence between (3.76) and (3.77) can be established as follows. First, we note that (3.76) can be expressed as

$$\begin{bmatrix} A_o^t\mathcal{P} \\ D_o^t\mathcal{P} \end{bmatrix} \mathcal{P}^{-1} \begin{bmatrix} A_o^t\mathcal{P} \\ D_o^t\mathcal{P} \end{bmatrix}^t - \begin{bmatrix} \mathcal{P} - d_s\mathcal{Q} & 0 \\ \bullet & \mathcal{Q} \end{bmatrix} < 0 \quad (3.87)$$

which by Schur complement is equivalent to

$$\begin{bmatrix} -\mathcal{P} & \mathcal{P}A_o\mathcal{P} & \mathcal{P}D_o \\ \bullet & d_s\mathcal{Q}-\mathcal{P} & 0 \\ \bullet & \bullet & -\mathcal{Q} \end{bmatrix} < 0 \tag{3.88}$$

Obviously, the equivalence between (3.76) and (3.77) is the same as that between (3.77) and (3.88). Hence, if (3.88) is verified, then (3.77) is true for $\mathcal{X} = \mathcal{X}^t = -\mathcal{P}$, $\mathcal{Y} = \mathcal{Z} \equiv 0$. On the other hand, if (3.77) is verified, then $\Xi_a = T^t\ \Xi_c\ T$ with

$$T = \begin{bmatrix} A_o & D_o \\ I & 0 \\ 0 & I \end{bmatrix} < 0 \tag{3.89}$$

completes the proof. ■

The result of **Theorem 3.28** has been developed in [23, 185, 208, 222] using alternative analytical directions.

3.4.5 Descriptor Model Transformation

Let $y(k)$ denote the state increment, that is

$$y(k) = x(k+1) - x(k) \tag{3.90}$$

then in line with the continuous-time case, system (3.69) can be represented by the following descriptor form

$$\begin{bmatrix} x(k+1) \\ 0 \end{bmatrix} = \begin{bmatrix} y(k) + x(k) \\ -y(k) + A_o\, x(k) - x(k) + D_o x(k-d(k)) \end{bmatrix} \tag{3.91}$$

Recall by successive iterations on (3.90) that

$$x(k-d(k)) = x(k) - \sum_{j=k-d(k)}^{k-1} y(j)$$

and letting

$$\xi(k) \triangleq \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}, \quad E^t = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \tilde{A}_o = A_o + D_o$$

it follows that

$$\begin{aligned} E\ \xi(k+1) &= \begin{bmatrix} I & I \\ \tilde{A}_o - I & -I \end{bmatrix} \xi(k) - \begin{bmatrix} 0 \\ D_o \end{bmatrix} \sum_{j=k-d(k)}^{k-1} y(j) \\ &= \bar{A}_o\ \xi(k) - \bar{D}_o \sum_{j=k-d(k)}^{k-1} y(j) \end{aligned} \tag{3.92}$$

where the initial conditions are characterized by

$$\xi(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \psi(0) \\ (A_o - I)\psi(0) - D_o\psi_{-d(0)} \end{bmatrix} \tag{3.93}$$

In short, if $x(k)$ is a solution of system (3.69), then $\xi(k) = \{x(k),\ \ y(k)\}$ is a solution of the free descriptor system (3.92) subject to (3.93) and the reverse is true. This is the essence of descriptor transformation.

Now we have the following result:

Theorem 3.29 *Consider system (3.69). If there exists continuous functional*

$$V(k) \triangleq V(x(k-d_M), \ldots, x(k), y(k-d_M), \ldots, y(k-1)) \tag{3.94}$$

such that

$$\begin{aligned} 0 \ \leq\ V(k) &\leq \omega \max\left\{ \max_{k-d^+ \leq j \leq k} |x(j)|^2, \max_{k-d^+ \leq j \leq k-1} |y(j)|^2 \right\} \\ V(k+1) - V(k) &\leq -\kappa\ |x(k)|^2 \end{aligned} \tag{3.95}$$

for x_k and y_k satisfying (3.92), then system (3.69) is asymptotically stable

Proof Summing up (3.95), it follows that

$$\sum_{j=0}^{k} V(k+1) - V(k) \ = \ V(k+1) - V(0) \ \leq \ -\kappa \sum_{j=0}^{k} |x(k)|^2$$

Since $x(k)$ and $y(k)$ are satisfying (3.92), then (3.95) implies that

$$\begin{aligned} |x(k)|^2 \ &\leq\ \sum_{j=0}^{k} |x(j)|^2 \ \leq\ \kappa^{-1} V(0) \\ &\leq\ \kappa^{-1}\omega\ \max\left\{ \max_{-d^+ \leq j \leq 0} |x(j)|^2\ ,\ \max_{-d^+ \leq j \leq -1} |x(j)|^2 \right\} \\ &\forall k\ \geq\ 0 \end{aligned} \tag{3.96}$$

If $x(k)$ is a solution of (3.69) and $x(k)$ is defined by (3.90), then $\{x(k), \ y(k)\}$ satisfies (3.91), (3.92), and (3.93) and hence (3.96) holds.

Note that (3.96) implies that if $\max_{j\in[-d_M,0]} \ |\psi(-j)|^2$ is sufficiently small, then $|x(k)|^2$ is sufficiently small and subsequently $\sum_{j=0}^{\infty} \ |x(j)|^2 \ < \ \infty$. Therefore $|x(j)|^2 \rightarrow 0$ as $j \rightarrow \infty$.

To derive tractable conditions for stability, we introduce the Lyapunov–Krasovskii functional

$$\begin{aligned} V(k) &= V_a(k) + V(k)_c + V(k)_e \\ V_a(k) &= x^t(k)\mathcal{P}_o x(k) = \xi^t(k)E^t\mathcal{P}E\xi(k) \\ &= \xi^t(k)E^t \begin{bmatrix} \mathcal{P}_o & 0 \\ \mathcal{P}_e & \mathcal{P}_c \end{bmatrix} E\xi(k), \ \ \mathcal{P}_o > 0 \end{aligned} \tag{3.97}$$

$$\begin{aligned} V_c(k) &= \sum_{p=-d_M}^{-1} \sum_{j=k+p}^{k-1} y^t(j)\mathcal{Q}y(j), \ \ \mathcal{Q} > 0 \\ V_e(k) &= \sum_{j=k-d}^{k-1} x^t(k)\mathcal{W}x(k), \ \ \mathcal{W} > 0 \end{aligned} \tag{3.98}$$

We observe that $V(k)$ is constructed from three terms: $V_a(k)$ signifies necessary and sufficient conditions for the stability of discrete descriptor system without delay [221], $V_c(k)$ corresponds to delay-dependent criteria [216], and $V_e(k)$ is common for delay-independent stability conditions.

For simplicity in exposition, we introduce the following matrix expressions:

$$\mathcal{W}_q = \begin{bmatrix} \mathcal{W} & 0 \\ \bullet & d_M\mathcal{Q} \end{bmatrix}, \ \ \begin{bmatrix} \mathcal{Z} & \mathcal{Y} \\ \bullet & \mathcal{Q} \end{bmatrix} \geq 0 \ , \ \ \mathcal{Z} \in \Re^{2n\times 2n}, \ \ \mathcal{Y} \in \Re^{2n\times n} \tag{3.99}$$

The following theorem establishes LMI-based sufficient conditions for asymptotic stability of system (3.69):

Theorem 3.30 *Consider system (3.69) and the delay factor $d_k = d$ being an unknown constant satisfying $0 \le d(k) \le d_M$. Given matrices $0 < \mathcal{Q} = \mathcal{Q}^t \in \Re^{n\times n}$, $0 < \mathcal{W} = \mathcal{W}^t \in \Re^{n\times n}$, this system is asymptotically stable if there exist matrices $0 < \mathcal{P} = \mathcal{P}^t \in \Re^{2n\times 2n}$, $\mathcal{Y} \in \Re^{2n\times n}$, and $\mathcal{Z} \in \Re^{2n\times 2n}$ satisfying (3.99) and the LMI*

$$\Upsilon \stackrel{\Delta}{=} \begin{bmatrix} \Upsilon_o & \mathcal{Y} - \bar{A}_o^t\mathcal{P}\bar{D}_o \\ \bullet & -\mathcal{W} + \bar{D}_o^t\mathcal{P}\bar{D}_o \end{bmatrix} < 0 \tag{3.100}$$

where

$$\Upsilon_o = \bar{A}_o^t\mathcal{P}\bar{A}_o - E^t\mathcal{P}E + \mathcal{W}_q + d_M\mathcal{Z} + \mathcal{Y}E + E^t\mathcal{Y}^t$$

The proof of this theorem can be found in [221] and alternative forms can be found in [22, 67, 71, 267]. We record that the developed results have been obtained by using the bounding inequality (3.20).

3.4.6 Improved Stability Methods

Much like the continuous time-delay systems, improved delay-dependent stability criteria can be developed by constructing more appropriate Lyapunov–Krasovskii functionals. In these criteria, trade-off arises between the extra added components and the use of bounding inequalities. One such criterion is developed in [82, 85, 284] based on the LKF

$$
\begin{aligned}
\tilde{V}(k) &= V_o(k) + V_a(k) + V_c(k) + V_s \\
V_o &= x^t(k)\mathcal{P}x(k), \quad V_a(k) = \sum_{m=k-d(k)}^{k-1} x^t(m)\mathcal{Q}x(m) \\
V_c &= \sum_{s=2-d_M}^{1-d_m} \sum_{m=k+s-1}^{k-1} x^t(m)\mathcal{Q}x(m) \\
V_s &= \sum_{s=-d_M}^{-1} \sum_{m=k+s}^{k-1} \delta x^t(m)\mathcal{Z}\delta x(m), \quad \delta x = x(k+1) - x(k)
\end{aligned}
\tag{3.101}
$$

and invoking the inequality (3.20). It is summarized by the following theorem

Theorem 3.31 *The discrete-delay system (3.69) subject to (3.70) is asymptotically stable if there exist matrices $\mathcal{P} > 0$, $\mathcal{Q} > 0$, $\mathcal{Z}$, $\mathcal{Y}$, and $\mathcal{W}$ satisfying the following*

$$
\begin{bmatrix}
\Omega & -(A_o + D_o)^t\mathcal{P}D_o - \mathcal{Y} + \mathcal{W}^t & -A_o^t\mathcal{P}D_o - \mathcal{Y} & d_M(A_o - I)^t\mathcal{Z} \\
\bullet & -\mathcal{Q} - \mathcal{W} - \mathcal{W}^t & -D_o^t\mathcal{P}D_o - \mathcal{W} & d_M D_o^t\mathcal{Z} \\
\bullet & \bullet & -\mathcal{Z} & 0 \\
\bullet & \bullet & \bullet & -\mathcal{Z}
\end{bmatrix}
< 0 \tag{3.102}
$$

where

$$
\Omega = A_o^t\mathcal{P}(A_o + D_o) + (A_o + D_o)^t\mathcal{P}A_o + \mathcal{Y} + \mathcal{Y}^t - \mathcal{P} + d_s\mathcal{Q}
$$

A further improvement is attained by choosing the following LKF:

$$\begin{aligned}
\widehat{V}_k &= V_{ok} + V_{ak} + V_{ck} + V_{mk} + V_{nk} \\
V_{ok} &= x^t(k)\mathcal{P}x(k), \; V_{ak} = \sum_{j=k-d(k)}^{k-1} x^t(j)\mathcal{Q}x(j) \\
V_{ck} &= \sum_{k-d_M}^{k-1} x^t(j)\mathcal{R}x(j) \\
V_{nk} &= \sum_{m=-d_M}^{-1} \sum_{j=k+m}^{k-1} \delta x^t(j)(\mathcal{W}_a + \mathcal{W}_c)\delta x(j) \\
V_{mk} &= \sum_{m=-d_M+1}^{-d_m} \sum_{j=k+m}^{k-1} x^t(j)\mathcal{Q}x(j)
\end{aligned} \tag{3.103}$$

together with the free-weighting matrix method, where $0 < \mathcal{P}, \; 0 < \mathcal{Q}, \; 0 < \mathcal{R}, \; 0 < \mathcal{W}_a, \; 0 < \mathcal{W}_c$ are weighting matrices of appropriate dimensions. The first term in (3.103) is standard to the delayless nominal system while the second and fifth correspond to the delay-dependent conditions. The third and fourth terms are added to compensate for the enlargement in the time interval from $(k-1 \rightarrow d-d_k)$ to $(k-1 \rightarrow d-d_M)$. Introduce

$$\begin{aligned}
\delta x(k) &= x(k+1) - x(k), \; x(k-d(k)) = x(k) - \sum_{j=k-d_k}^{k-1} \delta x(j) \\
\delta x(k) &= (A_o - I)x(k) + D_o\, x(k-d(k)), \; \overline{d} = (d_M - d_m + 1)
\end{aligned} \tag{3.104}$$

The following theorem provides the desired result:

Theorem 3.32 *Given the bounds $d_M > 0$ and $d_m > 0$. System (3.69) subject to (3.70) is delay-dependent asymptotically stable if there exist weighting matrices $0 < \mathcal{P}, \; 0 < \mathcal{Q}, \; 0 < \mathcal{R}, \; 0 < \mathcal{W}_a, \; 0 < \mathcal{W}_c$, and slack variable matrices $\mathcal{M}, \; \mathcal{S}, \; \mathcal{Z}$ satisfying the following LMI*

$$\Omega = \begin{bmatrix} \bar{\Omega} + \Omega_a + \Omega_a^t + \Omega_c & \Omega_z \\ \bullet & -\Omega_w \end{bmatrix} < 0 \tag{3.105}$$

where

$$\begin{aligned}
\bar{\Omega} &= \begin{bmatrix} \Omega_o & \Omega_m & 0 \\ \bullet & \Omega_s & 0 \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix} \\
\Omega_o &= A_o^t\mathcal{P}A_o - \mathcal{P} + \overline{d}\mathcal{Q} + \mathcal{R}, \; \Omega_m = A_o^t\mathcal{P}D_o, \; \Omega_s = D_o^t\mathcal{P}D_o - \mathcal{Q} \\
\Omega_a &= \begin{bmatrix} \mathcal{M} + \mathcal{Z} & \mathcal{S} - \mathcal{M} & -\mathcal{S} - \mathcal{Z} \end{bmatrix}
\end{aligned}$$

$$\Omega_c = d_M \Omega_{cc}^t \, (\mathcal{W}_a + \mathcal{W}_c) \, \Omega_{cc}, \;\; \Omega_{cc} = \left[A_o - I \;\; D_o \;\; I \right]$$
$$\Omega_z = \left[\sqrt{d_M}\mathcal{M} \; \sqrt{d_M - d_m}\mathcal{S} \; \sqrt{d_M}\mathcal{Z} \right], \;\; \Omega_w = \text{diag}\left[\mathcal{W}_a \; \mathcal{W}_a \; \mathcal{W}_c \right] \tag{3.106}$$

Along same direction of thought, another improvement is achieved by choosing the following LKF:

$$\begin{aligned}
\widehat{V}_k &= V_{ok} + V_{ak} + V_{ck} + V_{mk} + V_{nk} + V_{sk} \\
V_{ok} &= x^t(k)\mathcal{P}x(k), \;\; V_{ak} = \sum_{j=k-d(k)}^{k-1} x^t(j)\mathcal{Q}x(j) \\
V_{ck} &= \sum_{k-d_m}^{k-1} x^t(j)\mathcal{R}_a x(j) + \sum_{k-d_M}^{k-1} x^t(j)\mathcal{R}_c x(j) \\
V_{nk} &= \sum_{m=-d_M}^{-d_m-1} \sum_{j=k+m}^{k-1} \delta x^t(j)\mathcal{S}\delta x(j) \\
V_{mk} &= \sum_{m=-d_M+1}^{-d_m} \sum_{j=k+m}^{k-1} x^t(j)\mathcal{Q}x(j) \\
V_{sk} &= \sum_{m=-d_M}^{-1} \sum_{j=k+m}^{k-1} \delta x^t(j)\mathcal{W}\delta x(j)
\end{aligned} \tag{3.107}$$

where $0 < \mathcal{P}, \; 0 < \mathcal{Q}, \; 0 < \mathcal{W}, \; 0 < \mathcal{R}_a, \; 0 < \mathcal{R}_c$ are weighting matrices of appropriate dimensions. The following result is due to [435]:

Theorem 3.33 *Given the bounds $d_M > 0$ and $d_m > 0$. System (3.69) subject to (3.70) is delay-dependent asymptotically stable if there exist weighting matrices $0 < \mathcal{P}, \; 0 < \mathcal{Q}, \; 0 < \mathcal{S}, \; 0 < \mathcal{R}_a, \; 0 < \mathcal{R}_c, \; \mathcal{W}$ and slack variable matrices $\mathsf{L}_a, \; \mathsf{M}_a, \; \mathsf{N}_a, \; \mathsf{L}_c, \; \mathsf{M}_c, \; \mathsf{N}_c$ satisfying the following LMI*

$$\Omega = \begin{bmatrix} \Phi_o & \Phi_a \\ \bullet & -\Phi_s \end{bmatrix} < 0 \tag{3.108}$$

where

$$
\begin{aligned}
\Phi_o &= \begin{bmatrix} \Phi_{o1} & \Phi_{o2} & \mathsf{M}_a & -\mathsf{L}_a \\ \bullet & \Phi_{o3} & \mathsf{M}_c & -\mathsf{L}_c \\ \bullet & \bullet & -\mathcal{R}_a & 0 \\ \bullet & \bullet & \bullet & -\mathcal{R}_c \end{bmatrix} \\
\Phi_a &= \begin{bmatrix} \sqrt{d_M - d_m}\mathsf{L}_a & \sqrt{d_M - d_m}\mathsf{M}_a & \sqrt{d_M - d_m}\mathsf{N}_a \\ \sqrt{d_M - d_m}\mathsf{L}_c & \sqrt{d_M - d_m}\mathsf{M}_c & \sqrt{d_M - d_m}\mathsf{N}_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Phi_s &= \begin{bmatrix} -\mathcal{S} - \mathcal{W} & 0 & 0 \\ \bullet & -\mathcal{S} & 0 \\ \bullet & \bullet & -\mathcal{W} \end{bmatrix} \\
\Phi_{o1} &= A_o^t \mathcal{P} A_o - \mathcal{P} + \overline{d}\mathcal{Q} + \mathcal{R}_a + \mathcal{R}_c + (d_M - d_m)(A_o - I)^t \mathcal{S}(A_o - I) \\
&\quad + d_M (A_o - I)^t \mathcal{W}(A_o - I) + \mathsf{N}_a + \mathsf{N}_a^t \\
\Phi_{o2} &= A_o^t \mathcal{P} D_o + (d_M - d_m)(A_o - I)^t \mathcal{S} D_o + d_M (A_o - I)^t \mathcal{W} D_o \\
&\quad + \mathsf{L}_a + \mathsf{M}_a - \mathsf{N}_a + \mathsf{N}_c^t \\
\Phi_{o3} &= D_o^t \mathcal{P} D_o - \mathcal{Q} + (d_M - d_m) D_o^t \mathcal{S} D_o + + d_M D_o^t \mathcal{W} D_o \\
&\quad + \mathsf{L}_c^t - \mathsf{M}_c - \mathsf{M}_c^t - \mathsf{N}_c - \mathsf{N}_c^t
\end{aligned} \tag{3.109}
$$

Remark 3.34 Consideration of **Theorem 3.33** emphasizes the effective use of the LKF (3.107) and the free-weighting matrix technique thereby yielding an improved delay-dependent stability condition for the discrete time-delay system in (3.69). We note that, due to the introduction of the following two terms

$$
\sum_{k-d_m}^{k-1} x^t(j)\mathcal{R}_a x(j), \quad \sum_{m=-d_M}^{-d_m-1} \sum_{j=k+m}^{k-1} \delta x^t(j)\mathcal{S}\delta x(j)
$$

the result in **Theorem 3.33** is less conservative than those in [67, 82, 85, 125, 153].

3.4.7 *Simulation Examples*

Illustrative Example D

The example is used [185] and has the following matrices

$$
A_o = \begin{bmatrix} 0.6 & 0 \\ 0.35 & 0.7 \end{bmatrix}, \quad D_o = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}
$$

The results of simulation by different methods are summarized in Table 3.5.

Table 3.5 Computational summary: example 4

Method	d_m	d_M
[67]	2	11
[185]	2	10
[202]	2	13

Illustrative Example E

The example is used [125] and has the following data

$$A_o = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad D_o = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}$$

The values of the upper bound on d_M for various d_m are given in Table 3.6.

Table 3.6 Computational summary: example 5

d_m	2	4	6	10	12
[85]	7	8	9	12	13
[82]	13	13	14	15	17
[125]	17	17	18	20	21
[435]	18	18	20	21	23

Remark 3.35 As a closing point, one has to look for a measure of the computational complexity of the LMI-based stability conditions in order to better evaluate the various methods. A proposed measure would be in terms of

1. the number of scalar variables N_s, and
2. the number of rows in the main LMI matrix N_r.

For a symmetric $n \times n$ matrix, the number of scalar variables $\mathsf{N}_s = \frac{1}{2}n(n+1)$ while for arbitrary $n \times n$ matrix, the number of rows $\mathsf{N}_r = Rn$, where R corresponds to the number of row blocks. In the case of using **MATLAB LMI** solver [74], the computational complexity is $\mathbf{O}\left(\mathsf{N}_s^3\, \mathsf{N}_r\right)$ and, alternatively, on using the **LMI** solver **SeDuMi** [362] the computational complexity is $\mathbf{O}\left(\mathsf{N}_s^2\, \mathsf{N}_r^{2.5} + \mathsf{N}_r^{3.5}\right)$.

3.5 Notes and References

Indeed, there is voluminous literature on time-delay systems in terms of numerous papers and articles, particularly on delay-dependent analysis. It is hoped that this introductory chapter has succeeded in motivating the readers to the upcoming topics. We have reviewed some existing methods and provided new ones for delay-dependent stability for a class of nominally linear continuous-time systems with time-varying delays. Appropriate Lyapunov functionals have been constructed to exhibit the delay-dependent dynamics. Delay-dependent stability analysis has been presented in terms of theorems and we have provided some remarks and comments whenever deemed appropriate. To further follow up on the subject, the interested readers are referred to [32, 54, 65–67, 69, 71, 114, 121, 123, 124, 127, 128, 152, 155, 171, 214, 265, 373, 391, 392, 404, 418], and their references.

Chapter 4
Switched Systems

4.1 Introduction

This chapter is concerned with the main ingredients and basic notions of switched systems. For simplicity in exposition, we present the relevant topics and materials in general perspective.

4.2 Switched Systems: Overview

The past two decades have witnessed a great deal of activity in the study of switched dynamical systems. Such systems, which behave in continuous time at some levels and in discrete time at others, are at present ubiquitous. Apart from the more traditional application areas of control engineering such as aerospace and automotive engineering, they are also appearing with increasing frequency in biological systems, computer science, and computer communication networks. We record that continuous dynamical systems have been studied extensively in control theory and mathematics and discrete distributed systems have been investigated in computer science. However, the problems arising at the confluence of the two subject areas have raised a wide spectrum of questions, many of which are still not well understood. During the past 15 years, the work accomplished has been well documented in a number of monographs [192, 366], survey articles [19, 41, 47], and special issues [357–359].

Loosely speaking, switched time-delay systems are *hybrid* in the sense that the state trajectory evolution is governed by different functional dynamical equations over different polyhedral partitions $\{X_j\}$ of the state-space X. That is, the STD system modeled as an ensemble of subsystems, each of which is a valid representation of the system over a set of each partitions.

In this section, we present a summary of the basic concepts of related issues of switched systems and switched time-delay systems.

M.S. Mahmoud, *Switched Time-Delay Systems*,
DOI 10.1007/978-1-4419-6394-9_4,

4.2.1 Dynamic Model

In general, a *switched system* is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

$$\begin{aligned} \delta x(t) &= f_\sigma(x(t), u(t), v(t)), \quad x(t_o) = x_o \\ y(t) &= g_\sigma(x(t), w(t)) \end{aligned} \tag{4.1}$$

where $x(t)$ is the state, $u(t)$ is the controlled input, $y(t)$ is the measured output, $v(t)$ and $w(t)$ stand for the external signals such as perturbations, σ is the piecewise constant signal taking values from an index set $M \triangleq \{1, ..., m\}$, $f_k,\ k \in M$ are vector fields, and $g_k,\ k \in M$ are vector functions, while the symbol δ denotes the derivative operator in the continuous time $(that\ is,\ \delta x(t) = \dot{x}(t))$ and the shift forward operator in discrete time $(that\ is,\ \delta x(t) = x(t+1))$. By requesting a switching signal to be piecewise constant, we mean that the switching signal $\sigma(t)$ has finite number of discontinuities on any finite interval of $\Re^+$, the set of nonnegative real numbers. This actually corresponds to no-chattering requirement for the continuous-time switched systems; note that this is not an issue in the discrete-time case. Figure 4.1 is a schematic diagram of the switched system architecture.

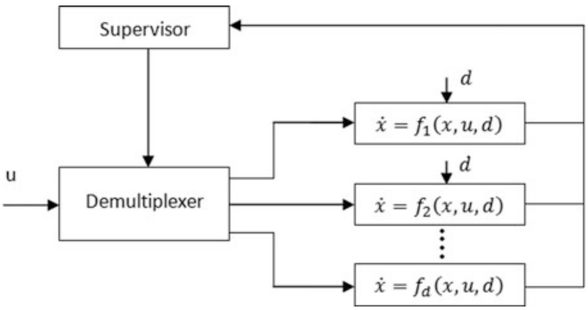

Fig. 4.1 Schematic of switched system

It is easy to recognize that a switched system is basically multimodel in nature. Each individual component model

$$\begin{aligned} \delta x(t) &= f_j(x(t), u(t), V(t)), \quad x(t_o) = x_o \\ y(t) &= g_j(x(t), w(t)), \quad j \in M \end{aligned} \tag{4.2}$$

is said to be a *subsystem* or *mode* of the switched system. Besides the subsystems, the switched system also consists of a switching device, usually called the *supervisor*. The supervisor produces the switching rule σ, denoting the *switching signal* or *switching law*, which orchestrates the switching among the subsystems.

4.2.2 Model and Definitions

Let $\mathcal{C} = \mathbf{C}([-\tau, 0], \Re^n)$ be a Banach space of continuous functions with the norm

$$||\psi||_\tau = \sup_{\tau \leq \theta \leq 0} ||\psi(\theta)||$$

Given an initial time t_o, an initial function $\psi \in \mathbf{C}$ and a switching sequence $\{(i_o, t_o),\ (i_1, t_1), ...,\ (i_j, t_j), ...\}$ where $i_k \in \mathbf{S} = \{1, 2, ..., s\}, 0 < s < \infty$.

Likewise, a *switched time-delay system* is composed of a family of time-delay subsystems and a rule that governs the switching among them, and is mathematically described by

$$\begin{aligned} \delta x(t) &= f_\sigma(x(t), u(t), \tau(t), d(t)), \quad x(t_o) = x_o \\ y(t) &= g_\sigma(x(t), \tau(t), w(t)) \end{aligned} \tag{4.3}$$

where $\tau(t)$ is the time delay and the remaining quantities are as above. Each individual time-delay component model

$$\begin{aligned} \delta x(t) &= f_k(x(t), u(t), \tau(t), d(t)), \quad x(t_o) = x_o \\ y(t) &= g_k(x(t), \tau(t), w(t)), \quad k \in \mathbf{S} \end{aligned} \tag{4.4}$$

is said to be a *time-delay subsystem* or *time-delay mode* of the switched system. In the sequel, it is assumed that the delay $\tau(t)$ is a differentiable time-varying function satisfying

$$0 < \tau(t) \leq \varrho, \quad \dot{\tau}(t) \leq \mu \tag{4.5}$$

where the bounds ϱ and μ are known constant scalars. Sometimes the bounding relation $\mu < 1$ [181, 216, 301] is used. Alternatively, depending on the problem formulation, the delay $\tau(t)$ is considered as a time-varying function satisfying

$$0 < \tau(t) \leq \varrho \tag{4.6}$$

where the bounds ϱ is a known constant scalar.

From information processing standpoint, the time-delay subsystems represent the low-level *local* dynamics governed by FDEs, whereas the supervisor is the high-level coordinator producing the switches among local dynamics. Thus, the dynamics of the STD system is determined by both the time-delay subsystems and the switching signal (Fig. 4.2).

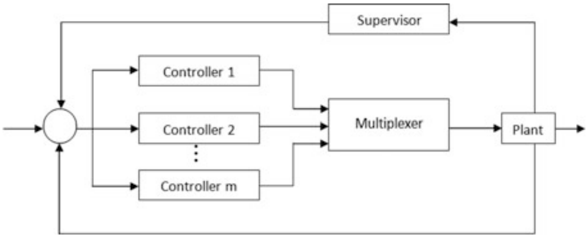

Fig. 4.2 Schematic of multiple-controller system

A switching signal may depend on the time, its own past value, the state/output, and/or an external signal as well

$$\sigma(t+) = \varphi(t, \sigma(t), x(t)/y(t), r(t)) \quad \forall t \tag{4.7}$$

where $r(t)$ is an external signal produced by other devices, $\sigma(t+) = \lim_{s \downarrow t} \sigma(s)$ in continuous time and $\sigma(t+) = \sigma(t+1)$ in discrete time.

In the case $\sigma(t) = j$, then we say that the jth subsystem is *active at time t*. It is quite evident that at any instant there is one (and only one) active subsystem.

Over the time interval $[t_o,\ t_1)$, define $x : [t_o,\ t_1) \rightarrow \Re^n$ as the state segment and $\beta : [t_o,\ t_1) \rightarrow \mathbf{S}$ as a typical switching signal. Obviously, x is absolutely continuous and σ is piecewise constant. The pair $(x(.),\ \beta(.))$ gives a characterization of the solution of system (4.4) subject to (4.5) via switching signal (4.7) at x_o, for almost all $t \in [t_o,\ t_1)$. This means that the solution is specified for all $t \in [t_o,\ t_1)$ except for possibly a set of isolated instants in continuous time and for all integers in $[t_o,\ t_1)$ in discrete time. Accordingly, we had, in fact, excluded any impulse in the state and input variables.

Throughout the book, we focus most of the time on a special but very important class of switched time-delay systems where all the subsystems are linear time-invariant and the switching signals are governed by deterministic processes. Added cone-bounded nonlinear perturbations will be considered in some instants.

Without delays, these systems are termed as linear switched systems and are described by

$$\begin{aligned} \delta x(t) &= A_i x(t) + B_i u(t) + \Gamma_i w(t) \\ y(t) &= C_i x(t) + \Phi_i w(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.8}$$

where A_i, B_i, C_i, Γ_i, and Φ_i are linear mappings (matrices) in appropriate spaces. The nominal system is the system free of disturbances, that is

$$\begin{aligned} \delta x(t) &= A_i x(t) + B_i u(t) \\ y(t) &= C_i x(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.9}$$

If no control input is imposed on the system, then the system is said to be a switched autonomous system, or an unforced switched system. The unforced switched linear system is described by

$$\begin{aligned} \delta x(t) &= A_i x(t) \\ y(t) &= C_i x(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.10}$$

As a short-hand notation, we denote system (4.9) by $\Sigma(A_i, B_i, C_i)_{\mathbf{S}}$. Similarly, we denote by $\Sigma(A_i, B_i)_{\mathbf{S}}$, $\Sigma(C_i, A_i)_{\mathbf{S}}$, and $\Sigma(A_i)_{\mathbf{S}}$ the switched system without output and/or input, respectively. In the case that we need to distinguish between continuous time and discrete time, we simply denote $\Sigma_c(A_i, B_i, C_i)_{\mathbf{S}}$ for continuous-time systems and $\Sigma_d(A_i, B_i, C_i)_{\mathbf{S}}$ for discrete-time systems.

Incorporating the delays, these systems are termed as linear switched time-delay (STD) systems and are described by

$$\begin{aligned} \delta x(t) &= A_i x(t) + D_i x(t-\tau) + B_i u(t) + \Gamma_i w(t) \\ y(t) &= C_i x(t) + \Phi_i w(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.11}$$

where A_i, B_i, C_i, Γ_i, and Φ_i are linear mappings (matrices) in appropriate spaces. The nominal STD system is the system free of disturbances, that is

$$\begin{aligned} \delta x(t) &= A_i x(t) + D_i x(t-\tau) + B_i u(t) \\ y(t) &= C_i x(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.12}$$

If no control input is imposed on the system, then the system is said to be an autonomous STD system, or an unforced STD system. The unforced linear STD system is described by

$$\begin{aligned} \delta x(t) &= A_i x(t) + D_i x(t-\tau) \\ y(t) &= C_i x(t), \quad i \in \mathbf{S} \end{aligned} \tag{4.13}$$

In a similar way, as a short-hand notation, we denote system (4.12) by $\Sigma(A_i, B_i, C_i, D_i)_{\mathbf{S}}$. Similarly, we denote by $\Sigma(A_i, B_i, D_i)_{\mathbf{S}}$, $\Sigma(C_i, A_i, D_i)_{\mathbf{S}}$, and $\Sigma(A_i, D_i)_M$ the STD system without output and/or input, respectively. In the case that we need to distinguish between continuous time and discrete time, we simply denote $\Sigma_c(A_i, B_i, C_i, D_i)_{\mathbf{S}}$ for continuous-time systems and $\Sigma_d(A_i, B_i, C_i, D_i)_{\mathbf{S}}$ for discrete-time systems.

In later parts of the book, we focus on model (4.11) subject to time-delay pattern (4.5) or (4.6) and the switching signal (4.7). In the remainder of this chapter, we review the properties and features of model (4.10).

4.2.3 Arbitrary Switching

A survey of basic problems in stability and design of switched systems has been proposed recently in [193]. Among the large variety of problems encountered in practice, one can study the existence of a switching rule that ensures stability of the switched system. One can also assume that the switching sequence is not known a priori and look for stability results under arbitrary switching sequences. One can also consider some useful class of switching sequences, see [192, 193, 196], and the references therein. By studying stability analysis and control synthesis of switched systems under arbitrary switching sequences amounts to looking at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the switched system under consideration. To evaluate the interest of this approach for control design problems, usually one concentrates on the state or output feedback design problem. By feedback design, we mean the design of state or output feedback gains for each subsystem such that the closed-loop switched system is asymptotically stable.

For internal stability, we focus on the model

$$\dot{x}(t) = A_i x(t), \quad i \in \mathbf{S} \tag{4.14}$$

describing a family of continuous-time linear time-invariant (LTI) systems or

$$x(k+1) = A_i x(k), \quad i \in \mathbf{S}, \quad k \in I^+ \tag{4.15}$$

describing a family of discrete-time linear time-invariant (LTI) systems, where the state $x \in \Re^n$ and $A_i \in \Re^{n\times n}$, $\forall\, i \in \mathbf{S}$.

It is obvious that the origin $x_e \equiv 0$ is an equilibrium (may be unstable) for the systems described in (4.14) and (4.15). The main concern in dealing with switched systems is to understand the conditions that can guarantee the stability of the system. It is interesting to know that even when all the subsystems are exponentially stable, the switched systems may have divergent trajectories for certain switching signals. Another remarkable fact is that one may carefully switch between unstable subsystems to make the switched system exponentially stable [47].

Briefly stated, it is suggested that the stability of switched systems depends not only on the dynamics of each subsystem but also on the properties of switching signals. Therefore, the stability study of switched systems can be roughly divided into two kinds of problems:

1. One is the stability analysis of switched systems under given switching signals (may be arbitrary, slow switching, etc.);
2. The other is the synthesis of stabilizing switching signals for a given collection of dynamical systems.

Both problems will be addressed in the subsequent sections and chapters. For the stability analysis problem of switched systems, the crucial question is whether the switched system is stable when there is no restriction on the switching signals.

This problem is usually called *stability analysis under arbitrary switching*. For this problem, all the subsystems are required to be asymptotically stable. On the one hand, even when all the subsystems of a switched system are exponentially stable, it is still possible to construct a divergent trajectory from any initial state for such a switched system. It therefore concluded, in general, that the foregoing subsystems stability assumption is not sufficient to assure stability for the switched systems under arbitrary switching, except for some special cases, such as all the subsystems are pairwise commutative $A_i A_j = A_j A_i, \ \forall i, j \in N$ [426], A_i-symmetric $A_i = A_i^t, \ \forall i, j \in \mathbf{S}$, [422, 424] or normal $A_i A_i^t = A_i^t A_i, \ \forall i, j \in \mathbf{S}$) [306].

On the other hand, if there exists a common Lyapunov function for all the subsystems, then the stability of the switched system is guaranteed under arbitrary switching. This paves a possible way to solve this problem, and much efforts have been focused on the common quadratic Lyapunov functions (CQLFs). Obviously, the existence of a CQLF for all its subsystems assures the quadratic stability of the switched system. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and therefore, has attracted much research efforts due to its importance in practice. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities (LMIs) [27]. Namely there exists a positive definite symmetric matrix $P \in \Re^{n \times n}$, such that

$$PA_i + A_i P < 0, \quad i \in \mathbf{S} \tag{4.16}$$

for the continuous-time case, or

$$A_i^t P A_i - P < 0, \quad i \in \mathbf{S} \tag{4.17}$$

for the discrete-time case, hold simultaneously.

It is worth pointing out that the existence of a CQLF is only sufficient for the stability of arbitrary switching systems. In [192], there are examples of systems that do not have a CQLF, but are exponentially stable under arbitrary switching. Some necessary and sufficient conditions for the asymptotic stability of switched linear systems under arbitrary switching signals are developed in [196]. This result shows that *the asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems, for which several strong stability conditions exist*.

To illustrate this result, we recall a robust stability result for linear time-variant systems with polytopic uncertainty

$$x(k+1) = A_k x(k), \quad k \in \overline{A} \stackrel{\Delta}{=} \mathbf{conv}\{A_1, \ A_2, ..., A_N\} \tag{4.18}$$

where $\mathbf{conv}\{.\}$ stands for convex combinations. In other words, the state matrix A_k of the above linear time-variant system (4.18) is constructed by a convex combinations (with time-variant coefficients) of all the subsystems' state matrices of the switched linear system (4.15). The following preliminary result is recalled [19]:

Lemma 4.1 *The linear time-variant system (4.18) is robustly asymptotically stable if and only if there exists a finite integer* **n** *such that*

$$||A_{i_1}\ A_{i_2}...A_{i_n}||_\infty < 1$$

for all **n**-*tuple* $A_{i_j}\ \in \{A_1, A_2, A_N\}$, *where* $j = 1, ..., \mathbf{n}$

Based on the above lemma, a necessary and sufficient condition for the asymptotic stability of switched linear systems (2) can be expressed by the following theorem [196].

Theorem 4.2 *A switched linear system*

$$x(k+1) = A_{\sigma(k)}\ x(k), \quad A_{\sigma(k)}\ \in\ \{A_1,\ A_2,\ A_N\}$$

is asymptotically stable under arbitrary switching if and only if there exists a finite integer **n** *such that*

$$||A_{i_1}\ A_{i_2}...A_{i_n}||_\infty < 1$$

for all **n**-*tuple* $A_{i_j}\ \in \{A_1,\ A_2,\ A_N\}$, *where* $j = 1, ..., \mathbf{n}$

The sufficiency of the above condition is implied by **Lemma 4.1**, and the necessity can be shown by contradiction. It is interesting to notice that this condition coincides with the necessary and sufficient condition for the robust asymptotic stability for polytopic uncertain linear time-variant systems (4.18). In turn, the following equivalence relationship between these two problems is established.

Lemma 4.3 *The following statements are equivalent:*

1. *The switched linear system*

$$x(k+1) = A_{\sigma(k)}\ x(k), \quad A_{\sigma(k)}\ \in\ \{A_1,\ A_2,\ A_N\}$$

   is asymptotically stable under arbitrary switching;
2. *The linear time-variant system*

$$x(k+1) = A_k x(k), \quad k \in \overline{A} \triangleq \mathbf{conv}\{A_1,\ A_2, ..., A_N\}$$

   is robustly asymptotically stable;
3. *there exists a finite integer* **n** *such that*

$$||A_{i_1}\ A_{i_2}...A_{i_n}||_\infty < 1$$

   for all **n**-*tuple* $A_{i_j}\ \in \{A_1,\ A_2,\ A_N\}$, *where* $j = 1, ..., \mathbf{n}$

It is quite interesting that the study of robust stability of a polytopic uncertain linear time-variant system, which has infinite number of possible dynamics (modes), is equivalent to considering only a finite number of its vertex dynamics in an arbitrary switching system. Note that this is not a surprising result since this fact has already been implied by the finite vertex stability criteria for robust stability in the literature [296].

4.2.4 Average Dwell Time

The motivation for studying switched systems comes partly from the fact that switched systems and switched multicontroller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. The problems encountered in switched systems can be classified into three categories [357]. The first one is to find conditions that guarantee that the switched systems are asymptotically stable under any switching signal. The second one, which is of interest in this section, is to identify certain useful classes of switching signals for which the switched system is asymptotically stable. The third one is to construct a switching signal that makes the switched systems asymptotically stable.

We have learned earlier that there are classes of switched systems, including closed-loop multiple controller systems, that may fail to preserve stability under arbitrary switching, but may be stable under some sort of restricted switching signals. *Restricted switching* may arise naturally from the physical constraints of the system, that is, in the automobile gear switching, particular switching sequence/order (from first gear to the second gear, etc.) must be followed [47].

There are cases when one may have some knowledge about possible switching logic in a switched system, that is, partitions of the state space and their induced switching rules. This knowledge may imply restrictions on the switching signals. For example, there must exist certain bound on the time interval between two successive switchings, which may be due to the fact that the state trajectories have to spend some finite length of time in traveling from the initial set to certain guard sets, if these two sets are separated. With such kind of a priori knowledge about the switching signals, we can derive stronger stability results for a given hybrid system than in the arbitrary switching case, where we use, by necessity, worst-case arguments.

In the sequel, we look at the case when the switching signals are restricted, and our problem is to study the stability of the switched systems under these restricted switching signals. We seek to evaluate what restrictions should be put on the switching signals in order to guarantee the stability of switched systems. The restrictions on switching signals may be either time-domain restrictions (that is, dwell-time, average dwell-time switching signals that will be defined below) or state-space restrictions (that is, abstractions from partitions of the state space). In this regard, the distinction between time-controlled switching signals (trajectory independent) and trajectory-dependent switching signals is significant.

Careful consideration of typical systems [47] where divergent trajectories are generated through switching between two stable systems, one may notice that the unboundedness is caused by the failure to absorb the energy increase caused by the switching. In addition, when there is an unstable subsystem (that is, controller failure or sensor fault), if one either stays too long at or switches too frequently to the unstable subsystem, the stability may be lost. Therefore, a natural issue to address concerns the restriction on the switching signal to some constrained subclasses. Intuitively, if one stays at stable subsystems long enough and switches less frequently, that is, *slow switching*, one may trade off the energy increase caused by switching or unstable modes, and maintain stability.

These ideas are proved to be reasonable and are captured by concepts like dwell time and average dwell time switching proposed by Morse and Hespanha; see for example [133, 134, 426].

A positive constant $\mathsf{T}_d \in \Re$ *is called the dwell time of a switching signal if the time interval between any two consecutive switchings is no smaller than* T_d*.*

In principle, it is always possible to maintain stability when all the subsystems are stable and switching is slow enough, in the sense that is sufficiently large [302]. It really does not matter if one occasionally has a smaller dwell time between switching, provided this does not occur too frequently.

This concept is captured by the notion of *average dwell time* in [133]: *A positive constant* T_d *is called the average dwell time for a switching signal* $\sigma(t)$ *if*

$$\mathsf{N}_\sigma(t,\eta) \;\le\; \mathsf{N}_o + \frac{t-\eta}{\mathsf{T}_a}$$

holds for all $t \ge \eta \ge 0$ *and some scalar* $\mathsf{N}_o \ge 0$*, where* $\mathsf{N}_\sigma(t,\eta)$ *denotes the number of mode switches of a given switching signal* σ *over the interval* (η, t)*.*

Here the constant T_a is called the *average dwell time* and N_o the chatter bound. The reason for a switching signal that satisfies the foregoing inequality is considered as having an average dwell time no less than T_a because

$$\mathsf{N}_\sigma(t,\eta) \;\le\; \mathsf{N}_o + \frac{t-\eta}{\mathsf{T}_a} \iff \frac{t-\eta}{\mathsf{N}_\sigma(t,\eta)-\mathsf{N}_o} \;\ge\; \mathsf{T}_a$$

which means that, on average, the *dwell time* between any two consecutive switchings is no smaller than T_a. It was shown in [133] that if all the subsystems are exponentially stable then the switched system remains exponentially stable, provided that the average dwell time is sufficiently large.

It is clear that switching signals with bounded (fixed) dwell time also have bounded average dwell time by definition. Therefore, the average dwell time scheme characterizes a larger class of stable switching signals than (fixed) dwell time scheme. Interested readers may refer to [134] for further details and a recent review on this topic.

The stability results for slow switching can be extended to the discrete-time switched systems, where the dwell time T_d or average dwell time T_a is counted as the number of sampling periods, and similar results can be developed. In addition, it is possible to extend the discrete-time results to the case where both stable and unstable subsystems coexist. When one considers unstable dynamics, slow switching (that is, long enough dwell or average dwell time) is not sufficient for stability; it is also required to make sure that the switched system does not spend too much time in the unstable subsystems. The reason to consider unstable subsystems in switched systems is because there are cases where switching to unstable subsystems becomes unavoidable; such is the case, that is, when a failure occurs or therefore a packet drops outs in communication.

4.2.5 Lyapunov Functions

Construction of Lyapunov functions is a fundamental problem in system theory; internal stability of the system under consideration is concluded if an appropriate (continuous and differentiable) Lyapunov function is shown to exist. Conceptually, when looking at an STD system, perhaps the simplest solution would be a *common quadratic* Lyapunov function, that is, a quadratic function which is a global Lyapunov function for the subsystems comprising the hybrid system. It turns out that the construction of such a Lyapunov function is an NP-hard problem even when the subsystems are linear time-invariant [20].

It should be emphasized that intrinsic discontinuous nature of a switched system strongly suggests using multiple Lyapunov-like functions concatenated together to produce a nontraditional (piecewise continuous and piecewise differentiable) Lyapunov function. Using multiple Lyapunov functions (MLF's) to form a single nontraditional Lyapunov function offers much greater freedom and infinitely more possibilities for demonstrating stability, for constructing a nontraditional Lyapunov function, and for achieving the stabilization of the switched system (4.3).

It has been demonstrated in [28] that the conservatism introduced by a *global* Lyapunov function V can be reduced by searching for a set $\{V_j\}$ of local Lyapunov functions and by ensuring that the Lyapunov functions *match* in the sense that the values of the Lyapunov functions $\{V_j\}$ and $\{V_m\}$ are equal when the state trajectory leaves a cell $\{X_j\}$ and enters a cell $\{X_m\}$, where $\{V_j\}$ is a local Lyapunov function in the cell $\{X_j\}$ and $\{V_m\}$ is a local Lyapunov function in the cell $\{X_m\}$.

4.2.6 Converse Lyapunov Theorem

When dealing with globally uniformly asymptotically stable and locally uniformly exponentially stable continuous-time switched systems with arbitrary switching signals, a converse Lyapunov theorem was derived in [43]. The result was that such arbitrary switching system admits a common Lyapunov function, as summarized by the following theorem:

Theorem 4.4 *If the switched system is globally uniformly asymptotically stable and, in addition, uniformly exponentially stable, the family has a common Lyapunov function.*

The converse Lyapunov theorem was extended in [288] to switched nonlinear systems that are globally uniformly asymptotically stable with respect to a compact forward invariant set. Although these converse Lyapunov theorems justify the common Lyapunov function method being pursued, they also suggest that the common Lyapunov function may not necessarily be quadratic. Based on the equivalence between the asymptotic stability of arbitrary switching linear systems and the robust stability of polytopic uncertain linear time-variant systems, some well-established converse Lyapunov theorems can be introduced for arbitrary switching linear systems [296] as follows:

Theorem 4.5 *If the switched linear system $x(k+1) = A_i \; x(k), \;\; k \in I^+, i \in M$ is exponentially stable under arbitrary switching, then it has a strictly convex, homogeneous (of second order) common Lyapunov function of a quasi-quadratic form $V(x) = x^t L(x) x$, where $L(x) = L^t(x) = L(\nu x)$ for all nonzero $x \in \Re^n$ and $nu \in \Re$.*

Restricting attention to include only polyhedral Lyapunov functions (also known as piecewise linear Lyapunov function) [18] as the following result was pointed out.

Theorem 4.6 *If a switched linear system is asymptotically stable under arbitrary switching signals, then there exists a polyhedral Lyapunov function, which is monotonically decreasing along the switched system's trajectories.*

Theorems 4.5 and **4.6** have the following advantages. First, it shows that one may focus on polyhedral Lyapunov functions without loss of generality. Second, there exist automated computational methods to calculate polyhedral Lyapunov functions.

Finding conditions to guarantee stability under all possible switching signals is also of practical importance. For example, multiple-controller schemes are often employed to satisfy different performance requirements. When one designs multiple controllers for a plant, a desirable property is that switching between these controllers does not cause instability. The benefit of this property is that there is no need to worry about stability when switching among controllers and one can focus on gaining better performance. In this regard, it was shown in [135] that it is possible to guarantee such a good property for multiple controller design in certain cases.

4.3 Some Representative Examples

In this section, we give some representative examples of switched systems along with simulation studies to demonstrate some pertinent features.

4.3.1 Car Transmission System

The simplified dynamics of a car (mass m) with an automatic transmission having velocity v on a road inclined at angle α is

$$\begin{aligned}\dot{v} &= -\frac{k}{m}\, v^2 \,\text{sign}(v) - g\ \sin(\alpha) + \frac{G_{\sigma(t)}}{m}\, T \\ \omega &= G_{\sigma(t)}\ v\end{aligned} \tag{4.19}$$

where the discrete state $G_{\sigma(t)}\ \in \{G_1,\ G_2,\ G_3,\ G_4\}, G_1 > G_2 > G_3 > G_4$ are the transmission gear ratios normalized by the wheel radius R, k is an appropriate constant, ω is the angular velocity of the motor, and T is the torque generated by the engine, an input to the model. The discrete state transition function is

$$\sigma(t^+) \triangleq \begin{cases} i+1, & if\ \sigma(t) = i \neq 4 \text{ and } v = \frac{\omega_h}{G_i} \\ i, & if\ \sigma(t) = i+1 \geq 2 \text{ and } v = \frac{\omega_\ell}{G_{i+1}} \end{cases}$$

where ω_h and ω_ℓ are preset angular velocities of the engine.

A PI cruise controller (of the torque) that must also compensate for the nonlinear load forces is given by

$$T = T_p + T_I + \frac{k}{G_{\sigma(t)}}\, v^2 \,\text{sign}(v)$$

for a reference velocity v_{ref} and a proportional control $T_p = K_{\sigma(t)}\ (v_{\text{ref}} - v)$.

This leads to combined/reduced vehicle cruise controller dynamics:

$$\begin{aligned}\dot{v} &= \frac{G_{\sigma(t)}}{m}\left(K_{\sigma(t)}\ (v_{\text{ref}} - v) + T_I\right) - g\ \sin(\alpha) \\ \dot{T}_I &= \frac{K_{\sigma(t)}}{T_R}\ (v_{\text{ref}} - v)\end{aligned} \tag{4.20}$$

The constant T_R is chosen to balance fast convergence with small overshoot; the discrete gains $K_{\sigma(t)}\ \in \{K_1,\ K_2,\ K_3,\ GK_4\}$ are chosen to insure a smooth ride and satisfy: $G_i\ K_i = G_{i+1}, K_{i+1}$

The initial condition is

(a) reset to zero for new v_ref inputs and
(b) for any change in the discrete state $\sigma(t)$ at t_k say , the state $T_I\left(t_k^+\right)$ is reset discontinuously (a so-called state jump) so that: $G_{\sigma(t_k^-}i\ \ T_I\left(t_k^-\right) = G_{\sigma(t_k^+}i\ T_I\left(t_k^+\right)$ also to ensure a smooth ride.

Let $M = 1500,\ T_R = 40,\ G_{\sigma(t)}\ \in \{50,\ 32,\ 20,\ 14\}, G_{\sigma(t)}K_{\sigma(t)} = 187.5,$ $K_{\sigma(t)}\ \in \{3.75,\ 35.86,\ 9.37,\ 13.39\}, G_{\sigma(t)}K_{\sigma(t)} = 187.5$ and $v_{ref} = 30\,\text{m/s}$

The S-function was used in simulink to simulate this system, see Fig. 4.3. Among the advantages of using the S-function for simulating hybrid system are as follows: (1) Only one block in simulink is needed to simulate both continuous and discrete dynamics along with all decision rules and constraints. (2) Both continuous and discrete functions are separately treated and called in the simulation process. These functions express the differential and difference equations, along with the logic associated for each in normal Matlab language format. (3) All Matlab functions can be used as part of the model. Hence, a variety of models can be found: linear switched system, nonlinear, mixed linear and nonlinear systems, etc. (4) The S-function is treated as a block in simulink, hence feedback can be used to stabilize the system.

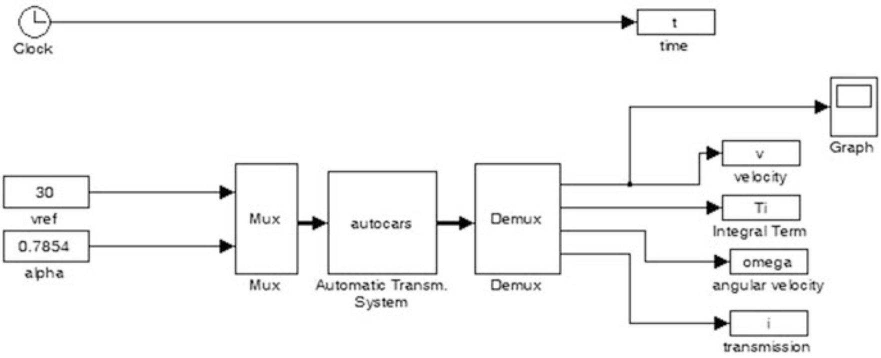

Fig. 4.3 A simulink diagram

The system was simulated and was found to operate as desired. The S-function works as follows:

(a) The number of inputs, outputs, continuous states, and discrete states. (b) Calling the continuous differential equation functions and updating the results. (c) Calling the discrete difference equation functions and updating the results. (d) Finding the states after integration of continuous differential equations and recursive calculations of discrete difference equations. (e) Populating the results and continuing the cycle of reading and writing.

The step response for $v_{ref} = 30\,\mathrm{m/s}$ is plotted in Fig. 4.4. The closed-loop response clearly shows stability despite the undamped cycles.

4.3.2 Autonomous Switched System

Consider the autonomous state dynamics $\dot{x}(t) = A_{\sigma(t)}\, x(t)$, where

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^t \Re^2, \ \sigma \in \{1,\ 2\}$$
$$A_1 = \begin{bmatrix} -1 & 100 \\ 10 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \tag{4.21}$$

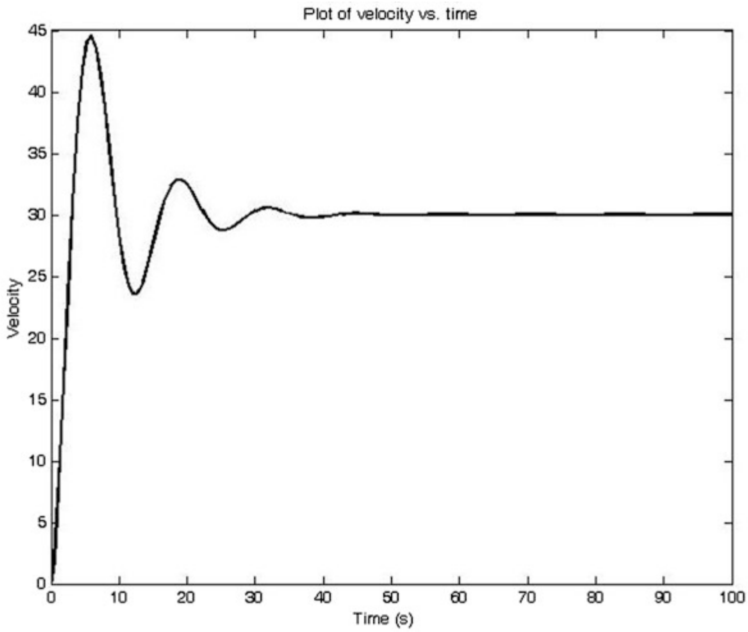

Fig. 4.4 Velocity response due to step input

Both A_1 and A_2 are stable, having identical eigenvalues $\lambda_{1,2} = -1 \pm j\sqrt{1000}$. Define the switching function $\sigma(t)$ as follows:

$$\sigma(t^+) \triangleq \begin{cases} 1, & \text{if } \sigma(t) = 2 \text{ and } x_2(t) = \frac{-1}{k}\, x_1(t) \\ 2, & \text{if } \sigma(t) = 1 \text{ and } x_2(t) = k\, x_1(t) \end{cases}$$

State flow machine was used simulate this example in simulink as shown in Fig. 4.5. In the state-flow machine, states corresponding to $\sigma(t) = 1$ and $\sigma(t) = 2$ need to be defined. As shown in Fig. 4.6, each state box can have entry actions. Moving from one state to another is called a transition and is blocked by a condition. Transitions require input information; namely the states of the plant. The simulink state-space model is reassigned the A matrix after every transition, and hence the dynamic hybrid model is simulated.

For any given initialization, the switching function $\sigma(t)$ specifies a rule with memory for switching the dynamic motion of the system between A_1 and A_2. For $k = -0.2$ and arbitrary nonzero initial condition, state trajectories diverge. This is depicted in Fig. 4.7, which is consistent with the results reported in [47]. This illuminates the observation that switching between two asymptotically stable systems can produce an unstable trajectory.

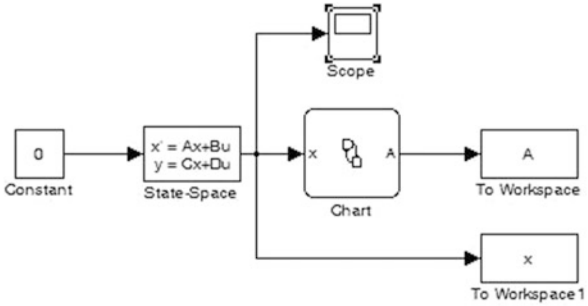

Fig. 4.5 A simulink diagram

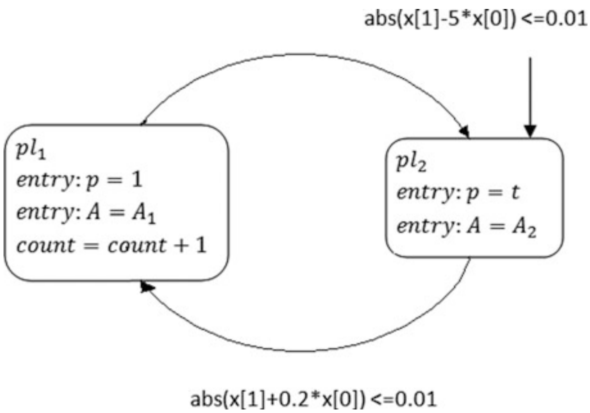

Fig. 4.6 A state flow chart

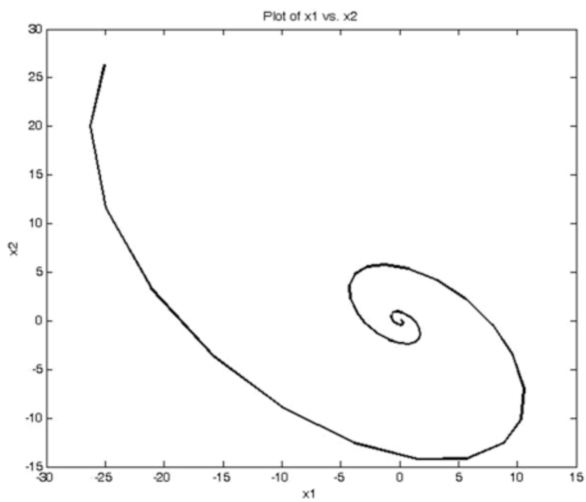

Fig. 4.7 State plot x_1 vs. x_2

4.3.3 Another Switched System

Consider the autonomous state dynamics $\dot{x}(t) = A_{\sigma(t)}\, x(t)$, where

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^t \ \Re^2, \ \sigma \ \in \ \{1,\ 2\}$$
$$A_1 \ = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix} \tag{4.22}$$

Both A_1 and A_2 are unstable as A_1 has zero eigenvalues and $\lambda(A_2) = 0.5 \pm j\sqrt{3}$. Define the switching function $\sigma(t)$ as follows:

$$\sigma(t^+) \triangleq \begin{cases} 1, & \text{if } \sigma(t^-) = 2 \text{ and } x_2(t) = -0.25\, x_1(t) \\ 2, & \text{if } \sigma(t^-) = 1 \text{ and } x_2(t) = 0.5\, x_1(t) \end{cases}$$

In Fig. 4.8, the state responses using S-function are plotted and shown that the system is unstable.

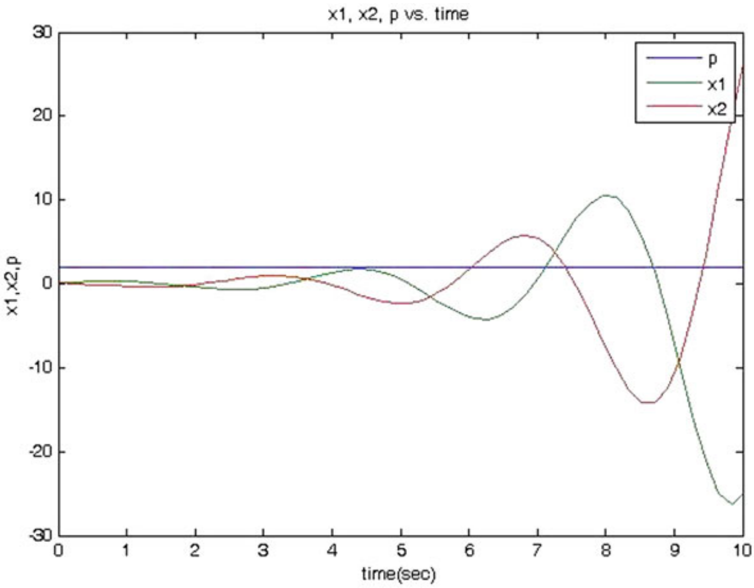

Fig. 4.8 State trajectories

4.3.4 Simplified Longitudinal Dynamics of an Aircraft

A highly simplified longitudinal dynamics of an aircraft can take the form:

$$\begin{aligned}\dot{x} &= \begin{bmatrix}\dot{q}\\ \dot{\alpha}\end{bmatrix} = f(x, u, \sigma)\\ &= \begin{bmatrix}-1 & 10\\ 1 & -1\dot{\alpha}\end{bmatrix}\begin{bmatrix}q\\ \alpha\end{bmatrix} + \begin{bmatrix}-1\\ 0.1\dot{\alpha}\end{bmatrix} u_{\sigma(t)}\end{aligned} \tag{4.23}$$

where $\alpha \leq \alpha_M$ is the constrained angle of attack and q is the pitch rate. The output is

$$\begin{bmatrix}\alpha\\ n_z\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -300\dot{\alpha}\end{bmatrix}\begin{bmatrix}q\\ \alpha\end{bmatrix} + \begin{bmatrix}0\\ 30\dot{\alpha}\end{bmatrix} u_{\sigma(t)} \tag{4.24}$$

where $\sigma(t) \in \{1,\ 2\}$, N_z is the normal acceleration, and the control variable $u_{\sigma(t)}$ is the angle of the elevator measured down from the horizontal with the aircraft. The control objective is twofold:

track the pilot's reference normal acceleration while maintaining the safety constraint that the angle of attack must be less than α_m. To simultaneously achieve both objectives (to the extent possible), we define a switched 'max control law':

$$\sigma(t^+) = \varphi(x(t),\ u(t),\ \sigma(t)) = \text{argmax}_i\ (u_i))$$

where

$$u_1 = -F\ x + k_1\ \alpha_m, \quad u_2 = -G\ x + k_2\ r(t)$$

Here, u_1 is the output of a controller designed to stabilize the aircraft about α_m, and u_2 is a control designed to make n_z track $r(t)$. Roughly, the max control law acts to track the pilot's reference using the elevator except when to do so would cause the safety constraint to be violated.

Let

$$r(t) = 0, \quad u = \max(-F\ x + k_1\ \alpha_m, \quad -G\ x + k_2\ r(t))$$

The closed-loop equations with the maximum control law are as follows:

$$\begin{aligned}\dot{x} &= A\ x + B\ \max(-F\ x + k_1\ \alpha_m, \quad G\ x)\\ &= (A - BG)x + B\ \max((G - F)\ x + k_1\ \alpha_m, \quad 0)\end{aligned} \tag{4.25}$$

The analysis presupposes that the feedback gain matrices F, G are designed so that $(A - BG),\ (A - BF)$ are stable and provide the necessary performance. This is possible because the controllability is equivalent to the ability to reassign the eigenvalues of A by state feedback.

To simulate this problem, we need to locate the poles of the closed-loop systems $(A - BG),\ (A - BF)$. This can be done using place function in Matlab. We choose the closed-loop eigenvalues to be at -1, -4.1623 for both systems. This gives $F = G\ [-4.6248 - 14.6248]$.

We will simulate the case for $r(t)$. Next, we simulate the system using S-function technique as depicted in Fig. 4.9, which yields the output and input responses of the system shown in Fig. 4.10.

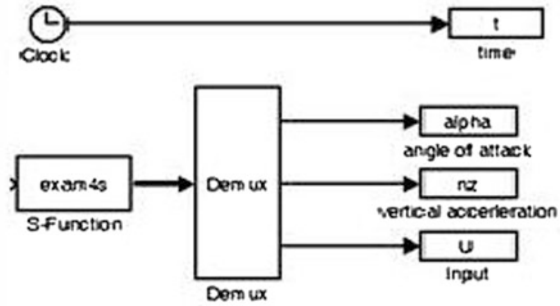

Fig. 4.9 A simulink diagram

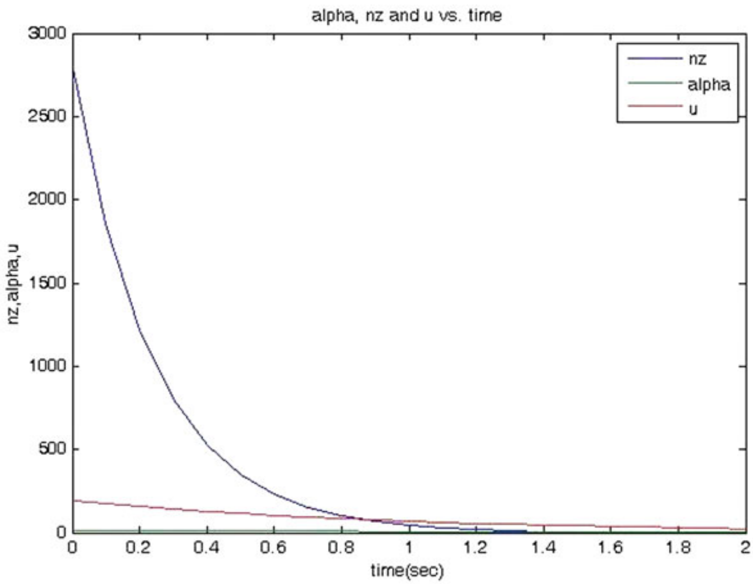

Fig. 4.10 State variable trajectories

Remark 4.7 The results of the foregoing examples brought up a crucial issue. That switching between two asymptotically stable systems as above can occur in the control of several dynamic systems. In this regard, two questions arise:

(a) What classes of stable systems admit a stable-state trajectory for all switching sequences and
(b) What switching sequences always result in stable trajectories?

If there exists a common Lyapunov function for a set of stable A-matrices, the resulting system is stable for all switching sequences [193], which answers question a).

A partial answer to question b) is intuitive: If switching among asymptotically stable systems is slow enough, one would expect a stable response. Stability here is characterized by a traditional Lyapunov function that measures the system energy. Mathematically, $V(.)$ is continuous and differentiable, $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. Further, if $\dot{V} < 0, \ x \neq 0,$ then the state will converge to zero, implying local stability (global stability if is radially unbounded) [116].

4.4 $\mathcal{L}_2$ Gain Analysis and Synthesis

Many physical systems can be represented by hybrid models containing continuous and discrete states that affect their dynamic behavior. For example, a variety of power systems [389], chemical processes [41], and mechanical systems [47], and many others can be modeled as hybrid systems. A particular class of hybrid systems, which is of interest in this work, is the one composed of many discrete subsystems and a rule that governs the switching between these subsystems. This class of switched systems has received great attention in the past decade because of the fast development in computing technologies, which helped improve the efficiency of switching between systems or controllers. For example, in [30, 42, 47, 174, 193, 292] and the references cited therein, the stability and control synthesis of switched hybrid systems have been investigated. In [47], the authors present an introduction of the concept of switched systems, the challenges associated with the stability of switches systems, and an overview of the major results in the Lyapunov stability of finite-dimensional hybrid systems. In [42], the stability of switched discrete systems is studied by checking for the existence of switched Lyapunov function for the system under consideration. In [28], multiple Lyapunov functions for the stability analysis of continuous hybrid systems is investigated, and the use of iterated function systems (IFS) as a tool for Lagrange stability is examined. Also, a survey of switched systems' problems has been proposed in [193].

In this section, the results of [292, 333, 424] are extended further to the discrete-time case using the result of [217] is provided. Specifically, the paper presents a criterion for uniform quadratic stability and $\mathcal{H}_\infty$ stabilization of a class of uncertain switched systems. In this class, the parametric uncertainties are represented by a real convex-bounded polytopic model. The problems of $\mathcal{L}_2$ gain analysis and control synthesis for a class of linear discrete-time switched systems with convex bounded parameter uncertainties in all system matrices are investigated. The main thrust is based on the constructive use of an appropriate switched Lyapunov functions. The $\mathcal{L}_2$ gain analysis is utilized to characterize conditions under which the linear switched system with polytopic uncertainties is uniformly quadratically stable with an $\mathcal{L}_2$ gain smaller that a prescribed constant level. Then, control synthesis is used to design switched feedback schemes, based on state-, output-measurements, or by using dynamic output feedback, to guarantee that the corresponding closed-loop system enjoys the uniform quadratic stability with an $\mathcal{L}_2$ gain smaller that a prescribed constant level. All the developed results are expressed in terms of convex optimization over LMIs and tested on representative examples.

4.4.1 Switched Gain Analysis

We consider a class of discrete-time linear switched systems described by

$$(\Sigma_J): \quad x_{k+1} = A_\sigma x_k + B_\sigma u_k + \Gamma_\sigma w_k, \sigma \in \mathbf{N} \tag{4.26}$$

$$z_k = C_\sigma x_k + D_\sigma u_k + \Phi_\sigma w_k \tag{4.27}$$

$$y_k = L_\sigma x_k \tag{4.28}$$

where $x_k \in \Re^n$ is the state; $u_k \in \Re^m$ is the control input; $w_k \in \Re^q$ is the exogenous disturbance; $y_k \in \Re^p$ is the measured output; $z_k \in \Re^r$ is the controlled output. Following [174], model (4.26) represents the continuous (state) portion of linear hybrid systems. The particular mode σ at any given time instant may be a selective procedure characterized by a switching rule of the form

$$\sigma_{k+1} = \delta(\sigma_k, x_k), \delta : \mathbf{N} \times \Re^n \to \mathbf{N} \tag{4.29}$$

The function $\delta(.)$ is usually defined using a partition of the continuous state space [333]. Let $\mathcal{S}$ denote the set of all selective rules. Therefore, the linear hybrid system under consideration is composed of N subsystems, each of which is activated at particular switching instant. For a switching mode $j \in \mathbf{N}$, the associated matrices $A_j, ..., \Phi_j$ contain uncertainties represented by a real convex-bounded polytopic model of the type

$$\begin{bmatrix} A_j & B_j & \Gamma_j \\ C_j & D_j & \Phi_j \end{bmatrix} \triangleq \left\{ = \sum_{p=1}^{M_j} \lambda_{jp} \begin{bmatrix} A_{jp} & B_{jp} & \Gamma_{jp} \\ C_{jp} & D_{jp} & \Phi_{jp} \end{bmatrix}, \quad j \in \mathbf{N} \right\} \tag{4.30}$$

where $\lambda_j = (\lambda_{j1}, \lambda_{j2}, ..., \lambda_{jM_j}) \in \Lambda_j$ belongs to the unit simplex of M_j vertices

$$\Lambda_j \triangleq \left\{ \lambda_j : \sum_{p=1}^{M_j} \lambda_{jp} = 1 \ , \ \lambda_{jk} \geq 0 \right\} \tag{4.31}$$

where $A_{jp}, ..., \Phi_{jp}, p = 1, ..., M_j$ are known real constant matrices of appropriate dimensions, which describe the jth nominal subsystem. Distinct from (4.26–4.28) is the free switched system

$$(\Sigma_{Jo}): \quad x_{k+1} = A_\sigma x_k + \Gamma_\sigma w_k \tag{4.32}$$

$$z_k = C_\sigma x_k + \Phi_\sigma w_k \tag{4.33}$$

we have the following definitions:

Definition 4.8 *Switched system* (Σ_{Jo}) *is said to be* **uniformly quadratically stable (UQS)** *if there exist a Lyapunov functional* $V(x,k) > 0$, *a constant* $\varepsilon > 0$ *such that for all admissible uncertainties satisfying (4.30 and 4.31) and arbitrary switching rule* $\sigma(.)$ *activating subsystem* $j \in \boldsymbol{N}$ *at instant* $k+1$ *and subsystem* i *at instant* k, *the Lyapunov functional difference* $\Delta V(x_k,k)$ *satisfies* $\Delta V(x_k,k) \stackrel{\Delta}{=} V(x_{k+1}, k+1) - V(x_k,k) \leq -\varepsilon\ x_k^t x_k\ ,\ \forall x_k \neq 0$.

Definition 4.9 *Given a scalar* $\gamma \geq 0$, *the* $\mathcal{L}_2$ *gain* $\mathcal{G}$ *of switched system* (Σ_{Jo}) *over* $\mathcal{S}$ *is*

$$\mathcal{G} \stackrel{\Delta}{=} \inf\{\gamma \geq 0 : ||z_k||_2 < \gamma^2\ ||w_k||_2,\ \forall\sigma \in \mathcal{S}, \forall\ \lambda_j \in \Lambda_j,\ j \in \mathbf{N}\}$$

Definition 4.10 *Switched system* (Σ_{Jo} *is said to be* **uniformly quadratically stable (UQS) with an** $\mathcal{L}_2$ **gain** $\mathcal{G} < \gamma$ *if for all switching signal vector* $\sigma \in \mathcal{S}$ *and for all admissible uncertainties satisfying (4.30 and 4.31) it is UQS and* $\forall w_k \neq 0$, $||z_k||_2 < \gamma^2\ ||w_k||_2$.

Our purpose in this section is to develop criteria for uniform quadratic stability and stabilization of system (Σ_J) and examine their robustness, then design appropriate $\mathcal{L}_2$ feedback controllers that guarantee stability with a prescribed performance.

Lemma 4.11 *Switched system* (Σ_{Jo}) *is UQS with an* $\mathcal{L}_2$ *gain* $\mathcal{G} < \gamma$ *there exist a scalar* $\gamma > 0$ *and a quadratic Lyapunov functional* $V_\sigma(x,k) > 0$ *such that for all switching rules* $\sigma \in \mathcal{S}$, *the Lyapunov functional difference* $\Delta V(x_k,k)$ *along the solutions of (4.32 and 4.34) satisfies*

$$\Delta V_\sigma(x_k,k) + z_k^t z_k - \gamma^2\ w_k^t w_k < 0 \tag{4.34}$$

Proof That switched system (Σ_{Jo}) is UQS follows directly from (4.34). Now by summing up (4.34) over the range $0 \longrightarrow q,\ \forall q \in \mathbf{N}$, it follows that

$$V_\sigma(x_{q+1}, q+1) - V(0, x_o) + \sum_{p=0}^{q} \left(z_p^t z_p - \gamma^2\ w_p^t w_p \right)\ <\ 0$$

Since $V_\sigma(x_{q+1}, q+1) \geq 0$, $x_o = 0$, it follows that $\sum_{p=0}^{q} z_p^t z_p < \gamma^2 \sum_{p=0}^{q}\ w_p^t w_p\ < 0$ and by **Definition 4.10**, switched system (Σ_{Jo}) is UQS with an $\mathcal{L}_2$ gain smaller than γ.

In the sequel, we consider the following quadratic Lyapunov functional

$$V_\sigma(x_k,k)\ \stackrel{\Delta}{=}\ x_k^t\ P_\sigma x_k,\quad 0 < P_\sigma^t = P_\sigma,\ \sigma \in \mathcal{S} \tag{4.35}$$

Remark 4.12 We note from **Definition 4.9** that the $\mathcal{L}_2$ gain $\mathcal{G}$ under arbitrary switching can be looked as the worst-case energy amplitude gain for switched system (4.30 and 4.31) over all possible inputs, switching signals, and all admissible uncertainties. The functional (4.35) is called a switched Lyapunov function (SLF) since it has the same switching signals as system (4.32 and 4.33), which is known to yield less conservative results than the constant Lyapunov functional $x_k^t P x_k$.

The following theorem summarizes the first result.

Theorem 4.13 *The following statements are equivalent:*

(A) There exists an SLF of the type (4.35) with $\sigma \in \mathcal{S}$ and a scalar $\gamma > 0$ such that switched system (4.32 and 4.33) is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$.
(B) There exist matrices $0 < P_i^t = P_i, \ 0 < X_j^t = X_j, \ i \in \boldsymbol{N}, \ j \in \boldsymbol{N}$, and a scalar $\gamma > 0$ satisfying the LMIs

$$\begin{bmatrix} -P_i & 0 & A_{ip}^t & C_{ip}^t \\ \bullet & -\gamma^2 I & \Gamma_{ip}^t & \Phi_{ip}^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, \quad (i, j) \in \boldsymbol{N} \times \boldsymbol{N}, \quad p \in \{1, ..., M_i\} \tag{4.36}$$

Proof (A) $\Rightarrow$ (B) Suppose that there exist a constant $\gamma > 0$ and a switched Lyapunov function of the type (4.35) satisfying (4.34). Let the switching rule $\sigma(.)$ activates subsystem $j \in \mathbf{N}$ at instant $k + 1$ and subsystem i at instant k. Thus

$$\begin{aligned} \Delta V_\sigma(x_k, k) + z_k^t z_k - \gamma^2 \ w_k^t w_k &= x_{k+1}^t P_j x_{k+1} - x_k^t P_i x_k + z_k^t z_k - \gamma^2 \ w_k^t w_k \\ = \Big[x_k^t A_i^t + w_k^t \Gamma_i^t\Big] P_j \Big[A_i x_k + \Gamma_i w_k\Big] &+ \Big[x_k^t C_i^t + w_k^t \Phi_i^t\Big]\Big[C_i x_k + \Phi_i w_k\Big] \\ -x_k^t P_i x_k - \gamma^2 \ w_k^t w_k & \\ < 0 & \end{aligned} \tag{4.37}$$

Since (4.37) holds for arbitrary switching, it follows on using (4.31) that for any vectors $x_k \neq 0, \ w_k \neq 0$ that for all $(i, j) \in \mathbf{N} \times \mathbf{N}$

$$\begin{bmatrix} -P_i + A_i^t P_j A_i + C_i^t C_i & A_i^t P_j \Gamma_i + C_i^t \Phi_i \\ \bullet & -\gamma^2 I + \Gamma_i^t P_j \Gamma_i + \Phi_i^t \Phi_i \end{bmatrix} < 0 \tag{4.38}$$

By Schur complements operations, inequality (4.38) can be put into the form

$$\begin{bmatrix} -P_i & 0 & A_i^t P_j & C_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t P_j & \Phi_i^t \\ \bullet & \bullet & -P_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, \quad (i, j) \in \mathbf{N} \times \mathbf{N} \tag{4.39}$$

Applying the congruent transformation $\mathfrak{C} = \mathrm{diag}[I, \quad I, \quad X_j, \quad I], \; X_j = P_j^{-1}$, we readily obtain

$$\begin{bmatrix} -P_i & 0 & A_i^t & C_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, (i, j) \in \mathbf{N} \times \mathbf{N} \tag{4.40}$$

Upon using vertex representation (4.30) and (4.31), we get (4.36) from (4.40).
(B) $\Rightarrow$ **(A)** Follows by reversing the steps in the proof and applying (**Lemma 2**, [210]) to system (4.32) and (4.33) for all modes $(i, j) \in \mathbf{N} \times \mathbf{N}$ and using (4.31).

Remark 4.14 It should be observed that in LMI (4.36) the system matrices are readily separated from the Lyapunov matrices. The optimal $\mathcal{L}_2$ gain of switched system (4.32 and 4.33) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned} & \quad\; Minimize \;\; \gamma \\ & s.t. \quad LMIs \; (4.36), (i, j) \in \mathbf{N} \times \mathbf{N}, \;\; p \in \{1, ..., M_i\} \\ & P_i > 0, Y_j > 0, \gamma > 0, \end{aligned}$$

which can be conveniently solved by existing software [74].

Remark 4.15 A special case of **Theorem 4.13** is now provided.

Corollary 4.16 *The following statements are equivalent:*

(A) *There exists an SLF of the type (4.35) with $\sigma \in \mathcal{S}$ such that switched system (4.32 and 4.33) with polytopic representation (4.30 and 4.31) is UQS.*
(B) *There exist matrices $0 < P_i^t = P_i, \;\; 0 < X_j^t = X_j, \;\; i \in N, \;\; j \in N$ and a scalar $\gamma > 0$ satisfying the LMIs*

$$\begin{bmatrix} -P_i & A_{ip}^t & C_{ip}^t \\ \bullet & -X_j & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0, (i, j) \in N \times N, p \in \{1, ..., M_i\} \tag{4.41}$$

In the nominal case $(M_i = 1, \;\; \forall i)$ LMI (4.41) provides an alternative stability test that requires fewer matrix variables to manipulate than the result in [42].

Extending on the last section, we examine here the problem of switched control synthesis using either switched state-feedback or output-feedback design schemes.

4.4.2 Switched State Feedback

With reference to system (4.26 and 4.27), we consider that the arbitrary switching rule $\sigma(.)$ activate subsystem i at instant k. Our objective herein is to design a switched state feedback $u_k = K_i \; x_k$ at $i \in \mathbf{N}$ mode such that the closed-loop system

$$
\begin{aligned}
(\Sigma_{Js}) : \quad x_{k+1} &= [A_i + B_i K_i] x_k + \Gamma_i w_k \\
&= \bar{A}_i x_k + \Gamma_i w_k \qquad (4.42)\\
z_k &= [C_i + D_i K_i] x_k + \Phi_i w_k \\
&= \bar{C}_i x_k + \Phi_i w_k \qquad (4.43)
\end{aligned}
$$

is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$. The following theorem summarizes the main result.

Theorem 4.17 *Switched system (4.42) and (4.43) is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$ if there exist matrices $0 < X_i^t = X_i, \; Z_i, \; 0 < X_j^t = X_j, \; (i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ and a scalar $\gamma > 0$ satisfying the LMIs*

$$
\begin{bmatrix} -X_i & 0 & X_i A_i^t + Z_i^t B_i^t & X_i C_i^t + Z_i^t D_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, (i, j) \in \boldsymbol{N} \times \boldsymbol{N} \qquad (4.44)
$$

Moreover, the gain matrix is given by $K_i = Z_i X_i^{-1}$.

Proof It follows from **Theorem 4.13** that switched system (4.42) and (4.43) is UQS if $\forall (i, j) \in \mathbf{N} \times \mathbf{N}$ there exist matrices $0 < P_i = P_i^t, \; 0 < X_j = X_j^t$ such that

$$
\begin{bmatrix} -P_i & 0 & \bar{A}_i^t & \bar{C}_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} =
$$

$$
\begin{bmatrix} -P_i & 0 & A_i^t + K_i^t B_i^t & C_i^t + K_i^t D_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0. \qquad (4.45)
$$

Applying the congruent transformation $[X_i, \; I, \; I, \; I]$ to LMIs (4.45) with $X_i = P_i^{-1}, \; Z_i = K_i X_i$, we immediately obtain (4.44). ■

Remark 4.18 The optimal switched state feedback with $\mathcal{L}_2$ gain for system (4.42 and 4.43) can be determined by solving the following convex minimization problem over LMIs:

$$
\begin{aligned}
& Minimize \;\; \gamma \\
& s.t. \quad LMIs \; (4.44), \quad (i, j) \in \mathbf{N} \times \mathbf{N} \\
& Z_i, \quad X_i > 0, \quad X_j > 0, \quad \gamma > 0
\end{aligned}
$$

In the case of polytopic representation (4.30) and (4.31), the corresponding convex minimization problem takes the form

$$\begin{aligned} &Minimize\ \ \gamma \\ &wrt\ \ Z_i,\ \ X_i > 0,\ \ X_j > 0,\ \ \gamma > 0 \end{aligned}$$

$$\begin{bmatrix} -X_i & X_i A_{ip}^t + Z_i^t B_{ip}^t & \Gamma_i & 0 \\ \bullet & -\gamma^2 I & \Gamma_{ip}^t & \Phi_{ip}^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, (i, j) \in \mathbf{N} \times \mathbf{N}, p \in \{1, ..., M_i\}$$

4.4.3 Switched Static Output Feedback

The objective now is to design a switched output feedback $u_k = G_i\ \ y_k$ at mode $i \in \mathbf{N}$ such that the closed-loop system

$$\begin{aligned} (\Sigma_{Js}):\quad x_{k+1} &= [A_i + B_i G_i L_i] x_k + \Gamma_i w_k \\ &= \hat{A}_i x_k + \Gamma_i w_k \end{aligned} \tag{4.46}$$

$$\begin{aligned} z_k &= [C_i + D_i G_i L_i] x_k + \Phi_j w_k \\ &= \hat{C}_i x_k + \Phi_i w_k \end{aligned} \tag{4.47}$$

is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$. To facilitate further development, we consider initially the case where the set of output matrices $L_i,\ \ i \in \mathbf{N},\ \ p \in \{1, ..., M_j\}$ are assumed to be of full row rank. This case can be fullfilled by deleting redundant measurement components of the output variable y_k. Therefore, it follows from **Theorem 4.13** that switched system (4.46) and (4.47) is UQS if $\forall (i, j) \in \mathbf{N} \times \mathbf{N}$ there exist matrices $0 < P_i = P_i^t,\ 0 < X_j = X_j^t$ such that

$$\begin{bmatrix} -P_i & 0 & \hat{A}_i^t & \hat{C}_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} =$$

$$\begin{bmatrix} -P_i & 0 & A_i^t + L_i^t G_i^t B_i^t & C_i^t + L_i^t G_i^t D_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{4.48}$$

Using the congruent transformation $[X_i,\ \ I,\ \ I,\ \ I]$ to LMIs (4.48) with $X_i = P_i^{-1},\ \ L_i X_i = E_i L_i,\ \ R_i = G_i E_i,$ then there exist matrices $0 < X_i = X_i^t,\ 0 < X_j = X_j^t,\ 0 < E_i = E_i^t,\ R_i$ such that

$$\begin{bmatrix} -X_i & 0 & X_i A_i^t + L_i^t R_i^t B_i^t & X_i C_i^t + L_i^t R_i^t D_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{4.49}$$

$$L_i X_i = E_i L_i \tag{4.50}$$

We note that the presence of matrix equality in (4.50) renders the computations of (4.49) and (4.50) using MATLAB-LMI Toolbox [74] rather costly. Therefore, one is encouraged to convert (4.49) and (4.50) into true LMIs. With this in mind, we recall that the use of singular value decomposition (SVD) can express the output matrix L_i in the form

$$L_i \;=\; \mathcal{U}_i \; [\Lambda_{pi}, 0] \; \mathcal{V}_i^t \tag{4.51}$$

where $\mathcal{U}_i \in \mathbf{R}^{p\times p}$, $\mathcal{V}_i \in \mathbf{R}^{n\times n}$ are unitary matrices and $\Lambda_{pi} \in \mathbf{R}^{p\times p}$ is a diagonal matrix with positive diagonal elements in decreasing order. The conversion to LMIs can now be accomplished by the following theorem:

Theorem 4.19 *Given a matrix* $L_i \in \boldsymbol{R}^{p\times n}$, $rank[L_i] = p$ *and let* $0 < X_i = X_i^t \in \boldsymbol{R}^{n\times n}$. *Then there exists a matrix* $0 < E_i \in \boldsymbol{R}^{p\times p}$ *such that*

$$L_i X_i = E_i L_i \tag{4.52}$$

if and only if

$$X_i = \mathcal{V}_i \begin{bmatrix} X_{iu} & 0 \\ \bullet & X_{iv} \end{bmatrix} \mathcal{V}_i^t, \quad X_{iu} \in \boldsymbol{R}^{p\times p}, \quad X_{iv} \in \boldsymbol{R}^{(n-p)\times(n-p)} \tag{4.53}$$

Proof When $p = n$, it readily evident with L_i being nonsingular that (4.52) is solvable for X_i. Now, consider that $p < n$. It follows from (4.51) and (4.52) and the properties of $\mathcal{V}_i$ that

$$\begin{aligned} & E_i \, \mathcal{U}_i \; [\Lambda_{pi}, 0] \; \mathcal{V}_i^t = \mathcal{U}_i \; [\Lambda_{pi}, 0] \; \mathcal{V}_i^t X_i \Longrightarrow \\ & [E_i \, \mathcal{U}_i \Lambda_{pi}, 0] = [\mathcal{U}_i \Lambda_{pi}, 0] \; \mathcal{V}_i^t X_i \mathcal{V}_i \end{aligned} \tag{4.54}$$

On letting

$$\begin{aligned} X_i \;&=\; \mathcal{V}_i \begin{bmatrix} X_{iu} & X_{id} \\ \bullet & X_{iv} \end{bmatrix} \mathcal{V}_i^t \\ X_{iu} &\in \mathbf{R}^{\times p}, X_{iv} \in \mathbf{R}^{(n-p)\times(n-p)}, X_{id} \in \mathbf{R}^{p\times(n-p)} \end{aligned} \tag{4.55}$$

it follows that (4.54) is equivalent to

$$[E_i \mathcal{U}_i \Lambda_{pi}, 0] = [\mathcal{U}_i \Lambda_{pi} X_{iu}, \mathcal{U}_i \Lambda_{pi} X_{id}] \tag{4.56}$$

The solvability of (4.56) with respect to X_i holds if and only if

$$\mathcal{U}_i \Lambda_{pi} X_{id} \equiv 0 \Longrightarrow X_{id} \equiv 0 \text{ and } E_i \mathcal{U}_i \Lambda_{pi} = \mathcal{U}_i \Lambda_{pi} X_{iu}$$

which completes the proof. ■

It is significant to observe that **Theorem 4.19** substitutes the matrix equation (4.50) by structural selection of the matrix variable X_i. Incorporating this result into **Theorem 4.19**, we have thus established the following result:

Theorem 4.20 *Consider switched system (4.42) and (4.43) with $w \equiv 0$ subject to the output feedback control $u_k = G_i \; y_k$ with output matrix L_i having the SVD form $L_i = \mathcal{U}_i \; [\Lambda_{pi}, 0] \; \mathcal{V}_i^t$, $\Lambda_{pi} \in \boldsymbol{R}^{p \times p}$. The resulting closed-loop system is UQS if there exist matrices $0 < X_{iu} = X_{iu}^t \in \Re^{p \times p}$, $0 < X_{iv} = X_{iv}^t \in \boldsymbol{R}^{(n-p)\times(n-p)}$, $0 < X_j = X_j^t$, $0 < E_i = E_i^t$, R_i such that for all $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ the LMIs*

$$\begin{bmatrix} -X_i & 0 & X_i A_i^t + L_i^t R_i^t B_i^t & X_i C_i^t + L_i^t R_i^t D_i^t \\ \bullet & -\gamma^2 I & \Gamma_i^t & \Phi_i^t \\ \bullet & \bullet & -X_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{4.57}$$

have a feasible solution. Moreover, the feedback gain is given by

$$G_i = R_i \mathcal{U}_i \Lambda_{pi} X_{iu}^{-1} \Lambda_{pi}^{-1} \mathcal{U}_i^{-1}, \; i \in \boldsymbol{N}$$

Remark 4.21 The optimal switched static output feedback with $\mathcal{L}_2$ gain for system (4.46 and 4.47) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned} & Minimize \;\; \gamma \\ & Subject\ to \quad LMIs\ (4.57), \quad (i, j) \in \mathbf{N} \times \mathbf{N} \\ & X_{iu}, \quad X_{iv}, \quad R_i, \;\; X_j > 0, \quad \gamma > 0 \end{aligned}$$

which can be conveniently solved by the existing software [74].

4.4.4 Switched Dynamic Output Feedback

Now we direct attention to the more general case and employ at every mode $i \in \mathbf{N}$ a switched dynamic output-feedback scheme of the form:

$$\begin{aligned} (\Sigma_C): \qquad \zeta_{k+1} &= A_{ci} \zeta_k + B_{ci} y_k \\ u_k &= C_{ci} \zeta_k \end{aligned} \tag{4.58}$$

Augmenting controller (4.58) to switched system (4.26), (4.27), and (4.28) and defining the composite vector $\xi_k^t = \begin{bmatrix} x_k^t & \zeta_k^t \end{bmatrix}$, we get the closed-loop system

$$(\Sigma_{JC}): \quad \begin{aligned} \xi_{k+1} &= \mathcal{A}_i \xi_k + \bar{\Gamma}_i w_k \\ z_k &= \mathcal{C}_i \xi_k + \Phi_i w_k \end{aligned} \tag{4.59}$$

where the respective matrices are given by

$$\mathcal{A}_i = \begin{bmatrix} A_i & B_j C_{ci} \\ B_{ci} L_i & A_{ci} \end{bmatrix}, \quad \bar{\Gamma}_i = \begin{bmatrix} \Gamma_j \\ B_{ci} \Phi_j \end{bmatrix}, \quad \mathcal{C}_i = [C_i \quad D_i C_{ci}] \tag{4.60}$$

Application of **Theorem 4.13** shows that switched system (4.60) is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$ if there exist matrices $0 < \mathcal{P}_i^t = \mathcal{P}_i, \; 0 < \mathcal{Y}_j^t = \mathcal{Y}_j, \; j \in \mathbf{N}$ and a scalar $\gamma > 0$ satisfying the LMIs

$$\begin{bmatrix} -\mathcal{P}_i & 0 & \mathcal{A}_i^t & \mathcal{C}_i^t \\ \bullet & -\gamma^2 I & \bar{\Gamma}_i^t & \Phi_i^t \\ \bullet & \bullet & -\mathcal{Y}_j & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, \quad (i, j) \in \mathbf{N} \times \mathbf{N} \tag{4.61}$$

Introducing the shorthand

$$\mathcal{P}_i = \begin{bmatrix} \mathcal{P}_{si} & 0 \\ 0 & \mathcal{P}_{ci} \end{bmatrix}, \quad \mathcal{X}_i \triangleq \mathcal{P}_i^{-1} = \begin{bmatrix} \mathcal{X}_{si} & 0 \\ 0 & \mathcal{X}_{ci} \end{bmatrix}, \quad \mathcal{Y}_j = \begin{bmatrix} \mathcal{Y}_{sj} & 0 \\ 0 & \mathcal{Y}_{cj} \end{bmatrix} \tag{4.62}$$

we have the following result:

Theorem 4.22 *Consider switched system (4.59) and (4.60) with output matrix L_i having the SVD form $L_i = \mathcal{U}_i \; [\Lambda_{pi}, \;\; 0] \; \mathcal{V}_i^t, \; \Lambda_{pi} \in \boldsymbol{R}^{p \times p}$. This system is UQS with an $\mathcal{L}_2$ gain $\mathcal{G} < \gamma$ if there exist matrices $0 < \mathcal{X}_{siu}^t = \mathcal{X}_{siu}, 0 < \mathcal{X}_{siv}^t = \mathcal{X}_{siv}, 0 < \mathcal{X}_{ci}^t = \mathcal{X}_{ci}, 0 < \mathcal{Y}_{si}^t = \mathcal{Y}_{si}, 0 < \mathcal{Y}_{ci}^t = \mathcal{Y}_{ci}, \Omega_{ci}, \Pi_{cj}, \Upsilon_{ci}, \Psi_{ci}, (i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ and a scalar $\gamma > 0$ satisfying the systems LMIs*

$$\begin{bmatrix} -\mathcal{X}_{si} & 0 & 0 & \mathcal{X}_{si} A_i^t & L_i^t \Omega_{cj}^t & \mathcal{X}_{si} C_i^t \\ \bullet & -\mathcal{X}_{ci} & 0 & \Lambda_{ci}^t B_i^t & \Upsilon_{cj}^t & \Pi_{ci}^t D_i^t \\ \bullet & \bullet & -\gamma^2 I & \Gamma_i^t & \Psi_{ci}^t & \Phi_i^t \\ \bullet & \bullet & \bullet & -\mathcal{Y}_{sj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{Y}_{sj} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0, \quad (i, j) \in \boldsymbol{N} \times \boldsymbol{N} \tag{4.63}$$

Moreover, the gain matrices are given by

$$A_{ci} = \Upsilon_{ci} \; \mathcal{X}_{ci}^{-1}, \quad B_{cj} = \Omega_{cj} \; \mathcal{U}_i^{-1} \Lambda_{pi}^{-1} \mathcal{X}_{siu}^{-1} \Lambda_{pi} \mathcal{U}_i, \quad C_{ci} = \Pi_{ci} \; \mathcal{X}_{ci}^{-1} \tag{4.64}$$

Proof Follows from **Theorems 4.13–4.19** by applying the congruent transformation $[\mathcal{X}_i, \; I, \; I, \; I]$ to LMIs (4.61) with $\mathcal{X}_i = \mathcal{P}_i^{-1}$ and expanding the result using

(4.62) along with the matrix substitutions $A_{ci}\ \mathcal{X}_{ci} = \Upsilon_{ci},\ L_i \mathcal{X}_{si} = \Xi_{si} L_i,\ \Omega_{ci} = B_{ci}\Xi_{ci},\ \Xi_i = \mathcal{U}_i \Lambda_{pi} \mathcal{X}_{siu} \Lambda_{pi}^{-1} \mathcal{U}_i^{-1},\ C_{ci}\mathcal{X}_{ci} = \Pi_{cj}$.

Remark 4.23 The optimal switched dynamic output feedback with $\mathcal{L}_2$ gain for system (4.59 and 4.60) subject to the polytopic representation (4.30 and 4.31) can be determined by solving the following convex minimization problem over LMIs:

$$Minimize\ \ \gamma$$
$$wrt\ \ \mathcal{X}_{siu} > 0,\ \mathcal{X}_{siv} > 0,\ \mathcal{X}_{cj} > 0,\ \mathcal{Y}_{sj} > 0,\ \mathcal{Y}_{cj} > 0,\ \Omega_{cj},\ \Upsilon_{cj},\ \Pi_{cj},\ \gamma > 0$$

$$\begin{bmatrix} -\mathcal{X}_{si} & 0 & 0 & \mathcal{X}_{si}A_{ip}^t & L_{ip}^t\Omega_{cj}^t & \mathcal{X}_{si}C_{ip}^t \\ \bullet & -\mathcal{X}_{ci} & 0 & \Pi_{ci}^t B_{ip}^t & \Upsilon_{cj}^t & \Lambda_{ci}^t D_{ip}^t \\ \bullet & \bullet & -\gamma^2 I & \Gamma_i^t & \Psi_{ci}^t & \Phi_{ip}^t \\ \bullet & \bullet & \bullet & -\mathcal{Y}_{sj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{Y}_{sj} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0,$$
$$\forall\ (i, j) \in \mathbf{N} \times \mathbf{N},\ \ p \in \{1, ..., M_i\}$$

4.4.5 Numerical Examples

Two examples will be given in the sequel:

Illustrative Example A

In this example, we consider a discrete model of water pollution described by dynamical system of the type (4.42) and (4.43) with multiple modes. In terms of our terminology, each mode represents a particular equilibrium operating point. We wish to design a switched-state feedback control for this system based on **Theorem 4.17**. Switching occurs between three modes described by the following coefficients:

Mode 1:

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 & 1 \\ 0.6 & 1 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix},\ \Phi_1 = \begin{bmatrix} 0.6 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix},\ L_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix},\ \Phi_2 = \begin{bmatrix} 0.3 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.8 & 0.3 \end{bmatrix},\ L_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}$$

Mode 3:

$$A_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.6 & 0.3 \end{bmatrix}, \ \Gamma_3 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \ B_3 = \begin{bmatrix} 2 & 0.9 \\ 0.7 & 2 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, \ \Phi_3 = \begin{bmatrix} 0.1 \end{bmatrix}, \ D_3 = \begin{bmatrix} 0.9 & 0.3 \end{bmatrix}, \ L_3 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}$$

The number of vertices at the respective modes and weightings are taken as $M_j = 3$, $\lambda_{1k} = 0.3$, $\lambda_{2k} = 0.5$, $\lambda_{3k} = 0.2$, $k = 1, .., 3$. Invoking the software environment [75], the feasible solution of LMIs (4.44) is given by

$$X_1 = \begin{bmatrix} 5.8731 & 0 \\ 0 & 5.8731 \end{bmatrix}, \ X_2 = \begin{bmatrix} 5.3925 & 0 \\ 0 & 5.3925 \end{bmatrix}, \ X_3 = \begin{bmatrix} 5.2141 & 0 \\ 0 & 5.2141 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} -0.6717 & 0.1189 \\ 1.8608 & -3.9669 \end{bmatrix}, \ Z_2 = \begin{bmatrix} -1.7867 & 0.2830 \\ -0.9088 & -0.0476 \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} -2.8880 & -0.2209 \\ -1.2026 & -0.2208 \end{bmatrix}$$

Since $K_i = Z_i X_i^{-1}$, the control gains become

$$K_1 = \begin{bmatrix} -0.1144 & 0.0202 \\ 0.3168 & -0.6754 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.3313 & 0.0525 \\ -0.1685 & -0.0088 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -0.5539 & -0.0424 \\ -0.2306 & -0.0423 \end{bmatrix}$$

Illustrative Example B

Similar to illustrative example A, we consider here another discrete model of water pollution described by dynamical system of the type (4.42) and (4.43) with multiple modes. Again, each mode represents a particular equilibrium operating point. We wish to design a switched static-output feedback control law for this water system based on **Theorem 4.20**. Switching occurs between three modes described by the following coefficients:

Mode 1:

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{bmatrix}, \ \Gamma_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \ B_1 = \begin{bmatrix} 2 & 1 \\ 0.6 & 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, \ \Phi_1 = \begin{bmatrix} 0.6 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}, \ L_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix}, \; \Phi_2 = \begin{bmatrix} 0.3 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0.8 & 0.3 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}$$

Mode 3:

$$A_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.6 & 0.3 \end{bmatrix}, \; \Gamma_3 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \; B_3 = \begin{bmatrix} 2 & 0.9 \\ 0.7 & 2 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, \; \Phi_3 = \begin{bmatrix} 0.1 \end{bmatrix}, \; D_3 = \begin{bmatrix} 0.9 & 0.3 \end{bmatrix}, \; L_3 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}$$

Invoking the software environment [74], the feasible solution of LMIs (4.57) is given by

$$X_1 = \begin{bmatrix} 2.0255 & 0 \\ 0 & 2.0255 \end{bmatrix}, \; X_2 = \begin{bmatrix} 1.9232 & 0 \\ 0 & 1.9232 \end{bmatrix}, \; X_3 = \begin{bmatrix} 1.8327 & 0 \\ 0 & 1.8327 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} -1.5708 & 2.0324 \\ 0.5787 & -1.4950 \end{bmatrix}, \; R_2 = \begin{bmatrix} -0.7030 & 0.0502 \\ -0.4344 & -0.0523 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} -0.3482 & -0.7446 \\ -0.2057 & -0.5837 \end{bmatrix}$$

Using the SVD of $L_i, \; i = 1, \ldots, 3$, the control gains become

$$G_1 = \begin{bmatrix} -0.7755 & 1.0034 \\ 0.2857 & -0.7381 \end{bmatrix}, \; G_2 = \begin{bmatrix} -0.3655 & 0.0261 \\ -0.2259 & -0.0272 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} -0.1900 & -0.4063 \\ -0.1122 & -0.3185 \end{bmatrix}$$

4.5 Notes and References

In addition to the numerous papers and articles on switched systems, there are few reference books that might have some connection to the topics to be discussed in this book. This includes the fundamental references [20, 192, 193, 357–359, 366]. The material covered in this chapter draws heavily on the excellent papers [47, 133, 134, 196, 426]. We have examined $\mathcal{L}_2$ gain analysis and control synthesis for a class of linear switched systems with convex-bounded parameter uncertainties in the system matrices using an appropriate switched Lyapunov functional. LMI-based feasibility conditions have been developed to ensure that the linear switched system with polytopic uncertainties is uniformly quadratically stable with an $\mathcal{L}_2$

gain smaller that a prescribed constant level. Switched feedback schemes have been designed using state measurements, output measurements, and by using dynamic output feedback, to guarantee that the corresponding closed-loop system enjoys the uniform quadratic stability with an $\mathcal{L}_2$ gain smaller than a prescribed constant level. All the developed results have been expressed in terms of convex optimization over LMIs and have been tested on representative examples.

Together with the foregoing chapter, this introductory chapter is hoped to have succeeded in motivating the readers to the upcoming topics and in paving the way to study the interesting topics of stability, stabilization, control design, and filtering switched time-delay systems.

Chapter 5
Switched Time-Delay Systems

5.1 Introduction

This chapter is concerned with the main ingredients and basic notions of switched time-delay systems. For simplicity of exposition, we present the relevant topics and materials of both switched systems at large and time-delay systems in particular. Therefore, the chapter is divided into two major sections: the first section gives an overview about switched time-delay systems and the second presents an overview of piecewise-affine systems.

5.2 Switched Time-Delay Systems

In this manner, a *switched time-delay system* is recognized to be composed of a family of time-delay subsystems and a rule that governs the switching among them, and is mathematically described by

$$\begin{aligned} \delta x(t) &= f_\sigma(x(t), u(t), \tau(t), v(t)), \quad x(t_o) = x_o \\ y(t) &= g_\sigma(x(t), \tau(t), w(t)) \end{aligned} \tag{5.1}$$

where $\tau(t)$ is the time-delay factor and the remaining quantities are standard in state space representation. Each individual time-delay component model

$$\begin{aligned} \delta x(t) &= f_j(x(t), u(t), \tau(t), d(t)), \quad x(t_o) = x_o \\ y(t) &= g_k(x(t), \tau(t), w(t)), \quad j \in \mathbf{S} \end{aligned} \tag{5.2}$$

is said to be a *time-delay subsystem* or *time-delay mode* of the switched system. In the sequel, it is assumed that the delay $\tau(t)$ is a differentiable time-varying function satisfying

$$0 < \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu \tag{5.3}$$

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where the bounds ϱ and μ are known constant scalars. Sometimes the bounding relation $\mu < 1$ [181, 216, 301] is used.

Alternatively, depending on the problem formulation, the delay $\tau(t)$ is considered as a time-varying function satisfying

$$0 < \tau(t) \leq \varrho \tag{5.4}$$

where the bounds ϱ is a known constant scalar.

By and large, the time-delay subsystems represent the low-level local dynamics governed by FDEs, while the supervisor is the high-level coordinator producing the switches among local dynamics. Thus, the dynamics of the switched time-delay (STD) system is determined by both the time-delay subsystems and the switching signal.

A switching signal may depend on the time, its own past value, the state/output, and/or an external signal as well:

$$\sigma(t+) = \varphi(t, \sigma(t), x(t)/y(t), r(t)) \quad \forall t \tag{5.5}$$

where $r(t)$ is an external signal produced by other devices, $\sigma(t+) = \lim_{s \downarrow t} \sigma(s)$ in continuous time, and $\sigma(t+) = \sigma(t+1)$ in discrete time.
In the case $\sigma(t) = j$, we say that the jth subsystem is *active at time t*. It is quite evident that at any instant there is one (and only one) active subsystem.

In the remainder of the book, we focus on model (5.2) subject to time-delay pattern (5.3) or (5.4) and the switching signal (5.5).

5.2.1 Multiple Lyapunov Functions

Construction of Lyapunov functions is a fundamental problem in system theory; internal stability of the system under consideration is concluded if an associated Lyapunov function is shown to exist. Conceptually, when looking at STD system, perhaps the simplest solution would be a *common quadratic* Lyapunov function, that is a quadratic function which is a global Lyapunov function for the subsystems comprising the hybrid system. It turns out that the construction of such a Lyapunov function is an NP-hard problem even when the subsystems are linear time invariant [20]. The conservatism introduced by a *global* Lyapunov function V can be reduced by searching for a set $\{V_j\}$ of local Lyapunov functions and by ensuring that the Lyapunov functions *match* in the sense that the values of the Lyapunov functions $\{V_j\}$ and $\{V_m\}$ are equal when the state trajectory leaves a cell $\{X_j\}$ and enters a cell $\{X_m\}$, where $\{V_j\}$ is a local Lyapunov function in the cell $\{X_j\}$ and $\{V_m\}$ is a local Lyapunov function in the cell $\{X_m\}$ [28].

5.2.2 Switched-Stability Analysis

Recently, fundamental development of stability analysis of switched systems has been made in the control community (see, for example, [28, 192, 366]). In the literature, switched time-delays (STD) systems appear in applications whenever switching and time delay coexist in either system modeling or signal transmission. Due to the interaction between continuous dynamics and discrete dynamics and because of the impact of time delays, the behavior of STD systems is usually much more compounded than that of switched systems or delay systems. To date, there are a few results reported on such systems [169, 370, 397]. We recall from that there are three basic issues associated with the problems of stability and design of switched systems without delays [192]; similar problems also exist in the study of STD systems, namely

- finding conditions of stabilizability under arbitrary switching,
- identifying the useful class of stabilizing switching signals, and
- constructing a stabilizing switching signal

In the sequel, we focus on constructing stabilizing switching signal; it is well known that on the premises of Hurwitz convex combination, a linear switched system without delay is asymptotically stable under the switching law designed by the single Lyapunov function method [192]. It turns out [169] that such result still holds for linear switched systems with constant delay if the delay is sufficiently small. Also, a method of quantifying the delay bound was given. Alternatively, it has simultaneously been pointed out that the delay bound obtained by their method is comparatively conservative and improving the theoretical result is an open problem. Thus the first question is how to get a less conservative criterion. In [169], the case of constant delay is only considered. So, the immediate question is that for the case of time-varying delay, whether a similar result can be obtained. These questions motivate the write-up of this part of the book.

Therefore, the stability problem for a class of switched time-delay system with time-varying delay is considered hereafter. Consider for the time being the class of STD systems in the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-\tau(t)), \quad x_{t_o} = \psi(\alpha), \ \alpha \in [-\tau, 0] \tag{5.6}$$

where $x(t) \in \Re^n$ denotes the state vector; $\sigma(t) : [0, \infty) \rightarrow \mathbf{S} = \{1, 2, ..., s\}$ is the switching signal which depends on time t or state $x(t)$, A_j and B_j are constant matrices for $j \in \mathbf{S}$, $\psi(\alpha)$ is a continuously differentiable initial function, $\tau(t) > 0$ and $x_t = x(t+\alpha)$, $\alpha \in [-\tau, 0]$, and the term $\tau(t)$ denotes the time-varying delay satisfying either of the following patterns:

$$\begin{aligned} \text{Case 1}: &\quad 0 \le \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu < 1 \\ \text{Case 2}: &\quad 0 \le \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu \end{aligned} \tag{5.7}$$

where the bounds ϱ, μ are known constants in order to guarantee smooth growth of the state trajectories. The following switching sequence

$$\left\{ x_{t_o};\ (j_o, t_o), ..., (j_k, t_k), ..., | j_k \in \mathbf{S},\ k = 0, 1, ... \right\}$$

corresponds to the switching signal $\sigma(t)$, meaning that the jth subsystem is activated when $t \in [t_k, t_{k+1})$. Observe that when $\tau(t) \equiv 0$, stability of system (5.6) is equivalent to $(A_j + A_{dj})$ being Hurwitz. Bearing this in mind, we proceed further and recall the following condition

Hurwitz convex combination condition: *There exists a Hurwitz linear convex combination Λ of $(A_j + A_{dj})$, that is,*

$$\Lambda = \sum_{j=1}^{s} \beta_j \, (A_j + A_{dj}) \tag{5.8}$$

where $0 < \beta_j < 1$ *and* $\sum_{j=1}^{s} \beta_j = 1$.

It follows since Λ is Hurwitz that there exists a matrix $\mathcal{P} > 0$ such that

$$\mathcal{P}\Lambda + \Lambda^t\mathcal{P} + \mathcal{Q} = 0, \quad \mathcal{Q} > 0 \tag{5.9}$$

Given a pair $\mathcal{P}, \ \mathcal{Q}$ satisfying (5.9), let us introduce the following set

$$\Omega_j = \left\{ x \in \Re^n | x^t[\mathcal{P}(A_j + A_{dj}) + (A_j + A_{dj})^t\mathcal{P}]x \ \le \ - x^t \mathcal{Q} x \right\} \tag{5.10}$$

for each $j \in \mathbf{S}$. As presented in [169], it is readily seen that

$$\Re^n = \bigcup_{j=1}^{n} \Omega_j$$

Based thereon, we construct the following switching regions:

$$\hat{\Omega}_1 = \Omega_1, \quad \hat{\Omega}_2 = \Omega_1 / \left(\Omega_2 \bigcap \hat{\Omega}_1 \right), \ldots.,$$
$$\hat{\Omega}_j = \Omega_j / \left(\Omega_j \bigcap \bigcup_{m=1}^{j-1} \hat{\Omega}_m \right), \ldots., \hat{\Omega}_s = \Omega_j / \left(\Omega_s \bigcap \bigcup_{m=1}^{s-1} \hat{\Omega}_m \right)$$

Hence

$$\bigcup_{j=1}^{s} \hat{\Omega}_j = \Re^n, \quad \hat{\Omega}_j \bigcap \hat{\Omega}_r = \oslash$$

and the switching law takes the form

$$\sigma(t) = \ j, \ \text{when } x(t) \ \in \ \hat{\Omega}_j \tag{5.11}$$

Since in the case $\tau(t) \equiv 0$, system (5.6) reduces to the delayless system $\dot{x}(t) = (A_j + A_{dj})x(t)$. It is well known that the condition of Hurwitz convex combination can guarantee the asymptotic stability of system $\dot{x}(t) = (A_j + A_{dj})x(t)$ under switching law (5.11). The following analysis will show that for a certain delay bound, the result still holds for system (5.6). The analytical treatment relies on **Lemma 13.5** and leads to the following result:

Lemma 5.1 *There exist a constant* $\mu > 0$ *and matrices* $\mathcal{P} > 0$, $\mathcal{R} > 0$, $\mathcal{Z} > 0$, $\mathcal{X}^j = \begin{bmatrix} \mathcal{X}_a^j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix}$, $j \in \mathbf{S}$ *and some matrices* $\mathcal{Y}_j$, $\mathcal{T}_j$ *with appropriate dimensions such that the following LMIs*

$$\Pi_j = \begin{bmatrix} \Pi_a^j & \Pi_c^j & \varrho(A_j + A_{dj})^t \mathcal{Z} \\ \bullet & \Pi_o^j & 0 \\ \bullet & \bullet & -\varrho \mathcal{Z} \end{bmatrix} > 0 \tag{5.12}$$

$$\Theta_j = \begin{bmatrix} \mathcal{X}_a^j & \mathcal{X}_c^j & \mathcal{Y}_j + \mathcal{P}A_{dj} + \varrho(A_j + A_{dj})^t \mathcal{Z} A_{dj} \\ \bullet & \mathcal{X}_o^j & \mathcal{T}_j \\ \bullet & \bullet & \mathcal{Z} - \varrho^2 A_{dj}^t \mathcal{Z} A_{dj} \end{bmatrix} \geq 0 \tag{5.13}$$

$$\mathcal{P}\Lambda + \Lambda^t \mathcal{P} < 0 \tag{5.14}$$

hold for any given $\varrho \in (0, \widehat{\varrho}]$ *and any* $j \in \mathbf{S}$ *where* A_j, A_{dj} *are given in (5.6) and* Λ *is a Hurwitz matrix defined in (5.8) and*

$$\begin{aligned} \Pi_a^j &= \Lambda^t \mathcal{P} + \mathcal{P}\Lambda + \mathcal{Y}_j + \mathcal{Y}_j^t + \mathcal{R} + \varrho \mathcal{X}_a^j \\ \Pi_c^j &= -\mathcal{Y}_j + \mathcal{T}_j^t + \varrho \mathcal{X}_c^j \\ \Pi_o^j &= -\mathcal{T}_j^t + \mathcal{T}_j - (1-\mu)\mathcal{R} + \varrho \mathcal{X}_o^j \end{aligned} \tag{5.15}$$

Proof Initially, given two matrices $0 < \mathcal{R}^t = \mathcal{R}$, $0 < \mathcal{Z}^t = \mathcal{Z}$, consider $\mathcal{Y}_j \equiv 0$, $\mathcal{T}_j \equiv 0$. Since Λ is Hurwitz matrix, there exists a matrix $0 < \mathcal{P}^t = \mathcal{P}$ such that $\mathcal{P}\Lambda + \Lambda^t \mathcal{P} + 2\,\mathcal{R} = 0$ implying that LMI (5.14) is satisfied. In addition, Π_j and Θ_j reduce to the following form:

$$\widehat{\Pi}_j = \begin{bmatrix} -\mathcal{R}_j + \varrho \mathcal{X}_a^j & \varrho \mathcal{X}_c^j & \varrho(A_j + A_{dj})^t \mathcal{Z} \\ \bullet & -(1-\mu)\mathcal{R} + \varrho \mathcal{X}_o^j & 0 \\ \bullet & \bullet & -\varrho \mathcal{Z} \end{bmatrix}$$

$$\widehat{\Theta}_j = \begin{bmatrix} \mathcal{X}_a^j & \mathcal{X}_c^j & \mathcal{P}A_{dj} + \varrho(A_j + A_{dj})^t \mathcal{Z} A_{dj} \\ \bullet & \mathcal{X}_o^j & 0 \\ \bullet & \bullet & \mathcal{Z} - \varrho^2 A_{dj}^t \mathcal{Z} A_{dj} \end{bmatrix}$$

Our task now is to show that there exists a constant $\widehat{\varrho} > 0$ such that for any $\varrho \in (0, \widehat{\varrho}]$, $\widehat{\Pi}_j < 0$, $\widehat{\Theta}_j \geq 0$ hold. By the **S**-procedure, it is known that there exists a small constant $0 < \varrho_*^j,\ j \in \mathbf{S}$, such that for $\varrho \in (0,\ \widehat{\varrho}],\ \mathcal{Z} - \varrho^2 A_{dj}^t \mathcal{Z} A_{dj} > 0$. For some $\mathcal{P} > 0,\ \mathcal{R} > 0,\ \mathcal{Z} > 0$ and for all $\varrho \in \left[0,\ \varrho_*^j\right]$, there must exist a matrix Ξ_j such that

$$\Xi_j \geq \widehat{\Xi}_j = [\mathcal{P} A_{dj} + \varrho(A_j + A_{dj})^t \mathcal{Z} A_{dj}] \times \left(\mathcal{Z} - \varrho^2 A_{dj}^t \mathcal{Z} A_{dj}\right)^{-1} [\mathcal{P} A_{dj} + \varrho(A_j + A_{dj})^t \mathcal{Z} A_{dj}]^t$$

More importantly, there exists a matrix

$$\mathcal{X}^j = \begin{bmatrix} \mathcal{X}_a^j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix} \geq 0, \quad j \in \mathbf{S}$$

such that

$$\begin{bmatrix} \mathcal{X}_a^j - \Xi_j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix} \geq 0$$

Correspondingly, it holds that

$$\begin{bmatrix} \mathcal{X}_a^j - \widehat{\Xi}_j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix} \geq 0$$

By Schur complements, it is readily seen that for all $\varrho \in \left[0,\ \varrho_*^j,\ \ \widehat{\Theta}_j \geq 0\right]$. Once again by the $\mathbf{S}$-procedure [27], it holds that for the matrix

$$\widetilde{\Pi}_j = \begin{bmatrix} -\mathcal{R} & 0 \\ \bullet & -(1-\mu)\mathcal{R} \end{bmatrix} + \varrho \begin{bmatrix} (A_j + A_{dj})^t \mathcal{Z}(A_j + A_{dj}) + \mathcal{X}_a^j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix}$$

there exists a constant $0 < \varrho_+^j,\ j \in \mathbf{S}$, such that $\widetilde{\Pi}_j < 0$ for any $\varrho \in \left(0,\ \varrho_+^j\right]$. By Schur complements, we know that $\widehat{\Pi}_j < 0$. Define $\bar{\varrho} = \min_{m=1}^s \left\{\varrho_*^j,\ \varrho_+^j\right\}$. Then for all $\varrho \in (0,\ \widehat{\varrho}],$ we have $\widehat{\Pi}_j < 0$ and $\widehat{\Theta}_j \geq 0$ for $j \in \mathbf{S}$, which yields the desired result. This completes the proof. ■

Lemma 5.2 *There exist a constant $\mu > 0$ and matrices $\mathcal{P} > 0,\ \mathcal{Z} > 0,\ \mathcal{X}^j = \begin{bmatrix} \mathcal{X}_a^j & \mathcal{X}_c^j \\ \bullet & \mathcal{X}_o^j \end{bmatrix},\ j \in \mathbf{S}$ and some matrices $\mathcal{Y}_j,\ \mathcal{T}_j$ with appropriate dimensions such that LMIs (5.12) with $\mathcal{R} \equiv 0$ and LMIs (5.13) and (5.14) hold for any $j \in \mathbf{S}$ and any $\varrho \in (0,\ \widehat{\varrho}]$.*

Proof Follows directly from **Lemma 5.1** by setting $\mathcal{Y}_j = \mathcal{T}_j = \widehat{\mathcal{R}}$ and $\mathcal{P}\Lambda + \Lambda^t\mathcal{P} + 3\widehat{\mathcal{R}}$ for any $j \in \mathbf{S}$ and $\widehat{\mathcal{R}} > 0$. ■

5.2.3 *Illustrative Example A*

Consider the following piecewise linear time-delay system of the type (5.6) with $\sigma(t) \in \mathbf{S} = \{1, 2\}$ [372]:

$$A_1 = \begin{bmatrix} -2 & 2 \\ -20 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 10 \\ -4 & -2 \end{bmatrix}$$
$$A_{d1} = \begin{bmatrix} -1 & -7 \\ 23 & 6 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 4 & -5 \\ 1 & -8 \end{bmatrix}$$

This example was also treated in [169] in the case of constant delay $\dot{\tau} \equiv 0$. Observe that $A_j + A_{dj}, \;\; j = 1, 2$, is unstable. Taking $\beta_1 = 0.6, \;\; \beta_2 = 0.4$, from (5.8), we get

$$\Lambda = \begin{bmatrix} -1 & 1 \\ -0.6 & -1.6 \end{bmatrix}$$

Solving (5.12), (5.13), and (5.14) yields the maximum delay bound $\mu = 0.0202$ and

$$\mathcal{P} = \begin{bmatrix} 151.5293 & -16.8856 \\ -16.8856 & 184.9110 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 282.7959 & -84.4853 \\ -84.4853 & 625.4863 \end{bmatrix}$$

and the switching law will be given by (5.11). By applying **Lemma 5.1**, the system under consideration with $\mu \le 0.0202$ is asymptotically stable. Notice that the corresponding result in [169] was $\mu = 0.001573$. When $\dot{\tau} \neq 0$, it is found by **Lemma 5.1** that the system under consideration is asymptotically stable with $\mu \le 0.0176$.

5.3 Piecewise-Affine Systems

Thus far, we have learned from the foregoing sections that construction of Lyapunov functions is a fundamental problem in system theory. Its importance stems from the fact that the internal stability of a system is concluded if an associated Lyapunov function is shown to exist. This part of the book is concerned with such a construction for a class of switched systems in the sense that the state trajectory evolution is governed by different dynamical equations (or different difference equations) over different polyhedral partitions $\{X_j\}$ of the state space $\{X\}$, that is, the system is modeled by an ensemble of subsystems, each of which is a valid representation of the system over a set of such partitions. Motivating applications for the study of such systems is described in [13]. We knew before that the simplest solution is perhaps a *common quadratic Lyapunov function*, that is, a quadratic function which is a global Lyapunov function for the subsystems comprising the switched system. However, the existence of such a function is, in principle, an overly restrictive requirement to deduce the stability [156]. Moreover, the construction of such a Lyapunov function is an NP-hard problem even when the subsystems are linear time invariant.

The conservatism introduced by a global Lyapunov function V can be reduced by searching for a set $\{V_j\}$ of local Lyapunov functions and by ensuring that the Lyapunov functions match in the sense that the values of Lyapunov functions $\{V_m\}$ and $\{V_n\}$ are equal when the state trajectory leaves a cell $\{X_m\}$ and enters a cell $\{V_n\}$, where $\{V_m\}$ is a local Lyapunov function in the cell $\{X_m\}$ and $\{V_n\}$ is a local Lyapunov function in the cell $\{V_n\}$ (see [28]). In this context, an elegant result has been recently derived by [156] to construct Lyapunov functions when the subsystem dynamics are known to be affine time invariant; an independent interpretation of this result is given in [134]. For some practical applications, however, the piecewise-affine structure must be modified to address modeling uncertainties and time delays. For such systems, consequently, the stability conditions laid down by [156] get modified as we will demonstrate.

5.3.1 Continuous-Time Systems

In this section, we focus attention on piecewise-affine continuous-time systems. At start, the following definition from [156] introduces piecewise-affine (PWA) systems.

Definition 5.3 The class S_a of switched systems is defined by a family of ordinary differential equations as

$$\dot{x}(t) = A_j x(t) + a_j \quad \forall\, x(t)\ \mathsf{X}_j \tag{5.16}$$

where $A_j \in \Re^{n\times n},\ a_j \in \Re^n$, and $\{\mathsf{X}\}_{\mathsf{j}\in\mathsf{I}_+} \subset \Re^{\mathsf{n}}$ is a partition of the state space into a finite number of closed, and possibly unbounded, polyhedral cells with pairwise disjoint interior. The set of cells that include the origin is denoted by I_O, that is, $a_j = 0, \forall j \in I_O$, and its compliment is denoted I_O^c.

Next, we provide a definition for piecewise-affine time-delay (PWATD) systems.

Definition 5.4 The class S_c of switched systems is defined by a family of retarded differential equations as

$$\dot{x}(t) = A_j x(t) + A_{dj} x(t-\tau) + a_j \quad \forall\, x(t)\ \in\ \mathsf{X}_j \tag{5.17}$$

where $A_j \in \Re^{n\times n},\ A_{dj} \in \Re^{n\times n},\ a_j \in \Re^n$, and $\{\mathsf{X}\}_{\mathsf{j}\in\mathsf{I}_+} \subset \Re^{\mathsf{n}}$ is a partition of the state space into a finite number of closed, and possibly unbounded, polyhedral cells with pairwise disjoint interior. The set of cells that include the origin is denoted I_O, that is, $a_j = 0, \forall j \in I_O$ and its complement is denoted I_O^c. In system (5.7), the time delay τ could be either

- A constant factor (lag) satisfying $0 < \tau \in \Re$,
- A time-varying differentiable function satisfying $0 \leq \tau(t) \leq \varrho, \quad \dot{\tau}(t) \leq \mu$, or
- A time-varying interval differentiable function satisfying $0 < \varphi \leq \tau(t) \leq \varrho, \quad \dot{\tau}(t) \leq \mu$

where the lower bound φ, the upper bound ϱ, and the rate bound μ are known constants. The case $\mu < 1$ corresponds to slowly varying time delay.

For classes of system S_c with alternative delay patterns, our objective hereafter is to determine a set of computationally tractable analytical conditions under which S_c is stable. In preparation, we introduce the following notations:

$$\tilde{A}_j = \begin{bmatrix} A_j & a_j \\ 0 & 0 \end{bmatrix}, \ \tilde{E}_j = \begin{bmatrix} E_j \\ e_j \end{bmatrix}, \ \tilde{F}_j = \begin{bmatrix} F_j \\ f_j \end{bmatrix}, \ \tilde{A}_{dj} = \begin{bmatrix} A_{dj} & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} \begin{bmatrix} e_j \\ f_j \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall j \in I_+ \\ \tilde{E}_j \begin{bmatrix} x \\ 1 \end{bmatrix} &\geq 0, \quad \forall\, x\, \mathsf{X}_j, \quad \in \forall\ j \ \in I_+ \\ \tilde{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} &= \tilde{F}_m \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \forall\, x\, \mathsf{X}_j \cap \mathsf{X}_m, \quad \in \forall\ j, m \ \in I_+ \end{aligned} \tag{5.18}$$

The following result is due to [156]:

Theorem 5.5 *Consider matrices* $T = T^t, U_j = U_j^t, W_j = W_j^t$ *such that the elements of* $U_j, \ W_j$ *are nonnegative. Let* $\mathcal{P}_j \ \stackrel{\Delta}{=}, \ F_j^t \ T \ F_j, \ \forall \ j \in I_O$, *and* $\tilde{\mathcal{P}}_k \stackrel{\Delta}{=} \tilde{F}_k^t \ T \ \tilde{F}_k, \ \forall\ k \in I_O^c$, *satisfy*

$$\mathcal{P}_j A_j + \mathcal{P}_j A_j^t + E_j^t U_j E_j \ < \ 0 \tag{5.19}$$

$$\mathcal{P}_j - E_j^t W_j E_j \ > \ 0 \tag{5.20}$$

$$\tilde{\mathcal{P}}_k \tilde{A}_k + \tilde{\mathcal{P}}_k \tilde{A}_k^t + \tilde{E}_k^t U_j \tilde{E}_k \ < \ 0 \tag{5.21}$$

$$\tilde{\mathcal{P}}_j - \tilde{E}_k^t W_j \tilde{E}_k \ > \ 0 \tag{5.22}$$

for all $j \in I_O$ *and for all* $k \in I_O^c$. *Then, every piecewise continuous trajectory of* S_a *tends to zero exponentially.*

Note that to ensure that the local Lyapunov functions match on the cell boundaries, the predetermined matrices $\tilde{F}_j$ and $\tilde{F}_k$ were taken as the given variables [156] in the manner of (5.19). In addition, the elements of the matrix T are used as the free variables. Hence, condition (5.19) allows for a number of choices of $\tilde{F}_j$ and $\tilde{F}_k$ which might violate the matching condition, thereby incurring an unnecessarily high cost of computation. Indeed, this can be avoided by working directly with the local Lyapunov functions $\mathcal{P}_j$ and $\mathcal{P}_k$ as the unknown variables and by stipulating that $\mathcal{P}_j - \mathcal{P}_k = 2\ \text{herm}(F_{jk}\ \ K_{jk}),\ \ \forall\ j, k$, where the elements $K_{j,k}$ are known variables.

Remark 5.6 In general, it is difficult to deduce the cell containing $x(t-\tau)$ given that a particular cell contains $x(t)$ and, therefore, it is difficult to state the correct matching conditions for the local Lyapunov functions. This implies that state aggregation frequently employed in time-delay system analysis and design cannot be applied [105].

5.3.2 Solution of PWATD Continuous Systems

A solution to the stability of piecewise-affine continuous-time systems with time delay is provided by the following theorem:

Theorem 5.7 *Consider matrices* $T = T^t,\ U_j = U_j^t,\ W_j = W_j^t$ *such that the elements of* $U_j,\ W_j$ *are nonnegative. Let* $\Theta,\ \Upsilon,\ 0 < \mathcal{Q},\ 0 < \mathcal{G}$ *be parameter matrices,* $\mathcal{P}_j \stackrel{\Delta}{=},\ F_j^t\ T\ F_j,\ \forall\ j \in I_O$*, and* $\tilde{\mathcal{P}}_k \stackrel{\Delta}{=} \tilde{F}_k^t\ T\ \tilde{F}_k,\ \forall\ k \in I_O^c$*, satisfy the following inequalities:*

$$\Xi_j = \begin{bmatrix} \Xi_{oj} & \Xi_{nj} & -\mu\Theta & -\mu A_{oj}^t\mathcal{G} \\ \bullet & -\Xi_m & -\mu\Upsilon & -\mu A_{dj}^t\mathcal{G} \\ \bullet & \bullet & -\mu\mathcal{G} & 0 \\ \bullet & \bullet & \bullet & -\mu(2I-\mathcal{G}) \end{bmatrix} < 0 \tag{5.23}$$

$$\mathcal{P}_j - E_j^t W_j E_j > 0 \tag{5.24}$$

$$\tilde{\Xi}_j = \begin{bmatrix} \tilde{\Xi}_{oj} & \tilde{\Xi}_{nj} & -\tau\Theta & -\tau\tilde{A}_j^t\mathcal{G} \\ \bullet & -\Xi_m & -\tau\Upsilon & -\tau\tilde{A}_{dj}^t\mathcal{G} \\ \bullet & \bullet & -\tau\mathcal{G} & 0 \\ \bullet & \bullet & \bullet & -\mu(2I-\mathcal{G}) \end{bmatrix} < 0 \tag{5.25}$$

$$\tilde{\mathcal{P}}_j - \tilde{E}_k^t W_j \tilde{E}_k > 0 \tag{5.26}$$

for all $j \in I_O$ *and for all* $k \in I_O^c$ *where*

$$\begin{aligned} \tilde{\Xi}_{oj} &= \tilde{\mathcal{P}}\tilde{A}_j + \tilde{A}_j^t\tilde{\mathcal{P}} + \Theta + \Theta^t + \mathcal{Q} \\ \tilde{\Xi}_{nj} &= \tilde{\mathcal{P}}\tilde{A}_{dj} - \Theta + \Upsilon^t, \\ \Xi_m &= -\Upsilon - \Upsilon^t + (1-\mu)\mathcal{Q} \end{aligned} \tag{5.27}$$

Then, every piecewise continuous trajectory of S_c *tends to zero exponentially*

Proof It is readily seen from (5.24) that there exists a scalar $\omega > 0$ such that

$$\begin{bmatrix} \Xi_{oj} + \omega I & \Xi_{nj} & \Theta & \widehat{A}_{oj}^t \mathcal{G} \\ \bullet & -\Xi_m & -\Upsilon & \widehat{A}_{dj}^t \mathcal{G} \\ \bullet & \bullet & -\mathcal{G}/\mu & 0 \\ \bullet & \bullet & \bullet & -(2I - \mathcal{G})/\mu \end{bmatrix} < 0 \tag{5.28}$$

Therefore, for all $0 < \tau \in \Re$ we have

$$\Xi_\omega = \begin{bmatrix} \Xi_{oj} + \omega I & \Xi_{nj} & -\tau\Theta & \tau \widehat{A}_{oj}^t \mathcal{G} \\ \bullet & -\Xi_m & -\tau\Upsilon & \tau A_{dj}^t \mathcal{G} \\ \bullet & \bullet & -\tau\mathcal{G} & 0 \\ \bullet & \bullet & \bullet & -\tau(2I - \mathcal{G}) \end{bmatrix} < 0 \tag{5.29}$$

Consider the Lyapunov – Krasovskii functional (LKF):

$$\begin{aligned} V(t) &= V_o(t) + V_a(t) + V_m(t) \\ V_o(t) &= x^t(t)\mathcal{P}x(t), \ \ V_m(t) = \int_{t-\tau(t)}^{t} x^t(s)\mathcal{Q}x(s)\, \mathrm{d}s \\ V_a(t) &= \int_{-\mu}^{0} \int_{t+s}^{t} \dot{x}^t(\alpha)\mathcal{G}\dot{x}(\alpha)\mathrm{d}\alpha\ \mathrm{d}s \end{aligned} \tag{5.30}$$

where $0 < \mathcal{P} = \mathcal{P}^t,\ 0 < \mathcal{G} = \mathcal{G}^t,\ 0 < \mathcal{Q} = \mathcal{Q}^t$ are weighting matrices of appropriate dimensions. It is significant to observe that the third term accounts for delay dependency. Additionally, it can be easily verified that $V(t)$ is continuous in x and t, piecewise continuously differentiable in t, and

$$\alpha\ ||x|| \ \leq\ V(t)\ \leq\ \beta\ ||x||, \ \ \alpha > 0,\ \beta > 0$$

Note also that

$$0\ <\ x^t(t)\ E_j^t\ U_j\ E_j\ x(t), \ \ \forall\, x(t)\ \in\ \mathsf{X}_j \tag{5.31}$$

A straightforward computation gives the time derivative of $V(x)$ along the solutions of (5.17) as

$$\dot{V}_o(t) = 2x^t\mathcal{P}\dot{x} \tag{5.32}$$

On using the equality

$$2[x^t\Theta + x^t(t-\tau(t))\Upsilon]\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right] = 0 \tag{5.33}$$

and manipulating, we get

$$\dot{V}_o(t) = \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \Bigg[2x^t[\mathcal{P}\widehat{A}_{\Delta s} + \Theta]x + 2x^t[\mathcal{P}\widehat{A}_{d\Delta} - \Theta + \Upsilon^t]x(t-\tau) \\ -2x^t(t-\tau)\Upsilon x(t-\tau) - 2x^t\tau(t)\Theta\dot{x}(s) \\ -2x^t(t-\tau)\tau(t)\Upsilon\dot{x}(s) + 2x^t\mathcal{P}\Gamma_\Delta w(t) \Bigg] \mathrm{d}s \tag{5.34}$$

where Θ and Υ are appropriate relaxation matrices injected to facilitate the delay-dependence analysis

$$\begin{aligned} \dot{V}_a(t) &= \int_{-\mu}^{0} [\dot{x}^t(t)\mathcal{G}\dot{x}(t) - \dot{x}^t(t+s)\mathcal{G}\dot{x}(t+s)]\mathrm{d}s \\ &= \int_{t-\mu}^{t} [\dot{x}^t(t)\mathcal{G}\dot{x}(t) - \dot{x}^t(s)\mathcal{G}\dot{x}(s)]\mathrm{d}s \\ &= \mu\, \dot{x}^t(t)\mathcal{G}\dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}^t(s)\mathcal{G}\dot{x}(s)\mathrm{d}s \\ &= \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \Big[\mu\dot{x}^t(t)\mathcal{G}\dot{x}(t) - \tau(t)\dot{x}^t(s)\mathcal{G}\dot{x}(s) \Big] \end{aligned} \tag{5.35}$$

$$\begin{aligned} \dot{V}_m(t) &= x^t(t)\mathcal{Q}x(t) - (1-\dot{\tau})x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\ &\leq x^t(t)\mathcal{Q}x(t) - (1-\mu)x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\ &= \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \Bigg[x^t(t)\mathcal{Q}x(t) \\ &\quad -(1-\mu)x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \Bigg] \mathrm{d}s \end{aligned} \tag{5.36}$$

Finally, from (5.30), (5.31), (5.32), (5.33), (5.34), (5.35), and (5.36) with Schur complements, we have

$$\dot{V}(x)|_{(5.17)} \leq \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \chi^t(t,s)\ \Xi\ \chi(t,s)\ \mathrm{d}s \tag{5.37}$$

$$\chi(t,s) = \big[x^t(t)\ x^t(t-\tau(t))\ \dot{x}(s) \big] \tag{5.38}$$

where Ξ corresponds to Ξ_ω in (5.29) incorporating the inequality $-\mathcal{G}^{-1} \leq -(2I - \mathcal{G})$ (see Appendix) and $\dot{V}(x)|_{(5.17)}$ defines the Lyapunov derivative along the solutions of system (5.17). If $\Xi < 0$, there must be a small scalar $\omega > 0$ such that $\Xi + \mathrm{diag}[\omega,\ 0,\ 0,\ 0,\ 0,\ 0] \leq 0$. Then it follows from (5.37) that

$$\dot{V}(x)|_{(5.17)} < \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \chi^t(t,s)\mathrm{diag}[-\omega,\ 0,\ 0,\ 0,\ 0,\ 0]\chi(t,s)\ \mathrm{d}s \\ = -\omega\ ||x||^2 \tag{5.39}$$

This establishes the desired internal asymptotic stability. ■

Remark 5.8 It is obvious that **Theorem 5.7** is a natural generalization of the results of [156]. In fact, by setting $\tau \equiv 0,\ A_{dj} \equiv 0,\ \mathcal{Q} \equiv 0,\ \mathcal{G} \equiv 0,$ we can reproduce in **Theorem 5.7**; we readily obtain the results of [156]. We would like to assert that by choosing a different LKF, we expect to arrive at a different LMI-based stability condition. We leave this to the reader to verify this point. For example, a relevant LMI-based stability condition based on the small-gain theorem can be derived by setting $\mathcal{Q} = I$. More importantly, a lower bound on the maximum delay τ^* for which the system S_c is stable can be obtained by checking whether the conditions laid down by **Theorem 5.7** are satisfied as τ increases, starting with $\tau \equiv 0$: the least value τ^* for which the conditions laid down by **Theorem 5.7** are not satisfied is a conservative estimate of the maximum delay τ under which the system S_c is stable. Observe that by setting $\mu \equiv 0$, we obtain a solution for the constant-delay case.

5.3.3 Illustrative Example B

Consider the following piecewise linear time-delay system of the type (5.17) with the cell decomposition expressed by $E_j\, x \geq 0$, with

$$E_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},\ E_4 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

and the system matrices are expressed by

$$A_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},\ A_4 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$
$$A_{d1} = \begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix},\ A_{d3} = \begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}$$
$$A_{d2} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix},\ A_{d4} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$$

By applying **Theorem 5.7**, the estimated delay margin $\tau^* = 0.0156$.

5.3.4 Discrete-Time Systems

In this section, we examine discrete-time piecewise-affine (PWA) systems and provide a solution method for its stability. Under mild assumptions, discrete-time PWA systems are equivalent to interconnections of linear systems and finite automata

[355] to complementarity systems [129] and also hybrid systems in the mixed logic dynamical (MLD) form [13]. An important feature of a PWA model is that the state-update map can be discontinuous along the boundary of the regions. For instance, when considering PWA systems stemming from hybrid systems in the MLD form, discontinuities can arise from the representation of logic conditions.

Concerning the stability analysis of PWA systems, various algorithms with different degrees of conservativeness were presented in [293]. Similar to [156], where a particular class of continuous-time PWA system was considered (see also [331, 333, 342, 353], such procedures exploit piecewise quadratic (PWQ) Lyapunov functions that can be computed as the solution of a set of LMIs. For the sake of completeness, the main stability test of [293] is reported in a suitable form.

A class of linear discrete-time piecewise-affine (PWA) systems is defined by the state-space model:

$$x(k+1) = A_j x(k) + B_j u(k) + a_j \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \tag{5.40}$$

In the same manner, a class of linear discrete-time piecewise-affine time-delay (PWATD) systems is defined by the state-space model:

$$\begin{aligned} x(k+1) &= A_j x(k) + A_{dj} x(k-d(k)) + B_j u(k) + a_j \\ &\quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \end{aligned} \tag{5.41}$$

Introducing the following notations

$$\tilde{A}_j = \begin{bmatrix} A_j & a_j \\ 0 & 1 \end{bmatrix}, \; \tilde{B}_j = \begin{bmatrix} B_j \\ 0 \end{bmatrix}, \; \tilde{A}_{dj} = \begin{bmatrix} A_{dj} & 0 \\ 0 & 0 \end{bmatrix}$$

Then with $\tilde{x}(k) = [x^t(k) \quad 1]^t$ we rewrite (5.40) in the form

$$\tilde{x}(k+1) = \tilde{A}_j \tilde{x}(k) + \tilde{B}_j u(k) \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \tag{5.42}$$

and similarly (5.41) be rewritten into the compact form

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A} \tilde{x}(k) + \tilde{A}_{dj} \tilde{x}(k-d(k)) + \tilde{B}_j u(k) \\ &\quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \end{aligned} \tag{5.43}$$

where $x(k) \in \Re^n$ is the state and $u(k) \in \Re^m$ is the control input. The set $\mathbf{X} \subset \Re^{n+m}$ of every possible vector $[x^t(k) \quad u^t(k)]^t$ is either $\Re^{n+m}$ or a polyhedron containing the original, $\{X\}_{j=1}^{s}$ is a polyhedral partition of $\mathbf{X}$ (in the sense that each set X_j is a convex not necessarily closed polyhedron such that $X_j \cap X_j = \oslash, \ \forall j \neq m$, $\cup_{j=1}^{s} = \mathbf{X}$), and $a_j \in \Re^n$ are constant vectors. Much like the continuous case, we refer to each X_j as a *cell*. The delay factor $d(k)$ satisfies $d_m \leq d(k) \leq d_M$ where d_m, d_M are known delay bounds.

Moreover, in order to simplify the exposition, we assume that our cells are polyhedral defined by matrices F_j^x, F_j^u, f_j^x, and f_j^u as

$$X_j \triangleq \left\{[x^t(k) \ \ u^t(k)]^t : F_j^x x \geq f_j^x \ and \ F_j^u u \geq f_j^u\right\} \tag{5.44}$$

Additionally, the following notations are introduced

$$\begin{aligned}\bar{X}_j &\triangleq \left\{x : F_j^x x \geq f_j^x\right\},\\ S_j &\triangleq \left\{j : \exists\, x,\ u \ with \ x \in \bar{X}_j,\ [x^t(k) \ \ u^t(k)]^t \in X_j\right\}\end{aligned} \tag{5.45}$$

Note that S_j is the set of all indices j such that X_j is a cell containing a vector $[x^t(k) \ \ u^t(k)]^t$ for which the condition $x \in \bar{X}_j$ is satisfied. We denote with $\mathcal{I} = \{1, ..., s\}$ the set of indices of the cells X_j whereas the symbol $\mathcal{J} = \{1, ..., t\}$ will be used to denote the set of indices of the cells X_j. It is important to observe that

$$\bigcap_{j=1}^{t} S_j = \mathcal{I} \tag{5.46}$$

Furthermore, if cells X_j have the structure pointed out in (5.44) then the sets S_j are disjoint whereas if cells X_j have a more complicated structure (for instance, when mixed state-input constraints are used to define each cell X_j) then the sets S_j could be overlapping. In the latter case the results could become more conservative.

5.3.5 Stability of PWA Discrete Systems

When we focus on the stability of the origin, we consider autonomous PWA systems

$$\tilde{x}(k+1) = \tilde{A}_j \tilde{x}(k) \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in X_j \tag{5.47}$$

In [293] the stability of the origin of discrete-time PWA systems was characterized by using piecewise quadratic (PWQ) Lyapunov functions. In the following theorem the main result of [293] is presented for the case $a_j \equiv 0, \ \forall \in \mathcal{I}$:

Theorem 5.9 *Consider the system (5.47). If there exist matrices* $0 < \mathcal{P}_j^t = \mathcal{P}_j$, $\forall j \in \mathcal{I}$, *such that the positive-definite function* $V(x(k)) = x^t \mathcal{P}_j x, \ x \in \mathsf{X}_j$, *satisfies* $V(x(k+1)) - V(x(k)) < 0$, *then the origin of the PWA system (5.47) is exponentially stable and* $\lim_{k \to \infty} ||x(k)|| = 0$ *for all system trajectories fulfilling* $x(k) \in \mathbf{X}, \ \forall k \in \mathsf{N}_+$.

The Lyapunov function appearing in **Theorem 5.9** can be computed by solving the LMIs

$$A_j^t \mathcal{P}_m A_j - \mathcal{P}_j < 0, \quad \forall (j, m) \in \mathsf{S} \tag{5.48}$$

$$\mathcal{P}_m = \mathcal{P}_m^t > 0, \quad \forall (m) \in \mathcal{I} \tag{5.49}$$

where

$$\mathsf{S} \triangleq \Big\{ (j, m) : j, m \in \mathcal{I} \text{ and } \exists \ k \in \mathsf{N}_o, \ \exists \ x(k), \ x(k+1) \in \mathbf{X} \text{ such that } x(k) \in \mathsf{X}_j \text{ and } x(k+1) \in \mathsf{X}_m \Big\}$$

In other words, the set S contains all the ordered pairs of indices denoting the possible switches from cell j to cell m and it can be computed via reachability analysis for MLD systems [14]. Then, the inequalities (5.48) take into account all the admissible switches between different regions and guarantee that the Lyapunov function is decreasing along all possible state trajectories. When there exist matrices $\mathcal{P}_m$ such that the LMIs (5.48) and (5.49) are satisfied, the PWA system is termed PWQ-stable. We refer the interested reader to [293] for further details.

5.3.6 Stability of PWATD Discrete Systems

Extending on the previous section, we consider a class of piecewise-affine systems with time delay (PWATD):

$$\tilde{x}(k+1) = \tilde{A}_j \tilde{x}(k) + A_{dj} \tilde{x}(k - d(k)) + \tilde{B}_j u(k) \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \tag{5.50}$$

The delay $d(k)$ is unknown but restricted within the bounded range $d_j^* \leq d(k) \leq d_j^+$ where the limiting scalars $d_j^*, \ d_j^+$ are known. Let $\beta_j = \left(d_j^+ - d_j^* + 1 \right)$ representing the number of samples within the delay range $d_j^* \leq d(k) \leq d_j^+$. For stability purposes, we set $u(k) \equiv 0$ and introduce the following Lyapunov–Krasovskii functional (LKF):

$$V(x(k)) = x^t(k)\mathcal{P}_m x(k) + \sum_{s=k-d(k)}^{k-1} x^t(s)\mathcal{Q}_j x(s)$$
$$+ \sum_{s=2-d^+}^{1-d^*} \sum_{s=k+s-1}^{k-1} x^t(s)\mathcal{Q}_j x(s), \quad x \in \mathsf{X}_j \tag{5.51}$$

where $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{Q}_j^t = \mathcal{Q}_j$ are weighting matrices of appropriate dimensions. In the sequel, we consider the switching profile with $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{Q}_j^t = \mathcal{Q}_j$ at the kth instant and $0 < \mathcal{P}_m^t = \mathcal{P}_m$ at the $(k+1)$th instant. We establish the following stability result:

Theorem 5.10 *Given the delay sample number β_j, system (5.50) with $u \equiv 0$ is delay-dependent asymptotically stable if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{Q}_j^t = \mathcal{Q}_j,\ \forall j \in \mathcal{I}\ 0 < \mathcal{Y}_j^t = \mathcal{Y}_j \in \Re^{n_j \times n_j},\ 0 < \mathcal{W}_j^t = \mathcal{W}_j \in \Re^{n_j \times n_j}$ such that the LKF $V(x(k))$ in (5.51) satisfies*

$$\begin{bmatrix} -\mathcal{P}_j + \beta_j\mathcal{Q}_j & 0 & A_j^t\mathcal{P}_m A_{dj} \\ \bullet & -\mathcal{Q}_j & A_{dj}^t \\ \bullet & \bullet & -\mathcal{P}_m \end{bmatrix} < 0 \tag{5.52}$$

$$0 < \mathcal{P}_m = \mathcal{P}_m^t, \quad \forall (m) \in \mathcal{I} \tag{5.53}$$

Then the origin of the PWA system (5.47) is exponentially stable and $\lim_{k\to\infty} ||x(k)|| = 0$ *for all system trajectories fulfilling* $x(k) \in \mathbf{X},\ \forall k \in \mathsf{N}_+$

Proof A straightforward computation gives the first difference of $\Delta V_j(k) = V_j(k+1) - V_j(k)$ along the solutions of (5.50) with $u_j(k) \equiv 0$ as

$$\Delta V(x(k)) = [A_j\tilde{x}(k) + A_{dj}\tilde{x}(k-d(k))]^t\mathcal{P}_m[A_j\tilde{x}(k) + A_{dj}\tilde{x}(k-d(k))]$$
$$-\tilde{x}^t(k)\mathcal{P}_j\tilde{x}(k) + \tilde{x}^t(k)\mathcal{Q}_j\tilde{x}(k) - \tilde{x}^t(k-d(k))\mathcal{Q}_j\tilde{x}(k-d(k))$$
$$+ \sum_{s=k+1-d_j(k+1)}^{k-1} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s) - \sum_{s=k+1-d_j(k)}^{k-1} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s)$$
$$+(d^+ - d^*)\tilde{x}^t(k)\mathcal{Q}_j\tilde{x}(k) - \sum_{s=k+1-d_j^+)}^{k-d_j^*} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s) \tag{5.54}$$

Since

$$\sum_{s=k+1-d(k+1)}^{k-1} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s) = \sum_{s=k+1-d_j^*}^{k-1} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s)$$
$$+ \sum_{s=k+1-d(k+1)}^{k-d^*} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s)$$
$$\leq \sum_{s=k+1-d(k)}^{k-1} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s) + \sum_{s=k+1-d^+}^{k-d^*} \tilde{x}^t(s)\mathcal{Q}_j\tilde{x}(s) \tag{5.55}$$

Then using (5.55) in (5.54) and manipulating, we reach

$$\begin{aligned}\Delta V(x(k)) &\leq [A_j\tilde{x}(k) + A_{dj}\tilde{x}(k-d(k))]^t\mathcal{P}_m[A_j\tilde{x}(k) + A_{dj}\tilde{x}(k-d(k))] \\ &+ \tilde{x}^t(k)[(d^+ - d^* + 1)\mathcal{Q}_j - \mathcal{P}_j]\tilde{x}(k) - \tilde{x}^t(k-d(k))\mathcal{Q}_j\tilde{x}(k-d(k)) \\ &= \xi^t(k)\ \Xi_j\ \xi(k)\end{aligned} \tag{5.56}$$

where

$$\begin{aligned}\Xi_j &= \begin{bmatrix} A_j^t\mathcal{P}_mA_j + \beta_j\mathcal{Q}_j - \mathcal{P}_j & A_j^t\mathcal{P}_mA_{dj} \\ \bullet & A_{dj}^t\mathcal{P}_mA_{dj} - \mathcal{Q}_j \end{bmatrix} \\ \xi_j(k) &= \begin{bmatrix} \tilde{x}_j^t(k) & \tilde{x}_j^t(k-d(k)) \end{bmatrix}^t\end{aligned} \tag{5.57}$$

The sufficient condition of stability $\Delta V_j k < 0$ implies that $\Xi_j < 0$. By Schur complements, Ξ_j can be brought to the LMI (5.52) which concludes the proof. ■

Remark 5.11 The conservativeness of the LMI's conditions for stability analysis can be reduced by exploiting the so-called S-procedure [409], in order to avoid imposing $x^t\mathcal{P}_jx > 0$ for $[x^t \quad u^t]^t \in \mathsf{X}_m,\ j \neq m,$ see [293]. This modification was proposed in [156] for continuous-time PWA systems and can be easily generalized to the discrete-time case. We point out that similar modifications can be applied to all the analysis LMIs we derive in the following. It is important to highlight that with respect to the continuous-time approach of [156] in our discrete-time framework there is no need to guarantee the continuity of the Lyapunov function over the whole state space. This fact can determine a reduced degree of conservativeness of the results that we are going to present with respect to those presented in [156]. Finally, following the lead given in [156], discrete-time performance analysis results with a notably reduced degree of conservativeness could be performed. This will be demonstrated in the following chapters.

5.3.7 Synthesis of a Stabilizing State Feedback

In the following, we consider a piecewise linear state feedback with the structure

$$\begin{aligned} u(k) &= [K_j \ \ 0]\, \tilde{x}(k) \\ &= \tilde{K}_j\, \tilde{x}(k), \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \end{aligned} \tag{5.58}$$

By applying the controller (5.58) to the system (5.42) we achieve the closed-loop dynamic system

$$\begin{aligned} \tilde{x}(k+1) &= \mathcal{A}_j\, \tilde{x}(k), \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j \\ \mathcal{A}_j &= \tilde{A}_j + \tilde{B}_j \tilde{K}_j = \begin{bmatrix} A_j + B_j K_j & a_j \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{5.59}$$

It should be noted that the evolution of closed-loop system (5.59) depends on the *hidden* variable $u(k)$ since it influences the index j of the current cell X_j.

As customary for constrained systems, we assume that the state trajectories $[x^t(k)u^t(k)]^t$ generated by the control law (5.58) satisfy $[x^t(k)u^t(k)]^t \in \mathbf{X}, \forall k I_+$. We recall that in [293] the stability of the origin of PWA systems was characterized by using piecewise quadratic (PWQ) Lyapunov functions.

When designing the unknown controller gain K_j appearing in the inequalities (5.48), the set of all possible switches is generally not known in advance, and it could be necessary to consider all the pairs of indices in $\mathsf{S}_{\text{all}} = \mathcal{I} \times \mathcal{I}$ instead of S. Moreover, we note that the design of a controller of type (5.58) could be a very hard task because, at each time instant, the vector $u(k)$ has to be calculated by means of a control gain $K_{\bar{j}}$ whose index $\bar{j}$ is found on the basis of the admissibility condition

$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_{\bar{j}} \tag{5.60}$$

This implies that, in general, it is not possible to calculate $u(k)$ since the index $\bar{j}$ for which the condition (5.60) is satisfied is very hard to be known in advance. Therefore, the problem under consideration is turned into one of designing a controller with the following structure:

$$\begin{aligned} u(k) &= [K_m \ \ 0]\, \tilde{x}(k) \\ &= \tilde{K}_m\, \tilde{x}(k), \quad x(k) \in \bar{\mathsf{X}}_j \end{aligned} \tag{5.61}$$

Thus we consider a different control gain not for all the cells X_j with $j\ \mathcal{I}$ but for all cells $\bar{\mathsf{X}}_j$ with $m\ \mathcal{M}$. Despite this restricted controller structure, in order to design a control law of type (5.61) one must exploit a different Lyapunov matrix $\mathcal{P}_j$ for each X_j with $j\ \mathcal{I}$ to reduce the conservativeness.

In the sequel, we consider the problem of finding a state feedback control law of type (5.61) for the system (5.42). For this purpose, we start from the analysis condition (5.48) rewritten for the closed-loop system

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{\mathcal{A}}_{jm}\tilde{x}(k) \quad \forall \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathsf{X}_j, \quad x(k) \in \bar{\mathsf{X}}_j \\ \tilde{\mathcal{A}}_{jm} &= \tilde{A}_j + \tilde{B}_j\tilde{K}_m \end{aligned} \tag{5.62}$$

More precisely, equation (5.48) rewritten for the closed-loop system (5.62) assumes the form

$$\tilde{\mathcal{A}}^t_{jm}\mathcal{P}_s\tilde{\mathcal{A}}_{jm} - \mathcal{P}_j < 0, \quad \forall m \in \mathcal{M}, \ \forall j \in \mathsf{X}_j, \ \forall\, (s, j) \in \mathsf{S}_{\text{all}} \tag{5.63}$$

$$\mathcal{P}_j = \mathcal{P}^t_j > 0, \quad \forall j \in \mathcal{I} \tag{5.64}$$

Inequalities (5.64) and (5.64) represent a closed-loop stability condition. By defining $\mathcal{W}_j = \mathcal{P}_j^{-1}$, we rewrite (5.63) in the form

$$\begin{bmatrix} -\mathcal{W}_j & \mathcal{W}_j\tilde{\mathcal{A}}^t_{jm} \\ \bullet & -\mathcal{W}_s \end{bmatrix} < 0, \quad \forall m \in \mathcal{M}, \ \forall j \in \mathsf{X}_j, \ \forall\, (s, j) \in \mathsf{S}_{\text{all}} \tag{5.65}$$

Applying a convex analysis procedure, we arrive at the following result:

Theorem 5.12 *Consider the discrete PWA system (5.40). There exists a state-feedback control law of the type (5.58) guaranteeing piecewise quadratic PWQ stability if there exist matrices* $0 < \mathcal{W}^t_j = \mathcal{W}_j, \ j \in \mathcal{I}$, *and matrices* $\mathcal{G}_m, \ \mathcal{Y}_m, \ \forall m \in \mathcal{M}$, *such that* $\forall m \in \mathcal{M}, \ \forall j \in \mathsf{X}_j, \ \forall\, (s, j) \in \mathsf{S}_{\text{all}}$

$$\begin{bmatrix} \mathcal{W}_j - \mathcal{G}_m - \mathcal{G}^t_m & \mathcal{G}^t_m A^t_j + \mathcal{Y}^t_m B^t_j \\ \bullet & -\mathcal{W}_s \end{bmatrix} < 0, \tag{5.66}$$

The feedback gains K_m *are given by* $K_m = \mathcal{Y}_m, \ \mathcal{G}_m^{-1}$

5.3.8 Illustrative Example C

Consider the following system [13]

$$\begin{aligned} x(k+1) &= 0.8\begin{bmatrix} \cos(\sigma(k)) & -\sin(\sigma(k)) \\ \sin(\sigma(k)) & \cos(\sigma(k)) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k), \quad u(k) \in [-1, 1], \\ \sigma(k) &= \begin{cases} \pi/3 & \text{if } [1 \ \ 0]x(k) \geq 0 \\ -\pi/3 & \text{if } [1 \ \ 0]x(k) < 0 \end{cases} \end{aligned}$$

Using four additional auxiliary variables, the foregoing model was converted to PWA system with six cells X_j and two cells $\bar{\mathsf{X}}_j$. Observe that the output $y(k)$ coincides with the first state $x_1(k)$, which in turn represents the variables used to define the switching structure of the system. By applying **Theorem 5.12**, the closed-loop simulation is displayed in Fig. 5.1 (closed-loop state simulation), Fig. 5.2 (control input), and Fig. 5.3 (switching history) (Fig. 5.4).

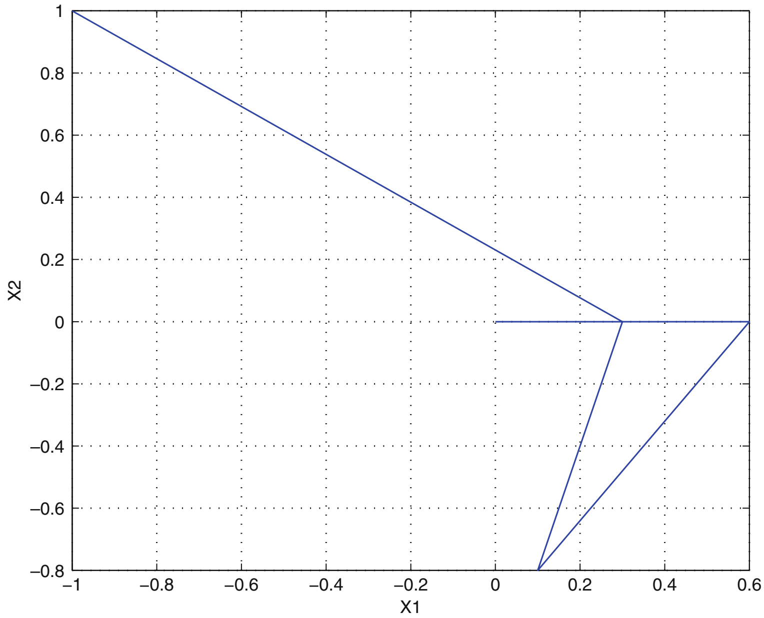

Fig. 5.1 Closed-loop state simulation

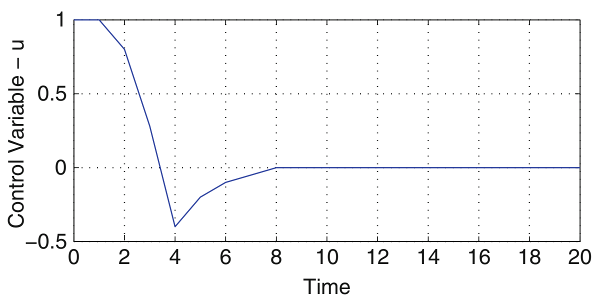

Fig. 5.2 Control input

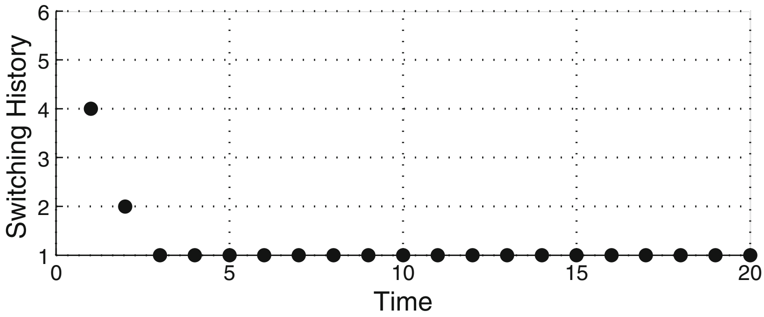

Fig. 5.3 Switching pattern

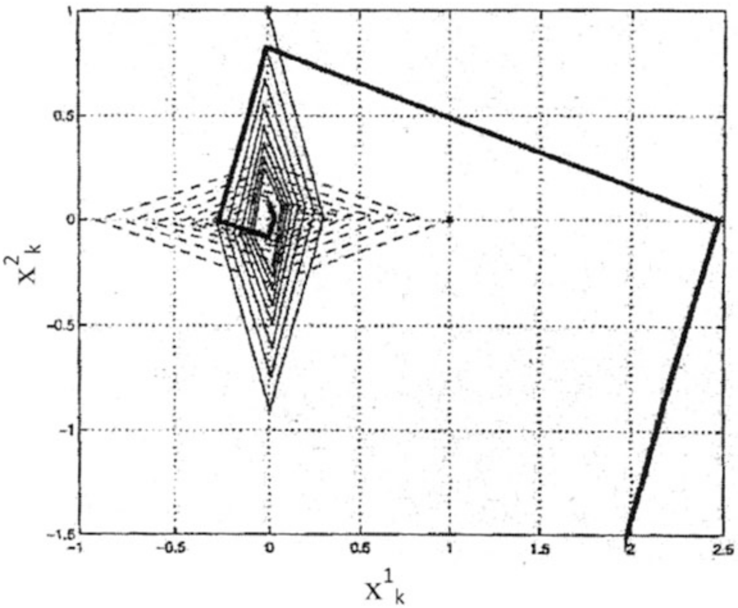

Fig. 5.4 Phase plane of example C

5.4 Notes and References

At the end of our initial tour into the fascinating field of switched time-delay systems, we dwelled on two distinct regions. The first region is concerned with the characterization of two-level models whereby the first level is subsumed of time-delay systems and at the second level there is a coordinator sending our switched signals to harmonize their motion. The second region deals with piecewise-affine continuous-time and discrete-time systems. For further detailed views and modeling directions, the reader is advised to consult [26, 40, 41, 47, 95, 131, 176].

Part III
Switched Stabilization and Control

Chapter 6
Three-Term Stabilization Schemes

It becomes increasingly evident that delays are the main causes of instability and poor performance in dynamical systems and frequently encountered in various engineering and physical systems. Stability analysis and control design of time-delay systems have attracted the attention of numerous investigators, see [24, 221, 237, 338] and their references. Some recent views pertaining to the problems of robust stability analysis and robust stabilization of uncertain time-delay systems have been reported, see [65, 181, 188, 238, 257] and their references.

In Chapter 4, it turns out that the choice of an appropriate Lyapunov–Krasovskii functional (LKF) and the introduction of additional parameters are crucial for developing sufficient stability conditions based on linear matrix inequalities (LMIs). General LKF forms might lead to a complicated system of inequalities [265] and therefore approaches to construct new and effective LKF forms are needed. In this regard, stability criteria for linear state-delay systems can be broadly classified into two categories: delay independent, which are applicable to delays of arbitrary size [214], and delay dependent, which include information on the size of the delay, see [66] and their references. Several model transformation methods and parameterization schemes have been derived in the literature to derive delay-dependent stability conditions, see [22, 65, 66, 181, 188, 198, 218, 238, 257, 301, 373, 392] and their references.

From the previous chapters, we learned that switched systems are a class of dynamical systems formed by several subsystems (continuous or discrete time) and a rule that governs the switching among these subsystems. Recently, the basic problems of stability and control have received increasing interests [28, 41, 42, 47, 174, 193, 292, 424, 427] and the references cited therein. Among the large variety of problems investigated in the literature is the stability analysis and feedback control synthesis of switched systems under arbitrary switching sequences. Recent reported results are found in [56] using multiple Lyapunov functions for nonlinear systems, in [42] employing switched Lyapunov functions, and in [426] utilizing dwell-time properties. Of particular interest in this chapter is the class of STD systems which have widespread engineering applications, including network control systems [170] and power systems [291]. We cover both continuous-time and discrete-time systems with particular emphasis on three-term stabilization schemes. In the continuous-time case, these schemes correspond to proportional-integral-derivative (PID) structure

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whereas in the discrete-time case these schemes are represented by proportional-summation-difference (PSD) structure.

6.1 Continuous-Time Systems

Among the feedback design methods, state-derivative feedback methods have been used to design controllers for several system applications, see [1, 6, 53, 337] and their references. The interest in these methods stems from the fact that it is easier in several practical applications to obtain state-derivative signals than the state signals. These include, but not limited to, mechanical systems [1], car suspension systems [337], and bridge cable vibration [53]. In these applications, the effect of the delay elements was not taken into during the modeling process despite the presence of several physical systems possessing delay phenomena, such as water quality in streams [179], power systems [386], CSTR with recycling [211], combustion in motor chambers [442], to name a few.

In this section, we developed a three-term feedback stabilization of linear STD systems. We focus on the problems of delay-dependent $\mathcal{H}_\infty$ stabilization using proportional-integral-derivative (PID) under arbitrary switching and for different time-delay patterns. Several special cases are derived for nominal and polytopic models. New parametrized LMI characterization for PID feedback stabilization are established.

6.1.1 Problem Statement

We consider the following class of linear switched time-delay (STD) systems:

$$\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-\tau) + B_{\sigma(t)}u(t) + \Gamma_{\sigma(t)}w(t), \quad x(\phi) = \omega(\phi) \\
z(t) &= G_{\sigma(t)}x(t) + \Phi_{\sigma(t)}w(t), \quad \phi \in [-\tau, 0]
\end{aligned} \tag{6.1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ is the control input, $w(t) \in \Re^q$ is the disturbance input, which belongs to $\mathcal{L}_2[0, \infty)$, $z(t) \in \Re^q$ is the observed output, $\sigma(t) : [0, \infty) \longrightarrow \mathbf{S} = \{1, 2, \ldots, s\}$ is the switching signal and $\tau > 0$ is a time-delay factor. The initial condition $\omega(\phi)$ is a differentiable vector-valued function on $[-\tau, 0]$. The matrices $A_\sigma \in \Re^{n\times n}$, $B_\sigma \in \Re^{n\times p}$, $G_\sigma \in \Re^{q\times n}$, $F_\sigma \in \Re^{q\times p}$, $A_{d\sigma} \in \Re^{n\times n}$, and $\Gamma_\sigma \in \Re^{n\times q}$, $\Phi_\sigma \in \Re^{q\times q}$ are real and known constant matrices.

It should be emphasized from the theory of delay differential equations [108, 109] that the existence of the solutions of a nonswitched linear delay system is guaranteed by a continuous and piecewise differentiable initial condition. This is carried over to linear switched-delay systems since the state does not experience any jump at the switching instants.

Define the indication function

$$\xi(t) = \left[\xi_1(t), \; .... \; \xi_s(t)\right]^t, \quad \xi_i(t) = \begin{cases} 1, & \sigma(t) = i \\ 0, & \text{otherwise} \end{cases}$$

Then, the STD system (6.1) can be written as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{s} \xi_i(t) A_i x(t) + \sum_{i=1}^{s} \xi_i(t) A_{di} x(t-\tau) + \sum_{i=1}^{s} \xi_i(t) B_i u(t) \\ &\quad + \sum_{i=1}^{s} \xi_i(t) \Gamma_i w(t) \\ z(t) &= \sum_{i=1}^{s} \xi_i(t) G_i x(t) + \sum_{i=1}^{s} \xi_i(t) \Phi_i w(t) \end{aligned} \tag{6.2}$$

Of prime interest in this section is to find constant matrix gains $K_{oi} \in \Re^{p \times n}$, $K_{si} \in \Re^{p \times n}$, $\forall i \in \mathbf{S}$ such that the following conditions hold:

1. Matrices $\left(I + \sum_{i=1}^{s} \xi_i(t) B_i K_{oi}\right)$, $\forall i \in \mathbf{S}$ have full rank.
2. Using the proportional-integral-derivative (PID) feedback control

$$\begin{aligned} u(t) \; &= \; \sum_{i=1}^{s} \xi_i(t) K_{si} \; x(t) \; - \sum_{i=1}^{s} \xi_i(t) K_{oi} \; \dot{x}(t) \\ &\quad + \sum_{i=1}^{s} \xi_i(t) K_{ai} \int_{t-\varrho}^{t} x(s) \mathrm{d}s \end{aligned} \tag{6.3}$$

the closed-loop system (6.2) under control (6.3) is delay-dependent asymptotically stable for possible patterns of the delay τ. We recall that condition (1) above is meant to ensure the solvability of the problem. By similarity to the conventional control methods, here the role of the proportional gain K_{si}, $i \in \mathbf{N}$ is mainly to ensure that the system is internally stable whereas the role K_{oi}, K_{ai}, $i \in \mathbf{N}$ is to meet the control objectives.

Applying control (6.3) to system (6.2) yields the closed-loop system

$$\begin{aligned} \left(I + \sum_{i=1}^{s} \xi_i(t) B_i K_{oi} \right) \; \dot{x}(t) &= \sum_{i=1}^{s} \xi_i(t) (A_i - B_i K_{si}) x(t) + \sum_{i=1}^{s} \xi_i(t) A_{di} x(t-\tau) \\ &\quad + \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \int_{t-\varrho}^{t} x(s) \mathrm{d}s + \sum_{i=1}^{s} \xi_i(t) \Gamma_i w(t) \\ &= \sum_{i=1}^{s} \xi_i(t) (A_{si} x(t) + \sum_{i=1}^{s} \xi_i(t) A_{di} x(t-\tau) \\ &\quad + \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \int_{t-\varrho}^{t} x(s) \mathrm{d}s \end{aligned}$$

$$+\sum_{i=1}^{s}\xi_i(t)\Gamma_i w(t), \quad A_{si} = A_i + B_i K_{si}$$

$$z(t) = \sum_{i=1}^{s}\xi_i(t)G_i x(t) + \sum_{i=1}^{s}\xi_i(t)\Phi_i w(t) \tag{6.4}$$

When the matrices $\left\{\left(I + \sum_{i=1}^{s}\xi_i(t)B_i K_{oi}\right)\right\}_{i=1}^{s}$ have full rank, then the closed-loop system (6.4) has a well-defined structure in the form

$$\begin{aligned}\dot{x}(t) &= \left(I + \sum_{i=1}^{s}\xi_i(t)B_i K_{oi}\right)^{-1} \left\{\sum_{i=1}^{s}\xi_i(t)(A_{si}x(t) + \sum_{i=1}^{s}\xi_i(t)A_{di}x(t-\tau)\right.\\ &\quad \left. + \sum_{i=1}^{s}\xi_i(t)B_i K_{ai}\int_{t-\varrho}^{t} x(s)\mathrm{d}s + \sum_{i=1}^{s}\xi_i(t)\Gamma_i w(t)\right\},\\ z(t) &= \sum_{i=1}^{s}\xi_i(t)G_i x(t) + \sum_{i=1}^{s}\xi_i(t)\Phi_i w(t)\end{aligned} \tag{6.5}$$

which describe an integro-delay system. In the sequel, we seek to determine the gains
$K_{oi}, \ K_{pi}, \ K_{si}, i \in \mathbf{S}$ for the two cases:

Case 1: τ is a continuous function satisfying for all $t \geq 0$

$$0 \ \leq \ \tau(t) \ \leq \ \varrho$$

Case 2: The time-delay τ is a differentiable time-varying function satisfying

$$0 < \tau(t) \leq \varrho, \quad \dot{\tau}(t) \ \leq \ \mu \tag{6.6}$$

where the bounds ϱ and μ are known.

6.1.2 Model Transformation

To deal with the integro-delay system (6.5), we introduce

$$\theta(t) = \sum_{i=1}^{s}\xi_i(t)\int_{t-\varrho}^{t} x(s)\mathrm{d}s \tag{6.7}$$

such that

$$\dot{\theta}(t) = \sum_{i=1}^{s} \xi_i(t) x(t) - \sum_{i=1}^{s} \xi_i(t) x(t-\varrho) \tag{6.8}$$

Now, append (6.8) to system (6.5) and define

$$\zeta(t) \stackrel{\Delta}{=} \left[x^t(t)\ \theta^t(t) \right]^t, \quad F_i = \left(I + \sum_{i=1}^{s} \xi_i(t) B_i K_{oi} \right)^{-1}$$

we get the augmented system

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{pi}\ \zeta(t) + \sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{ci}\ \zeta(t-\varrho) \\
&\quad + \sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{di}\ \zeta(t-\tau) + \sum_{i=1}^{s} \xi_i(t) \hat{\Gamma}_{pi} w(t) \qquad (6.9) \\
z(t) &= \left[\sum_{i=1}^{s} \xi_i(t) G_{oi} \quad 0 \right] \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \\
&\quad + \left[\sum_{i=1}^{s} \xi_i(t) G_{di} \quad 0 \right] \begin{bmatrix} x(t-\tau) \\ \theta(t-\tau) \end{bmatrix} + \sum_{i=1}^{s} \xi_i(t) \Phi_i w(t) \\
&= \sum_{i=1}^{s} \xi_i(t) \hat{G}_{oi} \zeta(t) + \sum_{i=1}^{s} \xi_i(t) \hat{G}_{di} \zeta(t-\tau) + \sum_{i=1}^{s} \xi_i(t) \Phi_i w(t) \qquad (6.10)
\end{aligned}$$

where

$$\begin{aligned}
\sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{pi} &= \begin{bmatrix} F_i \sum_{i=1}^{s} \xi_i(t) A_{si} & F_i \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \\ \sum_{i=1}^{s} \xi_i(t) I & 0 \end{bmatrix} \\
\sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{ci} &= \begin{bmatrix} 0 & 0 \\ -\sum_{i=1}^{s} \xi_i(t) I & 0 \end{bmatrix}, \quad \sum_{i=1}^{s} \xi_i(t) \mathcal{A}_{di} = \begin{bmatrix} F_i \sum_{i=1}^{s} \xi_i(t) A_{di} & 0 \\ 0 & 0 \end{bmatrix} \\
\sum_{i=1}^{s} \xi_i(t) \hat{\Gamma}_{pi} &= \begin{bmatrix} F_i \sum_{i=1}^{s} \xi_i(t) \Gamma_i \\ 0 \end{bmatrix} w(t) \qquad (6.11)
\end{aligned}$$

which is essentially now a two-time-delay system.

6.1.3 $\mathcal{H}_\infty$ Stabilization: Unknown Continuous Delay

We consider that the time-delay factor τ is an unknown constant, corresponding to *Case 1)*. The following theorem establishes a delay-independent LMI-based condition for proportional-integral-derivative (PID) feedback stabilization with $\mathcal{H}_\infty$ performance bound:

Theorem 6.1 *Consider the time-delay pattern of Case 1. System (6.1) under PID feedback control*

$$u(t) = \sum_{i=1}^{s} \xi_i(t) K_{si} x(t) - \sum_{i=1}^{s} \xi_i(t) K_{oi} \dot{x}(t) - \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \int_{t-\varrho}^{t} x(s) \mathrm{d}s$$

is delay-independent asymptotically stabilizable with $\mathcal{H}_\infty$ performance bound γ if there exist matrices

$$\{\mathcal{X}_{xi}\}_{i=1}^{s},\ \{\mathcal{Y}_i\}_{i=1}^{s},\ \mathcal{F},\ \mathcal{Z}_x,\ \mathcal{Q}_x,\ \{\mathcal{W}_i\}_{i=1}^{s},\ \{\mathcal{R}_i\}_{i=1}^{s},\ \forall (i,s) \in \mathbf{S}$$

such that the following LMI

$$\begin{bmatrix} \Pi_{ai} & \Pi_{cs} & \Pi_{ei} & \bar{\Gamma}_i + (\mathcal{X}_{xi} + B_i \mathcal{Y}_i) \bar{G}_{oi}^t & (\mathcal{X}_{xi} + B_i \mathcal{Y}_i) \hat{G}_{oi}^t & \Pi_{fi} & \Pi_{gi} \\ \bullet & -\Pi_{ds} & 0 & \bar{G}_{di}^t & \hat{G}_{di}^t & 0 & 0 \\ \bullet & \bullet & -\Pi_{ds} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t \Phi_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} \end{bmatrix} < 0 \tag{6.12}$$

has a feasible solution, where

$$\begin{aligned}
\Pi_{ai} &= \begin{bmatrix} A_i \mathcal{X}_{xi} + \mathcal{X}_{xi} A_i^t + A_i \mathcal{Y}_i^t B_i^t + B_i \mathcal{Y}_i A_i^t + B_i \mathcal{W}_i + \mathcal{W}_i^t B_i^t & I + B_i \mathcal{R}_i + \mathcal{R}_i^t B_i^t \\ & -F + 2I \end{bmatrix} \\
\Pi_{cs} &= \left[\begin{bmatrix} A_{di} \mathcal{Z}_{xs} & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{Q}_{xs} \end{bmatrix} \right] \\
\Pi_{ds} &= diag\left[\begin{bmatrix} \mathcal{Z}_{xs} & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} \mathcal{Q}_{xs} & 0 \\ 0 & I \end{bmatrix} \right], \ \bar{\Gamma}_i = \begin{bmatrix} \Gamma_i \\ 0 \end{bmatrix} \\
\Pi_{ei} &= [\mathcal{X}_{xi} + B_i \mathcal{Y}_i \quad \mathcal{X}_{xi} + B_i \mathcal{Y}_i] \\
\bar{G}_{oi}^t &= \begin{bmatrix} G_{oi}^t \Phi_i \\ 0 \end{bmatrix}, \ \bar{G}_{di}^t = \begin{bmatrix} G_{di}^t \Phi_i \\ 0 \end{bmatrix}, \ \hat{G}_{oi}^t = \begin{bmatrix} G_{oi}^t \\ 0 \end{bmatrix}, \ \hat{G}_{di}^t = \begin{bmatrix} G_{di}^t \\ 0 \end{bmatrix} \\
\Pi_{fi} &= \begin{bmatrix} B_i (\mathcal{W}_i + \mathcal{Y}_i) \\ 0 \end{bmatrix}, \quad \Pi_{gi} = \begin{bmatrix} B_i (\mathcal{R}_i + \mathcal{Y}_i) \\ 0 \end{bmatrix}
\end{aligned} \tag{6.13}$$

Moreover, the feedback gains are given by

$$K_{oi} = \mathcal{Y}_i \mathcal{X}_{xi}^{-1}, \ K_{si} = \mathcal{W}_i \mathcal{X}_{xi}^{-1}, \ K_{ai} = \mathcal{R}_i \mathcal{X}_{xi}^{-1}$$

Proof First, we establish the asymptotic stability of the closed-loop system (6.9) with $w(.) \equiv 0$

$$\begin{aligned}\dot{\zeta}(t) &= \sum_{i=1}^{s} \xi_i(t)\mathcal{A}_{pi}\ \zeta(t) \\ &\quad + \sum_{i=1}^{s} \xi_i(t)\mathcal{A}_{ci}\ \zeta(t-\varrho) + \sum_{i=1}^{s} \xi_i(t)\mathcal{A}_{di}\ \zeta(t-\tau) \end{aligned} \tag{6.14}$$

Define the selective Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned} V_1(t) &= \zeta^t(t)\left(\sum_{i=1}^{s} \xi_i(t)\mathcal{P}_i\right)\zeta(t) + \int_{t-\tau}^{t} \zeta^t(s)\mathcal{Z}^{-1}\zeta(s)\mathrm{d}s \\ &\quad + \int_{t-\varrho}^{t} \zeta^t(s)\mathcal{Q}^{-1}\zeta(s)\mathrm{d}s \\ \mathcal{P}_i &= \begin{bmatrix} \mathcal{P}_{xi} & 0 \\ 0 & I \end{bmatrix}, \ \mathcal{Z} = \begin{bmatrix} \mathcal{Z}_x & 0 \\ 0 & I \end{bmatrix}, \ \ \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_x & 0 \\ 0 & I \end{bmatrix}, \quad i \in \mathbf{S} \end{aligned} \tag{6.15}$$

We note that the form of matrices $\mathcal{P}_i, \ \mathcal{Q}, \ \mathcal{Z}$ is not restrictive since any non-unity value in the lower rows would not affect the subsequent analysis. Differentiating $V_1(t)$ along the solutions of (6.14), we get

$$\begin{aligned} \dot{V}_1(t)|_{(6.9)} &= \zeta^t(t)\left(\sum_{i=1}^{s} \xi_i(t)\mathcal{P}_i\mathcal{A}_{si} + \mathcal{A}_{si}^t\mathcal{P}_i\right)\zeta(t) \\ &\quad + \zeta^t(t)\xi_i(t)\mathcal{Z}^{-1}\zeta(t) + \zeta^t(t)\xi_i(t)\mathcal{Q}^{-1}\zeta(t) \\ &\quad + 2\zeta^t(t)\left(\sum_{i=1}^{s} \xi_i(t)\mathcal{P}_i\mathcal{A}_{di}\right)\zeta(t-\tau) + 2\zeta^t(t)\left(\sum_{i=1}^{s} \xi_i(t)\mathcal{P}_i\mathcal{A}_{ci}\right)\zeta(t-\varrho) \\ &\quad - \zeta^t(t-\tau)\xi_i(t)\mathcal{Z}^{-1}\zeta(t-\tau) - \zeta^t(t-\varrho)\xi_i(t)\mathcal{Q}^{-1}\zeta(t-\varrho) \end{aligned} \tag{6.16}$$

It follows that for any nonzero vector $x(t)$ and the particular case $\xi_i(t) = 1$ and $\xi_{m\neq i}(t) = 0$. Therefore, with some algebraic manipulations, we have from (6.16):

$$\begin{aligned} \dot{V}_1(t)|_{(6.9)} &= \eta^t(t)\ \Omega_{is}\ \eta(t), \\ \eta(t) &= \begin{bmatrix} \zeta^t(t) & \zeta^t(t-\tau) & \zeta^t(t-\varrho) \end{bmatrix}^t \\ \Omega_{is} &= \begin{bmatrix} \mathcal{P}_i\mathcal{A}_{si} + \mathcal{A}_{si}^t\mathcal{P}_i + \mathcal{Z}^{-1} + \mathcal{Q}^{-1} & \mathcal{P}_s\mathcal{A}_{di} & \mathcal{P}_s\mathcal{A}_{ci} \\ \bullet & -\mathcal{Z}^{-1} & 0 \\ \bullet & \bullet & -\mathcal{Q}^{-1} \end{bmatrix} \end{aligned} \tag{6.17}$$

That $\dot{V}_1(t)|_{(6.9)} < 0$ it implies that $\Omega_{is} < 0, \ \forall \ (i, s) \ \in \ \mathbf{S}$. Now, let

$$T_1 = \begin{bmatrix} (I + B_i K_{oi})\mathcal{P}_{xi}^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$\mathcal{P}_i^{-1} = \mathcal{X}_i = \begin{bmatrix} \mathcal{X}_{xi} & 0 \\ 0 & I \end{bmatrix} \tag{6.18}$$

Applying the congruent transformation

$$\text{diag}\left[\, T_1 \ T_2 \ T_2 \,\right]$$

to Ω_{is} with $\mathcal{Y}_i = K_{oi}\mathcal{X}_{xi}, \ \mathcal{W}_i = K_{si}\mathcal{X}_{xi}, \ \mathcal{R}_i = K_{ai}\mathcal{X}_{xi}$ and making use of the algebraic inequalities $\forall \ i \ \in \ \mathbf{S}$

$$\begin{aligned} \left(\mathcal{W}_i\mathcal{X}_{xi}^{-1}\mathcal{Y}_i^t + \mathcal{Y}_i\mathcal{X}_{xi}^{-1}\mathcal{W}_i^t\right) &\le \left(\mathcal{W}_i + \mathcal{Y}_i\right)\mathcal{X}_{xi}^{-1}\left(\mathcal{W}_i^t + \mathcal{Y}_i^t\right), \\ \left(\mathcal{R}_i\mathcal{X}_{xi}^{-1}\mathcal{Y}_i^t + \mathcal{Y}_i\mathcal{X}_{xi}^{-1}\mathcal{R}_i^t\right) &\le \left(\mathcal{R}_i + \mathcal{Y}_i\right)\mathcal{X}_{xi}^{-1}\left(\mathcal{R}_i^t + \mathcal{Y}_i^t\right) \end{aligned} \tag{6.19}$$

it follows from (6.12) by the Schur complements formula that the asymptotic stability of the closed-loop system (6.5) is established.

Consider the performance measure

$$J = \int_0^\infty \left(\sum_{i=1}^N \xi_i(t) \left[z^t(s)z(s) - \gamma^2 w^t(s)w(s) \right] \right) \mathrm{d}s$$

For any $w(t) \in \mathcal{L}_2(0, \infty) \neq 0$ and zero initial condition $x(0) = 0$, hence $V(0) = 0$, we have

$$\begin{aligned} J &= \int_0^\infty \left(\sum_{i=1}^N \xi_i(t) \left[z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_1(t)|_{(6.9)} \right] \right) \mathrm{d}s \\ &\quad - \dot{V}_1(t)|_{(6.9)} \\ &\le \int_0^\infty \left(\sum_{i=1}^N \xi_i(t) \left[z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_1(t)|_{(6.9)} \right] \right) \mathrm{d}s \end{aligned}$$

where $\dot{V}_1(t)|_{(6.9)}$ defines the Lyapunov derivative along the solutions of system (6.9). Under arbitrary switching, we get

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \; + \dot{V}_1(s)|_{(6.9)} = \bar{\eta}^t(s)\Xi_{is}\bar{\eta}(s) =$$

$$\Xi_{is} = \begin{bmatrix} \begin{matrix} \mathcal{P}_i\mathcal{A}_{si} + \mathcal{A}_{si}^t\mathcal{P}_i + \\ \mathcal{Z}^{-1} + \mathcal{Q}^{-1} + \hat{G}_{oi}^t\hat{G}_{oi} \end{matrix} & \mathcal{P}_s\mathcal{A}_{di} + \hat{G}_{oi}^t\hat{G}_{di} & \mathcal{P}_s\mathcal{A}_{ci} & \mathcal{P}_i\bar{\Gamma}_i + \bar{G}_{oi}^t \\ \bullet & -\mathcal{Z}^{-1} + \hat{G}_{di}^t\hat{G}_{di} & 0 & \bar{G}_{di}^t \\ \bullet & \bullet & -\mathcal{Q}^{-1} & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t\Phi_i \end{bmatrix}$$

$$\bar{\eta}(s) = \begin{bmatrix} x^t(s) & x^t(s-\tau) & x^t(s-\varrho) & w^t(s) \end{bmatrix}^t \tag{6.20}$$

Using (6.18), we apply the congruent transformation

$$\mathrm{diag}\begin{bmatrix} T_1 & T_2 & T_2 & T_2 \end{bmatrix}, \;\; \mathcal{P}_i^{-1} := \mathcal{X}_i = \begin{bmatrix} \mathcal{X}_{xi} & 0 \\ 0 & I \end{bmatrix}$$

to Ξ_{is} with

$$\mathcal{Y}_i = K_{oi}\mathcal{X}_{xi}, \;\; \mathcal{W}_i = K_{si}\mathcal{X}_{xi}, \;\; \mathcal{R}_i = K_{ai}\mathcal{X}_{xi}$$

and making use of the algebraic inequalities (6.19), it readily follows from LMI (6.12) and Schur complement operations that

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \; + \dot{V}_1(s)|_{(6.9)} < 0$$

for arbitrary $s \in [t, \infty)$, which implies for any $w(t) \in \mathcal{L}_2(0, \infty) \neq 0$ that $J < 0$ leading to $||z(t)||_2 < \gamma \; ||w(t)||_2$ and the proof is completed. ■

Remark 6.2 The optimal delay-independent asymptotically stabilizable controller can be determined by solving the following convex minimization problem over LMIs:

$$\begin{array}{c} Minimize \;\; \gamma \\ wrt \;\; \mathcal{X}_{xi} > 0, \; \mathcal{Y}_i, \; \mathcal{W}_i, \; \mathcal{Z}_x > 0, \; \mathcal{Q}_x > 0, \; \gamma > 0, \; \forall (i, s) \in \mathbf{S} \\ subject \; to \quad LMI \; (6.12) \end{array}$$

Remark 6.3 A connection to **Theorem 6.1** for the no-switching case $i = 1$ can be found in [216] (pp. 88–95) when examining linear uncertain systems with input delays. However, the analytical treatment and the final results here are basically different.

Remark 6.4 Had we used only a state-feedback stabilization

$$u(t) = \sum_{i=1}^{s} \xi_i(t) K_{si} \; x(t)$$

we would have obtained the following result:

Theorem 6.5 *Consider the time-delay pattern of Case 1. System (6.1) under state feedback* $u(t) = \sum_{i=1}^{s} \xi_i(t) K_{si}\ x(t)$ *is delay-independent asymptotically stabilizable with* $\mathcal{H}_\infty$ *performance bound* γ *if there exist matrices*

$$\{\mathcal{X}_i\}_{i=1}^{s},\ \{\mathcal{Y}_i\}_{i=1}^{s},\ \mathcal{Z}_x,\ \{\mathcal{W}_i\}_{i=1}^{s},\ \forall(i,s)\in\mathbf{S}$$

such that the following LMI

$$\begin{bmatrix} \bar{\Pi}_{ai} & \Pi_{cs} & \bar{\Pi}_{ei} & \bar{\Gamma}_i + (\mathcal{X}_{xi} + B_i\mathcal{Y}_i)\bar{G}_{oi}^t & (\mathcal{X}_{xi} + B_i\mathcal{Y}_i)\hat{G}_{oi}^t \\ \bullet & -\Pi_{ds} & 0 & \bar{G}_{di}^t & \hat{G}_{di}^t \\ \bullet & \bullet & -\Pi_{ds} & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t\Phi_i & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{6.21}$$

has a feasible solution, where

$$\begin{aligned} \bar{\Pi}_{ai} &= \begin{bmatrix} A_i\mathcal{X}_{xi} + \mathcal{X}_{xi}A_i^t + B_i\mathcal{W}_i + \mathcal{W}_i^t B_i^t & 0 \\ I & 0 \end{bmatrix},\quad \bar{\Pi}_{ei} = [\mathcal{X}_{xi} \quad \mathcal{X}_{xi}] \\ \Pi_{cs} &= \left[\begin{bmatrix} A_{di}\mathcal{Z}_x & 0 \\ 0 & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{Q}_x \end{bmatrix} \right], \\ \Pi_{ds} &= \operatorname{diag}\left[\begin{bmatrix} \mathcal{Z}_x & 0 \\ 0 & I \end{bmatrix},\quad \begin{bmatrix} \mathcal{Q}_x & 0 \\ 0 & I \end{bmatrix} \right] \end{aligned} \tag{6.22}$$

Moreover, the feedback gain is given by

$$K_{si} = \mathcal{W}_i\mathcal{X}_i^{-1}$$

Proof The proof of this theorem can be readily obtained by parallel development to **Theorem 6.1** with $\mathcal{Y}_i \equiv 0,\ \mathcal{R}_i \equiv 0$. ■

Remark 6.6 By setting $A_{di} \equiv 0,\ \forall i \in \mathbf{S}$, we obtain the linear controlled switched delayless system

$$\begin{aligned} \dot{x}(t) &= \left(I + \sum_{i=1}^{s} \xi_i(t) B_i K_{oi} \right)^{-1} \left\{ \sum_{i=1}^{s} \xi_i(t)(A_{si}x(t) + \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \int_{t-\varrho}^{t} x(s)\mathrm{d}s \right. \\ &\quad \left. + \sum_{i=1}^{s} \xi_i(t)\Gamma_i w(t) \right\} \\ z(t) &= \sum_{i=1}^{s} \xi_i(t) G_i x(t) + \sum_{i=1}^{s} \xi_i(t)\Phi_i w(t) \end{aligned} \tag{6.23}$$

for which **Theorem 6.1** specializes to the following important result

Corollary 6.7 *Consider the time-delay pattern of Case 1. System (6.23) under PID feedback control*

$$u(t) = \sum_{i=1}^{s} \xi_i(t) K_{si} x(t) - \sum_{i=1}^{s} \xi_i(t) K_{oi} \dot{x}(t) - \sum_{i=1}^{s} \xi_i(t) B_i K_{ai} \int_{t-\varrho}^{t} x(s) \mathrm{d}s$$

is delay-independent asymptotically stabilizable with $\mathcal{H}_\infty$ performance bound γ if there exist matrices

$$\{\mathcal{X}_{xi}\}_{i=1}^{s}, \ \{\mathcal{Y}_i\}_{i=1}^{s}, \ \mathcal{Z}_x, \ \mathcal{Q}_x, \ \mathcal{W}_i\}_{i=1}^{s}, \ \forall (i, s) \in \mathbf{S}$$

such that the following LMI

$$\begin{bmatrix} \Pi_{ai} & \widehat{\Pi}_{cs} & \Pi_{ei} & \Pi_{vi} & \Pi_{wi} & \Pi_{fi} & \Pi_{gi} \\ \bullet & -\widehat{\Pi}_{ds} & 0 & \bar{G}_{di}^t & \hat{G}_{di}^t & 0 & 0 \\ \bullet & \bullet & -\Pi_{di} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t \Phi_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} \end{bmatrix} < 0 \qquad (6.24)$$

has a feasible solution, where

$$\begin{aligned} \Pi_{ai} &= \begin{bmatrix} A_i \mathcal{X}_{xi} + \mathcal{X}_{xi} A_i^t + A_i \mathcal{Y}_i^t B_i^t + B_i \mathcal{Y}_i A_i^t & \\ +B_i \mathcal{W}_i + \mathcal{W}_i^t B_i^t & B_i \mathcal{R}_i + \mathcal{R}_i^t B_i^t \\ I & 0 \end{bmatrix} \\ \widehat{\Pi}_{cs} &= \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{Q}_x \end{bmatrix}, \quad \Pi_{ds} = \begin{bmatrix} \mathcal{Q}_x & 0 \\ 0 & I \end{bmatrix} \\ \Pi_{vi} &= \bar{\Gamma}_i + (\mathcal{X}_{xi} + B_i \mathcal{Y}_i) \bar{G}_{oi}^t \\ \Pi_{wi} &= (\mathcal{X}_{xi} + B_i \mathcal{Y}_i) \hat{G}_{oi}^t \end{aligned} \qquad (6.25)$$

and $\bar{\Gamma}_i$, $\bar{G}_{oi}^t$, $\bar{G}_{di}^t$, and $\hat{G}_{oi}^t$ are given in (6.11). Moreover, the feedback gains are given by

$$K_{oi} = \mathcal{Y}_i \mathcal{X}_{xi}^{-1}, \ K_{si} = \mathcal{W}_i \mathcal{X}_{xi}^{-1}, \ K_{ai} = \mathcal{R}_i \mathcal{X}_{xi}^{-1}$$

Remark 6.8 When the model matrices of system (6.1) are partially known, we assume that

$$\Xi_i := \{A_i, A_{di}, B_i, \Gamma_i, G_i, \Phi_i\} \ \in \ \Lambda_i \qquad (6.26)$$

where Λ_i is a given convex-bounded polyhedral domain described by vertices as follows:

$$\Lambda_i := \left\{ \Xi_i(\lambda_i) | \Xi_i(\lambda_i) = \sum_{j=1}^{M} \lambda_{ij} \Omega_{ij},\ \sum_{j=1}^{s} \lambda_{ij} = 1\ \lambda_{ij} \geq 0 \right\} \tag{6.27}$$

$$\Xi_{ij} \overset{\Delta}{=} \{A_{ij}, A_{dij}, B_{ij}, \Gamma_{ij}, G_{ij}, G_{dij}, \Phi_{ij}\} \tag{6.28}$$

In this regard, we are in a position to establish the following corollary:

Corollary 6.9 *Consider the time-delay of Case 1. System (6.1) with the polytopic representation (6.26), (6.27), and (6.28) under PID feedback control (6.3) is robustly delay-independent asymptotically stabilizable with $\mathcal{H}_\infty$ performance bound γ if there exist matrices*

$$\{\mathcal{X}_{xi}\}_{i=1}^{s},\ \{\mathcal{Y}_i\}_{i=1}^{s},\ \mathcal{Z}_x,\ \mathcal{Q}_x,\ \{\mathcal{W}_i\}_{i=1}^{s},\ \forall (i,s) \in \mathbf{S},\ \ j = 1\ldots, M$$

such that the following LMIs

$$\begin{bmatrix} \Pi_{aij} & \Pi_{csj} & \Pi_{eij} & \Pi_{vij} & \Pi_{wij} & \Pi_{gij} & \\ \bullet & -\Pi_{ds} & 0 & \bar{G}^t_{dij} & \hat{G}^t_{dij} & 0 & 0 \\ \bullet & \bullet & -\Pi_{di} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi^t_{ij}\Phi_{ij} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} \end{bmatrix} < 0 \tag{6.29}$$

have a feasible solution, where

$$\Pi_{vij} = \bar{\Gamma}_{ij} + (\mathcal{X}_{xi} + B_{ij}\mathcal{Y}_i)\bar{G}^t_{oij},$$

$$\Pi_{wij} = (\mathcal{X}_{xi} + B_{ij}\mathcal{Y}_i)\hat{G}^t_{oij},$$

$$\Pi_{aij} = \begin{bmatrix} \begin{matrix} A_{ij}\mathcal{X}_{xi} + \mathcal{X}_{xi}A^t_{ij} + A_{ij}\mathcal{Y}^t_i B^t_{ij} \\ +B_{ij}\mathcal{Y}_i A^t_{ij} + B_{ij}\mathcal{W}_i + \mathcal{W}^t_i B^t_{ij} \end{matrix} & B_{ij}\mathcal{R}_i + \mathcal{R}^t_i B^t_{ij} \\ I & 0 \end{bmatrix}$$

$$\Pi_{csj} = \left[\begin{bmatrix} A_{dij}\mathcal{Z}_x & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{Q}_x \end{bmatrix} \right]$$

$$\Pi_{ds} = \mathrm{diag}\left[\begin{bmatrix} \mathcal{Z}_x & 0 \\ 0 & I \end{bmatrix},\ \begin{bmatrix} \mathcal{Q}_{xs} & 0 \\ 0 & I \end{bmatrix} \right],\ \bar{\Gamma}_{ij} = \begin{bmatrix} \Gamma_{ij} \\ 0 \end{bmatrix}$$

$$\Pi_{eij} = [\mathcal{X}_{xi} + B_{ij}\mathcal{Y}_i \quad \mathcal{X}_{xi} + B_{ij}\mathcal{Y}_i],$$

$$\bar{G}^t_{oij} = \begin{bmatrix} G_{oij}\Phi_{ij} \\ 0 \end{bmatrix},\ \bar{G}^t_{dij} = \begin{bmatrix} G_{dij}\Phi_{ij} \\ 0 \end{bmatrix},\ \hat{G}^t_{oij} = \begin{bmatrix} G_{oij} \\ 0 \end{bmatrix},\ \hat{G}^t_{dij} = \begin{bmatrix} G_{dij} \\ 0 \end{bmatrix}$$

$$\Pi_{fij} = \begin{bmatrix} B_{ij}(\mathcal{W}_i + \mathcal{Y}_i) \\ 0 \end{bmatrix}, \quad \Pi_{gij} = \begin{bmatrix} B_{ij}(\mathcal{R}_i + \mathcal{Y}_i) \\ 0 \end{bmatrix} \tag{6.30}$$

Moreover, the feedback gains are given by

$$K_{oi} = \mathcal{Y}_i \mathcal{X}_i^{-1}, \; K_{si} = \mathcal{W}_i \mathcal{X}_i^{-1}, \; K_{ai} = \mathcal{R}_i \mathcal{X}_i^{-1}$$

Grouping **Remarks 6.2–6.8** together evidently illuminates the generality and flexibility of the foregoing stabilization approach to linear switched time-delay systems.

6.1.4 $\mathcal{H}_\infty$ Stabilization: Time-Varying Delays

In this section, we address *Case 2* where the time delay is a continuous time-varying function and proceed to establish new LMI characterization for delay-dependent stabilization by proportional-integral-derivative (PID) feedback. Initially, recall the standard Leibniz–Newton formula

$$\zeta(t - \tau(t)) = \sum_{i=1}^{s} \xi_i(t)\zeta(t) - \sum_{i=1}^{s} \xi_i(t) \int_{t-\tau(t)}^{t} \dot{\zeta}(s)\mathrm{d}s \tag{6.31}$$

We consider the transformed closed-loop system (6.9) and establish the following theorem:

Theorem 6.10 *Consider the time-delay pattern of Case 2. System (6.9), (6.10), and (6.11) is delay-dependent asymptotically stabilizable with $\mathcal{H}_\infty$ performance bound γ if there exist matrices*

$$\{\mathcal{X}_{xi}\}_{i=1}^{s}, \; \{\mathcal{Y}_i\}_{i=1}^{s}, \; \{\mathcal{W}_i\}_{i=1}^{s}, \; \mathcal{Q}_x, \; \mathcal{Z}_x, \; \{\mathcal{S}_{ji}\}_{i=1}^{s}, j = 1, \ldots, 4$$
$$\{\mathcal{N}_{ki}\}_{i=1}^{s}, k = 1, \ldots, 8, \; \forall (i, s) \in \mathbf{S}$$

such that the following LMI

$$\begin{bmatrix} \widetilde{\Pi}_{ai} & \widetilde{\Pi}_{ci} & \widetilde{\Pi}_{ei} & \widetilde{\Pi}_{vi} & \widetilde{\Pi}_{wi} & \widetilde{\Pi}_{fi} & \widetilde{\Pi}_{gi} & \widetilde{\Pi}_{hi} \\ \bullet & -\widetilde{\Pi}_{di} & 0 & \bar{G}_{di}^t & \hat{G}_{di}^t & 0 & 0 & 0 \\ \bullet & \bullet & -\widetilde{\Pi}_{di} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t \Phi_i & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{xi} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\widetilde{\Pi}_{mi} \end{bmatrix} < 0 \tag{6.32}$$

has a feasible solution for all $\tau(t) \le \varrho, \; \dot{\tau}(t) \le \mu$, where

$$
\begin{aligned}
\widetilde{\Pi}_{vi} &= \bar{\Gamma}_i + (\mathcal{X}_{xi} + B_i\mathcal{Y}_i)\bar{G}^t_{oi} \\
\widetilde{\Pi}_{wi} &= (\mathcal{X}_{xi} + B_i\mathcal{Y}_i)\hat{G}^t_{oi} \\
\widetilde{\Pi}_{ai} &= \begin{bmatrix} \widetilde{\Pi}_{aai}A_i & B_i\mathcal{R}_i + \mathcal{R}^t_i B^t_i \\ I & 0 \end{bmatrix} \\
\widetilde{\Pi}_{aai} &= A_i\mathcal{X}_{xi} + \mathcal{X}_{xi}A^t_i + A_i\mathcal{Y}^t_i B^t_i + B_i\mathcal{Y}_i A^t_i + B_i\mathcal{W}_i + \mathcal{W}^t_i B^t_i \\
&\quad + \mathcal{X}_{xi}\mathcal{S}_{1i} + B_i\mathcal{S}_{2i} + \mathcal{S}^t_{1i}\mathcal{X}_{xi} + \mathcal{S}_{2i}B^t_i \\
\widetilde{\Pi}_{cai} &= \begin{bmatrix} A_{di}\mathcal{Z}_x - \mathcal{X}_{xi}\mathcal{S}_{3i} - B_i\mathcal{S}_{3i} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{\Pi}_{cci} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{Q}_{xs} \end{bmatrix} \\
\widetilde{\Pi}_{ci} &= \begin{bmatrix} \widetilde{\Pi}_{cai} & \widetilde{\Pi}_{cci} \end{bmatrix} \\
\widetilde{\Pi}_{dai} &= \begin{bmatrix} \mathcal{S}_{4i} + \mathcal{S}^t_{4i} + (1-\mu)\mathcal{Z}_x & 0 \\ 0 & I \end{bmatrix}, \quad \widetilde{\Pi}_{dci} = \begin{bmatrix} \mathcal{Q}_{xs} & 0 \\ 0 & I \end{bmatrix} \\
\widetilde{\Pi}_{di} &= \mathrm{diag}\begin{bmatrix} \widetilde{\Pi}_{dai}, & \widetilde{\Pi}_{dci} \end{bmatrix} \\
\widetilde{\Pi}_{ei} &= [\mathcal{X}_{xi} + B_i\mathcal{Y}_i \quad \mathcal{X}_{xi} + B_i\mathcal{Y}_i], \\
\bar{\Gamma}_i &= \begin{bmatrix} \Gamma_i \\ \\ 0 \end{bmatrix}, \quad \bar{G}^t_{oi} = \begin{bmatrix} G^t_{oi}\Phi_i \\ \\ 0 \end{bmatrix}, \quad \bar{G}^t_{di} = \begin{bmatrix} G^t_{di}\Phi_i \\ \\ 0 \end{bmatrix} \\
\hat{G}^t_{oi} &= \begin{bmatrix} G^t_{oi} \\ \\ 0 \end{bmatrix}, \quad \hat{G}^t_{di} = \begin{bmatrix} G^t_{di} \\ 0 \end{bmatrix} \\
\widetilde{\Pi}_{fi} &= \begin{bmatrix} B_i(\mathcal{W}_i + \mathcal{Y}_i) \\ \\ 0 \end{bmatrix}, \quad \Pi_{gi} = \begin{bmatrix} B_i(\mathcal{R}_i + \mathcal{Y}_i) \\ \\ 0 \end{bmatrix} \\
\widetilde{\Pi}_{hi} &= \begin{bmatrix} \widetilde{\Pi}_{hi1} & \widetilde{\Pi}_{hi2} \end{bmatrix} \\
\widetilde{\Pi}_{hi1} &= \begin{bmatrix} \mathcal{X}_{xi}\mathcal{N}_{1i} + B_i\mathcal{N}_{2i} & \mathcal{X}_{xi}\mathcal{N}_{3i} + B_i\mathcal{N}_{4i} \\ I & 0 \end{bmatrix} \\
\widetilde{\Pi}_{hi2} &= \begin{bmatrix} \mathcal{X}_{xi}\mathcal{N}_{5i} + B_i\mathcal{N}_{6i} & \mathcal{X}_{xi}\mathcal{N}_{7i} + B_i\mathcal{N}_{8i} \\ I & 0 \end{bmatrix} \\
\widetilde{\Pi}_{fi} &= \mathrm{diag}\begin{bmatrix} \widetilde{\Pi}_{fai}, & \widetilde{\Pi}_{fai} \end{bmatrix}, \quad \widetilde{\Pi}_{fai} = \begin{bmatrix} \mathcal{M}_x & 0 \\ 0 & I \end{bmatrix}
\end{aligned} \tag{6.33}
$$

Moreover, the PID feedback gains are given by

$$K_{oi} = \mathcal{Y}_i\mathcal{X}^{-1}_{xi}, \;\; K_{si} = \mathcal{W}_i\mathcal{X}^{-1}_{xi}$$

Proof Consider the selective Lyapunov–Krasovskii functional (LKF):

$$V_2(t) = V_{a2}(t) + V_{b2}(t) + V_{c2}(t) + V_{d2}(t)$$

$$
\begin{aligned}
V_{a2}(t) &= \zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\right)\zeta(t), \quad V_{b2}(t) = \int_{t-\tau}^{t} \zeta^t(s)\mathcal{Z}^{-1}\zeta(s)\mathrm{d}s \\
V_{c2}(t) &= \int_{t-\varrho}^{t} \zeta^t(s)\mathcal{Q}^{-1}\zeta(s)\mathrm{d}s \\
V_{d2}(t) &= \int_{-\tau}^{0}\int_{t+\phi}^{t} \dot{\zeta}^t(s)\mathcal{M}^{-1}\dot{\zeta}(s)\mathrm{d}s\mathrm{d}\phi \\
\mathcal{P}_i &= \begin{bmatrix} \mathcal{P}_{xi} & 0 \\ 0 & I \end{bmatrix} \mathcal{Z} = \begin{bmatrix} \mathcal{Z}_x & 0 \\ 0 & I \end{bmatrix}, \ \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_x & 0 \\ 0 & I \end{bmatrix} \\
\mathcal{M} &= \begin{bmatrix} \mathcal{M}_x & 0 \\ 0 & I \end{bmatrix}, \quad i \in \mathbf{S}
\end{aligned} \tag{6.34}
$$

With (6.31) in mind and setting $w(.) \equiv 0$, evaluation of the derivative $\dot{V}_2(t)$ along the solutions of (6.9) yields

$$
\begin{aligned}
\dot{V}_{a2}(t)|_{(6.9)} &= 2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\left\{\mathcal{A}_{pi}\zeta(t) + \mathcal{A}_{ci}\ \zeta(t-\varrho)\right\}\right) \\
&\quad +2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\mathcal{A}_{di}\ \zeta(t-\tau)\right) \\
&= 2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\left[\mathcal{A}_{pi} + \mathcal{A}_{di}\right]\right)\zeta(t) \\
&\quad -2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\mathcal{A}_{di}\right)\int_{t-\tau(t)}^{t} \dot{\zeta}(\phi)\mathrm{d}\phi \\
&\quad +2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\mathcal{A}_{ci}\right)\ \zeta(t-\varrho) \\
&= 2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\left[\mathcal{A}_{pi} + \mathcal{A}_{di}\right]\right)\zeta(t) \\
&\quad +2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\left[\Theta_i - \mathcal{P}_i\mathcal{A}_{di}\right]\right)\int_{t-\tau(t)}^{t} \dot{\zeta}(s)\mathrm{d}s \\
&\quad +2\,\zeta^t(t-\tau)\left(\sum_{i=1}^{s}\xi_i(t)\Psi_i\right)\int_{t-\tau(t)}^{t} \dot{\zeta}(s)\mathrm{d}s \\
&\quad +2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\mathcal{A}_{ci}\right)\ \zeta(t-\varrho)
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg[2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\Theta_i\right)\int_{t-\tau(t)}^{t}\dot{\zeta}(\phi)\mathrm{d}\phi \\
&+2\,\zeta^t(t-\tau)\left(\sum_{i=1}^{s}\xi_i(t)\Psi_i\right)\int_{t-\tau(t)}^{t}\dot{\zeta}(\phi)\mathrm{d}\phi\Bigg] \\
=\;&\frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\Bigg[2\,\zeta^t(t)\left(\sum_{i=1}^{N}\xi_i(t)\Big[\mathcal{P}_i\mathcal{A}_{pi}+\Theta_i\Big]\right)\zeta(t) \\
&+2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\Big[\mathcal{P}_i\mathcal{A}_{di}-\Theta_i+\Psi_i^t\Big]\right)\zeta(t-\tau) \\
&-2\,\zeta^t(t-\tau)\left(\sum_{i=1}^{s}\xi_i(t)\Psi_i\right)\zeta(t-\tau) \\
&-2\,\tau(t)\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\Theta_i\right)\dot{\zeta}(\phi) \\
&-2\,\tau(t)\,\zeta^t(t-\tau)\left(\sum_{i=1}^{s}\xi_i(t)\Psi_i\right)\dot{\zeta}(s) \\
&+2\,\zeta^t(t)\left(\sum_{i=1}^{s}\xi_i(t)\mathcal{P}_i\mathcal{A}_{ci}\right)\zeta(t-\varrho)\Bigg]\mathrm{d}\phi
\end{aligned}
\tag{6.35}
$$

where Θ_i, Ψ_i are relaxation matrices injected to facilitate the delay-dependent analysis. Moreover,

$$
\begin{aligned}
\dot{V}_{b2}(t)|_{(6.9)} &= \zeta^t(t)\mathcal{Z}^{-1}\zeta(t)-(1-\dot{\tau})\zeta^t(t-\tau)\xi_i(t)\mathcal{Z}^{-1}\zeta(t-\tau) \\
&\le \zeta^t(t)\xi_i(t)\mathcal{Z}^{-1}\zeta(t)-(1-\mu)\zeta^t(t-\tau)\xi_i(t)\mathcal{Z}^{-1}\zeta(t-\tau) \\
&= \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\Bigg[\zeta^t(t)\left(\sum_{i=1}^{N}\xi_i(t)\mathcal{Z}_i^{-1}\right)\zeta(t) \\
&\quad -(1-\mu)\zeta^t(t-\tau)\mathcal{Z}^{-1}\zeta(t-\tau)\Bigg]
\end{aligned}
\tag{6.36}
$$

$$
\begin{aligned}
\dot{V}_{c2}(t)|_{(6.9)} &= \zeta^t(t)\xi_i(t)\mathcal{Q}^{-1}\zeta(t)-\zeta^t(t-\rho)\mathcal{Q}^{-1}\zeta(t-\rho) \\
&= \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\Bigg[\zeta^t(t)\mathcal{Q}^{-1}\zeta(t) \\
&\quad -\zeta^t(t-\rho)\xi_i(t)\mathcal{Q}^{-1}\zeta(t-\rho)\Bigg]\mathrm{d}\phi
\end{aligned}
\tag{6.37}
$$

and

$$
\begin{aligned}
\dot{V}_{d2}(t)|_{(6.9)} &= \int_{-\tau}^{0} \Big\{ \dot{\zeta}^t(t)\xi_i(t)\mathcal{M}^{-1}\dot{\zeta}(t) - \dot{\zeta}^t(t+\phi)\xi_i(t)\mathcal{M}^{-1}\dot{\zeta}(t+\phi) \Big\} \mathrm{d}\phi \\
&= \int_{t-\tau}^{t} \Big\{ \dot{\zeta}^t(t)\xi_i(t)\mathcal{M}_i^{-1}\Big)\dot{\zeta}(t) - \dot{\zeta}^t(\phi)\Big(\sum_{i=1}^{N}\xi_i(t)\mathcal{M}_i^{-1}\Big)\dot{\zeta}(\phi)\Big\}\mathrm{d}\phi \\
&= \int_{t-\tau}^{t} \Big\{ \Big[\mathcal{A}_{pi}\zeta(t) + \mathcal{A}_{ci}\zeta(t-\varrho) + \mathcal{A}_{di}\zeta(t-\tau)\Big]^t \mathcal{M}^{-1} \\
&= \Big[\mathcal{A}_{pi}\zeta(t) + \mathcal{A}_{ci}\zeta(t-\varrho) + \mathcal{A}_{di}\zeta(t-\tau)\Big] \\
&\quad -\dot{\zeta}^t(\phi)\mathcal{M}^{-1}\dot{\zeta}(\phi)\Big\}\mathrm{d}\phi \\
&= \frac{1}{\tau(t)} \int_{t-\tau}^{t} \Big\{ \zeta^t(t)\tau(t)\mathcal{A}_{pi}\mathcal{M}^{-1}\mathcal{A}_{pi}\zeta(t) \\
&\quad +\zeta^t(t-\varrho)\tau(t)\mathcal{A}_{ci}^t\mathcal{M}^{-1}\mathcal{A}_{ci}\zeta(t-\varrho) \\
&\quad +\zeta^t(t-\tau)\tau(t)\mathcal{A}_{di}^t\mathcal{M}^{-1}\mathcal{A}_{di}\zeta(t-\tau) \\
&\quad +2\zeta^t(t)\tau(t)\mathcal{A}_{pi}^t\mathcal{M}^{-1}\mathcal{A}_{ci}\zeta(t-\rho) \\
&\quad +2\zeta^t(t)\tau(t)\mathcal{A}_{pi}^t\mathcal{M}^{-1}\mathcal{A}_{di}\zeta(t-\tau) \\
&\quad +2\zeta^t(t-\rho)\tau(t)\mathcal{A}_{ci}^t\mathcal{M}^{-1}\mathcal{A}_{di}\zeta(t-\tau) \\
&\quad -\dot{\zeta}^t(\phi)\tau(t)\mathcal{M}^{-1}\dot{\zeta}(\phi)\Big\}\mathrm{d}\phi
\end{aligned}
\tag{6.38}
$$

It follows that for nonzero vectors $\zeta(t),\ \zeta(t-\tau),\ \zeta(t-\rho)$, and the particular case $\xi_i(t) = 1$ and $\xi_{m\neq i}(t) = 0$. Therefore, with some algebraic manipulations, we get from (6.34), (6.35), (6.36), (6.37), and (6.38)

$$
\dot{V}_2(t)|_{(6.9)} = \frac{1}{\tau(t)} \int_{t-\tau}^{t} \chi(t,\phi)\ \Xi_{is}\ \chi(t,\phi)\ \mathrm{d}\phi \tag{6.39}
$$

where

$$
\begin{aligned}
\chi(t,\phi) &= \big[\, \zeta^t(t) \;\; \zeta^t(t-\tau) \;\; \zeta^t(t-\rho) \;\; \dot{\zeta}^t(\phi) \,\big]^t \\
\Xi_{is} &= \begin{bmatrix} \Xi_{ais} & \Xi_{bis} & \Xi_{cis} & -\tau\Theta_i \\ \bullet & \Xi_{dis} & \Xi_{eis} & -\tau\Psi_i \\ \bullet & \bullet & \Xi_{fis} & 0 \\ \bullet & \bullet & \bullet & -\tau\mathcal{M}^{-1} \end{bmatrix} \\
\Xi_{ais} &= \mathcal{P}_i\mathcal{A}_{pi} + \mathcal{A}_{pi}^t\mathcal{P}_i + \Theta_i + \Theta_i^t + \mathcal{Z}^{-1} + \mathcal{Q}^{-1} + \tau(t)\mathcal{A}_{pi}^t\mathcal{M}^{-1}\mathcal{A}_{pi} \\
\Xi_{bis} &= \mathcal{P}_s\mathcal{A}_{di} - \Theta_i + \Psi_i^t + \tau\mathcal{A}_{pi}^t\mathcal{M}^{-1}\mathcal{A}_{di}
\end{aligned}
\tag{6.40}
$$

$$\begin{aligned}\Xi_{cis} &= \mathcal{P}_s\mathcal{A}_{ci} + \tau\mathcal{A}^t_{pi}\mathcal{M}^{-1}\mathcal{A}_{ci}\\ \Xi_{dis} &= -\Psi_i - \Psi_i^t - (1-\mu)\mathcal{Z}_s + \tau\mathcal{A}^t_{di}\mathcal{M}^{-1}\mathcal{A}_{di}, \quad \Xi_{eis} = \tau\mathcal{A}^t_{di}\mathcal{M}_i^{-1}\mathcal{A}_{ci}\\ \Xi_{fis} &= -\mathcal{Q}^{-1} + \tau\mathcal{A}^t_{ci}\mathcal{M}^{-1}\mathcal{A}_{ci}\end{aligned} \tag{6.41}$$

When $\Xi_{is} < 0, \ \forall(i,s) \in \mathbf{S},$ we infer from (6.39) that $\dot{V}_2(t) < 0$ for any $\chi(t,\phi) \neq 0$ and all $\tau \le \varrho$. By Schur complement operations, we express (6.40) for all $0 < \tau < \varrho$ as $\mathcal{M}_i > 0$ and

$$\begin{bmatrix} \hat{\Xi}_{ais} & \mathcal{P}_s\mathcal{A}_{di} - \Theta_i + \Psi_i^t & \mathcal{P}_s\mathcal{A}_{ci} & -\tau\Theta_i & \tau\mathcal{A}^t_{pi}\\ \bullet & -\Psi_i - \Psi_i^t - (1-\mu)\mathcal{Z}^{-1} & 0 & -\tau\Psi_i & \tau\mathcal{A}^t_{di}\\ \bullet & \bullet & -\mathcal{Q}^{-1} & 0 & \tau\mathcal{A}^t_{ci}\\ \bullet & \bullet & \bullet & -\mathcal{M}^{-1} & 0\\ \bullet & \bullet & \bullet & \bullet & -\mathcal{M}\end{bmatrix} \le$$

$$\begin{bmatrix} \hat{\Xi}_{ais} & \mathcal{P}_s\mathcal{A}_{di} - \Theta_i + \Psi_i^t & \mathcal{P}_s\mathcal{A}_{ci} & -\varrho\Theta_i & \varrho\mathcal{A}^t_{pi}\\ \bullet & -\Psi_i - \Psi_i^t - (1-\mu)\mathcal{Z}^{-1} & 0 & -\varrho\Psi_i & \varrho\mathcal{A}^t_{di}\\ \bullet & \bullet & -\mathcal{Q}^{-1} & 0 & \varrho\mathcal{A}^t_{ci}\\ \bullet & \bullet & \bullet & -\mathcal{M}^{-1} & 0\\ \bullet & \bullet & \bullet & \bullet & -\mathcal{M}\end{bmatrix} := \widetilde{\Xi}_{is} < 0 \tag{6.42}$$

where

$$\hat{\Xi}_{ais} = \mathcal{P}_i\mathcal{A}_{pi} + \mathcal{A}^t_{pi}\mathcal{P}_i + \Theta_i + \Theta_i^t + \mathcal{Z}^{-1} + \mathcal{Q}^{-1} \tag{6.43}$$

Using (6.18), we apply the congruent transformation

$$diag\begin{bmatrix} T_1 & T_2 & T_2 & T_2 & T_2\end{bmatrix}, \quad \mathcal{P}_i^{-1} \triangleq \mathcal{X}_i = \begin{bmatrix}\mathcal{X}_{xi} & 0\\ 0 & I\end{bmatrix}$$

to $\widetilde{\Xi}_{is}$ with

$$\mathcal{Y}_i = K_{oi}\mathcal{X}_{xi}, \ \mathcal{W}_i = K_{si}\mathcal{X}_{xi}, \ \mathcal{R}_i = K_{ai}\mathcal{X}_{xi}$$

and making use of the algebraic inequalities (6.19), it follows by the Schur complements formula that the asymptotic stability of the closed-loop system (6.9), (6.10), and (6.11) is established.

Next, consider the performance measure

$$J = \int_0^\infty \left(\sum_{i=1}^N \xi_i(t)\Big[z^t(s)z(s) - \gamma^2 w^t(s)w(s)\Big]\right) \mathrm{d}s$$

For any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ and zero initial condition $x(0) = 0$, we have

$$J = \int_0^\infty \left(\sum_{i=1}^{N} \xi_i(t) \Big[z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_2(t)|_{(6.9)} \Big] \right) \mathrm{d}s$$
$$- \dot{V}_2(t)|_{(6.9)}$$
$$\leq \int_0^\infty \left(\sum_{i=1}^{N} \xi_i(t) \Big[z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_2(t)|_{(6.9)} \Big] \right) \mathrm{d}s$$

where $\dot{V}_2(t)|_{(6.9)}$ defines the Lyapunov derivative along the solutions of system (6.9). Proceeding, we get under arbitrary switching and Schur complement operations

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_2(s)|_{(6.9)} = \widetilde{\eta}^t(t,\phi)\overline{\Xi}_{is}\widetilde{\eta}(t,\phi)$$

$$\overline{\eta}^t(t,\phi) = \begin{bmatrix} \zeta(t) \\ \zeta(t-\tau) \\ \zeta(t-\rho) \\ \dot{\zeta}(\phi) \\ w(t) \end{bmatrix}, \quad \overline{\Xi}_{is} = \begin{bmatrix} & \widetilde{\Xi}_{ais} & & \mathcal{P}_i\bar{\Gamma}_i + \bar{G}^t_{oi} & \hat{G}^t_{oi} \\ & & & \bar{G}^t_{di} & \hat{G}^t_{di} \\ & & & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I + \Phi_i^t\Phi_i & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} \tag{6.44}$$

Using (6.18), we apply the congruent transformation

$$\mathrm{diag}\big[\, T_1 \ T_1 \ T_2 \ T_2 \ T_2 \ T_2 \ T_2 \,\big], \quad \mathcal{P}_i^{-1} := \mathcal{X}_i = \begin{bmatrix} \mathcal{X}_{xi} & 0 \\ 0 & I \end{bmatrix}$$

to $\overline{\Xi}_{is}$ with $\mathcal{Y}_i = K_{oi}\mathcal{X}_{xi}, \ \mathcal{W}_i = K_{si}\mathcal{X}_{xi}, \ \mathcal{R}_i = K_{ai}\mathcal{X}_{xi}$, and making use of the algebraic inequalities (6.19), it readily follows from LMI (6.32) and Schur complement operations that

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}_2(s)|_{(6.9)} < 0$$

for arbitrary $s \in [t,\infty)$, which implies for any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ that $J < 0$ leading to $||z(t)||_2 < \gamma \ ||w(t)||_2$ and the proof of $\mathcal{H}_\infty$ performance bound is completed. ■

6.1.5 Simulation Examples

In this section, we will demonstrate the application of the foregoing analytical results on some typical systems examples.

Illustrative Example A

A model of combustion in rocket motor chambers [442] is considered here for feedback stabilization. This model represents a liquid monopropellant rocket motor with a pressure feeding system. Under the assumption of nonsteady flow and lumped lag factor, an appropriate linearized model can be in the form (6.1, 6.2, and 6.3) with the following coefficients:

$$A_i = \begin{bmatrix} \rho_i - 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\zeta J} \\ -\frac{p}{2J(1-\zeta)} & 0 & -\frac{1}{J(1-\zeta_i)} & -\frac{1}{J(1-\zeta_i)} \\ 0 & \frac{1}{E_e} & -\frac{1}{E_e} & 0 \end{bmatrix}, \; A_{di} = \begin{bmatrix} -\rho_i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0 \\ \frac{1}{\zeta_i J} \\ 1 \\ 0 \end{bmatrix}, \; \Gamma_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; G_i = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \; \Psi_i = [0.4]$$

Subscript i corresponds to mode of operation as in Table 6.1.

Table 6.1 Data of illustrative example A

Mode	ρ_i	ζ
1	0.95	0.105
2	1.05	0.100
3	1.15	0.110

With ζ being the fractional length for pressure supply, J is the line inertia, E_e is the line elasticity parameter, p is the ratio of steady-state pressure and steady-state injector pressure drop, and ρ is the pressure exponent of the combustion process. For simulation purposes, the nominal values taken are $p = 1.02, \; J = 2, \; E_e = 0.95$. Implementation of the developed theorems was accomplished using the LMI-solver Scilab-5.1 and the ensuing results are summarized in Table 6.2. The results show that the PID feedback strategy provides improved stabilization for the switched model of combustion in rocket motor chambers under arbitrary switching among operating modes.

Table 6.2 Computational results of illustrative example A

Method	$K_o(K_s)$	Matrix gain				γ
Theorem 6.1	K_{o1}	0.4936	−1.1205	−0.3986	−0.2135	3.554
	K_{s1}	1.8172	−0.9677	−0.6329	0.3558	
	K_{a1}	−0.0195	1.0025	−0.2448	0.3558	
	K_{o2}	0.4825	−1.1195	−0.3877	−0.3014	
	K_{s2}	1.7299	−1.1076	−0.6158	0.3702	
	K_{a2}	−0.0538	0.9785	−0.3762	0.3558	
	K_{o3}	0.5104	−1.1306	−0.4115	−0.2985	
	K_{s3}	1.8384	−1.1213	−0.7045	0.4104	
	K_{a3}	−0.1004	0.8856	−0.2448	0.2755	
Theorem 6.5	K_{s1}	2.3456	−1.2333	1.3611	−0.6775	7.045
	K_{s2}	2.3502	−1.2287	1.4102	0.6548	
	K_{s3}	2.3611	−1.2205	1.4126	−0.6643	

Illustrative Example B

For all practical purposes, it is crucial to preserve the standards of water quality in streams. This can be measured by the concentrations of some water biochemical constituents. Let $z(t),\ q(t)$ be the concentrations per unit volume of biological oxygen demand (BOD) and dissolved oxygen (DO), respectively, at time t. Under the simplifying assumptions [179, 254] that the stream has a constant flow rate and the water is well mixed and there exists a $\tau > 0$ such that the (BOD,DO) concentrations entering at time t are equal to the corresponding concentrations τ time units ago. Employing a linearization of the mass balance concentrations about an equilibrium operating point and using representative data on a single reach of the River Nile, the growth of (BOD,DO) can then be expressed as

$$\begin{aligned}
\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} &= \begin{bmatrix} -1.285 & 0 \\ -3.263 & -1.975 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} -0.15 & 0 \\ 0 & -0.10 \end{bmatrix} \begin{bmatrix} p(t-\tau) \\ q(t-\tau) \end{bmatrix} \\
&+ \begin{bmatrix} 1.2 & 0 \\ 0 & 1.4 \end{bmatrix} \begin{bmatrix} u_p(t) \\ u_q(t) \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} w_p(t) \\ w_q(t) \end{bmatrix} \\
\begin{bmatrix} z_p(t) \\ z_q(t) \end{bmatrix} &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} -0.15 & 0 \\ 0 & -0.10 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} \phi_p(t) \\ \phi_q(t) \end{bmatrix}
\end{aligned}$$

The feasible solution of **Theorem 6.10** with $\varrho = 2,\ \mu = 1$ yields the delay-dependent PID controller of the form

$$\begin{aligned}
\begin{bmatrix} u_{z1}(t) \\ u_{q1}(t) \end{bmatrix} &= \begin{bmatrix} 0.8837 & -0.0307 \\ -0.1235 & 0.0315 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} - \begin{bmatrix} -0.9945 & -0.0513 \\ 0.3505 & -0.0265 \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0.7113 & -0.0114 \\ 0.0023 & -0.0405 \end{bmatrix} \begin{bmatrix} \int_{t-2}^{t} p(s)\mathrm{d}s \\ \int_{t-2}^{t} q(s)\mathrm{d}s \end{bmatrix} \\
\begin{bmatrix} u_{z2}(t) \\ u_{q2}(t) \end{bmatrix} &= \begin{bmatrix} 0.9088 & -0.1034 \\ -0.1332 & 0.0023 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} - \begin{bmatrix} -1.1306 & -0.0661 \\ 0.5123 & -0.0265 \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0.3887 & -0.0206 \\ -0.1661 & 0.1115 \end{bmatrix} \begin{bmatrix} \int_{t-2}^{t} p(s)\mathrm{d}s \\ \int_{t-2}^{t} q(s)\mathrm{d}s \end{bmatrix}
\end{aligned}$$

which renders the water quality system asymptotically stable with $\gamma = 2.5115$.

Illustrative Example C

A continuous-time model used in resilience control studies [221] is considered here where the associated matrices are

$$A_1 = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.2 & 0 & 0.6 & 0 \\ 0 & -1 & 0 & -0.7 \\ 0 & -0.8 & 0 & -1.3 \\ 0.1 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3 & 0 & 1 & -11 \\ 4 & 0 & -1 & 1 \\ -1 & 0 & -3 & -1 \\ -1 & 0 & 3 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.3 & 0 & 0.4 & 0 \\ 0 & -0.8 & 0 & -0.5 \\ 0 & -0.7 & 0 & -1.1 \\ 0.3 & 0 & 0.6 & 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

In Tables 6.3 and 6.4, a summary of the computational results of applying **Theorem 6.10** is presented.

An overall summary to be recorded from the foregoing examples is that the three-term stabilization approach provides flexibility and guarantees a lowest performance bound.

Table 6.3 Computational results of example C

Method	$K_o(K_s)$	Matrix gain				Matrix norm	γ
Theorem 6.10	K_{o1}	−0.4539	−0.6875	−0.4539	−0.5389	1.6028	1.653
		−0.3725	−0.6348	−0.8746	−0.4215		
	K_{s1}	−0.6025	−0.8305	−0.3592	−0.8884	1.6877	
		−0.5309	−0.9421	0.0523	−0.1375		
	K_{a1}	−0.3515	−0.7204	−0.4082	−0.7765	0.9875	
		−0.4367	−0.8436	0.0724	−0.1425		
	K_{o2}	−0.5014	−0.7511	−0.4818	−0.6355	1.8668	
		−0.6021	−0.6802	−1.0065	−0.5545		
	K_{s2}	−0.6501	−1.0549	−0.5092	−0.6734	1.7971	
		−0.5911	−0.8881	0.1421	−0.2345		
	K_{a2}	−0.7555	−0.9135	−0.0524	−1.2564	1.3245	
		−0.4545	−1.0021	0.0624	−0.1566		

Table 6.4 A Summary of results-illustrative example C

Method	Mode	$\|\|K_o\|\|$	$\|\|K_s\|\|$	$\|\|K_a\|\|$	γ
Theorem 6.10	1	1.6028	1.6877	1.4305	3.247
	2	1.8668	1.7971	1.9513	
Theorem 6.1	1	2.4377	3.1445	2.3529	8.688
	2	2.6658	3.2312	2.5145	
[216]	1	4.6028	4.6445		23.766
[221]	1		5.037		15.455

Remark 6.11 In practice, there are two additional stabilization schemes: proportional integral (PI) and proportional derivative (PD). The gains can be readily generated from the developed setup by setting $K_{oi} \equiv 0$ and $K_{ai} \equiv 0$, respectively, in the foregoing theorems.

6.2 Discrete-Time Systems

This section will address the problem of control design of switched systems in the discrete-time domain and looks at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the switched system under consideration with and without unknown time delay. Two different $\mathcal{H}_\infty$ switched controller design schemes are established based on the state-feedback and proportional-summation-difference (PSD) feedback designs. In the state feedback, an improved LMI-based method is provided. By PSD feedback, a three-term feedback controller gains is designed for each subsystem such that the closed-loop discrete-time switched system is asymptotically stable. In both cases, appropriate Lyapunov–Krasovskii functionals (LKFs) are constructed and efficient parametrized characterizations are established in terms of feasibility testing of linear matrix inequalities (LMIs).

6.2.1 Introduction

We know that switched systems have hybrid features comprising a family of subsystems described by continuous-time or discrete-time dynamics, and a rule specifying the switching among them [28, 42, 47]. The switching rule, determined by time or system state, or both, or other supervisory logic decision, yields different switching signals and decides the categories of switched systems; see for example, [42, 192, 424] and the references therein. A survey of basic problems in stability and design of switched systems has been proposed recently in [193]. A basic fact in switched systems theory is that among the large variety of problems encountered in practice, one can study the existence of a switching rule that ensures stability of the switched system. One can also assume that the switching sequence is not known a priori and look for stability results under arbitrary switching sequences. One can also consider some useful class of switching sequences see, for instance, [192]. The applications using switched systems theory and practical examples include modeling of networked control systems (NCS) [196], stirred tank reactor [56], and wind turbine regulation [186].

In this section, we are interested in control synthesis of discrete-time switched systems under arbitrary switching sequences, which in some sense complement the foregoing section. The reasons of considering discrete-time switched systems have been enumerated in [424]. A multimodal dynamical system, for example, may be composed of several discrete-time dynamical subsystems due to its physical structure, and even when all subsystems are of continuous time, the case of considering sampled-data control for the entire system can be dealt with in the framework of discrete-time switched systems. Furthermore, experience showed that the extension from continuous-time switched systems to discrete-time ones is not obvious in most cases, and the results may be quite different, as also pointed out in [196].

The approach followed in this section looks at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the switched system under consideration. To evaluate the interest of this approach for control design problems, we concentrate on the state-feedback and proportional-summation-difference (PSD) feedback design problems, deferring other possible design methods to later developments in subsequent chapters.

6.2.2 Problem Statement

We consider the following class of switched discrete-time systems with time-varying delays:

$$\begin{aligned} x(k+1) &= A_\sigma x(k) + D_\sigma x(k-d(k)) + B_\sigma u(k) + \Gamma_\sigma \omega(k) \\ z(k) &= G_\sigma x(k) + H_\sigma x(k-d(k)) + \Phi_\sigma \omega(k) \end{aligned} \tag{6.45}$$

where $x(k) \in \Re^n$ is the state vector , $u(k) \in \Re^m$ is the control input, $\omega(k) \in \Re^q$ is the disturbance input which belongs to $\ell_2[0,\infty)$, $z(k) \in \Re^q$ is the observed output, $\sigma : \Re_+ = [0,\infty) \rightarrow \mathbf{S} = \{1,\ldots,s\}$ is the switching signal, which is assumed to be piecewise constant function available in real time with N being the number of modes of the switched system and the scalar $d(k)$ is a time-delay factor satisfying $0 < d_m \leq d(k) \leq d_M$, where d_m and d_M are known bounding factors. The initial condition $\omega(\phi)$ is a differentiable vector-valued function on $[-d, 0]$. At an arbitrary discrete time k, the switching signal σ is dependent on k, $x(k)$ or both, or other switching rules.

The matrices of each mode $A_j \in \Re^{n\times n}$, $B_j \in \Re^{n\times m}$, $G_j \in \Re^{q\times n}$, $H_j \in \Re^{q\times n}$, $D_j \in \Re^{n\times n}$, and $\Gamma_j \in \Re^{n\times q}$, $\Phi_j \in \Re^{q\times q}$ are real and known constant matrices describing the jth system

$$\begin{aligned} x(k+1) &= A_j x(k) + D_j x(k-d(k)) + B_j u(k) + \Gamma_j \omega(k) \\ z(k) &= G_j x(k) + H_j x(k-d(k)) + \Phi_j \omega(k) \end{aligned} \tag{6.46}$$

Remark 6.12 It should be noted that system (6.45) designates a class of discrete-time systems with multimodes and unknown time delay. This means that system (6.45) is constrained to jump among the N vertices of matrix polytope

$$\left\{(A_j)_{j=1}^{s},\ (B_j)_{j=1}^{s},\ (D_j)_{j=1}^{s},\ (G_j)_{j=1}^{s},\ (H_j)_{j=1}^{s},\ (\Gamma_j)_{j=1}^{s},\ (\Phi_j)_{j=1}^{s}\right\}$$

Stability and stabilization problems of this class systems render several challenging issues to the control engineers and designers. In the control design literature with $y(k) \in \Re^p$ being the measured output, most of the developed methods focused on either state feedback $u(k) = K_o\ x(k)$, which is a proportional control with single unknown gain matrix K_o, dynamic output feedback using observer-based controllers $\hat{x}(k+1) = A_o x(k) + D_o x(k-d(k)) + B_o u(k) + K_s C_o (x(k) - \hat{x}(k)), u(k) = K_o\ x(k)$ where the unknown gain matrices are two: K_o and K_s, dynamic output feedback scheme with strictly proper transfer function $\xi(k+1) = A_c \xi(k) + B_c y(k),\ u(k) = C_c\ \xi(k)$ corresponding to three unknown gain matrices $A_c,\ B_c$, and C_c or dynamic output feedback scheme with proper transfer function $\xi(k+1) = A_c \xi(k) + B_c y(k),\ u(k) = C_c\ \xi(k) + D_c\ y(k)$ corresponding to four unknown gain matrices $A_c,\ B_c,\ C_c$, and D_c. In all of the foregoing cases, different computational techniques were offered [82, 120]. Apart from the state-feedback case, the developed techniques are computationally demanding and some of them are iterative in nature. The degree of success of these techniques to systems with time-varying delays is generally limited.
In this section, we are interested in control-feedback synthesis of discrete delayed switched systems under arbitrary switching sequences. The approach followed in this note looks at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the closed-loop discrete switched system under consideration. To evaluate the interest of this approach for switched control design problems, we address both single-term (state-feedback) and three-term (proportional-summation-difference (PSD)) feedback design problems. By feedback control, we mean the design of single-term or three-terms feedback gains for each system such that the closed-loop switched system is asymptotically stable. The results proposed in this work can be considered as a trade off between highly conservative results (those using a single quadratic Lyapunov function) and less conservative but those numerically hard to check.

We seek the development of improved stabilization schemes for system (6.45). These schemes should possess reduced-order computational requirements. Toward our goal, we first provide an improved LMI-based state-feedback stabilization.

6.2.3 State-Feedback $\mathcal{H}_\infty$ Stabilization

Let the state-feedback control be $u(k) = K_{oj}\ x(k)$. The closed-loop feedback system becomes

$$
\begin{aligned}
x(k+1) &= A_{sj}x(k) + D_j x(k-d(k)) + \Gamma_j \omega(k) \\
z(k) &= G_j x(k) + H_j x(k-d(k)) + \Phi_j \omega(k) \\
A_{sj} &= A_j + B_j K_{oj}
\end{aligned} \tag{6.47}
$$

where $K_{sj} \in \Re^{n\times n}$ is the state-feedback gain matrix to be determined. Our goal is to establish tractable conditions guaranteeing closed-loop asymptotic stability of the origin ($x = 0$) for system (6.47). The underlying notion is that system (6.47) is globally asymptotically stable if there is a Lyapunov–Krasovskii function V, which is a positive–definite function, decrescent, and radially unbounded, and its first difference ΔV is negative definite along the solutions of (6.47), thereby proving global asymptotic stability.

In the sequel we let $\beta = (d^+ - d^* + 1)$, which represents the number of samples within the delay range $d^* \le d(k) \le d^+$. The main result of subsystem stability is given by the following theorem:

Theorem 6.13 *Given the delay sample number β. System (6.47) is delay-dependent asymptotically stable if there exist matrices $0 < \mathcal{X}_s^t = \mathcal{X}_s$, $0 < \mathcal{X}_j^t = \mathcal{X}_j$, $0 < \mathcal{W}_j^t = \mathcal{W}_j$, $\mathcal{Y}_j$ and a scalar $\gamma > 0$ such that the following convex optimization problem is feasible for all $(j, s) \in \mathbf{S} \times \mathbf{S}$*

$$
\min_{\mathcal{X}_s, \mathcal{W}_s, \mathcal{Y}_{js}} \gamma^2, \quad \text{subject to}
$$

$$
\widehat{\Pi} = \begin{bmatrix}
\beta\mathcal{W}_s - \mathcal{X}_j & 0 & \mathcal{X}_j G_j^t \Phi_j & \mathcal{X}_j A_j^t + \mathcal{Y}_j B_j^t & \mathcal{X}_j G_j^t \\
\bullet & -\mathcal{W}_j & \mathcal{X}_j H_j^t \Phi_j & \mathcal{X}_j D_j^t & \mathcal{X}_j H_j^t \\
\bullet & \bullet & -\gamma^2 I + \Phi_j^t \Phi_j & \Gamma_j^t & 0 \\
\bullet & \bullet & \bullet & -\mathcal{X}_s & 0 \\
\bullet & \bullet & \bullet & \bullet & -I
\end{bmatrix} < 0 \tag{6.48}
$$

Moreover, the state-feedback gain is given by

$$
K_{sj} = \mathcal{Y}_j \mathcal{X}_j^{-1}.
$$

Proof We start by introducing the Lyapunov–Krasovskii functional at mode j (LKF):

$$
\begin{aligned}
V_s(k) &= V_{so}(k) + V_{sa}(k) + V_{sc}(k) \\
V_{so}(k) &= x^t(k)\mathcal{P}_j x(k), \; V_{sa}(k) = \sum_{m=k-d(k)}^{k-1} x^t(m)\mathcal{Q}_j x(m) \\
V_{sc}(k) &= \sum_{s=2-d_M}^{1-d_m} \sum_{m=k+s-1}^{k-1} x^t(m)\mathcal{Q}_j x(m)
\end{aligned} \tag{6.49}
$$

where $0 < \mathcal{P}_j^t = \mathcal{P}_j, \ 0 < \mathcal{Q}^t = \mathcal{Q}$ are weighting matrices of appropriate dimensions. A straightforward computation gives the first difference of $\Delta V(k) = V(k+1) - V(k)$ along the solutions of (6.2) $\forall (j,s) \in \mathbf{S} \times \mathbf{S}$ as

$$\begin{aligned}\Delta V_{so}(k) &= x^t(k+1)\mathcal{P}_s x_j(k+1) - x^t(k)\mathcal{P}_j x_j(k) \\ &= [A_{sj}x(k) + D_j x(k-d(k)) + \Gamma_j \omega(k)]^t \mathcal{P}_s \\ &\quad \times [A_{sj}x(k) + D_j x(k-d(k)) + \Gamma_j \omega(k)] - x^t(k)\mathcal{P}_j x_j(k) \end{aligned} \tag{6.50}$$

$$\begin{aligned}\Delta V_{sa}(k) &= x^t(k)\mathcal{Q}_j x(k) - x^t(k-d_j(k))\mathcal{Q}_j x(k-d_j(k)) \\ &\quad + \sum_{m=k+1-d(k+1)}^{k-1} x^t(m)\mathcal{Q}_j x(m) - \sum_{m=k+1-d(k)}^{k-1} x^t(m)\mathcal{Q}_j x(m) \end{aligned} \tag{6.51}$$

$$\Delta V_{sc}(k) = (d_M - d_m)x^t(k)\mathcal{Q}_j x(k) - \sum_{m=k+1-d_M}^{k-d^*} x^t(m)\mathcal{Q}_j x(m) \tag{6.52}$$

Observe from (6.51) that

$$\begin{aligned}\sum_{m=k+1-d(k+1)}^{k-1} x^t(m)\mathcal{Q}_j x(m) &= \sum_{m=k+1-d_M}^{k-1} x^t(m)\mathcal{Q}_j x_j(m) \\ &\quad + \sum_{m=k+1-d(k+1)}^{k-d_M} x^t(m)\mathcal{Q}_j x(m) \\ &\le \sum_{m=k+1-d(k)}^{k-1} x^t(m)\mathcal{Q}_j x_j(m) \\ &\quad + \sum_{m=k+1-d_M}^{k-d_m} x^t(m)\mathcal{Q}_j x(m) \end{aligned} \tag{6.53}$$

Then using (6.53) into (6.51) and manipulating, we reach

$$\begin{aligned}\Delta V_{sa}(k) &\le x^t(k)\mathcal{Q}_j x(k) - x^t(k-d(k))\mathcal{Q}_j x(k-d(k)) \\ &\quad + \sum_{m=k+1-d_M}^{k-d_m} x^t(m)\mathcal{Q}_j x(m) \end{aligned} \tag{6.54}$$

Taking into consideration (6.50), (6.52), and (6.54), the following upper bound for $\Delta V_s(k)$ can be obtained

$$\begin{aligned}\Delta V_s(k) &\le [A_{sj}x(k) + D_j x(k-d(k)) + \Gamma_j \omega(k)]^t \mathcal{P}_s \\ &\quad \times [A_{sj}x(k) + D_j x(k-d(k)) + \Gamma_j \omega(k)] \end{aligned}$$

$$+ x^t(k)[\beta \mathcal{Q}_j - \mathcal{P}_s]x(k) - x^t(k-d(k))\mathcal{Q}_j x(k-d(k))$$
$$= \zeta^t(k)\ \Xi\ \zeta(k) \qquad (6.55)$$

where

$$\Xi = \begin{bmatrix} A_{sj}^t \mathcal{P}_s A_{sj} + \beta \mathcal{Q}_j - \mathcal{P}_j & A_{sj}^t \mathcal{P}_s D_j & A_{sj}^t \mathcal{P}_s \Gamma_j \\ \bullet & D_j^t \mathcal{P}_s D_j - \mathcal{Q}_j & D_j^t \mathcal{P}_s \Gamma_j \\ \bullet & \bullet & \Gamma_j^t \mathcal{P}_s \Gamma_j \end{bmatrix}$$
$$\zeta(k) = \left[x^t(k)\ x^t(k-d(k))\ \omega^t(k) \right]^t \qquad (6.56)$$

Note that in (6.55), the case when $j = s$ indicates that the discrete switched system is described by the jth mode, while the case $j \neq s$ illustrates that the discrete switched system is at the switching time from mode j to mode s; see [42] for more details.

The sufficient condition of internal stability implies $\Delta V k < 0$ with $\omega(k) \equiv 0$ implies that $\Xi < 0$ when $\Gamma_o \equiv 0$.

Next, consider the performance measure

$$J = \sum_{j=0}^{\infty} \left(z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j) \right)$$

For any $\omega(k) \in \ell_2(0,\infty) \neq 0$ and zero initial condition $x(0) = 0$, (hence $V_s(0) = 0$), we have

$$J = \sum_{j=0}^{\infty} \left(z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j)\ + \Delta V_s|_{(6.47)} \right) - \sum_{j=0}^{\infty} \Delta V_j|_{(6.47)}$$
$$= \sum_{j=0}^{\infty} \left(z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j)\ + \Delta V_s|_{(6.47)} \right) - V_\infty$$
$$\leq \sum_{j=0}^{\infty} \left(z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j)\ + \Delta V_s|_{(6.47)} \right) \qquad (6.57)$$

where $\Delta V_s|_{(6.47)}$ defines the Lyapunov difference along the solutions of system (6.47). Proceeding like the foregoing section and considering (6.45) and (6.57), it can easily shown by algebraic manipulation that

$$z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j)\ + \Delta V_s|_{(6.47)}$$
$$= [G_j x(j) + H_j x(j-d(j)) + \Phi_j \omega(j)]^t [G_j x_j + H_j x(j-d(j)) + \Phi_j \omega(j)]$$
$$-\gamma^2 \omega^t(j)\omega(j)$$
$$= \xi_j^t\ \Omega\ \xi_j \qquad (6.58)$$

where

$$\Omega = \begin{bmatrix} \Omega_o & A_{sj}^t \mathcal{P} D_j + G_j^t H_j & A_{sj}^t \mathcal{P}_s \Gamma_j + G_j^t \Phi_j \\ \bullet & D_j^t \mathcal{P}_s D_j - \mathcal{Q}_j + H_j^t H_j & D_j^t \mathcal{P}_s \Gamma_j + G_{do}^t \Phi_j \\ \bullet & \bullet & \Gamma_j^t \mathcal{P}_s \Gamma_j - \gamma^2 I + \Phi_j^t \Phi_j \end{bmatrix} \tag{6.59}$$

$$\Omega_o = A_{sj}^t \mathcal{P}_s A_{sj} + \beta \mathcal{Q}_j - \mathcal{P}_j + G_j^t G_j \tag{6.60}$$

By Schur complements, we express Ω into the form

$$\widehat{\Omega} = \begin{bmatrix} \beta \mathcal{Q}_j - \mathcal{P}_j + G_j^t G_j & G_j^t H_j & G_j^t \Phi_j & A_{sj}^t \\ \bullet & -\mathcal{Q}_j + H_j^t H_j & H_j^t \Phi_j & D_j^t \\ \bullet & \bullet & -\gamma^2 I + \Phi_j^t \Phi_j & \Gamma_j^t \\ \bullet & \bullet & \bullet & -\mathcal{P}_s^{-1} \end{bmatrix} \tag{6.61}$$

To convexify matrix $\widehat{\Omega}$, we define

$$\mathcal{X}_s = \mathcal{P}_s^{-1},\ \mathcal{X}_j = \mathcal{P}_j^{-1},\ \mathcal{W}_j = \mathcal{P}_j^{-1} \mathcal{Q}_j \mathcal{P}_j^{-1},\ K_{sj} \mathcal{X}_j = \mathcal{Y}_j$$

Upon applying the congruence transformation

$$\mathrm{diag}[\mathcal{X}_j, \quad \mathcal{X}_j, \quad I, \quad I]$$

to $\widehat{\Omega}$, we finally obtain the form (6.48). This leads to

$$z_j^t z_j - \gamma^2 w_j^t w_j \ + \Delta V_j|_{(6.47)} < 0$$

for arbitrary $j \in [0, \infty)$, which implies for any $w(j) \in \ell_2(0, \infty) \neq 0$ that $J < 0$. This eventually leads to $||z_k||_2 \ < \ \gamma \ ||w_k||_2$ and hence the proof is completed. ■

Remark 6.14 It should be noted that the derivation of LMI lower bound d_m and the upper bound d_M account for extreme cases of delay factors stemming from physical consideration. Seeking computational convenience and effectiveness, the solutions to the problems of stability analysis and control synthesis are cast into convex optimization in terms of linear matrix inequalities (LMIs) that are handled using interior-point minimization algorithms. These algorithms have been recently coded into efficient numerical software.

Had we considered the following class of delay-free discrete-time systems with state feedback:

$$\begin{aligned} x(k+1) &= A_{sj} x(k) + \Gamma_j \omega(k), \\ z(k) &= G_j x(k) + \Phi_j \omega(k) \\ A_{sj} &= A_j + B_j K_{oj} \end{aligned} \tag{6.62}$$

Then **Theorem 6.13** specializes to

Corollary 6.15 *System (6.62) is delay-dependent asymptotically stable if there exist matrices* $0 < \mathcal{X}_s^t = \mathcal{X}_s$, $\mathcal{Y}$ *and a scalar* $\gamma > 0$ *such that the following convex optimization problem is feasible for all* $(j, s) \in \mathbf{S} \times \mathbf{S}$

$$\min_{\mathcal{X}_j, \mathcal{X}_s, \mathcal{Y}_j} \gamma^2, \quad \text{subject to}$$

$$\overline{\Pi} = \begin{bmatrix} -\mathcal{X}_j & \mathcal{X}_j G_j^t \Phi_j & \mathcal{X}_j A_j^t + \mathcal{Y}_j B_j^t & \mathcal{X}_j G_j^t \\ \bullet & -\gamma^2 I + \Phi_j^t \Phi_j & \Gamma_j^t & 0 \\ \bullet & \bullet & -\mathcal{X}_s & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{6.63}$$

Moreover, the state-feedback gain is given by

$$K_{oj} = \mathcal{Y}_j \mathcal{X}_j^{-1}$$

Remark 6.16 It is significant to observe that **Corollary 6.15** provides improved state-feedback control design result, which extends over similar existing methods [279, 436].

6.2.4 Proportional-Summation-Difference (PSD) Stabilization

Instead of using standard dynamic output feedback schemes, we take in the sequel a departure from this research direction and proceed to tackle the design problem in a direct way. The following feedback controller is proposed:

$$\begin{aligned} u(k) &= K_{oj}\ x(k) + K_{dj}\ (x(k) - x(k-1)) + K_{sj} \sum_{s=0}^{k} x(s) \\ &= K_{oj}\ x(k) + K_{dj}\ \delta(k) + K_{sj}\ \sigma(k) \end{aligned} \tag{6.64}$$

Observe that controller (6.64) consists of three terms: a proportional term $K_o\ x(k)$, a difference term $K_d\ \ \delta(k)$, and a summation term $K_s\ \ \sigma(k)$, and henceforth is labeled PSD controller. Therefore, it resembles a discrete version of the conventional proportional-integral-derivative (PID) controller for continuous-time systems, see the foregoing section. Applying (6.64) to system (6.45), we get the closed-loop system:

$$\begin{aligned} x(k+1) &= A_{sj} x(k) + B_j [K_{dj}\ \delta(k) + K_{sj}\ \sigma(k)] + D_j x(k - d(k)) + \Gamma_j \omega(k) \\ z(k) &= G_j x(k) + H_j x(k - d(k)) + \Phi_j \omega(k) \end{aligned} \tag{6.65}$$

The main design result is established by the following theorem

Theorem 6.17 *Given the delay sample number* β_j. *System (6.65) is delay-dependent asymptotically stable if there exist matrices* $0 < \mathcal{X}^t = \mathcal{X}$, $0 < \mathcal{W}^t = \mathcal{W}$,

$0 < \mathcal{S}^t = \mathcal{S},\ 0 < \mathcal{Z}^t = \mathcal{Z},\ \mathcal{Y},\ K_d,\ K_s$, *and a scalars* $\gamma > 0$ *such that the following convex optimization problem is feasible*

$$\min_{\mathcal{X}_s, \mathcal{W}_s, \mathcal{Y}_{js}} \gamma^2, \quad \text{subject to}$$

$$\widehat{\Pi} = \begin{bmatrix} \widehat{\Pi}_{oj} & \widehat{\Pi}_{aj} & \widehat{\Pi}_{vj} \\ \bullet & \widehat{\Pi}_{cj} & \widehat{\Pi}_{wj} \\ \bullet & \bullet & \widehat{\Pi}_{zj} \end{bmatrix} < 0 \tag{6.66}$$

where

$$\widehat{\Pi}_o = \begin{bmatrix} -\mathcal{X}_s + \beta\mathcal{W}_s & 0 & \mathcal{X}A^t_{sj} + \mathcal{Y}^t_{js}B^t_j \\ \bullet & -\mathcal{W}_s & \mathcal{X}_s D^t_j \\ \bullet & \bullet & \mathcal{X}_s - I \end{bmatrix}$$

$$\widehat{\Pi}_{cj} = \begin{bmatrix} -\gamma^2 I + \Phi^t_j \Phi_j & 0 & 0 \\ \bullet & -\mathcal{S} & 0 \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix},\ \widehat{\Pi}_{aj} = \begin{bmatrix} \mathcal{X}_s G^t_j \Phi_j & 0 & 0 \\ \mathcal{X}_s H^t_j \Phi_j & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_{vj} = \begin{bmatrix} \mathcal{X}_s A^t_j + \mathcal{Y}^t_{js} B^t_j & \mathcal{X}_s A^t_j + \mathcal{Y}^t_{js} B^t_j - I & \mathcal{X}_s A^t_j + \mathcal{Y}^t_{js} B^t_j \\ \mathcal{X}_s D^t_j & \mathcal{X}_s D^t_j & \mathcal{X}_s D^t_j \\ I & I & I \end{bmatrix}$$

$$\Pi_{wj} = \begin{bmatrix} \Gamma^t_j & \Gamma^t_j & \Gamma^t_j \\ K^t_{dj} B^t_j & K^t_{dj} B^t_j & K^t_{dj} B^t_j \\ K^t_{sj} B^t_j & K^t_{sj} B^t_j & K^t_{sj} B^t_j + I \end{bmatrix}$$

$$\Pi_s = \text{diag}[-\mathcal{X} \quad -2\mathcal{X} + \mathcal{S} \quad -2\mathcal{X} + \mathcal{Z}] \tag{6.67}$$

Moreover, the PSD feedback gains are given by

$$K_{oj} = \mathcal{Y}_j \mathcal{X}_j^{-1},\quad K_{dj},\quad K_{sj}$$

Proof Extending on (6.49), we introduce the augmented LKF :

$$V_c(k) = V_{so}(k) + V_{sa}(k) + V_{sc}(k) + \delta^t(k)\mathcal{S}\delta(k) + \sigma^t(k)\mathcal{Z}\sigma(k) \tag{6.68}$$

where $0 < \mathcal{S}^t = \mathcal{S},\ 0 < \mathcal{Z}^t = \mathcal{Z}$ are weighting matrices. It is straightforward to show that

$$\begin{aligned} \Delta V_{so}(k) = {} & [A_{sj}x(k) + B_j(K_{dj}\delta(k) + K_{sj}\sigma(k)) + D_j x(k - d(k)) + \Gamma_j \omega(k)]^t \mathcal{P} \\ & \times [A_{sj}x(k) + B_j(K_{dj}\delta(k) + K_{sj}\sigma(k)) + D_j x(k - d(k)) + \Gamma_j \omega(k)] \\ & - x^t(k)\mathcal{P}x_j(k) \end{aligned} \tag{6.69}$$

Observe that $\Delta V_{sc}(k)$ and $\Delta V_{sa}(k)$ are given by (6.52) and (6.52), respectively. So that

$$\begin{aligned}\Delta V_c(k) = & V_{so}(k) + V_{sa}(k) + V_{sc}(k) + [x(k+1) - x(k)]^t \mathcal{S}[x(k+1) - x(k)] \\ & -\delta^t(k)\mathcal{S}\delta(k) + [x(k+1) + \sigma(k)]^t \mathcal{Z}[x(k+1) + \sigma(k)] \\ & -\sigma^t(k)\mathcal{Z}\sigma(k) \end{aligned} \tag{6.70}$$

From (6.65), we have

$$\begin{aligned} x(k+1) - x(k) = & (A_{sj} - I)x(k) + D_j x(k - d(k)) + B_j[K_{dj}\ \delta(k) + K_{sj}\ \sigma(k)] \\ & + \Gamma_j \omega(k) \end{aligned} \tag{6.71}$$

$$\begin{aligned} x(k+1) + \sigma(k) = & A_{sj}x(k) + B_j K_{dj}\ \delta(k) + (B_j K_{sj} + I)\ \sigma(k) + D_j x(k - d(k)) \\ & + \Gamma_j \omega(k) \end{aligned} \tag{6.72}$$

Substituting (6.71) and (6.72) into (6.70), using (6.52), (6.54), and (6.69), we cast $\Delta V_c(k)$ into the form

$$\begin{aligned} \Delta V_c(k) & \leq \xi^t(k)\ \Pi_j\ \xi(k), \\ \xi(k) & = \left[x^t(k)\ \ x^t(k - d(k))\ \ \omega^t(k)\ \ \delta^t(k)\ \ \sigma^t(k) \right]^t \end{aligned} \tag{6.73}$$

where

$$\begin{aligned} \Pi_j &= \begin{bmatrix} \Pi_{oj} & \Pi_{aj} & \Pi_{vj} \\ \bullet & \Pi_{cj} & \Pi_{wj} \\ \bullet & \bullet & \Pi_{sj} \end{bmatrix} \\ \Pi_{aj} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \Pi_{cj} = \begin{bmatrix} -\mathcal{S} & 0 \\ \bullet & -\mathcal{Z} \end{bmatrix} \\ \Pi_{oj} &= \begin{bmatrix} \beta\mathcal{Q} - \mathcal{P}_s & 0 & A_{sj}^t \\ \bullet & -\mathcal{Q} & D_j^t \\ \bullet & \bullet & \mathcal{P}_j^{-1} \end{bmatrix}, \ \Pi_{vj} = \begin{bmatrix} A_{sj}^t & A_{sj}^t - I & A_{sj}^t \\ D_j^t & D_j^t & D_j^t \\ I & I & I \end{bmatrix} \\ \Pi_{wj} &= \begin{bmatrix} \Gamma_j^t & \Gamma_j^t & \Gamma_j^t \\ K_{dj}^t B_j^t & K_{dj}^t B_j^t & K_{dj}^t B_j^t \\ K_{sj}^t B_j^t & K_{sj}^t B_j^t & K_{sj}^t B_j^t + I \end{bmatrix} \\ \Pi_{sj} &= \mathrm{diag}[-\mathcal{P}_j^{-1} \quad -\mathcal{S}^{-1} \quad -\mathcal{Z}^{-1}] \end{aligned} \tag{6.74}$$

The sufficient condition of stability $\Delta V_c(k) < 0$ implies that $\Pi < 0$. Following parallel development to the foregoing section and considering the performance measure

$$J = \sum_{j=0}^{\infty} \left(z^t(j)z(j) - \gamma^2 \omega^t(j)\omega(j) \right)$$

we arrive at

$$\begin{aligned} & z^t(j)z(j) - \gamma^2\omega^t(j)\omega(j) \ + \Delta V_s|_{(6.47)} \\ = & [G_j x(j) + H_j x(j-d(j)) + \Phi_j \omega(j)]^t [G_j x_j + H_j x(j-d(j)) + \Phi_j \omega(j)] \\ & -\gamma^2 \omega^t(j)\omega(j) \\ = & \xi_j^t \ \widetilde{\Pi} \ \xi_j \end{aligned} \tag{6.75}$$

where $\Delta V_c(k)|_{(6.65)}$ defines the Lyapunov difference along with the solutions of system (6.65) and

$$\begin{aligned} \widetilde{\Pi}_j &= \begin{bmatrix} \widetilde{\Pi}_{oj} & \widetilde{\Pi}_{aj} & \Pi_{vj} \\ \bullet & \widetilde{\Pi}_{cj} & \Pi_{wj} \\ \bullet & \bullet & \Pi_{sj} \end{bmatrix}, \ \widetilde{\Pi}_{aj} = \begin{bmatrix} G_j^t \Phi_j & 0 & 0 \\ H_j^t \Phi_j & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \widetilde{\Pi}_{cj} &= \begin{bmatrix} -\gamma^2 I + \Phi_j^t \Phi_j & 0 & 0 \\ \bullet & -\mathcal{S} & 0 \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix} \\ \widetilde{\Pi}_o &= \begin{bmatrix} \beta \mathcal{Q} - \mathcal{P}_s + G_j^t G_j & G_j^t H_j & A_{sj}^t \\ \bullet & -\mathcal{Q} + H_j^t H_j & D_o^t \\ \bullet & \bullet & \mathcal{P}^{-1} - I \end{bmatrix} \end{aligned} \tag{6.76}$$

Using the linearizations

$$\mathcal{X}_s = \mathcal{P}_s^{-1}, \ \mathcal{W}_s = \mathcal{P}_s^{-1} \mathcal{Q} \mathcal{P}_s^{-1}, \ K_{oj} \mathcal{X}_s = \mathcal{Y}$$

along with the algebraic inequalities

$$-\mathcal{X} \mathcal{S}^{-1} \mathcal{X} \le 2\mathcal{X} - \mathcal{S}, \ -\mathcal{X} \mathcal{Z}^{-1} \mathcal{X} \le 2\mathcal{X} - \mathcal{Z}$$

we first expand $\widetilde{\Pi}_j$ via Schur complements and then apply the congruence transformation

$$[\mathcal{X}, \ \mathcal{X}, \ I, \ I, \ I, \ I, \ I, \ \mathcal{X}, \ \mathcal{X}]$$

Finally we obtain the desired LMI (6.66). This leads to

$$z_j^t z_j - \gamma^2 w_j^t w_j \ + \Delta V_j|_{(6.65)} < 0$$

for arbitrary $j \in [0, \infty)$, which implies for any $w(j) \in \ell_2(0, \infty) \neq 0$ that $J < 0$. This eventually leads to $||z_k||_2 \ < \ \gamma \ ||w_k||_2$ and hence the proof is completed. ■

Remark 6.18 In the following corollary, an interesting case is derived:

Corollary 6.19 *System (6.62) is delay-dependent asymptotically stable if there exist matrices* $0 < \mathcal{X}_s^t = \mathcal{X}_s, \ 0 < \mathcal{S}^t = \mathcal{S}, \ 0 < \mathcal{Z}^t = \mathcal{Z}, \ \mathcal{Y}_{js}, \ K_d, \ K_s$, *and a scalar*

$\gamma > 0$ *such that the following convex optimization problem is feasible*

$$\min_{\mathcal{X}_s, \mathcal{Y}_{js}} \gamma^2, \quad subject\ to$$

$$\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_{oj} & \widehat{\Pi}_{aj} & \widehat{\Pi}_{vj} \\ \bullet & \widehat{\Pi}_{cj} & \widehat{\Pi}_{wj} \\ \bullet & \bullet & \widehat{\Pi}_{z} \end{bmatrix} < 0 \tag{6.77}$$

where

$$\overline{\Pi}_{oj} = \begin{bmatrix} -\mathcal{X}_s & \mathcal{X}_s A_j^t + \mathcal{Y}_{js}^t B_j^t \\ \bullet & \mathcal{X}_s - I \end{bmatrix}, \quad \widehat{\Pi}_{aj} = \begin{bmatrix} \mathcal{X}_s G_j^t \Phi_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\Pi}_{cj} = \begin{bmatrix} -\gamma^2 I + \Phi_j^t \Phi_j & 0 & 0 \\ \bullet & -\mathcal{S} & 0 \\ \bullet & \bullet & -\mathcal{Z} \end{bmatrix} \tag{6.78}$$

Moreover, the PSD feedback gains are given by

$$K_s = \mathcal{Y}\mathcal{X}^{-1}, \quad K_d, \quad K_s$$

This can be obtained from **Theorem 6.17** *by setting* $D_o \equiv 0,\ E_d \equiv 0,\ G_{do} \equiv 0.$ *Much like the continuous-time case, there are two additional cases corresponding to proportional-summation (PS) and proportional-difference (PD) controllers. These can be readily derived by setting* K_d *and* K_s*, respectively, in all of the foregoing results*.

Illustrative Example D

Consider a linear discrete-time delay system in the form (6.45) consisting of two subsystems with the following coefficients:

$$A_1 = \begin{bmatrix} 0.7001 & 0.0002 \\ 0.0799 & 0.9505 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1501 & 0.0001 \\ -0.1001 & -0.1002 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.6001 \\ -0.5011 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.1005 \\ -0.0204 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.2010 & 0.1011 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -0.4989 & 0.3001 \end{bmatrix}, \quad \Phi_1 = 0.8005$$

$$A_2 = \begin{bmatrix} 0.7001 & 0.0002 \\ -0.0799 & 0.9001 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1401 & 0.0001 \\ -0.0401 & -0.0501 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -0.7001 \\ 0.4001 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.0805 \\ -0.0104 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.4001 & -0.1001 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.2001 & -0.3001 \end{bmatrix}, \quad \Phi_1 = 0.4001$$

Using the LMI-solver Scilab-5.1.1, the feasible solution of **Theorem 6.13** with $d_m = 2$ is given by

$$\gamma = 2.8765, \; d_M = 8$$
$$K_{o1} = \begin{bmatrix} -0.0693 & -0.0871 \end{bmatrix}, \; K_{o12} = \begin{bmatrix} -0.3015 & 0.2415 \end{bmatrix}$$

On the contrary, the feasible solution of **Theorem 6.17** is summarized by

$$\gamma = 1.1743, \; d_M = 8$$
$$K_{o1} = \begin{bmatrix} -0.0582 & -0.0883 \end{bmatrix}, \; K_{d1} = \begin{bmatrix} -0.0945 & 0.0688 \end{bmatrix}$$
$$K_{s1} = \begin{bmatrix} 0.2142 & -0.1562 \end{bmatrix}$$
$$K_{o2} = \begin{bmatrix} -0.0713 & -0.0914 \end{bmatrix}, \; K_{d2} = \begin{bmatrix} -0.0905 & 0.0935 \end{bmatrix}$$
$$K_{s2} = \begin{bmatrix} 0.2906 & -0.0181 \end{bmatrix}$$

The closed-loop state trajectories under state feedback are plotted in Fig. 6.1 and those under PSD feedback control are plotted in Fig. 6.2. It should be observed that although all the state trajectories settle down to zero level, the state trajectories under PSD feedback control are more damped with smaller time to settlement. This emphasizes the effectiveness of the PSD feedback control since it has three degrees of freedom or feedback gains.

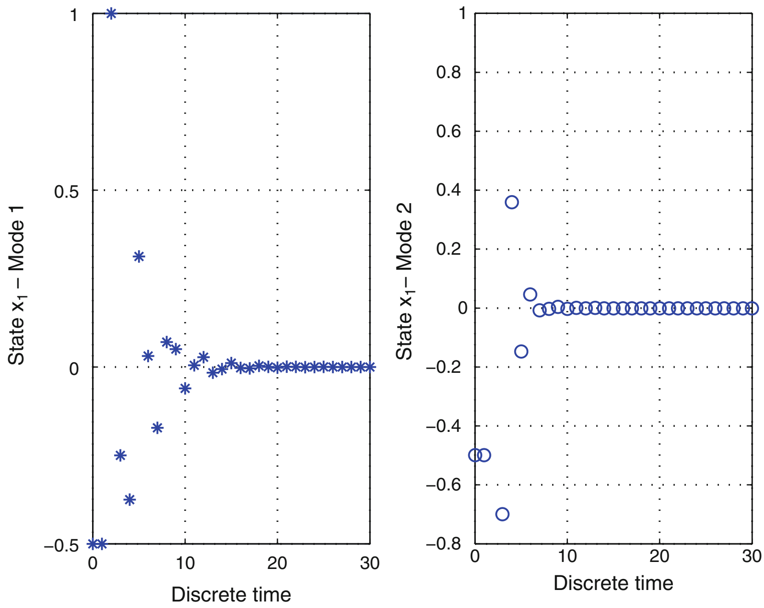

Fig. 6.1 State trajectories under state-feedback controller

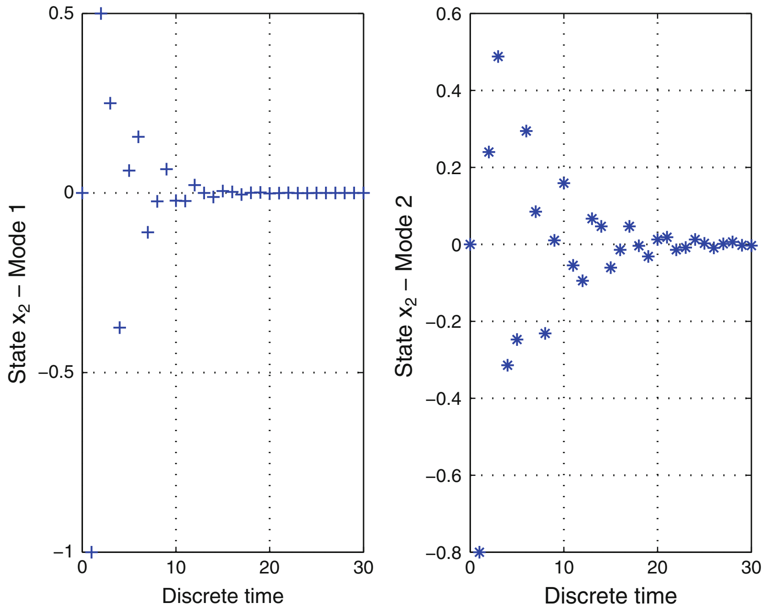

Fig. 6.2 State trajectories under PSD feedback controller

6.3 Notes and References

In view of the numerous papers and articles on switched time-delay systems, we have selected to cover in this chapter the synthesis of three-term stabilization methods. This corresponds to proportional-integral-derivative (PID) feedback in the continuous-time format and to proportional-summation-difference (PSD) feedback, respectively. The main vehicle has been the constructive use of Lyapunov–Krasovskii functional under arbitrary switching. Different stabilization and feedback control methods are examined in the next chapters.

Chapter 7
Delay-Dependent Switched Control

7.1 Continuous-Time Systems

In this chapter, we continue the discussion about delay-dependent switched feedback techniques and compare among their merits, features, and computational requirements. We pay equal attention to both continuous-time and discrete-time systems.

7.1.1 Introduction

Among the large variety of problems investigated in the literature is the stability analysis and feedback control synthesis of switched systems under arbitrary switching sequences. Recent reported results are found in [56] using multiple Lyapunov functions for nonlinear systems and in [42] employing switched Lyapunov functions. Of particular interest in this paper is the class of switched time-delay (STD) systems, which have widespread engineering applications including network control systems [170] and power systems [47].

Some theoretical studies were recently conducted for STD systems. In [425], $\mathcal{L}_2$-gain properties for a class of symmetric HTD systems are examined under arbitrary switching. In [396], the focus was on asymptotic stability conditions for STD systems. In [26–28, 31], feedback control design results are developed for discrete-time STD systems. While [23, 25] treated constant delays thereby leading to delay-independent results, the work of [370] studied the stability and $\mathcal{L}_2$-gain problems of STD systems with time-varying delays. They have not discussed feedback stabilization.

In this chapter, we focus on the robust problems of delay-dependent stability, performance analysis, and $\mathcal{H}_2/\mathcal{H}_\infty$ stabilization for STD systems under arbitrary switching as well as average-dwell time. Improved solutions to these problems in terms of feasibility testing of linear matrix inequalities (LMIs) are developed based on selective Lyapunov–Krasovskii functionals (LKFs) for linear STD systems. We consider the time-delay factor as a differentiable time-varying function satisfying some bounding relations and derive the solution for nominal and polytopic models as well as identifying several existing results as special cases. Robust control

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synthesis is used to design switched feedback schemes, based on state feedback, to guarantee that the corresponding closed-loop system enjoys the delay-dependent robust stability with an $\mathcal{L}_2$ gain smaller that of a prescribed constant level.

7.1.2 Problem Statement

We consider the following class of linear switched time-delay systems:

$$\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-\tau) + B_{\sigma(t)}u(t) + \Gamma_{\sigma(t)}w(t)\\
z(t) &= G_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) + \Phi_{\sigma(t)}w(t)\\
y(t) &= C_{\sigma(t)}x(t) + F_{\sigma(t)}u(t) + \Psi_{\sigma(t)}w(t)\\
x(\phi) &= \beta(\phi), \quad \phi \in [-\tau, 0]
\end{aligned} \tag{7.1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $w(t) \in \Re^q$ is the disturbance input which belongs to $\mathcal{L}_2[0,\infty)$, $y(t) \in \Re^p$ is the measured output, and $z(t) \in \Re^q$ is the controlled output. The matrices $A_\sigma \in \Re^{n\times n}$, $B_\sigma \in \Re^{n\times m}$, $G_\sigma \in \Re^{q\times n}$, $D_\sigma \in \Re^{q\times m}$, $F_\sigma \in \Re^{p\times m}$, $A_{d\sigma} \in \Re^{n\times n}$, $\Phi_\sigma \in \Re^{q\times q}$, $\Psi_\sigma \in \Re^{p\times q}$, $\Gamma_\sigma \in \Re^{n\times q}$ are real and known constant matrices.

Extending on [177], model (7.1) represents the continuous (state) portion of linear hybrid systems. $\sigma(t) : [0,\infty) \longrightarrow \mathcal{S} = \{1, 2, \ldots, S\}$ is the switching signal that excites a particular mode at any given time instant. It may be determined via selective procedure leading to a partition of the continuous-state space [333]. Let $\mathcal{S}$ denote the set of all selective rules. Therefore, the linear hybrid system under consideration is composed of S subsystems; each of which is activated at particular switching instant. For a switching mode $i \in \mathcal{S}$, the associated matrices are $\{A_i, \ldots, \Psi_i\}$.

Now, define the indication function

$$\xi(t) = \left[\, \xi_1(t), \ldots \xi_N(t) \,\right]^t, \quad \xi_i(t) = \begin{cases} 1, & \sigma(t) = i \\ 0, & \text{otherwise} \end{cases}$$

Then, the hybrid time-delay system (7.1) can be written as

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{S} \xi_i(t) A_i x(t) + \sum_{i=1}^{S} \xi_i(t) A_{di} x(t-\tau) + \sum_{i=1}^{S} \xi_i(t) B_i u(t)\\
&\quad + \sum_{i=1}^{S} \xi_i(t) \Gamma_i w(t)\\
y(t) &= \sum_{i=1}^{S} \xi_i(t) C_i x(t) + \sum_{i=1}^{S} \xi_i(t) D_i u(t) + \sum_{i=1}^{S} \xi_i(t) \Psi_i w(t)\\
z(t) &= \sum_{i=1}^{S} \xi_i(t) G_i x(t) + \sum_{i=1}^{S} \xi_i(t) F_i u(t) + \sum_{i=1}^{S} \xi_i(t) \Phi_i w(t)
\end{aligned} \tag{7.2}$$

We investigate the problems of delay-dependent analysis and control synthesis for a class of linear continuous-time switched systems with time-varying delays. Constructive use of switched Lyapunov functional is the main vehicle for deriving the main results. For a switching mode $i \in \mathcal{S}$, the associated matrices $\{A_i, ..., \Psi_i\}$ contain uncertainties represented by a real convex-bounded polytopic model of the type

$$\begin{bmatrix} A_i & A_{di} & B_i & \Gamma_i \\ C_i & 0 & D_i & \Psi_j \\ G_i & 0 & F_i & \Phi_i \end{bmatrix} := \left\{ \sum_{p=1}^{M_i} \lambda_{ip} \begin{bmatrix} A_{ip} & A_{di} & B_{ip} & \Gamma_{ip} \\ C_{ip} & 0 & D_{ip} & \Psi_{ip} \\ G_{ip} & 0 & D_{ip} & \Phi_{ip} \end{bmatrix}, \quad i \in \mathcal{S} \right\} \tag{7.3}$$

where $\lambda_i = (\lambda_{i1}, \lambda_{i2}, ..., \lambda_{iM_i}) \in \Lambda_i$ belongs to the unit simplex of M_i vertices

$$\Lambda_i \stackrel{\Delta}{=} \left\{ \lambda_i : \sum_{p=1}^{M_i} \lambda_{ip} = 1 \ , \ \lambda_{ik} \geq 0 \right\} \tag{7.4}$$

where $\{A_{ip}, ..., \Phi_{ip}, \ \ p = 1, ..., M_i\}$ are known real constant matrices of appropriate dimensions which describe the jth nominal subsystem.

The delay factor $\tau(t)$ in system (7.2) is time varying and continuously uniformly bounded, $\tau(t) \in [0, \tau^*]$

Remark 7.1 The state delay $\tau(t)$ appearing in the switched system dynamics (7.2) are frequently encountered in several system applications, including networked control systems, chemical processes, population dynamics, and economic systems [237]. It should be emphasized from the theory of delay differential equations that the existence of the solutions of a nonswitched linear delay system is guaranteed by a continuous and piecewise differentiable initial condition. This is carried over to linear switched delay systems since the state does not experience any jump at the switching instants.

In the absence of control input ($u(.) \equiv 0$), system (7.2) reduces to a free switched system

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{S} \xi_i(t) A_i x(t) + \sum_{i=1}^{S} \xi_i(t) A_{di} x(t-\tau) + \sum_{i=1}^{S} \xi_i(t) \Gamma_i w(t) \\ y(t) &= \sum_{i=1}^{S} \xi_i(t) C_i x(t) + \sum_{i=1}^{S} \xi_i(t) \Psi_i w(t) \\ z(t) &= \sum_{i=1}^{S} \xi_i(t) G_i x(t) + \sum_{i=1}^{S} \xi_i(t) \Phi_i w(t) \end{aligned} \tag{7.5}$$

In the sequel, we consider two performance measures:

(A) The $\mathcal{H}_2$ performance measure:

$$\mathsf{J}_2 \;=\; \left[\int_0^{\infty} y^t(s) y(s) \, \mathrm{d}s \right] \tag{7.6}$$

(B) The $\mathcal{H}_\infty$ performance measure:

$$\mathsf{J}_\infty(w) = \int_0^{\infty} [z^t(s) z(s) - \gamma^2 \; w^t(s) w(s)] \, \mathrm{d}s \tag{7.7}$$

for a prescribed scalar $\gamma > 0$. The objective of this paper is to develop delay-dependent methods for asymptotic stability and switched feedback control design of system (7.2) using the foregoing measures.

7.1.3 Delay-Dependent Stability

In this section, a model transformation will be used to exhibit the delay-dependent dynamics. We introduce the following state transformation

$$\beta(t) = x(t) + \int_{t-\tau}^{t} \sum_{i=1}^{S} \xi_i(t) A_{di} \; x(s) \mathrm{d}s \tag{7.8}$$

into (7.2) to yield

$$\begin{aligned} \dot{\beta}(t) &= \sum_{i=1}^{S} \xi_i(t) \tilde{A}_i \; x(t) + \sum_{i=1}^{S} \xi_i(t) B_i u(t) + \sum_{i=1}^{S} \xi_i(t) \Gamma_i w(t), \\ \tilde{A}_i &= A_i + A_{di} \end{aligned} \tag{7.9}$$

Define the augmented state vector $\zeta^t(t) = [\beta^t(t) \quad x^t(t)]$. By combining (7.8) and (7.9), we obtain the transformed system

$$\begin{aligned} \dot{\zeta}(t) \;&=\; \sum_{i=1}^{S} \xi_i(t) \Lambda_i \zeta(t) + \int_{t-\tau}^{t} \sum_{i=1}^{S} \xi_i(t) \Upsilon_i \; \zeta(s) \mathrm{d}s + \sum_{i=1}^{S} \xi_i(t) E_1 B_i u(t) \\ &\quad + \sum_{i=1}^{S} \xi_i(t) \bar{\Gamma}_i w(t) \\ y(t) &= \sum_{i=1}^{S} \xi_i(t) \bar{C}_i \zeta(t) + \sum_{i=1}^{S} \xi_i(t) \bar{C}_{di} \zeta(t-\tau) + \sum_{i=1}^{S} \xi_i(t) D_i u(t) \\ &\quad + \sum_{i=1}^{S} \xi_i(t) \Psi_i w(t) \end{aligned}$$

$$
\begin{aligned}
z(t) &= \sum_{i=1}^{S} \xi_i(t)\bar{G}_i\zeta(t) + \sum_{i=1}^{S} \xi_i(t)\bar{G}_{di}\zeta(t-\tau) + \sum_{i=1}^{S} \xi_i(t)F_i u(t) \\
&\quad + \sum_{i=1}^{S} \xi_i(t)\Phi_i w(t) \\
\zeta(t) &= \bar{\phi}(t),\ t \in [-2\tau, 0]
\end{aligned}
\tag{7.10}
$$

where for $i \in \mathcal{S}$

$$
\begin{aligned}
\bar{\Gamma} &= \begin{bmatrix} \Gamma_i \\ 0 \end{bmatrix},\ E_1 = \begin{bmatrix} I \\ 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 \\ I \end{bmatrix},\ \bar{G}_i^t = \begin{bmatrix} 0 \\ G_i^t \end{bmatrix},\ \bar{G}_{di}^t = \begin{bmatrix} 0 \\ G_{di}^t \end{bmatrix} \\
\Lambda_i &= \begin{bmatrix} 0 & \tilde{A}_i \\ -I & I \end{bmatrix},\ \Upsilon_i = \begin{bmatrix} 0 & 0 \\ 0 & A_{di} \end{bmatrix},\ \bar{C}_i^t = \begin{bmatrix} 0 \\ C_i^t \end{bmatrix}\ \bar{C}_{di}^t = \begin{bmatrix} 0 \\ C_{di}^t \end{bmatrix}
\end{aligned}
\tag{7.11}
$$

For convenience, we introduce the matrices

$$
\bar{\mathcal{P}}_i = U\mathcal{P}_i\ ,\ U = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ \mathcal{P}_i = \begin{bmatrix} P_{\sigma i} & P_{oi} \\ P_{di} & P_{xi} \end{bmatrix},\quad i \in \mathcal{S}
\tag{7.12}
$$

Two theorems are established in the sequel to show that the stability behavior of system (7.2) (or equivalently (7.10)) is related to the existence of a positive definite solution of a family of linear matrix inequalities (LMIs).

Theorem 7.2 *Given the delay bound $\tau^* > 0$. System (7.10) is robustly stable if there exist matrices $\{\mathcal{P}\}_i^N, \{Q\}_s^N, P_{\sigma i} > 0,\ P_{di},\ P_{xi}$ and scalar $\gamma > 0$ satisfying the system of LMIs $\forall (i, s) \in \mathcal{S}$*

$$
\begin{bmatrix}
\Pi_{ai} & \Pi_{bi} & \Pi_{ci} & \bar{G}_i^t \\
\bullet & -\tau^* Q_s & 0 & 0 \\
\bullet & \bullet & -\gamma^2 I & \Phi_i \\
\bullet & \bullet & \bullet & -I
\end{bmatrix} < 0
\tag{7.13}
$$

where

$$
\begin{aligned}
\Pi_{ai} &= \begin{bmatrix} -P_{di} - P_{di}^t & -P_{xi} + P_{di}^t + P_{\sigma i}^t \tilde{A}_i \\ \bullet & P_{xi} + P_{xi}^t + P_{oi}^t \tilde{A}_i + \tilde{A}_i^t P_{oi} + \tau Q_s \end{bmatrix}, \\
\Pi_{bi} &= \begin{bmatrix} \tau^* P_{ds}^t A_{di} \\ \tau^* P_{xs}^t A_{di} \end{bmatrix},\ \Pi_{ci} = \begin{bmatrix} P_{\sigma i}^t \bar{\Gamma}_i \\ 0 \end{bmatrix}
\end{aligned}
\tag{7.14}
$$

Proof First, we establish the asymptotic stability of system (7.2). Let the selective Lyapunov functional $V(\cdot)$ of the transformed system Σ_o be selected as

$$
V(\zeta) = V_o(\zeta) + V_d(\zeta)
$$

$$
\begin{aligned}
V_o(\zeta) &= \zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\bar{\mathcal{P}}_i\right)\zeta(t),\ \mathcal{P}_i > 0,\ i\ \in\ \mathcal{S} \\
V_d(\zeta) &= \int_{t-\tau(t)}^{t}\int_{t+\phi}^{t}\zeta^t(s)E_2\left(\sum_{i=1}^{S}\xi_i(t)\mathcal{Q}_i\right)E_2^t\zeta(s)\ \mathrm{d}\alpha\mathrm{d}s \\
&\quad \mathcal{Q}_i > 0,\ i\ \in\ \mathcal{S}
\end{aligned}
\tag{7.15}
$$

Since the weighting matrices $\mathcal{P}_i > 0,\ \ \mathcal{Q}_i > 0,$ it follows that $V(\zeta) > 0$. Now using (7.10), (7.11), and (7.12), we get

$$
\begin{aligned}
\dot{V}_o(\zeta) &= 2\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\mathcal{P}_i^t\ U\right)\dot{\zeta}(t) = 2\beta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_{\sigma i}^t\right)\dot{\beta}(t) \\
&= 2\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_i^t\right)\begin{bmatrix}\dot{\beta}(t) \\ 0\end{bmatrix} \\
&= 2\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_i^t\right)\begin{bmatrix}\sum_{i=1}^{S}\xi_i(t)(\tilde{A}_i x(t) + \Gamma_i w(t)) \\ -\beta(t) + x(t) + \int_{t-\tau}^{t}\sum_{i=1}^{S}\xi_i(t)A_{di}\ x(s)\mathrm{d}s\end{bmatrix} \\
&= 2\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_i^t\Lambda_i\right)\zeta(t) + 2\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_i^t\bar{\Gamma}_i\right)w(t) \\
&\quad + 2\zeta^t(t)\int_{t-\tau}^{t}\left(\sum_{i=1}^{S}\xi_i(t)\ \mathcal{P}_i^t\Upsilon_i\right)\zeta(\theta)\ \mathrm{d}\theta \\
\dot{V}_d(\zeta) &= \tau\ \zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)E_2\mathcal{Q}_i E_2^t\right)\zeta^t(t) \\
&\quad - \int_{t-\tau}^{t}\zeta^t(t)\left(\sum_{i=1}^{S}\xi_i(t)E_2\mathcal{Q}_i E_2^t\right)\zeta(t)\ \mathrm{d}\theta
\end{aligned}
\tag{7.16}
$$

Under arbitrary switching [42], it follows for any nonzero vector $x(t)$ that a particular case is $\xi_i(t) = 1,\ \xi_{m\neq i}(t) = 0,\ \xi_s(t-\tau) = 1$, and $\xi_{m\neq s}(t-\tau) = 0$. Therefore, with some algebraic manipulations, it follows from (7.7) and (7.15), and (7.16) that

$$
\begin{aligned}
&\dot{V}(\zeta) + z^t(t)z(t) - \gamma^2 w^t(t)w(t) \\
&\quad = 2\zeta^t(t)\Lambda_i^t\mathcal{P}_i\zeta(t) + 2\zeta^t(t)\mathcal{P}_i^t\bar{\Gamma}_i w(t) \\
&\qquad + \tau\ \zeta^t(t)E_2\mathcal{Q}_s E_2^t\zeta^t(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathcal{P}_s^t\Upsilon_i\zeta(\theta)\ \mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{t-\tau}^{t} \zeta^t(\theta) E_2 \mathcal{Q}_s E_2^t \zeta(\theta)\, \mathrm{d}\theta - \gamma^2\, w^t(s) w(s) \\
&+ \Big[\bar{G}_i \zeta(t) + \Phi_i w(t)\Big]^t \Big[\bar{G}_i \zeta(t) + \Phi_i w(t)\Big]
\end{aligned}
\tag{7.17}
$$

Using the algebraic inequality $2\rho^t \pi \;\le\; \rho^t\ \Psi\ \rho + \pi^t\ \Psi^{-1}\ \pi$ for any real vectors π, ρ, and any matrix $\Psi^t = \Psi > 0$ with appropriate dimensions, we have

$$
\begin{aligned}
&2 \int_{t-\tau}^{t} \zeta^t(t) \mathcal{P}_s^t \Upsilon_s \zeta(\theta)\, \mathrm{d}\theta = 2 \int_{t-\tau}^{t} \zeta^t(t) \mathcal{P}_s^t E_2 A_{di} x(\theta)\, \mathrm{d}\theta \\
&\le \tau \zeta^t(t) \mathcal{P}_s^t E_2 A_{di} \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}_s^t \zeta(t) + \int_{t-\tau}^{t} x^t(s) \mathcal{Q}_s x(s)\, \mathrm{d}s \\
&< \tau \zeta^t(t) \mathcal{P}_s^t E_2 A_{di} \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}_s \zeta(t) + \int_{t-\tau}^{t} \zeta^t(t) E_2 \mathcal{Q}_s E_2^t \zeta(\theta)\, \mathrm{d}\theta
\end{aligned}
\tag{7.18}
$$

In terms of $\xi^t = [\zeta^t \quad w^t]$, it follows from (7.17) and (7.18) on using Schur complements that

$$
\begin{aligned}
&\dot{V}(\zeta) + z^t(t) z(t) - \gamma^2 w^t(t) w(t) \\
&\quad < 2\zeta^t(t) \Lambda_i^t \mathcal{P}_i \zeta(t) + 2\zeta^t(t) \mathcal{P}_i^t \bar{\Gamma}_i w(t) + \tau\ \zeta^t(t) E_2 \mathcal{Q}_s E_2^t \zeta^t(t) \\
&\quad + \Big[\bar{G}_i \zeta(t) + \Phi_i w(t)\Big]^t \Big[\bar{G}_i \zeta(t) + \Phi_i w(t)\Big] \\
&\quad + \tau \zeta^t(t) \mathcal{P}_s^t E_2 A_{di} \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}_s \zeta(t) - \gamma^2\ w^t(s) w(s) \\
&\quad = \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix}^t \Omega_i \begin{bmatrix} \zeta(t) \\ w(t) \end{bmatrix} \quad \xi^t\ \Omega_i\ \xi \\
&\Omega_i = \begin{bmatrix} \Xi_i + \bar{G}_i^t \bar{G}_i & \mathcal{P}_i^t \bar{\Gamma}_i + \bar{G}_i^t \Phi_i \\ \bullet & -\gamma^2 I + \Phi_i^t \Phi_i \end{bmatrix} \\
&\Xi_i = \Lambda_i^t \mathcal{P}_i + \mathcal{P}_i^t \Lambda_i + \tau\ E_2 \mathcal{Q}_s E_2^t + \tau \mathcal{P}_s^t E_2 A_{di} \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}_s
\end{aligned}
\tag{7.19}
$$

Thus, it follows that $\mathsf{J}_\infty(w) < 0$ ([27], pp. 91) if the following LMI holds:

$$
\begin{bmatrix}
\Lambda_i^t \mathcal{P}_i + \mathcal{P}_i^t \Lambda_i + \tau\ E_2 \mathcal{Q}_s E_2^t & \tau \mathcal{P}_s^t E_2 A_{di} & \mathcal{P}_i^t \bar{\Gamma}_i & \bar{G}_i^t \\
\bullet & -\tau \mathcal{Q}_s & 0 & 0 \\
\bullet & \bullet & -\gamma^2 I & \Phi_i \\
\bullet & \bullet & 0 & -I
\end{bmatrix} < 0
\tag{7.20}
$$

By taking $w(t) \equiv 0$, $\Gamma \equiv 0$, $\Phi \equiv 0$, we obtain the following inequality from (7.20)

$$
\dot{V}(\zeta) < \zeta^t(t) \Big[\Lambda_i^t \mathcal{P}_i + \mathcal{P}_i^t \Lambda_i + \tau\ E_2 \mathcal{Q}_s E_2^t + \bar{G}_i^t \bar{G}_i
$$

$$+ \tau \mathcal{P}_s^t E_2 A_{di} \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}_s \Big] \zeta(t) \tag{7.21}$$

whose right side is always negative under (7.20). Thus we conclude that $\dot{V}(\zeta) < 0$ for all $\zeta \neq 0$. This implies that $x \to 0$ as $t \to \infty$. By Schur complements to LMI (7.20) and using (7.10), (7.11), and (7.12) we obtain LMIs (7.13) and (7.14) for all $\tau \leq \tau^*$ and hence system (7.10) is asymptotically stable with disturbance attenuation $\gamma > 0$. ■

Remark 7.3 It is known [237] that the descriptor model addressed here is a slow-type state-transformation where the dynamics of x is faster than σ in the augmented vector ζ and the relative dynamics of state components are implicit in the analysis. This is in contrast of the descriptor approach of [66] which is a fast-type state transformation employed for nonswitching systems. We note that the application of the algebraic inequality $2\rho^t \pi \leq \rho^t \Psi \rho + \pi^t \Psi^{-1} \pi$ has not introduced additional matrix variables. Alternatively, methods based on the use of modified inequality like [324, 370] hinges on the incorporation of an extra term in the Lyapunov–Krasovskii functional plus the introduction of three additional matrix variables and a nonstrict matrix inequality. In this regard, the computational load utilizing these methods would generally be costlier.

Illustrative Example A

To demonstrate the advantages of the new transformation, consider a switched systems composed of two time-delay models given by

$$A_1 = \begin{bmatrix} -1 & 0.5 \\ -0.5 & -1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_1^t = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad \Phi_1 = 0.4$$

$$A_2 = \begin{bmatrix} -1.1 & 0.4 \\ -0.4 & -0.9 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_2^t = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \quad \Phi_2 = 0.6$$

Table 7.1 summarizes the computational results of different pairs (γ, τ^*) from which it is clear that as τ^* is decreased, the minimum value of γ rendering feasible solutions increases.

Table 7.1 Results of illustrative example A

Method	γ	τ^*
Theorem 7.2	1.469	0.325
	2.856	0.271
	3.473	0.228

Illustrative Example B

A switched continuous time-delay model consists of two identical models with the data coefficient given by

$$
\begin{aligned}
& i = 1, 2 \\
& A_i = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{di} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0.2 \\ 1.3 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \\
& C_i = [1 \quad 0] \;, \; G_i = [0.1 \quad 0.1] \;, \; D_i = 0.4 \;, \; F_o = 0.6 \;, \; \Phi = 0.5
\end{aligned}
$$

Application of **Theorem 7.2** shows that this switched system is asymptotically stable with disturbance level $\gamma = 3.247$ for all $\tau \leq 2.642$.

7.1.4 State-Feedback Design

We now exploit the results developed in the foregoing section to design a delay-dependent switched-state feedback controller based on different criteria. Common to the design methods is the use of the feedback law

$$
\begin{aligned}
u(t) = \sum_{i=1}^{S} \xi_i(t)\; K_i x(t) = [0 \qquad \sum_{i=1}^{S} \xi_i(t)\; K_i] \zeta(t) \\
\stackrel{\Delta}{=} \sum_{i=1}^{S} \xi_i(t)\; \bar{K}_i \zeta(t)
\end{aligned}
\tag{7.22}
$$

to derive the closed-loop system

$$
\begin{aligned}
(\Sigma_k): \quad \dot{\zeta}(t) \; &= \; \sum_{i=1}^{S} \xi_i(t)\; \Lambda_{ki} \zeta(t) + \int_{t-\tau}^{t} \sum_{i=1}^{S} \xi_i(t)\; \Upsilon_i \; \zeta(s) \mathrm{d}s \\
&+ \sum_{i=1}^{S} \xi_i(t)\; \bar{\Gamma}_i w(t) \\
y(t) = \sum_{i=1}^{S} \xi_i(t)\; \bar{C}_{ki} \zeta(t), &\quad z(t) = \sum_{i=1}^{S} \xi_i(t)\; \bar{G}_{ki} \zeta(t) \\
&+ \sum_{i=1}^{S} \xi_i(t)\; \Phi_i w(t) \\
\zeta(t) = \bar{\phi}(t), \; t \in [-2\tau, 0]
\end{aligned}
\tag{7.23}
$$

where

$$\begin{aligned}\Lambda_{ki} &= \begin{bmatrix} 0 & \tilde{A}_i + B_i K_i \\ -I & I \end{bmatrix} = \Lambda_i + E_1 B_i \bar{K}_i, \\ \bar{C}_{ki} &= \bar{C}_i + D_i \bar{K}, \ \ \bar{G}_k = \bar{G}_i + F_i \bar{K}_i \end{aligned} \tag{7.24}$$

For convenience, we introduce the block matrix

$$\mathcal{Y}_i = \mathcal{P}_i^{-1} = \begin{bmatrix} \mathcal{Y}_{\sigma i} & \mathcal{Y}_{oi} \\ \mathcal{Y}_{di} & \mathcal{Y}_{xi} \end{bmatrix} \tag{7.25}$$

7.1.5 $\mathcal{H}_\infty$ Feedback Design

By considering the $\mathcal{H}_\infty$ performance (7.7), the following theorems summarize the main results:

Theorem 7.4 *Given the delay bound $\tau^* > 0$. The $\mathcal{H}_\infty$ state-feedback controller (7.22) renders system (Σ_k) asymptotically stable with a disturbance attenuation level γ for all $w(t) \in \mathcal{L}_2[0, \infty)$ if there exist matrices $\{\mathcal{Y}\}_1^N$, $\{\bar{\mathcal{Q}}\}_1^N$, $\{\mathcal{Z}\}_1^N$, $\{\mathcal{L}\}_1^N$ satisfying the system of LMIs*

$$\begin{bmatrix} \Omega_{ai} & \Omega_{bi} & \Omega_{ci} & \Omega_{di} & \mathcal{Y}_{di}^t \\ \bullet & -\mathcal{Q}_s & 0 & 0 & \mathcal{Y}_{xi}^t \\ \bullet & \bullet & -\gamma^2 I & \Phi_i & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{Q}_s \end{bmatrix} < 0 \tag{7.26}$$

subject to

$$\begin{aligned} \begin{bmatrix} (\mathcal{L}_i + \mathcal{L}_i^t)/2 & (\mathcal{Z}_i + \mathcal{Y}_{xi}^t)/2 \\ \bullet & (\mathcal{Y}_{di} + \mathcal{Y}_{di}^t)/2 \end{bmatrix} &\geq 0 \\ \begin{bmatrix} (\mathcal{L}_i - \mathcal{L}_i^t)/2 & (\mathcal{Z}_i - \mathcal{Y}_{xi}^t)/2 \\ \bullet & (\mathcal{Y}_{di} - \mathcal{Y}_{di}^t)/2 \end{bmatrix} &\geq 0 \end{aligned} \tag{7.27}$$

where

$$\begin{aligned} \Omega_{ai} &= \begin{bmatrix} \mathcal{Y}_{di} \tilde{A}_i^t + \tilde{A}_i \mathcal{Y}_{di} + B_i \mathcal{Z}_i + \mathcal{Z}_i^t B_i^t & -\mathcal{Y}_{\sigma i}^t + \mathcal{Y}_{di}^t + \tilde{A}_i \mathcal{Y}_{xi} + B_i \mathcal{L}_i \\ \bullet & \mathcal{Y}_{xi}^t + \mathcal{Y}_{xi} - \mathcal{Y}_{oi}^t - \mathcal{Y}_{oi} \end{bmatrix} \\ \Omega_{di} &= \begin{bmatrix} \mathcal{Y}_{di}^t G_i^t + \mathcal{Z}_i^t F_i^t \\ \mathcal{Y}_{xi}^t G_i^t + \mathcal{L}_i^t F_i^t \end{bmatrix}, \quad \Omega_{bi} = \begin{bmatrix} 0 \\ \tau A_{di} \mathcal{Q}_s \end{bmatrix}, \quad \Omega_{ci} = \begin{bmatrix} \Gamma_i \\ 0 \end{bmatrix} \end{aligned} \tag{7.28}$$

The switched $\mathcal{H}_\infty$-controller feedback gain is given by $K_i = \mathcal{Z}_i \mathcal{Y}_{di}^{-1}$, $i \in \mathcal{S}$

Proof It follows from **Theorem 7.2** that system (Σ_k) is asymptotically stable with a disturbance attenuation level γ if the following LMI

$$\begin{bmatrix} \Lambda_{ki}^t \mathcal{P}_i + \mathcal{P}_i^t \Lambda_{ki} + \tau\ E_2 \mathcal{Q}_s E_2^t & \tau \mathcal{P}_s^t E_2 A_{di} & \mathcal{P}_i^t \bar{\Gamma}_i & \bar{G}_{ki}^t \\ \bullet & -\tau \mathcal{Q}_s & 0 & 0 \\ \bullet & \bullet & -\gamma^2 I & \Phi_i \\ \bullet & \bullet & 0 & -I \end{bmatrix} < 0 \qquad (7.29)$$

has a feasible solution. Using (7.24) and (7.25) and applying the congruence transformation

$$\mathbb{C} = \text{diag}[\mathcal{P}_i^{-1},\ I,\ I,\ I],\ \mathcal{P}_i^{-1} = \mathcal{Y}_i,\ i \in \mathcal{S}$$

under arbitrary switching, LMI (7.29) becomes

$$\begin{bmatrix} \mathcal{Y}_i^t \Lambda_{ki}^t + \Lambda_{ki} \mathcal{Y}_i + \tau\ \mathcal{Y}_i^t E_2 \mathcal{Q}_s E_2^t \mathcal{Y}_i & \tau E_2 A_{di} & \bar{\Gamma}_i & \mathcal{Y}_i^t \bar{G}_{ki}^t \\ \bullet & -\tau \mathcal{Q}_s & 0 & 0 \\ \bullet & \bullet & -\gamma^2 I & \Phi_i \\ \bullet & \bullet & 0 & -I \end{bmatrix} < 0 \quad (7.30)$$

Defining $\bar{\mathcal{Q}}_s = \mathcal{Q}_s^{-1}$ allows us to deal with the quadratic terms like $Y_{xi}^t \mathcal{Q}_s^{-1} Y_{xi}$ via Schur complements. Then introducing the linearizations $K_i \mathcal{Y}_{xi} = \mathcal{L}_i$ and $K_i \mathcal{Y}_{di} = \mathcal{Z}_i$, which constrain the choice of $\mathcal{L}_i$ and $\mathcal{Z}_i$ via the inequality

$$\begin{bmatrix} \mathcal{L}_i & \mathcal{Z}_i \\ \mathcal{Y}_{xi} & \mathcal{Y}_{di} \end{bmatrix} \geq 0$$

to limit the selection of the gain K_i to single value. To put the foregoing inequality in a standard LMI, we express it in the form (7.27). Finally, using (7.11) and (7.25) with some standard manipulations, we readily obtain LMIs (7.26) – subject to (7.28). ■

Remark 7.5 Indeed, there are other ways to handle the multiple values of the state-feedback gain through the introduction of relaxation variables, invoking bilinear matrix inequalities or iterative LMI procedure. It is felt however that the imposed constraint (7.27) provides less conservative results.

7.1.6 $\mathcal{H}_2$ Feedback Design

By considering the $\mathcal{H}_2$ performance (7.6), the following theorems summarize the main results:

Theorem 7.6 *Given the delay bound $\tau^* > 0$. In the absence of input disturbance $w(t) \equiv 0$, the switched state-feedback controller (7.22) is an $\mathcal{H}_2$-optimal controller for system (Σ_k) if there exist matrices $\{\mathcal{Y}\}_1^N,\ \{\mathcal{Q}\}_1^N,\ \{\mathcal{Z}\}_1^N,\ \{\mathcal{L}\}_1^N$ satisfying the system of LMIs*

$$\begin{bmatrix} \Sigma_{ai} & \Sigma_{bi} & \Sigma_{fi} \\ \bullet & -\mathcal{Q}_s & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.31}$$

subject to

$$\begin{bmatrix} (\mathcal{L}_i + \mathcal{L}_i^t)/2 & (\mathcal{Z}_i + \mathcal{Y}_{xi}^t)/2 \\ \bullet & (\mathcal{Y}_{di} + \mathcal{Y}_{di}^t)/2 \end{bmatrix} + \begin{bmatrix} (\mathcal{L}_i - \mathcal{L}_i^t)/2 & (\mathcal{Z}_i - \mathcal{Y}_{xi}^t)/2 \\ \bullet & (\mathcal{Y}_{di} - \mathcal{Y}_{di}^t)/2 \end{bmatrix} \geq 0 \tag{7.32}$$

where

$$\Sigma_{ai} = \begin{bmatrix} \mathcal{Y}_{di}\tilde{A}_i^t + \tilde{A}_i\mathcal{Y}_{di} + B_i\mathcal{Z}_i + \mathcal{Z}_i^t B_i^t & -\mathcal{Y}_{\sigma i}^t + \mathcal{Y}_{di}^t + \tilde{A}_i\mathcal{Y}_{xi} + B_i\mathcal{L}_i \\ \bullet & \mathcal{Y}_{xi}^t + \mathcal{Y}_{xi} - \mathcal{Y}_{oi}^t - \mathcal{Y}_{oi} \end{bmatrix}$$
$$\Sigma_{bi} = \begin{bmatrix} 0 \\ \tau A_{di}\mathcal{Q}_s \end{bmatrix}, \quad \Sigma_{fi} = \begin{bmatrix} \mathcal{Y}_{di}^t C_i^t + \mathcal{Z}_i^t D_i^t \\ \mathcal{Y}_{xi}^t C_i^t + \mathcal{L}_i^t D_i^t \end{bmatrix} \tag{7.33}$$

The $\mathcal{H}_2$-controller feedback gain is given by $K_i = \mathcal{Z}_i\mathcal{Y}_{di}^{-1}, \; i \in \boldsymbol{N}$. An upper bound on the $\mathcal{H}_2$ performance measure is given by

$$\begin{aligned} \mathsf{J}_2 \;\leq\; \mathsf{J}^+ \stackrel{\Delta}{=} \Bigg[& \zeta^t(0) \sum_{i=1}^{S} \xi_i(t)\, \bar{\mathcal{P}}_i \zeta(0) \\ & + \tau \int_{-\tau}^{0} \zeta^t(s) E_2 \sum_{s=1}^{N} \xi_s(t)\, \mathcal{Q}_s E_2^t \zeta(s)\, \mathrm{d}s \Bigg]^{1/2} \end{aligned} \tag{7.34}$$

Proof To establish the system stability, we consider the Lyapunov functional (7.14) for system (Σ_k) with $w \equiv 0$. As a consequence of **Theorem 7.2**, it is not difficult to see that

$$\dot{V}(\zeta) \;<\; \zeta^t(t)\Big[\Lambda_{ki}^t\mathcal{P}_i + \mathcal{P}_i^t\Lambda_{ki} + \tau\; E_2\mathcal{Q}_s E_2^t + \tau\mathcal{P}^t E_2 A_d \mathcal{Q}_s^{-1} A_{di}^t E_2^t \mathcal{P}\Big]\zeta(t) \tag{7.35}$$

For asymptotic stability and since $\bar{C}_{ki}^t \bar{C}_{ki} \geq 0$, it is sufficient that

$$\begin{bmatrix} \Lambda_{ki}^t\mathcal{P} + \mathcal{P}^t\Lambda_{ki} + \tau\; E_2\mathcal{Q}_s E_2^t & \tau\mathcal{P}^t E_2 A_{di} & \bar{C}_{ki}^t \\ \bullet & -\mathcal{Q}_s & 0 \\ \bullet & \bullet & -I \end{bmatrix} \;<\; 0 \tag{7.36}$$

and thus we conclude that $\dot{V}(\zeta) < 0$ for all $\zeta \neq 0$ and $\dot{V}(\zeta) \leq 0$ for all ζ. Now, by applying the congruence transformation $\mathfrak{C} = diag\Big[\mathcal{P}_i^{-1}, \;\; I, \;\; I, \;\; I\Big], \; \mathcal{P}_i^{-1} = \mathcal{Y}_i$

to (7.36) along with the linearizations $K_i \mathcal{Y}_{xi} = \mathcal{L}_i,\ K_i \mathcal{Y}_{di} = \mathcal{Z}_i$ and using (7.11) and (7.25), we readily obtain LMI (7.31) subject to (7.28) and (7.33).

Next, we examine the $\mathcal{H}_2$ performance. By considering (7.35) and (7.36), it follows that

$$\dot{V}(\zeta) \;<\; -y^t(t)y(t)$$

Integrating both sides over the range $[0, t_f]$, we obtain

$$\begin{aligned} -\int_0^{t_f} y^t(r)y(r)\mathrm{d}r &> \zeta^t(t_f)\sum_{i=1}^{S}\xi_i(t)\ \bar{\mathcal{P}}_i\zeta(t_f) - \zeta^t(0)\sum_{i=1}^{S}\xi_i(t)\ \bar{\mathcal{P}}_i\zeta(0) \\ &+ \int_{t_f-\tau}^{t_f}\int_{-\theta}^{t_f}\zeta^t(s)E_2\sum_{s=1}^{N}\xi_s(t)\ \mathcal{Q}_s E_2^t\zeta(s)\ \mathrm{d}s\mathrm{d}\theta \\ &+ \int_{-\tau}^{0}\int_{-\theta}^{0}\zeta^t(s)E_2\sum_{s=1}^{N}\xi_s(t)\ \mathcal{Q}_s E_2^t\zeta(s)\ \mathrm{d}s\mathrm{d}\theta \end{aligned} \tag{7.37}$$

In view of the asymptotic stability of system (Σ_k) and letting $t_f \longrightarrow \infty$, we have

$$\zeta^t(t_f)\sum_{i=1}^{S}\xi_i(t)\ \bar{\mathcal{P}}_i\zeta(t_f) \longrightarrow 0$$

$$\int_{t_f-\tau}^{t_f}\int_{-\theta}^{t_f}\zeta^t(s)E_2\sum_{s=1}^{N}\xi_s(t)\ \mathcal{Q}_s E_2^t\zeta(s)\ \mathrm{d}s\mathrm{d}\theta \longrightarrow 0$$

and therefore (7.37) reduces to

$$\begin{aligned} \int_0^{\infty} y^t(r)y(r)\mathrm{d}r &\le \zeta^t(0)\sum_{i=1}^{S}\xi_i(t)\ \bar{\mathcal{P}}_i\zeta(0) \\ &\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t}\zeta^t(s)E_2\sum_{s=1}^{N}\xi_s(t)\ \mathcal{Q}_s E_2^t\zeta(s)\ \mathrm{d}s\mathrm{d}\theta \Longrightarrow \\ ||y||_2 &\le \Bigg[\zeta^t(0)\sum_{i=1}^{S}\xi_i(t)\ \bar{\mathcal{P}}_i\zeta(0) \\ &\quad + \tau\int_{-\tau}^{0}\zeta^t(s)E_2\sum_{s=1}^{N}\xi_s(t)\ \mathcal{Q}_s E_2^t\zeta(s)\ \mathrm{d}s\Bigg]^{1/2} \\ &= \mathsf{J}^+ \end{aligned} \tag{7.38}$$

and the proof is completed. ■

Remark 7.7 Had we considered the delayed-state feedback-control law

$$u(t) = \sum_{i=1}^{S} \xi_i(t)\ K_{di}\ x(t-\tau)$$

we would have used the state transformation as

$$\beta(t) = x(t) + \int_{t-\tau}^{t} \sum_{i=1}^{S} \xi_i(t)\ [A_{di} + B_i K_{di}]\ x(s)\mathrm{d}s \tag{7.39}$$

On substituting (7.39) into (7.1), it yields

$$\dot{\beta}(t) = \sum_{i=1}^{S} \xi_i(t)\ [\tilde{A}_i + B_i K_{di}]\ x(t) + \sum_{i=1}^{S} \xi_i(t)\ \Gamma_i w(t) \tag{7.40}$$

A little algebra gives the transformed system:

$$\begin{aligned}
(\Sigma_{dk}):\quad \dot{\zeta}(t) \ &= \ \sum_{i=1}^{S} \xi_i(t)\ \Lambda_{dki}\zeta(t) + \int_{t-\tau}^{t} \sum_{i=1}^{S} \xi_i(t)\ \Upsilon_i\ \zeta(s)\mathrm{d}s \\
&\quad + \sum_{i=1}^{S} \xi_i(t)\ \bar{\Gamma} i w(t) \\
z(t) &= \sum_{i=1}^{S} \xi_i(t)\ \bar{G}_i \zeta(t) + \sum_{i=1}^{S} \xi_i(t)\ \Phi_i w(t) \\
\zeta(s) &= \bar{\kappa}(s),\ \ s \in [-2\tau, 0]
\end{aligned} \tag{7.41}$$

where

$$\Lambda_{dki} = \begin{bmatrix} 0 & \tilde{A}_i + B_i K_{di} \\ -I & I \end{bmatrix} \tag{7.42}$$

Taking into account the matrices of (7.24) and (7.25), we could have established a result parallel to that of **Theorems 7.4** and **7.6** by using $K_i \to K_{di},\ i \in \mathcal{S}$. It is thus not surprising to find that the results on both instantaneous and delayed state feedback are equivalent. This, in fact, strengthens the state transformations (7.8) or (7.39) as vehicles to derive pertinent delay-dependent dynamic models.

7.1.7 Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ Design

Extending on the foregoing results, we consider below the simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ control design problem [161, 168], which can be formally phrased as

Determine a state-feedback controller that achieves the minimization of $\mathcal{H}_2$ performance measure (7.5) and satisfying an $\mathcal{H}_\infty$ norm bound within a scalar γ.

Technically, the control objective is to minimize the output energy of $y(t)$ satisfying the prescribed $\mathcal{H}_\infty$ norm bound of the feedback system from $w(t)$ to $z(t)$. A solution to the delay-dependent simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ control for switched time-delay systems is established by the following theorem:

Theorem 7.8 *Given the delay bound $\tau^* > 0$ and a prescribed constant $\gamma > 0$. The switched state-feedback controller (7.22) with gain matrix $K_i = \mathcal{Z}_i \mathcal{Y}_{di}^{-1}, \ i \in \mathcal{S}$ is a simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ controller satisfying the performance measure (7.7) for system (Σ_k) if there exist matrices $\{\mathcal{Y}\}_1^N, \ \{\mathcal{Q}\}_1^N, \ \{\mathcal{Z}\}_1^N, \ \{\mathcal{L}\}_1^N, \ \mathcal{W}, \ \mathcal{X}, \ \{\mathcal{R}\}_1^N$ such that the system of generalized eigenvalue problem*

$$\min \Big[\lambda + Tr(\mathcal{W}) \Big]$$

$$\text{subject to LMIs (7.26), (7.27), and (7.28), (7.31), (7.32), and (7.33)}$$

$$\begin{bmatrix} -\mathcal{W} & \mathcal{X}^t \\ \bullet & \mathcal{R}_s \end{bmatrix} < 0 \ , \ \begin{bmatrix} -\lambda & \bar{\phi}^t(0)E_1 \\ \bullet & \mathcal{Y}_{xi} \end{bmatrix} < 0, \quad \forall (i,s) \in \mathcal{S} \tag{7.43}$$

has a feasible solution

Proof On observing that

$$\zeta^t(0) \sum_{i=1}^{S} \xi_i(t) \ \bar{\mathcal{P}}_i \zeta(0) \ \stackrel{\Delta}{=} \ \lambda \Longrightarrow$$

$$-\lambda + \bar{\phi}^t(0)E_1 \sum_{i=1}^{S} \xi_i(t) \ \mathcal{Y}_{\sigma i}^{-1} E_1^t \bar{\phi}(0) \ < \ 0 \tag{7.44}$$

and in similar way using the cyclic properties of matrix trace

$$\begin{aligned} &\int_{-\tau}^{0} \zeta^t(s) E_2 \sum_{i=1}^{S} \xi_s(t) \ \mathcal{Q}_s E_2^t \zeta(s) \mathrm{d}s \\ &= \int_{-\tau(0)}^{0} Tr \Big[x^t(s) \sum_{i=1}^{S} \xi_s(t) \ \mathcal{Q}_s x(s) \Big] \mathrm{d}s \\ &= Tr \Big[\mathcal{X}\mathcal{X}^t \sum_{i=1}^{S} \xi_s(t) \ \mathcal{Q}_s \Big] \end{aligned}$$

$$= Tr\left[\mathcal{X}^t \sum_{i=1}^{S} \xi_s(t)\, \mathcal{R}_s^{-1}\mathcal{X}\right] < Tr(\mathcal{W}) \Longrightarrow$$

$$-\mathcal{W} + \mathcal{X}^t \sum_{i=1}^{S} \xi_s(t)\, \mathcal{R}_s^{-1}\mathcal{X} < 0 \tag{7.45}$$

where $\mathcal{X}\mathcal{X}^t = \int_{-\tau(0)}^{0} x^t(s)x(s)\mathrm{d}s$. Under arbitrary switching and utilizing the results of **Theorems 7.4** and **7.6** and achieving the objective of simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ control leads to the minimization of $\left[\lambda + Tr(W)\right]$ subject to LMIs (7.26), (7.27), and (7.28) and (7.31), (7.32), and (7.33). Relations (7.44) and (7.45) are expressed by LMI (7.43), which completes the proof. ■

Illustrative Example C

The following switched time-delay model is considered for state-feedback design:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$G_1 = [1 \quad 0],\ D_1 = 0.4\ ,\ F_1 = 0.2\ ,\ \Phi_1 = 0.4$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$G_2 = [1 \quad 0],\ D_2 = 0.6\ ,\ F_2 = 0.4\ ,\ \Phi_2 = 0.6$$

In Table 7.2, we provide the results of some methods to compute the state feedback gain

It is interesting to see that the computational results of Table 7.2 are in full agreement with those of the foregoing examples.

Table 7.2 Results of Example 4.1

Method	γ	τ^*	K_i	
Theorem 7.4	2.201	1.882	−0.1045	−1.591
			−0.2284	−1.824
Theorem 7.4	1.671	2.648	−3.435	−0.764
			−1.318	−1.753
Theorem 7.6	1.823	1.963	−3.358	−1.024
			−1.273	−1.842
Theorem 7.6	1.588	2.745	−2.667	−1.405
			−2.143	−1.426
Theorem 7.8	2.051	1.875	−1.462	−1.981
			−1.534	−1.955
Theorem 7.8	1.444	3.125	−1.502	−1.977
			−1.604	−1.988

Remark 7.9 The results developed in this paper can be extended in a straightforward manner to the case of linear time-delay systems with real convex polytopic uncertainties of the form:

$$\begin{bmatrix} A_o & A_d & B_o & C_o & G_o \end{bmatrix} \in \Xi \triangleq \Big\{ \begin{bmatrix} A_o(\alpha) & A_d(\alpha) & B_o(\alpha) & C_o(\alpha) & G_o(\alpha) \end{bmatrix}$$
$$= \sum_{j=1}^{\upsilon} \alpha_j \begin{bmatrix} A_{oj} & A_{dj} & B_{oj} & C_{oj} & G_{oj} \end{bmatrix}, \sum_{j=1}^{\upsilon} \alpha_j = 1 \ , \ \alpha_j \geq 0 \Big\}$$

by simply developing the different theorems at each of the υ vertices and solving the resulting system of LMIs. Along similar lines, dynamic output feedback controllers could be designed. Research results on these topics and others will be reported shortly.

7.2 Discrete-Time Systems

Switched models containing continuous and discrete states that affect their dynamic behavior are frequently occurring in many physical systems, including a variety of power systems [389], chemical processes [41], and mechanical systems [47]. A wide class of switched systems composed of several discrete subsystems and a rule that governs the switching between these subsystems has received great attention in the past decade because of the fast development in computing technologies, which helped improve the efficiency of switching between systems or controllers [28, 42, 47, 174, 193] and the references cited therein.

Delay-dependent methods are usually developed to take the information about time delay into consideration in the process of controller design. By and large, *delay-dependent* methods are regarded as more practical and yield less conservative designs. Results pertaining to discrete-time systems with state delay are found in [25, 159, 213, 273, 345, 393] for nonswitching systems and in [41, 50, 217, 432, 437, 439] for topics on classes of switching systems.

7.2.1 Introduction

In [368], a descriptor-transformation plus a free-weighting equation are employed to study a class of linear switched discrete-time systems with mode-dependent bounded delays. Delay-independent LMI-based stability and state-feedback stabilization conditions are derived in [298]. Robust $\mathcal{H}_\infty$ control and stabilization based on multiple Lyapunov functions is provided in [149] for a class of linear switched continuous-time systems. The results of [50] extended the method of [368] to time-varying delays and bypassed the shortcomings of the descriptor transformation; however, the solution conditions are expressed into nonstrict LMIs. Later on in [372], the stability analysis was undertaken for a class of differentiable time-delay functions thereby generalized the results of [50, 368]. However, employing a bounding inequality paves the way to ample extensions. In [169], the focus was on formulating a switching rule to stabilize a given switched system with time delay using a common Lyapunov functional method. By using a suitable discretization, the work of [138] converted a linear switched continuous-time system into a linear switched delay-dependent discrete-time system and hence developed a robust control synthesis. Recent developments can be found in [76, 88, 90, 130, 297], where delay-dependent stability method was provided in [76] for sufficiently small delay. In [90], stabilization strategies for a class of switched discrete-time systems based on trajectory independent and trajectory dependent were developed using the notion of average dwell time. For a class of continuous-time systems with time-varying delays, methods for exponential switched stability and stabilization methods were presented in [130] using Riccati-like equations. In [297], the problem of designing state-feedback controllers for a class of continuous-time systems was addressed using arbitrary switching rules. It appears from the available results thus far that stability and stabilization problems of classes of nonlinear switched discrete-time systems have received little attention.

Therefore, in this section, inspired by the results of [279, 424, 438], we study a class of nominally linear switched discrete-time systems with time-varying delays, bounded nonlinearities satisfying some Lipschitz conditions and real convex bounded parametric uncertainties in all system matrices. Specifically, the problems of robust delay-dependent $\mathcal{L}_2$ gain analysis and control synthesis are investigated for Lipschitz-type nonlinear switched systems under arbitrary switching sequences. We develop new criteria for delay-dependent stability and feedback stabilization for such class of switched state-delay systems. The main vehicle is the constructive use of an appropriate switched Lyapunov functional coupled with *Finsler's Lemma* and a free-weighting parameter matrix. The delay-dependent $\mathcal{L}_2$ gain analysis is utilized to characterize linear matrix inequality (LMI)-based conditions under which the linear switched state-delay system with polytopic uncertainties is robustly asymptotically stable with an $\mathcal{L}_2$ gain smaller than a prescribed constant level. Then, robust control synthesis is used to design switched-feedback schemes, based on state-, output-measurements or by using dynamic output feedback, to guarantee that the corresponding closed-loop system enjoys the delay-dependent robust stability with an $\mathcal{L}_2$ gain smaller than a prescribed constant level. Several significant results for classes of discrete-time switched systems are derived as special cases.

7.2.2 Problem Statement

We consider a class of nonlinear switched discrete-time systems with state-delay described by

$$\begin{aligned} x_{k+1} &= A_\sigma x_k + A_{d\sigma} x_{k-d_k} + B_\sigma u_k + \Gamma_\sigma w_k \\ &\quad + f_\sigma(x_k, k) + h_\sigma(x(k-d_k), k), \;\; \sigma \in \mathcal{S} \end{aligned}$$

$$x_k = \alpha_k, \quad k = -\bar{d}, -\bar{d}+1, ..., 0 \tag{7.46}$$

$$y_k = C_\sigma x_k + C_{d\sigma} x_{k-d_k} \tag{7.47}$$

$$z_k = G_\sigma x_k + G_{d\sigma} x_{k-d_k} + D_\sigma u_k + \Phi_\sigma w_k \tag{7.48}$$

where $x_k \in \Re^n$ is the state; $u_k \in \Re^m$ is the control input; $w_k \in \Re^q$ is the disturbance input which belongs to $\ell_2[0, \infty)$, $y_k \in \Re^p$ is the measured output; $z_k \in \Re^r$ is the controlled output. The state delay d_k appearing in the hybrid system dynamics is frequently encountered in several system applications, including networked control systems, chemical processes, population dynamics, and economic systems [216]. In the sequel, it is assumed that d_k is time varying and satisfying $\underline{d} \le d_k \le \bar{d}$, where the bounds $\underline{d} > 0$ and $\bar{d} > 0$ known are constant scalars. The initial condition sequence $\{\alpha_k, \ k = -\bar{d}, -\bar{d}+1, ..., 0\}$ is given. The unknown functions $f_\sigma = f_\sigma(x_k, k) \in \Re^n$, $h_\sigma = h_\sigma(x_k, k) \in \Re^n$ are vector-valued time-varying nonlinear perturbations with $f_\sigma(0, t) = 0$, $h_\sigma(0, t) = 0 \ \forall \ t$ and satisfy the following Lipschitz condition for all $(x, k), (\hat{x}, k) \in \Re^n \times \Re$:

$$\begin{aligned} ||f_\sigma(x_k, k) - f_\sigma(\hat{x}_k, k)|| &\le \alpha \ ||F(x_k - \hat{x}_k)|| \\ ||h_\sigma(x_{k-d_k}, k) - h_\sigma(\hat{x}_{k-d_k}, k)|| &\le \beta \ ||H(x_{k-d_k} - \hat{x}_{k-d_k})|| \end{aligned} \tag{7.49}$$

for some constant $\beta > 0$ and $F \in \Re^{n \times n}$, $H \in \Re^{n \times n}$ are constant matrices. Note as a consequence of (7.49), we have

$$||f_\sigma(x_k, k)|| \le \alpha \ ||F \ x_k||, \quad ||h_\sigma(x_{k-d_k}, k)|| \le \beta \ ||H \ x_{k-d_k}|| \tag{7.50}$$

Equivalently stated, condition (7.49) implies that

$$\begin{aligned} &\left[f_\sigma^t(x_k, k) f_\sigma(x_k, k) - \alpha^2 x_k^t F^t F x_k \right] \le 0 \\ &\left[h_\sigma^t(x_{k-d_k}, k) h_\sigma(x_{k-d_k}, k) - \beta^2 x_{k-d_k}^t H^t H x_{k-d_k} \right] \le 0 \end{aligned} \tag{7.51}$$

Extending on [174], model (7.46), (7.47), and (7.48) represents the continuous (state) portion of a nonlinear hybrid system. The particular mode σ at any given time instant may be a selective procedure characterized by a switching rule of the form

$$\sigma_{k+1} = \delta(\sigma_k, x_k), \quad \delta : \mathcal{S} \times \Re^n \to \mathcal{S} \tag{7.52}$$

The function $\delta(.)$ is usually defined using a partition of the continuous-state space. Let $\mathcal{S}$ denote the set of all selective rules. Therefore, the linear hybrid system under consideration is composed of N subsystems; each of which is activated at a particular switching instant. For a switching mode $i \in \mathcal{S}$, the associated matrices $\{A_i, ..., \Phi_i\}$ contain uncertainties represented by a real convex bounded polytopic model of the type

$$\left(\begin{bmatrix} A_i & B_i & A_{di} & \Gamma_i \\ C_i & C_{di} & & \\ G_i & G_{di} & D_i & \Phi_i \end{bmatrix}, \begin{bmatrix} f_i \\ h_i \end{bmatrix}\right) \triangleq \left\{ \left(\sum_{p=1}^{M_i} \lambda_{jp} \begin{bmatrix} A_{ip} & B_{ip} & A_{di} & \Gamma_{ip} \\ C_{ip} & C_{dip} & & \\ G_{ip} & G_{dip} & D_{ip} & \Phi_{ip} \end{bmatrix} \right.\right.$$
$$\left.\left. \sum_{p=1}^{M_i} \lambda_{ip} \begin{bmatrix} f_{ip} \\ h_{ip} \end{bmatrix} \right), \quad i \in \mathcal{S} \right\} \tag{7.53}$$

where $\lambda_i = (\lambda_{i1}, \lambda_{i2}, ..., \lambda_{iM_i}) \in \Lambda_j$ belongs to the unit simplex of M_i vertices

$$\Lambda_j \triangleq \left\{ \lambda_j : \sum_{p=1}^{M_i} \lambda_{ip} = 1 \ , \ \lambda_{ip} \geq 0 \right\} \tag{7.54}$$

where $A_{ip}, ..., \Phi_{ip}, \ \ p = 1, ..., M_i$ are known real constant matrices of appropriate dimensions, which describe the linear portion of the ith nominal subsystem:

$$x_{k+1} = A_j x_k + A_{dj} x_{k-d_k} + B_j u_k + \Gamma_j w_k + f_j(x_k, k) + h_j(x(k - d_k), k)$$
$$x_k = \alpha_k, \quad k = -\bar{d}, -\bar{d} + 1, ..., 0 \tag{7.55}$$
$$y_k = C_j x_k + C_{dj} x_{k-d_k} \tag{7.56}$$
$$z_k = G_j x_k + G_{dj} x_{k-d_k} + D_j u_k + \Phi_j w_k \tag{7.57}$$

Remark 7.10 As indicated in [438], the polytopic-type uncertainty in (7.53) and (7.54) can be used to describe the parametric uncertainty more precisely, and cover wider classes of uncertainties than the norm-bounded uncertainty. In fact, (7.53) and (7.54) is a generalization of the so-called matching condition. It is important to note that if the lower and upper delay bounds in the become identical, that is, $\underline{d} = \bar{d} = d$, then the time delay becomes a constant delay. Also, if d_k only changes when the system mode is switched then the time delay becomes a mode-dependent constant delay; thus the time-varying delay considered here covers the previous two cases.

Definition 7.11 *Systems (7.46), (7.47), and (7.48) with $w(.) \equiv 0, u(.) \equiv 0$ is said to be delay-dependent asymptotically stable, if it is asymptotically stable in the sense of Lyapunov for the prescribed delay range $\underline{d} \leq d_k \leq \bar{d}$. If this occurs for all admissible uncertainties satisfying (7.53) and (7.54), then it is called delay-dependent robustly asymptotically stable. If in addition it satisfies $||z_k||_2 < \gamma ||w_k||_2$, then it achieves a prescribed disturbance attenuation level γ.*

Definition 7.12 *Systems (7.46), (7.47), and (7.48) under the feedback control $u(t) = U(x(t))$ with $w(.) \equiv 0$ is said to be delay-dependent asymptotically stabilizable, if the closed-loop system is asymptotically stable in the sense of Lyapunov for the prescribed delay range $\underline{d} \leq d_k \leq \bar{d}$. If this takes place for all admissible uncertainties satisfying (7.53) and (7.54), then it is called delay-dependent robustly asymptotically stabilizable. If in addition it satisfies $||z_k||_2 < \gamma ||w_k||_2$, then it achieves a prescribed disturbance attenuation level γ.*

Our purpose hereafter, in the light of **Definitions 7.11** and **7.12**, is to develop robust criteria for delay-dependent asymptotic stability and stabilization of system (Σ_J) and then design appropriate $\mathcal{L}_2$ feedback controllers that guarantee robust delay-dependent asymptotic stability with a prescribed performance measure.

7.2.3 Delay-Dependent $\mathcal{L}_2$ Gain Analysis

In this section, we derive robust criteria for delay-dependent asymptotic stability of system (Σ_J). The major thrust is based on the fundamental stability theory of Lyapunov, which states that for asymptotic stability, it suffices to find a Lyapunov function candidate $V_\sigma(x_k, k) > 0,\ \forall x_k \neq 0,\ k \in \mathbb{Z}$ satisfying $\Delta V_\sigma(x_k, k) = V_\sigma(x_{k+1}, k+1) - V_\sigma(x_k, k) < 0$. We apply this theorem hereafter under arbitrary switching. The following theorem summarizes the main result.

Theorem 7.13 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched systems (7.46), (7.47), and (7.48) with $u_k \equiv 0$ is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{P}_s^t = \mathcal{P}_s,\ \widehat{X}_j,\ 0 < \mathcal{Q}^t = \mathcal{Q},\ 0 < \mathcal{W}^t = \mathcal{W},\ \{\mathcal{M}_j,\ j = 1, ..., 5\},\ \forall (j, s) \in \mathcal{S}$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$*

$$\widehat{X}_j \widetilde{A}_j + \widetilde{A}_j^t \widehat{X}_j^t + \widetilde{P}_{js} < 0 \tag{7.58}$$

$$\widetilde{P}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\Xi_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 & G_j^t \\ \bullet & \bullet & -\Xi_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 & G_{dj}^t \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.59}$$

$$\Xi_2 = \mathcal{P}_j - (\bar{d} - \underline{d} + 1)\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t - \sigma\alpha^2 F^t F$$

$$\Xi_3 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q} - \kappa\beta^2 H^t H,\ \widetilde{A}_j = \begin{bmatrix} -I & A_j & A_{dj} & I & I & 0 & \Gamma_j \end{bmatrix} \tag{7.60}$$

Proof In the sequel, we use $\xi_m = x_{m+1} - x_m$ and consider the following switched Lyapunov–Krasovskii functional

$$V(k) \stackrel{\Delta}{=} V_a(k) + V_b(k) + V_c(k) + V_d(k)$$

$$V_a(k) = x_k^t\ P_j x_k, \quad V_b(x_k, k) = \sum_{j=k-d_k}^{k-1} x_j^t\ \mathcal{Q} x_j$$

$$V_c(k) = \sum_{m=-\bar{d}+2}^{-\underline{d}+1} \sum_{j=k+m-1}^{k-1} x_j^t \mathcal{Q} x_j, \quad V_d(k) = \sum_{m=-\bar{d}}^{-1} \sum_{j=k+m}^{k-1} \xi_j^t \mathcal{W} \xi_j$$

$$0 < P_j^t = P_j, \; 0 < \mathcal{Q}^t = \mathcal{Q}, \; 0 < \mathcal{W}^t = \mathcal{W} \tag{7.61}$$

Define $\Delta V(k) = V(k+1) - V(k)$, along with the solution of (7.55) we obtain for all $(j, s) \in \mathcal{S} \times \mathcal{S}$

$$\Delta V_a(k) = x_{k+1}^t P_s x_{k+1} - x_k^t P_j x_k \tag{7.62}$$

$$\begin{aligned}
\Delta V_b(k) &= \sum_{m=k-d_{k+1}+1}^{k} x_m^t \mathcal{Q} x_m - \sum_{j=k-d_k}^{k-1} x_j^t \mathcal{Q} x_j \\
&= x_k^t \mathcal{Q} x_k - x_{k-d_k}^t \mathcal{Q} x_{k-d_k} + \sum_{m=k-d_{k+1}+1}^{k-1} x_m^t \mathcal{Q} x_m \\
&\quad - \sum_{m=k-d_k+1}^{k-1} x_m^t \mathcal{Q} x_m \\
&\leq x_k^t \mathcal{Q} x_k - x_{k-d_k}^t \mathcal{Q} x_{k-d_k} + \sum_{m=k-\bar{d}+1}^{k-\underline{d}} x_m^t \mathcal{Q} x_m
\end{aligned} \tag{7.63}$$

$$\Delta V_c(k) = (\bar{d} - \underline{d})\, x_k^t \mathcal{Q} x_k - \sum_{m=k-\bar{d}+1}^{k-\underline{d}} x_m^t \mathcal{Q} x_m \tag{7.64}$$

$$\Delta V_d(k) \leq \bar{d}(x_{k+1} - x_k)^t \mathcal{W}(x_{k+1} - x_k) - \bar{d} \sum_{m=k-\bar{d}}^{k-1} \xi_m^t \mathcal{W} \xi_m \tag{7.65}$$

Note that in (7.62), the case when $j = s$ shows that the switched system is described by the jth mode, whereas the case $j \neq s$ represents that the switched system is at the switching times from mode j to mode s [42].

Since $x_{k-d_k} = x_k - \sum_{m=k-d_k}^{k-1} \xi_m$, then for free-weighting parameter matrices $\mathcal{M}_p$, $p = 1, \ldots, 5$, we have

$$\begin{aligned}
\widehat{x}(k, m) &= \left[x_{k+1}^t \; x_k^t \; x_{k-d_k}^t \; f^t \; h^t \; \xi_m^t \right]^t, \; \widehat{\mathcal{M}} = \left[\mathcal{M}_1^t \; \mathcal{M}_2^t \; \mathcal{M}_3^t \; \mathcal{M}_4^t \; \mathcal{M}_5^t \; 0 \right]^t \\
\widehat{\mathcal{S}} &= \left[0 \; I \; -I \; 0 \; 0 \; -d_k I \right]
\end{aligned} \tag{7.66}$$

such that the following equation holds

$$2 \sum_{j=k-d_k}^{k-1} \widehat{x}^t(k, m)\, \widehat{\mathcal{M}} \widehat{\mathcal{S}} \widehat{x}(k, m) = 0 \tag{7.67}$$

On considering (7.62), (7.63), (7.64), and (7.65) in the light of (7.61) for $d_k \leq \bar{d},\ w_k \equiv 0,$ it is not difficult to show that $\Delta V(k) < 0$ is equivalent to the following set of inequalities:

$$\sum_{m=k-d_k}^{k-1} \widehat{x}^t(k,m)\, \widehat{\mathcal{P}}_{sj}\widehat{x}(k,m) < 0, \quad (s,j) \in \mathcal{N} \times \mathcal{N}$$

$$\widehat{\mathcal{P}}_{js} = \begin{bmatrix} -\mathcal{P}_j + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{7.68}$$

More importantly, in view of (7.46) with $u_k \equiv 0,\ w_k \equiv 0$, we have

$$\widehat{A}_j\, \widehat{x}(k,m) = 0 \tag{7.69}$$

where $\widehat{A}_j$ corresponds to $\widetilde{A}_j$ in (7.60) with $\Gamma_j \equiv 0$. Application of the Finsler's Lemma to (7.68) and (7.69) with $\widehat{x}(k,j) \equiv x,\ \widehat{\mathcal{P}}_{sj} \equiv \mathcal{P},\ \widehat{A}_s \equiv \mathcal{Z}^t,\ \widehat{X}_s \equiv \mathcal{B},$ and and taking into account (7.51) via the S-procedure, we readily obtain LMI (7.58) as desired, which establishes the asymptotic stability.

Consider the performance measure

$$J_K = \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j \right)$$

For any $w_k \in \ell_2(0,\infty) \neq 0$ and zero initial condition $x_o = 0$, we have

$$J_K \leq \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j + \Delta V(j) \right)$$

Standard algebraic manipulation using (7.48) leads to

$$\begin{aligned} z_j^t z_j - \gamma^2 w_j^t w_j + \Delta V(j) &= \widetilde{x}^t(k,m)\, \widetilde{\mathcal{P}}_{js}\, \widetilde{x}(k,m) \\ \widetilde{x}(k,m) &= \left[\widehat{x}^t(k,m) \quad w_k^t \right]^t \end{aligned} \tag{7.70}$$

and $\widetilde{\mathcal{P}}_{js}$ is given by (7.59). It follows from [279] for the switched system (7.46), (7.47), and (7.48) to be asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ that $z_j^t z_j - \gamma^2 w_j^t w_j + \Delta V(j) < 0,\ \forall j \in \{0, K\}$ holds for arbitrary switching, which in turn implies that $J_K < 0$. The desired asymptotic stability result is achieved by Finsler's Lemma and LMI (7.58) subject to (7.59). ■

In the sequel, we provide several robust stability results in terms of the following corollaries:

Corollary 7.14 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched system (7.46), (7.47), and (7.48) with $u_k \equiv 0$ and vertex representation (7.53) and (7.54) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{P}_s^t = \mathcal{P}_s,\ \widehat{X}_j,\ 0 < \mathcal{Q}^t = \mathcal{Q},\ 0 < \mathcal{W}^t = \mathcal{W},\ \{\mathcal{M}_j,\ j = 1, ..., 5\},\ (j, s) \in \{1, 2, ...\}$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMIs for $(j, s) \in \mathcal{S}$*

$$\widehat{X}_j \widetilde{A}_{jp} + \widetilde{A}_{jp}^t \widehat{X}_j^t + \widetilde{\mathcal{P}}_{jps} < 0 \tag{7.71}$$

$$\widetilde{P}_{jps} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\Xi_{2p} & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 & G_{jp}^t \\ \bullet & \bullet & -\Xi_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 & G_{djp}^t \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \Phi_{jp}^t \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.72}$$

$$\begin{aligned} \Xi_{2p} &= P_{jp} - (\bar{d} - \underline{d} + 1)\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t - \sigma\alpha^2 F^t F, \\ \widetilde{A}_{jp} &= \begin{bmatrix} -I & A_{jp} & A_{djp} & I & I & 0 & \Gamma_{jp} \end{bmatrix} \end{aligned} \tag{7.73}$$

Proof Obtained from **Theorem 7.13** by using the vertex representation (7.53) and (7.54) to get (7.72) from (7.59).
blacksquare

Corollary 7.15 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched system (7.46), (7.47), and (7.48) with $u_k \equiv 0$ is delay-dependent asymptotically stable if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{P}_s^t = \mathcal{P}_s,\ \widehat{X}_j,\ 0 < \mathcal{Q}^t = \mathcal{Q},\ 0 < \mathcal{W}^t = \mathcal{W},\ \{\mathcal{M}_j,\ j = 1, ..., 5\},\ \forall (j, s) \in \mathcal{S}$ and scalars $\sigma > 0,\ \kappa > 0$ satisfying the following LMIs for $(j, s) \in \mathcal{S}$*

$$\widehat{X}_j \overline{A}_j + \overline{A}_j^t \widehat{X}_j^t + \widetilde{\mathcal{P}}_{js} < 0 \tag{7.74}$$

$$\widetilde{P}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{7.75}$$

$$\overline{A}_j = \begin{bmatrix} -I & A_j & A_{dj} & I & I & 0 \end{bmatrix} \tag{7.76}$$

Proof Obtained from **Theorem 7.13** by substituting $w_k \equiv 0$, $\Gamma_i \equiv 0$, and $\Phi \equiv 0$.

Remark 7.16 We note from that the $\mathcal{L}_2-$ gain under arbitrary switching can be looked as the worst-case energy amplitude gain for switched system (7.53) and (7.54) over all possible inputs, switching signals, and all admissible uncertainties. The functional (7.61) is called a switched Lyapunov functional (SLF) since it has the same switching signals as system (7.46), (7.47), and (7.48), which is known to yield less conservative results than the constant Lyapunov functional

$$x_k^t P x_k x_k^t + \sum_{j=k-d_k}^{k-1} x_j^t \, \mathcal{Q} x_j + \sum_{m=-\bar{d}+2}^{-\underline{d}+1} \sum_{j=k+m-1}^{k-1} x_j^t \, \mathcal{Q} x_j + \sum_{m=-\bar{d}}^{-1} \sum_{j=k+m}^{k-1} \xi_j^t \, W \xi_j .$$

.

Remark 7.17 Among the novel features of the developed approach is the arbitrary selection of the matrix $\widetilde{X}_j$, which helps much in the feedback stabilization later on as well as in the numerical simulation. Another feature is the strong delay dependence of the stability criteria through the upper bound $\bar{d}$ and the number of delayed samples as represented by $\bar{d} - \underline{d} + 1$.

Remark 7.18 The main stability result is derived from feasibility testing in the enlarged state space in contrast with similar techniques [50, 138, 217, 273, 279, 368, 432]. The novelty of our approach relies on the deployment of *Finsler's Lemma* in conjunction with a set of free-weighting matrices without using bounding techniques to ensure that the system matrices are readily separated from the Lyapunov matrices. This decoupling feature simplifies numerical implementation and, as will be shown in the subsequent sections, paves the way to flexible feedback stabilization synthesis. A simple comparison would support our intuition that the LMI results are less conservative and in the nonswitching case are superior than the existing methods [345]. The optimal $\mathcal{L}_2 - gain$ of switched system (7.46), (7.47), and (7.48) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{array}{c} \text{Minimize } \gamma \\ \text{s.t. LMIs } (7.58) \text{ and } (7.59), \ \forall \, (j,s) \in \mathcal{S} \times \mathcal{S} \\ \mathcal{P}_j, \ \mathcal{P}_s, \ \widehat{X}_j, \ \mathcal{Q}, \ \mathcal{W}, \ \{\mathcal{M}_j, \ j = 1, ..., 5\}, \ \forall (j,s) \in \mathcal{S}, \ \ \gamma > 0, \ \sigma > 0, \ \kappa > 0 \end{array}$$

which can be conveniently solved by the existing software [74].

Stability results for the linear switched system

$$\begin{aligned} x_{k+1} &= A_j x_k + A_{dj} x_{k-d_k} + B_j u_k + \Gamma_j w_k \\ x_k &= \alpha_k, \quad k = -\bar{d}, -\bar{d}+1, ..., 0 \end{aligned} \tag{7.77}$$

$$y_k = C_j x_k + C_{dj} x_{k-d_k} + \Psi_j w_k \tag{7.78}$$

$$z_k = G_j x_k + G_{dj} x_{k-d_k} + D_j u_k + \Phi_j w_k \tag{7.79}$$

is provided by the following theorem

Theorem 7.19 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched system (7.46), (7.47), and (7.48) with $u_k \equiv 0$ is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j,\ 0 < \mathcal{P}_s^t = \mathcal{P}_s,\ \widehat{X}_j,\ 0 < \mathcal{Q}^t = \mathcal{Q},\ 0 < \mathcal{W}^t = \mathcal{W},\ \{\mathcal{M}_j,\ j = 1, ..., 5\},\ (j, s) \in \{1, 2, ...\}$ and a scalar $\gamma > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$*

$$\widehat{X}_j \mathsf{A}_j + \mathsf{A}_j^t \widehat{X}_j^t + \widetilde{\mathcal{P}}_{js} < 0 \tag{7.80}$$

$$\widetilde{P}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\hat{\Xi}_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & -\bar{d}\mathcal{M}_2 & G_j^t \\ \bullet & \bullet & -\Xi_3 & -\bar{d}\mathcal{M}_3 & G_{dj}^t \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.81}$$

$$\hat{\Xi}_2 = P_j - (\bar{d} - \underline{d} + 1)\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t,$$
$$\hat{\Xi}_3 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q},\ \mathsf{A}_j = \begin{bmatrix} -I & A_j & A_{dj} & 0 & \Gamma_j \end{bmatrix} \tag{7.82}$$

Proof Obtained from **Theorem 7.13** by substituting $f \equiv 0,\ h \equiv 0$. ■

Illustrative Example D

Consider the following second-order system where the switching occurs between four modes described by the following coefficients

$$A_1 = \begin{bmatrix} 0.7 & 0.09 \\ 0 & 0.35 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} 0.25 \\ 0.15 \end{bmatrix},\ G_{d1} = \begin{bmatrix} -0.1 \\ -0.01 \end{bmatrix},\ \Phi_1 = 0.01$$
$$||f_1(x_k, k)|| \le \alpha_1 ||x_k||,\ ||h_1(x_k, k)|| \le \beta_1 ||x_{k-d_k}||,\ |\alpha_1| \le 0.1,\ |\beta_1| \le 0.1$$
$$A_2 = \begin{bmatrix} 0.41 & 0.11 \\ 0 & 0.97 \end{bmatrix},\ A_{d3} = \begin{bmatrix} 0 & 0.05 \\ 0 & -0.15 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 0.2 \\ -0.02 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix},\ G_{d2} = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix},\ \Phi_2 = 0.02$$
$$||f_2(x_k, k)|| \le \alpha_2 ||x_k||,\ ||h_2(x_k, k)|| \le \beta_2 ||x_{k-d_k}||,\ |\alpha_2| \le 0.3,\ |\beta_2| \le 0.3$$
$$A_3 = \begin{bmatrix} 0.6 & 0.02 \\ 0 & 0.49 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -0.1 & 0.01 \\ -0.1 & -0.1 \end{bmatrix},\ \Gamma_3 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix}$$
$$G_3 = \begin{bmatrix} 0.17 \\ 0.19 \end{bmatrix},\ G_{d3} = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix},\ \Phi_3 = 0.02$$
$$||f_3(x_k, k)|| \le \alpha_1 ||x_k||,\ ||h_3(x_k, k)|| \le \beta_3 ||x_{k-d_k}||,\ |\alpha_3| \le 0.4,\ |\beta_3| \le 0.2$$
$$A_4 = \begin{bmatrix} -0.33 & 0.22 \\ 0 & -0.45 \end{bmatrix},\ A_{d4} = \begin{bmatrix} 0 & 0.25 \\ 0 & -0.05 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 0.1 \\ -0.02 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix}, \; G_{d4} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \; \Phi_4 = 0.02$$

$$||f_4(x_k, k)|| \le \alpha_4 ||x_k||, \;\; ||h_4(x_k, k)|| \le \beta_4 ||x_{k-d_k}||, \; |\alpha_4| \le 0.5, \;\; |\beta_4| \le 0.4$$

In the implementation of **Theorem 7.13**, we start by assigning a value for the lower bound $\underline{d}$ and seek the maximum allowable value for the upper bound $\bar{d}$ rendering feasible solution. For each pair $(\underline{d}, \bar{d})$, we record the performance bound γ. A summary of the computations of applying **Theorem 7.13** such that the above switched system is asymptotically stable is depicted in Table 7.3.

Table 7.3 Computational summary of illustrative example D

$\underline{d}$	$\bar{d}$	γ
2	6	2.145
3	8	2.663
4	10	2.874
5	12	3.114
6	13	3.219
4	16	2.874
8	18	3.534

7.2.4 Switched Feedback Design

Extending on the last section, we examine here the problem of switched feedback stabilization using either switched state-feedback or output-feedback design schemes.

With reference to system (7.46), (7.47), and (7.48), we consider that the arbitrary switching rule $\sigma(.)$ activate subsystem j at instant k. Our objective herein is to design a switched state feedback $u_k = K_j \; x_k$ at $i \in \mathcal{S}$ mode such that the closed-loop system

$$\begin{aligned} x_{k+1} &= [A_j + B_j K_j] x_k + A_{dj} x_{k-d_k} + \Gamma_j w_k + f_j(x_k, k) + h_j(x(k-d_k), k) \\ &= \bar{A}_j x_k + A_{dj} x_{k-d_k} + \Gamma_j w_k \end{aligned} \tag{7.83}$$

$$\begin{aligned} z_k &= [G_j + D_j K_j] x_k + G_{dj} x_{k-d_k} + \Phi_j w_k \\ &= \bar{G}_j x_k + G_{dj} x_{k-d_k} + \Phi_j w_k \end{aligned} \tag{7.84}$$

is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$. It follows from Theorem 7.13 that system (7.83) and (7.84) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j, \;\; 0 < \mathcal{P}_s^t = \mathcal{P}_s, \;\; \widehat{X}_j, \;\; 0 < \mathcal{Q}^t = \mathcal{Q}, \;\; 0 < \mathcal{W}^t = \mathcal{W}, \;\; \{\mathcal{M}_j, \;\; j = 1, ..., 7\}, \;\; (j, s) \in \{1, 2, ...\}$ and scalars $\gamma > 0, \;\; \sigma > 0, \;\; \kappa > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$

$$\widetilde{X}_j \mathcal{A}_j + \mathcal{A}_j^t \widetilde{X}_j^t + \mathcal{P}_{js} < 0 \tag{7.85}$$

$$\mathcal{P}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\Xi_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 & \bar{G}_j^t \\ \bullet & \bullet & -\Xi_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 & G_{dj}^t \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.86}$$

$$\mathcal{A}_j = \left[-I \;\; \bar{A}_j \;\; A_{dj} \;\; I \;\; I \;\; 0 \;\; \Gamma_j \right] \tag{7.87}$$

The following theorem states the main result on switched state feedback

Theorem 7.20 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched system (7.83) and (7.84) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < X_j^t = X_j,\ 0 < X_s^t = X_s,\ Y_j,\ 0 < \mathcal{S}^t = \mathcal{S},\ 0 < \mathcal{R}^t = \mathcal{R},\ \Upsilon,\ \Lambda,\ \{\Theta_j,\ j = 1, .., 7\},\ (j, s) \in \{1, 2, ..\}$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$*

$$\begin{bmatrix} -\Delta_{1s} & \Delta_{2j} & \Delta_{3j} & \Lambda & \Lambda & -\bar{d}\Theta_1 & \Gamma_j & 0 \\ \bullet & -\Delta_{4j} & \Delta_5 & \Theta_4^t & \Theta_5^t & -\bar{d}\Theta_2 & \Lambda G_j^t & \Theta_6 \\ \bullet & \bullet & -\Delta_6 & -\Theta_4^t & -\Theta_5^t & -\bar{d}\Theta_3 & \Lambda G_{dj}^t & 0 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Theta_4 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Theta_5 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.88}$$

$$\begin{aligned} &\Delta_{1s} = \Lambda + \Lambda^t + X_s - \bar{d}\mathcal{S},\ \Delta_{2j} = A_j\Lambda^t + B_jY_j + \Theta_1 - \bar{d}\mathcal{S} \\ &\Delta_{4j} = X_j - (\bar{d} - \underline{d} + 1)\mathcal{R} - \bar{d}\mathcal{S} - \Theta_2 - \Theta_2^t,\ \Delta_{3j} = A_{dj}\Lambda^t - \Theta_1 \\ &\Delta_6 = \Theta_3 + \Theta_3^t + \mathcal{R} - \Theta_7,\ \Delta_5 = -\Theta_2 - \Theta_2^t \end{aligned} \tag{7.89}$$

Moreover, the switched state-feedback gain is given by $K_j = Y_j\Upsilon^{-t}$.

Proof Define $\widetilde{X}_j = \left[\Upsilon^t \ 0\,0\,0\,0\,0\,0 \right]^t,\ \Upsilon \in \Re^{n\times n}$ and let $Y_j = K_j\Upsilon^t$. Applying the congruence transformation

$$T = \text{diag}[\Lambda,\ \Lambda,\ \Lambda,\ \Lambda,\ \Lambda, \Lambda,\ I],\ \Lambda = \Upsilon^{-1}$$

to inequality (7.85) using (7.86) and (7.87) and the linearizations

$$X_s = \Upsilon^{-t}P_s\Upsilon^{-1},\ \mathcal{S} = \Upsilon^{-t}\mathcal{W}\Upsilon^{-1},\ \{\Theta\}_1^5 = \Upsilon^{-t}\{\mathcal{M}\}_1^5\Upsilon^{-1},\ X_j = \Upsilon^{-t}P_j\Upsilon^{-1},$$
$$\Theta_6 = \sigma\alpha\Upsilon^{-t}F^t,\ \Theta_7 = \kappa\beta^t\Upsilon^{-t}H^tH\Upsilon^{-1}$$

we immediately obtain LMI (7.88) subject to (7.89). ∎

Remark 7.21 The optimal switched state feedback with $\mathcal{L}_2 - gain$ for system (7.83 and 7.84) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned} &\text{Minimize } \gamma \\ &\text{s. t. LMIs } (7.88), \quad \forall\, (j,s) \in \mathcal{S} \times \mathcal{S} \\ &X_j,\ X_s,\ Y_j,\ \mathcal{S},\ \mathcal{R},\ \Upsilon,\ \Lambda,\ \{\Theta_j,\ j = 1,..7\},\ \gamma > 0,\ \sigma > 0,\ \kappa > 0 \end{aligned}$$

In the case of polytopic representation (4.30) and (4.31), the corresponding convex minimization problem takes the form

$$\begin{aligned} &\text{Minimize } \gamma \\ &\text{wrt } X_j,\ X_s,\ Y_j,\ \mathcal{S},\ \mathcal{R},\ \Upsilon,\ \Lambda,\ \{\Theta\}_1^7,\ \gamma > 0,\ \sigma > 0,\ \kappa > 0 \end{aligned}$$

the LMIs

$$\begin{bmatrix} -\Delta_{1s} & \Delta_{2j} & \Delta_{3j} & \Lambda & \Lambda & -\bar{d}\Theta_1 & \Gamma_{jp} & 0 \\ \bullet & -\Delta_{4j} & \Delta_5 & \Theta_4^t & \Theta_5^t & -\bar{d}\Theta_2 & \Lambda G_{jp}^t & \Theta_6 \\ \bullet & \bullet & -\Delta_6 & -\Theta_4^t & -\Theta_5^t & -\bar{d}\Theta_3 & \Lambda G_{dj}^t & 0 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Theta_4 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Theta_5 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_{jp}^t & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

$\forall\ (j,s) \in \mathcal{S} \times \mathcal{S}$ and $p = 1,\ldots,M_j$ where $\Delta_{2jp} = A_{jp}\Lambda^t + B_{jp}Y_j + \Theta_1 - \bar{d}\mathcal{S},\ \Delta_{3jp} = A_{djp}\Lambda^t - \Theta_1$

A state-feedback design for the linear switched system (7.77), (7.78), and (7.79) is established below:

Theorem 7.22 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Linear switched system (7.77), (7.78), and (7.79) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < X_j^t = X_j,\ 0 < X_s^t = X_s,\ Y_j,\ 0 < \mathcal{S}^t = \mathcal{S},\ 0 < \mathcal{R}^t = \mathcal{R},\ \Upsilon,\ \Lambda,\ \{\Theta_j,\ j = 1,\ldots,5\},\ (j,s) \in \{1,2,\ldots\}$ and a scalar $\gamma > 0$ satisfying the LMIs for $(j,s) \in \mathcal{S}$*

$$\begin{bmatrix} -\Delta_{1s} & \Delta_{2j} & \Delta_{3j} & -\bar{d}\Theta_1 & \Gamma_j \\ \bullet & -\Delta_{4j} & \Delta_5 & -\bar{d}\Theta_2 & \Lambda G_j^t \\ \bullet & \bullet & -\Delta_6 & -\bar{d}\Theta_3 & \Lambda G_{dj}^t \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} < 0 \tag{7.90}$$

$$\Delta_{1s} = \Lambda + \Lambda^t + X_s - \bar{d}\mathcal{S},\ \Delta_{2j} = A_j\Lambda^t + B_jY_j + \Theta_1 - \bar{d}\mathcal{S}$$

$$\Delta_{4j} = X_j - (\bar{d} - \underline{d} + 1)\mathcal{R} - \bar{d}\mathcal{S} - \Theta_2 - \Theta_2^t, \ \Delta_{3j} = A_{dj}\Lambda^t - \Theta_1$$
$$\Delta_6 = \Theta_3 + \Theta_3^t + \mathcal{R}, \ \Delta_5 = -\Theta_2 - \Theta_2^t \tag{7.91}$$

Moreover, the switched state-feedback gain is given by $K_j = Y_j \Upsilon^{-t}$.

The objective now is to design a switched output feedback $u_k = G_j \ y_k$ at mode $j \in \mathcal{S}$ such that the closed-loop system

$$\begin{aligned} x_{k+1} &= [A_j + B_j G_j C_j]x_k + [A_{dj} + B_j G_j C_{dj}]x_{k-d_k} + \Gamma_j w_k \\ &= \check{A}_j x_k + \check{A}_{dj} x_{k-d_k} + \Gamma_j w_k \end{aligned} \tag{7.92}$$
$$\begin{aligned} z_k &= [G_j + D_j G_j C_j]x_k + [G_{dj} + D_j G_j C_{dj}]x_{k-d_k} + \Phi_j w_k \\ &= \check{G}_j x_k + \check{G}_{dj} x_{k-d_k} + \Phi_j w_k \end{aligned} \tag{7.93}$$

is delay-dependent asymptotically with an $\mathcal{L}_2-$ gain $< \gamma$. To proceed further, we invoke the following assumption:

Assumption 7.1 *The set of output matrices* $\{C_j, \ j = 1, ..., N\}$ *are of full row rank.*

It is worth noting that this case can be fulfilled by deleting redundant measurement components of the output variable y_k. Therefore, subject to **Assumption 1**, it follows from **Theorem 7.13** that switched system (7.92) and (7.93) is delay-dependent asymptotically stable if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j, \ 0 < \mathcal{P}_s^t = \mathcal{P}_s, \ \widehat{\mathcal{X}}_j, \ 0 < \mathcal{Q}^t = \mathcal{Q}, \ 0 < \mathcal{W}^t = \mathcal{W}, \ \{\mathcal{M}_j, \ j = 1, \ldots, 5\}, \ \forall (j, s) \in \mathcal{S}$ and scalars
$\gamma > 0, \ \sigma > 0, \ \kappa > 0$ satisfying the following LMIs for all $(j, s) \in \mathcal{S}$

$$\widetilde{X}_j \widehat{\mathcal{A}}_j + \widehat{\mathcal{A}}_j^t \widetilde{X}_j^t + \widehat{\mathcal{P}}_{js} < 0 \tag{7.94}$$

$$\widehat{\mathcal{P}}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\mathcal{W} & \mathcal{M}_1 - \bar{d}\mathcal{W} & -\mathcal{M}_1 & 0 & 0 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\widehat{\Xi}_2 & -\mathcal{M}_2 - \mathcal{M}_2^t & \mathcal{M}_4^t & \mathcal{M}_5^t & -\bar{d}\mathcal{M}_2 & \check{G}_j^t \\ \bullet & \bullet & -\widehat{\Xi}_3 & -\mathcal{M}_4^t & -\mathcal{M}_5^t & -\bar{d}\mathcal{M}_3 & \check{G}_{dj}^t \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{M}_4 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{M}_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.95}$$

$$\widehat{\Xi}_2 = \mathcal{P}_j - (\bar{d} - \underline{d} + 1)\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t - \sigma\alpha^2 F^t F$$
$$\widehat{\Xi}_3 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q} - \kappa\beta^2 H^t H, \ \widehat{\mathcal{A}}_j = \begin{bmatrix} -I \ \check{A}_j \ \check{A}_{dj} \ I \ I \ 0 \ \Gamma_j \end{bmatrix} \tag{7.96}$$

The following theorem states the main result on switched static output feedback

Theorem 7.23 *Given* $\bar{d} > 0$ *and* $\underline{d} > 0$. *Switched system (7.92) and (7.93) is delay-dependent asymptotically stable with an* $\mathcal{L}_2 - gain < \gamma$ *it follows that there exist matrices* $0 < \mathcal{X}_j^t = \mathcal{X}_j, \ 0 < \mathcal{X}_s^t = \mathcal{X}_s, \ \mathcal{Y}_j, \ 0 < \mathcal{S}^t = \mathcal{S}, \ 0 < \mathcal{R}^t =$

$\mathcal{R}$, Υ, Λ, L_{1j}, L_{2j}, E_1, E_2, $\{\Theta_j,\ j = 1,\dots,7\}$, $(j,s) \in \{1,2,...\}$ and scalars $\gamma > 0$, $\sigma > 0$, $\kappa > 0$ satisfying the following LMIs for $(j,s) \in \mathcal{S}$

$$\begin{bmatrix} -\Delta_{1s} & \widehat{\Delta}_{2j} & \widehat{\Delta}_{3j} & \Lambda & \Lambda & -\bar{d}\Theta_1 & \Gamma_j & 0 \\ \bullet & -\Delta_{4j} & \Delta_5 & \Theta_4^t & \Theta_5^t & -\bar{d}\Theta_2 & \widehat{\Delta}_{7j} & \Theta_6 \\ \bullet & \bullet & -\Delta_6 & -\Theta_4^t & -\Theta_5^t & -\bar{d}\Theta_3 & \widehat{\Delta}_{8j} & 0 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Theta_4 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Theta_5 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.97}$$

$$C_j\Lambda = E_1C_j,\quad C_{dj}\Lambda = E_2C_{dj} \tag{7.98}$$

$$\widehat{\Delta}_{2j} = A_j\Lambda + B_jL_{1j}C_j + \Theta_1 - \bar{d}\mathcal{S},\quad \widehat{\Delta}_{3j} = A_{dj}\Lambda^t + B_jL_{2j}C_{dj} - \Theta_1$$
$$\widehat{\Delta}_{7j} = \Lambda G_j^t + C_j^tL_{1j}^tD_j^t,\quad \widehat{\Delta}_{8j} = \Lambda G_{dj}^t + C_{dj}^tL_{2j}^tD_j^t \tag{7.99}$$

where Δ_{1s}, Δ_{4j}, Δ_5, Δ_6 are given in (7.89). Moreover, the switched static output-feedback gain is given by $G_j = L_{1j}E_1^{-1}$.

Proof Define $\widetilde{X}_j = \begin{bmatrix} \Upsilon^t & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t$, $\Upsilon \in \Re^{n\times n}$. Now let

$$C_j\Lambda = E_1C_j,\ L_{1j} = G_jE_1,\ C_{dj}\Lambda = E_2C_{dj},\ L_{2j} = G_jE_2$$

where $E_k \in R^{p\times p}$. In the spirit of [279], it is easy to show under **Assumption 1** that the matrix E is nonsingular.[1] By applying the congruence transformation

$$T = \text{diag}[\Lambda,\ \Lambda,\ \Lambda,\ \Lambda,\ \Lambda, \Lambda,\ I],\ \Lambda = \Upsilon^{-1}$$

to inequality (4.48) using (7.95) and (7.96) and the linearizations

$$\mathcal{X}_s = \Upsilon^{-t}\mathcal{P}_s\Upsilon^{-1},\ \mathcal{S} = \Upsilon^{-t}\mathcal{W}\Upsilon^{-1},\ \{\Theta\}_1^5 = \Upsilon^{-t}\{\mathcal{M}\}_1^5\Upsilon^{-1},\ \mathcal{X}_j = \Upsilon^{-t}\mathcal{P}_j\Upsilon^{-1},$$
$$\Theta_6 = \sigma\alpha\Upsilon^{-t}F^t,\ \Theta_7 = \kappa\beta^t\Upsilon^{-t}H^tH\Upsilon^{-1}$$

we immediately obtain the LMI (7.97). ■

Remark 7.24 The optimal switched static output feedback with $\mathcal{L}_2 - gain$ for system (7.83) and (7.84) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned} &\text{Minimize}\ \ \gamma \\ \text{s. t. LMIs}\ \ &(7.97)\text{–}(7.98),\quad \forall\,(j,s) \in \mathcal{S}\times\mathcal{S} \end{aligned}$$

[1] This follows since $p \ge rank[E] \ge rank[EC_j] = rank[C_j\Lambda] \ge rank[(C_j\Lambda)\Lambda^{-1}] = rank[C_j] = p$

$$X_j,\ X_s,\ Y_j,\ \mathcal{S},\ \mathcal{R},\ \Upsilon,\ \Lambda,\ L_{1j},\ L_{2j},\ E_1,\ E_2,\ \{\Theta_j,\ j=1,\ldots,7\},\\ \gamma>0,\ \sigma>0,\ \kappa>0$$

In the case of polytopic representation (7.53) and (7.54), the corresponding convex minimization problem takes the form

$$\begin{aligned}&\text{Minimize}\ \ \gamma\\ &\text{wrt}\ X_j,\ X_s,\ Y_j,\ \mathcal{S},\ \mathcal{R},\ \Upsilon,\ \Lambda,\ L_{1j},\ L_{2j},\ E_1,\ E_2,\ \{\Theta\}_1^7,\\ &\gamma>0,\ \sigma>0,\ \kappa>0\end{aligned}$$

the LMIs $\forall\ (j,s)\in\mathcal{S}\times\mathcal{S}$ and $p=1,\ldots,M_j$

$$\begin{bmatrix} -\Delta_{1s} & \widehat{\Delta}_{2j} & \widehat{\Delta}_{3j} & \Lambda & \Lambda & -\bar{d}\Theta_1 & \Gamma_j & 0\\ \bullet & -\Delta_{4j} & \Delta_5 & \Theta_4^t & \Theta_5^t & -\bar{d}\Theta_2 & \widehat{\Delta}_{7jp} & \Theta_6\\ \bullet & \bullet & -\Delta_6 & -\Theta_4^t & -\Theta_5^t & -\bar{d}\Theta_3 & \widehat{\Delta}_{8jp} & 0\\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Theta_4 & 0 & 0\\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Theta_5 & 0 & 0\\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t & 0\\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0\\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

$$\begin{aligned}&C_j\Lambda=E_1C_j,\quad C_{dj}\Lambda=E_2C_{dj}\\ &\widehat{\Delta}_{2jp}=A_{jp}\Lambda+B_{jp}L_{1j}C_{jp}+\Theta_1-\bar{d}\mathcal{S},\ \ \Delta_{3jp}=A_{djp}\Lambda^t+B_{jp}L_{2j}C_{djp}-\Theta_1\\ &\widehat{\Delta}_{7jp}=\Lambda G_{jp}^t+C_{jp}^tL_{1j}^tD_{jp}^t,\ \ \widehat{\Delta}_{8jp}=\Lambda G_{djp}^t+C_{djp}^tL_{2j}^tD_{jp}^t\end{aligned}$$

Theorem 7.25 *Given $\bar{d}>0$ and $\underline{d}>0$. Switched linear system (7.77), (7.78), and (7.79) with static output-feedback $u_k=G_j\ y_k$ is delay-dependent asymptotically stable with an $\mathcal{L}_2-gain<\gamma$ it follows that there exist matrices $0<\mathcal{X}_j^t=\mathcal{X}_j,\ 0<\mathcal{X}_s^t=\mathcal{X}_s,\ \mathcal{Y}_j,\ 0<\mathcal{S}^t=\mathcal{S},\ 0<\mathcal{R}^t=\mathcal{R},\ \Upsilon,\ \Lambda,\ L_{1j},\ L_{2j},\ E_1,\ E_2,$ $\{\mathcal{N}_j,\ j=1,\ldots,5\},\ (j,s)\in\{1,2,...,S\}$ and a scalar $\gamma>0$ satisfying the LMIs for $(j,s)\in\mathcal{S}$*

$$\begin{bmatrix} -\Delta_{1s} & \widehat{\Delta}_{2j} & \widehat{\Delta}_{3j} & -\bar{d}\Theta_1 & \Gamma_j\\ \bullet & -\Delta_{4j} & \Delta_5 & -\bar{d}\Theta_2 & \widehat{\Delta}_{7j}\\ \bullet & \bullet & -\Delta_6 & -\bar{d}\Theta_3 & \widehat{\Delta}_{8j}\\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Lambda\Phi_j^t\\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} < 0 \tag{7.100}$$

$$C_j\Lambda=E_1C_j,\quad C_{dj}\Lambda=E_2C_{dj} \tag{7.101}$$

Moreover, the switched static output-feedback gain is given by $G_j=L_{1j}E_1^{-1}$.

Next, we examine the case of dynamic output feedback using a switched observer-based controller and employ it at every mode $j\in\mathcal{S}$ of the form:

$$\begin{aligned} \mu_{k+1} &= A_j \mu_k + B_j u_k + G_j [y_k - C_j \ \mu_k] \\ u_k &= K_j \mu_k \end{aligned} \tag{7.102}$$

where the gain matrices $G_j \in \Re^{n \times p}$ and $K_j \in \Re^{m \times n}$ are to be determined. Connecting controller (7.102) to switched system (7.77), (7.78), and (7.79) and defining the composite vector $\widehat{x}_k^t = \begin{bmatrix} \mu_k^t & x_k^t - \mu_k^t \end{bmatrix}$, we get the closed-loop system

$$\begin{aligned} \widehat{x}_{k+1} &= \overline{A}_j \widehat{x}_k + \overline{A}_{dj} \widehat{x}_{k-d_k} + \overline{\Gamma}_j w_k + \bar{f}_j + \bar{h}_j \\ z_k &= \overline{G}_j \widehat{x}_k + \overline{G}_{dj} \widehat{x}_{k-d_k} + \Phi_j w_k \end{aligned} \tag{7.103}$$

where the respective matrices are given by

$$\begin{aligned} &\overline{A}_j = \begin{bmatrix} A_j + B_j K_j & G_j C_j \\ 0 & A_j - G_j C_j \end{bmatrix}, \ \overline{A}_{dj} = \begin{bmatrix} 0 & 0 \\ 0 & A_{dj} - G_j C_{dj} \end{bmatrix}, \ \overline{\Gamma}_j = \begin{bmatrix} 0 \\ \Gamma_j \end{bmatrix} \\ &\bar{f}_j = \begin{bmatrix} 0 \\ f_j \end{bmatrix}, \ \bar{h}_j = \begin{bmatrix} 0 \\ h_j \end{bmatrix}, \ \overline{G}_j = [G_j + D_j K_j \quad G_j] \\ &\overline{G}_{dj} = [G_{dj} \quad G_{dj}] \end{aligned} \tag{7.104}$$

Application of **Theorem 7.13** shows that switched system (7.103) is delay-dependent asymptotically stable with an $\mathcal{L}_2-$ gain $< \gamma$ if there exist matrices $0 < \mathcal{P}_j^t = \mathcal{P}_j, \ 0 < \mathcal{P}_s^t = \mathcal{P}_s, \ \widehat{\mathcal{X}}_j, \ 0 < \widehat{\mathcal{Q}}^t = \widehat{\mathcal{Q}}, \ 0 < \widehat{\mathcal{W}}^t = \widehat{\mathcal{W}}, \ \{\mathcal{N}_j, \ j = 1, \ldots, 5\}, \ (j, s) \in \{1, 2, \ldots\}$ and scalars $\gamma > 0, \ \sigma > 0, \ \kappa > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$

$$\widehat{\mathcal{X}}_j \widetilde{\mathcal{A}}_j + \widetilde{\mathcal{A}}_j^t \widehat{\mathcal{X}}_j^t + \widetilde{\mathcal{P}}_{js} < 0 \tag{7.105}$$

$$\widetilde{\mathcal{P}}_{js} = \begin{bmatrix} -\mathcal{P}_s + \bar{d}\widehat{\mathcal{W}} & \mathcal{N}_1 - \bar{d}\widehat{\mathcal{W}} & -\mathcal{N}_1 & 0 & 0 & -\bar{d}\mathcal{N}_1 & 0 \\ \bullet & -\Pi_2 & -\mathcal{N}_2 - \mathcal{N}_2^t & \mathcal{N}_4^t & \mathcal{N}_5^t & -\bar{d}\mathcal{N}_2 & \overline{G}_j^t \\ \bullet & \bullet & -\Pi_3 & -\mathcal{N}_4^t & -\mathcal{N}_5^t & -\bar{d}\mathcal{N}_3 & \overline{G}_{dj}^t \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\mathcal{N}_4 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\mathcal{N}_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\widehat{\mathcal{W}} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \tag{7.106}$$

$$\begin{aligned} \widetilde{\mathcal{A}}_j &= \begin{bmatrix} -I \ \overline{A}_j \ \overline{A}_{dj} \ I \ I \ 0 \ \overline{\Gamma}_j \end{bmatrix} \\ \Pi_2 &= \mathcal{P}_j - (\bar{d} - \underline{d} + 1)\widehat{\mathcal{Q}} - \bar{d}\widehat{\mathcal{W}} - \mathcal{N}_2 - \mathcal{N}_2^t - \sigma\alpha^2 \bar{F}^t \bar{F} \\ \Pi_3 &= \mathcal{N}_3 + \mathcal{N}_3^t + \widehat{\mathcal{Q}} - \kappa\beta^2 \bar{H}^t \bar{H} \end{aligned} \tag{7.107}$$

where $\{\mathcal{N}\}_1^5$ are a set of free-parameter matrices that play the same role for the composite system (7.103) as $\{\mathcal{M}\}_1^5$ do for system (7.77), (7.78), and (7.79). Note that $\widehat{\mathcal{X}}_j, \ \widehat{\mathcal{Q}}, \ \widehat{\mathcal{W}}$ have dimensions compatible with system (7.103). To facilitate further development, define $\widehat{\mathcal{X}}_j = \begin{bmatrix} \widehat{\Upsilon}^t \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}^t, \ \widehat{\Upsilon} \in \Re^{2n \times 2n}$. Now, in order to benefit from the foregoing design results, we conveniently express $\widehat{\Upsilon}$ and $\widehat{\Upsilon}^{-1}$ in the form

$$\widehat{\Upsilon} = \begin{bmatrix} \Upsilon_s & 0 \\ \Upsilon_o & \Upsilon_c \end{bmatrix}, \; \widehat{\Upsilon}^{-1} = \widehat{\Lambda} = \begin{bmatrix} \Lambda_s & 0 \\ \Upsilon_o & \Lambda_c \end{bmatrix}, \; \mathcal{P}_j = \begin{bmatrix} \mathcal{P}_{1j} & 0 \\ \mathcal{P}_{2j} & \mathcal{P}_{3j} \end{bmatrix}$$
$$\widehat{\mathcal{S}} = \begin{bmatrix} \mathcal{S}_1 & 0 \\ \mathcal{S}_2 & \mathcal{S}_3 \end{bmatrix}, \; \widehat{\mathcal{R}} = \begin{bmatrix} \mathcal{R}_1 & 0 \\ \mathcal{R}_2 & \mathcal{R}_3 \end{bmatrix}, \; \Psi_k = \begin{bmatrix} \Psi_{1k} & 0 \\ \Psi_{2k} & \Psi_{3k} \end{bmatrix}$$
$$\mathcal{X}_j = \mathcal{P}_j^{-1} = \begin{bmatrix} \mathcal{X}_{1j} & 0 \\ \mathcal{X}_{2j} & \mathcal{X}_{3j} \end{bmatrix} [6pt] \tag{7.108}$$

The following design result is established:

Theorem 7.26 *Given $\bar{d} > 0$ and $\underline{d} > 0$. Switched system (7.83) and (7.84) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist matrices $0 < \mathcal{X}_j^t = \mathcal{X}_j$, $0 < \mathcal{X}_s^t = \mathcal{X}_s$, Y_j, $0 < \widehat{\mathcal{S}}^t = \widehat{\mathcal{S}}$, $0 < \widehat{\mathcal{R}}^t = \widehat{\mathcal{R}}$, $\widehat{\Upsilon}$, $\widehat{\Lambda}$, E_s, Ω_1, Ω_2, $\{\Psi_j, \; j = 1, \ldots, 7\}$, $(j, s) \in \{1, 2, ...\}$ and scalars $\gamma > 0$, $\sigma > 0$, $\kappa > 0$ satisfying the LMIs for $(j, s) \in \mathcal{S}$*

$$\begin{bmatrix} -\Sigma_{1s} & \Sigma_{2j} & \Sigma_{3j} & \widehat{\Lambda} & \widehat{\Lambda} & -\bar{d}\Psi_1 & \overline{\Gamma}_j & 0 \\ \bullet & -\Sigma_4 & -\Sigma_5 & \Psi_4^t & \Psi_5^t & -\bar{d}\Psi_2 & \Sigma_{7j} & \Psi_6 \\ \bullet & \bullet & -\Sigma_6 & -\Psi_4^t & -\Psi_5^t & -\bar{d}\Psi_3 & \Sigma_{8j} & 0 \\ \bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Psi_4 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Psi_5 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.109}$$

$$\Sigma_{1s} = \begin{bmatrix} \Lambda_s + \Lambda_s^t + \mathcal{X}_{1s} + \bar{d}\mathcal{S}_1 & \Lambda_o^t \\ \Lambda_o + \mathcal{X}_{2s} + \bar{d}\mathcal{S}_2 & \Lambda_c + \Lambda_c^t + \mathcal{X}_{3s} - \bar{d}\mathcal{S}_1 \end{bmatrix}$$
$$\Sigma_{2j} = \begin{bmatrix} \Psi_{11} + A_j\Lambda_s^t + B_jY_j - \bar{d}\mathcal{S}_1 & A_j\Lambda_o^t + \Omega_1 \\ \Psi_{21} - \bar{d}\mathcal{S}_2 & A_j\Lambda_c^t - L_{1j}C_j \end{bmatrix}$$
$$\Sigma_{3j} = \begin{bmatrix} -\Psi_{11} & 0 \\ -\Psi_{12} & -\Psi_{13} + A_{dj}\Lambda_c^t - G_jE_c\Lambda_c^t \end{bmatrix}$$
$$\Sigma_5 = \begin{bmatrix} \Psi_{21} + \Psi_{21}^t & 0 \\ \Psi_{22} + \Psi_{22}^t & \Psi_{23} + \Psi_{23}^t \end{bmatrix}$$
$$\Sigma_4 = \begin{bmatrix} \mathcal{P}_{1s} - \widehat{d}\mathcal{S}_1 - \mathcal{R}_1 - \Psi_{21} - \Psi_{21}^t & 0 \\ \mathcal{P}_{2s} - \widehat{d}\mathcal{S}_2 - \mathcal{R}_2 - \Psi_{22} - \Psi_{22}^t & \mathcal{P}_{3s} - \widehat{d}\mathcal{S}_3 - \mathcal{R}_3 - \Psi_{23} - \Psi_{23}^t \end{bmatrix},$$
$$\Sigma_6 = \begin{bmatrix} \Psi_{31} + \Psi_{31}^t + \mathcal{R}_1 - \Psi_{71} & 0 \\ \Psi_{32} + \Psi_{32}^t + \mathcal{R}_2 - \Psi_{72} & \Psi_{33} + \Psi_{33}^t + \mathcal{R}_3 - \Psi_{73} \end{bmatrix},$$
$$\Sigma_{7j} = \begin{bmatrix} \Lambda_s G_j^t + Y_j^t D_j^t \\ \Lambda_o G_j^t + \Omega_2 \end{bmatrix}, \; \Sigma_{8j} = \begin{bmatrix} \Lambda_s G_{dj}^t \\ \Lambda_o G_{dj}^t + \Lambda_c G_{dj}^t \end{bmatrix} \tag{7.110}$$

where $\widehat{d} = (\bar{d} - \underline{d} + 1)$. *Moreover, the gain matrices are given by*

$$K_j = Y_j \Lambda_s^{-t}, \quad G_j = L_{1j} E_s^{-1} \tag{7.111}$$

Proof Applying the congruence transformation

$$T = \mathrm{diag}[\widehat{\Lambda},\ \widehat{\Lambda},\ \widehat{\Lambda},\ \widehat{\Lambda},\ \widehat{\Lambda},\ \widehat{\Lambda},\ I]$$

to inequality (7.105) using (7.106) and (7.107) and the linearizations

$$\begin{aligned}
&X_s = \Upsilon^{-t} P_s \Upsilon^{-1},\ \mathcal{S} = \Upsilon^{-t} \mathcal{W} \Upsilon^{-1},\ \{\Theta\}_1^5 = \Upsilon^{-t}\{\mathcal{M}\}_1^5 \Upsilon^{-1},\ X_j = \Upsilon^{-t} P_j \Upsilon^{-1} \\
&\Psi_6 = \sigma\alpha \widehat{\Lambda} \bar{F}^t,\ \Psi_7 = \kappa\beta^2 \widehat{\Lambda} \bar{H}^t \bar{H} \widehat{\Lambda}^t \\
&\Omega_1 = B_j K_j \Lambda_o^t + G_j C_j \Lambda_c^t,\ \Omega_2 = \Lambda_o G_j^t K_j + \Lambda_c G_j^t
\end{aligned}$$

we immediately obtain LMI (7.109) subject to (7.110). ■

Remark 7.27 The optimal switched dynamic output feedback with $\mathcal{L}_2 - gain$ for system (7.103) and (7.104) subject to the polytopic representation (7.53) and (7.54) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{gathered}
\text{Minimize} \quad \gamma \\
\text{wrt } \mathcal{X}_j, \mathcal{X}_s, Y_j, \widehat{\mathcal{S}}, \widehat{\mathcal{R}}, \widehat{\Lambda}, E_s,\ \Omega_1, \Omega_2, \{\Psi_j,\ j = 1, \ldots, 7\}, \forall (j, s) \in \mathcal{S}, \sigma, \kappa
\end{gathered}$$

$$\begin{bmatrix}
-\Sigma_{1s} & \Sigma_{2jp} & \Sigma_{3jp} & \widehat{\Lambda} & \widehat{\Lambda} & -\bar{d}\Psi_1 & \overline{\Gamma}_{jjp} & 0 \\
\bullet & -\Sigma_4 & -\Sigma_5 & \Psi_4^t & \Psi_5^t & -\bar{d}\Psi_2 & \Sigma_{7jp} & \Psi_6 \\
\bullet & \bullet & -\Sigma_6 & -\Psi_4^t & -\Psi_5^t & -\bar{d}\Psi_3 & \Sigma_{8jp} & 0 \\
\bullet & \bullet & \bullet & -\sigma I & 0 & -\bar{d}\Psi_4 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\kappa I & -\bar{d}\Psi_5 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_{jp}^t & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I
\end{bmatrix} < 0$$

$$\Sigma_{2jp} = \begin{bmatrix} \Psi_{11} + A_{jp}\Lambda_s^t + B_{jp}Y_j - \bar{d}\mathcal{S}_1 & A_{jp}\Lambda_o^t + \Omega_1 \\ \Psi_{21} - \bar{d}\mathcal{S}_2 & A_{jp}\Lambda_c^t - L_{1j}C_{jp} \end{bmatrix}$$

$$\Sigma_{7jp} = \begin{bmatrix} \Lambda_s G_{jp}^t + Y_j^t D_{jp}^t \\ \Lambda_o G_{jp}^t + \Omega_2 \end{bmatrix}, \quad \Sigma_{8jp} = \begin{bmatrix} \Lambda_s G_{djp}^t \\ \Lambda_o G_{dj}^t + \Lambda_c G_{djp}^t \end{bmatrix}$$

$$\Sigma_{3jp} = \begin{bmatrix} -\Psi_{11} & 0 \\ -\Psi_{12} & -\Psi_{13} + A_{djp}\Lambda_c^t - G_{jp}E_c\Lambda_c^t \end{bmatrix}$$

Finally, we have the following result:

Theorem 7.28 *Given* $\bar{d} > 0$ *and* $\underline{d} > 0$. *Linear switched system (7.77), (7.78), and (7.79) is delay-dependent asymptotically stable with an* $\mathcal{L}_2 - gain < \gamma$ *if there* *exist matrices* $0 < \mathcal{X}_j^t = \mathcal{X}_j,\ 0 < \mathcal{X}_s^t = \mathcal{X}_s,\ Y_j,\ 0 < \widehat{\mathcal{S}}^t = \widehat{\mathcal{S}},\ 0 < \widehat{\mathcal{R}}^t = \widehat{\mathcal{R}},\ \widehat{\Upsilon},\ \widehat{\Lambda},\ E_s,\ \Omega_1,\ \Omega_2,\ \{\Psi_j,\ j = 1, \ldots, 5\},\ \forall (j, s) \in \mathcal{S}$ *and a scalar* $\gamma > 0$ *satisfying the LMIs for* $(j, s) \in \mathcal{S}$

$$\begin{bmatrix} -\Sigma_{1s} & \bar{\Sigma}_{2j} & \Sigma_{3j} & -\bar{d}\Psi_1 & \overline{\Gamma}_j \\ \bullet & -\Sigma_4 & -\Sigma_5 & -\bar{d}\Psi_2 & \Sigma_{7j} \\ \bullet & \bullet & -\Sigma_6 & -\bar{d}\Psi_3 & \Sigma_{8j} \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} < 0 \tag{7.112}$$

$$\bar{\Sigma}_{2j} = \begin{bmatrix} \Psi_{11} + A_j\Lambda_s^t + B_j Y_j - \bar{d}\mathcal{S}_1 & A_j\Lambda_o^t \\ \Psi_{21} - \bar{d}\mathcal{S}_2 & A_j\Lambda_c^t - L_{1j}C_j \end{bmatrix}$$

$$\bar{\Sigma}_{7j} = \begin{bmatrix} \Lambda_s G_j^t + Y_j^t D_j^t \\ \Lambda_o G_j^t \end{bmatrix} \tag{7.113}$$

where $\widehat{d} = (\bar{d} - \underline{d} + 1)$ *and* $\Sigma_{1s},\ \Sigma_{3j},\ \Sigma_4,\ \Sigma_5,\ \Sigma_6,\ \Sigma_{7j},\ \Sigma_{8j}$ *are given by (7.110). Moreover, the gain matrices are given by*

$$K_j = Y_j \Lambda_s^{-t}, \quad G_j = L_{1j} E_s^{-1} \tag{7.114}$$

In the next section, we consider some examples for numerical implementation and compare the results with the existing methods.

Illustrative Example E

Here, we consider a discrete model of the type (7.46), (7.47), and (7.48) with multiple modes. In terms of our terminology, each mode represents a particular equilibrium operating point. We wish to design a switched feedback control for this system. Switching taking place between the modes is described by the following coefficients:

Mode 1

$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.6 & 0.3 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 & 0.9 \\ 0.7 & 2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ C_{d1} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix},\ G_{d1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix},\ \Phi_1 = [0.1],\ D_1 = \begin{bmatrix} 0.9 & 0.3 \end{bmatrix}$$

$$||f_1(x_k, k)|| \le \alpha_1 ||x_k||, \quad ||h_1(x_k, k)|| \le \beta_1 ||x_{k-d_k}||$$

$$|\alpha_1| \le 0.15, \quad |\beta_1| \le 0.15$$

Mode 2

$$A_2 = \begin{bmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.6 & 0 \\ 0.2 & 0.3 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},\ B_2 = \begin{bmatrix} 2 & 1 \\ 0.6 & 1 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_{d2} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix},\ G_{d2} = \begin{bmatrix} 0 & 0.5 \end{bmatrix},\ \Phi_2 = [0.6],\ D_2 = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}$$
$$||f_2(x_k, k)|| \leq \alpha_2 ||x_k||,\quad ||h_2(x_k, k)|| \leq \beta_2 ||x_{k-d_k}||$$
$$|\alpha_2| \leq 0.25,\quad |\beta_2| \leq 0.25$$

Mode 3

$$A_3 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix},\ A_{d3} = \begin{bmatrix} -0.5 & 0.1 \\ 0 & -0.4 \end{bmatrix},\ \Gamma_3 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ C_{d3} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$G_3 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix},\ G_{d3} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix},\ \Phi_3 = [0.3],\ D_3 = \begin{bmatrix} 0.8 & 0.3 \end{bmatrix}$$
$$||f_3(x_k, k)|| \leq \alpha_3 ||x_k||,\quad ||h_3(x_k, k)|| \leq \beta_3 ||x_{k-d_k}||$$
$$|\alpha_3| \leq 0.35,\quad |\beta_3| \leq 0.35$$

For simulation purposes, we select $\underline{d} = 2$ and implementing the LMI solver Scilab 5.1.1, a feasible solution of the convex optimization problem given in **Remark 4.18** for the case of state feedback is attained for $\bar{d} = 9,\ \gamma = 0.9875$. The ensuing results are given by

$$K_1 = \begin{bmatrix} -0.1144 & 0.0202 \\ 0.3168 & -0.6754 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.3313 & 0.0525 \\ -0.1685 & -0.0088 \end{bmatrix}$$
$$K_3 = \begin{bmatrix} -0.5539 & -0.0424 \\ -0.2306 & -0.0423 \end{bmatrix}$$

For the purpose of comparison, we provide in Table 7.4 a computational summary of applying several published methods. It is quite clear that the developed switching state-feedback control strategy provides a better performance bound in contrast to the existing methods.

Table 7.4 Computational summary of state-feedback design: illustrative example E

Method	$\underline{d}$	$\bar{d}$	γ
[50]	6	11	3.4682
[174]	4	13	3.8335
[298]	4	10	2.3650
[368]	3	15	1.9874
Theorem 7.22	2	9	0.9875

We next consider the convex optimization problem given in **Remark 7.24** for the static output feedback. A feasible solution is reached with $\underline{d} = 4$ and $\bar{d} = 12$. The corresponding gains are given by

$$
\begin{aligned}
G_1 &= \begin{bmatrix} -0.3550 & 0.1341 \\ -0.6522 & 0.6754 \end{bmatrix}, \; G_2 = \begin{bmatrix} -0.4756 & 0.2511 \\ -0.0985 & 0.1235 \end{bmatrix} \\
G_3 &= \begin{bmatrix} -0.0756 & -0.1948 \\ -0.4416 & 0.4512 \end{bmatrix}
\end{aligned}
$$

Finally, we attend to the observer-based output feedback. The results of the feasible computations are summarized as follows:

$$
\begin{aligned}
\underline{d} &= 3, \quad \bar{d} = 14 \\
G_1 &= \begin{bmatrix} -0.2453 & 0.4131 \\ -0.6522 & 0.6754 \end{bmatrix}, \; K_1 = \begin{bmatrix} -0.4675 & 0.1251 \\ -0.1859 & 0.5114 \end{bmatrix} \\
G_2 &= \begin{bmatrix} -0.0756 & -0.1948 \\ -0.4407 & 0.2856 \end{bmatrix} K_2 = \begin{bmatrix} -0.3145 & 0.1341 \\ -0.6522 & 0.6754 \end{bmatrix} \\
G_3 &= \begin{bmatrix} -0.5226 & 0.4825 \\ -0.1385 & 0.5008 \end{bmatrix}, \; K_3 = \begin{bmatrix} -0.1336 & -0.1608 \\ -0.4006 & 0.5217 \end{bmatrix}
\end{aligned}
$$

Illustrative Example F

Here, we consider a discrete model of the type (7.46), (7.47), and (7.48) with two modes with the basic linearized data from [50]. We wish to design a switched feedback control for this system. Switching occurring between the modes is described by the following coefficients:

Mode 1

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.09 & 1.00 \\ 0.00 & 1.2 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} -0.03 & 0 \\ 0.08 & 0.05 \end{bmatrix}, \; \Gamma_1 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \\
B_1 &= \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\
G_1 &= \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, \; \Phi_1 = [0.1], \; D_1 = \begin{bmatrix} 0.9 & 0.3 \end{bmatrix} \\
C_{d1} &= \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}, \; G_{d1} = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix} \\
||f_1(x_k, k)|| &\leq \alpha_1 ||x_k||, \quad ||h_1(x_k, k)|| \leq \beta_1 ||x_{k-d_k}|| \\
|\alpha_1| &\leq 0.1, \quad |\beta_1| \leq 0.2
\end{aligned}
$$

Mode 2

$$A_2 = \begin{bmatrix} 1.2 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \ A_{d2} = \begin{bmatrix} 0.03 & 0.08 \\ 0 & -0.05 \end{bmatrix}, \ \Gamma_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}, \ \Psi_2 = [0.3], \ D_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$$
$$C_{d2} = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix}, \ G_{d2} = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}$$
$$||f_2(x_k, k)|| \leq \alpha_2||x_k||, \quad ||h_2(x_k, k)|| \leq \beta_2||x_{k-d_k}||$$
$$|\alpha_2| \leq 0.2, \quad |\beta_1| \leq 0.1$$

By selecting $\underline{d} = 2$ and implementing the LMI solver Scilab 4, a feasible solution of the convex optimization problem given in **Remark 4.18** for the case of state feedback is attained for $\bar{d} = 11, \ \gamma = 0.9875, \ \sigma = 1, \ \kappa = 1$. The obtained results are given by

$$K_1 = \begin{bmatrix} 0.0905 & -1.0896 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7154 & -0.0375 \end{bmatrix}$$

In Fig. 7.1, the closed-loop state trajectories are plotted.

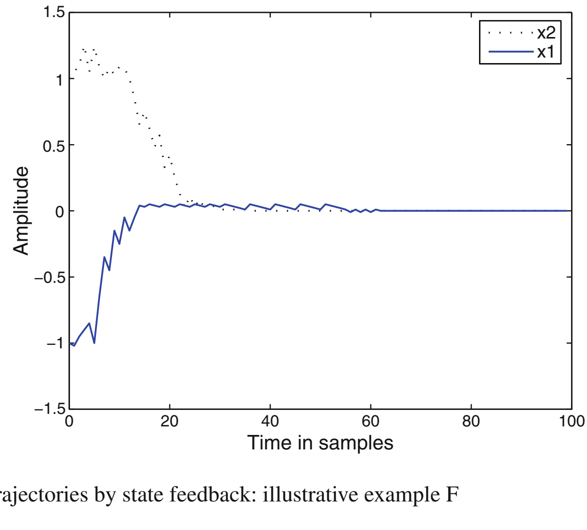

Fig. 7.1 State trajectories by state feedback: illustrative example F

Next, we consider the convex optimization problem given in **Remark 7.24** for the static output feedback. A feasible solution is reached with $\underline{d} = 4$ and $\bar{d} = 12$. The corresponding gains are given by

$G_1 = \begin{bmatrix} 0.6524 & -0.6752 \end{bmatrix}, \ G_2 = \begin{bmatrix} -0.4516 & -0.0755 \end{bmatrix}$. Similarly, the closed-loop state trajectories under switched output feedback are plotted in Fig. 7.2. It is readily evident from the displayed results that the switched feedback controllers (state and output) are quite effective in stabilizing the nonlinear switched system.

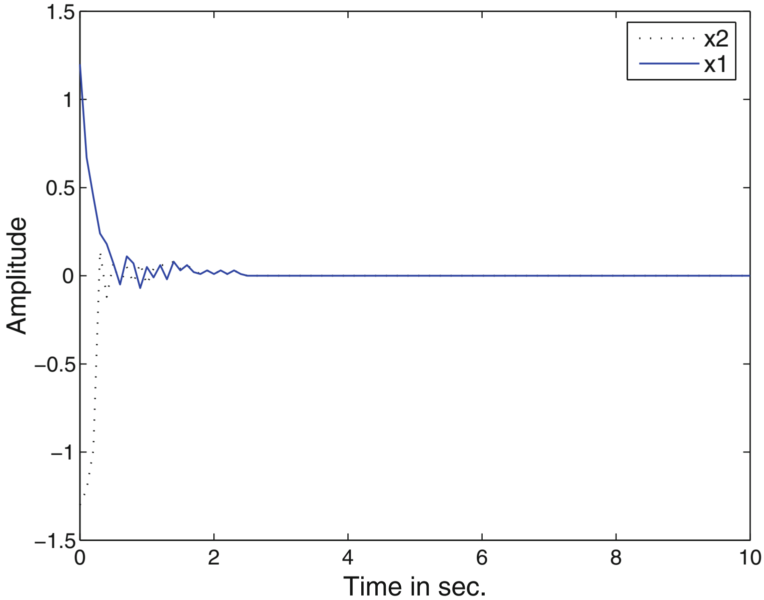

Fig. 7.2 State trajectories by output feedback: illustrative example F

Finally, we attend to the observer-based output feedback. The results of the feasible computations are summarized as follows:

$$\underline{d} = 3, \quad \bar{d} = 14$$

$$G_1 = \begin{bmatrix} -0.2453 & 0.4131 \\ -0.6522 & 0.6754 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -0.4675 & 0.1251 \\ -0.1859 & 0.5114 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -0.0756 & -0.1948 \\ -0.4407 & 0.2856 \end{bmatrix} K_2 = \begin{bmatrix} -0.3145 & 0.1341 \\ -0.6522 & 0.6754 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} -0.5226 & 0.4825 \\ -0.1385 & 0.5008 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -0.1336 & -0.1608 \\ -0.4006 & 0.5217 \end{bmatrix}$$

7.3 Multi-Controller Structure

During the past decade, problems of stability and control of switched systems have received increasing interests [42, 47, 174, 192] and the references cited therein. Reported results under arbitrary switching are found in [56] using multiple Lyapunov functions for nonlinear systems and in [42] employing switched Lyapunov functions. Of particular interest in this chapter is the class of switched time delay

(STD) systems, which have widespread engineering applications, including network control systems and power systems [47]. More recently, some theoretical studies were conducted for STD systems including [26, 223–226, 238–241, 252, 270–275, 278–280, 283–285] where different design methods were developed.

In this paper a switched-state feedback control is designed to deal with a class of continuous-time systems subject to linear fractional parametric uncertainty and interval time delays. Looked at in this light, the results of [88, 89] are generalized to cope with switched time-delay systems. An improved Lyapunov–Krasovskii functional is constructed to derive robust delay-dependent switching policies. The problem is treated as multi-controller configurations and a switched feedback approach is developed to jointly determine the feedback gains and the switching rule while minimizing a suitable guaranteed cost. The developed results are tested on representative examples.

7.3.1 Problem Statement

We consider hereafter the following switched time-delay system with full parametric uncertainties

$$\begin{aligned}\dot{x}(t) &= A_{\xi(x)}x(t) + D_{\xi(x)}x(t-\tau_{\xi(x)}) + M_{\xi(x)}q(t) \\ &\quad + \Gamma_{\xi}(x)w(t) \qquad (7.115)\\ p(t) &= E_{\xi(x)}x(t) + H_{\xi(x)}x(t-\tau_{\xi(x)}) + L_{\xi(x)}q(t) \qquad (7.116)\\ z(t) &= C_{\xi(x)}x(t) + G_{\xi(x)}x(t-\tau_{\xi(x)}) \\ q(t) &= \Delta p(t), \quad \Delta \in \boldsymbol{\Delta} \qquad (7.117)\end{aligned}$$

where $x(t) \in \Re^n$ is the state vector , $u(t) \in \Re^m$ is the control input , $w(t) \in \Re^v$ is the exogenous input, $z(t) \in \Re^v$ is the observed output, and the function $\xi(x) : \Re^n \to \mathsf{N} = \{1, ..., N\}$ is a switching rule. The vectors $p(t) \in \Re^s$, $q(t) \in \Re^r$ are internal variables. For all $j \in \mathsf{N}$, the matrices $A_j \in \Re^{n\times n}$, $D_j \in \Re^{n\times n}$, $M_j \in \Re^{n\times r}$, $C_j \in \Re^{v\times n}$, $G_j \in \Re^{v\times n}$, $E_j \in \Re^{s\times n}$, $H_j \in \Re^{s\times n}$, $L_j \in \Re^{s\times r}$, and $\Gamma_j \in \Re^{n\times v}$ are real and known constants and $\tau_j(t)$ denotes an interval time-varying delay satisfying

$$0 < \tau_{mj} \le \tau_j \le \tau_{Mj}, \quad \dot{\tau_j}(t) \le \mu_j, \quad j \in \mathsf{N} \qquad (7.118)$$

where the bounds $\tau_{mj} > 0$, $\tau_{Mj} > 0$ are known constants. Observe that whenever deemed necessary, the variables (p, q) could be easily eliminated when $W_j \neq 0$. The initial condition $x(s) = \kappa(s)$, $s \in [-\tau_M, 0]$ is a differentiable vector-valued function. Matrix Δ represents the parametric uncertainty, which belongs to the set $\boldsymbol{\Delta}$ defined by

$$\boldsymbol{\Delta} = \{\boldsymbol{\Delta} \in \Re^{r\times s} : ||\boldsymbol{\Delta}||_\infty \le 1\} \qquad (7.119)$$

Remark 7.29 It should be noted that the model described by (7.115), (7.116), and (7.117) represents a class of linear continuous-time systems with mode-dependent interval delays and a linear-fractional transformation (LFT) structure [216]. Under arbitrary switching and special characterizations of $\Delta \in \mathbf{\Delta}$, different methods have been proposed in the literature to design feedback stabilization such that the closed-loop system remains asymptotically stable [280] and the references therein. In the case of free-delay systems, excellent results are recently reported in [88–90, 92] and [196].

For systems without delays, an approach was proposed in [342] and the references therein, where given a set of N state-feedback gain matrices $\mathsf{K} : \{K_1, ..., K_N\}$, a switching function $\xi(x) : \Re^n \to \mathsf{K}$ is determined such that the state feedback switched control

$$u(t) = K_{\xi(x(t))}\, x(t) \tag{7.120}$$

assures the global asymptotic stability of the closed-loop time-varying system. The matrices $\mathsf{K} : \{K_1, ..., K_N\}$ were supposed to be given by the designer prior to the determination of the stabilizing switching function $\xi(.)$. Further improvements were attained [150] for the stabilizing problem. One of the basic modeling issues of switched systems [366] is the multi-controller configurations. Following this trend, a switched feedback approach was addressed in [89, 92] for linear systems with LFT parametric uncertainties to jointly determine the feedback gains and the switching rule while minimizing a suitable guaranteed cost. Our approach in this work extends the results of [89, 92] for the class of linear time-delay systems (7.115), (7.116), and (7.117).

7.3.2 Robust Delay-Dependent Switching Control

The objective now is to determine a switching rule $\xi(x)$ that guarantees the global asymptotic stability at the equilibrium point $x = 0$ and a suitable minimal value of the performance criteria $J(\xi)$ defined by

$$J(K_1, ..., K_N, \xi) = \max_{\Delta \in \mathbf{\Delta}} \sum_{j=1}^{N} \left\| z_j^t(t) z_j(t) \right\|_2^2 \tag{7.121}$$

is attained. Toward our goal, we consider the Lyapunov–Krasovskii functional $\mathsf{L}(x) = \min_{j \in \mathsf{N}} V_j(x)$, where

$$V_j(x) = x^t(t)\mathsf{P}_j x(t) + \int_{t-\varrho}^{t} x^t(s)\mathsf{R}_j x(s)\mathrm{d}s$$
$$+ \int_{-\varrho}^{0} \int_{t+s}^{t} \dot{x}^t(\alpha)\mathsf{Q}_j \dot{x}(\alpha)\mathrm{d}\alpha \; \mathrm{d}s$$

$$+\varphi_j \int_{-\varphi_j}^{0} \int_{t+s}^{t} \dot{x}_j^t(\alpha) \mathsf{W}_j \dot{x}(\alpha) \mathrm{d}\alpha \mathrm{d}s$$
$$+ (\varrho_j - \varphi_j) \int_{-\varrho_j}^{-\varphi_j} \int_{t+s}^{t} \dot{x}^t(\alpha) \mathsf{S}_j \dot{x}(\alpha) \mathrm{d}\alpha$$
$$+ \int_{t-\tau_j(t)}^{t} x^t(s) \mathsf{Z}_j x(s) \mathrm{d}s \qquad (7.122)$$

Since $\mathsf{L}(x)$ is not differentiable, we will work with the Dini derivative [160] defined by

$$D^+ \mathsf{L}(x) = \lim_{h \to 0^+} \sup \frac{\mathsf{L}(x(t+h)) - \mathsf{L}(x(t))}{h}$$

Observe in case of the switching rule $\xi(x) = j, \;\; j \in \mathsf{N},$ we have

$$D^+ \mathsf{L}(x) = \min_{j \in \mathsf{N}} (\mathsf{L}(x) = V_j(x(t))$$

where $V_j(x(t)$ is specified in (7.122). The next theorem provides a method to meet our objective

Theorem 7.30 *Given the delay bounds* $\varphi_j >, \varrho_j > 0, \mu_j > 0$ *for all* $j \in \mathsf{N}$. *If there exist matrices* $\mathsf{P}_j > 0, \mathsf{Q}_j > 0, \mathsf{R}_j > 0, \mathsf{S}_j > 0, \mathsf{Z}_j > 0, \;\; j \in \mathsf{N}$ *such that the following LMI holds for all* $j \in \mathsf{N}$

$$\Omega_j = \begin{bmatrix} \Omega_{aj} & \Omega_{cj} & \Omega_{zj} \\ \bullet & -\Omega_{ej} & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0 \qquad (7.123)$$

where

$$\Omega_{aj} = \begin{bmatrix} \Omega_{aaj} & \mathsf{W}_j & \Omega_{acj} \\ \bullet & -\Omega_{sj} & \mathsf{S}_j \\ \bullet & \bullet & -\Omega_{caj} \end{bmatrix}, \;\; \Omega_{zj} = \begin{bmatrix} C_j^t \\ 0 \\ G_j^t \end{bmatrix}$$
$$\Omega_{cj} = \begin{bmatrix} \mathsf{W}_j & \mathsf{P}_j M_j + E_j^t L_j & \varrho_j A_j^t \mathsf{W}_j & (\varrho_j - \varphi j) A_j^t \mathsf{W}_j \\ 0 & 0 & 0 & 0 \\ \mathsf{S}_j & H_j^t L_j & \varrho_j D_j^t \mathsf{W}_j & (\varrho_j - \varphi_j) D_j^t \mathsf{W}_j \end{bmatrix}$$
$$\Omega_{ej} = \begin{bmatrix} \Omega_{eaj} & \Omega_{ecj} \\ \bullet & \Omega_{eej} \end{bmatrix}, \;\; \Omega_{eej} = \begin{bmatrix} \mathsf{W}_j & 0 \\ \bullet & \mathsf{W}_j \end{bmatrix}$$
$$\Omega_{eaj} = \begin{bmatrix} \mathsf{R}_j + \mathsf{S}_j & 0 \\ \bullet & -L_j^t L_j + I \end{bmatrix}$$
$$\Omega_{ecj} = \begin{bmatrix} 0 & 0 \\ -\varrho_j M_j^t \mathsf{W}_j & -(\varrho_j - \varphi_j) M_j^t \mathsf{W}_j \end{bmatrix}$$

$$\begin{aligned}\Omega_{aaj} &= \mathsf{P}_j A_j + A_j^t \mathsf{P}_j + \mathsf{Q}_j + \mathsf{R}_j + \mathsf{Z}_j - \mathsf{W}_j + E_j^t E_j \\ \Omega_{sj} &= \mathsf{Q}_j + \mathsf{W}_j + \mathsf{S}_j, \ \Omega_{acj} = \mathsf{P}_j D_j + E_j^t H_j \\ \Omega_{caj} &= (1-\mu_j)\mathsf{Z}_j + 2\mathsf{S}_j - H_j^t H_j \end{aligned} \tag{7.124}$$

then the switching rule $\xi(x) = arg \min_{j \in \mathsf{N}} V_j(x)$ *with* $V_j(x)$ *being given in (7.122) renders the equilibrium solution* $x = 0$ *of the system (7.115), (7.116), and (7.117) globally asymptotically stable and*

$$\begin{aligned} J(\xi) &= \max_{\Delta \in \mathbf{\Delta}} \int_0^\infty z^t(t) z(t) \mathrm{d}t \\ &< \min_{j \in \mathsf{N}} \kappa^t \Big[\mathsf{P}_j + \varrho_j(\mathsf{R}_j + \mathsf{Z}_j) \Big] \kappa \end{aligned} \tag{7.125}$$

Proof Introduce the set of optimal indices

$$\mathsf{I}(x) = \{ j \in \mathsf{N} : V_j(x) = \mathsf{L}(x) \}$$

and let the switching rule be $\xi(x(t)) = j, \ t \geq 0$ for some $j \in \mathsf{N}$. In the appendix, the Dini derivative $\mathsf{D}^+\mathsf{L}(x)$ with respect to an arbitrary trajectory of (7.115), (7.116), and (7.117) is computed and is shown to have the form

$$\begin{aligned} \mathsf{D}^+\mathsf{L}(x) &= \min_{j \in \mathsf{N}} Y_j(x(t)) \\ Y_j(x(t)) &\leq \chi^t(t) \Xi_j \chi(t) + \varphi^2 \dot{x}^t(t) \mathsf{W}_j \dot{x}_j(t) \\ &\quad + (\varrho - \varphi)^2 \dot{x}^t(t) \mathsf{S}_j \dot{x}(t) \end{aligned} \tag{7.126}$$

where $\Xi_j, \ \ \chi(t)$ are given in the appendix. To complete our effort, we take into consideration (7.115) subject to $\Delta \in \mathbf{\Delta}$ or equivalently $\Delta^t \Delta \leq I$. Together with (7.117) and $\beta(t) = [x^t(t) \ x^t(t-\tau) \ q(t)]^t$, we get

$$\beta^t(t) \begin{bmatrix} E_j^t E_j & E_j^t H_j & E_j^t L_j \\ \bullet & H_j^t H_j & H_j^t L_j \\ \bullet & \bullet & L_j^t L_j - I \end{bmatrix} \beta(t) \geq 0 \tag{7.127}$$

Next, considering LMI (7.123) and applying the Schur complement for $\xi(x(t)) = j$ and using inequality (7.127), it follows that $\mathsf{D}^+ L(x) < -z^t z$. This holds true since $V_m(x) \geq V_j(x) = \mathsf{L}(x)$ for all $m \in \mathsf{N}$ and all $j \in \mathsf{I}(x)$. This emphasizes the stabilizing feature of the switching rule $\xi(x)$, which in turn means that the origin is globally asymptotically stable equilibrium point. Moreover, since from (7.123) and (7.124) $\mathsf{D}^+\mathsf{L}(x) < -z^t z$ for all $\Delta \in \mathbf{\Delta}$, it follows upon integration over the period $[0, \infty)$ and observing that $V_j(x)|_{t \to \infty} = 0, \ x(0) = \phi$, we obtain (7.125). ∎

Remark 7.31 When the conditions of **Theorem 7.30** are met, meaning a feasible solution exists, then the global asymptotic stability is guaranteed by the switching rule $\xi(x) = arg\min_{j \in \mathsf{N}} V_j(x)$. A guaranteed cost associated with $J(\xi)$ is also provided. It is interesting to observe that when LMI (7.123) is feasible for all $j \in \mathsf{N}$, it consequently implies the validity of the switching signal being kept constant $\xi(t) = j \in \mathsf{N}, \ \forall t \geq 0$. Such a result is pleasing and it is eventually interpreted as the stability of system (7.115), (7.116), and (7.117) is preserved under constant switching. In turn, this corresponds to $||z||^2/||q||^2 < 1$, see [216].

Remark 7.32 It is interesting to observe that when LMI (7.123) is feasible for all $j \in \mathsf{N}$, it consequently implies the validity of the switching signal being kept constant $\xi(t) = j, \ j \in \mathsf{N}, \ \forall t \geq 0$. Such a result is pleasing and it is eventually interpreted as the stability of system (7.115), (7.116), and (7.117) is preserved under constant switching. In turn, this corresponds to $||z||^2/||q||^2 < 1$, see [278].

Remark 7.33 We emphasize that the bound μ_j might take any value, which, due to the presence of LMI variable S_j, allows for slow ($\mu_j < 1$) and fast ($\mu_j > 1$) time-delay patterns. The case of time-delay pattern $0 < \tau_j \leq \tau_{Mj} \ \dot{\tau}(t)_j \leq \mu_j, \ j \in \mathsf{N}$ could be easily derived from **Theorem 7.30** by setting $\varphi_j \equiv 0, \ \mathsf{W}_j \equiv 0 \ \forall j \in \mathsf{N}$.

7.3.3 Delay-Dependent Switching Control Design

In this section, we seek to develop robust delay-dependent control strategy for the system

$$\dot{x}(t) = A_j x(t) + D_j x(t - \tau_j) + M_j q(t) + \Gamma_j w(t) + B_j u(t) \tag{7.128}$$

$$p(t) = E_j x(t) + H_j x(t - \tau_j) + L_j q(t) + T_j u(t) \tag{7.129}$$

$$z(t) = C_j x(t) + G_j x(t - \tau_j) + F_j u(t)$$

$$q(t) = \Delta p(t), \quad \Delta \in \mathbf{\Delta} \tag{7.130}$$

using the state-feedback switched control (7.120) and associated with the cost

$$J(K_1, ..., K_N, \xi) = \max_{\Delta \in \mathbf{\Delta}} \sum_{j=1}^{N} \left|\left| z_j^t(t) z_j(t) \right|\right|_2^2 \tag{7.131}$$

Where for each $j = 1, ..., N$ the trajectory $z_j(t)$ is the response of the closed-loop system associated with the input signal $w(t) = e_j \delta(t)$, with e_j being the jth column of identity matrix. The closed-loop system becomes

$$\dot{x}(t) = \hat{A}_j x(t) + D_j x(t - \tau_j) + M_j q(t) + \Gamma_j w(t) \tag{7.132}$$

$$p(t) = \hat{E}_j x(t) + H_j x(t - \tau_j) + L_j q(t) \tag{7.133}$$

$$z(t) = \hat{C}_j x(t) + G_j x(t - \tau_j)$$

$$q(t) = \Delta p(t), \quad \Delta \in \mathbf{\Delta} \tag{7.134}$$

$$\hat{A}_j = A_j + B_j K_j, \;\; \hat{E}_j = E_j + T_j K_j$$

$$\hat{C}_j = C_j + F_j K_j \tag{7.135}$$

for which the asymptotic stability is governed by the feasibility of the LMI

$$\begin{bmatrix} \hat{\Omega}_{aj} & \hat{\Omega}_{cj} & \hat{\Omega}_{zj} \\ \bullet & -\hat{\Omega}_{ej} & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.136}$$

where $\hat{\Omega}_{aj}, ..., \hat{\Omega}_{ej}$ correspond to $\Omega_{aj}, ..., \Omega_{aj}$ with $\hat{A}_j, \hat{E}_j, \hat{C}_j$ replacing A_j, E_j, C_j, respectively. The main design result is established by the next theorem

Theorem 7.34 *Given the delay bounds $\varphi_j >, \varrho_j > 0, \mu_j > 0$ for all $j \in \mathsf{N}$. If there exist matrices $\mathsf{X}_j > 0, \mathsf{Y}_j, \widehat{\mathsf{Q}}_j > 0, \widehat{\mathsf{R}}_j > 0, \widehat{\mathsf{S}}_j > 0, \widehat{\mathsf{Z}}_j > 0, \;\; j \in \mathsf{N}$ such that the following LMI holds for all $j \in \mathsf{N}$*

$$\Omega_j = \begin{bmatrix} \Pi_{aj} & \Pi_{cj} & \Pi_{zj} & \Pi_{vj} \\ \bullet & -\Pi_{ej} & 0 & \Pi_{wj} \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{7.137}$$

where

$$\begin{bmatrix} \Pi_{aaj} & \widehat{\mathsf{W}}_j & D_j \mathsf{X}_j \\ \bullet & -\Pi_{sj} & \mathsf{S}_j \\ \bullet & \bullet & -\Pi_{caj} \end{bmatrix}, \;\; \Pi_{zj} = \begin{bmatrix} \mathsf{X}_j C_j^t + \mathsf{Y}_j^t F_j^t \\ 0 \\ \mathsf{X}_j G_j^t \end{bmatrix}$$

$$\Pi_{cj} = \left[\Pi_{c1j} \; \Pi_{c2j} \right], \;\; \Pi_{cj} = \begin{bmatrix} \widehat{\mathsf{W}}_j & M_j \\ 0 & 0 \\ \widehat{\mathsf{S}}_j & 0 \end{bmatrix}, \;\; \Pi_{wj} = \begin{bmatrix} 0 \\ L_j^t \end{bmatrix}$$

$$\Pi_{c2j} = \begin{bmatrix} \varrho_j (\mathsf{X}_j A_j^t + \mathsf{Y}_j^t B_j^t) & (\varrho_j - \varphi_j) \left(\mathsf{X}_j A_j^t + \mathsf{Y}_j^t B_j^t \right) \\ 0 & 0 \\ \varrho_j \mathsf{X}_j D_j^t & (\varrho_j - \varphi_j) \mathsf{X}_j D_j^t \end{bmatrix}$$

$$\Pi_{ej} = \begin{bmatrix} \Pi_{eaj} & \Pi_{ecj} \\ \bullet & \Pi_{eej} \end{bmatrix}, \;\; \Pi_{vj} = \begin{bmatrix} \mathsf{X}_j E_j^t + \mathsf{Y}_j^t T_j^t \\ 0 \\ \mathsf{X}_j H_j^t \end{bmatrix}$$

$$\Pi_{eej} = \begin{bmatrix} \widehat{\mathsf{W}}_j - 2\mathsf{X}_j & 0 \\ \bullet & \widehat{\mathsf{W}}_j - 2\mathsf{X}_j \end{bmatrix}, \;\; \Pi_{eaj} = \begin{bmatrix} \widehat{\mathsf{R}}_j + \widehat{\mathsf{S}}_j & 0 \\ \bullet & I \end{bmatrix}$$

$$\Pi_{ecj} = \begin{bmatrix} 0 & 0 \\ -\varrho_j M_j^t & -(\varrho_j - \varphi_j) M_j^t \end{bmatrix}$$

$$\Pi_{aaj} = A_j\mathsf{X}_j + B_j\mathsf{Y}_j + \mathsf{X}_jA_j^t + \mathsf{Y}_j^tB_j^t$$
$$+\widehat{\mathsf{Q}}_j + \widehat{\mathsf{R}}_j + \widehat{\mathsf{Z}}_j - \widehat{\mathsf{W}}_j$$
$$\Pi_{sj} = \widehat{\mathsf{Q}}_j + \widehat{\mathsf{W}}_j + \widehat{\mathsf{S}}_j,\ \Pi_{caj} = (1-\mu_j)\widehat{\mathsf{Z}}_j + 2\widehat{\mathsf{S}}_j \tag{7.138}$$

then the switching rule $\xi(x) = arg\min_{j\in\mathsf{N}} V_j(x)$ *with* $V_j(x)$ *being given in (7.122) and the switched matrix gain* $\mathsf{K}_j = \mathsf{Y}_j\mathsf{X}_j^{-1}$ *render the equilibrium solution* $x = 0$ *of the system (7.115), (7.116), and (7.117) globally asymptotically stable and*

$$J(K_1, ..., K_N\xi) < \min_{j\in\mathsf{N}} Tr\left(\Gamma_o^t\mathsf{X}_j^{-1}\Gamma_o\right) \tag{7.139}$$

Proof Consider LMI (7.136) with system (7.132), (7.133), (7.134), and (7.135). Applying the congruent transformation

$$T = [\mathsf{X}_j,\ \mathsf{X}_j,\ \mathsf{X}_j,\ \mathsf{X}_j,\ I,\ I,\ I,\ I,\ I],\ \mathsf{X}_j = \mathsf{P}_j^{-1}$$

along with the change of variables $\mathsf{Y}_j = \mathsf{K}_j\mathsf{X}_j, \widehat{\mathsf{W}}_j = \mathsf{X}_j\mathsf{W}_j\mathsf{X}_j, \widehat{\mathsf{Q}}_j = \mathsf{X}_j\mathsf{Q}_j\mathsf{X}_j, \widehat{\mathsf{Z}}_j = \mathsf{X}_j\mathsf{Z}_j\mathsf{X}_j, \widehat{\mathsf{R}}_j = \mathsf{X}_j\mathsf{R}_j\mathsf{X}_j, \widehat{\mathsf{S}}_j = \mathsf{X}_j\mathsf{S}_j\mathsf{X}_j,$ and Schur complements convert LMI (7.136) into LMI (7.137) subject to (7.138). Next, the output $z_j(t)$ of system (7.132), (7.133), (7.134), and (7.135) with zero initial condition and input $w(t) = e_j\delta(t)$ can be determined from the same system with zero input and initial condition $x_o = \Gamma_o e_j,\ \ j = 1, ..., N$. Thus we have from (7.131) that

$$J(K_1, ..., K_N\xi) \le \max_{\Delta\in\boldsymbol{\Delta}} \left|\left|z_j^t(t)z_j(t)\right|\right|_2^2$$

which by **Theorem 7.30** can be put into

$$J(K_1, ..., K_N\xi) < \sum_{j=1}^{N} \min_{j\in\mathsf{N}} (\Gamma_o e_j)^t\mathsf{P}_j(\Gamma_o e_j)$$

By the trace properties, we reach

$$J(K_1, ..., K_N\xi) < \min_{j\in\mathsf{N}} Tr\left(\Gamma_o^t\mathsf{X}_j^{-1}\Gamma_o\right)$$

This in turn completes the proof. ■

Illustrative Example G

A standard water-quality model [179] is described by growth of biological oxygen demand (BOD) and dissolved oxygen (DO), respectively, at time t. Under simplifying assumptions, employing a linearization procedure and using representative data on a single reach of the River Nile [255] about three different operating points, the growth of (BOD,DO) can then be cast into the form (7.115), (7.116), and (7.117):

$$A_o = \begin{bmatrix} -1.285 & 0 \\ -3.263 & -1.975 \end{bmatrix}, \; D_o = \begin{bmatrix} -0.15 & -0.05 \\ -0.2 & 0.10 \end{bmatrix}$$
$$B_o = \text{diag}\begin{bmatrix} 1.2 & 1.4 \end{bmatrix}, \; \Gamma_o = \text{diag}\begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$$
$$C_1 = \text{diag}\begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \; G_1 = \text{diag}\begin{bmatrix} -0.15 & -0.10 \end{bmatrix}$$
$$E_1 = \text{diag}\begin{bmatrix} 0.3 & 0.1 \end{bmatrix}, \; H_1 = \text{diag}\begin{bmatrix} -0.10 & -0.15 \end{bmatrix}$$
$$M_1 = \text{diag}\begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \; L_1 = \text{diag}\begin{bmatrix} -0.15 & -0.12 \end{bmatrix}$$
$$C_2 = \text{diag}\begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \; G_2 = \text{diag}\begin{bmatrix} -0.05 & -0.05 \end{bmatrix}$$
$$E_2 = \text{diag}\begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, \; H_2 = \text{diag}\begin{bmatrix} -0.10 & -0.10 \end{bmatrix}$$
$$M_2 = \text{diag}\begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \; L_2 = \text{diag}\begin{bmatrix} -0.15 & -0.12 \end{bmatrix}$$
$$C_3 = \text{diag}\begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, \; G_3 = \text{diag}\begin{bmatrix} -0.15 & -0.10 \end{bmatrix}$$
$$E_3 = \text{diag}\begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, \; H_3 = \text{diag}\begin{bmatrix} -0.15 & -0.10 \end{bmatrix}$$
$$M_3 = \text{diag}\begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \; L_3 = \text{diag}\begin{bmatrix} -0.15 & -0.12 \end{bmatrix}$$

where the subscript o indicates common data and $1, 2, 3$ means the corresponding operating point. With $F_1 = I_2, \; F_2 = 0.4I_2, \; F_3 = 0.6I_2, \; T_1 = 0.8I_2, \; F_2 = 0.3I_2, \; F_3 = 0.5I_2$, the feasible solution of **Theorem 7.34** yields feedback gains:

$$K_1 = \begin{bmatrix} 0.9237 & -0.0307 \\ -0.1145 & 0.0415 \end{bmatrix}, \; K_2 = \begin{bmatrix} -1.1145 & -0.0513 \\ 0.3415 & -0.0265 \end{bmatrix}$$
$$K_3 = \begin{bmatrix} 0.9088 & -0.1104 \\ -0.1222 & 0.1018 \end{bmatrix}$$

which renders the water-quality system asymptotically stable with a guaranteed cost $J(K_1, K_2, K_3, \xi) \leq 4.4765$. In Figs. 7.3 and 7.4, the ensuing trajectories of the water-quality states and controls are depicted.

7.3.4 Appendix

Computing the Dini derivative $\mathsf{D}^+ L(x)$ with respect to an arbitrary trajectory of (7.115), (7.116), and (7.117) is computed [160] to yield $\mathsf{D}^+ L(x) = \min_{j \in \mathsf{N}} Y_j(x(t))$, where

$$\begin{aligned} Y_j(x(t)) \leq\; & 2x^t \mathsf{P}_j [A_j x(t) + D_j x(t - \tau_j) \\ & + M_j q(t)] + x^t(t) \mathsf{Q}_j x(t) \\ & - x^t(t - \varphi_j) \mathsf{Q}_j x(t - \varphi_j) + x^t(t) \mathsf{R}_j x(t) \\ & + x^t(t) \mathsf{Z}_j x(t) - x^t(t - \varrho_j) \mathsf{R}_j x(t - \varrho_j)] \\ & - (1 - \mu)\; x^t(t - \tau_j) \mathsf{Z}_j x_j t - \tau_j)] \end{aligned}$$

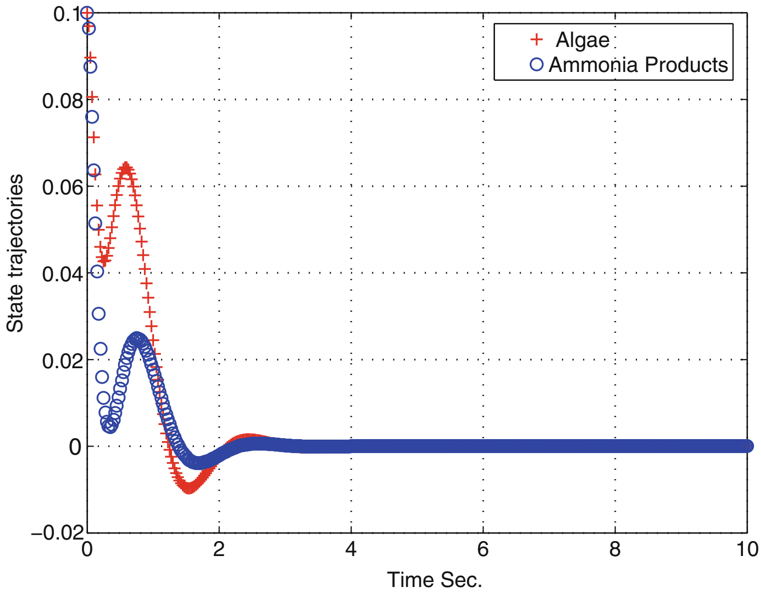

Fig. 7.3 State trajectories under switched feedback

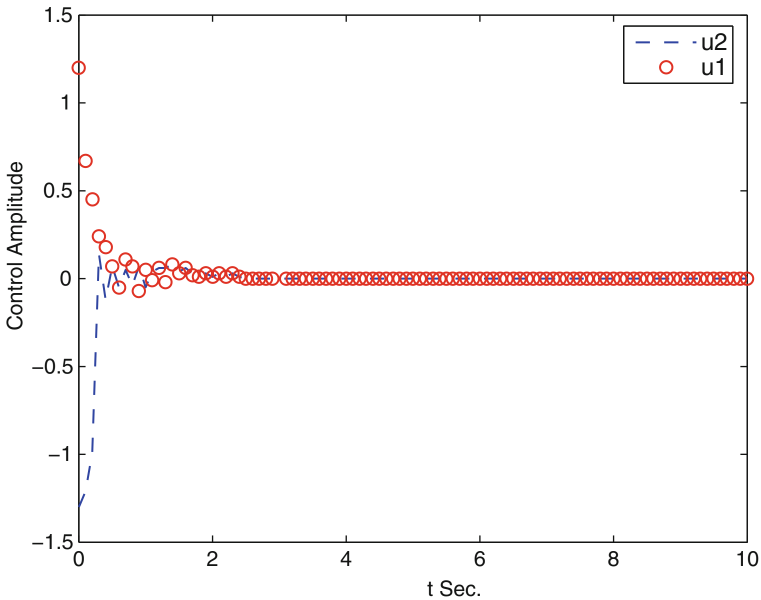

Fig. 7.4 Control trajectories under switched feedback

$$+ \varphi_j^2 \dot{x}^t(t) \mathsf{W}_j \dot{x}(t) - \varphi_j \int_{t-\varphi_j}^{t} \dot{x}^t(s) \mathsf{W}_j \dot{x}(s) \mathrm{d}s$$
$$+ (\varrho_j - \varphi_j)^2 \dot{x}^t(t) \mathsf{S}_j \dot{x}(t)$$
$$- (\varrho_j - \varphi_j) \int_{t-\varrho_j}^{t-\varphi_j} \dot{x}^t(s) \mathsf{S}_j \dot{x}(s) \mathrm{d}s \tag{7.140}$$

Applying **Lemma 13.3**, we get

$$- \varphi_j \int_{t-\varphi_j}^{t} \dot{x}^t(s) \mathsf{W}_j \dot{x}(s) \mathrm{d}s \le -[x(t) - x(t-\varphi_j)]^t \ \mathsf{W}_j$$
$$[x(t) - x(t-\varphi_j)] \tag{7.141}$$

$$- (\varrho_j - \varphi_j) \int_{t-\varrho_j}^{t-\varphi_j} \dot{x}^t(\alpha) \mathsf{S}_j \dot{x}(\alpha) \mathrm{d}\alpha$$
$$= -(\varrho_j - \varphi_j) \Big[\int_{t-\tau_j}^{t-\varphi_j} \dot{x}^t(\alpha) \mathsf{S}_j \dot{x}(\alpha) \mathrm{d}\alpha$$
$$+ \int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha) \mathsf{S}_j \dot{x}_j(\alpha) \mathrm{d}\alpha \Big]$$
$$\le -(\tau_j - \varphi_j) \Big[\int_{t-\tau_j}^{t-\varphi} \dot{x}^t(\alpha) \mathsf{S}_j \dot{x}(\alpha) \mathrm{d}\alpha \Big]$$
$$- (\varrho_j - \tau_j) \Big[\int_{t-\varrho_j}^{t-\tau_j} \dot{x}^t(\alpha) \mathsf{S}_j \dot{x}(\alpha) \mathrm{d}\alpha \Big]$$
$$\le -\Big(\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \mathrm{d}\alpha \Big) \mathsf{S}_j \Big(\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \mathrm{d}\alpha \Big)$$
$$- \Big(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}^t(\alpha) \mathrm{d}\alpha \Big) \mathsf{S}_j \Big(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}^t(\alpha) \mathrm{d}\alpha \Big)$$
$$= -[x(t-\varphi_j) - x(t-\tau_j)]^t \mathsf{S}_j [x(t-\varphi_j) - x(t-\tau_j)]$$
$$- [x(t-\tau_j) - x(t-\varrho_j)]^t \mathsf{S}_j [x(t-\tau_j) - x(t-\varrho_j)] \tag{7.142}$$

On combining (7.140), (7.141), and (7.142), we get

$$Y_j(x(t)) \le \chi^t(t) \Xi_j \chi(t) + \varphi_j^2 \dot{x}^t(t) \mathsf{W}_j \dot{x}_j(t)$$
$$+ (\varrho_j - \varphi_j)^2 \dot{x}^t(t) \mathsf{S}_j \dot{x}(t) \tag{7.143}$$
$$\chi(t) = \left[x^t(t) \ x^t(t-\varphi_j) \ x^t(t-\tau_j) \ x^t(t-\varrho_j) \ q(t) \right]^t$$
$$\Xi_j = \begin{bmatrix} \Xi_{oj} & \mathsf{W}_j & \mathsf{P}_j D_j & \mathsf{W}_j & \mathsf{P}_j M_j \\ \bullet & -\Xi_{aj} & \mathsf{S}_j & 0 & 0 \\ \bullet & \bullet & -\Xi_{cj} & \mathsf{S}_j & 0 \\ \bullet & \bullet & \bullet & -\mathsf{R}_j - \mathsf{S}_j & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \end{bmatrix} \tag{7.144}$$

$$\Xi_{oj} = \mathsf{P}_j A_j + A_j^t \mathsf{P}_j + \mathsf{Q}_j + \mathsf{R}_j + \mathsf{Z}_j - \mathsf{W}_j,$$
$$\Xi_{aj} = \mathsf{Q}_j + \mathsf{W}_j + \mathsf{S}_j, \ \Xi_{cj} = (1 - \mu_j)\mathsf{Z}_j + 2\mathsf{S}_j \quad (7.145)$$

7.4 Notes and References

In the chapter, we have investigated the problems of robust delay-dependent $\mathcal{L}_2$ gain analysis and feedback-control synthesis for a class of nonlinear switched discrete-time systems with time-varying delays and real convex-bounded parametric uncertainties in all system matrices under arbitrary switching sequences. Then, we developed new criteria for such class of nonlinear switched state-delay systems based on the constructive use of an appropriate switched Lyapunov functionals coupled with *Finsler's Lemma* and a free-weighting parameter matrices. LMI characterization of delay-dependent conditions are established under which the nonlinear switched delay system is robustly asymptotically stable with an $\mathcal{L}_2 - gain$ smaller than a prescribed constant level. Finally, we designed switched-feedback schemes, based on state-, output-measurements, or by using dynamic-output feedback to guarantee that the corresponding switched closed-loop system enjoys the delay-dependent asymptotic stability with an $\mathcal{L}_2$ gain smaller than a prescribed constant level.

All the developed results have been expressed in terms of convex optimization over LMIs and tested on representative examples.

Part IV
Switched Filtering

Chapter 8
Delay-Dependent Switched Filtering

In this chapter, the filtering problem for a class of discrete-time switched systems with state delays is thoroughly investigated. We will focus on discrete-time systems. Attention will be equally focused on the design of stable filters guaranteeing different prescribed performance criteria including the $\mathcal{L}_2$ sense and in the $\mathcal{L}_2 - \mathcal{L}_\infty$ sense. In all cases, switched Lyapunov functionals are employed to derive sufficient conditions for the solvability of the filtering problem and expressed in terms of linear matrix inequalities (LMIs).

8.1 $\mathcal{H}_\infty$ Filter Design

The problem of $\mathcal{H}_\infty$ filtering for a class of discrete-time switched systems with state delays is investigated in this section. Attention is focused on the design of a stable filter guaranteeing a prescribed noise attenuation level in the $\mathcal{H}_\infty$ sense. By using switched Lyapunov functionals, sufficient conditions for the solvability of this problem are obtained in terms of linear matrix inequalities (LMIs), by solving which a desired $\mathcal{H}_\infty$ filter can be constructed.

8.1.1 Introduction

It is well known that state estimation has been widely studied and has found many practical applications during the past decades. When a priori information on the external noise is not precisely known, the celebrated Kalman filtering scheme is no longer applicable. In this case, $\mathcal{H}_\infty$ filter was introduced in [57], where the noise signal was assumed to be energy bounded and the main objective was to minimize the $\mathcal{H}_\infty$ norm of the filtering error system [78, 282, 346, 394, 408]. When time delays are taken into account in a system, linear matrix inequality-based (LMI-based) results on the $\mathcal{H}_\infty$ filtering problem have also been reported in the literature; see, for example, [79, 106, 347, 381, 393] and the references therein.

Recently, the control synthesis of switched systems has been extensively investigated and many methodologies have been used in the study of switched systems

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[42, 52, 56, 86, 170, 191, 441]. For example, multiple Lyapunov functions were employed to establish certain general Lyapunov-like results for nonlinear switched systems [56]; dwell-time and average dwell-time approaches were employed to study the stability and disturbance attenuation of switched systems [377, 426]; piecewise Lyapunov function approach was adopted in [156, 388]; and a switched Lyapunov function method has been applied in [42] to study the stability problem of discrete-time switched systems.

On the contrary, time delays are the inherent features of many physical process and the big sources of instability and poor performances. Switched systems with time delays have strong engineering background in network control systems [170] and power systems [291]. More recently, some theoretical studies were conducted for switched systems with time delays [370, 395, 425]. Till date, to the best of the authors' knowledge, the $\mathcal{H}_\infty$ filtering problem has not been addressed for time-delayed switched systems. In this paper, an $\mathcal{H}_\infty$ filtering design is developed using switched Lyapunov functional approach for discrete-time switched systems with time delay. The filtering design solution is facilitated by introducing some additional instrumental matrix variables. These additional matrix variables decouple the Lyapunov and the system matrices, which makes the filtering design feasible.

8.1.2 Problem Formulation

Consider the following discrete-time switched system with state delay :

$$\Sigma_0 : \qquad x_{k+1} = \sum_{i=1}^{S} \alpha_i(k) A_i x_k + \sum_{i=1}^{S} \alpha_i(k) A_{di} x_{k-d} + \sum_{i=1}^{S} \alpha_i(k) B_i \omega_k \tag{8.1}$$

$$y_k = \sum_{i=1}^{S} \alpha_i(k) C_i x_k + \sum_{i=1}^{S} \alpha_i(k) C_{di} x_{k-d} + \sum_{i=1}^{S} \alpha_i(k) D_i \omega_k \tag{8.2}$$

$$z_k = \sum_{i=1}^{S} \alpha_i(k) G_i x_k \tag{8.3}$$

where $x_k \in R^n$ is the state, $y_k \in R^r$ is the measured output, $z_k \in R^q$ is the signal to be estimated, $\omega_k \in R^p$ is the disturbance input, which is assumed to belong to $l_2[0, \infty)$, and the positive integer d denotes the known state delay. $\alpha_i(k)$ is the switching signal:

$$\alpha_i : Z^+ \longrightarrow \{0, 1\}, \qquad \sum_{i=1}^{S} \alpha_i(k) = 1, \qquad k \in Z^+ = \{0, 1, \cdots\}$$

which specifies which subsystem will be activated at certain discrete time. A_i, A_{di}, B_i, C_i, C_{di}, D_i, and G_i are system matrices with compatible dimensions.

Here we are interested in designing a filter described by

$$\Sigma_f: \qquad \hat{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k) A_{fi} \hat{x}_k + \sum_{i=1}^{S} \alpha_i(k) B_{fi} y_k \tag{8.4}$$

$$\hat{z}_k = \sum_{i=1}^{S} \alpha_i(k) C_{fi} \hat{x}_k \tag{8.5}$$

where $\hat{x}_k \in R^n$ and $\hat{z}_k \in R^q$, the matrices A_{fi}, B_{fi}, and C_{fi} are to be determined. Augmenting the model of Σ_0 to include the system Σ_f, we obtain the following system (called filtering error system):

$$\Sigma_c: \qquad \tilde{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k) \tilde{A}_i \tilde{x}_k + \sum_{i=1}^{S} \alpha_i(k) \tilde{A}_{di} \tilde{x}_{k-d} + \sum_{i=1}^{S} \alpha_i(k) \tilde{B}_i \omega_k \tag{8.6}$$

$$\tilde{z}_k = \sum_{i=1}^{S} \alpha_i(k) \tilde{C}_i \tilde{x}_k \tag{8.7}$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ B_{fi} C_i & A_{fi} \end{bmatrix}, \ \tilde{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ B_{fi} C_{di} & 0 \end{bmatrix}, \ \tilde{B}_i = \begin{bmatrix} B_i \\ B_{fi} D_i \end{bmatrix},$$
$$\tilde{x}_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T, \quad \tilde{z}_k = z_k - \hat{z}_k, \ \tilde{C}_i = \begin{bmatrix} G_i & -C_{fi} \end{bmatrix} \tag{8.8}$$

Our objective is to develop a filter in the form of (8.4) and (8.5) such that the following specifications are met for the filtering error system Σ_c:

(H1): The filtering error system Σ_c is globally asymptotically stable when $\omega_k = 0$.
(H2): The filtering error system Σ_c guarantees, under zero-initial condition, $\|\tilde{z}_k\|_2 \le \gamma \|\omega_k\|_2$ for all nonzero $\omega_k \in l_2[0, \infty)$ and a given positive constant γ.

In the sequel, we will refer systems satisfying (H1) and (H2) as stable and with $\mathcal{H}_\infty$ norm bound γ.

Remark 8.1 The robust filter design problem for switched systems has been investigated in [86], where the minimax linear filters are developed for discrete-time systems whose dynamics switches are within a finite set of stochastic behaviors. In this paper, our attention is focused on the design of delay-independent robust $\mathcal{H}_\infty$ filters for the system Σ_0 under arbitrary switching signal.

8.1.3 Stability and Performance Analysis

This section gives a new characterization involving switched Lyapunov functional for the filtering error system Σ_c to be stable and with $\mathcal{H}_\infty$ norm bound γ.

Theorem 8.2 *The filtering error system Σ_c is stable and with $\mathcal{H}_\infty$ norm bound γ, if there exist matrices $\{P_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ for all $\{i, j, l\} \in \mathcal{S} = \{1, 2, \cdots, S\}$ such that*

$$\begin{bmatrix} -P_j^{-1} & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 \\ \bullet & -P_i + Q_i & 0 & 0 & \tilde{C}_i^t \\ \bullet & \bullet & -Q_l & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{8.9}$$

where $*$ denotes the corresponding transposed block matrix due to symmetry.

Proof First, we establish the stability of system (6). When $\omega_k = 0$, (8.6) becomes

$$\tilde{x}_{k+1} = \sum_{i=1}^{S} \alpha_i(k)\tilde{A}_i\tilde{x}_k + \sum_{i=1}^{S} \alpha_i(k)\tilde{A}_{di}\tilde{x}_{k-d} \tag{8.10}$$

Define

$$V_k = \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)P_i\right)\tilde{x}_k + \sum_{s=k-d}^{k-1} \tilde{x}_s^t \left(\sum_{i=1}^{S} \alpha_i(s)Q_i\right)\tilde{x}_s \tag{8.11}$$

Then

$$\begin{aligned} \Delta V_k|_{(8.10)} &= V_{k+1} - V_k \\ &= \tilde{x}_{k+1}^t \left(\sum_{i=1}^{S} \alpha_i(k+1)P_i\right)\tilde{x}_{k+1} - \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)P_i\right)\tilde{x}_k \\ &\quad + \tilde{x}_k^t \left(\sum_{i=1}^{S} \alpha_i(k)Q_i\right)\tilde{x}_k - \tilde{x}_{k-d^t T} \left(\sum_{i=1}^{S} \alpha_i(k-d)Q_i\right)\tilde{x}_{k-d} \end{aligned}$$

It follows that for any nonzero vector $\tilde{x}_k$ and the particular case $\alpha_i(k) = 1$, $\alpha_{r\neq i}(k) = 0, \alpha_j(k+1) = 1, \alpha_{r\neq j}(k+1) = 0, \alpha_l(k-d) = 1, \alpha_{r\neq l}(k-d) = 0$. Then, we have

$$\Delta V_k|_{(8.10)} = \eta_k^t \left(\begin{bmatrix} \tilde{A}_i^t \\ \tilde{A}_{d^t} \end{bmatrix} P_j \begin{bmatrix} \tilde{A}_i & \tilde{A}_{di} \end{bmatrix} + \begin{bmatrix} -P_i + Q_i & 0 \\ 0 & -Q_l \end{bmatrix} \right) \eta_k$$

where $\eta_k = \left[\, \tilde{x}_k^t \;\, \tilde{x}_{k-d^t} \,\right]^t$. By the Schur complement formula, it follows from (8.9) that $\Delta V_k|_{(8.10)} < 0$, which establishes the stability of system (8.10).

Let

$$J_K = \sum_{k=0}^{K-1} \left(\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^t \omega_k \right)$$

where K is an arbitrary positive integer. For any nonzero $\omega_k \in l_2[0, \infty)$ and zero initial condition $\tilde{x}_0 = 0$, one has

$$\begin{aligned} J_K &= \sum_{k=0}^{K-1} \left(\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \right) - V_K \\ &\le \sum_{k=0}^{K-1} \left(\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \right) \end{aligned}$$

where $\Delta V_k|_{(8.6)}$ defines the increment of V_k along the solution of system (8.6). It is noted that

$$\begin{aligned} &\tilde{z}_k^t \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} \\ &\quad = \tilde{\eta}_k^t \begin{bmatrix} \tilde{A}_i^t \\ \tilde{A}_{di}^t \\ \tilde{B}_i^t \end{bmatrix} P_j \left[\, \tilde{A}_i \;\, \tilde{A}_{di} \;\, \tilde{B}_i \,\right] \tilde{\eta}_k \\ &\qquad + \tilde{\eta}_k^t \begin{bmatrix} -P_i + Q_i + \tilde{C}_i^T \tilde{C}_i & 0 & 0 \\ 0 & -Q_l & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \tilde{\eta}_k \end{aligned} \tag{8.12}$$

where

$$\tilde{\eta}_k = \left[\, \tilde{x}_k^t \;\, \tilde{x}_{k-d^t} \;\, \omega_k^t \,\right]^t$$

It follows from (8.9) and Schur complement that

$$\tilde{z}_k^T \tilde{z}_k - \gamma^2 \omega_k^t \omega_k + \Delta V_k|_{(8.6)} < 0$$

which implies, for any K, $J_K < 0$. Then one has that for any nonzero $\omega_k \in l_2[0, \infty)$, $\|\tilde{z}_k\|_2 < \gamma \|\omega_k\|_2$ ■.

Motivated by the idea in [44], we present the following theorem.

Theorem 8.3 *The filtering error system Σ_c is stable and with H_∞ norm bound γ, if there exist matrices $\{R_i\}_{i=1}^N$, $\{\Psi_i\}_{i=1}^N$, and Ω for all $\{i, j, l\} \in \mathcal{S} = \{1, 2, \cdots, S\}$ such that*

$$\begin{bmatrix} -R_j & \tilde{A}_i \Omega & \tilde{A}_{di} \Omega & \tilde{B}_i & 0 & 0 \\ \bullet & R_i - (\Omega + \Omega^T) & 0 & 0 & \Omega^T \tilde{C}_i^T & \Omega^T \\ \bullet & \bullet & \Psi_l - (\Omega + \Omega^T) & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \tag{8.13}$$

Proof Suppose that (8.13) holds, then it is easy to see from (8.13) that

$$(R_i - \Omega)^t R_i^{-1} (R_i - \Omega) \geq 0$$

which implies

$$-\Omega^t R_i^{-1} \Omega \leq R_i - (\Omega + \Omega^t)$$

Similarly, we can get $-\Omega^t \Psi_i^{-1} \Omega \leq \Psi_i - (\Omega + \Omega^T)$. Then, (8.13) is transformed into

$$\begin{bmatrix} -R_j & \tilde{A}_i \Omega & \tilde{A}_{di} \Omega & \tilde{B}_i & 0 & 0 \\ \bullet & -\Omega^T R_i^{-1} \Omega & 0 & 0 & \Omega^T \tilde{C}_i^T & \Omega^T \\ \bullet & \bullet & -\Omega^T \Psi_l^{-1} \Omega & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \tag{8.14}$$

Pre-multiplying (8.14) by

$$\mathrm{diag}\{I, \Omega^{-t}, \Omega^{-t}, I, I, I\}$$

and post-multiplying by

$$\mathrm{diag}\{I, \Omega^{-1}, \Omega^{-1}, I, I, I\}$$

then (8.13) is transformed into

$$\begin{bmatrix} -R_j & \tilde{A}_i & \tilde{A}_{di} & \tilde{B}_i & 0 & 0 \\ \bullet & -R_i^{-1} & 0 & 0 & \tilde{C}_i^T & I \\ \bullet & \bullet & -\Psi_l^{-1} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_i \end{bmatrix} < 0 \tag{8.15}$$

Notice that $R_i = P_i^{-1}, \Psi_i = Q_i^{-1}$. Then, by using the Schur complement formula we can see that (8.15) is equivalent to (8.9). The proof is completed. ∎

Remark 8.4 With the introduction of a new additional matrix Ω, we obtain a sufficient condition in which the matrices R_i and Ψ_i are not involved in any product with matrices $\tilde{A}_i$, $\tilde{A}_{di}$, $\tilde{B}_i$, and $\tilde{C}_i$. This makes a filter design feasible.

8.1.4 Filter Design

In this section, we will present a sufficient condition for the existence of H_∞ filter in the form of (8.4) and (8.5), and show how to construct a filter based on **Theorem 8.2**.

Theorem 8.5 *Consider system Σ_0 and given a constant $\gamma > 0$. If there exist matrices $0 < R_{1j} = R_{1j}^t$, $0 < R_{3j} = R_{3j}^t$, $0 < X_{1m} = X_{1m}^t$, $0 < X_{3m} = X_{3m}^t$ and R_{2j}, X_{2m}, Z_i, Y_i, H_i, L_i, M_i, S_i such that the following inequality holds:*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 & 0 \\ \bullet & \widehat{\Theta}_{22} & 0 & 0 & \Theta_{25} & \Theta_{26}^t \\ \bullet & \bullet & \widehat{\Theta}_{33} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Theta_{66} \end{bmatrix} < 0 \tag{8.16}$$

where

$$\Theta_{11} = \begin{bmatrix} R_{1j} & R_{2j} \\ \bullet & R_{3j} \end{bmatrix}, \quad \Theta_{12} = \begin{bmatrix} Z_i A_i & Z_i A_i \\ Y_i A_i + H_i C_i + L_i & Y_i A_i + H_i C_i \end{bmatrix}$$

$$\Theta_{13} = \begin{bmatrix} Z_i A_{di} & Z_i A_{di} \\ Y_i A_{di} + H_i C_{di} & Y_i A_{di} + H_i C_{di} \end{bmatrix}, \quad \Theta_{14} = \begin{bmatrix} Z_i B_i \\ Y_i B_i + H_i D_i \end{bmatrix}$$

$$\Theta_{22} = \begin{bmatrix} R_{1i} & R_{2i} \\ \bullet & R_{3i} \end{bmatrix}, \quad \Theta_{25} = \begin{bmatrix} G_i^t - S_i^t \\ G_i^t \end{bmatrix}, \quad \Theta_{26} = \begin{bmatrix} Z_i & Z_i \\ Y_i + M_i & Y_i \end{bmatrix}$$

$$\Theta_{33} = \begin{bmatrix} X_{1m} & X_{2m} \\ \bullet & X_{3m} \end{bmatrix}, \quad \Theta_{66} = \begin{bmatrix} X_{1i} & X_{2i} \\ \bullet & X_{3i} \end{bmatrix}$$

$$\widehat{\Theta}_{22} = \Theta_{22} - \Theta_{26} - \Theta_{26}^t, \quad \widehat{\Theta}_{33} = \Theta_{33} - \Theta_{26} - \Theta_{26}^t$$

then, there exists a filter in the form of (8.4) and (8.5) such that the filtering error system Σ_c is asymptotically stable with $\mathcal{H}_\infty$ norm bound γ. Moreover, if LMI (8.16) has a feasible solution, then the filter matrix

$$\mathcal{F} := \begin{bmatrix} A_{fi} & B_{fi} \\ C_{fi} & 0 \end{bmatrix} \tag{8.17}$$

can be constructed by

$$\mathcal{F} := \begin{bmatrix} V_i^{-1} L_i M_i^{-1} V_i & V_i^{-1} H_i \\ S_i M_i^{-1} V_i & 0 \end{bmatrix} \tag{8.18}$$

Proof Suppose the inequality (8.16) holds. It can be obtained that

$$\begin{bmatrix} Z_i + Z_i^t & Z_i + Y_i^t + M_i^t \\ \bullet & Y_i + Y_i^t \end{bmatrix} > \begin{bmatrix} R_{1i} & R_{2i}^t \\ \bullet & R_{3i} \end{bmatrix} > 0 \tag{8.19}$$

which implies that matrices Z_i and Y_i are nonsingular. Pre-multiplying (8.19) by $\begin{bmatrix} I & -I \end{bmatrix}$ and post-multiplying the result by $\begin{bmatrix} I & -I \end{bmatrix}^t$, one obtains

$$-M_i - M_i^t > 0 \tag{8.20}$$

which implies that M_i is also nonsingular. Hence there exist nonsingular matrices U_i and V_i satisfying $M_i = V_i U_i$ such that (8.16) holds.

Let

$$\begin{aligned} \Pi_i^t &= \begin{bmatrix} Z_i & 0 \\ Y_i & V_i \end{bmatrix}, \ \Omega \Pi_i = \begin{bmatrix} I & I \\ U_i & 0 \end{bmatrix} \\ H_i &= V_i B_{fi}, \ L_i = V_i A_{fi} U_i, \ S_i = C_{fi} U_i, \ M_i = V_i U_i \\ R_j &= \Pi_i^{-t} \Psi_{11} \Pi_i^{-1}, \ R_i = \Pi_i^{-t} \Psi_{22} \Pi_i^{-1} \\ \Phi_m &= \Pi_i^{-t} \Psi_{33} \Pi_i^{-1}, \ \Phi_i = \Pi_i^{-t} \Psi_{66} \Pi_i^{-1} \end{aligned} \tag{8.21}$$

By (8.8) and (8.21), one has

$$\begin{aligned} \Pi_i^t \tilde{A}_i \Omega \Pi_i &= \Psi_{12}, \ \Pi_i^t \tilde{A}_{di} \Omega \Pi_i = \Psi_{13}, \ \Pi_i^t \tilde{B}_i = \Psi_{14} \\ \tilde{C}_i \Omega \Pi_i &= \Psi_{25}^t, \ \Pi_i^t \tilde{A}_i \Omega \Pi_i = \Psi_{26} \end{aligned} \tag{8.22}$$

Pre-multiplying (8.13) by

$$\text{diag} \begin{bmatrix} \Pi_i^t & \Pi_i^t & \Pi_i^t & I & I & \Pi_i^t \end{bmatrix}$$

and post-multiplying the result by

$$\text{diag} \begin{bmatrix} \Pi_i & \Pi_i & \Pi_i & I & I & \Pi_i \end{bmatrix}$$

and using (8.21) and (8.22), we readily obtain (8.16). Finally, it is not difficult to verify from (8.21) that the filter matrices are given by (8.18), which completes the proof.

Remark 8.6 The filter expressed in the form of (8.4) and (8.5) not only guarantees analytical properties, such as stability and guaranteed $\mathcal{H}_\infty$ performance of the filtering error system Σ_c, but is itself a switched system.

Remark 8.7 By using the techniques in [30] and [444], the result of **Theorem 8.3** can be readily extended to the discrete-time switched systems with state delay, which contain norm-bounded parameter uncertainties or linear fractional form parameter uncertainties.

8.1.5 Illustrative Example A

Consider the system Σ_0 with $N = 2$ and

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.4 & 0.05 \\ 0 & -0.35 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.025 & 0 \\ -0.1 & -0.35 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.34 \\ -0.3 \end{bmatrix} \\
C_{d1} &= \begin{bmatrix} 0.02 \; 0 \end{bmatrix}, \; D_1 = 0.02, \; G_1 = \begin{bmatrix} 0.24 \; 0.23 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.29 \; 0.15 \end{bmatrix} \\
A_2 &= \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.05 & -0.1 \\ 0 & 0.15 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.1 \\ -1 \end{bmatrix} \\
C_{d2} &= \begin{bmatrix} 0 \; 0.017 \end{bmatrix}, \; D_2 = 0.015, \; G_2 = \begin{bmatrix} 0.2 \; 0.1 \end{bmatrix}, \; C_2 = \begin{bmatrix} -0.19 \; 0.17 \end{bmatrix}
\end{aligned}$$

The purpose here is to design a filter such that the filtering error system is stable and with a given $\mathcal{H}_\infty$ norm bound γ. Here the performance level is chosen as $\gamma = 0.6$. By using the Matlab LMI Control Toolbox to solve LMI (8.16), we can get a feasible set of solutions. By **Theorem 8.3**, a filter in the form of (8.4) and (8.5) as follows:

$$\begin{aligned}
A_{f1} &= \begin{bmatrix} 0.3497 & -0.5481 \\ 0.1094 & -0.1653 \end{bmatrix}, \; B_{f1} = \begin{bmatrix} -8.3430 \\ 4.3427 \end{bmatrix}, \; C_{f1} = \begin{bmatrix} -0.0030 \; -0.0758 \end{bmatrix} \\
A_{f2} &= \begin{bmatrix} -0.1385 & -0.0975 \\ 0.0049 & 0.0157 \end{bmatrix}, \; B_{f2} = \begin{bmatrix} -4.9351 \\ -1.4790 \end{bmatrix}, C_{f2} = \begin{bmatrix} -0.0059 \; -0.0282 \end{bmatrix}
\end{aligned}$$

The simulation results of the state responses of the plant and filter are, respectively, given in Figs. 8.1 and 8.2, where the initial conditions $x_0 = \begin{bmatrix} 1.0 \; -0.8 \end{bmatrix}^t$ and $\hat{x}_0 = \begin{bmatrix} 0 \; 0 \end{bmatrix}^t$, respectively, and the noise signal is chosen as $\omega_k = 1/(k+1)$, which belongs to $l_2[0, \infty)$. The simulation results of signal z_k and $\hat{z}_k$ are shown in Figs. 8.3 and 8.4. Figure 8.5 shows the simulation result of the filtering error $\tilde{z}_k = z_k - \hat{z}_k$. It is observed that the designed H_∞ filter meets the specified requirements, and works well.

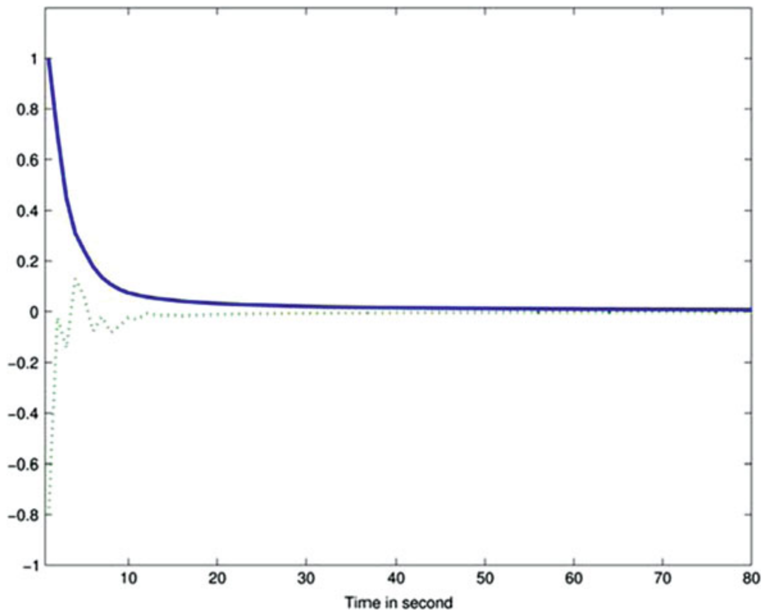

Fig. 8.1 Step response of plant states

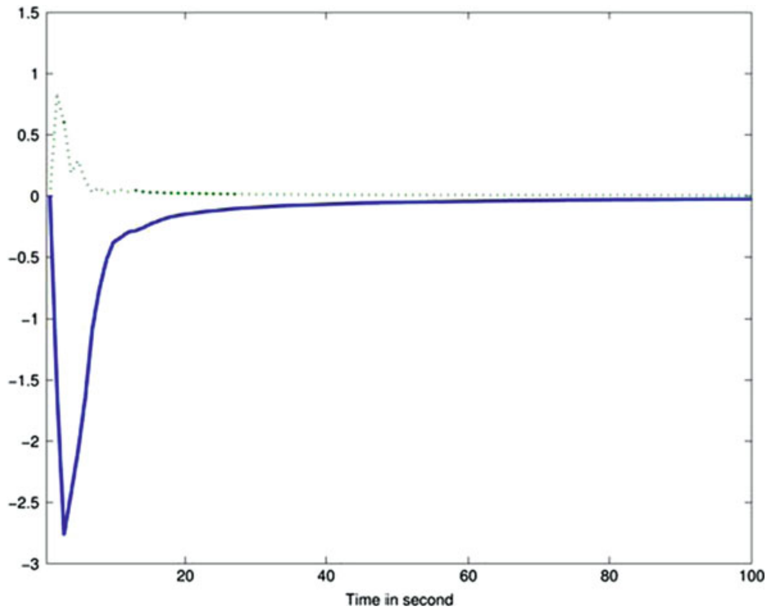

Fig. 8.2 Step response of plant states

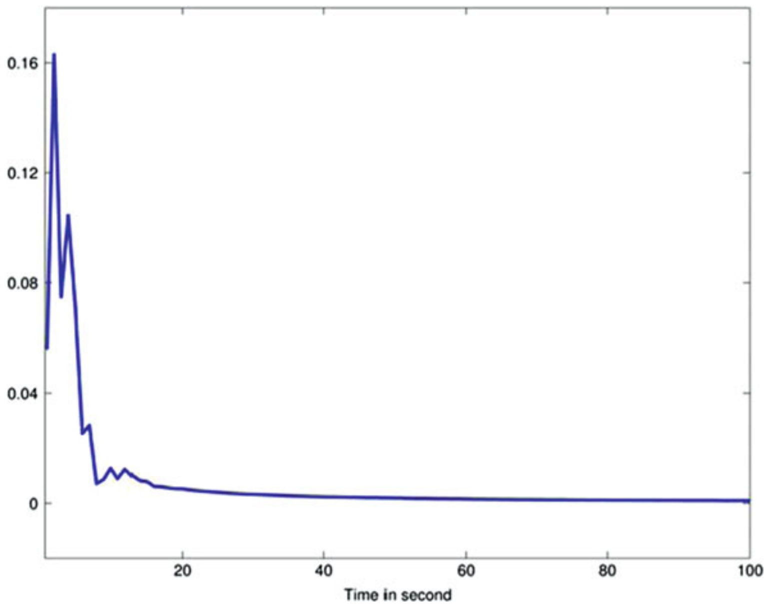

Fig. 8.3 Step response of plant states

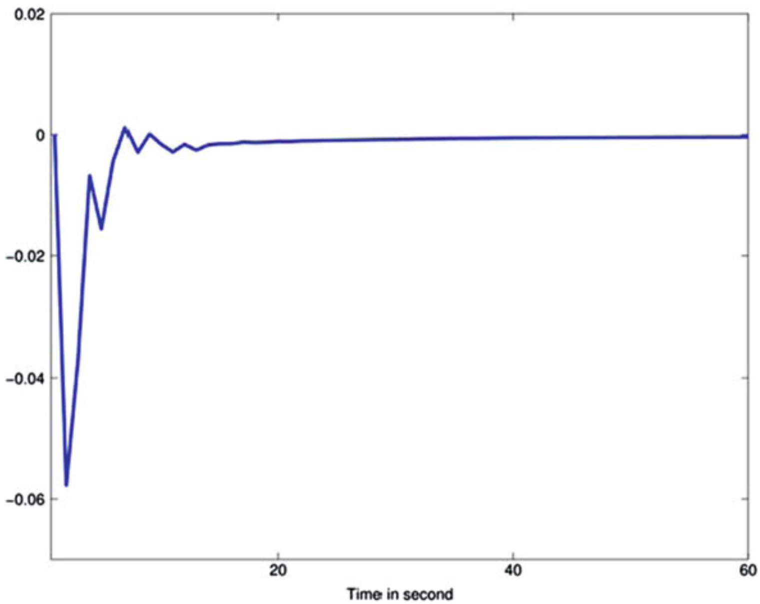

Fig. 8.4 Step response of plant states

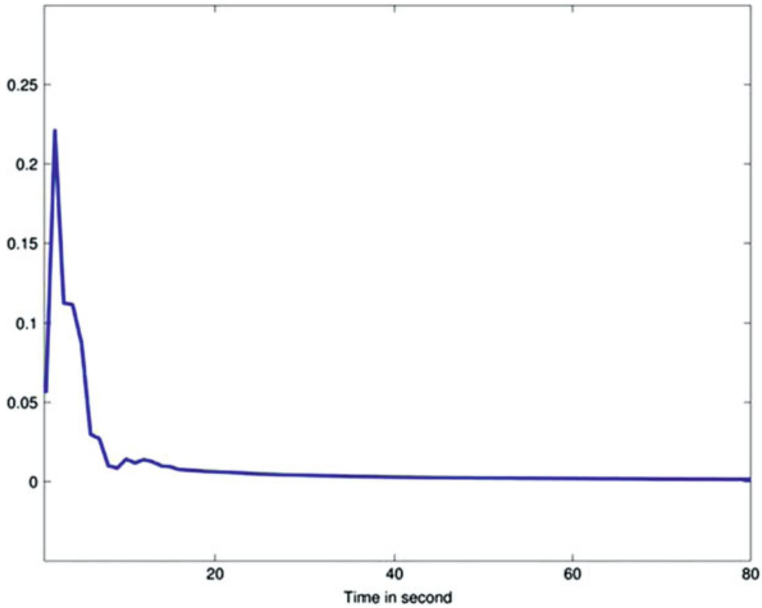

Fig. 8.5 Step response of plant states

8.2 Filter Design for Piecewise Systems

Broadly speaking, hybrid systems have proved to be an effective tool for multi-modeling, analysis, and design of a large number of evolving technological systems, in which digital devices interact with an analog environment. Systems of this type are common in embedded computation, robotics, mechatronics, avionics, and process control. Owing to the rapid advances in computer technology, hybrid systems are becoming increasingly relevant and important and consequently have attracted considerable research interests. A wide class of hybrid systems is *piecewise dynamical systems* for which some of the research results relevant to this study have been reported in [2, 63, 144, 293, 334] and their references. Common to these activities is the development of piecewise Lyapunov function approaches for stability analysis [156, 176, 313] and linear control design [118, 336, 431, 447] of piecewise continuous-time systems. In a parallel development, similar results are obtained for piecewise discrete-time linear systems [184, 293, 334, 376, 428]. For a class of piecewise discrete-time linear systems, the output feedback control problem has been investigated in [61] and the design of $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ filters are performed in [62] using observer-type filters (without parametric uncertainties or time delays). The solution is attained via the solution of a set of LMIs.

On another research front, the filtering problem has been the focal point of numerous research activities in the past four decades due to its central role in systems, control, and signal processing. The celebrated Kalman filter [3, 158, 352, 356] provides a recursive algorithm to minimize the variance of the state estimation error when the power spectral density of the process and measurement noise is known. During the past four decades, Kalman filtering techniques have found widespread applications in aerospace guidance, navigation, and control problems [213–221, 235, 250, 256, 257, 262, 263, 266, 267, 269, 287, 352, 356]. When a priori information on the external noise is not precisely known, Kalman filtering approach is no longer applicable. In such cases, $\mathcal{H}_\infty$ filtering was introduced [87, 305], in which the input signal is assumed to be energy bounded and the main objective is to minimize the energy of the estimation error for the worst possible bounded energy disturbance. The solution to this problem guarantees that the $\mathcal{L}_2$-induced norm from the noise signals to the filtering error will be less than a prescribed performance bound, where the noise are arbitrary energy-bounded signals. In the literature, there have been different approaches to solve $\mathcal{H}_\infty$ filtering problem [16, 67, 69, 71–216, 244, 245, 249, 250, 253, 254, 262–265, 269, 276, 277, 287, 305, 373, 438, 439]. When the systems are subjected to norm-bonded parametric uncertainties, robust $\mathcal{H}_\infty$ filtering were developed in [72] based on a Riccati equation approach and in [189] using a convex optimization approach. For systems with polytopic parameter uncertainties, linear matrix inequalities-based sufficient conditions were derived for robust $\mathcal{H}_\infty$ filters in [87, 317].

By contrast, the objective of $\mathcal{L}_2 - \mathcal{L}_\infty$ filtering problem is to minimize the peak value of the estimation error for all possible bounded energy disturbances. Hence, the $\mathcal{L}_2 - \mathcal{L}_\infty$ (energy-to-peak) filtering can be considered as a deterministic formulation of the Kalman filter [223, 318]. The class of robust filtering arose out of the desire to determine estimates of nonmeasurable state variables for dynamical systems with uncertain parameters. The past decade has witnessed major developments in robust filtering problem using various approaches [16, 305].

In recent years, research investigations into dynamical systems with time delays have been intensified and spread to several domains, including neural networks [35, 37, 194] and nonlinear systems [385, 390, 420]. In addition, the development of $\mathcal{H}_\infty$ filters and robust $\mathcal{H}_\infty$ filters were accomplished, leading to delay-independent and delay-dependent sufficient conditions [69, 217–223, 235–237, 250, 255–258, 266, 267, 278, 282]. By considering the developed conditions of $\mathcal{H}_\infty$ filters, it turns out that the results are generally conservative due to two sources: one introduced after using finite filters for infinite-dimensional systems like time-delay systems and the other source arose from uncertainties. To reduce overdesign conservatism, a new approach to $\mathcal{H}_\infty$ filtering was introduced using a bounded-real lemma (BRL) derived for the corresponding adjoint system. This approach was further refined in [69] using overbounding inequalities. In spite of the considerable advantages of the $\mathcal{H}_\infty$ filtering design results, it still entails some appreciable amount of conservatism due to the majorization procedure in filter design.

The design of robust $\mathcal{H}_\infty$ piecewise filters based on piecewise Lyapunov functional method for a class of piecewise discrete-time linear systems with time-varying delays has not been fully addressed before, which is very challenging. In this paper, we attend to this problem and consider the design of novel filters for a class of linear piecewise discrete-time systems with polytopic parametric uncertainties and time-varying delays. The time delays appear in the state as well as the output and measurement channels. We consider a general full-order filter that guarantees the desired estimation accuracy over the entire uncertainty polytope and accordingly develop two new types of filters by deploying piecewise Lyapunov–Krasovskii functional. The first filter is based on $\mathcal{H}_\infty$ criteria and the design incorporates new parametrization coupled with Finsler's Lemma to establish sufficient conditions for delay-dependent filter feasibility. The other one utilizes the $\mathcal{L}_2 - \mathcal{L}_\infty$ criteria and accomplishes the design via elegant use of Schur complement operations. In both cases, the filter gains are determined by solving linear matrix inequalities (LMIs).

8.2.1 Problem Statement and Definitions

We consider the following class of piecewise discrete-time linear (PDTL) systems:

$$x_{k+1} = A_j x_k + A_{dj} x_{k-d_k} + \Gamma_j \omega_k \tag{8.23}$$

$$y_k = C_j x_k + C_{dj} x_{k-d_k}$$

$$y_k \in \mathbf{\Omega}_j, \quad j = 1, 2, ..., r \tag{8.24}$$

$$z_k = G_j x_k + G_{dj} x_{k-d_k} + \Phi_j \omega_k \tag{8.25}$$

$$x_j = \psi_j, \quad j = -d_M, -d_M + 1, ..., 0 \tag{8.26}$$

where $\{\mathbf{\Omega}_j\}_{j \in S} \subseteq \Re^p$ denotes a partition of the output space into a number of closed polyhedral regions, with $\mathcal{S}$ being the index set of regions, $x_k \in \Re^n$ is the state vector, $\omega_k \in \Re^q$ is the disturbance input, which belongs to $\ell_2[0, \infty)$, $y_k \in \Re^p$ is the measured output and $z_k \in \Re^m$ is the signal to be estimated, $\{\psi_k,\ k = -d_M, -d_M + 1, ..., 0\}$ is a real-valued initial condition and $\{A_j, A_{dj}, \Gamma_j, C_j, C_{dj}, \Psi_j, G_j, G_{dj}, \Phi_j\}$ is the sth local model of the discrete system. In the sequel, we define the set

$$\Pi \stackrel{\Delta}{=} \{j, s | y_k \in \mathbf{\Omega}_j,\ \ y_{k+1} \in \mathbf{\Omega}_s\}$$

to represent all possible transitions from one region to itself or another region. In the sequel, it is assumed that the delay d_k is a time-varying function satisfying

$$d_m \le d_k \le d_M \tag{8.27}$$

where the lower bound $d_m > 0$ and the upper bound $d_M > 0$ are known constant scalars. For well-posedness of the problem and the subsequent results [144, 156, 336], we invoke the following assumptions:

Assumption 8.8 *The solution of the PDTL system (8.23), (8.24), (8.25), and (8.26) starting from any initial condition* ψ_k *is unique for all* $k > 0$.

Assumption 8.9 *When the state of the PDTL system (8.23), (8.24), (8.25), and (8.26) propagates from region* $\mathbf{\Omega}_j$ *to* $\mathbf{\Omega}_s$ *at time k, then the local model* $\mathbf{\Omega}_j$ *governs the system dynamics at that time.*

Assumption 8.10 *The state variables of the PDTL system (8.23), (8.24), (8.25), and (8.26) are bounded for every initial condition and all admissible disturbances.*

Definition 8.11 The energy-to-peak gain of system (8.23), (8.24), (8.25), and (8.26) is defined as

$$\sup_{0 \neq w \in \ell_2} \{||z_k||_{\ell_\infty} / ||w_k||_{\ell_2}\}$$

Remark 8.12 It should be noted that **Assumption 8.8** and **8.9** give a rule that characterize the piecewise state trajectories of the PDTL system (8.23), (8.24), (8.25), and (8.26). The partition is performed in the output space to ensure measurement consideration. Further details are presented in [144, 156, 336].

In case the PDTL system undergoes parametric uncertainties, we consider the following class of uncertain piecewise discrete-time linear (UPDTL) systems

$$x_{k+1} = A_{j\Delta} x_k + A_{dj\Delta} x_{k-d_k} + \Gamma_{j\Delta} \omega_k \tag{8.28}$$
$$y_k = C_{j\Delta} x_k + C_{dj\Delta} x_{k-d_k}$$
$$y_k \in \boldsymbol{\Omega}_j, \quad j = 1, 2, ..., r \tag{8.29}$$
$$z_k = G_{j\Delta} x_k + G_{dj\Delta} x_{k-d_k} + \Phi_{j\Delta} \omega_k \tag{8.30}$$

whose matrices contain uncertainties that belong to a real convex-bounded polytopic model of the type

$$\begin{bmatrix} A_{j\Delta} & A_{dj\Delta} & \Gamma_{j\Delta} \\ C_{j\Delta} & C_{dj\Delta} & \\ G_{j\Delta} & G_{dj\Delta} & \Phi_{j\Delta} \end{bmatrix} \triangleq \left\{ \begin{bmatrix} A_{j\lambda} & A_{dj\lambda} & \Gamma_{j\lambda} \\ C_{j\lambda} & C_{dj\lambda} & \\ G_{j\lambda} & G_{dj\lambda} & \Phi_{j\lambda} \end{bmatrix} = \right.$$
$$\left. \sum_{m=1}^{N} \lambda_m \begin{bmatrix} A_{jm} & A_{djm} & \Gamma_{jm} \\ C_{jm} & C_{djm} & \\ G_{jm} & G_{djm} & \Phi_{jm} \end{bmatrix}, \lambda \in \Lambda \right\} \tag{8.31}$$

where Λ is the unit simplex

$$\Lambda \triangleq \left\{ (\lambda_1, \cdots, \lambda_N) : \sum_{m=1}^{N} \lambda_m = 1 \ , \ \lambda_m \geq 0 \right\} \tag{8.32}$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A_j, ..., \Phi_j\}$ to imply generic system matrices and $\{A_{jm}, ..., \Phi_{jm}, \ \ m \in \mathcal{N}\}$ to represent the respective values at the vertices.

The objective of this paper is to develop delay-dependent methods for piecewise filtering of the class of PDTL systems of the type (8.23), (8.24), (8.25), and (8.26) and subsequently generalize them to the UPDTL systems (8.28), (8.29), and (8.30). We investigate this problem by recourse to linear filter structure. Specifically, we seek to design an estimate $\hat{z}_k$ of z_k given by the linear state-space realization:

$$\begin{aligned} \hat{x}_{k+1} &= A_{fj}\hat{x}_k + B_{fj}\ y_k, \quad \hat{x}(0) = 0, \ \ y_k \in \mathbf{\Omega}_j \\ \hat{z}_k &= G_{fj}\hat{x}_k \end{aligned} \tag{8.33}$$

In (8.33), $\hat{x}(t) \in \Re^n$ is the state vector of the filter, $\hat{z}(t) \in \Re^q$ is the estimate of $z(t)$ and $A_{fs} \in \Re^{n\times n},\ B_{fs} \in \Re^{n\times m},\ G_{fs} \in \Re^{q\times n}$ are unknown filter matrices to be determined in the sequel based on prescribed performance criteria.

8.2.2 Error Dynamics

In terms of the filtering error $\widetilde{z}_k := z_k - \hat{z}_k$ and the augmented state $\widetilde{x}_k := \begin{bmatrix} x_k^t & \hat{x}_k^t \end{bmatrix}^t$, we get from the PDTL system (8.23) and the piecewise filter (8.33) the error dynamic model described by

$$\begin{aligned} \widetilde{x}_{k+1} &= \widetilde{A}_j\ \widetilde{x}_k + \widetilde{A}_{dj}\ \widetilde{x}_{k-d_k} + \widetilde{\Gamma}_j\omega_k \\ \widetilde{y}_k &= \widetilde{C}_j\widetilde{x}_k + \widetilde{C}_{dj}\ \widetilde{x}_{k-d_k} \\ \widetilde{z}_k &= \widetilde{G}_j\widetilde{x}_k + \widetilde{G}_{dj}\ \widetilde{x}_{k-d_k} + \widetilde{\Phi}_j\omega_k, \quad y_k \in \mathbf{\Omega}_j \end{aligned} \tag{8.34}$$

where the associated matrices are given by

$$\begin{aligned} \widetilde{A}_j &= \begin{bmatrix} A_j & 0 \\ B_{fj}C_j & A_{fj} \end{bmatrix},\ \widetilde{\Gamma}_j = \begin{bmatrix} \Gamma_j \\ B_{fj}\Phi_j \end{bmatrix} \\ \widetilde{G}_j &= \begin{bmatrix} G_j & -G_{fj} \end{bmatrix},\ \widetilde{A}_{dj} = \begin{bmatrix} A_{dj} & 0 \\ B_{fj}C_{dj} & 0 \end{bmatrix} \\ \widetilde{C}_j &= \begin{bmatrix} C_j & 0 \end{bmatrix},\ \widetilde{C}_{dj} = \begin{bmatrix} C_{dj} & 0 \end{bmatrix} \\ \widetilde{G}_{dj} &= \begin{bmatrix} G_{dj} & 0 \end{bmatrix},\ \widetilde{\Phi}_j = \Phi_j \end{aligned} \tag{8.35}$$

In this regard, the piecewise filtering problem of the PDTL system under consideration can be phrased as follows: *Given the PDTL system (8.23), (8.24), (8.25), and (8.26) and the piecewise filter (8.33), it is desired to determine the unknown piecewise matrices* $\{A_{fj},\ B_{fj},\ G_{fj}\}$ *such that the filtered system (8.34) is asymptotically stable and a prescribed performance criterion is achieved for all admissible uncertainties satisfying (8.31) and (8.32).* Two performance criteria will be considered in the sequel:

(1) $\mathcal{H}_\infty$-performance meaning that for a given prescribed performance bound $\gamma_\infty > 0$, $||\widetilde{z}_k||_2 < \gamma_\infty||\omega_k||_2,\ \forall\omega \in \ell_2[0,\infty)$,

This means that the $\gamma_\infty-$*suboptimal* $\mathcal{H}_\infty$*-piecewise filtering problem is to find a piecewise filter such that energy-to-peak value gain of the filtered system from the disturbance* ω_k *to the filtering error* $\widetilde{z}_k$ *is less than* γ_∞.

(2) $\mathcal{L}_2 - \mathcal{L}_\infty$-performance meaning that for a given prescribed performance bound $\gamma_2 > 0$ $||\widetilde{y}_k||_\infty < \gamma_2||\omega_k||_2, \ \forall \omega \in \ell_2[0, \infty)$, and

This means that the γ_2-suboptimal generalized $\mathcal{H}_2$-piecewise filtering problem is to find a piecewise filter such that energy-to-peak value gain of the filtered system from the disturbance ω_k to the output filtering error $\widetilde{y}_k$ is less than γ_2.

8.2.3 Delay-Dependent Stability

In this section, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and ℓ_2 gain analysis of the singular filtered. The criteria include some parameter matrices aiming at expanding the range of applicability of the developed conditions. The major thrust is based on the fundamental stability theory of Lyapunov, which states that for asymptotic stability, it suffices to find a Lyapunov function candidate $V_\sigma(x_k, k) > 0, \ \forall x_k \neq 0, \ k \in \mathbf{N}$ satisfying $\Delta V_\sigma(x_k, k) = V_\sigma(x_{k+1}, k+1) - V_\sigma(x_k, k) < 0$. We apply this theorem hereafter for arbitrary switching.

8.2.4 Piecewise Lyapunov Functional

For convenience, we define $\hat{d} = (d_M - d_m + 1)$ as the number of delay samples. The following theorem summarizes the main result.

Theorem 8.13 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable if there exist matrices $0 < P_j^t = P_j, \ 0 < P_s^t = P_s, \ \widehat{X}_j, \ 0 < Q^t = Q, \ 0 < W^t = W, \ \{\mathcal{M}\}_1^3$ satisfying the LMIs for $(j, s) \in \Pi$*

$$\widehat{X}_j \overline{A}_j + \overline{A}_j^t \widehat{X}_j^t + \widetilde{P}_{js} < 0 \tag{8.36}$$

$$\widetilde{P}_{js} = \begin{bmatrix} -\Xi_{1s} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.37}$$

$$\Xi_{1s} = P_s - d_M\mathcal{W}, \ \Xi_2 = \mathcal{M}_1 - d_M\mathcal{W}$$

$$\Xi_{3j} = P_j - \hat{d}\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t$$

$$\Xi_4 = -\mathcal{M}_2 - \mathcal{M}_2^t, \ \Xi_5 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q}$$

$$\overline{A}_j = \begin{bmatrix} -I & \widetilde{A}_j & \widetilde{A}_{dj} & 0 \end{bmatrix} \tag{8.38}$$

Proof Let the switching rule $\sigma(.)$ have an activated subsystem $j \in \Pi$ at instant k then an activated subsystem $s \in \Pi$ at instant $k+1$. In the sequel, we use $\xi_m = \widetilde{x}_{m+1} - \widetilde{x}_m$ and consider the following switched Lyapunov–Krasovskii functional

$$
\begin{aligned}
V_\sigma(\widetilde{x}_k, k) &\stackrel{\Delta}{=} V_{a\sigma}(\widetilde{x}_k, k) + V_{b\sigma}(\widetilde{x}_k, k) + V_{c\sigma}(\widetilde{x}_k, k) + V_{d\sigma}(\widetilde{x}_k, k) \\
V_{a\sigma}(\widetilde{x}_k, k) &= \widetilde{x}_k^t \; P_\sigma \widetilde{x}_k, \quad V_{b\sigma}(\widetilde{x}_k, k) = \sum_{j=k-d_k}^{k-1} \widetilde{x}_j^t \; Q\widetilde{x}_j \\
V_{c\sigma}(\widetilde{x}_k, k) &= \sum_{m=-d_M+2}^{-d_m+1} \sum_{j=k+m-1}^{k-1} \widetilde{x}_j^t \; Q\widetilde{x}_j \\
V_{d\sigma}(\widetilde{x}_k, k) &= \sum_{m=-d_M}^{-1} \sum_{j=k+m}^{k-1} \xi_j^t \; W\xi_j \\
0 < P_\sigma^t = P_\sigma, &\quad 0 < Q^t = Q, \; 0 < W^t = W, \; \sigma \in \mathcal{S}
\end{aligned}
\tag{8.39}
$$

Define $\Delta V_\sigma(\widetilde{x}_k, k) = V_\sigma(\widetilde{x}_{k+1}, k+1) - V_\sigma(\widetilde{x}_k, k)$, along the solution of (8.23) we obtain

$$
\Delta V_{a\sigma}(\widetilde{x}_k, k) = \widetilde{x}_{k+1}^t P_s \widetilde{x}_{k+1} - \widetilde{x}_k^t P_j \widetilde{x}_k \tag{8.40}
$$

$$
\begin{aligned}
\Delta V_{b\sigma}(\widetilde{x}_k, k) &= \sum_{m=k-d_{k+1}+1}^{k} \widetilde{x}_m^t \; Q\widetilde{x}_m - \sum_{j=k-d_k}^{k-1} \widetilde{x}_j^t \; Q\widetilde{x}_j \\
&= \widetilde{x}_k^t \; Q\widetilde{x}_k - \widetilde{x}_{k-d_k}^t \; Q\widetilde{x}_{k-d_k} + \sum_{m=k-d_{k+1}+1}^{k-1} \widetilde{x}_m^t \; Q\widetilde{x}_m \\
&\quad - \sum_{m=k-d_k+1}^{k-1} \widetilde{x}_m^t \; Q\widetilde{x}_m \\
&\leq \widetilde{x}_k^t \; Q\widetilde{x}_k - \widetilde{x}_{k-d_k}^t \; Q\widetilde{x}_{k-d_k} + \sum_{m=k-\bar{d}+1}^{k-\underline{d}} \widetilde{x}_m^t \; Q\widetilde{x}_m
\end{aligned}
\tag{8.41}
$$

$$
\Delta V_{c\sigma}(\widetilde{x}_k, k) = (d_M - d_m)\, \widetilde{x}_k^t \; Q\widetilde{x}_k - \sum_{m=k-d_M+1}^{k-d_m} \widetilde{x}_m^t \; Q\widetilde{x}_m \tag{8.42}
$$

$$
\begin{aligned}
\Delta V_{d\sigma}(\widetilde{x}_k, k) &\leq \bar{d}(\widetilde{x}_{k+1} - \widetilde{x}_k)^t \; W(\widetilde{x}_{k+1} - \widetilde{x}_k) \\
&\quad - d_M \sum_{m=k-d_M}^{k-1} \xi_m^t \; W\xi_m
\end{aligned}
\tag{8.43}
$$

Since $\widetilde{x}_{k-d_k} = \widetilde{x}_k - \sum_{m=k-d_k}^{k-1} \xi_m$, then for arbitrary parameter matrices (a set of free-weighting matrices) $\mathcal{M}_p$, $p = 1, \ldots, 5$, we have

$$
\begin{aligned}
\widehat{x}(k, m) &= \begin{bmatrix} \widetilde{x}_{k+1}^t \; \widetilde{x}_k^t \; \widetilde{x}_{k-d_k}^t \; \xi_m^t \end{bmatrix}^t \\
\widehat{\mathcal{M}} &= \begin{bmatrix} \mathcal{M}_1^t \; \mathcal{M}_2^t \; \mathcal{M}_3^t \; 0 \end{bmatrix}^t \\
\widehat{\mathcal{S}} &= \begin{bmatrix} 0 \; I \; -I \; -d_k I \end{bmatrix}
\end{aligned}
\tag{8.44}
$$

such that the following equation holds

$$2 \sum_{j=k-d_k}^{k-1} \widehat{x}^t(k,m)\ \widehat{\mathcal{M}}\widehat{\mathcal{S}}\widehat{x}(k,m) = 0 \tag{8.45}$$

On considering (8.40), (8.41), (8.42), and (8.43) in the light of (8.39) for $d_k \le \bar{d}$, $w_k \equiv 0$, it is not difficult to show that $\Delta V(x_k, k) < 0$ is equivalent to the following set of inequalities:

$$\sum_{m=k-d_k}^{k-1} \widehat{x}^t(k,m)\ \widetilde{\mathcal{P}}_{sj}\widehat{x}(k,m) < 0, \quad (s,j) \in \mathbf{N} \times \mathbf{N} \tag{8.46}$$

More importantly, in view of (10.45) with $u_k \equiv 0$, $w_k \equiv 0$, we have

$$\overline{A}_j\ \widehat{x}(k,m) = 0 \tag{8.47}$$

where $\widetilde{\mathcal{P}}_{sj}$, $\widetilde{A}_j$ are given by (8.37) and (8.38), respectively. Application of Finsler's **Lemma A.12** (from the Appendix) to (8.46) and (8.47) with $\widehat{x}(k,j) \equiv x$, $\widetilde{\mathcal{P}}_{sj} \equiv \mathcal{P}$, $\widetilde{A}_s \equiv \mathcal{Z}^t$, $\widehat{X}_s \equiv \mathcal{B}$, we readily obtain LMI (8.37) as desired, which establishes the asymptotic stability. ■

8.2.5 Robust Stability

Corollary 8.14 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) and vertex representation (8.31) and (8.32) is delay-dependent asymptotically stable if there exist matrices $0 < P_j^t = P_j$, $0 < P_s^t = P_s$, $\widehat{X}_j$, $0 < Q^t = Q$, $0 < W^t = W$, $\{\mathcal{M}\}_1^3$ satisfying the LMIs for $\forall (j,s) \in \Pi$*

$$\widehat{X}_j\overline{A}_{jp} + \overline{A}_{jp}^t\widehat{X}_j^t + \widetilde{\mathcal{P}}_{jps} < 0 \tag{8.48}$$

$$\widetilde{\mathcal{P}}_{jps} = \begin{bmatrix} -\Xi_1 & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_3 & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.49}$$

$$\overline{A}_{jp} = \begin{bmatrix} -I & \widetilde{A}_{jp} & \widetilde{A}_{djp} & 0 \end{bmatrix} \tag{8.50}$$

Proof Obtained from Theorem (8.13) by using the polytopic representation (8.31) and (8.32) to get (8.49) from (8.36).

8.2.6 Common Lyapunov Functional

In the special case of using a common Lyapunov functional, the ensuing delay-dependent stability results are summarized by the following corollaries:

Corollary 8.15 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable if there exist matrices* $0 < P_j^t = P_j,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^3$ *satisfying the LMIs for* $\forall (j,s) \in \Pi$

$$\widehat{X}_j \overline{A}_j + \overline{A}_j^t \widehat{X}_j^t + \widetilde{P}_j < 0 \tag{8.51}$$

$$\widetilde{P}_j = \begin{bmatrix} -\Xi_{1j} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.52}$$

$$\Xi_{1j} = P_j - \bar{d}\mathcal{W},\ \Xi_2 = \mathcal{M}_1 - \bar{d}\mathcal{W}$$
$$\Xi_{3j} = P_j - \hat{d}\mathcal{Q} - \bar{d}\mathcal{W} - \mathcal{M}_2 - \mathcal{M}_2^t$$
$$\Xi_4 = -\mathcal{M}_2 - \mathcal{M}_2^t,\ \Xi_5 = \mathcal{M}_3 + \mathcal{M}_3^t + \mathcal{Q}$$
$$\overline{A}_j = \begin{bmatrix} -I\ \widetilde{A}_j\ \widetilde{A}_{dj}\ 0 \end{bmatrix} \tag{8.53}$$

Corollary 8.16 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) and vertex representation (8.31) and (8.32) are delay-dependent asymptotically stable if there exist matrices* $0 < P_j^t = P_j,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^3,\ \forall (j) \in \Pi$ *satisfying the LMIs for* $\forall (j) \in \Pi$

$$\widehat{X}_j \overline{A}_{jp} + \overline{A}_{jp}^t \widehat{X}_j^t + \widetilde{\mathcal{P}}_{jp} < 0 \tag{8.54}$$

$$\widetilde{P}_{jp} = \begin{bmatrix} -\Xi_{1j} & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 \\ \bullet & -\Xi_{3j} & \Xi_4 & -\bar{d}\mathcal{M}_2 \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} \end{bmatrix} \tag{8.55}$$

$$\overline{A}_{jp} = \begin{bmatrix} -I\ \widetilde{A}_{jp}\ \widetilde{A}_{djp}\ 0 \end{bmatrix} \tag{8.56}$$

Remark 8.17 The main stability results are derived from feasibility testing in the enlarged state space in contrast with existing similar techniques [184, 368, 372, 438]. The novelty of our approach relies on the deployment of Finsler's Lemma in conjunction with a set of free-weighting matrices without using bounding techniques to ensure that the system matrices are readily separated from the Lyapunov matrices. This decoupling feature simplifies numerical implementation and, as will be shown in the subsequent sections, paves the way to flexible feedback stabilization synthesis. A simple comparison would support our intuition that the LMI results are less conservative and in the nonswitching case are superior than the existing methods [345, 354].

8.2.7 $\mathcal{H}_\infty$ Performance

Here, we consider the performance measure

$$J_{1K} = \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j \right)$$

The following theorem states the main result

Theorem 8.18 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched system (8.23), (8.24), and (8.25) with $u_k \equiv 0$ is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $0 < P_j^t = P_j,\ 0 < P_s^t = P_s,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^5$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall (j,s) \in \Pi$*

$$\widehat{X}_j \mathcal{A}_j + \mathcal{A}_j^t \widehat{X}_j^t + \widehat{P}_{js} < 0 \tag{8.57}$$

$$\widehat{P}_{js} = \begin{bmatrix} -\Xi_1 & \Xi_2 & -\mathcal{M}_1 & -\bar{d}\mathcal{M}_1 & 0 \\ \bullet & -\Xi_3 & \Xi_4 & -\bar{d}\mathcal{M}_2 & \widetilde{G}_j^t \\ \bullet & \bullet & -\Xi_5 & -\bar{d}\mathcal{M}_3 & \widetilde{G}_{dj}^t \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{W} & \widetilde{\Phi}_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} \tag{8.58}$$

$$\mathcal{A}_j = \left[-I \ \widetilde{A}_j \ \widetilde{A}_{dj} \ 0 \ \widetilde{\Gamma}_j \right] \tag{8.59}$$

where $\Xi_1,\ \ldots,\ \Xi_5$ are given in (8.38).

Proof For any $\omega_k \in \ell_2(0,\infty) \neq 0$ and zero initial condition $x_o = 0$, we have

$$J_{1K} \leq \sum_{j=0}^{K} \left(z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j \ + \Delta V_\sigma(x_j, j) \right)$$

Standard algebraic manipulation using (8.25) leads to

$$\begin{aligned} & z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j \ + \Delta V_\sigma(x_j, j) = \\ & \widetilde{x}^t(k,m)\, \widehat{\mathcal{P}}_{js}\, \widetilde{x}(k,m),\ \widetilde{x}(k,m) = \left[\widehat{x}^t(k,m) \quad \omega_k^t \right]^t \end{aligned} \tag{8.60}$$

and $\widehat{\mathcal{P}}_{js}$ is given by (8.57). It follows from [279] that for the switched system (8.23), (8.24), and (8.25) to be asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ it suffices that $z_j^t z_j - \gamma_\infty^2 \omega_j^t \omega_j \ + \Delta V_\sigma(x_j, j) < 0,\ \forall j \in \{0, K\}$ holds for arbitrary switching, which in turn implies that $J_{1K} < 0$. The desired result is achieved by Finsler's Lemma and LMI (8.37) subject to (8.36). ■

8.2.8 $\ell_2 - \ell_\infty$ Performance

Here, we consider the performance measure

$$J_{2K} = V_\sigma(x_K, K) - \sum_{j=0}^{K-1} \omega_j^t\, \omega_j$$

where K is an arbitrary positive integer. The following theorem states the desired stability result

Theorem 8.19 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable with generalized $\mathcal{H}_2 - gain < \gamma_2$ if there exist matrices $0 < P_j^t = P_j,\ 0 < P_s^t = P_s,\ \widehat{X}_j,\ 0 < Q^t = Q,\ 0 < W^t = W,\ \{\mathcal{M}\}_1^5$ and scalars $\gamma_2 > 0, \varepsilon_j > 0$ satisfying the LMIs for $\forall (j,s) \in \Pi$*

$$\begin{bmatrix} -P_s + \hat{d}Q & 0 & 0 & \widetilde{A}_j^t P_j & \bar{d}\left(\widetilde{A}_j^t - I\right)W \\ \bullet & -Q & 0 & \widetilde{A}_{dj}^t P_j & \bar{d}\widetilde{A}_{dj}^t W \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t P_j & \bar{d}\widetilde{\Gamma}_j^t W \\ \bullet & \bullet & \bullet & -P_j & 0 \\ \bullet & \bullet & \bullet & \bullet & -\bar{d}W \end{bmatrix} < 0 \tag{8.61}$$

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \widetilde{C}_j^t \\ \bullet & -\varepsilon_j I & \widetilde{C}_{dj}^t \\ \bullet & \bullet & -\gamma_2^2 I \end{bmatrix} < 0 \tag{8.62}$$

Proof For any sequence $0 \neq \omega_j \in \ell_2[0,\infty),\ j \in \{1,\ldots,K-1\}$ and zero initial condition $\tilde{x}_o = 0$, one has

$$J_{2K} = \sum_{j=0}^{K-1} \left[\Delta V_K|_{(8.23)} - \omega_j^t\, \omega_j \right] \tag{8.63}$$

Using (8.40) (8.41), (8.42), and (8.43) and manipulating, we get

$$J_{2K} = \begin{bmatrix} x_k \\ x_{k-d_k} \\ \omega_k \end{bmatrix}^t \Xi_{sj} \begin{bmatrix} x_k \\ x_{k-d_k} \\ \omega_k \end{bmatrix} \tag{8.64}$$

$$\Xi_{sj} = \begin{bmatrix} \Xi_{1sj} & \Xi_{2sj} & \Xi_{3sj} \\ \bullet & \Xi_{4sj} & \Xi_{5sj} \\ \bullet & \bullet & \Xi_{6sj} \end{bmatrix}$$

$$\Xi_{1sj} = \widetilde{A}_j^t P_s \widetilde{A}_j - P_j + (\bar{d} - \underline{d} + 1)Q + \bar{d}\left(\widetilde{A}_j^t - I\right)W(\widetilde{A}_j - I)$$

$$\Xi_{2sj} = \widetilde{A}_j^t P_s \widetilde{A}_{dj} + \bar{d}\left(\widetilde{A}_j^t - I\right)W\widetilde{A}_{dj}$$

$$\begin{aligned}
\Xi_{3sj} &= \widetilde{A}_j^t P_s \Gamma_j + \bar{d}\left(\widehat{A}_j^t - I\right) W \widetilde{\Gamma}_j \\
\Xi_{4sj} &= -Q + \widetilde{A}_{dj}^t P_s \widetilde{A}_{dj} + \bar{d} \widetilde{A}_{dj}^t W \widetilde{A}_{dj} \\
\Xi_{5sj} &= \widetilde{A}_{dj}^t P_s \widetilde{A}_{dj} + \bar{d} \widehat{A}_{dj}^t W \widetilde{A}_{dj}, \\
\Xi_{6sj} &= -I + \widetilde{\Gamma}_j^t P_s \widetilde{\Gamma}_j + \bar{d} \widetilde{\Gamma}_j^t W \widetilde{\Gamma}_j
\end{aligned} \tag{8.65}$$

By virtue of (8.61) and Schur complements, it is easy to see that $J_{2K} < 0$ for any K. Subsequently, for any $0 \neq \omega_j \in \ell_2[0, \infty)$, it follows that

$$V_K < \sum_{j=0}^{K-1} \omega_j^t \, \omega_j \tag{8.66}$$

In turn, Schur complements on LMI (8.62) and applying the **S**-procedure, it yields

$$\begin{bmatrix} -\gamma_2^2 \mathcal{P}_j + \widetilde{C}_j^t \widetilde{C}_j & \widetilde{C}_j^t \widetilde{C}_{dj} \\ \bullet & \widetilde{C}_{dj}^t \widetilde{C}_{dj} \end{bmatrix} < 0 \tag{8.67}$$

from which it is readily evident that

$$\widetilde{y}_K^t \widetilde{y}_K - \gamma_2^2 \, V_K < 0 \tag{8.68}$$

Finally, by LMIs (8.66) and (8.68), it follows that switched filtered system (8.34) has a generalized $\mathcal{H}_2$ norm bound γ_2. ■

Remark 8.20 We note from that the $\mathcal{L}_2 -$ gain under arbitrary switching can be looked as the worst-case energy amplitude gain for the switched system (8.23, 8.24, 8.25, and 8.26) over all possible inputs, switching signals, and all admissible uncertainties. The functional (8.39) is called a switched Lyapunov function (SLF) since it has the same switching signals as system (8.23), (8.24), and (8.25), which is known to yield less conservative results than the constant Lyapunov functional. A novel feature of the developed approach is the arbitrary selection of the matrix $\widetilde{X}_j$, which helps much in the feedback stabilization later on as well as in the numerical simulation.

Remark 8.21 The optimal $\mathcal{L}_2 - gain$ of switched system (8.23, 8.24, and 8.25) can be determined by solving the following convex minimization problem over LMIs:

$$\begin{aligned}
&\qquad Minimize \;\; \gamma \\
&s.t. \quad LMIs \; (8.36) - (8.37), \quad \forall \, (j, s) \in \mathbf{N} \times \mathbf{N} \\
&P_j, \; P_s, \; \widehat{X}_j, \; Q, \; W, \; \{\mathcal{M}\}_1^5, \; \forall (j, s), \;\; \gamma > 0, \; \sigma > 0, \; \kappa > 0
\end{aligned}$$

which can be conveniently solved by the existing LMI software.

8.2.9 $\mathcal{H}_\infty$ Filter Design

To facilitate further development, define

$$\widehat{\mathcal{X}}_j = \left[\, \widehat{\Upsilon}^t \ 0\ 0\ 0\ 0\ 0\ 0 \,\right]^t, \ \widehat{\Upsilon} \in \Re^{2n\times 2n}$$

Next, we express $\widehat{\Upsilon}$ and $\widehat{\Lambda} = \widehat{\Upsilon}^{-1}$ and other relevant matrices into the convenient form

$$\begin{aligned}
\widehat{\Upsilon} &= \begin{bmatrix} \Upsilon_s & 0 \\ \Upsilon_o & \Upsilon_c \end{bmatrix}, \ \widehat{\mathcal{R}} = \begin{bmatrix} \mathcal{R}_1 & 0 \\ \mathcal{R}_2 & \mathcal{R}_3 \end{bmatrix}, \ \widehat{\Lambda} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_1 & \Lambda_2 \end{bmatrix} \\
\widehat{\mathcal{S}} &= \begin{bmatrix} \mathcal{S}_1 & 0 \\ \mathcal{S}_2 & \mathcal{S}_3 \end{bmatrix}, \ \Psi_k = \begin{bmatrix} \Psi_{1k} & 0 \\ \Psi_{2k} & \Psi_{3k} \end{bmatrix} \\
\mathcal{P}_j &= \begin{bmatrix} \mathcal{P}_{1j} & 0 \\ \mathcal{P}_{2j} & \mathcal{P}_{3j} \end{bmatrix}, \ \mathcal{X}_j = \mathcal{P}_j^{-1} = \begin{bmatrix} \mathcal{X}_{1j} & 0 \\ \mathcal{X}_{2j} & \mathcal{X}_{3j} \end{bmatrix}
\end{aligned} \tag{8.69}$$

The following design result is established:

Theorem 8.22 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.69). Switched filtered system (8.34) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $\{\mathcal{X}_{kj}\}_{k=1}^3$, $\{\mathcal{X}_{ks}\}_{k=1}^3$, $\{\mathcal{S}_k\}_1^3$, $\{\mathcal{R}_k\}_1^3$, B_{fj}, $\{\Psi_k\}_1^5$, $\{\mathcal{Y}_{kj}\}_{k=1}^3$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall(j,s) \in \Pi$*

$$\begin{bmatrix} -\Sigma_{1s} & \Sigma_{2j} & \Sigma_{3j} & -\bar{d}\Psi_1 & \widetilde{\Gamma}_j \\ \bullet & -\Sigma_4 & -\Sigma_5 & -\bar{d}\Psi_2 & \Sigma_{7j} \\ \bullet & \bullet & -\Sigma_6 & -\bar{d}\Psi_3 & \Sigma_{8j} \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} < 0, \tag{8.70}$$

$$\begin{aligned}
\Sigma_{1s} &= \begin{bmatrix} \Lambda_1 + \Lambda_1^t + \mathcal{X}_{1s} + \bar{d}\mathcal{S}_1 & \Lambda_1^t \\ \Lambda_1 + \mathcal{X}_{2s} + \bar{d}\mathcal{S}_2 & \Lambda_2 + \Lambda_2^t + \mathcal{X}_{3s} - \bar{d}\mathcal{S}_1 \end{bmatrix} \\
\Sigma_{2j} &= \begin{bmatrix} \Psi_{11} + A_j\Lambda_1^t - \bar{d}\mathcal{S}_1 & A_j\Lambda_1^t \\ \Psi_{21} - \bar{d}\mathcal{S}_2 + \mathcal{Y}_1 & \mathcal{Y}_1 + \mathcal{Y}_2^t + \Psi_{31} - \bar{d}\mathcal{S}_3 \end{bmatrix} \\
\Sigma_{7j} &= \begin{bmatrix} \Lambda_1 G_j^t \\ \Lambda_1 G_j^t - \mathcal{Y}_{4j} \end{bmatrix}, \ \Sigma_{8j} = \begin{bmatrix} \Lambda_1 G_{dj}^t \\ \Lambda_1 G_{dj}^t \end{bmatrix} \\
\Sigma_{3j} &= \begin{bmatrix} -\Psi_{11} + A_{dj}\Lambda_1^t & A_{dj}\Lambda_1^t \\ -\Psi_{12} + \mathcal{Y}_3 & -\Psi_{13} + \mathcal{Y}_3 \end{bmatrix} \\
\Sigma_5 &= \begin{bmatrix} \Psi_{21} + \Psi_{21}^t & 0 \\ \Psi_{22} + \Psi_{22}^t & \Psi_{23} + \Psi_{23}^t \end{bmatrix}, \ \Sigma_4 = \begin{bmatrix} \Sigma_{41} & 0 \\ \Sigma_{42} & \Sigma_{43} \end{bmatrix} \\
\Sigma_6 &= \begin{bmatrix} \Psi_{31} + \Psi_{31}^t + \mathcal{R}_1 & 0 \\ \Psi_{32} + \Psi_{32}^t + \mathcal{R}_2 & \Psi_{33} + \Psi_{33}^t + \mathcal{R}_3 \end{bmatrix} \\
\Sigma_{41} &= \mathcal{P}_{1s} - \widehat{d}\mathcal{S}_1 - \mathcal{R}_1 - \Psi_{21} - \Psi_{21}^t,
\end{aligned}$$

$$\Sigma_{42} = \mathcal{P}_{2s} - \widehat{d}\mathcal{S}_2 - \mathcal{R}_2 - \Psi_{22} - \Psi_{22}^t,$$
$$\Sigma_{43} = \mathcal{P}_{3s} - \widehat{d}\mathcal{S}_3 - \mathcal{R}_3 - \Psi_{23} - \Psi_{23}^t \tag{8.71}$$

Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \tag{8.72}$$

Proof Applying the congruence transformation

$$\text{diag}[\widehat{\Lambda}, \ \widehat{\Lambda}, \ \widehat{\Lambda}, \ \widehat{\Lambda}, \ I]$$

to inequality (8.58) using (8.57), and (8.59), and the linearizations

$$X_s = \Upsilon^{-t} P_s \Upsilon^{-1}, \ \mathcal{S} = \Upsilon^{-t}\mathcal{W}\Upsilon^{-1}, \ \mathcal{Y}_{j3} = B_{fj}C_{dj}\Lambda_1^t$$
$$X_j = \Upsilon^{-t} P_j \Upsilon^{-1}, \ \mathcal{Y}_{j1} = B_{fj}C_j\Lambda_1^t, \ \mathcal{Y}_{j2} = \Lambda_2 A_{fj}$$
$$\mathcal{Y}_{4j} = \Lambda_2 G_{fj}^t, \ \{\Psi\}_1^5 = \Upsilon^{-t}\{\mathcal{M}\}_1^5\Upsilon^{-1}$$

we immediately obtain LMI (8.70) subject to (8.71). ■

A special design procedure based on the common Lyapunov functional is given below:

Corollary 8.23 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.69). Switched filtered system (8.34) is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_\infty$ if there exist matrices $\{\mathcal{X}_{kj}\}_{k=1}^3$, $\{\mathcal{S}_k\}_1^3$, $\{\mathcal{R}_k\}_1^3$, B_{fj}, $\{\Psi_k\}_1^5$, $\{\mathcal{Y}_{kj}\}_{k=1}^3$, $\forall(j,s) \in \boldsymbol{N}$ and a scalar $\gamma_\infty > 0$ satisfying the LMIs for $\forall(j) \in \Pi$*

$$\begin{bmatrix} -\Sigma_{1j} & \Sigma_{2j} & \Sigma_{3j} & -\bar{d}\Psi_1 & \widetilde{\Gamma}_j \\ \bullet & -\Sigma_4 & -\Sigma_5 & -\bar{d}\Psi_2 & \Sigma_{7j} \\ \bullet & \bullet & -\Sigma_6 & -\bar{d}\Psi_3 & \Sigma_{8j} \\ \bullet & \bullet & \bullet & -\bar{d}\mathcal{S} & \Phi_j^t \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty^2 I \end{bmatrix} < 0 \tag{8.73}$$

$$\Sigma_{1j} = \begin{bmatrix} \Lambda_1 + \Lambda_1^t + \mathcal{X}_{1j} + \bar{d}\mathcal{S}_1 & \Lambda_1^t \\ \Lambda_1 + \mathcal{X}_{2j} + \bar{d}\mathcal{S}_2 & \Lambda_2 + \Lambda_2^t + \mathcal{X}_{3j} - \bar{d}\mathcal{S}_1 \end{bmatrix} \tag{8.74}$$

where Σ_{2j}, ..., Σ_{43} are given by (8.71). Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \tag{8.75}$$

8.2.10 $\ell_2 - \ell_\infty$ Filter Design

Initially, we recall the following result:

Lemma 8.24 *The matrix inequality*

$$-\mathcal{M} + \mathcal{N}\,\Omega^{-1}\,\mathcal{N}^t \ < \ 0 \tag{8.76}$$

holds for some $0 < \Omega = \Omega^t \in \Re^{n\times n}$, *if and only if*

$$\begin{bmatrix} -\mathcal{M} & \mathcal{N}\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} < \ 0 \tag{8.77}$$

holds for some matrices $\mathcal{X} \in \Re^{n\times n}$ *and* $\mathcal{Z} \in \Re^{n\times n}$.

Proof ($\Longrightarrow$) By Schur complements, inequality (8.76) is equivalent to

$$\begin{bmatrix} -\mathcal{M} & \mathcal{N}\Omega^{-1} \\ \bullet & -\Omega^{-1} \end{bmatrix} < \ 0 \tag{8.78}$$

Setting $\mathcal{X} = \mathcal{X}^t = \mathcal{Z} = \Omega^{-1}$, we readily obtain inequality (8.77).
($\Longleftarrow$) Since the matrix $[I \quad \mathcal{N}]$ is of full rank, we obtain

$$\begin{aligned} \begin{bmatrix} I \\ \mathcal{N}^t \end{bmatrix}^t \begin{bmatrix} -\mathcal{M} & \mathcal{N}\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} \begin{bmatrix} I \\ \mathcal{N}^t \end{bmatrix} &< \ 0 \Longleftrightarrow \\ -\mathcal{M} + \mathcal{N}\,\mathcal{Z}\,\mathcal{N}^t &< \ 0 \Longleftrightarrow, \\ -\mathcal{M} + \mathcal{N}\,\Omega^{-1}\,\mathcal{N}^t &< \ 0, \quad \mathcal{Z} = \Omega^{-1} \end{aligned} \tag{8.79}$$

which completes the proof. ■

In preparation for the filter design, we use **Lemma 8.24** to introduce relaxation variables and establish the theorem below:

Theorem 8.25 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27). Switched filtered system (8.34) is delay-dependent asymptotically stable with* $\ell_2 - \ell_\infty \ < \gamma_2$ *if there exist matrices* $\{\mathcal{X}\}_{i=1}^N$, $\mathcal{Y}$, $\mathcal{G}$, $\mathcal{F}$ $\forall (i, j, s) \ \in \ \Pi$ *and scalars* $\gamma_2 > 0$, $\varepsilon_j > 0$ *such that the LMIs*

$$\begin{bmatrix} -\mathcal{X}_s & 0 & 0 & \mathcal{G}^t\widetilde{A}_j^t & \bar{d}\mathcal{G}^t\left(\widetilde{A}_j^t - I\right) & \bar{d}\mathcal{F} \\ \bullet & -\mathcal{F} - \mathcal{F}^t + \mathcal{Y} & 0 & \mathcal{G}^t\widetilde{A}_{dj}^t & \bar{d}\mathcal{G}^t\widetilde{A}_{dj}^t & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_i^t & \bar{d}\widetilde{\Gamma}_i^t & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{G} - \mathcal{G}^t + \bar{d}\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{F} - \mathcal{F}^t + \hat{d}\mathcal{Y} \end{bmatrix} < \ 0 \tag{8.80}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \widetilde{C}_j\mathcal{G} \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\mathcal{G} - \mathcal{G}^t + \mathcal{X}_j \end{bmatrix} < \ 0 \tag{8.81}$$

have a feasible solution.

Proof Applying the congruent transformations

$$[\mathcal{X}_s,\ I,\ I,\ \mathcal{X}_j,\ I,\ I]$$

to LMI (8.61) and

$$[\mathcal{X}_j,\ I,\ I]$$

to LMI (8.62), respectively, with $\mathcal{X}_i = \mathcal{P}_i^{-1},\ i = j, s$ and Schur complements, it yields

$$\begin{bmatrix} -\mathcal{X}_s & 0 & 0 & \mathcal{X}_s\widetilde{A}_j^t & \bar{d}\left(\widetilde{A}_j^t - I\right) & \bar{d}\mathcal{X}_s\mathcal{Q} \\ \bullet & -\mathcal{Q} & 0 & \widetilde{A}_{dj}^t & \bar{d}\widetilde{A}_{dj}^t & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X}_j & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\bar{d}W^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\bar{d}\mathcal{Q} \end{bmatrix} < 0 \tag{8.82}$$

$$\begin{bmatrix} -\mathcal{X}_j & 0 & \mathcal{X}_j\widetilde{C}_j^t \\ \bullet & -\varepsilon_j I & \widetilde{C}_{dj}^t \\ \bullet & \bullet & -\gamma_2^2 I \end{bmatrix} < 0 \tag{8.83}$$

When (8.80) and (8.81) hold, it is not difficult to infer that $0 < \mathcal{X}_j < \mathcal{G} + \mathcal{G}^t$. The inequality $(\mathcal{X}_j - \mathcal{G})^t\mathcal{X}_j^{-1}(\mathcal{X}_j - \mathcal{G}) \geq 0$ implies that $-\mathcal{G}^t\mathcal{X}_j^{-1}\mathcal{G} \leq \mathcal{X}_j - (\mathcal{G} + \mathcal{G}^t)$ and in the same way, the inequality $(\mathcal{Y} - \mathcal{F})^t\mathcal{Y}^{-1}(\mathcal{Y} - \mathcal{F}) \geq 0$ implies that $-\mathcal{F}^t\mathcal{Y}^{-1}\mathcal{F} \leq \mathcal{Y} - (\mathcal{F} + \mathcal{F}^t)$. Alternatively, it follows from **Lemma A.2** that there exist matrices $\mathcal{G},\ \mathcal{F},\ \mathcal{Y}_{i=1}^N$ such that LMIs (8.80) and (8.81) are readily obtained. ∎

Next, to determine the unknown matrices of the piecewise filter we proceed and define the following matrices

$$\mathcal{X}_k = \begin{bmatrix} \mathcal{X}_{1k} & 0 \\ \mathcal{X}_{2k} & \mathcal{X}_{3k} \end{bmatrix},\ k = s, j,\ \mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 \\ \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix}$$
$$\Psi_k = \begin{bmatrix} \Psi_{1k} & 0 \\ \Psi_{2k} & \Psi_{3k} \end{bmatrix},\ \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 & 0 \\ \mathcal{F}_1 & \mathcal{F}_2 \end{bmatrix} \tag{8.84}$$

and the linearizations

$$\mathcal{D}_{1j} = \mathcal{G}_2^t A_{fj}^t,\ \mathcal{D}_{2j} = \mathcal{G}_1^t C_j^t B_{fj}^t + \mathcal{G}_1^t A_{fj}^t$$

The following design results are established.

Theorem 8.26 *Given $d_M > 0$ and $d_m > 0$ subject to (8.27) and the matrices in (8.84). Switched filtered system (8.34) is delay-dependent asymptotically stable with* *an* $\mathcal{L}_2 - \mathcal{L}_\infty < \gamma_2$ *if there exist matrices* $\{\mathcal{X}_{kj}\}_{k=1}^3,\ \{\mathcal{X}_{ks}\}_{k=1}^3,\ \{\mathcal{S}_k\}_1^3,\ \{\mathcal{R}_k\}_1^3,\ B_{fj},\ \{\Psi_k\}_1^5,\ \{\mathcal{Y}_{kj}\}_{k=1}^3,\ \forall (j,s) \in \boldsymbol{N}$ *and scalars* $\gamma_2 > 0, \varepsilon_j > 0$ *satisfying the LMIs for* $\forall (j,s) \in \Pi$

$$\begin{bmatrix} -\Pi_{1s} & 0 & 0 & \Pi_{2j} & \bar{d}\Pi_{3j} & \bar{d}\mathcal{F} \\ \bullet & -\Pi_4 & 0 & -\Pi_4 & -\bar{d}\Pi_4 & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t & -\bar{d}\widetilde{\Gamma}_j^t & 0 \\ \bullet & \bullet & \bullet & -\Pi_{1j} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Pi_6 \end{bmatrix} < 0 \tag{8.85}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \Pi_7 \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\Pi_8 \end{bmatrix} < 0 \tag{8.86}$$

$$\begin{aligned} \Pi_{1s} &= \begin{bmatrix} \mathcal{X}_{1s} & 0 \\ \mathcal{X}_{2s} & \mathcal{X}_{3s} \end{bmatrix},\ \Pi_{2j} = \begin{bmatrix} \mathcal{G}_1^t A_j^t & \mathcal{D}_{2j} \\ 0 & \mathcal{D}_{1j} \end{bmatrix} \\ \Pi_{3j} &= \begin{bmatrix} \mathcal{G}_1^t \left(A_j^t - I \right) & \mathcal{D}_{2j} \\ 0 & \mathcal{D}_{1j} - \mathcal{G}_2^t \end{bmatrix},\ \Pi_7 = \begin{bmatrix} C_j \mathcal{G}_1 \\ 0 \end{bmatrix} \\ \Pi_4 &= \begin{bmatrix} \mathcal{F}_1 + \mathcal{F}_1^t - \mathcal{Y}_1 & 0 \\ \mathcal{F}_2 + \mathcal{F}_2^t - \mathcal{Y}_2 & \mathcal{F}_3 + \mathcal{F}_3^t - \mathcal{Y}_3 \end{bmatrix} \\ \Pi_5 &= \begin{bmatrix} \mathcal{G}_1 + \mathcal{G}_1^t - \bar{d}\mathcal{Z}_1 & 0 \\ \mathcal{G}_2 + \mathcal{G}_2^t - \bar{d}\mathcal{Z}_2 & \mathcal{G}_3 + \mathcal{G}_3^t - \bar{d}\mathcal{Z}_3 \end{bmatrix} \\ \Pi_6 &= \begin{bmatrix} \mathcal{F}_1 + \mathcal{F}_1^t - \bar{d}\mathcal{Y}_1 & 0 \\ \mathcal{F}_2 + \mathcal{F}_2^t - \bar{d}\mathcal{Y}_2 & \mathcal{F}_3 + \mathcal{F}_3^t - \bar{d}\mathcal{Y}_3 \end{bmatrix} \\ \Pi_8 &= \begin{bmatrix} \mathcal{G}_1 + \mathcal{G}_1^t - \mathcal{X}_{1j} & 0 \\ \mathcal{G}_2 + \mathcal{G}_2^t - \mathcal{X}_{2j} & \mathcal{G}_3 + \mathcal{G}_3^t - \mathcal{G}_{3j} \end{bmatrix} \end{aligned} \tag{8.87}$$

Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1} \mathcal{Y}_{j2},\quad B_{fj},\quad G_{fj} = \mathcal{Y}_{4j} \Lambda_2^{-t} \tag{8.88}$$

A special design procedure based on a common Lyapunov functional is given below:

Corollary 8.27 *Given* $d_M > 0$ *and* $d_m > 0$ *subject to (8.27) and the matrices in (8.84). Switched filtered system (8.34) is delay-dependent asymptotically stable with an* $\ell_2 - \ell_\infty < \gamma_2$ *if there exist matrices* $\{\mathcal{X}_{kj}\}_{k=1}^3,\ \{\mathcal{S}_k\}_1^3,\ \{\mathcal{R}_k\}_1^3,\ B_{fj},\ \{\Psi_k\}_1^5,\ \{\mathcal{Y}_{kj}\}_{k=1}^3,\ \forall (j,s) \in \boldsymbol{N}$ *and scalars* $\gamma_2 > 0, \varepsilon_j > 0$ *satisfying the LMIs for* $\forall (j,s) \in \Pi$

$$\begin{bmatrix} -\Pi_{1j} & 0 & 0 & \Pi_{2j} & \bar{d}\Pi_{3j} & \bar{d}\mathcal{F} \\ \bullet & -\Pi_4 & 0 & -\Pi_4 & -\bar{d}\Pi_4 & 0 \\ \bullet & \bullet & -I & \widetilde{\Gamma}_j^t & -\bar{d}\widetilde{\Gamma}_j^t & 0 \\ \bullet & \bullet & \bullet & -\Pi_{1j} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_5 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Pi_6 \end{bmatrix} < 0 \tag{8.89}$$

$$\begin{bmatrix} -\gamma_2^2 I & \widetilde{C}_{dj} & \Pi_7 \\ \bullet & -\varepsilon_j I & 0 \\ \bullet & \bullet & -\Pi_8 \end{bmatrix} < 0 \tag{8.90}$$

$$\Pi_{1j} = \begin{bmatrix} \mathcal{X}_{1j} & 0 \\ \mathcal{X}_{2j} & \mathcal{X}_{3j} \end{bmatrix} \tag{8.91}$$

where $\Pi_{2j}, \ldots, \Pi_8$ are given by (8.87). Moreover, the gain matrices are given by

$$A_{fj} = \Lambda_2^{-1}\mathcal{Y}_{j2}, \quad B_{fj}, \quad G_{fj} = \mathcal{Y}_{4j}\Lambda_2^{-t} \tag{8.92}$$

8.2.11 Illustrative Example B

Consider the following system of the type (8.23), (8.24), and (8.25) where the switching occurs between four modes described by the following coefficients:

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.7 & 0.09 \\ 0 & 0.35 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; \Gamma_1 = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix} \\
G_1 &= \begin{bmatrix} 0.25 \\ 0.15 \end{bmatrix}, \; G_{d1} = \begin{bmatrix} -0.1 \\ -0.01 \end{bmatrix}, \; \Phi_1 = 0.01 \\
C_1 &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \; C_{d1} = \begin{bmatrix} -0.1 & 0 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0.41 & 0.11 \\ 0 & 0.97 \end{bmatrix}, \; A_{d3} = \begin{bmatrix} 0 & 0.05 \\ 0 & -0.15 \end{bmatrix}, \; \Phi_2 = 0.02 \\
G_2 &= \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix}, \; G_{d2} = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.2 \\ -0.02 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, \; C_{d2} = \begin{bmatrix} 0 & -0.1 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 0.6 & 0.02 \\ 0 & 0.49 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} -0.1 & 0.01 \\ -0.1 & -0.1 \end{bmatrix}, \; \Phi_3 = 0.02 \\
G_3 &= \begin{bmatrix} 0.17 \\ 0.19 \end{bmatrix}, \; G_{d3} = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \; \Gamma_3 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} \\
C_3 &= \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \; C_{d3} = \begin{bmatrix} -0.1 & 0 \end{bmatrix} \\
A_4 &= \begin{bmatrix} -0.33 & 0.22 \\ 0 & -0.45 \end{bmatrix}, \; A_{d4} = \begin{bmatrix} 0 & 0.25 \\ 0 & -0.05 \end{bmatrix}, \; \Phi_4 = 0.02
\end{aligned}$$

$$G_4 = \begin{bmatrix} 0.22 \\ 0.13 \end{bmatrix}, \; G_{d4} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \; \Gamma_4 = \begin{bmatrix} 0.1 \\ -0.02 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 0.3 \; 0.7 \end{bmatrix}, \; C_{d4} = \begin{bmatrix} -0.1 \; 0.1 \end{bmatrix}$$

and the corresponding two sets $\{j = 1 \; if \; y_k < 1\}$ and $\{j = 2 \; if \; y_k \geq 1\}$ respectively.

A computational summary of applying **Theorem 8.22** and **Corollary 8.23** and using the tools of [17], such that the above piecewise system is asymptotically stable is depicted in Table 8.1. The piecewise filter matrices are given by

$$A_{f1} = \begin{bmatrix} -0.8118 & -0.2795 \\ 0.2105 & -0.7467 \end{bmatrix}, \; B_{f1} = \begin{bmatrix} -0.7833 \\ -1.2554 \end{bmatrix}$$
$$G_{f1} = \begin{bmatrix} -1.3024 \; -0.1185 \end{bmatrix}$$
$$A_{f2} = \begin{bmatrix} 0.7767 & -0.2665 \\ 0.1905 & -0.6885 \end{bmatrix}, \; B_{f2} = \begin{bmatrix} -0.8452 \\ -1.3725 \end{bmatrix}$$
$$G_{f2} = \begin{bmatrix} -1.4513 \; -0.1335 \end{bmatrix}$$
$$A_{f3} = \begin{bmatrix} -0.7467 & -0.2835 \\ 0.2019 & 0.7645 \end{bmatrix}, \; B_{f3} = \begin{bmatrix} -1.3675 \\ -0.9008 \end{bmatrix}$$
$$G_{f3} = \begin{bmatrix} -0.2025 \; -1.4366 \end{bmatrix}$$
$$A_{f4} = \begin{bmatrix} 0.8258 & -0.2193 \\ 0.2005 & -0.7534 \end{bmatrix}, \; B_{f4} = \begin{bmatrix} -1.5364 \\ -0.8111 \end{bmatrix}$$
$$G_{f4} = \begin{bmatrix} -1.4448 \; -0.2167 \end{bmatrix}$$

The state x and filtered state $\hat{x}$ trajectories using $\mathcal{H}_\infty$-performance are plotted in Figs. 8.6 and 8.7.

It is quite evident the developed piecewise $\mathcal{H}_\infty$ filter gives improved performance.

Turning to the implementation of **Theorem 8.26** and **Corollary 8.27** such that the piecewise discrete-time system under consideration is asymptotically stable, comparison of the feasible results is presented in Table 8.2 and the corresponding state x and filtered state $\hat{x}$ trajectories using $\mathcal{L}_2 - \mathcal{L}_\infty$-performance are plotted in Figs. 8.8 and 8.9.

The foregoing results come in support with the effectiveness of our filtering approach.

Table 8.1 A summary of $\mathcal{H}_\infty$-performance bound: illustrative example B

$\underline{d}$	$\bar{d}$	*The*.8.22	*Coro*.8.23
2	6	2.145	2.335
3	9	2.774	3.021
4	11	3.182	3.664
5	13	3.534	4.875
6	13	3.732	6.438

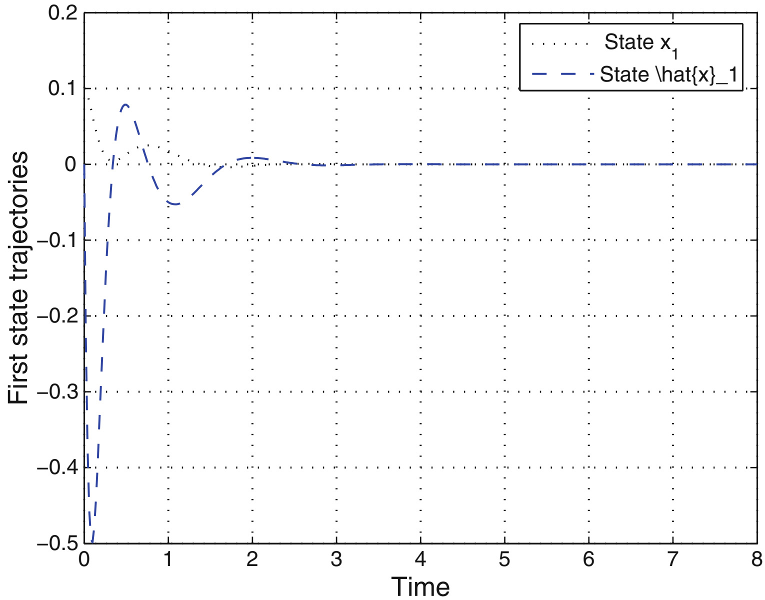

Fig. 8.6 Plot of x_1 and $\hat{x}_1$ versus time: $\mathcal{H}_\infty$ filter

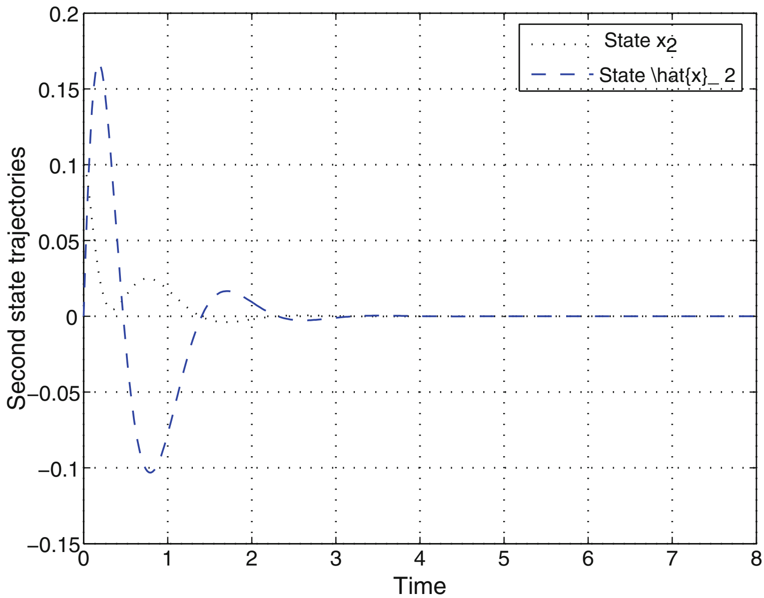

Fig. 8.7 Plot of x_2 and $\hat{x}_2$ versus time: $\mathcal{H}_\infty$ filter

Table 8.2 A summary of $\ell_2 - \ell_\infty$-performance bound: illustrative example B

$\underline{d}$	$\bar{d}$	$The.8.26$	$Coro.8.27$
2	6	3.015	3.532
3	9	3.684	4.021
4	11	5.182	6.224
5	13	6.534	7.694
6	13	6.732	9.015

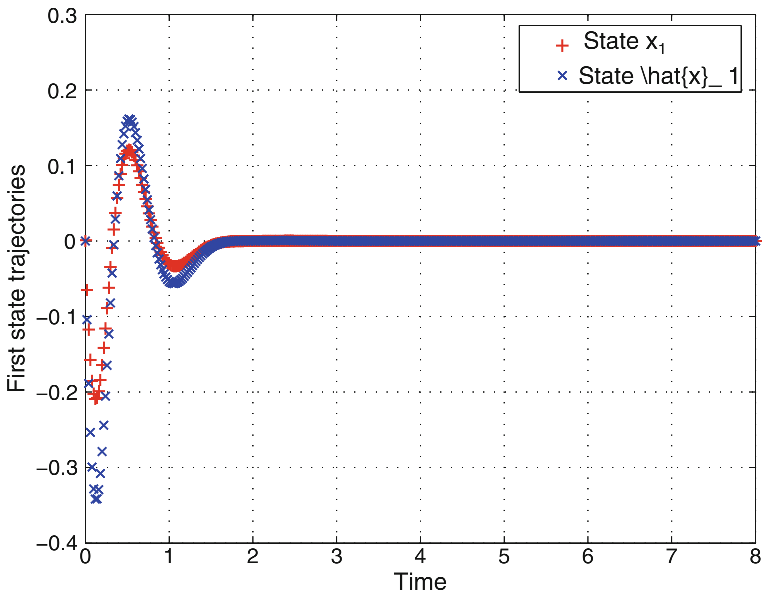

Fig. 8.8 Plot of x_1 and $\hat{x}_1$ versus time: $\mathcal{L}_2 - \mathcal{L}_\infty$ filter

8.2.12 Illustrative Example C

Consider a third-order system of the type (8.23), (8.24), and (8.25) where the switching occurs between two modes described by the following coefficients:

$$Mode\ 1 = y_k \ \geq \ 0$$

$$A_1 = \begin{bmatrix} 0 & 0.2 & 0.3 \\ -0.3 & 0 & 0.2 \\ -0.1 & 0.4 & 0 \end{bmatrix}, \quad G_{d1} = \begin{bmatrix} -0.1 \\ 0 \\ -0.01 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -0.3 \\ 0 \\ 0.7 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & -0.2 & 0.4 \\ 0 & 0.2 & -0.3 \\ 0.5 & 0.1 & 0 \end{bmatrix}$$

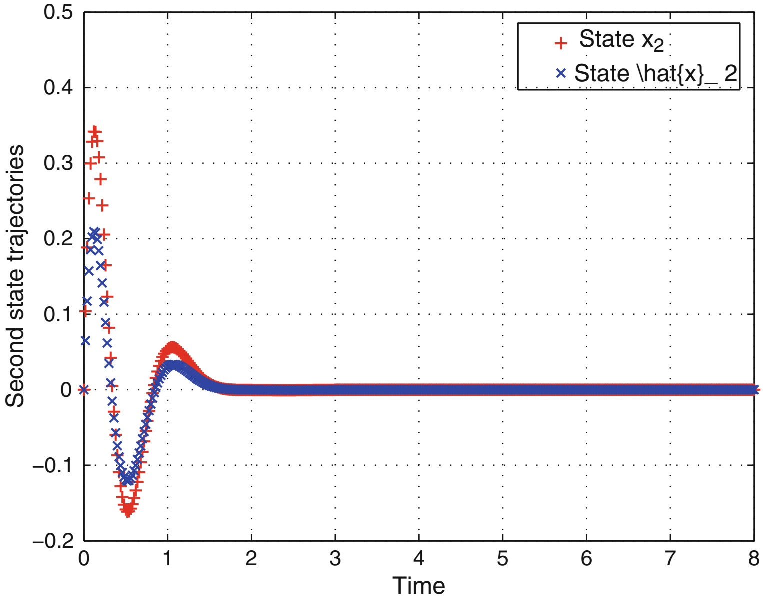

Fig. 8.9 Plot of x_2 and $\hat{x}_2$ versus time: $\mathcal{L}_2 - \mathcal{L}_\infty$ filter

$$
\begin{aligned}
C_1 &= \begin{bmatrix} 0.8 \\ 0.2 \\ 0.2 \end{bmatrix}, \; C_{d1} = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \; \Gamma_1 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0 \end{bmatrix} \\
\Phi_1 &= 0.1
\end{aligned}
$$

$$
\begin{aligned}
Mode\ 2 = y_k \ &\leq \ 0 \\
A_2 &= \begin{bmatrix} 0.3 & 0.2 & 0 \\ 0.3 & 0 & 0.5 \\ 0 & 0.4 & -0.1 \end{bmatrix}, \; G_{d2} = \begin{bmatrix} -0.1 \\ 0.1 \\ 0 \end{bmatrix} \\
G_2 &= \begin{bmatrix} 0.8 \\ -0.2 \\ 0.3 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} 0.1 & 0.2 & -0.4 \\ 0 & 0.2 & -0.5 \\ 0 & -0.1 & 0.3 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0.7 \\ 0.1 \\ 0.4 \end{bmatrix}, \; C_{d2} = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.1 \end{bmatrix}, \; \Gamma_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0.4 \end{bmatrix} \\
\Phi_2 &= 0.3
\end{aligned}
$$

and the corresponding two sets $\{j = 1 \ if \ \ y_k < 0\}$ and $\{j = 2 \ if \ \ y_k \geq 0\}$, respectively.

The piecewise filter matrices such that the above piecewise system is asymptotically stable are given by

$$A_{f1} = \begin{bmatrix} -0.6110 & 1.5279 & -1.4688 \\ -1.6789 & 3.6755 & -3.1187 \\ -1.6333 & 4.1765 & -3.8337 \end{bmatrix}$$
$$B_{f1} = \begin{bmatrix} -0.3753 \\ -0.4955 \\ -0.4675 \end{bmatrix}$$
$$G_{f1} = \begin{bmatrix} -2.8874 & -0.8225 & 2.8795 \end{bmatrix}$$
$$A_{f2} = \begin{bmatrix} -1.6022 & 2.7952 & -2.4268 \\ -3.9458 & 5.3355 & -5.1167 \\ -3.6443 & 6.1385 & -34.9837 \end{bmatrix}$$
$$B_{f2} = \begin{bmatrix} -0.8883 \\ -3.0495 \\ -3.0465 \end{bmatrix}$$
$$G_{f2} = \begin{bmatrix} 1.2098 & 0.0224 & -0.2615 \end{bmatrix}$$

In Tables 8.3 and 8.4, computational summaries of applying **Theorem 8.22–Corollary 8.23** for $\mathcal{H}_\infty$-filter and **Theorem 8.26–Corollary 8.27** for $\ell_2 - \ell_\infty$-filter are depicted.

Once again, the foregoing results come in support with the effectiveness of our filtering approach.

Table 8.3 A summary of $\mathcal{H}_\infty$-performance bound: illustrative example C

$\underline{d}$	$\bar{d}$	$The.8.22$	$Coro.8.23$
2	6	0.889	1.035
3	8	0.924	1.044
4	10	0.965	1.067
5	12	0.977	1.095
6	14	0.989	1.105

Table 8.4 A summary of $\mathcal{L}_2 - \mathcal{L}_\infty$-performance bound: illustrative example C

$\underline{d}$	$\bar{d}$	$The.8.26$	$Coro.8.27$
2	6	0.975	1.045
3	8	0.986	1.076
4	10	1.015	1.088
5	12	1.117	1.096
6	14	1.229	1.107

8.3 Notes and References

In this chapter, novel delay-dependent filtering design approaches have been developed for a class of linear piecewise discrete-time systems with convex-bounded parametric uncertainties and time-varying delays appearing in the state as well as the output and measurement channels. The filters have linear full-order structure and guarantee the desired estimation accuracy over the entire uncertainty polytope. We have used switched Lyapunov functionals and introduced some additional instrumental matrix variables to pave the way toward deriving sufficient conditions for the asymptotic stability of the filtering error system.

The desired accuracy has been assessed in terms of either $\mathcal{H}_\infty$-performance or $\ell_2 - \ell_\infty$ criteria. A new parametrization procedure based on a combined Finsler's Lemma and piecewise Lyapunov–Krasovskii functional has been established to yield sufficient conditions for delay-dependent filter feasibility. The filter gains have been subsequently determined by solving a convex optimization problem over LMIs. In comparison to the existing design methods, the developed methodology has been shown to yield the least conservative measures since all previous overdesign limitations are almost eliminated. By means of simulation examples, the advantages of the developed technique have been readily demonstrated.

Chapter 9
Switched Kalman Filtering

In this chapter, the problem of Kalman filtering for a class of switched systems with state delays is investigated. Both discrete-time and continuous-time representations are treated. In both cases, attention is focused on the design of a stable filter guaranteeing a prescribed noise attenuation level in the $\mathcal{H}_\infty$ sense. By using an appropriate switched estimation scheme, sufficient conditions for the solvability of this problem are obtained in terms of algebraic Riccati equations (AREs), which, when solved, a desired $\mathcal{H}_\infty$ filter can be constructed.

9.1 Discrete Switched Delay System

The problem of optimal filtering has been well studied for more than three decades in various branches of science and engineering. Much focus has been directed to dynamical systems subject to stationary Gaussian input and measurement noise processes [3]. The celebrated Kalman filtering provides a solution to this problem. When the available plant model contains uncertain parameters, the robust state estimation problem comes into the scene for which several techniques have been proposed; see [326, 328–356, 360–372, 374–393, 399, 400] and the references cited therein. On another front of research, uncertain systems with state delay have received increasing interests in recent years [207, 208]. Most of the research efforts have been concentrated on robust stability and stabilization; see [54, 216] and the references cited therein. The problem of estimating the state of uncertain system with state delay has been overlooked despite its importance for control and signal processing.

We consider in this section the state estimation problem for linear switched discrete-time systems with norm-bounded parameter uncertainties and constant state delay. This delay factor arises naturally in different engineering fields [216]. It could result from constant processing delays as in digital systems, inherent gestation lags as in production systems, or finite transit time as in industrial mills. Indeed, this delay is among the main sources of instability in control systems. A related problem is the design of deterministic observers with unknown inputs [60] using algebraic methods. Here, we address the state estimator design problem such that the

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estimation error covariance has a guaranteed bound for all admissible uncertainties and state delay. The approach hinges upon the application of the multi-estimation structure depicted in Fig. 9.1 The main tool for solving the foregoing problem is the Riccati equation approach. It is shown that the stabilizing solution of robust Kalman filtering is given in terms of two algebraic Riccati equations. The existence of the solutions hinges on the quadratic stability of the uncertain system. In principle, all the developed results can be cast into the framework of linear matrix inequalities to yield a satisfactory solution (not necessarily stabilizing) [216].

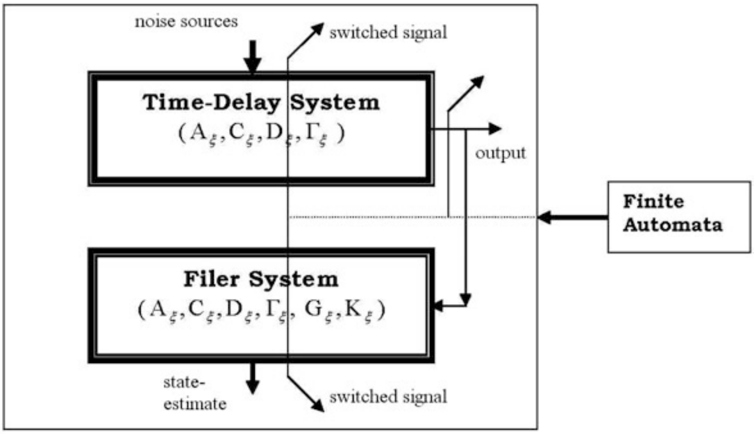

Fig. 9.1 Multi-estimator structure: discrete

9.1.1 Problem Formulation

We consider a class of switched time-delay systems represented by the discrete model:

$$
\begin{aligned}
x_{k+1} &= [A_{\xi(k)} + \Delta A_{\xi(k)}]\, x_k + D_{\xi(k)}\, x_{k-\tau\xi(k)} + \Gamma_{\xi(k)}\, w_k \\
&= A_{\xi(k),\Delta}\, x_k + D_{\xi(k)}\, x_{k-\tau\xi(k)} + \Gamma_{\xi(k)}\, w_k
\end{aligned} \tag{9.1}
$$

$$
\begin{aligned}
y_k &= [C_{\xi(k)} + \Delta C_{\xi(k)}]\, x_k + v_k \\
&= C_{\xi(k),\Delta}\, x_k + v_k
\end{aligned} \tag{9.2}
$$

$$
z_k = C_{1,\xi(k)}\, x_k \tag{9.3}
$$

$$
\xi(k) \in \mathbf{S} = \{1, \ldots, s\} \tag{9.4}
$$

where in (9.1), (9.2), (9.3), and (9.4), $x_k \in \Re^n$ is the state, $y_k \in \Re^p$ is the measured output, $z_k \in \Re^q$ is a linear combination of the state variables to be estimated, and $w_k \in \Re^r$ and $v_k \in \Re^p$ are, respectively, the process and measurement noise sequences. The matrices $A_{\xi(k)} \in \Re^{n\times n}$, $D_{\xi(k)} \in \Re^{n\times n}$, $\Gamma_{\xi(k)} \in \Re^{n\times r}$, and

$C_{\xi(k)} \in \Re^{p\times n}$ are the nominal plant matrices, which are allowed to depend on the system mode $\xi(k)$ with $A_{\xi(k)}$ being invertible for all modes. Here, $\tau_{\xi(k)}$ is a known constant scalar depending on the system mode and representing the amount of delay in the state. As in the previous chapters, the switching signal $\xi(k)$ defines the system mode at the discrete instant k, which determines the current system dynamics and the associated measurements and input vectors. The signal $\xi(k)$ can be generated by finite-discrete automata. The matrices $\Delta A_{\xi(k)}$ and $\Delta C_{\xi(k)}$ represent time-varying parametric uncertainties given by

$$\begin{bmatrix} \Delta A_{\xi(k)} \\ \Delta C_{\xi(k)} \end{bmatrix} = \begin{bmatrix} H_{1,\xi(k)} \\ H_{2,\xi(k)} \end{bmatrix} \Delta_{\xi(k)}\ E_{\xi(k)} \tag{9.5}$$

where $H_{1,\xi(k)} \in \Re^{n\times\alpha}$, $H_{2,\xi(k)} \in \Re^{m\times\alpha}$, and $E_{\xi(k)} \in \Re^{\beta\times n}$ are known matrices at every mode $\xi(k)$ and $\Delta_{\xi(k)} \in \Re^{\alpha\times\beta}$ is an unknown matrix satisfying

$$\Delta^t_{\xi(k)}\ \Delta_{\xi(k)} \quad \leq \quad I \quad k = 0, 1, 2... \tag{9.6}$$

The initial condition is specified as $\langle x_o, \phi(s)\rangle$, where $\phi(.) \in \ell_2[-\tau, 0]$. The vector x_o is assumed to be a zero-mean Gaussian random vector. The following standard assumptions on x_o and the noise sequences $\{w_{\xi(k)}\}$ and $\{v_{\xi(k)}\}$, are assumed:

$$(\mathbf{a})\ \ \mathbf{E}[w_{\xi(k)}] = 0,\ \ \mathbf{E}\left[w_{\xi(k)} w^t_{\xi(j)}\right] = W_{\xi(k)}\ \delta(k-j),\ W_k > 0\ \forall k, j \tag{9.7}$$

$$(\mathbf{b})\ \ \mathbf{E}[v_{\xi(k)}] = 0,\ \ \mathbf{E}\left[v_{\xi(k)} v^t_{\xi(j)}\right] = V_{\xi(k)}\ \delta(k-j),\ V_k > 0\ \forall k, j \tag{9.8}$$

$$(\mathbf{c})\ \ \mathbf{E}\left[w_{\xi(k)} v^t_{\xi(j)}\right] = 0,\ \mathbf{E}\left[x_o w^t_{\xi(k)}\right] = 0,\ \mathbf{E}\left[x_o v^t_{\xi(k)}\right] = 0\ \forall k, j \tag{9.9}$$

$$(\mathbf{d})\ \ \mathbf{E}\left[x_o x^t_o\right] = R_o \tag{9.10}$$

where $\mathbf{E}[.]$ stands for the mathematical expectation and

$$\delta(s) = \begin{cases} 1 & s = 0 \\ 0 & \text{otherwise} \end{cases}$$

is the Dirac function.

It is interesting to observe that system (9.1), (9.2), (9.3), and (9.4) can be expressed as the composition of two subsystems: the first is a deterministic time-delay subsystem described by

$$\bar{x}_{k+1} = A_{\xi(k)}\ \bar{x}_k + B_{\xi(k)}\ u_{\xi(k),k} + D_{\xi(k)}\ \bar{x}_{k-\tau\xi(k)} \tag{9.11}$$

$$\bar{y}_k = C_{\xi(k)}\ \bar{x}_k,\quad \bar{x}(0) = 0 \tag{9.12}$$

and the other is a stochastic autonomous subsystem given by

$$\tilde{x}_{k+1} = A_{\xi(k)}\ \tilde{x}_k + D_{\xi(k)}\ \tilde{x}_{k-\tau\xi(k)} + \Gamma_{\xi(k)}\ w_k \tag{9.13}$$

$$\tilde{y}_k = C_{\xi(k)} \, \tilde{x}_k + v_{\xi(k),k}, \quad \bar{x}(0) = x_o \tag{9.14}$$

such that

$$x_{k+1} = \bar{x}_{k+1} + \tilde{x}_{k+1} \tag{9.15}$$

$$y_k = \bar{y}_k + \tilde{y}_k \tag{9.16}$$

Such an approach is very useful when $D_{\xi(k)} \equiv 0$ corresponding to the delay-free case [21]. We follow hereafter the approach developed in [287].

9.1.2 Switched State Estimation

Given the measurement sequences $y_{\xi(0)}, ..., y_{\xi(k)}$, the switched state estimation of interest is to determine at each discrete step k an unbiased linear estimation $\hat{x}_k$ of the unknown system state x_k with the following properties:

- The error model of the state estimation should be stable in the sense of Lyapunov.
- The variances of all components of the estimation error $x_k - \hat{x}_k$ should not exceed any finite bound.
- The estimation method must be applicable for all switching sequences $\xi(0), ..., \xi(k)$.

The switching state estimation under consideration could be solved by the block diagram depicted in Fig. 9.1. Observe that the current state estimate $\hat{x}_{k+1}$ is generated by

$$\hat{x}_{k+1} = G_{\xi(k)} \, \hat{x}_k + K_{\xi(k)} y_{\xi(k)} \tag{9.17}$$

where $G_{\xi(k)} \in \Re^{n\times n}$ and $K_{\xi(k)} \in \Re^{n\times p}$ are appropriately selected gain matrices such that there exists a matrix $\Psi \geq 0$ satisfying

$$\begin{aligned} \mathbf{E}[\{x_k - \hat{x}_k\}\{x_k - \hat{x}_k\}^t] &= \mathbf{E}\left[e_k e_k^t\right] \\ &\leq \Psi \end{aligned} \tag{9.18}$$

Note that (9.18) implies

$$\begin{aligned} \mathbf{E}[\{x_k - \hat{x}_k\}^t\{x_k - \hat{x}_k\}] &= \mathbf{E}\left[e_k^t e_k\right] \\ &\leq tr(\Psi) \end{aligned} \tag{9.19}$$

In this case, the estimator (9.17) is said to provide a guaranteed cost (GC) matrix Ψ.

It should be noted that if the signal $\xi(k)$ changes, the last state estimate $\hat{x}_k$ generated by the estimator for the system mode $\xi(k-1)$ is now utilized as initial state estimate for the estimator designed for the new system mode $\xi(k)$.

9.1.3 Robust Linear Filtering

We proceed to analyze the switched state estimator (9.17) by defining

$$G_{\xi(k)} \;=(A_{\xi(k)}+\delta A_{\xi(k)}) \;-\; K_{\xi(k)}C_{\xi(k)} \tag{9.20}$$

where $\delta A_{\xi(k)}$ and $K_{\xi(k)}$ are now the unknown matrices to be determined later on. Using (9.1), (9.2) and (9.20) to express the dynamics of the state estimator in the form

$$\begin{aligned}\hat{x}_{k+1} &= [(A_{\xi(k)}+\delta A_{\xi(k)})-K_{\xi(k)}C_k]\hat{x}_k \\ &\quad + K_{\xi(k)}[C_{\xi(k)}x_k+v_k]\end{aligned} \tag{9.21}$$

Introducing the augmented state vector

$$\zeta_k = \begin{bmatrix} x_k \\ x_k-\hat{x}_k \end{bmatrix} = \begin{bmatrix} x_k \\ e_k \end{bmatrix} \in \Re^{2n} \tag{9.22}$$

Then, it follows from (9.1) and (9.21) that

$$\begin{aligned}\zeta_{k+1} &= [\mathsf{A}_{\xi(k)}+\mathsf{H}_{\xi(k)}\Delta_{\xi(k)}\mathsf{E}_{\xi(k)}]\ \zeta_k+\mathsf{D}_{\xi(k)}\ \xi_{k-\tau_{\xi(k)}}+\mathsf{B}_{\xi(k)}\ \eta_k \\ &= \mathsf{A}_{\xi(k),\Delta}\ \zeta_k+\mathsf{D}_{\xi(k)}\ \xi_{k-\tau_{\xi(k)}}+\mathsf{B}_{\xi(k)}\ \eta_k\end{aligned} \tag{9.23}$$

where η_k is a stationary zero-mean noise signal with identity covariance matrix and

$$\begin{aligned}\mathsf{A}_{\xi(k)} &= \begin{bmatrix} A_{\xi(k)} & 0 \\ -\delta A_{\xi(k)} & (A_{\xi(k)}+\delta A_{\xi(k)})-K_{\xi(k)}C_{\xi(k)} \end{bmatrix} \\ \mathsf{D}_{\xi(k)} &= \begin{bmatrix} D_{\xi(k)} & 0 \\ D_{\xi(k)} & 0 \end{bmatrix}, \quad \eta_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix}\end{aligned} \tag{9.24}$$

$$\mathsf{B}_k\mathsf{B}_k^t = \begin{bmatrix} \Gamma_{\xi(k)}W_{\xi(k)}\Gamma_{\xi(k)}^t & 0 \\ 0 & W_{\xi(k)}+K_{\xi(k)}V_{\xi(k)}K_{\xi(k)}^t \end{bmatrix} \tag{9.25}$$

$$\mathsf{H}_{\xi(k)} = \begin{bmatrix} H_{1,\xi(k)} \\ H_{1,\xi(k)}-K_{\xi(k)}H_{2,\xi(k)} \end{bmatrix}, \quad \mathsf{E}_{\xi(k)} = [E_{\xi(k)} \quad 0] \tag{9.26}$$

Definition 9.1 Estimator (9.17) is said to be a **switched quadratic estimator (SQE)** at mode $\xi(k)$ associated with a sequence of matrices $\{\Omega_{\xi(k)}\} > 0$ for system (9.1) and (9.2) if there exist a sequence of scalars $\{\lambda_{\xi(k)}\} > 0$ and a sequence of matrices $\{\Omega_{\xi(k)}\}$ such that

$$0 \;<\; \Omega_{\xi(k)} \;=\; \begin{bmatrix} \Omega_{1,\xi(k)} & \Omega_{3,\xi(k)} \\ \Omega_{3,\xi(k)}^t & \Omega_{2,\xi(k)} \end{bmatrix} \tag{9.27}$$

satisfying the algebraic matrix inequality

$$(1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k),\Delta}\ \Omega_{\xi(k)}\ \mathsf{A}^t_{\xi(k),\Delta} - \Omega_{\xi(k+1)} + \\ \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\ \Omega_{k-\tau_{\xi(k)}}\ \mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)} \ \leq\ 0, \qquad k\ \geq\ 0 \quad (9.28)$$

for all admissible uncertainties satisfying (9.5) and (9.6)

The following result shows that if (9.17) is a **SQE** for system (9.1) and (9.2) with cost matrix $\Omega_{\xi(k)}$, then $\Omega_{\xi(k)}$ defines an upper bound for the filtering error covariance, that is,

$$\mathbf{E}\left[e_k\ e^t_k\right]\ \leq\ \Omega_{2,\xi(k)},\quad \forall k\ \geq\ 0 \quad (9.29)$$

Theorem 9.2 *Consider the time-delay system (9.1) and (9.2) satisfying (9.5) and (9.6) and with known initial state. Suppose there exists a solution* $\Omega_{\xi(k)} = \Omega^t_{\xi(k)} \geq 0$ *to inequality (21) for some* $\lambda_k > 0$ *and for all admissible uncertainties. Then the estimator (9.17) provides an upper bound for the filtering error covariance, that is,*

$$\boldsymbol{E}\ \left[e_k\ e^t_k\right]\ \leq\ [0\quad I]\Omega_{\xi(k)}[0\quad I]^t \qquad \forall k\ \geq\ 0 \quad (9.30)$$

Proof Suppose that estimator (9.17) is a **QE** with cost matrix $\Omega_{\xi(k)}$. By evaluating the one-step ahead covariance matrix $\Sigma_{\zeta,\xi(k+1)} = \mathbf{E}\left[\zeta_{k+1}\ \zeta^t_{k+1}\right]$, we get

$$\begin{aligned}\Sigma_{\zeta,\xi(k+1)} &= \mathbf{E}[\mathsf{A}_{\xi(k),\Delta}\ \xi_k + \mathsf{D}_{\xi(k)}\zeta_{k-\tau} + \mathsf{B}_{\xi(k)}\eta_k]\times \\ &\quad [\mathsf{A}_{\xi(k),\Delta}\zeta_k + \mathsf{D}_k\zeta_{k-\tau_{\xi(k)}} + \mathsf{B}_{\xi(k)}\eta_k]^t \\ &= \mathbf{E}\left[\mathsf{A}_{\xi(k),\Delta}\zeta_k\zeta^t_k\mathsf{A}_{\xi(k),\Delta}\right] + \mathbf{E}\left[\mathsf{A}_{\xi(k),\Delta}\zeta_k\zeta^t_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)}\right] \\ &\quad +\mathbf{E}\left[\mathsf{D}_{\xi(k)}\zeta_{k-\tau_{\xi(k)}}\zeta^t_k\mathsf{A}_{\xi(k),\Delta}\right] \\ &\quad +\mathbf{E}\left[\mathsf{D}_{\xi(k)}\zeta_{k-\tau_{\xi(k)}}\zeta^t_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)}\right] + \mathbf{E}\left[\mathsf{B}_{\xi(k)}\eta_k\eta^t_k\mathsf{B}^t_{\xi(k)}\right] \quad (9.31)\end{aligned}$$

Note that

$$\begin{aligned}&\mathsf{D}_{\xi(k)}\mathbf{E}\left[\zeta_{k-\tau_{\xi(k)}}\zeta^t_{\xi(k)}\right]\mathsf{A}^t_{\xi(k),\Delta} + \mathsf{A}_{\xi(k),\Delta}\mathbf{E}\left[\zeta_k\zeta^t_{k-\tau_{\xi(k)}}\right]\ \mathsf{D}^t_{\xi(k)} \\ &\quad\leq\ \lambda_{\xi(k)}\ \mathsf{A}_{\xi(k),\Delta}\mathbf{E}\left[\zeta_k\ \zeta^t_k\right]\mathsf{A}^t_{\xi(k),\Delta} \\ &\qquad +\lambda^{-1}_{\xi(k)}\mathsf{D}_{\xi(k)}\mathbf{E}\left[\zeta_{k-\tau_{\xi(k)}}\ \zeta^t_{k-\tau_{\xi(k)}}\right]\mathsf{D}^t_{\xi(k)} \quad (9.32)\end{aligned}$$

Using inequality (9.32) into (9.31) and arranging terms, we get

$$\begin{aligned}\Sigma_{\zeta,\xi(k+1)}\ &\leq\ (1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k),\Delta}\Sigma_{\zeta,\xi(k)}\mathsf{A}^t_{\xi(k),\Delta} \\ &\quad +\ \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_k\ \Sigma_{\zeta,k-\tau_{\xi(k)}}\ \mathsf{D}^t_k + \mathsf{B}_k\mathsf{B}^t_{\xi(k)} \quad (9.33)\end{aligned}$$

Letting $\Xi_{\xi(k)} = \Sigma_{\zeta,\xi(k)} - \Omega_{\xi(k)}$ with $e_k = x_k - \hat{x}_k$ and considering inequalities (21) and (25), we get

$$\Xi_{\xi(k+1)} \leq \\ (1+\lambda_{\xi(k)})\mathsf{A}_{\xi(k)}\Xi_{\xi(k)}\mathsf{A}^t_{\xi(k)} + \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_k\Xi_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)} \tag{9.34}$$

By considering that the state is known over the period $[-\tau_{\xi(k)}, 0]$, it justifies letting $\Sigma_{\zeta,\xi(k)} = 0 \ \forall k \in [-\tau_{\xi(k)}, 0]$. Then it follows from (9.34) that $\Xi_{\xi(k)} \leq 0$ for $k > 0$; that is, $\Sigma_{\zeta,\xi(k)} \leq \Omega_{\xi(k)}$ for $k > 0$. Hence, $\mathbf{E}\left[e_k e_k^t\right] \leq [0 \ \ I]\Omega_{\xi(k)}[0 \ \ I]^t \ \forall k \geq 0$. ■

9.1.4 A Design Approach

Motivated by the recent results of robust filtering theory [15, 399, 400], we employ hereafter a Riccati equation approach to solve the robust Kalman filtering for switched time-delay systems. To this end, we define matrices $0 < \mathsf{P}_{\xi(k)} = \mathsf{P}^t_{\xi(k)} \in \Re^{n\times n}$; $0 < \mathsf{S}_{\xi(k)} = \mathsf{S}^t_{\xi(k)} \in \Re^{n\times n}$ as the solutions of the Riccati difference equations (RDEs):

$$\begin{aligned}
\mathsf{P}_{\xi(k+1)} = \ & (1+\lambda_{\xi(k)})\{A_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}A^t_{\xi(k)}\} \\
& + \left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)} \\
\mathsf{P}_{k-\tau_{\xi(k)}} = \ & 0 \quad \forall \ k \ \in [0, \tau_{\xi(k)}] \qquad\qquad (9.35) \\
\mathsf{S}_{\xi(k+1)} = \ & (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} \\
& + (1+\lambda_{\xi(k)})\delta A_{\xi(k)}\left(I+\mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
& + (1+\lambda_{\xi(k)})\mu_{\xi(k)}\hat{A}_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} \\
& + (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
& - \hat{M}^t_{\xi(k)}\left(\hat{\Gamma}_{\xi(k)}+\hat{V}_{\xi(k)}\right)^{-1}\hat{M}_{\xi(k)} \\
& + \left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}P_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)} \\
S_{k-\tau_{\xi(k)}} = \ & 0 \quad \forall \ k \ \in [0, \tau] \qquad\qquad (9.36)
\end{aligned}$$

where $\mu_{\xi(k)} > 0, \lambda_{\xi(k)} > 0$ are scaling parameters such that $P^{-1}_{\xi(k)} - \mu^{-1}_{\xi(k)}E_{\xi(k)}E^t_{\xi(k)} > 0$ and the matrices $\hat{A}_{\xi(k)}, \delta A_{\xi(k)}, \hat{V}_{\xi(k)}, \hat{W}_{\xi(k)}, \hat{\Gamma}_{\xi(k)}$, and $\hat{M}_{\xi(k)}$ are given by

$$Y_{\xi(k)} = E^t_{\xi(k)}\left(I - \mu_{\xi(k)}E_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\right)^{-1}E_{\xi(k)} \tag{9.37}$$

$$\hat{W}_{\xi(k)} = W_{\xi(k)} + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)}H_{1,\xi(k)}H^t_{1,\xi(k)} \tag{9.38}$$

$$\hat{V}_{\xi(k)} = V_{\xi(k)} + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)} H_{2,\xi(k)} H^t_{2,\xi(k)} \tag{9.39}$$

$$\mathcal{T}_{\xi(k)} = (1+\lambda_{\xi(k)})(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)}) \left(I + \mu_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)}\right) A^t_{\xi(k)}$$

$$\begin{aligned} \mathcal{Z}_{\xi(k)} &= (1+\lambda_{\xi(k)}) \hat{M}^t_{\xi(k)} \left(\hat{\Gamma}_{\xi(k)} + \hat{V}_{\xi(k)}\right)^{-1} \\ &\quad \times \left(C_k \mathsf{S}_k (I + \mu_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)}) A^t_{\xi(k)} + \mu^{-1}_{\xi(k)} H_{2,\xi(k)} H^t_{1,\xi(k)}\right) \end{aligned} \tag{9.40}$$

$$\begin{aligned} \mathcal{X}_{\xi(k)} &= (1+\lambda_{\xi(k)})\mu_{\xi(k)} A_{\xi(k)} \mathsf{S}_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)} A^t_{\xi(k)} \\ &\quad + (1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)} H_{1,\xi(k)} H^t_{1,\xi(k)} \\ &\quad + \left(1 + \lambda^{-1}_{\xi(k)}\right) D_{\xi(k)} \mathsf{P}_{k-\tau_{\xi(k)}} D^t_{\xi(k)} \end{aligned} \tag{9.41}$$

$$\hat{\Gamma}_{\xi(k)} = (1+\lambda_{\xi(k)}) C_{\xi(k)} \mathsf{S}_{\xi(k)} C^t_{\xi(k)} \tag{9.42}$$

$$\hat{A}_{\xi(k)} = A_{\xi(k)} + \delta A_{\xi(k)}, \quad \delta A_{\xi(k)} = \mathcal{T}^{-1}_{\xi(k)} \left(\mathcal{X}_{\xi(k)} + \mathcal{Z}_{\xi(k)}\right) \tag{9.43}$$

$$\begin{aligned} \hat{M}_{\xi(k)} &= (1+\lambda_{\xi(k)}) \Big[C_{\xi(k)} \mathsf{S}_{\xi(k)} A^t_{\xi(k)} + \mu_{\xi(k)} \mathsf{S}_{\xi(k)} Y_{\xi(k)} \mathsf{P}_{\xi(k)} \delta A^t_{\xi(k)} \\ &\quad + \mu_{\xi(k)} H_{2,\xi(k)} H^t_{1,\xi(k)} \Big] \end{aligned} \tag{9.44}$$

Note that the assumption that $A_{\xi(k)}$, being invertible for all k, is needed for the existence of $\mathcal{T}_{\xi(k)}$ and $\delta A_{\xi(k)}$. Let the $(\lambda, \mu)-$parametrized switched estimator be expressed as

$$\begin{aligned} \hat{x}_{k+1} &= \left(A_{\xi(k)} + \mathcal{T}^{-1}_{\xi(k)} (\mathcal{X}_{\xi(k)} + \mathcal{Z}_{\xi(k)})\right) \hat{x}_k \\ &\quad + K_{\xi(k)} [y_k - C_{\xi(k)} \hat{x}_k] \end{aligned} \tag{9.45}$$

where the Kalman gain matrix $K_{\xi(k)} \in \Re^{n \times m}$ is to be determined. The following theorem summarizes the main result:

Theorem 9.3 *Consider system (9.1) and (9.2) satisfying the uncertainty structure (9.5) and (9.6) with zero initial condition and* $A_{\xi(k)}$ *being invertible* $\forall \xi(k) \in \mathbf{S}$. *Suppose the process and measurement noises satisfy (9.7), (9.8), (9.9), and (9.10). For some* $\mu_{\xi(k)} > 0$, $\lambda_{\xi(k)} > 0$, $\xi(k) \in \mathbf{S}$ *let* $0 < \mathsf{P}_{\xi(k)} = P^t_{\xi(k)}$ *and* $0 < \mathsf{S}_{\xi(k)} = \mathsf{S}^t_{\xi(k)}$ *be the solutions of RDEs (9.35) and (9.36), respectively. Then the* $(\lambda, \mu)-$*parametrized estimator (9.45) is an* **SQE estimator with GC**

$$\boldsymbol{E}[\{\hat{x}_k - x_k\}^t \{\hat{x}_k - x_k\}] \;\; \leq \;\; tr\ (\mathsf{S}_{\xi(k)}) \tag{9.46}$$

Moreover, the gain matrix K is given by

$$K_{\xi(k)} = \hat{M}^t_{\xi(k)} \left(\hat{\Gamma}_{\xi(k)} + \hat{V}_{\xi(k)}\right)^{-1} \tag{9.47}$$

Proof Let

$$\mathsf{X}_{\xi(k)} = \begin{bmatrix} \mathsf{P}_{\xi(k)} & \mathsf{S}_{\xi(k)} \\ \mathsf{S}_{\xi(k)} & \mathsf{S}_k \end{bmatrix} \tag{9.48}$$

where $\mathsf{P}_{\xi(k)}$ and $\mathsf{S}_{\xi(k)}$ are the positive-definite solutions to (9.35) and (9.36) at mode $\xi(k)$, respectively. By using the following standard inequalities (see the Appendix)

- For any real matrices Σ_1 , Σ_2, and Σ_3 with appropriate dimensions and $\Sigma_3^t\ \Sigma_3 \ \le\ I$, it follows that

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^t \Sigma_3^t \Sigma_1^t \ \le\ \alpha^{-1}\ \Sigma_1 \Sigma_1^t + \alpha\ \Sigma_2^t \Sigma_2 \qquad \forall \alpha > 0 \tag{9.49}$$

- Let $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ and $0 < R = R^t$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \le I$. Then for any $\rho > 0$ satisfying $\rho \Sigma_2^t \Sigma_2 < R$, the following matrix inequality holds:

$$\begin{aligned} &(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}\left(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t\right) \ \le \\ &\rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3\left(R - \rho\Sigma_2^t\Sigma_2\right)^{-1}\ \Sigma_3^t \end{aligned} \tag{9.50}$$

combining (9.35), (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44), it is a simple task to show that

$$\begin{aligned} &(1+\lambda_{\xi(k)})\Big[\mathsf{A}_{\xi(k)}\mathsf{X}_{\xi(k)}\mathsf{A}_{\xi(k)}^t + \mu_{\xi(k)}\mathsf{A}_{\xi(k)}\mathsf{X}_{\xi(k)}\mathsf{E}_k^t \\ &[I - \mu_{\xi(k)}\mathsf{E}_{\xi(k)}\mathsf{X}_{\xi(k)}\mathsf{E}_{\xi(k)}^t]^{-1}\mathsf{E}_{\xi(k)}\mathsf{X}_{\xi(k)}\mathsf{A}_{\xi(k)}^t\Big] \\ &\quad -\mathsf{X}_{k+1} + (1+\lambda_{\xi(k)})\mu_{\xi(k)}^{-1}\mathsf{H}_{\xi(k)}\mathsf{H}_{\xi(k)}^t + \mathsf{B}_{\xi(k)}\mathsf{B}_{\xi(k)}^t \\ &\quad + \left(1+\lambda_{\xi(k)}^{-1}\right)\mathsf{D}_k\ \mathsf{X}_{k-\tau_{\xi(k)}}\ \mathsf{D}_{\xi(k)}^t \\ &= \begin{bmatrix} \Pi_{1,\xi(k)} & \Pi_{3,\xi(k)} \\ \bullet & \Pi_{2,\xi(k)} \end{bmatrix} = 0 \end{aligned} \tag{9.51}$$

where $\Pi_{1,\xi(k)} \in \Re^{n\times n}$, $\Pi_{2,\xi(k)} \in \Re^{n\times n}$, $\Pi_{3,\xi(k)} \in \Re^{n\times n}$ and $\mathsf{A}_k, \mathsf{B}_k, \mathsf{H}_k, \mathsf{D}_k$ are given by (9.24), (9.25), and (9.26). One way to verify this is to expand (9.51) using (9.24), (9.25), and (9.26) and (9.48) to yield

$$\begin{aligned} \Pi_{1,\xi(k)} &= (1+\lambda_{\xi(k)})\{A_{\xi(k)}\left(I + \mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}A_{\xi(k)}^t\} \\ &\quad + \left(1+\lambda_{\xi(k)}^{-1}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D_{\xi(k)}^t + \hat{W}_{\xi(k)} - \mathsf{P}_{\xi(k+1)} \end{aligned} \tag{9.52}$$

$$\begin{aligned} \Pi_{2,\xi(k)} &= (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\left(I + \mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{S}_{\xi(k)}\hat{A}_{\xi(k)}^t \\ &\quad +(1+\lambda_{\xi(k)})\delta A_{\xi(k)}\left(I + \mu_{\xi(k)}\mathsf{P}_{\xi(k)}Y_{\xi(k)}\right)\mathsf{P}_{\xi(k)}\delta A_{\xi(k)}^t \end{aligned}$$

$$
\begin{aligned}
&+ (1+\lambda_{\xi(k)})\mu_{\xi(k)}\hat{A}_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\left(I - \mu_{\xi(k)}E_{\xi(k)}P_{\xi(k)}E^t_{\xi(k)}\right)^{-1} \\
&\times E_{\xi(k)}\mathsf{S}_{\xi(k)}\hat{A}^t_{\xi(k)} + (1+\lambda_{\xi(k)})\hat{A}_{\xi(k)}\mu_{\xi(k)}\mathsf{S}_{\xi(k)}Y_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
&-\hat{M}^t_{\xi(k)}\left(\hat{\Gamma}_{\xi(k)} + \hat{V}_{\xi(k)}\right)^{-1}\hat{M}_{\xi(k)} - \mathsf{S}_{\xi(k+1)} \\
&+ \left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} + \hat{W}_{\xi(k)}
\end{aligned} \tag{9.53}
$$

$$
\begin{aligned}
\Pi_{3,\xi(k)} = &-(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
&+(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{S}_{\xi(k)}\left[A^t_{\xi(k)} + \delta A^t_{\xi(k)} - C^t_{\xi(k)}K^t_{\xi(k)}\right] \\
&-\mu_{\xi(k)}(1+\lambda_{\xi(k)})A_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)} \\
&\times\left(I - \mu_{\xi(k)}E_{\xi(k)}\mathsf{P}_{\xi(k)}E^t_{\xi(k)}\right)^{-1}E_{\xi(k)}\mathsf{P}_{\xi(k)}\delta A^t_{\xi(k)} \\
&+\left(1+\lambda^{-1}_{\xi(k)}\right)D_{\xi(k)}\mathsf{P}_{k-\tau_{\xi(k)}}D^t_{\xi(k)} - \mathsf{S}_{\xi(k+1)} \\
&+(1+\lambda_{\xi(k)})\mu^{-1}_{\xi(k)}H_{1,\xi(k)}\left[H^t_{1,\xi(k)} - H_{2,\xi(k)}K^t_{\xi(k)}\right]
\end{aligned} \tag{9.54}
$$

By setting $\Pi_{1,\xi(k)} \equiv 0$ in (9.52) and using (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44) we immediately obtain (9.35). Next, we enforce $\Pi_2 \equiv 0$ in (9.53). By using (9.39), (9.40), (9.41), (9.42), (9.43), and (9.44) with some standard matrix manipulations, we define $K_{\xi(k)}$ as in (9.47) to yield (9.36). Finally, by using (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), and (9.42) in (9.54) and setting $\delta A_{\xi(k)}$ as in (9.43), we find that $\Pi_{3,\xi(k)} \equiv 0$.

Now, using the results of [45], it is easy to see on using inequality (9.50) with some algebraic manipulations that (9.51) implies that

$$
\begin{aligned}
&(1+\lambda_{\xi(k)})[\mathsf{A}_{\xi(k)} + \mathsf{H}_{\xi(k)}\Delta_{\xi(k)}\mathsf{E}_{\xi(k)}]\mathsf{X}_{\xi(k)}[\mathsf{A}_{\xi(k)} + \mathsf{H}_{\xi(k)}\Delta_{\xi(k)}\mathsf{E}_{\xi(k)}]^t \\
&- \mathsf{X}_{\xi(k+1)} + \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\mathsf{X}_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)} \\
&= \left(1+\lambda_{\xi(k)}\right)\mathsf{A}_{\xi(k),\Delta}\mathsf{X}_{\xi(k)}\mathsf{A}^t_{\xi(k),\Delta} - \mathsf{X}_{\xi(k+1)} \\
&+ \left(1+\lambda^{-1}_{\xi(k)}\right)\mathsf{D}_{\xi(k)}\mathsf{X}_{k-\tau_{\xi(k)}}\mathsf{D}^t_{\xi(k)} + \mathsf{B}_{\xi(k)}\mathsf{B}^t_{\xi(k)} \leq 0
\end{aligned} \tag{9.55}
$$

$\forall\ \Delta_{\xi(k)} : \Delta^t_{\xi(k)}\ \Delta_{\xi(k)} \leq I \quad \forall\ \xi(k) \in \mathbf{S}$.

It follows from **Theorem 9.2** that (9.45) is a quadratic estimator and

$$
\mathbf{E}\left[e_k e^t_k\right] = \mathbf{E}[0\ \ I]\mathsf{X}_{\xi(k)}[0\ \ I]^t \leq \mathsf{S}_{\xi(k)}
$$

which implies that $\mathbf{E}\left[e^t_k\ e_k\right] \leq tr\ (\mathsf{S}_{\xi(k)})$. ■

Remark 9.4 From the foregoing analysis, it is seen that our results are independent of the size of the delay. This might be considered to yield a conservative design method. However, as shown in the simulation example, the developed switched estimation method works well for a wide range of the delay factor $\tau_{\xi(k)}$.

Remark 9.5 In the case of systems without uncertainties and delay factors, that is, $H_{1,\xi(k)} = 0,\ H_{2,\xi(k)} = 0,\ E_{\xi(k)} = 0,\ D_{\xi(k)} = 0$, it can be easily shown that

$$\begin{aligned} Y_{\xi(k)} &= 0; \quad \mathcal{X}_{\xi(k)} = 0; \quad \hat{W}_{\xi(k)} = W_{\xi(k)} \\ \mathcal{T}_{\xi(k)} &= (1+\lambda_{\xi(k)})(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)})A^t_{\xi(k)} \\ \mathcal{Z}_{\xi(k)} &= (1+\lambda_{\xi(k)})^2 A_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} \\ &\quad \times \left[(1+\lambda_{\xi(k)})C_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} + V_{\xi(k)}\right]^{-1} C_{\xi(k)}\mathsf{S}_{\xi(k)}A^t_{\xi(k)} \end{aligned}$$

Now, in terms of $\mathsf{L}_k = \mathsf{P}_k - \mathsf{S}_k$ and

$$\begin{aligned} \Psi_{\xi(k)} &= \mathsf{S}_{\xi(k)}C^t_{\xi(k)}\left[(1+\lambda_{\xi(k)})C_{\xi(k)}\mathsf{S}_{\xi(k)}C^t_{\xi(k)} + V_{\xi(k)}\right]^{-1} C_{\xi(k)}\mathsf{S}_{\xi(k)} \\ \Phi_{\xi(k)} &= A_{\xi(k)}\Psi_{\xi(k)}A^t_{\xi(k)} \\ \mathcal{R}_{\xi(k)} &= A^{-t}_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)})^{-1}, \quad \hat{A}_{\xi(k)} = (1+\lambda_{\xi(k)})\mathcal{R}_{\xi(k)}\Phi_{\xi(k)} \end{aligned}$$

we manipulate (9.35) and (9.36) to reach

$$\begin{aligned} \mathsf{L}_{\xi(k+1)} &= (1+\lambda_{\xi(k)})\left[A_{\xi(k)}\mathsf{L}_{\xi(k)}A^t_{\xi(k)} + \Lambda_{\xi(k)}\right], \quad \mathsf{L}_{k-\tau_{\xi(k)}} = 0 \quad \forall k \in [0, \tau_{\xi(k)}] \\ \Lambda_{\xi(k)} &= \Phi_{\xi(k)} \\ &\quad - (1+\lambda_{\xi(k)})\Big[A_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})\Phi^t_{\xi(k)}\mathcal{R}^t_k\xi(k) \\ &\quad + \mathcal{R}_{\xi(k)}\Phi_{\xi(k)}(\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})A^t_{\xi(k)} \\ &\quad + (1+\lambda_{\xi(k)})\mathcal{R}_{\xi(k)}\Phi_{\xi(k)}(2\mathsf{P}_{\xi(k)} - \mathsf{L}_{\xi(k)})\Phi^t_{\xi(k)}\mathcal{R}^t_{\xi(k)}\Big] \end{aligned} \tag{9.56}$$

By iterating on (9.35) and (9.56), it follows that $\mathsf{L}_{\xi(k)} = \mathsf{P}_{\xi(k)} - \mathsf{S}_{\xi(k)} > 0\ \forall \xi(k) \in$ **S**.

It can be shown in the general case that manipulation of (9.35), (9.36), (9.37), (9.38), (9.39), (9.40), (9.41), (9.42), (9.43), (9.44), and (9.45) yields

$$\begin{aligned} \mathsf{L}_{\xi(k+1)} &= (1+\lambda_{\xi(k)})[A_{\xi(k)}(I + \mu_{\xi(k)}\mathsf{L}_{\xi(k)}Y_{\xi(k)})\mathsf{L}_{\xi(k)}A^t_{\xi(k)} + \Upsilon_{\xi(k)}], \\ \mathsf{L}_{k-\tau_{\xi(k)}} &= 0 \quad \forall k \in [0, \tau_{\xi(k)}] \end{aligned}$$

In this case, Υ_k depends on the system matrices

$$A_{\xi(k)}, H_{1,\xi(k)}, H_{2,\xi(k)}, D_{\xi(k)}, C_{\xi(k)}, \mathsf{P}_{\xi(k)}$$

Remark 9.6 Note that $\mathsf{P}_{\xi(k)}$ does not depend on the filter matrices, and the structure of $\mathsf{X}_{\xi(k)}$ is identical to that of the joint covariance matrix of the state of a certain system and its standard $\mathcal{H}_2$-optimal estimator. By similarity to the standard $\mathcal{H}_2$−optimal filter, an estimate of z_k in (9.3) will be given by $\hat{z}_k = C_{1,\xi(k)}\hat{x}_k$.

Remark 9.7 In the delay-free case ($D_{\xi(k)} \equiv 0$), we suppress the parameter $\lambda_{\xi(k)}$ and observe that (9.45) reduces to the recursive Kalman filter for the system

$$\begin{aligned} x_{k+1} &= \hat{A}_{\xi(k)}\, x_k + \hat{w}_k \\ y_k &= C_{\xi(k)}\, x_k + \hat{v}_k \end{aligned} \tag{9.57}$$

where $\hat{w}_k$ and $\hat{v}_k$ are zero-mean white noise sequences with covariance matrices $\hat{W}_{\xi(k)}$ and $\hat{\Gamma}_{\xi(k)}$, respectively, and having cross-covariance matrix $\hat{M}_{\xi(k)}$. Hence, the approach to robust filtering in **Theorem 9.3** generalizes the results of [287, 399, 400] to switched systems and corresponds to designing a standard Kalman filter for a related discrete-time system which captures all admissible uncertainties and time delay, but does not involve parameter uncertainties. In this regard, the matrix $\delta A_{\xi(k)}$ reflects the effect of uncertainties $(\Delta A_{\xi(k)}, \Delta C_{\xi(k)})$ and time-delay factor $D_{\xi(k)}$ on the structure of the filter.

9.1.5 *Steady-State Robust Filter*

In the sequel, we investigate the asymptotic properties of the recursive Kalman filter developed in the foregoing section. For this purpose, we consider that the switched signals are independent of the discrete instants. In this case, the uncertain switched time-delay system is given by

$$\begin{aligned} x_{k+1} &= [A_\xi + H_{1,\xi}\Delta_{k,\xi}E_\xi]\, x_k + D_\xi\, x_{k-\tau_\xi} + w_k \\ &= A_{\xi,\Delta}\, x_k + D_\xi\, x_{k-\tau_\xi} + w_k \end{aligned} \tag{9.58}$$

$$\begin{aligned} y_k &= [C_\xi + H_{2,\xi}\Delta_{k,\xi}E_\xi]\, x_k + v_k \\ &= C_{\xi,\Delta}\, x_k + v_k \end{aligned} \tag{9.59}$$

$$\xi \in \bar{\mathbf{S}} = \{1, \ldots, s\} \tag{9.60}$$

where now the switching signal ξ defines the system mode, which determines the current system dynamics and the associated measurements and input vectors. The signal ξ can be generated by finite-discrete automata but has a constant value independent of the discrete instant k. In addition, $\Delta_{k,\xi}$ satisfies (9.6). In the sequel, we assume that A_ξ is a Schur matrix; that is, $|\lambda(A_\xi)| < 1$. The matrices $A_\xi \in \Re^{n\times n}$, $C_\xi \in \Re^{m\times n}$ are mode-dependent constant matrices representing the nominal plant. The uncertain parameter matrix $\Delta_{k,\xi}$ is, however, time varying. In this regard, the objective is to design a switched shift-invariant a priori estimator of the form

$$\hat{x}_{k+1} = \hat{A}_\xi \hat{x}_k + K_\xi\, [y_k - C_\xi \hat{x}_k] \tag{9.61}$$

that achieves the following asymptotic performance bound

$$\lim_{k\to\infty} \mathbf{E}\left\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^t\right\} \;\leq\; S_\xi \tag{9.62}$$

Theorem 9.8 *Consider the uncertain time-delay system (9.58), (9.59), and (9.60) with A_ξ being invertible at every mode ξ. If for some scalars $\mu_\xi > 0, \lambda_\xi > 0,\ \xi \in \bar{\mathbf{S}}$, there exist stabilizing solutions $P_\xi \geq 0,\ S_\xi \geq 0$ for the AREs*

$$\begin{aligned} P_\xi =&\ (1+\lambda_\xi)\{A_\xi \left(I + \mu_\xi P_\xi Y_\xi\right) P_\xi A_\xi^t\} \\ &+ (1+\lambda_\xi^{-1}) D_\xi P_\xi D_\xi^t + \hat{W}_\xi \end{aligned} \tag{9.63}$$

$$\begin{aligned} S_\xi =&\ (1+\lambda_\xi)\hat{A}_\xi \left(I + \mu_\xi S_\xi Y_\xi\right) S_\xi \hat{A}_\xi^t \\ &+ (1+\lambda_\xi)\delta A_\xi \left(I + \mu_\xi P_\xi Y_\xi\right) P_\xi \delta A_\xi^t \\ &+ (1+\lambda_\xi)\mu_\xi \hat{A}_\xi P_\xi Y_\xi S_\xi \hat{A}_\xi^t + (1+\lambda_\xi)\hat{A}_\xi \mu_\xi S_\xi Y_\xi P_\xi \delta A_\xi^t \\ &- \hat{M}_\xi^t \left(\hat{\Gamma}_\xi + \hat{V}_\xi\right)^{-1} \hat{M}_\xi \end{aligned} \tag{9.64}$$

$$Y_\xi = E_\xi^t \left(I - \mu_\xi E_\xi P_\xi E_\xi^t\right)^{-1} E_\xi$$

$$\hat{W}_\xi = W_\xi + (1+\lambda_\xi)\mu_\xi^{-1} H_{1,\xi} H_{1,\xi}^t \tag{9.65}$$

$$\hat{V}_\xi = V_\xi + (1+\lambda_\xi)\mu_\xi^{-1} H_{2,\xi} H_{2,\xi}^t, \quad \hat{\Gamma}_\xi = (1+\lambda_\xi) C_\xi S_\xi C_\xi^t \tag{9.66}$$

$$\hat{M}_\xi = (1+\lambda_\xi)[C_\xi S_\xi A_\xi^t + \mu_\xi S_\xi Y_\xi P_\xi \delta A_\xi^t + \mu H_{2,\xi} H_{1,\xi}^t] \tag{9.67}$$

Then the estimator (9.61) is a stable switched quadratic (SSQ) estimator and achieves (9.62) with

$$\hat{A}_\xi = A_\xi + \delta A_\xi, \quad \delta A_\xi = \mathcal{T}_\xi^{-1} \left(\mathcal{X}_\xi + \mathcal{Z}_\xi\right) \tag{9.68}$$

$$K_\xi = \hat{M}^t \left\{\hat{\Gamma} + \hat{V}\right\}^{-1}, \quad \mathcal{T}_\xi = (1+\lambda_\xi)(P_\xi - S_\xi)\left(I + \mu_\xi Y_\xi P_\xi\right) A_\xi^t \tag{9.69}$$

$$\begin{aligned} \mathcal{Z}_\xi =&\ (1+\lambda_\xi)\hat{M}_\xi^t \left(\hat{\Gamma}_\xi + \hat{V}_\xi\right)^{-1} \\ &\times \left(C_\xi S_\xi (I + \mu_\xi Y_\xi P_\xi) A_\xi^t + \mu_\xi^{-1} H_{2,\xi} H_{1,\xi}^t\right) \end{aligned} \tag{9.70}$$

$$\begin{aligned} \mathcal{X}_\xi =&\ (1+\lambda_\xi)\mu_\xi A_\xi S_\xi Y_\xi P_\xi A_\xi^t + (1+\lambda_\xi)\mu_\xi^{-1} H_{1,\xi} H_{1,\xi}^t \\ &+ \left(1 + \lambda_\xi^{-1}\right) D_\xi S_\xi D_\xi^t \end{aligned} \tag{9.71}$$

Proof To examine the stability of the closed-loop system, we augment (9.58), (9.59), and (9.60) with ($w_k = 0, v_k = 0$) to obtain

$$\zeta_{k+1} = \mathsf{A}_{\xi,\Delta} \zeta_k + \mathsf{D}_\xi \zeta_{k-\tau_\xi}$$

$$= \begin{bmatrix} A_{\xi,\Delta} & 0 \\ A_{\xi,\Delta} - \hat{A}_\xi - K_\xi(C_{\xi,\Delta} - C_\xi) & \hat{A}_\xi - K_\xi C_\xi \end{bmatrix} \zeta_k + \begin{bmatrix} D_\xi & 0 \\ D_\xi & 0 \end{bmatrix} \zeta_{k-\tau_\xi} \tag{9.72}$$

Introduce a discrete Lyapunov – Krasovskii functional

$$V_{k,\xi} = \zeta_k^t X_\xi \zeta_k + \sum_{j=k-\tau_\xi}^{k-1} \zeta_j^t \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_j \tag{9.73}$$

for some $\lambda_\xi > 0$. By evaluating the first-order difference $\Delta V_{k,\xi} = V_{k+1,\xi} - V_{k,\xi}$ along the trajectories of (9.72) and arranging terms, we get

$$\begin{aligned} \Delta V_{k,\xi} &= \zeta_k^t \left[\mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi\right] \zeta_k + \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{A}_{\xi,\Delta} \zeta_k + \zeta_k^t \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} \\ &\quad + \zeta_{k-\tau_\xi}^t \mathsf{D}^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} + (1+\lambda^{-1}) \zeta_k^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_k \\ &\quad - (1+\lambda^{-1}) \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} \\ &\le \zeta_k^t \left[\mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi\right] \zeta_k \\ &\quad + \lambda^{-1} \zeta_{k-\tau_\xi}^t \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi \zeta_{k-\tau_\xi} + \lambda \zeta_k^t \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} \zeta_k \\ &\quad + \zeta_{k-\tau}^t \mathsf{D}^t X \mathsf{D} \xi_{k-\tau} - \xi_{k-\tau}^t (1+\lambda^{-1}) \mathsf{D}^t X \mathsf{D} \xi_{k-\tau} \\ &= \zeta_k^t \left[(1+\lambda_\xi) \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi\right] \zeta_k \end{aligned} \tag{9.74}$$

Sufficient condition of asymptotic stability $\Delta V_{k,\xi} < 0$, $\zeta_k \neq 0$ is implied by

$$(1+\lambda_\xi) \mathsf{A}_{\xi,\Delta}^t X_\xi \mathsf{A}_{\xi,\Delta} - X_\xi + \left(1 + \lambda_\xi^{-1}\right) \mathsf{D}_\xi^t X_\xi \mathsf{D}_\xi < 0 \tag{9.75}$$

Now select X_ξ as

$$X_\xi = \begin{bmatrix} P_\xi & S_\xi \\ S_\xi & S_\xi \end{bmatrix} \tag{9.76}$$

with P_ξ and S_ξ being the stabilizing solutions of (9.63) and (9.64), respectively. Following a similar procedure as in the proof of **Theorem 9.3** and in view of **Definition 9.1**, it follows in the steady state as $k \to \infty$ that the augmented system (9.72) is asymptotically stable. The guaranteed performance $\mathbf{E}[e_k e_k^t] \le S$ follows from similar lines of argument as in the proof of **Theorem 9.3**. ■

Remark 9.9 Note that the invertibility of A_ξ is needed for the existence of $\mathcal{T}_\xi$ *and* δA_ξ. In the switched delayless case $D_\xi \equiv 0$, it follows from (9.59) and (9.62) with $\hat{W}_\xi = B_\xi \bar{B}_\xi^{\,t}$ that

$$P_\xi = (1+\lambda_\xi)\{A_\xi P_\xi A_\xi^t + A_\xi P_\xi \left[\left(\mu_\xi^{-1} I + E_\xi P_\xi E_\xi^t\right)^{-1} P_\xi A_\xi^t\right\} + \hat{W}_\xi \quad (9.77)$$

which is a bounded real lemma equation for the system

$$\Sigma_\xi = \left(A_\xi \sqrt{1+\lambda_\xi}, \bar{B}_\xi, E_\xi, 0\right)$$

Suppose that for $\mu_\xi = \mu^+$, *the ARE* (9.77) *admits a solution* $P_\xi = P^+$. This implies that the $\mathcal{H}_\infty$-norm of Σ_ξ is less than $(\mu^+)^{-1/2}$. It then follows, given a λ_ξ, that system (9.58) and (9.59) is quadratically stable for some $\mu_\xi \le \mu^+$.

9.1.6 Simulation Example

Consider the following discrete time-delay system with two operational modes

Mode 1

$$\begin{aligned} x_{k+1} &= \left(\begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.004 & 0.4 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \Delta_k [0.5 \ \ 0.4 \ \ 0.2]\right) x_k \\ &\quad + \begin{bmatrix} -0.1 & 0 & 0 \\ 0.05 & -0.2 & 0.1 \\ 0 & 0 & -0.1 \end{bmatrix} x_{k-3} + w_k \\ y_k &= \left(\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \Delta_k [0.5 \ \ 0.4 \ \ 0.2]\right) x_k + v_k \end{aligned}$$

Mode 2

$$\begin{aligned} x_{k+1} &= \left(\begin{bmatrix} 0.3 & 0 & -0.1 \\ 0.002 & 0.5 & 0.2 \\ 0 & -0.1 & 0.7 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix} \Delta_k [0.3 \ \ 0.2 \ \ 0.1]\right) x_k \\ &\quad + \begin{bmatrix} -0.3 & 0 & 0 \\ 0.02 & -0.1 & -0.1 \\ 0 & 0 & -0.2 \end{bmatrix} x_{k-3} + w_k \\ y_k &= \left(\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 2.0 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} \Delta_k [0.3 \ \ 0.5 \ \ 0.3]\right) x_k + v_k \end{aligned}$$

which is of the type (9.1)-(9.2) with three units of delay. We further assume that $W_1 = I,\ W_2 = 0.8I,\ V_1 = 0.2I,\ V_2 = 0.3I$. To determine the Kalman gains, we solve (9.63) and (9.64) with the aid of (9.65), (9.66), (9.67), (9.68), (9.69), (9.70), and (9.71) for selected values of λ, μ. The numerical computation is basically of the form of iterative schemes and the results for a typical case of $\mu_1 = \mu_2 = 0.7, \lambda_1 = 0.3,\ \lambda_2 = 0.4$ are given by

$$P_1 = \begin{bmatrix} 0.141 & 0.005 & 0.003 \\ 0.005 & 0.255 & 0.175 \\ 0.003 & 0.175 & 0.501 \end{bmatrix}, \ S_1 = 10^{-4} \begin{bmatrix} 0.284 & -1.17 & -2.966 \\ -1.17 & 4.813 & 12.208 \\ -2.966 & 12.208 & 30.962 \end{bmatrix}$$

$$K_1 = 10^{-6} \begin{bmatrix} -0.841 & -1.309 \\ 3.463 & 5.388 \\ 8.782 & 13.665 \end{bmatrix}, \ \hat{A}_1 = \begin{bmatrix} 0.331 & -0.019 & -0.034 \\ -0.277 & 0.254 & 0.175 \\ -1.641 & -0.897 & 0.961 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.137 & 0.004 & 0.003 \\ 0.004 & 0.215 & 0.166 \\ 0.003 & 0.166 & 0.498 \end{bmatrix}, \ S_2 = 10^{-4} \begin{bmatrix} 0.277 & -1.23 & -2.887 \\ -1.23 & 4.813 & 11.768 \\ -2.887 & 11.768 & 31.102 \end{bmatrix}$$

$$K_2 = 10^{-6} \begin{bmatrix} -0.798 & -1.287 \\ 4.114 & 4.987 \\ 7.978 & 14.121 \end{bmatrix}, \ \hat{A}_2 = \begin{bmatrix} 0.342 & -0.023 & -0.042 \\ -0.281 & 0.261 & 0.183 \\ -1.666 & -0.928 & 0.973 \end{bmatrix}$$

The developed estimator is indeed asymptotically stable since

$$\lambda(\hat{A})_1 = \{0.3020, 0.4800, 0.7650\} \in (0, 1)$$
$$\lambda(\hat{A})_1 = \{0.7844, 0.3052, 0.4864\} \in (0, 1)$$

The guaranteed cost over the two modes is 36.059×10^{-4}. Several simulation studies have been carried out to examine the performance of the steady-state Kalman filter. In Table 9.1, the guaranteed cost is presented for selected values of the scaling parameters (μ, λ). It is readily evident that the scaling parameters (μ, λ) have a crucial impact on the optimality of the guaranteed cost. This is equally true for specified μ while changing λ or given λ and varying μ.

For the purpose of comparison, a standard Kalman filter was designed for the tw-mode nominal delayless system under consideration by setting $\Delta_k \equiv 0, x_{k-3} \equiv 0$. Then, we applied the developed robust Kalman filter and the standard Kalman filter with $\Delta_k = 0$, $\Delta_k = 0.2$, $\Delta_k = -0.2$ and retained the delayed state. The resulting filtering costs for both filtering schemes are provided in Table 9.2. Again, it is clearly shown that the developed robust Kalman filter outperforms the standard nominal Kalman filter in the presence of uncertainty and delay factor.

Together, Tables 9.1 and 9.2 demonstrate the superior performance of the developed robust Kalman filter.

In the next section, we look at the switched linear Kalman filter for a class of continuous-time state-delay systems with norm-bounded uncertain parameters. Essentially, this is the continuous analog of the foregoing section.

9.2 Continuous Switched Delay System

State estimation forms an integral part of control systems theory. Estimating the state variables of a dynamic model is important to help in improving our knowledge about different systems for the purpose of analysis and control design. The seminal Kalman filtering algorithm [3] is the optimal estimator over all

Table 9.1 The guaranteed cost (GC) vs. the scaling parameters (μ, λ)

Mode 1						
$\lambda = 0.2$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	219.241	141.706	37.029	27.583	65.742	120.333
$\lambda = 0.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	173.601	121.016	31.125	25.113	60.142	101.471
$\lambda = 0.8$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	96.711	101.315	29.451	24.881	51.371	89.116
$\lambda = 1.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	109.345	117.996	38.222	24.003	61.332	110.541
$\lambda = 2.7$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	165.124	122.236	46.113	35.723	72.119	121.171
$\lambda = 3.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	259.943	165.176	51.152	41.907	77.139	151.454
Mode 2						
$\lambda = 0.2$						
μ	0.21	0.43	0.71	1.1	1.48	2.31
$GC \times 10^{-4}$	219.241	141.706	37.029	27.583	65.742	120.333
$\lambda = 0.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	173.601	121.016	31.125	25.113	60.142	101.471
$\lambda = 0.8$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	96.711	101.315	29.451	24.881	51.371	89.116
$\lambda = 1.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	109.345	117.996	38.222	24.003	61.332	110.541
$\lambda = 2.7$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	165.124	122.236	46.113	35.723	72.119	121.171
$\lambda = 3.4$						
μ	0.2	0.45	0.7	1.1	1.5	2.3
$GC \times 10^{-4}$	259.943	165.176	51.152	41.907	77.139	151.454

Table 9.2 Comparison between the nominal kalman filter

	Actual cost		
Filter	$\Delta_k = -0.8$	$\Delta_k = 0$	$\Delta_k = 0.8$
Nominal kalman filter	72.034×10^{-2}	45.147×10^{-4}	235.654×10^{-2}
Robust kalman filter	66.802×10^{-4}	61.147×10^{-4}	67.113×10^{-4}

possible linear ones and gives unbiased estimates of the unknown state vectors under the conditions that the system and measurement noise processes are mutually independent Gaussian distributions. Robust state estimation arises out of the desire to estimate unmeasurable state variables when the plant model has uncertain parameters. In [16], a Kalman filtering approach has been studied with an $\mathcal{H}_\infty - norm$ constraint. For linear systems with norm-bounded parameter uncertainty, the robust estimation problem has been addressed in [329, 398, 399] and the references cited therein, where $\mathcal{H}_\infty$-estimators have been constructed. On another front of research, uncertain systems with state delay have received increasing interests in recent years [206, 344]. When dealing with continuous-time systems with state delay, there have been three basic approaches [206]:

- Infinite-dimensional systems theory, which is based on embedding the class of TLS into a larger class of dynamical systems for which the state evolution is described by appropriate operators in infinite-dimensional spaces;
- Algebraic systems theory, in which the evolution of delay-differential systems is provided in terms of linear systems over rings; and
- Functional differential systems, by incorporating the influence of the hereditary effects of system dynamics on the rate of change of the system and it provides an appropriate mathematical structure in which the system state evolves either in finite-dimensional space or in function space.

In this section, we follow the third approach for convenient representation and numerical compatibility.

The purpose of this section is to consider the state-estimation problem for a class of linear continuous-state delay systems with norm-bounded parameter uncertainties. Specifically, we address the state-estimator design problem such that the estimation error covariance has a guaranteed bound for all admissible uncertainties. The approach hinges upon the application of the multi-estimation structure depicted in Fig. 9.2. The main tool for solving the foregoing problem is

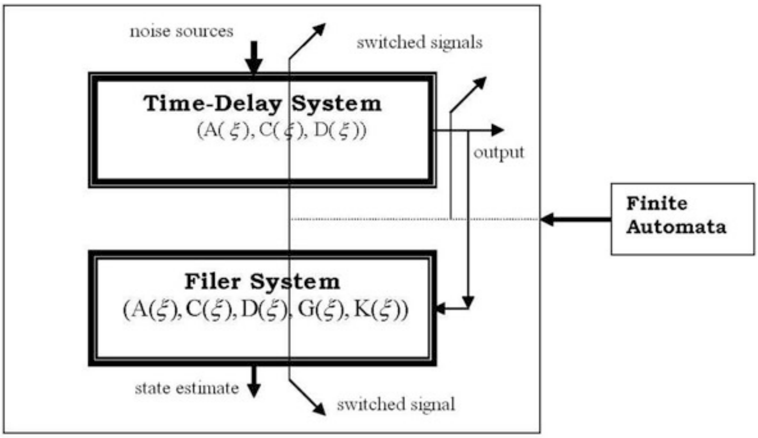

Fig. 9.2 Multi-estimator structure: continuous

the Riccati equation approach. Both time-varying and steady-state robust Kalman filtering are considered.

9.2.1 Problem Formulation

We consider a class of uncertain switched time-delay systems represented by

$$\begin{aligned}\dot{x}(t) &= [A(\xi(t)) + \Delta A(\xi(t))]x(t) + A_d(\xi(t))x(t-\tau_{\xi(t)}) + w(t)\\ &= A_{\Delta,\xi(t)}x(t) + A_d(\xi(t))x(t-\tau_{\xi(t)}) + w(t) \qquad (9.78)\\ y(t) &= [C(\xi(t)) + \Delta C(\xi(t))]x(t) + v(t)\\ &= C_{\Delta,\xi(t)}x(t) + v(t) \qquad (9.79)\end{aligned}$$

where $x(t) \in \Re^n$ is the state, $y(t) \in \Re^m$ is the measured output, and $w(t) \in \Re^n$ and $v(t) \in \Re^m$ are, respectively, the process and measurement noises. In (9.78) and (9.79), $A(\xi(t)) \in \Re^{n\times n}$, $A_d(\xi(t)) \in \Re^{n\times n}$, and $C(\xi(t)) \in \Re^{m\times n}$ are mode-dependent piecewise-continuous matrix functions. The switching rule $\xi(t)$ is not known a priori but we assume its instantaneous value is available in real time for practical implementations. Define the indicator function

$$\begin{aligned}\xi(t) &= [\xi_1(t), ..., \alpha_s(t)]^t, \quad \forall\, j \in \mathbf{S}\\ \xi_j(t) &= \begin{cases} = 1 & \text{when system (9.78) is in the jth mode,}\\ = 0 & \text{otherwise}\end{cases} \qquad (9.80)\end{aligned}$$

Here, $\tau_{\xi(t)}$ is a mode-dependent constant scalar representing the amount of time lag in the state. The matrices $\Delta A(\xi(t))$ and $\Delta C(\xi(t))$ represent time-varying parametric uncertainties, which are of the form

$$\begin{bmatrix}\Delta A(\xi(t))\\ \Delta C(\xi(t))\end{bmatrix} = \begin{bmatrix}H(\xi(t))\\ H_c(\xi(t))\end{bmatrix}\Delta(\xi(t))\; E(\xi(t)) \qquad (9.81)$$

where $H(\xi(t)) \in \Re^{n\times\alpha}$, $H_c(\xi(t)) \in \Re^{m\times\alpha}$, and $E(\xi(t)) \in \Re^{\beta\times n}$ are known piecewise-continuous matrix functions and $\Delta(\xi(t)) \in \Re^{\alpha\times\beta}$ is an unknown matrix with Lebesgue measurable elements satisfying

$$\Delta^t(\xi(t))\Delta(\xi(t)) \quad \leq \quad I \qquad \forall\, t \qquad (9.82)$$

The initial condition is specified as $\langle x(0), x(s)\rangle = \langle x_o, \phi(s)\rangle$, where $\phi(\cdot) \in \mathcal{L}_2[-\tau_{\xi(t)}, 0]$ which is assumed to be a zero-mean Gaussian random vector. The following standard assumptions on noise statistics are recalled:

Assumption 9.10 $\forall t\ ,\ s \geq 0$

$$(\mathbf{a})\mathbf{E}[w(t)] = 0;\quad \mathbf{E}[w(t)w^t(s)] = W(t)\delta(t-s);\ W(t) > 0 \tag{9.83}$$

$$(\mathbf{b})\mathbf{E}[v(t)] = 0;\quad \mathbf{E}[v(t)v^t(s)] = V(t)\delta(t-s);\ V(t) > 0 \tag{9.84}$$

$$(\mathbf{c})\mathbf{E}[x(0)w^t(t)] = 0;\quad \mathbf{E}[x(0)v^t(t)] = 0 \tag{9.85}$$

$$(\mathbf{d})\mathbf{E}[w(t)v^t(s)] = 0;\quad \mathbf{E}[x(0)x^t(0)] = R_o \tag{9.86}$$

where, as before, $\boldsymbol{E}[\cdot]$ *stands for the mathematical expectation and* $\delta(\cdot)$ *is the Dirac function.*

9.2.2 Robust Linear Filtering

Our objective is to design a stable switched-state estimator of the form

$$\dot{\hat{x}}(t) = G(\xi(t))\ \hat{x}(t) + K(\xi(t))\ y(t), \qquad \hat{x}(0) = 0 \tag{9.87}$$

where $G(\xi(t)) \in \Re^{n\times n}$ and $K(\xi(t)) \in \Re^{n\times m}$ are mode-dependent piecewise-continuous matrices to be determined such that there exists a matrix $\Psi \geq 0$ satisfying

$$\mathbf{E}[(x-\hat{x})(x-\hat{x})^t] \ \leq\ \Psi(\xi(t)),\ \forall\Delta : \Delta^t(\xi(t))\Delta(\xi(t)) \leq I \tag{9.88}$$

Note that (10) implies

$$\mathbf{E}[(x-\hat{x})^t(x-\hat{x})] \ \leq tr(\Psi)(\xi(t)),\ \forall\Delta(\xi(t)) : \Delta^t(\xi(t))\Delta(\xi(t)) \leq I \tag{9.89}$$

In this case, the switched estimator (9.87) is said to provide a guaranteed cost (GC) matrix Ψ.

Examination of the switched estimator proceeds by analyzing the estimation error

$$e(t) = x(t) - \hat{x}(t) \tag{9.90}$$

Substituting (9.78) and (9.87) into (9.90), we express the dynamics of the error in the form:

$$\begin{aligned}\dot{e}(\xi(t)) = G(\xi(t))e(t) + [A(\xi(t)) - G(\xi(t)) - K(\xi(t))C(\xi(t))]x(t)\\ +[\Delta A(\xi(t)) - K(\xi(t))\Delta C(\xi(t))]x(t)\\ +A_d(\xi(t))x(t-\tau_{\xi(t)}) + [w(t) - K(\xi(t))v(t)]\end{aligned} \tag{9.91}$$

By introducing the extended state vector

$$\zeta(t) \ = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \ \in \Re^{2n} \tag{9.92}$$

it follows from (9.78), (9.79), and (9.91) that

$$\begin{aligned}\dot{\zeta}(t) &= [\mathsf{A}(\xi(t)) + \mathsf{H}(\xi(t))\Delta(\xi(t))\mathsf{E}(\xi(t))]\zeta(t) + \mathsf{D}(\xi(t))\zeta(t-\tau_{\xi(t)}) + \mathsf{B}(\xi(t))\eta(t) \\ &= \mathsf{A}(\xi(t),\Delta)\zeta(t) + \mathsf{D}(\xi(t))\zeta(t-\tau_{\xi(t)}) + \mathsf{B}(\xi(t))\eta(t) \qquad (9.93)\end{aligned}$$

where $\eta(t)$ is a stationary zero-mean noise signal with identity covariance matrix and

$$\mathsf{A}(\xi(t)) = \begin{bmatrix} A(\xi(t)) & 0 \\ A(\xi(t)) - G(\xi(t)) - K(\xi(t))C(\xi(t)) & G(\xi(t)) \end{bmatrix} \qquad (9.94)$$

$$\mathsf{H}(\xi(t)) = \begin{bmatrix} H(\xi(t)) \\ H(\xi(t)) - K(\xi(t))H_c(\xi(t)) \end{bmatrix}$$

$$\mathsf{E}(\xi(t)) = [E(\xi(t)) \quad 0] \qquad (9.95)$$

$$\mathsf{BB}^t(\xi(t)) = \begin{bmatrix} W(\xi(t)) & W(\xi(t)) \\ W(\xi(t)) & W(\xi(t)) + K(\xi(t))V(\xi(t))K^t(\xi(t)) \end{bmatrix} \qquad (9.96)$$

$$\mathsf{D}(\xi(t)) = \begin{bmatrix} A_d(\xi(t)) & 0 \\ A_d(\xi(t)) & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \qquad (9.97)$$

Definition 9.11 *Estimator (9.87) is said to be a switched quadratic estimator (SQE) associated with a matrix $\Omega(\xi(t)) > 0$ for system (9.78) if there exists a scalar $\lambda(\xi(t)) > 0$ and a matrix*

$$0 < \Omega(\xi(t)) = \begin{bmatrix} \Omega_{1,\xi(t)} & \Omega_{3,\xi(t)} \\ \bullet & \Omega_{2,\xi(t)} \end{bmatrix} \qquad (9.98)$$

satisfying the algebraic inequality

$$\begin{aligned}&-\dot{\Omega}(\xi(t)) + \mathsf{A}_{\xi(t),\Delta}\Omega(\xi(t)) + \Omega(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))\ \Omega(t-\tau_{\xi(t)}) \\ &+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\Omega(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \le 0 \qquad (9.99)\end{aligned}$$

The next result shows that if (9.87) is an SQE for system (9.78) and (9.79) with cost matrix $\Omega(\xi(t))$, then $\Omega(\xi(t))$ defines an upper bound for the filtering error covariance, that is,

$$\mathbf{E}[e(t)e^t(t)] \le \Omega_{2,\xi(t)} \quad \forall\ (t,\xi(t)) \qquad (9.100)$$

for all admissible uncertainties satisfying (9.80) and (9.81).

Theorem 9.12 *Consider the time delay (9.78) and (9.79) satisfying (9.80) and (9.81) and with known initial state. Suppose there exists a solution $\Omega(\xi(t)) \ge 0$ to inequality (9.99) for some $\lambda(\xi(t)) > 0$ and for all admissible uncertainties. Then the switched estimator (9.87) provides an upper bound for the filtering error covariance, that is,*

$$\boldsymbol{E}[e(t)e^t(t)] \;\leq\; \Omega(\xi(t)) \tag{9.101}$$

Proof Suppose that the estimator (9.87) is an SQE with cost matrix $\Omega(\xi(t))$. By evaluating the derivative of the covariance matrix $\Sigma(\xi(t)) = \mathbf{E}[\zeta(t)\ \zeta^t(t)]$, we get

$$\begin{aligned}\dot{\Sigma}(\xi(t)) &= \mathsf{A}_{\xi(t),\Delta}\ \Sigma(\xi(t)) + \Sigma(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \mathsf{D}(\xi(t))\mathbf{E}[\zeta(t-\tau_{\xi(t)})\zeta^t(t)] \\ &+ \mathbf{E}[\zeta(t)\ \zeta^t(t-\tau_{\xi(t)})]\ \mathsf{D}^t(\xi(t)) \\ &+ \mathbf{E}[\eta(t)\ \zeta^t(t)] + \mathbf{E}[\zeta(t)\ \eta^t(t)]\end{aligned} \tag{9.102}$$

Using inequality (9.49), we get the following inequality:

$$\begin{aligned}&\mathsf{D}(\xi(t))\mathbf{E}[\zeta(t-\tau_{\xi(t)})\ \zeta^t(t)] + \mathbf{E}[\zeta(t)\ \zeta^t(t-\tau_{\xi(t)})]\ \mathsf{D}^t(\xi(t)) = \\ &\mathsf{D}(\xi(t))\Sigma(t-\tau_{\xi(t)}) + \Sigma(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) \leq \\ &\lambda(\xi(t))\ \Sigma(t-\tau_{\xi(t)}) + \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\Sigma(t-\tau_{\xi(t)})\mathsf{D}^t((\xi(t))\end{aligned} \tag{9.103}$$

Substituting (9.103) into (9.102) and arranging the terms, we obtain

$$\begin{aligned}\dot{\Sigma}(\xi(t)) \;&\leq\; \mathsf{A}_{\Delta,\xi(t)}(\xi(t))\ \Sigma(\xi(t)) + \Sigma(\xi(t))\mathsf{A}^t_{\Delta}(t) + \lambda(\xi(t))\ \Sigma(t-\tau_{\xi(t)}) \\ &+ \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\ \Sigma(t-\tau_{\xi(t)})\ \mathsf{D}^t(\xi(t)) +\ \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t))\end{aligned} \tag{9.104}$$

Combining (9.99) and (9.104) and letting $\Xi(\xi(t)) = \Sigma(\xi(t)) - \Omega(\xi(t))$, we obtain

$$\begin{aligned}\dot{\Xi}(\xi(t)) \;&\leq\; \mathsf{A}_{\xi(t),\Delta}\ \Xi(\xi(t)) + \Xi(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))\ \Xi(t-\tau_{\xi(t)}) \\ &+ \lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\ \Xi(t-\tau_{\xi(t)})\ \mathsf{D}^t(\xi(t))\end{aligned} \tag{9.105}$$

On considering that the state is known over the period $[-\tau_{\xi(t)}, 0]$ justifies letting $\Sigma(\xi(t))|_{t=0} = 0$. Hence, inequality (9.105) implies that $\Xi(\xi(t)) \;\leq\; 0 \quad \forall\, t \;>\; 0$, that is, $\Sigma(\xi(t)) \;\leq\; \Omega(\xi(t)) \;\; \forall\, t \;>\; 0$. Finally, it is obvious that

$$\mathbf{E}[e(t)e^t(t)] = [0 \quad I]\Sigma(\xi(t))\begin{bmatrix}0\\ I\end{bmatrix} \;\leq\; \Omega_{2,\xi(t)} \tag{9.106}$$

9.2.3 A Design Approach

In line of the discrete-time case, we employ hereafter a Riccati equation approach to solve the robust Kalman filtering for switched continuous time-delay systems. To this end, we define switched matrices $P(\xi(t)) = P^t(\xi(t)) \in \Re^{n\times n}$; $L(\xi(t)) = L^t(\xi(t)) \in \Re^{n\times n}$ as the solutions of the Riccati differential equations (RDEs):

$$\begin{aligned}
\dot{P}(\xi(t)) &= A(\xi(t))P(\xi(t)) + P(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))P(t-\tau_{\xi(t)}) \\
&\quad +\lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) + \hat{W}(\xi(t)) \\
&\quad +\mu(\xi(t))P(\xi(t))E^t(\xi(t))E(\xi(t))P(\xi(t)) \\
P(t-\tau_{(\xi(t))}) &= 0 \quad \forall\, t \ \in [0, \tau_{\xi(t)}] \qquad (9.107) \\
\dot{L}(\xi(t)) &= A(\xi(t))L(\xi(t)) + L(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))L(t-\tau_{\xi(t)}) \\
&\quad +\lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) + \hat{W}(\xi(t)) \\
&\quad +\mu(\xi(t))L(\xi(t))E^t(\xi(t))E(\xi(t))L(\xi(t)) \\
&\quad -\left[L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\right]\hat{V}^{-1}(\xi(t)) \\
&\quad [C(\xi(t))L(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H^t(\xi(t))] \\
L(t-\tau_{\xi(t)}) &= 0 \ \ \forall t \ \in [0, \tau_{\xi(t)}] \qquad (9.108)
\end{aligned}$$

where $\lambda(\xi(t)) > 0, \mu(\xi(t)) > 0 \forall t$ are scaling parameters and the matrices $\hat{A}(\xi(t))$, $\hat{V}(\xi(t))$, and $\hat{W}(\xi(t))$ are given by

$$\hat{W}(\xi(t)) = W(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H^t(\xi(t)) \qquad (9.109)$$

$$\hat{V}(\xi(t)) = V(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H_c^t(\xi(t)) \qquad (9.110)$$

$$\begin{aligned}
\hat{A}(\xi(t)) &= A(\xi(t)) + \delta A(\xi(t)) \\
&= A(\xi(t)) + \mu^{-1}(\xi(t))L^t(\xi(t))E^t(\xi(t))E(\xi(t)) \qquad (9.111)
\end{aligned}$$

Let the $(\lambda, \mu)-$parameterized switched estimator be expressed as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \left\{A(\xi(t)) + \mu^{-1}(\xi(t))L^t(\xi(t))E^t(\xi(t))E(\xi(t))\right\}\hat{x}(t) \\
&\quad +K(\xi(t))\left\{y(t) - C(\xi(t))\hat{x}(t)\right\} \qquad (9.112)
\end{aligned}$$

where the gain matrix $K(\xi(t)) \in \Re^{n\times m}$ is to be determined. The following theorem summarizes the main result:

Theorem 9.13 *Consider system (9.78) and (9.79) satisfying the uncertain structure (9.80) and (9.81) with zero initial condition. Suppose the process and measurement noises satisfy* **Assumption 9.10**. *For some $\mu(\xi(t)) > 0$, $\lambda(\xi(t)) > 0$, let $P(\xi(t)) = P^t(\xi(t))$ and $L(\xi(t)) = L^t(\xi(t))$ be the solutions of RDEs (9.107) and (9.108), respectively. Then the $(\lambda, \mu)-$parametrized estimator (31) is the SQE estimator with GC such that*

$$\boldsymbol{E}[\{x(t) - \hat{x}(t)\}^t \ \{x(t) - \hat{x}(t)\}] \ \leq \ tr[L(\xi(t))] \qquad (9.113)$$

Moreover, the gain matrix $K(\xi(t))$ is given by

$$K(\xi(t)) = \left\{L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\right\} \ \hat{V}^{-1}(\xi(t)) \quad (9.114)$$

Proof Let

$$X(\xi(t)) \;=\; \begin{bmatrix} P(\xi(t)) & L(\xi(t)) \\ L(\xi(t)) & L(\xi(t)) \end{bmatrix} \tag{9.115}$$

where $P(\xi(t))$ and $L(\xi(t))$ are the positive-definite solutions to (9.107) and (9.108), respectively. By combining (9.107), (9.108), (9.109), (9.110), (9.111), (9.112), and (9.113) with some standard matrix manipulations, it is easy to see that

$$\begin{aligned} &-\dot{X}(\xi(t)) + \mathsf{A}(\xi(t))X(\xi(t)) + X(\xi(t))\mathsf{A}^t(\xi(t)) + \lambda(\xi(t))X(t-\tau_{\xi(t)}) \\ &+\mu^{-1}(\xi(t))\mathsf{H}(\xi(t))\mathsf{H}^t(\xi(t)) + \mu(\xi(t))X(\xi(t))\mathsf{E}^t(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) \\ &+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))\; X(t-\tau_{\xi(t)})\; \mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \;=0 \end{aligned} \tag{9.116}$$

where $\mathsf{A}(\xi(t))$, $\mathsf{B}(\xi(t))$, $\mathsf{H}(\xi(t))$, $\mathsf{D}(\xi(t))$ are given by (9.94), (9.96), and (9.97). A simple comparison of (9.87) and (9.114) taking into consideration (9.111), (9.112), (9.113), and (9.114) and (9.116) shows that

$$G(\xi(t)) = \hat{A}(\xi(t)) - K(\xi(t))\; C(\xi(t))$$

By making use of a version of inequality (9.49) that for some $\mu(\xi(t)) \;>\; 0$, we have

$$\begin{aligned} &\mathsf{H}(\xi(t))\Delta(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) + X(\xi(t))\mathsf{E}^t(\xi(t))\Delta^t(\xi(t))\mathsf{H}^t(\xi(t)) \;\le \\ &\mu(\xi(t))\; X(\xi(t))\mathsf{E}^t(\xi(t))\mathsf{E}(\xi(t))X(\xi(t)) + \mu^{-1}(\xi(t))\mathsf{H}(\xi(t))\mathsf{H}^t(\xi(t)) \end{aligned} \tag{9.117}$$

Using (9.117), it is now a simple task to verify that (9.116) becomes

$$\begin{aligned} &-\dot{X}(\xi(t)) + \mathsf{A}_{\xi(t),\Delta}X(\xi(t)) + X(\xi(t))\mathsf{A}^t_{\xi(t),\Delta} + \lambda(\xi(t))X(t-\tau_{\xi(t)}) \\ &+\lambda^{-1}(\xi(t))\mathsf{D}(\xi(t))X(t-\tau_{\xi(t)})\mathsf{D}^t(\xi(t)) + \mathsf{B}(\xi(t))\mathsf{B}^t(\xi(t)) \;\le\; 0 \end{aligned} \tag{9.118}$$

$$\forall\Delta(\xi(t)) \colon \Delta^t(\xi(t))\;\Delta(\xi(t)) \;\le\; I \quad \forall(\xi(t)) \in \bar{\mathbf{S}} \tag{9.119}$$

By **Theorem 9.12**, it follows that for some $\mu(\xi(t)) > 0,\ \lambda(\xi(t)) > 0,$ that (9.112) is a switched quadratic estimator and $\mathbf{E}[e(t)e^t(t)] \;\le\; L(\xi(t))$. This implies that $\mathbf{E}[e^t(t)e(t)] \;\le\; tr[L(\xi(t))]$

Remark 9.14 It is known that the uncertainty representation (9.80) and (9.81) is not unique. We note that $H(\xi(t))$, $H_c(\xi(t))$ may be postmultiplied and $E(\xi(t))$ may be premultiplied by any unitary matrix since eventually this unitary matrix may be absorbed in $\Delta(\xi(t))$. It is significant to observe that such unitary multiplication does not affect the solution developed in this section.

Remark 9.15 Had we defined

$$X(\xi(t)) = \begin{bmatrix} P^{-1}(\xi(t)) & 0 \\ 0 & L(\xi(t)) \end{bmatrix} \tag{9.120}$$

we would have obtained

$$\begin{aligned} \dot{P}(\xi(t)) &= P(\xi(t))A(\xi(t)) + A^t(\xi(t))P(\xi(t)) + \lambda(\xi(t))P(t-\tau_{\xi(t)}) \\ &+ P(\xi(t))\hat{W}(\xi(t))P(\xi(t)) \\ &+ \lambda^{-1}(\xi(t))P(t-\tau_{\xi(t)})A_d(\xi(t))P^{-1}(t-\tau_{\xi(t)})A_d^t(\xi(t))P(t-\tau_{\xi(t)}) \\ &+ \mu(\xi(t))E^t(\xi(t))E(\xi(t)) \\ P(t-\tau_{\xi(t)}) &= 0 \quad \forall\, t \ \in [0, \tau_{\xi(t)}] \end{aligned} \tag{9.121}$$

$$\begin{aligned} \dot{L}(\xi(t)) &= A(\xi(t))L(\xi(t)) + L(\xi(t))A^t(\xi(t)) + \lambda(\xi(t))L(t-\tau_{\xi(t)}) \\ &+ \hat{W}(\xi(t)) + \lambda^{-1}(\xi(t))A_d(\xi(t))P(t-\tau_{\xi(t)})A_d^t(\xi(t)) \\ &+ \mu(\xi(t))L(\xi(t))E^t(\xi(t))E(\xi(t))L(\xi(t)) \\ &- \left[L(\xi(t))C^t(\xi(t)) + \mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))\right]\hat{V}^{-1}(\xi(t)) \\ &\quad [C(\xi(t))L(\xi(t)) + \mu^{-1}(\xi(t))H_c(\xi(t))H^t(\xi(t))] \\ L(t-\tau_{\xi(t)}) &= 0 \quad \forall\, t \ \in [0, \tau_{\xi(t)}] \end{aligned} \tag{9.122}$$

We note that (9.121), which is of a nonstandard form, although X(t) in (9.120) is frequently used in similar situations for the nonswitched delayless systems [329, 398, 399]. Indeed, the difficulty comes from the delay-term

$$\lambda^{-1}(\xi(t))P(t-\tau_{\xi(t)})A_d(\xi(t))P^{-1}(t-\tau_{\xi(t)})A_d^t(\xi(t))P(t-\tau_{\xi(t)})$$

This point emphasizes the fact that not every result of delayless systems are straightforwardly transformable to time delay systems.

Remark 9.16 It is interesting to observe that the switched estimator (9.113) is independent of the delay factor $\tau_{\xi(t)}$ and it reduces to the standard Kalman filtering algorithm in the case of systems without uncertainties and delay factor $H(\xi(t)) \equiv 0,\ H_c(\xi(t)) \equiv 0,\ E(\xi(t)) \equiv 0,\ A_d(\xi(t)) \equiv 0,\ \lambda(\xi(t)) \equiv 0.$

Remark 9.17 In the delay-free case $(A_d(\xi(t)) \equiv 0,\ \lambda(\xi(t)) \equiv 0)$, we observe that (9.114) reduces to the Kalman filter for the system

$$\dot{x}(t) = \hat{A}(\xi(t))\, x(t) + \hat{w}(t) \tag{9.123}$$

$$y(t) = C(\xi(t))\, x(t) + \hat{v}(t) \tag{9.124}$$

where $\hat{w}(t)$ and $\hat{v}(t)$ are zero-mean white noise sequences with covariance matrices $\hat{W}(t)$ and $\hat{V}(t)$, respectively, and having cross-covariance matrix $[\mu^{-1}(\xi(t))H(\xi(t))H_c^t(\xi(t))]$. Looked at in this light, the approach developed here before to robust filtering in **Theorem 9.13** corresponds to designing a standard Kalman filter for a related continuous-time system, which captures all admissible uncertainties and time delay, but does not involve parameter uncertainties. Indeed, the robust filter (9.113) using (9.108), (9.109), (9.110), (9.111), and (9.112) can be rewritten as

$$\dot{\hat{x}}(t) = [A(\xi(t)) + \delta A(\xi(t))]\, \hat{x}(t) + K(\xi(t))\, \{y(t) - C(\xi(t))\, \hat{x}(t)\} \qquad (9.125)$$

where $\delta A(\xi(t))$ is defined in (9.108) and it reflects the effect of uncertainties $\{\Delta A(\xi(t)), \Delta C(\xi(t))\}$ and time-delay factor $A_d((\xi(t))$ on the structure of the filter.

Remark 9.18 In view of the foregoing notes, it is readily evident that the results achieved by **Theorems 9.12** and **9.13** extend some of the existing robust estimation results to linear systems with state delay.

9.2.4 Steady-State Robust Filter

Now, we investigate the asymptotic properties of the Kalman filter developed previously, where the switching signal ξ is now independent of time. For this purpose, we consider the uncertain time-delay system

$$\begin{aligned} \dot{x}(t) &= [A(\xi) + H(\xi)\Delta(\xi)E(\xi)]x(t) + A_d(\xi)x(t-\tau_\xi) + w(t) \\ &= A_{\xi,\Delta}x(t) + A_d(\xi)x(t-\tau_\xi) + w(t) \qquad (9.126) \\ y(t) &= [C(\xi) + H_c(\xi)\Delta(\xi)E(\xi)]x(t) + v(t) \\ &= C_{\xi,\Delta}x(t) + v(t) \qquad (9.127) \end{aligned}$$

where $\Delta(\xi)$ satisfies (9.81). The matrices $A(\xi) \in \Re^{n\times n}$, $C(\xi) \in \Re^{m\times n}$ are mode-dependent constant matrices representing the nominal plant. It is assumed that $A(\xi),\ \xi \in \bar{\mathbf{S}}$ is Hurwitz. Our objective now is to design a switched time-invariant a priori estimator of the form:

$$\dot{\hat{x}}(t) = \hat{A}(\xi)\, \hat{x}(t) + K(\xi)\, [y(t) - C(\xi)\hat{x}(t)] \quad \hat{x}(0) = 0 \qquad (9.128)$$

that achieves the following asymptotic performance bound

$$\lim_{t\to\infty} \mathbf{E}\left\{[\hat{x}(t) - x(t)][\hat{x}(t) - x(t)]^t\right\} \quad \le \quad L(\xi) \qquad (9.129)$$

Theorem 9.19 *Consider the uncertain time-delay system (9.126) and (9.127) with $A(\xi)$ being Hurwitz. If for some scalars $\mu(\xi) > 0,\ \lambda(\xi) > 0,$ there exist stabilizing solutions for the AREs*

$$\begin{aligned} &A(\xi)P(\xi) + P(\xi)A^t(\xi) + \lambda(\xi)P(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) \\ &+ \mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi) = 0 \qquad (9.130) \end{aligned}$$

$$
\begin{aligned}
& A(\xi)L(\xi) + L(\xi)A^t(\xi) + \lambda(\xi)L(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) \\
& + \mu(\xi)L(\xi)E^t(\xi)E(\xi)L(\xi) - \\
& \left[L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \\
& [C(\xi)L(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)] = 0 \qquad (9.131)
\end{aligned}
$$

then the estimator (9.125) is a stable switched quadratic (SSQ) and achieves (9.129) with

$$
\begin{aligned}
\hat{W}(\xi) &= W + \mu^{-1}(\xi)H(\xi)H^t(\xi) \\
\hat{V}(\xi) &= V + \mu^{-1}(\xi)H_c(\xi)H_c^t(\xi) \qquad (9.132) \\
\hat{A}(\xi) &= A(\xi) + \delta A(\xi) \\
&= A(\xi) + \mu^{-1}(\xi)L^t(\xi)E^t(\xi)E(\xi) \\
K(\xi) &= \left\{L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right\}\ \hat{V}^{-1}(\xi) \qquad (9.133)
\end{aligned}
$$

for some $L(\xi) \geq 0$.

Proof To examine the stability of the closed-loop system, we augment (9.126), (9.127) and (9.128) with $(w(t) = 0, v(t) = 0)$, to obtain

$$
\begin{aligned}
\dot{\xi}(t) &= \mathsf{A}_{\xi,\Delta}\xi(t) + \mathsf{D}(\xi)\xi(t-\tau_\xi) \\
&= \begin{bmatrix} A_{\xi,\Delta} & 0 \\ A_{\xi,\Delta} - G(\xi) - K(\xi)C_{\xi,\Delta} & G(\xi) \end{bmatrix}\xi(t) \\
&+ \begin{bmatrix} A_d(\xi) & 0 \\ A_d(\xi) & 0 \end{bmatrix}\xi(t-\tau_\xi) \qquad (9.134)
\end{aligned}
$$

By a similar argument as in the proof of **Theorem 9.13**, it is easy to see that

$$
X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}_{\xi,\Delta}^t X(\xi) - \lambda(\xi)X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi) \ < \ 0 \qquad (9.135)
$$

where

$$
X(\xi) = \begin{bmatrix} P(\xi) & L(\xi) \\ L(\xi) & L(\xi) \end{bmatrix} \qquad (9.136)
$$

Introducing a Lyapunov – Krasovskii functional

$$
V(\xi) = \zeta^t(t)X(\xi)\zeta(t) + \int_{t-\tau_\xi}^{t} \zeta^t(\alpha)\lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(\alpha)\ d\alpha \qquad (9.137)
$$

and observe that $V(\xi) \ > \ 0$, for $\zeta(t) \ \neq 0$, for some $\lambda(\xi) > 0$ and $V(\xi) = 0$ when $\zeta = 0$. By differentiating the Lyapunov – Krasovskii functional (9.137) along the trajectories of system (9.134), we get

$$
\begin{aligned}
\dot{V}(\xi) &= \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&\quad + \zeta^t(t)X(\xi)\mathsf{D}(\xi)\zeta(t-\tau_\xi) + \zeta^t(t-\tau_\xi)\mathsf{D}^t(\xi)X(\xi)\zeta(t) \\
&\quad - \lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&\le \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&\quad + \lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&\quad - \lambda^{-1}(\xi)\zeta^t(t-\tau_\xi)\mathsf{D}(\xi)X(\xi)\mathsf{D}^t(\xi)\zeta(t-\tau_\xi) \\
&= \zeta^t(t)\Big[X(\xi)\mathsf{A}_{\xi,\Delta} + \mathsf{A}^t_{\xi,\Delta}X(\xi) + \lambda(\xi)X(\xi) + \lambda^{-1}(\xi)\mathsf{D}(\xi)X\mathsf{D}^t(\xi)\Big]\zeta(t) \\
&< \quad 0
\end{aligned}
\tag{9.138}
$$

which means that the augmented system (9.132) is asymptotically stable. In turn, this implies that (9.125) is SSQ. The guaranteed performance

$$
\mathbf{E}[e(t)e^t(t)] \;\le\; L(\xi) \tag{9.139}
$$

follows from similar lines of argument as in the proof of **Theorem 9.13**.

The next theorem provides LMI-based solution to the steady-state robust Kalman filter.

Theorem 9.20 *Consider the uncertain switched time-delay system (9.126) and (9.127) with $A(\xi)$ being Hurwitz. The estimator*

$$
\begin{aligned}
\dot{\hat{x}}(t) &= [A(\xi) + \mu^{-1}(\xi)L^t(\xi)E^t(\xi)E(\xi)]\hat{x}(t) \\
&\quad + \Big[L(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\Big]\hat{V}^{-1}(\xi)[y(t) - C(\xi)\hat{x}(t)]
\end{aligned}
\tag{9.140}
$$

where

$$
\hat{V}(\xi) = V + \mu^{-1}(\xi)H_c(\xi)H_c^t(\xi) \tag{9.141}
$$

is a stable switched quadratic and achieves (9.139) for some $L(\xi) \ge 0$ if for some scalars $\mu(\xi) > 0$, $\lambda(\xi) > 0$, there exist matrices $0 < Y(\xi) = Y^t(\xi)$ and $0 < X(\xi) = X^t(\xi)$ satisfying the LMIs

$$
\begin{bmatrix}
A(\xi)Y(\xi) + Y(\xi)A^t(\xi) + Q_y(Y,\lambda,(\xi)) & A_d(\xi)Y(\xi) & Y(\xi)E^t(\xi) \\
Y(\xi)A_d^t(\xi) & -\lambda(\xi)I & 0 \\
E(\xi)Y(\xi) & 0 & -\mu^{-1}(\xi)I
\end{bmatrix} < 0 \tag{9.142}
$$

$$
\begin{bmatrix}
A(\xi)X(\xi) + X(\xi)A^t(\xi) + Q_x(X,\lambda,(\xi)) & A_d(\xi)Y(\xi) & X(\xi)E^t(\xi) \\
Y(\xi)A_d^t(\xi) & -\lambda(\xi)I & 0 \\
E(\xi)X(\xi) & 0 & -\mu^{-1}(\xi)I
\end{bmatrix} < 0 \tag{9.143}
$$

where

$$Q_y(Y, \lambda, (\xi)) = \lambda(\xi)Y(\xi) + W + \mu^{-1}(\xi)H(\xi)H^t(\xi),$$
$$Q_x(X, \lambda, (\xi)) = \lambda(\xi)X(\xi) + W + \mu^{-1}(\xi)H(\xi)H^t(\xi) - \left[X(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \left[C(\xi)X(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)\right] \tag{9.144}$$

Proof By inequality (9.50) and (9.130) and (9.131), it follows that there exist matrices $0 < Y(\xi) = Y^t(\xi)$ and $0 < X(\xi) = X^t(\xi)$ satisfying the algebraic Riccati inequalities (ARIs):

$$A(\xi)Y(\xi) + Y(\xi)A^t(\xi) + \lambda(\xi)Y(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)Y(\xi)A_d^t(\xi) + \mu(\xi)Y(\xi)E^t(\xi)E(\xi)Y(\xi) < 0 \tag{9.145}$$

$$A(\xi)X(\xi) + X(\xi)A^t(\xi) + \lambda(\xi)X(\xi) + \hat{W}(\xi) + \lambda^{-1}(\xi)A_d(\xi)Y(\xi)A_d^t(\xi) + \mu(\xi)X(\xi)E^t(\xi)E(\xi)X(\xi) - \left[X(\xi)C^t(\xi) + \mu^{-1}(\xi)H(\xi)H_c^t(\xi)\right]\hat{V}^{-1}(\xi) \left[C(\xi)X(\xi) + \mu^{-1}(\xi)H_c(\xi)H^t(\xi)\right] < 0 \tag{9.146}$$

such that $Y(\xi) > P(\xi)$, $X(\xi) > L(\xi)$. Application of (9.50) to the ARIs (9.145) and (9.146) yields the LMIs (9.142) and (9.143).

Remark 9.21 It should be emphasized the AREs (9.130) and (9.131) do not have clear-cut monotonicity properties enjoyed by standard AREs. The main reason for this is the presence of the term $A_d(\xi)P(\xi)A_d^t(\xi)$.

Extending on the results [329], given τ_ξ, it follows that the uncertain time-delay system

$$\dot{x}(t) = [A(\xi) + H(\xi)\Delta(\xi)E(\xi)]x(t) + A_d(\xi)x(t - \tau_\xi) \tag{9.147}$$

is switched quadratically stable (SQS) if there exist matrices $0 < \bar{P}(\xi) = \bar{P}^t(\xi) \in \Re^{n\times n}$, $0 < \bar{R}(\xi) = \bar{R}^t(\xi) \in \Re^{n\times n}$ satisfying the ARI:

$$\bar{P}(\xi)A(\xi) + A^t(\xi)\bar{P}(\xi) + E^t(\xi)E(\xi) + \bar{P}(\xi)H(\xi)H^t(\xi)\bar{P}(\xi) + \bar{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\bar{P}(\xi) + \bar{R}(\xi) < 0 \tag{9.148}$$

The next theorem examines further properties of the positive-definite solution of the ARE (9.130).

Theorem 9.22 *If system (9.147) is (SQS), then there exist some $\mu(\xi) > 0, \lambda(\xi) > 0$ such that the ARE (9.130) admits a positive-definite solution $P(\xi) > 0$ for some $0 < R(\xi) = R^t(\xi)$. Furthermore, for a given $\mu(\xi) > 0, \lambda(\xi) > 0$ and $R(\xi) > 0$, if there exist $\bar{\mu}(\xi) > 0$, $\bar{\lambda}(\xi) > 0$ such that (9.130) admits a positive-definite solution $0 < \bar{P}(\xi) = \bar{P}^t(\xi)$ for some $0 < \bar{R}(\xi) = \bar{R}^t(\xi)$, then for any $\mu(\xi) \in$*

$(0, \bar{\mu}(\xi)]$, $\lambda(\xi) \in (0, \bar{\lambda}(\xi)]$, *the solution of (9.131)* $P(\xi) > 0$ *satisfies* $0 < P(\xi) \leq (\mu(\xi)/\bar{\mu}(\xi))\bar{P}(\xi)$ *for some* $0 < R(\xi) \leq (\lambda(\xi)/\bar{\lambda}(\xi))\bar{R}(\xi)$.

Proof Using (9.148), it follows for some $\mu(\xi) > 0$ that

$$\bar{P}(\xi)A(\xi) + A^t(\xi)\bar{P}(\xi) + E^t(\xi)E(\xi) + \bar{P}(\xi)[\mu(\xi)W + H(\xi)H^t(\xi)]\bar{P}(\xi) + \bar{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\bar{P}(\xi) + \bar{R}(\xi) < 0 \quad (9.149)$$

By setting $\widehat{P}(\xi) = \mu(\xi)\bar{P}(\xi)$, we get

$$\widehat{P}(\xi)A(\xi) + A^t(\xi)\widehat{P}(\xi) + \mu(\xi)E^t(\xi)E(\xi) + \widehat{P}(\xi)[W + \mu^{-1}(\xi)H(\xi)H^t(\xi)]\widehat{P}(\xi) + \mu^{-1}(\xi)\widehat{P}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi)\widehat{P}(\xi) + \mu(\xi)\bar{R}(\xi) < 0 \quad (9.150)$$

This implies that

$$A(\xi)\widehat{P}(\xi) + \widehat{P}(\xi)A^t(\xi) + \mu(\xi)\widehat{P}(\xi)E^t(\xi)E(\xi)\widehat{P}(\xi) + [W + \mu^{-1}(\xi)H(\xi)H^t(\xi)] + \mu^{-1}(\xi)A_d(\xi)\bar{R}^{-1}(\xi)A_d^t(\xi) + \mu(\xi)\widehat{P}(\xi)\bar{R}(\xi)\widehat{P}(\xi) < 0 \quad (9.151)$$

By letting $\bar{R}(\xi) = (\lambda(\xi)/\mu(\xi))\widehat{P}^{-1}(\xi)$ for some $\lambda(\xi) > 0$, it follows that there exist a positive-definite solution $P(\xi) > 0$ to the ARE

$$A(\xi)P(\xi) + P(\xi)A^t(\xi) + \lambda(\xi)P(\xi) + \mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi) + [W + \mu^{-1}(\xi)H(\xi)H^t(\xi)] + \lambda^{-1}(\xi)A_d(\xi)P(\xi)A_d^t(\xi) = 0 \quad (9.152)$$

The remaining part regarding the monotonicity of $\mu(\xi)$ and that $0 < P(\xi) < (\mu(\xi)/\bar{\mu}(\xi))\bar{P}(\xi)$ for some $0 < R(\xi) \leq (\lambda(\xi)/\bar{\lambda}(\xi))\bar{R}(\xi)$ follows by applying the results of [398].

Remark 9.23 For any pairs $(\lambda_1(\xi), \mu_1(\xi))$, $(\lambda_2(\xi), \mu_2(\xi)) \in (0, \bar{\lambda}(\xi)] \times (0, \bar{\mu}(\xi)]$,
$\lambda_1(\xi) \leq \lambda_2(\xi), \mu_1(\xi) \leq \mu_2(\xi)$, it follows from **Theorem 9.22** that

$$P(\xi, \mu_1)/\mu_1(\xi) \leq P(\xi, \mu_2)/\mu_2(\xi)$$

for some $R(\xi, \lambda_1) \leq R(\xi, \lambda_2)$. Thus

$$d^2P(\xi)/d\mu^2(\xi) \geq 0$$

This can also be justified by differentiating (9.130) using (9.132) and (9.133) to yield

$$A(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)} + \frac{d^2P(\xi)}{d\mu^2(\xi)}A^t(\xi) + \lambda(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)} + \lambda^{-1}(\xi)D(\xi)\frac{d^2P(\xi)}{d\mu^2(\xi)}D^t(\xi) + \frac{2}{\mu^3(\xi)}H_1(\xi)H_1^t(\xi) = 0 \qquad (9.153)$$

Since $A(\xi)$ is Hurwitz, it is obvious from (9.153) that

$$\frac{d^2P(\xi)}{d\mu^2(\xi)} \geq 0$$

By the same arguments, we have

$$A(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + \frac{d^2P(\xi)}{d\lambda^2(\xi)}A^t(\xi) + \lambda(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + 2\frac{dP(\xi)}{d\lambda(\xi)} + D(\xi)\left\{\frac{2P(\xi)}{\lambda^3(\xi)} - \frac{2P(\xi)}{\lambda^2(\xi)}\frac{dP(\xi)}{\lambda(\xi)} + \frac{1}{\lambda(\xi)}\frac{d^2P(\xi)}{d\lambda^2(\xi)}\right\}D^t(\xi)$$
$$\mu(\xi)P(\xi)E^t(\xi)E(\xi)P(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)} + \mu(\xi)\frac{d^2P(\xi)}{d\lambda^2(\xi)}E^t(\xi)E(\xi)P(\xi) + 2\mu(\xi)\frac{dP(\xi)}{d\lambda(\xi)}E^t(\xi)E(\xi)\frac{dP(\xi)}{d\lambda(\xi)} = 0 \qquad (9.154)$$

which leads to

$$\frac{d^2P(\xi)}{d\lambda^2(\xi)} \geq 0$$

Following a similar procedure, it can be shown that

$$\frac{d^2L(\xi)}{d\mu^2(\xi)} \geq 0, \quad \frac{d^2L(\xi)}{d\lambda^2(\xi)} \geq 0$$

Thus we conclude that $tr\ (L)(\xi)$ is a convex function over the region $(0, \bar{\lambda}(\xi)] \times (0, \bar{\mu}(\xi)]$. This indicates that a suboptimal robust Kalman filter can be obtained via convex optimization approach.

9.2.5 Numerical Simulation

For the purpose of illustrating the developed theory, we focus on the steady-state Kalman filtering and proceed to determine the estimator gains. Essentially, seek to solve (9.130), (9.131), (9.132), and (9.133) when $\lambda(\xi) \in [\lambda_1 \rightarrow \lambda_2]$, $\mu(\xi) \in [\mu_1 \rightarrow \mu_2]$, where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are given constants and $\xi \in \{1, 2, 3\}$. Initially, we observe that (9.130) depends on $P(\xi)$ only and it is not of the standard-forms of

AREs. On the contrary, (9.131) depends on both $L(\xi)$ and $P(\xi)$ and it can be put into the standard ARE form. For numerical simulation, we employ a Kronecker Product-like technique to reduce (9.130) into a system of nonlinear algebraic equations of the form

$$f(\alpha) = G\,\alpha + h(\alpha) + q \tag{9.155}$$

where $\alpha \in \Re^{n(n+1)/2}$ is a vector of the unknown elements of the P matrix. The algebraic equation (9.135) can then be solved using an iterative Newton – Raphson technique according to the rule

$$\alpha_{(i+1)} = \alpha_{(i)} - \gamma_{(i)}[G + \nabla_\alpha h(\alpha_{(i)})]^{-1} f(\alpha_{(i)}) \tag{9.156}$$

where i is the iteration index, $\alpha_{(o)} = 0$, $\nabla_\alpha h(\alpha)$ is the Jacobian of $h(\alpha)$, and the step-size $\gamma_{(i)}$ is given by $\gamma_{(i)} = 1/[||f(\alpha(i))|| + 1]$.

Given the solution of (9.130), we proceed to solve (9.131) using a standard hamiltonian/eigenvector method. All the computations are carried out using the Linear Algebra and System (L-A-S) software [5]. As a typical case, consider a time-delay system of the type (9.78)-(9.79) with

Mode 1

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0.5 \\ 1 & -3 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.4 \end{bmatrix} \\ W &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ C &= [1 \quad -3], \; E = [0.5 \quad 1], \; H_c = 2, \; V = 1 \end{aligned}$$

Mode 2

$$\begin{aligned} A &= \begin{bmatrix} -3 & 0 \\ 0.5 & -4 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.3 & -0.2 \\ 0 & 0.3 \end{bmatrix} \\ W &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} \\ C &= [0.7 \quad -2], \; E = [0.4 \quad 0.7], \; H_c = 1, \; V = 1 \end{aligned}$$

Mode 3

$$\begin{aligned} A &= \begin{bmatrix} -4 & 1 \\ 0 & -5 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.4 \end{bmatrix} \\ W &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; H = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ C &= [0.6 \quad -1], \; E = [0.3 \quad 0.8], \; H_c = 2, \; V = 0.7 \end{aligned}$$

A summary of the computational results is presented in Table 9.3 and from which we observe the following:

Table 9.3 Summary of some of the computational results

Mode	λ	μ	P		L		δA		K^t		$\lambda(\hat{A})$	
1	0.1	0.6	0.574	0.175	0.566	0.176	0.382	0.784	0.314	0.301	−3.227	−0.472
			0.175	0.457	0.176	0.464	0.459	0.919				
2	0.1	0.6	0.535	0.156	0.500	0.136	0.242	0.483	0.314	0.301	−3.249	−0.897
			0.156	0.431	0.136	0.421	0.306	0.612				
3	0.1	0.6	0.525	0.151	0.480	0.124	0.202	0.404	0.315	0.302	−3.258	−1.017
			0.151	0.425	0.124	0.409	0.261	0.523				
1	0.2	0.7	0.564	0.206	0.554	0.210	0.406	0.811	0.317	0.295	−3.279	−0.498
			0.206	0.374	0.210	0.386	0.409	0.818				
2	0.2	0.7	0.525	0.183	0.488	0.167	0.257	0.514	0.318	0.301	−3.293	−0.913
			0.183	0.350	0.167	0.346	0.269	0.537				
3	0.2	0.7	0.515	0.177	0.468	0.153	0.215	0.430	0.318	0.294	−3.298	−1.031
			0.177	0.344	0.153	0.334	0.228	0.456				
1	0.4	0.8	0.603	0.242	0.576	0.238	0.438	0.877	0.324	0.296	−3.302	−0.450
			0.242	0.357	0.238	0.367	0.405	0.810				
2	0.4	0.8	0.564	0.217	0.508	0.192	0.278	0.557	0.325	0.301	−3.311	−0.880
			0.217	0.335	0.192	0.328	0.265	0.530				
3	0.4	0.8	0.555	0.210	0.486	0.177	0.233	0.466	0.326	0.295	−3.415	−1.002
			0.210	0.330	0.177	0.316	0.225	0.449				

Table 9.3 (continued)

Mode	λ	μ	P		L		δA		K^t		$\lambda(\hat{A})$	
1	0.6	0.6	0.665	0.276	0.612	0.259	0.471	0.942	0.331	0.300	−3.311	−0.382
			0.276	0.369	0.259	0.372	0.418	0.836				
2	0.6	0.6	0.629	0.252	0.540	0.210	0.300	0.600	0.333	0.299	−3.318	−0.834
			0.252	0.348	0.210	0.333	0.274	0.548				
3	0.6	0.6	0.622	0.247	0.517	0.195	0.252	0.503	0.335	0.299	−3.321	−0.962
			0.247	0.344	0.195	0.321	0.232	0.464				
1	0.8	0.7	0.751	0.319	0.656	0.281	0.507	1.015	0.340	0.304	−3.318	−0.300
			0.319	0.392	0.281	0.384	0.437	0.874				
2	0.8	0.7	0.723	0.298	0.580	0.229	0.324	0.649	0.343	0.304	−3.324	−0.779
			0.298	0.374	0.229	0.344	0.287	0.574				
3	0.8	0.7	0.726	0.298	0.556	0.213	0.273	0.545	0.345	0.304	−3.326	−0.914
			0.298	0.372	0.213	0.332	0.244	0.487				
1	0.9	0.8	0.806	0.346	0.681	0.292	0.527	1.055	0.344	0.307	−3.321	−0.255
			0.346	0.407	0.292	0.392	0.448	0.897				
2	0.9	0.8	0.790	0.330	0.603	0.239	0.338	0.676	0.349	0.307	−3.326	−0.747
			0.330	0.392	0.239	0.352	0.294	0.589				
3	0.9	0.8	0.805	0.336	0.579	0.222	0.284	0.569	0.351	0.307	−3.328	−0.887
			0.336	0.393	0.222	0.339	0.250	0.501				

(1) For a given $\lambda \in [0.1 - 0.9]$, increasing μ by 50% results in 0.3% increase in $||K||$ (for small λ) and about 1.12% increase in $||K||$ when λ is relatively large.
(2) For a given μ, increasing λ from 0.1 to 0.9 causes $||K||$ to increase by about 5.35%.
(3) For $\mu < 0.6,\ \lambda in[0.1, 0.9]$, the estimator is unstable.
(4) Increasing (λ, μ) beyond $(1, 1)$ yields unstable estimator.

Therefore, we conclude that:

(1) The stable-estimator gains are practically insensitive to the (λ, μ) parameters.
(2) There is a finite range for (λ, μ) that guarantees stable performance of the developed Kalman filter.

9.3 Notes and References

We have considered in this chapter a robust Kalman filter for a class of switched continuous-time systems with norm-bounded uncertainties and unknown constant state delay. Both time-varying and steady-state filtering algorithms have been examined. The main results are contained in two parts: Part 1 includes **Theorems 9.2** and **9.3** that deal with time-varying problems on a finite horizon and Part 2 includes **Theorems 9.8–9.13** that treat the steady-state problem and its related properties. It has been established that the Kalman filter algorithm is related to solutions of two Riccati equations involving scalar parameters. Important properties of the robust filter have been delineated. It has been further shown that the guaranteed cost is a convex function of the scaling parameters. A numerical simulation is provided to illustrate the developed theory. The research results in the literature are few and hence researchers are encouraged to develop pertinent results.

Part V
Switched Decentralized Control

Chapter 10
Switched Decentralized Control

There are real-world systems consisting of coupled units or subsystems that directly interact with each other in a simple and predictable fashion to serve a common pool of objectives. When viewed as a whole, the resulting overall system often displays rich and complex behavior. Typical examples are found in electric power systems with strong interactions, water networks, which are widely distributed in space, traffic systems with many external signal or large-space flexible structures, to name a few, which are often termed *large-scale* or *interconnected* systems. It becomes increasingly evidently that the underlying notions of interconnected systems manifest the complexity as an essential and dominating problem in systems theory and practice and that the many associated problems cannot be tackled using one-shot approaches. Recent research investigations have revealed [8] that the crucial need for improved methodologies relies on dividing the analysis and synthesis of the overall system into independent or almost independent subproblems, searches for new ideas of dealing with the incomplete information about the system, for treating with the uncertainties, and for dealing with delays. System complexity frequently leads to severe difficulties that are encountered in the tasks of analyzing, designing, and implementing appropriate control methods. These difficulties arise mainly from the following well-known reasons: *dimensionality; information structure constraints; uncertainty; delays*. Pertinent results can be found in [7, 9, 10, 141, 201, 348].

In this chapter, we address the problems of robust decentralized stability and stabilization of classes of nonlinear interconnected systems. This class of interconnected systems consists of coupled nominally linear subsystems with unknown-but-bounded state delay. In one section, we treat discrete-time systems with arbitrary switched rules. We showed that multi-controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems with large uncertainties. We developed a delay-dependent decentralized structure that guarantees asymptotic stability with local disturbance attenuation on the subsystem level. Then, we constructed decentralized switched control schemes based on state feedback and dynamic output feedback to ensure stabilizability of the global system with ℓ_2-performance bound.

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10.1 Interconnected Discrete-Time Systems

For classes of switched time-delay (STD) systems, some of the stability and stabilization methods are given in [183, 295, 297, 351, 432]. In [351] a design procedure based on proportional plus delay control is presented for a class of flexible structures possessing multiple modes. In [298] constant delay factor is considered which render the results delay independent under arbitrary switchings. The case of time-varying delay was addressed in [183, 295] using appropriate parameter-dependent (switched) Lyapunov–Krasovskii functionals (LKFs). However, the results ignored some useful negative terms while evaluating the difference of the respective terms in the proposed LKF. Output-feedback $\mathcal{H}_\infty$ control for a class of switched linear discrete-time systems with time-varying delays was developed in [432].

From the published results on STD systems, we conclude that the study of switched linear systems provides additional insights into some long-standing problems, such as robust, adaptive, and intelligent control, gain scheduling, or multirate digital control. The recent results in switched systems have benefited many real-world systems such as power systems, automotive control, air traffic control, network and congestion control. One important problem in uncertain switched systems is the design of switching rules, which guarantee quadratic stability and performance, and such switching rules must be independent of uncertainties. A state-dependent switching rule satisfying this requirement, which is called the min-projection strategy, is presented in [148, 330]. The min-projection strategy introduced in [331, 332] as a simple stabilization method for systems composed of several subsystems. This motivates the need of multi-controller switched schemes for large-scale complex systems when implementing low-order local controllers. However, all these references deal with a centralized switching rule.

It appears that the problems of stability analysis and control design interconnected switched systems with time-varying delays have not been fully resolved thus far. In this section, both problems are addressed where we consider the subsystems representing the lower-level local dynamics governed by delayed difference equations, while the supervisor is the high-level coordinator producing the switches among the local dynamics. The dynamics of the global system is therefore determined by both the subsystem and the switching signal, which may depend on the time, its own past value, the state/output, and/or possibly an external signal. We deal with the problem of low-order $\mathcal{H}_\infty$ state-feedback or output-feedback controller design with a decentralized switching rule for a class of switching discrete-time interconnected systems, where we extend further the results of [26, 44, 224–234, 239–243, 246–248, 251, 252, 259–261, 268, 270–272, 274, 275, 279–281, 283–286, 288–325, 327–329, 331–334, 421] to the class of switched discrete-time systems with unknown-but-bounded state delay. In our work, we showed that multi-controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems and/or systems with large uncertainties. We developed a delay-dependent decentralized structure that guarantees asymptotic stability with local disturbance attenuation on the subsystem level. Then, we constructed decentralized switched control schemes based on state-feedback and dynamic output feedback to ensure stabilizability of the global system with ℓ_2-performance bound.

10.1.1 Problem Statement and Preliminaries

A class of nonlinear interconnected discrete-time systems with state-delay Σ composed of N coupled subsystems $\Sigma_j, \quad j \in \mathcal{N} = \{1, \ldots, N\}$, which is represented by

$$
\begin{aligned}
\Sigma_j :\; x_j(k+1) &= A_{j\xi} x_j(k) + D_{j\xi} x_j(k - d_j(k)) + B_{j\xi} u_j(k) \\
&\quad + \Gamma_{j\xi} \omega_j(k) + g_j(k, x(k), x(k - d(k))) \\
y_j(k) &= C_{j\xi} x_j(k) + F_{j\xi} x_j(k - d_j(k)) + \Psi_{j\xi} \omega_j(k) \\
z_j(k) &= G_{j\xi} x_j(k) + H_{j\xi} x_j(k - d_j(k)) + \Phi_{j\xi} \omega_j(k)
\end{aligned} \tag{10.1}
$$

where $k \in \mathrm{I}_+ \stackrel{\Delta}{=} \{0, 1, ...\}$ and the scalar $d_j(k)$ a time-varying delay is unknown but lies within the range $0 < d_{jm} \leq d_j(k) \leq d_{jM}$ where the lower bound $d_{jm} > 0$ and the upper bound $d_{jM} > 0$ being known constant scalars and the function $\xi = \xi(x_j, k) : \Re \times \mathrm{I}_+ \rightarrow \mathcal{S} = \{1, 2, ..., S\}$ is a switching rule within subsystem Σ_j which takes its values in the finite set of modes $\mathcal{S}$. This rule is selected for all j such that $\xi(x_j, k) = s$ implies that the sth switching mode is activated for the jth subsystem of the interconnected system. In this way, the matrices $\{A_\alpha, D_\alpha, ..., \Phi_\alpha\}$ take values, at arbitrary discrete instants, in the finite set of

$$
\{(A_{j1}, D_{j1}, ..., \Phi_{j1}), (A_{j2}, D_{j2}, ..., \Phi_{j2}), ..., (A_{jS}, D_{jS}, ..., \Phi_{jS})\}
$$

Thus the matrices $(A_{js}, D_{js}, ..., \Phi_{js})$ denotes the sth model of local subsystem j corresponding to operational mode s and hence (10.1) represents a time-controlled switched system [42]. Typically, the switching rule ξ is not known a priori but we assume its instantaneous value is available in real time for practical implementations. Define the indicator function

$$
\begin{aligned}
\alpha(k) &= [\alpha_1(k), ..., \alpha_N(k)]^t, \quad \forall\, j \in \mathcal{N} \\
\alpha_s(k) &= \begin{cases} = 1 \text{ when the } j\text{th subsystem (10.1) is in the sth mode,} \\ \qquad\qquad (A_{js}, D_{js}, ..., \Phi_{js}) \\ = 0 \qquad\qquad \text{otherwise} \end{cases}
\end{aligned} \tag{10.2}
$$

It is obvious that $\alpha_i(k) : \mathrm{I}_+ \rightarrow \{0, 1\},\ \sum_{j=1}^{N} \alpha_j(k) = 1,\ k \in \mathrm{I}_+,\ i \in \mathcal{N}$. Now we cast system (10.1) into the form

$$
\begin{aligned}
\Sigma_j: \quad x_j(k+1) &= \sum_{i=1}^{N} \alpha_i(k) \Big[A_{ji} x_j(k) + D_{ji} x_j(k - d_j(k)) + B_{ji} u_j(k) \\
&\quad + \Gamma_{ji} \omega_j(k) + g_j(k, x(k), x(k - d(k))) \Big] \\
y_j(k) &= \sum_{i=1}^{N} \alpha_i(k) \Big[C_{ji} x_j(k) + F_{ji} x_j(k - d_j(k)) \\
&\quad + \Psi_{ji} \omega_j(k) \Big] \\
z_j(k) &= \sum_{i=1}^{N} \alpha_i(k) \Big[G_{ji} x_j(k) + H_{ji} x_j(k - d_j(k)) \\
&\quad + \Phi_{ji} \omega_j(k) \Big]
\end{aligned}
\tag{10.3}
$$

where upon relating the local subsystems to the global system, we have

$$x(k) = \left(x_1^t(k), \ldots, x_N^t(k)\right)^t \in \Re^n, \; n = \sum_{j=1}^{N} n_j$$

$$x(k - d(k)) = \left(x_1^t(k - d_1(k)), \ldots, x_N^t(k - d_N(k))\right)^t \in \Re^n, \; n = \sum_{j=1}^{N} n_j$$

$$u(k) = \left(u_1^t(k), ..., u_N^t(k)\right)^t \in \Re^p, \; p = \sum_{j=1}^{N} p_j$$

$$y(k) = \left(y_1^t(k), ..., y_N^t(k)\right)^t \in \Re^m, \; m = \sum_{j=1}^{N} m_j$$

$$z(k) = \left(z_1^t(k), ..., z_N^t(k)\right)^t \in \Re^q, \; q = \sum_{j=1}^{N} q_j$$

being the state, delayed state, control input, measured output, and performance output vectors of interconnected (global) system Σ and $\omega(k) = \left(\omega_1^t(k), ..., \omega_N^t(k)\right)^t \in \Re^q$ is the disturbance input, which is assumed to belong to $\ell_2[0, \infty)$. It is significant to observe in the foregoing setup that there are N distinct switching rules where each subsystem has been assigned one local state-dependent switching rule that operates independently from other rules.

The associated matrices are real constants and modeled as

$$
\begin{aligned}
\mathcal{A}_s &= \text{diag}\{A_{1s}, \ldots, A_{Ns}\}, \; A_{js} \in \Re^{n_j \times n_j} \\
\mathcal{B}_s &= \text{diag}\{B_{1s}, \ldots, B_{Ns}\}, \; B_{js} \in \Re^{n_j \times p_j}
\end{aligned}
$$

$$\mathcal{D}_s = \text{diag}\{D_{1s}, \ldots, D_{Ns}\}, \;\; D_{js} \in \Re^{n_j \times n_j}$$
$$\mathcal{C}_s = \text{diag}\{C_{1s}, \ldots, C_{Ns}\}, \;\; C_{js} \in \Re^{q_j \times n_j}$$
$$\mathcal{H}_s = \text{diag}\{H_{1s}, \ldots, H_{Ns}\}, \;\; H_{js} \in \Re^{q_j \times p_j}$$

The function $g_j : \mathbb{I}_+ \times \Re^n \times \Re^n \to \Re^{n_j}$ is a piecewise-continuous vector function in its arguments and it satisfies the quadratic inequality

$$g_j^t(k, x(k), x(k-d(k)))g_j(k, x(k), x(k-d(k))) \leq \phi_j^2 \; x^t(k)\widetilde{E}_j^t \widetilde{E}_j x(k) + \psi_j^2 \; x^t(k-d(k))\widetilde{E}_{dj}^t \widetilde{E}_{dj} x(k-d(k)) \tag{10.4}$$

where $\phi_j > 0, \;\; \psi_j > 0$ are bounding parameters such that

$$\widetilde{\Phi} = diag\left\{\phi_j^2 I_{r_1}, \ldots, \phi_j^2 I_{r_N}\right\}, \;\; \widetilde{\Psi} = diag\left\{\psi_j^2 I_{s_1}, \ldots, \psi_j^2 I_{s_N}\right\}$$

with I_{m_j} being the $m_j \times m_j$ identity matrix. From (10.4) and the notation

$$g(k, x(k), x(k-d(k))) = \left[g_1^t(k, x(k), x(k-d(k))), \ldots, g_N^t(k, x(k), x(k-d(k)))\right]^t$$

it is always possible to find matrices $\Phi, \;\Psi$ such that

$$g^t(k, x(k), x(k-d(k)))g(k, x(k), x(k-d(k))) \leq x^t(k)E^t\Phi^{-1}Ex(k) + x^t(k-d(k))E_d^t\Psi^{-1}E_d x(k-d(k)) \tag{10.5}$$

where $E = diag\{E_1, \ldots, E_N\}$, $E_d = diag\{E_{d1}, \ldots, E_{dN}\}$, $\delta_j = \phi_j^{-2}$, $\nu_j = \psi_j^{-2}$, $\Phi = diag\{\delta_1 I_{r_1}, \ldots, \delta_N I_{r_N}\}$, $\Psi = diag\{\nu_{d1} I_{s_1}, \ldots, \nu_{dN} I_{s_N}\}$ with $E_j \in \Re^{r_j \times n_j}$, $E_{dj} \in \Re^{s_j \times n_j}$. Letting $\xi(k) = [x^t(k) \;\; x^t(k-d_j(k)) \;\; g^t(k, x(k), x(k-d(k)))]^t \stackrel{\Delta}{=} \left[\xi_1^t, \ldots, \xi_N^t\right]^t$. Then (10.5) can be conveniently written as

$$\xi^t \begin{bmatrix} -E^t\Phi^{-1}E & 0 & 0 \\ \bullet & -\,E_d^t\Psi^{-1}E_d & 0 \\ \bullet & \bullet & I \end{bmatrix} \xi \leq 0 \tag{10.6}$$

and in view of the block structure of matrices, it turns out for Σ_j that

$$\xi_j^t \begin{bmatrix} -\delta_j^{-2}E_j^tE_j & 0 & 0 \\ \bullet & -\nu_j^{-2}E_{dj}^tE_{dj} & 0 \\ \bullet & \bullet & I_j \end{bmatrix} \xi_j \leq 0 \tag{10.7}$$

Remark 10.1 It is significant to note that class of systems (10.3) is quite general in the context of switched time-delay systems as it includes state, measurement, and output-delays. This class emerges in many areas dealing with the applications functional difference equations or delay-difference equations [216] while preserving the nonlinear character of the models. These applications include cold rolling mills, decision-making processes, and manufacturing systems. Related results for a class of discrete-time systems with time-varying delays can be found in [25] where delay-independent stability and stabilization conditions are derived. It should be stressed that although we consider only the case of single time delay, extension to multiple time-delay systems can be easily attained using an augmentation procedure.

Remark 10.2 This section essentially develops a good conceptual framework for multi-controller state-dependent switching structure among smooth controllers with all the effort and computations being performed on the subsystem level thereby providing an efficient decentralized feedback control design guaranteeing the level of disturbance attenuation for the overall interconnected systems. In this work, we consider that the modes are represented by discrete-time linear systems with unknown-but-bounded delays. The subsystems thus represent the lower-level local dynamics governed by difference equations, while the supervisor is the high-level coordinator producing the switches among the local dynamics. The dynamics of the system is determined by both the subsystem and the switching signal. In general, a switching signal may depend on the time, its own past value, the state/output, and/or possibly an external signal.

10.1.2 Decentralized ℓ_2 Gain Analysis

In this section, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and ℓ_2 gain analysis. Introduce

$$\begin{aligned}\delta x_j(k) = x_j(k+1) - x_j(k) &= (A_{ji} - I)x_j(k) + D_{ji}\, x_j(k - d_j(k)) \\ &\quad + B_{ji}u_j(k) + \Gamma_{ji}\omega_j(k) \\ x_j(k - d_j(k)) &= x_j(k) - \sum_{s=k-d_j(k)}^{k-1} \delta x_j(k), \quad d_{js} = (d_{jM} - d_{jm} + 1) \qquad (10.8)\end{aligned}$$

To facilitate the delay-dependent analysis and feedback design, we consider the following switched Lyapunov–Krasovskii functional (SLKF):

$$\begin{aligned}V(k) &= \sum_{j=1}^{N} V_j(k), \;\; V_j(k) = V_{jo}(k) + V_{ja}(k) + V_{jc}(k) + V_{jm}(k) \\ &\quad + V_{jn}(k) + V_{js}(k) \\ V_{jo}(k) &= x_j^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{P}_{ji} x_j(k)\end{aligned}$$

$$V_{ja}(k) = \sum_{s=k-d_j(k)}^{k-1} x_s^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{Q}_{ji} x_s(k)$$

$$V_{jc}(k) = \sum_{k-d_{jm}}^{k-1} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{R}_{ji} x_j(k) + \sum_{k-d_{jM}}^{k-1} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{S}_{ji} x_j(k)$$

$$V_{jm}(k) = \sum_{m=-d_{jM}+1}^{-d_{jm}} \sum_{j=k+m}^{k-1} x_j^t(m) \sum_{i=1}^{N} \alpha_i(k) \mathcal{Q}_{ji} x_j(m)$$

$$V_{jn}(k) = \sum_{m=-d_{jM}}^{-d_{jm}-1} \sum_{j=k+m}^{k-1} \delta x_j^t(m) \sum_{i=1}^{N} \alpha_i(k) \mathcal{W}_{jai} \delta x_j(m)$$

$$V_{js}(k) = \sum_{m=-d_{jM}}^{-1} \sum_{j=k+m}^{k-1} \delta x_j^t(m) \sum_{i=1}^{N} \alpha_i(k) \mathcal{W}_{jci} \delta x_j(m) \tag{10.9}$$

where $\mathcal{P}_{ji},\ \mathcal{Q}_{ji},\ \mathcal{R}_{ji},\ \mathcal{S}_{ji},\ \mathcal{W}_{jai},\ \mathcal{W}_{jci},\ j \in \mathcal{N},\ i \in \mathcal{S}$ are weighting matrices of appropriate dimensions.

Remark 10.3 Note in the local Lyapunov functional $V_j(k)$ of (10.9) that the first term is standard to the delayless nominal systems while the second term and the first part of the fifth term together correspond to the delay-dependent conditions. The second part of the third term and the fourth terms are added to compensate for the enlargement in the time interval from $(k-1 \to d-d_j(k))$ to $(k-1 \to d-d_{jM})$. The introduction of

$$\sum_{k-d_{jm}}^{k-1} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{R}_{ji} x_j(k)$$

$$\sum_{k-d_{jM}}^{k-1} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k) \mathcal{S}_{ji} x_j(k)$$

$$\sum_{m=-d_{jM}}^{-d_{jm}-1} \sum_{j=k+m}^{k-1} \delta x_j^t(m) \sum_{i=1}^{N} \alpha_i(k) \mathcal{W}_{jai} \delta x_j(m)$$

and

$$\sum_{m=-d_{jM}}^{-1} \sum_{j=k+m}^{k-1} \delta x_j^t(m) \sum_{i=1}^{N} \alpha_i(k) \mathcal{W}_{jci} \delta x_j(m)$$

plus appropriate free-weighting matrices (to be introduced later on) serve in reducing the number of manipulated variables, a feature that improves the performance of the developed delay-dependent stability criterion. This is quite evident upon comparison with the LKFs in [298, 432].

The following theorem establishes the main LMI-based stability result for switched system (10.3):

Theorem 10.4 *Given the bounds* $d_{jM} > 0,\ d_{jm} > 0,\ j = 1, ..., N$. *The global system* Σ *with subsystem* Σ_j *given by (10.3) is delay-dependent asymptotically stable with* ℓ_2*-performance bound* γ_j *is delay-dependent asymptotically stable with an* $\mathcal{L}_2 - gain < \gamma_j$ *if there exist weighting matrices if there exist weighting matrices* $\mathcal{P}_{ji},\ \mathcal{P}_{js},\ \mathcal{Q}_{ji},\ \mathcal{R}_{ji},\ \mathcal{S}_{ji},\ \mathcal{W}_{jai},\ \mathcal{W}_{jci}$, *parameter matrices* $L_{ja},\ L_{jc},\ M_{ja},\ M_{jc}, N_{ja},\ N_{jc},\ ,\ \forall (i,s) \in \mathcal{S},\ \forall j \in \mathcal{N}$ *and scalars* $\gamma_j > 0$ *satisfying the following LMIs*

$$\Pi_{jsi} = \begin{bmatrix} \widehat{\Omega}_{jsi} & \widehat{\Omega}_{jwi} & \widehat{\Omega}_{jxi} \\ \bullet & -\widehat{\Omega}_{jzi} & 0 \\ \bullet & \bullet & -I_j \end{bmatrix} < 0 \tag{10.10}$$

where

$$\widehat{\Omega}_{jsi} = \begin{bmatrix} \tilde{\Omega}_{jio} & \tilde{\Omega}_{jia} & M_{ja} & -L_{ja} & A^t_{ji}\mathcal{P}_{js} & \tilde{\Omega}_{jic} \\ \bullet & \tilde{\Omega}_{jie} & M_{jc} & -L_{jc} & D^t_{ji}\mathcal{P}_{js} & \tilde{\Omega}_{jis} \\ \bullet & \bullet & -\mathcal{R}_{ji} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S}_{ji} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \mathcal{P}_{js} - I_j & \mathcal{P}_{js}\Gamma_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \tilde{\Omega}_{jiv} \end{bmatrix}$$

$$\begin{aligned}
\widetilde{\Omega}_{jio} &= A^t_{ji}\mathcal{P}_{js}A_{ji} + d_{js}\mathcal{Q}_{ji} - \mathcal{P}_{ji} + \mathcal{R}_{ji} + \mathcal{S}_{ji} + \phi^2_j E^t_j E_j \\
&\quad + (d_{jM} - d_{jm})(A_{ji} - I_j)^t \mathcal{W}_{jai}(A_{ji} - I_j) \\
&\quad + d_{jM}(A^t_{ji} - I_j)\mathcal{W}_{jci}(A_{ji} - I_j) + N_{ja} + N^t_{ja} \\
\widehat{\Omega}_{jia} &= A^t_{ji}\mathcal{P}_{js}D_{ji} + (d_{jM} - d_{jm})(A_{ji} - I)^t \mathcal{W}_{jai}D_{ji} \\
&\quad + d_{jM}\left(A^t_{ji} - I_j\right)\mathcal{W}_{jci}D_{ji} + L_{ja} - M_{ja} - N_{ja} + N^t_{jc} \\
\widetilde{\Omega}_{jic} &= A^t_{ji}\mathcal{P}_{js}\Gamma_{ji} + (d_{jM} - d_{jm})(A_{ji} - I)^t \mathcal{W}_{jai}\Gamma_{ji} \\
&\quad + d_{jM}\left(A^t_{ji} - I_j\right)\mathcal{W}_{jci}\Gamma_{ji} \\
\widetilde{\Omega}_{jie} &= D^t_{ji}\mathcal{P}_{js}D_{ji} - \mathcal{Q}_{ji} + (d_{jM} - d_{jm})D^t_{ji}\mathcal{W}_{jai}D_{ji} + d_{jM}D^t_{ji}\mathcal{W}_{jci}D_{ji} \\
&\quad + L_{jc} + L^t_{jc} - M_{jc} - M^t_{jc} - N_{jc} - N^t_{jc} + \psi^2_j E^t_{dj}E_{dj} \\
\widetilde{\Omega}_{jis} &= D^t_{ji}\mathcal{P}_{js}\Gamma_{ji} + (d_{jM} - d_{jm})D^t_{ji}\mathcal{W}_{jai}\Gamma_{ji} + d_{jM}D^t_{ji}\mathcal{W}_{jci}\Gamma_{ji} \\
\widetilde{\Omega}_{jiv} &= \Gamma^t_{ji}\mathcal{P}_{js}\Gamma_{ji} + (d_{jM} - d_{jm})\Gamma^t_{ji}\mathcal{W}_{jai}\Gamma_{ji} \\
&\quad + d_{jM}\Gamma^t_{ji}\mathcal{W}_{jci}\Gamma_{ji} - \gamma^2_j I_j
\end{aligned} \tag{10.11}$$

$$\widehat{\Omega}_{jwi} = \begin{bmatrix} \sqrt{d_{jM} - d_{jm}} L_{jai} & \sqrt{d_{jM} - d_{jm}} M_{jai} & \sqrt{d_{jM}} N_{jai} \\ \sqrt{d_{jM} - d_{jm}} L_{jci} & \sqrt{d_{jM} - d_{jm}} M_{jci} & \sqrt{d_{jM}} N_{jci} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\Omega}_{jzi} = diag \left[\mathcal{W}_{jai} + \mathcal{W}_{jci} \; \mathcal{W}_{jai} \; \mathcal{W}_{jci} \right]$$

$$\widehat{\Omega}_{jxi} = \left[G_{ji} \; G_{jdi} \; 0 \; 0 \; 0 \; \Phi_{ji} \right]^t \tag{10.12}$$

Proof Consider the selective LKF (10.9). A straightforward computation gives the first difference of $\Delta V_j(k) = V_j(k+1) - V_j(k)$ along the solutions of (10.3) with $u_j(k) \equiv 0$ as

$$\begin{aligned} \Delta V_{jo}(k) &= x_j^t(k+1) \sum_{i=1}^N \alpha_i(k+1) \mathcal{P}_{ji} x_j(k+1) - x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{P}_{ji} x_j(k) \\ &= \Big[A_{ji} x_j(k) + D_{ji} x_j(k - d_j(k)) + \Gamma_{ji} \omega_j(k) + g_j \Big]^t \\ &\quad \times \sum_{i=1}^N \alpha_i(k+1) \mathcal{P}_{ji} \times \\ &\quad \Big[A_{ji} x_j(k) + D_{ji} x_j(k - d_j(k)) + \Gamma_{ji} \omega_j(k) + g_j \Big] \\ &\quad - x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{P}_{ji} x_j(k) \end{aligned} \tag{10.13}$$

Also,

$$\begin{aligned} \Delta V_{ja}(k) = & \sum_{j=k-d_j(k+1)+1}^{k} x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{Q}_{ji} x_j(k) \\ & - \sum_{j=k-d_j(k)}^{k-1} x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{Q}_{ji} x_j(k) \end{aligned}$$

Noting that

$$\begin{aligned} & \sum_{j=k-d_j(k+1)+1}^{k-1} x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{Q}_{ji} x_j(k) = \\ & \sum_{j=k-d_{jm}+1}^{k-1} x_j^t(k) \sum_{i=1}^N \alpha_i(k) \mathcal{Q}_{ji} x_j(k) + \end{aligned}$$

$$\sum_{j=k-d_j(k+1)+1}^{k-d_{jm}} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k) \le \sum_{j=k-d_j(k)+1}^{k-1} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k) + \sum_{j=k-d_{jM}+1}^{k-d_{jm}} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k)$$

Thus, we have

$$\begin{aligned}\Delta V_{ja}(k) \le\ & x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k) \\ & - x_j^t(k-d_j(k)) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k-d_j(k)) \\ & + \sum_{j=k-d_{jM}+1}^{k-d_{jm}} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k)\end{aligned} \tag{10.14}$$

Similarly,

$$\begin{aligned}\Delta V_{jc}(k) =\ & x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_{ji} x_j(k) \\ & - x_j^t(k-d_{jm}) \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_{ji} x_j(k-d_{jm}) \\ & + x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{S}_{ji} x_j(k) \\ & - x_j^t(k-d_{jM}) \sum_{i=1}^{N} \alpha_i(k)\mathcal{S}_{ji} x_j(k-d_{jM})\end{aligned} \tag{10.15}$$

In addition,

$$\begin{aligned}\Delta V_{jm}(k) =\ & (d_{jM}-d_{jm}) x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(k) \\ & - \sum_{r=k-d_{jM}+1}^{k-d_{jm}} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(r)\end{aligned} \tag{10.16}$$

$$\Delta V_{jn}(k) = (d_{jM} - d_{jm})\delta x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jai}\delta x_j(k) - \sum_{r=k-d_{jM}}^{k-d_{jm}-1} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jai}x_j(r) \tag{10.17}$$

$$\Delta V_{js}(k) = d_{jM}\delta x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jci}\delta x_j(k) - \sum_{r=k-d_{jM}}^{k-1} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jci}x_j(r) \tag{10.18}$$

Finally, from (10.13), (10.14), (10.15), (10.16), (10.17), and (10.18) we have

$$\begin{aligned}
&\Delta V_j(k) \le \\
&\Big[A_{ji}x_j(k) + D_{ji}x_j(k - d_j(k)) + \Gamma_{ji}\omega_j(k) + g_j\Big]^t \sum_{i=1}^{N} \alpha_i(k+1)\mathcal{P}_{ji} \\
&\times \Big[A_{ji}x_j(k) + D_{ji}x_j(k - d_j(k)) + \Gamma_{ji}\omega_j(k) + g_j\Big] \\
&- x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{P}_{ji}x_j(k) + x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji}x_j(k) \\
&- x_j^t(k - d_j(k)) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji}x_j(k - d_j(k)) \\
&+ \sum_{j=k-d_{jM}+1}^{k-d_{jm}} x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji}x_j(k) + x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_{ji}x_j(k) \\
&- x_j^t(k - d_{jm}) \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_{ji}x_j(k - d_{jm}) \\
&+ x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{S}_{ji}x_j(k) \\
&- x_j^t(k - d_{jM}) \sum_{i=1}^{N} \alpha_i(k)\mathcal{S}_{ji}x_j(k - d_{jM}) \\
&+ (d_{jM} - d_{jm})x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji}x_j(k)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{r=k-d_{jM}+1}^{k-d_{jm}} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_{ji} x_j(r) \\
&+ (d_{jM} - d_{jm})\delta x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jai} \delta x_j(k) \\
&- \sum_{r=k-d_{jM}}^{k-d_{jm}-1} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jai} x_j(r) \\
&+ d_{jM}\delta x_j^t(k) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jci} \delta x_j(k) \\
&- \sum_{r=k-d_{jM}}^{k-1} x_j^t(r) \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{jci} x_j(r)
\end{aligned}
\tag{10.19}
$$

Since (10.19) has to be satisfied under arbitrary switching, it follows that this holds for the particular case $\alpha_i(k) = 1$, $\alpha_{m\neq i}(k) = 0$, $\alpha_j(k+1) = 1$, and $\alpha_{m\neq j}(k+1) = 0$. This implies for all $\delta x_j(k) \geq 0$ that

$$
\begin{aligned}
\Delta V_j(k) \leq\ & \\
& (A_{ji}x_j(k) + D_{ji}x_j(k - d_j(k)) + \Gamma_{ji}\omega_j(k) + g_j)^t \mathcal{P}_{ji} \\
& \times (A_{ji}x_j(k) + D_{ji}x_j(k - d_j(k)) + \Gamma_{ji}\omega_j(k) + g_j) \\
& - x_j^t(k)\mathcal{P}_{ji}x_j(k) + x_j^t(k)\mathcal{Q}_{ji}x_j(k) \\
& - x_j^t(k - d_j(k))\mathcal{Q}_{ji}x_j(k - d_j(k)) \\
& + \sum_{j=k-d_{jM}+1}^{k-d_{jm}} x_j^t(k)\mathcal{Q}_{ji}x_j(k) + x_j^t(k)\mathcal{R}_{ji}x_j(k) \\
& - x_j^t(k - d_{jm})\mathcal{R}_{ji}x_j(k - d_{jm}) \\
& + x_j^t(k)\mathcal{S}_{ji}x_j(k) \\
& - x_j^t(k - d_{jM})\mathcal{S}_{ji}x_j(k - d_{jM}) \\
& + (d_{jM} - d_{jm})x_j^t(k)\mathcal{Q}_{ji}x_j(k) \\
& - \sum_{r=k-d_{jM}+1}^{k-d_{jm}} x_j^t(r)\mathcal{Q}_{ji}x_j(r) \\
& + (d_{jM} - d_{jm})\delta x_j^t(k)\mathcal{W}_{jai}\delta x_j(k) \\
& - \sum_{r=k-d_{jM}}^{k-d_{jm}-1} x_j^t(r)\mathcal{W}_{jai}x_j(r)
\end{aligned}
$$

$$+ d_{jM} \delta x_j^t(k) \mathcal{W}_{jci} \delta x_j(k) - \sum_{r=k-d_{jM}}^{k-1} x_j^t(r) \mathcal{W}_{jci} x_j(r) \qquad (10.20)$$

To facilitate the delay-dependence analysis, we consider the following identities:

$$[2x^t(k) L_{ja} + 2x^t(k - d_j(k)) L_{jc}][x_j(k) - x_j(k - d_j(k)) - \sum_{k-d_j(k)}^{k-1} \delta x_j(k)] = 0$$

$$[2x^t(k) M_{ja} + 2x^t(k - d_j(k)) M_{jc}] \times$$

$$[x_j(k - d_j(k)) - x_j(k - d_{jM}) - \sum_{j=k-d_{jM}}^{k-d_j(k)-1} \delta x_j(k)] = 0$$

$$[2x^t(k) N_{ja} + 2x^t(k - d_j(k)) N_{jc}][x_j(k) - x_j(k - d_{jM}) - \sum_{j=k-d_{jM}}^{k-1} \delta x_j(k)] = 0 \qquad (10.21)$$

for arbitrary parameter matrices $L_{ja}, ..., N_{jc}$. Adding up (10.20) to (10.21) with some algebraic manipulations leads to

$$\Delta V_j(k) \le \zeta_j^t(k) \tilde{\Upsilon}_{jsi} \zeta_j(k) \qquad (10.22)$$

where

$$\zeta_j(k) = \left[x_j^t(k) \; x_j^t(k - d_j(k)) \; x_j^t(k - d_{jm}) \; x_j^t(k - d_{jM}) \; g_j^t \; \omega^t(k) \right]^t$$

$$\tilde{\Upsilon}_{jsi} = \tilde{\Omega}_{jsi} + (d_{jM} - d_{jm}) \mathsf{L}_j (\mathcal{W}_{jai} + \mathcal{W}_{jci})^{-1} \mathsf{L}_j^t + (d_{jM} - d_{jm}) \mathsf{M}_j (\mathcal{W}_{jai})^{-1} \mathsf{M}_j^t + d_{jM} \mathsf{N}_j (\mathcal{W}_{jci})^{-1} \mathsf{N}_j^t$$

$$\tilde{\Omega}_{jsi} = \begin{bmatrix} \tilde{\Omega}_{jio} & \tilde{\Omega}_{jia} & M_{ja} & -L_{ja} & A_{ji}^t \mathcal{P}_{js} & \tilde{\Omega}_{jic} \\ \bullet & \tilde{\Omega}_{jie} & M_{jc} & -L_{jc} & D_{ji}^t \mathcal{P}_{js} & \tilde{\Omega}_{jis} \\ \bullet & \bullet & -\mathcal{R}_{ji} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S}_{ji} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \mathcal{P}_{js} & \mathcal{P}_{js} \Gamma_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \tilde{\Omega}_{jiv} \end{bmatrix}$$

$$\tilde{\Omega}_{jio} = A_{ji}^t \mathcal{P}_{js} A_{ji} + d_{js} \mathcal{Q}_{ji} - \mathcal{P}_{ji} + \mathcal{R}_{ji} + \mathcal{S}_{ji} + (d_{jM} - d_{jm})(A_{ji} - I)^t \mathcal{W}_{jai}(A_{ji} - I) + d_{jM}(A_{ji} - I)^t \mathcal{W}_{jci}(A_{ji} - I) + N_{ja} + N_{ja}^t$$

$$\tilde{\Omega}_{jia} = A_{ji}^t \mathcal{P}_{js} D_{ji} + (d_{jM} - d_{jm})(A_{ji} - I)^t \mathcal{W}_{jai} D_{ji} + d_{jM}(A_{ji} - I)^t \mathcal{W}_{jci} D_{ji} + L_{ja} - M_{ja} - N_{ja} + N_{jc}^t$$

$$\begin{aligned}
\tilde{\Omega}_{jic} &= A_{ji}^t \mathcal{P}_{js} \Gamma_{ji} + (d_{jM} - d_{jm})(A_{ji} - I)^t \mathcal{W}_{jai} \Gamma_{ji} + d_{jM}(A_{ji} - I)^t \mathcal{W}_{jci} \Gamma_{ji} \\
\tilde{\Omega}_{jie} &= D_{ji}^t \mathcal{P}_{js} D_{ji} - \mathcal{Q}_{ji} + (d_{jM} - d_{jm}) D_{ji}^t \mathcal{W}_{jai} D_{ji} \\
&\quad + d_{jM} D_{ji}^t \mathcal{W}_{jci} D_{ji} + L_{jc} + L_{jc}^t - M_{jc} - M_{jc}^t - N_{jc} + - N_{jc}^t \\
\tilde{\Omega}_{jis} &= D_{ji}^t \mathcal{P}_{js} \Gamma_{ji} + D_{ji}^t (\mathcal{W}_{jai} + \mathcal{W}_{jci}) \Gamma_{ji} \\
\tilde{\Omega}_{jiv} &= \Gamma_{ji}^t \mathcal{P}_{js} \Gamma_{ji} + (d_{jM} - d_{jm}) \Gamma_{ji}^t \mathcal{W}_{jai} \Gamma_{ji} \\
&\quad + d_{jM} \Gamma_{ji}^t \mathcal{W}_{jci} \Gamma_{ji}
\end{aligned} \tag{10.23}$$

$$\begin{aligned}
\mathsf{L}_j &= \left[L_{ja}^t \; L_{jc}^t \; 0\;0\;0\;0 \right]^t \\
\mathsf{M}_j &= \left[M_{ja}^t \; M_{jc}^t \; 0\;0\;0\;0 \right]^t \\
\mathsf{N}_j &= \left[N_{ja}^t \; N_{jc}^t \; 0\;0\;0\;0 \right]^t
\end{aligned} \tag{10.24}$$

It is known that the sufficient condition of subsystem stability is $\Delta V_j k < 0$ implies that $\tilde{\Upsilon}_{ji} < 0$, which is true for an unconstrained case. To include the effect of quadratic constraint on uncertainties, we resort to the **S** procedure [27] to express inequalities (10.22) and (10.7) together into the form

$$\begin{aligned}
&\mathcal{P}_{js} > 0, \quad \mathcal{P}_{ji} > 0, \quad \sigma_j \ge 0 \\
&\bar{\Upsilon}_{jsi} = \bar{\Omega}_{jsi} + (d_{jM} - d_{jm}) \mathsf{L}_j (\mathcal{W}_{jai} + \mathcal{W}_{jci})^{-1} \mathsf{L}_j^t \\
&\qquad + (d_{jM} - d_{jm}) \mathsf{M}_j (\mathcal{W}_{jai})^{-1} \mathsf{M}_j^t + d_{jM} \mathsf{N}_j (\mathcal{W}_{jci})^{-1} \mathsf{N}_j^t < 0 \\
&\bar{\Omega}_{jsi} = \\
&\begin{bmatrix}
\tilde{\Omega}_{jio} + \sigma_j \phi_j^2 E_j^t E_j & \tilde{\Omega}_{jia} & M_{ja} & -L_{ja} & A_{ji}^t \mathcal{P}_{js} & \tilde{\Omega}_{jic} \\
\bullet & \tilde{\Omega}_{jie} + \sigma_j \psi_j^2 E_{dj}^t E_{dj} & M_{jc} & -L_{jc} & D_{ji}^t \mathcal{P}_{js} & \tilde{\Omega}_{jis} \\
\bullet & \bullet & -\mathcal{R}_{ji} & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -\mathcal{S}_{ji} & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \mathcal{P}_{js} - I_j & \mathcal{P}_{js} \Gamma_{ji} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \tilde{\Omega}_{jiv}
\end{bmatrix}
\end{aligned} \tag{10.25}$$

which describes nonstrict LMIs since $\sigma_j \ge 0$. Recalling from [27] that minimization under nonstrict LMIs corresponds to the same result as minimization under strict LMIs when both strict and nonstrict LMI constraints are feasible. Moreover, if there is a solution for (10.25) for $\sigma_j = 0$, there will also be a solution for some $\sigma_j > 0$ and sufficiently small ϕ_j, ψ_j. Therefore, we safely replace $\sigma_j \ge 0$ by $\sigma_j > 0$. Equivalently, we may further rewrite (10.25) in the form

$$\begin{aligned}
&\bar{\mathcal{P}}_{js} > 0, \quad \bar{\mathcal{P}}_{ji} > 0 \\
&\widetilde{\Upsilon}_{jsi} = \widetilde{\Omega}_{jsi} + (d_{jM} - d_{jm}) \mathsf{L}_j (\mathcal{W}_{jai} + \mathcal{W}_{jci})^{-1} \mathsf{L}_j^t \\
&\qquad + (d_{jM} - d_{jm}) \mathsf{M}_j (\mathcal{W}_{jai})^{-1} \mathsf{M}_j^t + d_{jM} \mathsf{N}_j (\mathcal{W}_{jci})^{-1} \mathsf{N}_j^t < 0
\end{aligned}$$

$$\widetilde{\Omega}_{jsi} = \begin{bmatrix} \widetilde{\Omega}_{jio} + \phi_j^2 E_j^t E_j & A_{ji}^t \bar{\mathcal{P}}_{js} D_{ji} & 0 & A_{ji}^t \bar{\mathcal{P}}_{js} & A_{ji}^t \bar{\mathcal{P}}_{js} \Gamma_{ji} \\ \bullet & \widetilde{\Omega}_{jic} + \psi_j^2 E_{dj}^t E_{dj} & 0 & D_{ji}^t \bar{\mathcal{P}}_{js} & D_{ji}^t \bar{\mathcal{P}}_{js} \Gamma_{ji} \\ \bullet & \bullet & -\mathcal{R}_{ji} & 0 & 0 \\ \bullet & \bullet & \bullet & \bar{\mathcal{P}}_{js} - I_j & \bar{\mathcal{P}}_{js} \Gamma_{ji} \\ \bullet & \bullet & \bullet & \bullet & \Gamma_{ji}^t \bar{\mathcal{P}}_{js} \Gamma_{ji} \end{bmatrix}$$

$$\begin{aligned} \widetilde{\Omega}_{jio} &= A_{ji}^t \bar{\mathcal{P}}_{js} A_{ji} + d_{js} \bar{\mathcal{Q}}_{ji} - \bar{\mathcal{P}}_{ji} + \bar{\mathcal{R}}_{ji} \\ \widetilde{\Omega}_{jic} &= D_{ji}^t \bar{\mathcal{P}}_{js} D_{ji} - \bar{\mathcal{Q}}_{ji} \\ \widetilde{\Omega}_{jie} &= D_{ji}^t \bar{\mathcal{P}}_{js} D_{ji} - \mathcal{Q}_{ji} + D_{ji}^t (\bar{\mathcal{W}}_{jai} + \bar{\mathcal{W}}_{jci}) D_{ji} \\ \widetilde{\Omega}_{jis} &= D_{ji}^t \bar{\mathcal{P}}_{js} \Gamma_{ji} + D_{ji}^t (\bar{\mathcal{W}}_{jai} + \bar{\mathcal{W}}_{jci}) \Gamma_{ji} \\ \widetilde{\Omega}_{jiv} &= \Gamma_{ji}^t \bar{\mathcal{P}}_{js} \Gamma_{ji} (d_{jM} - d_{jm}) \Gamma_{ji}^t \bar{\mathcal{W}}_{jai} \Gamma_{ji} \\ &\quad + d_{jM} \bar{\mathcal{W}}_{jci} \mathcal{W}_{jci} \Gamma_{ji} \end{aligned} \tag{10.26}$$

where

$$\bar{\mathcal{P}}_{ji} = \sigma_j^{-1} \mathcal{P}_{ji}, \; \bar{\mathcal{P}}_{js} = \sigma_j^{-1} \mathcal{P}_{js}, \; \bar{\mathcal{Q}}_{ji} = \sigma_j^{-1} \mathcal{Q}_{ji},$$
$$\bar{\mathcal{R}}_{ji} = \sigma_j^{-1} \mathcal{R}_{ji}, \; \bar{\mathcal{W}}_{jai} = \sigma_j^{-1} \mathcal{W}_{jai}, \; \bar{\mathcal{W}}_{jci} = \sigma_j^{-1} \mathcal{W}_{jci}$$

By setting $G_{ji} \equiv 0$, $H_{ji} \equiv 0$, $\Phi_{ji} \equiv 0$, letting $\bar{\mathcal{P}}_{ji} \to \mathcal{P}_{ji}, ..., \bar{\mathcal{W}}_{jci} \to \mathcal{W}_{jci}$ without abuse of notations and applying Schur complements, it is readily seen that (10.10) corresponds to (10.26), which provides the robust stability of the nonlinear interconnected system (10.1) under the constraint (10.2) with maximal ϕ_j, ψ_j.

Next, consider the performance measure

$$J = \sum_{j=0}^{\infty} \left(z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \right)$$

For any $\omega_j(k) \in \ell_2(0, \infty) \neq 0$ and zero initial condition $x_{jo} = 0$ (hence $V_j(0) = 0$), we have

$$\begin{aligned} J &= \sum_{j=0}^{\infty} \left(z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \; + \Delta V_j(k)|_{(10.3)} - \sum_{j=0}^{\infty} \Delta V_j(k)|_{(10.3)} \right) \\ &= \sum_{j=0}^{\infty} \left(z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \; + \Delta V_j(k)|_{(10.3)} \right) - V_j(\infty) \\ &\le \sum_{j=0}^{\infty} \left(z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \; + \Delta V_j(k)|_{(10.3)} \right) \end{aligned} \tag{10.27}$$

where $\Delta V_j(k)|_{(10.3)}$ defines the Lyapunov difference along the solutions of system (10.3) with $u_j(k) \equiv 0$. On proceeding as before and considering

(10.1), (10.26), and (10.27), it can easily shown by algebraic manipulations that

$$z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \ + \varDelta V_j(k)|_{(10.3)} = \chi_j^t(k) \ \varPi_{jsi} \ \chi_j^t(k) \tag{10.28}$$

where $\varPi_{jsi}$ is given in (10.10) by Schur complements. It is readily seen that

$$z_j^t(k) z_j(k) - \gamma^2 \omega_j^t(k) \omega_j(k) \ + \varDelta V_j(k)|_{(10.3)} < 0$$

for arbitrary $j \in [0, \infty)$, which implies for any $\omega_j(k) \in \ell_2(0, \infty) \neq 0$ that $J < 0$. This eventually leads to $||z_j(k)||_2 < \gamma \ ||\omega_j(k)||_2$ and hence the proof is completed. ■

Remark 10.5 As we learnt from the last chapter on water-quality application, the lower bound d_m and the upper bound d_M account for extreme cases of light and heavy waste dump loadings, respectively. System stability and stabilization of water and related resources systems are generally expressed in terms of algebraic Riccati inequalities (ARIs). Seeking computational convenience and effectiveness, the solutions to the problems of stability analysis and control synthesis are cast into convex optimization in terms of linear matrix inequalities (LMIs) that are handled using interior-point minimization algorithms. These algorithms have been recently coded into efficient numerical software [74]. It is remarked that LMIs and ARIs are equivalent [27]; however, parameter tuning intrinsic to the ARIs can be avoided by using the framework of feasibility testing of LMIs.

Remark 10.6 In comparison with the published results on stability methods for switched time-delay systems [174, 183, 295, 298, 432], it is significant to notice that **Theorem 10.4** has a more general setting as it deals with time-varying delays while [183, 298] deal with constant delay. In addition, it provides least-conservative LMI-based delay-dependent stability criteria and it employs reduced number of LMI variables. It can be calculated that the condition of **Theorem 10.4** involves $6n^2 + 6n$ variables as opposed to $16.5n^2 + 1.5n$ required by [295]. Moreover, it does not rely on overbounding relations and inequalities as used in [432] but rather deploys finite number of free-weighting matrices.

10.1.3 Switched State-Feedback Control

Next, we address the feedback control problem for the interconnected discrete-time systems $\varSigma$ by focusing the design effort on the subsystem $\varSigma_j$ as given by (10.3). The goal is to find global decentralized feedback switching controllers and a decentralized switching rule asymptotically stabilizing the system formed by subsystems $\varSigma_j, \ i \in \mathcal{S}, \ j \in \mathcal{N}$. This decentralized feedback controllers are composed of N local feedback controllers and each equipped with the corresponding local switching rule. In the sequel, a decentralized switched scheme is considered based on state-feedback measurements.

With reference to system (10.3), we seek to design a switched state feedback

$$u_j(k) = \sum_{i=1}^{N} \alpha_i(k) K_{ji}\ x_j(k), \quad i \in \mathcal{S}, \ j \in \mathcal{N}$$

that guarantees the controlled switched system achieves a prescribed performance level, where $K_{ji} \in \Re^{p_j \times n_j}$ is the local state-feedback gain matrix at the mode i. Letting $\mathcal{A}_{ji} = A_{ji} + B_{ji} K_{ji}$, it is readily seen from **Theorem 10.4** that the closed-loop switched system

$$\begin{aligned} \Sigma_j: \quad x_j(k+1) &= \sum_{i=1}^{N} \alpha_i(k) \Big[\mathcal{A}_{ji} x_j(k) + D_{ji} x_j(k - d_j(k)) + \Gamma_{ji} \omega_j(k) \\ &\quad + g_j(k, x(k), x(k-d(k))) \Big] \\ z_j(k) &= \sum_{i=1}^{N} \alpha_i(k) \Big[G_{ji} x_j(k) + H_{ji} x_j(k - d_j(k)) + \Phi_{ji} \omega_j(k) \Big] \end{aligned} \tag{10.29}$$

is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_j$ if there exist weighting matrices $\mathcal{P}_{ji}, \ \mathcal{P}_{js}, \ \mathcal{Q}_{ji}, \ \mathcal{R}_{ji}, \ \mathcal{W}_{jai}, \mathcal{W}_{jci},$ parameter matrices $L_{ja}, \ L_{jc}, \ M_{ja}, \ M_{jc}, \ N_{ja}, \ N_{jc}, \ \forall (i,s) \in \mathcal{S}, \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs

$$\breve{\Pi}_{jsi} = \begin{bmatrix} \breve{\Omega}_{jsi} & \widetilde{\Omega}_{jwi} & \widetilde{\Omega}_{jxi} \\ \bullet & -\widetilde{\Omega}_{jzi} & 0 \\ \bullet & \bullet & -I_j \end{bmatrix} < 0 \tag{10.30}$$

where

$$\breve{\Omega}_{jsi} = \begin{bmatrix} \breve{\Omega}_{jio} & \breve{\Omega}_{jia} & M_{ja} & -L_{ja} & A^t_{ji} \mathcal{P}_{js} & 0 \\ \bullet & -\breve{\Omega}_{jie} & M_{jc} & -L_{jc} & D^t_{ji} \mathcal{P}_{js} & 0 \\ \bullet & \bullet & -\mathcal{R}_{ji} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S}_{ji} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \mathcal{P}_{js} - I_j & \mathcal{P}_{js} \Gamma_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix}$$

$$\breve{\Pi}_{jsi} = \begin{bmatrix} \mathcal{A}^t_{ji} & \phi_j E^t_j & 0 & \sqrt{d_{jM} - d_{jm}}(\mathcal{A}^t_{ji} - I_j) & \sqrt{d_{jM}}(\mathcal{A}^t_{ji} - I_j) \\ D^t_{ji} & 0 & \psi_j E^t_{dj} & \sqrt{d_{jM} - d_{jm}} D^t_{ji} & \sqrt{d_{jM}} D^t_{ji} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Gamma^t_{ji} & 0 & 0 & \sqrt{d_{jM} - d_{jm}} \Gamma^t_{ji} & \sqrt{d_{jM}} \Gamma^t_{ji} \end{bmatrix}$$

$$\breve{\Pi}_{jsi} = \text{diag}\left[\mathcal{P}^{-1}_{js} \; I_j \; I_j \; \mathcal{W}^{-1}_{jai} \; \mathcal{W}^{-1}_{jci} \right]$$

$$\breve{\Omega}_{jio} = d_{js}\mathcal{Q}_{ji} - \mathcal{P}_{ji} + \mathcal{R}_{ji} + \mathcal{S}_{ji} + N_{ja} + N^t_{ja}$$

$$\breve{\Omega}_{jia} = L_{ja} - M_{ja} - N_{ja} + N^t_{jc}$$

$$\breve{\Omega}_{jie} = \mathcal{Q}_{ji} - L_{jc} - L^t_{jc} + M_{jc} + M^t_{jc} + N_{jc} + N^t_{jc} \tag{10.31}$$

where $\widetilde{\Omega}_{jwi}, ..., \widetilde{\Omega}_{jzi}$ are given by (10.65). It is convenient for feedback design to work with LMI (11.8) as it includes explicit terms of the closed-loop system matrix $\mathcal{A}_{ji}$, a feature which paves the way to systematic convex analysis leading to the determination of the unknown gain matrices $\{K_{ji}\}$, $i \in \mathcal{S}$. Based thereon, the following theorem states the main design result:

Theorem 10.7 *Given the bounds $d_{jM} > 0$, $d_{jm} > 0$, $j \in \mathcal{N}$. Then system (10.3) with state-feedback controller $u_j(k) = \sum_{i=1}^{N} \alpha_i(k) K_{ji}\ x_j(k)$, $i \in \mathcal{S}$, $j \in \mathcal{N}$ is delay-dependent asymptotically stable with ℓ_2-performance bound γ if there exist parameter matrices $\mathcal{X}_{ji}, \mathcal{Y}_{ji}, \Theta_{jqi}, \Theta_{jri}, \Theta_{jsi}, \Theta_{jwai}, \Theta_{jwci}, \Theta_{jla}, \Theta_{jlc}, \Theta_{jma}, \Theta_{jmc}, \Theta_{jna}, \Theta_{jnc}$ and scalars $\gamma_j > 0$, $\forall (j, s) \in \mathcal{N}$, $\forall (i) \in \mathcal{S}$ satisfying the following LMIs*

$$\breve{\Pi}_{jsi} = \begin{bmatrix} \widetilde{\Upsilon}_{joi} & \widetilde{\Upsilon}_{jai} & \widetilde{\Upsilon}_{jci} & \widetilde{\Upsilon}_{jei} \\ \bullet & -\widetilde{\Upsilon}_{jsi} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\widetilde{\Upsilon}_{jvi} \end{bmatrix} < 0 \tag{10.32}$$

where

$$\widetilde{\Upsilon}_{joi} = \begin{bmatrix} \Theta_{ji1} & \Theta_{ji2} & \Theta_{jma} & -\Theta_{jla} & \mathcal{X}_{js} A^t_{ji} + \mathcal{Y}^t_{ji} B^t_{ji} & 0 \\ \bullet & -\Theta_{ji3} & \Theta_{jmc} & -\Theta_{jlc} & \mathcal{X}_{js} D^t_{ji} & 0 \\ \bullet & \bullet & -\Theta_{jri} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\Theta_{jsi} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{X}_{js} + I_j & \Gamma_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma^2_j I_j \end{bmatrix}$$

$$\Theta_{jiq} = \left[\Theta_{jiq1} \; \Theta_{jiq2} \right]$$

$$\Theta_{jiq1} = \begin{bmatrix} \mathcal{X}_{js}A^t_{ji} + \mathcal{Y}^t_{ji}B^t_{ji} & \phi_j \mathcal{X}_{js}E^t_j & 0 \\ \mathcal{X}_{js}D^t_{ji} & 0 & \psi_j \mathcal{X}_{js}E^t_{dj} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Gamma^t_{ji} & 0 & 0 \end{bmatrix}$$

$$\Theta_{jiq2} =$$

$$\begin{bmatrix} \sqrt{d_{jM} - d_{jm}}(\mathcal{X}_{js}A^t_{ji} - \mathcal{X}_{js} + \mathcal{Y}^t_{ji}B^t_{ji}) & \sqrt{d_{jM}}(\mathcal{X}_{js}A^t_{ji} - \mathcal{X}_{js} + \mathcal{Y}^t_{ji}B^t_{ji}) \\ \sqrt{d_{jM} - d_{jm}}\mathcal{X}_{js}D^t_{ji} & \sqrt{d_{jM}}\mathcal{X}_{js}D^t_{ji} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sqrt{d_{jM} - d_{jm}}\Gamma^t_{ji} & \sqrt{\mathrm{d}_{jM}}\Gamma^t_{ji} \end{bmatrix}$$

$$\breve{\Pi}_{jti} = \mathrm{diag}\left[\, \mathcal{X}_{js} \;\; I_j \;\; I_j \;\; 2\mathcal{X}_{js} - \Theta_{jwa} \;\; 2\mathcal{X}_{js} - \Theta_{jwc} \,\right]$$

$$\Theta_{ji1} = d_{js}\Theta_{jqi} - \mathcal{X}_{ji} + \Theta_{jri} + \Theta_{jsi} + \Theta_{jna} + \Theta^t_{jna}$$

$$\Theta_{ji2} = \Theta_{jla} - \Theta_{jma} - \Theta_{jna} + \Theta^t_{jnc},$$

$$\Theta_{ji3} = \Theta_{jqi} - \Theta_{jlc} - \Theta^t_{jlc} + \Theta_{jmc} + \Theta^t_{jmc} + \Theta_{jnc} + \Theta^t_{jnc}$$

$$\widetilde{\Upsilon}_{jai} = \begin{bmatrix} \sqrt{d_{jM} - d_{jm}}\Theta_{jla} & \sqrt{d_{jM} - d_{jm}}\Theta_{jma} & \sqrt{d_{jM}}\Theta_{jna} \\ \sqrt{d_{jM} - d_{jm}}\Theta_{jlc} & \sqrt{d_{jM} - d_{jm}}\Theta_{jmc} & \sqrt{d_{jM}}\Theta_{jnc} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{\Upsilon}_{jci} = \mathrm{diag}\left[\, \Theta_{jwai} + \Theta_{jwci} \;\; \Theta_{jwai} \;\; \Theta_{jwci} \,\right]$$

$$\widetilde{\Upsilon}_{jsi} = \left[\, \mathcal{X}_{js}G_{ji} \;\; \mathcal{X}_{js}G_{jdi} \;\; 0 \; 0 \; 0 \;\; \Phi_{ji} \,\right]^t \tag{10.33}$$

Moreover, the feedback gain matrix is given by $K_{ji} = \mathcal{Y}_{ji}\mathcal{X}^{-1}_{js}$

Proof Define $\mathcal{X}_{js} = \mathcal{P}^{-1}_{js}$ and using the congruent transformation

$$T_{js} = \mathrm{diag}[T_{js1} \quad T_{js2}]$$

$$T js1 = \mathrm{diag}[\mathcal{X}_{js},\; \mathcal{X}_{js},\; \mathcal{X}_{js},\; \mathcal{X}_{js},\; \mathcal{X}_{js},\; I_j],$$

$$T js2 = \mathrm{diag}[I_j,\; \mathcal{X}_{js},\; \mathcal{X}_{js},\; \mathcal{X}_{js},\; I_j,\; \mathcal{X}_{js},\; I_j,\; I_j,\; I_j,\; I_j][-6pt]$$

into LMI (10.30) using (10.31) along with the linearizations

$$\Theta_{jqi} = \mathcal{X}_{js}\mathcal{Q}_{ji}\mathcal{X}_{js},\;\; \Theta_{jri} = \mathcal{X}_{js}\mathcal{R}_{ji}\mathcal{X}_{js},\;\; \Theta_{jsi} = \mathcal{X}_{js}\mathcal{S}_{ji}\mathcal{X}_{js},$$

$$\Theta_{jna} = \mathcal{X}_{js}N_{ja}\mathcal{X}_{js},\;\; \Theta_{jnc} = \mathcal{X}_{js}N_{jc}\mathcal{X}_{js},\;\; \Theta_{jla} = \mathcal{X}_{js}L_{ja}\mathcal{X}_{js},$$

$$\Theta_{jlc} = \mathcal{X}_{js}L_{jc}\mathcal{X}_{js},\;\; \Theta_{jma} = \mathcal{X}_{js}M_{ja}\mathcal{X}_{js},\;\; \Theta_{jmc} = \mathcal{X}_{js}M_{jc}\mathcal{X}_{js},$$

$$\Theta_{jwai} = \mathcal{X}_{js}\mathcal{W}_{jai}\mathcal{X}_{js},\;\; \Theta_{jwci} = \mathcal{X}_{js}\mathcal{W}_{jci}\mathcal{X}_{js}, \tag{10.34}$$

We then deploy the algebraic inequalities

$$(\mathcal{X}_{js} - \mathcal{X}_{js}\mathcal{W}_{jai}\mathcal{X}_{js})^t(\mathcal{X}_{js}\mathcal{W}_{jai}\mathcal{X}_{js})^{-1}(\mathcal{X}_{js} - \mathcal{X}_{js}\mathcal{W}_{jai}\mathcal{X}_{js}) \geq 0$$

which leads to

$$\mathcal{W}_{jai}^{-1} \geq 2\mathcal{X}_{js} - \Theta_{jwai}$$

and similarly

$$\mathcal{W}_{jci}^{-1} \geq 2\mathcal{X}_{js} - \Theta_{jwci}$$

We finally cast $T_{js}^t \breve{\Pi}_{jsi} T_{js}$ into the LMI (10.32) with (10.33) as desired. ■

10.1.4 Switched Dynamic Output Feedback

Next, we direct attention to the design of a switched dynamic output feedback of the form

$$\begin{aligned} \hat{x}_j(k+1) &= \sum_{i=1}^{N} \alpha_i(k) L_{ji} \hat{x}_j(k) + \sum_{i=1}^{N} \alpha_i(k) K_{oji} y_j(k) \\ u_k &= \sum_{i=1}^{N} \alpha_i(k) K_{cji} \hat{x}_j(k) \end{aligned} \tag{10.35}$$

where $\hat{x}_j \in \Re^{s_j}$ is the controller state vector and $L_{ji} \in \Re^{s_j \times s_j}$, $K_{oji} \in \Re^{s_j \times p_j}$, $K_{cji} \in \Re^{m_j \times s_j}$ are the unknown controller gain matrices. Observe that the controller (10.35) is a general linear switched dynamic system. Connecting this controller to system (10.3) yields the augmented switched closed-loop system:

$$\begin{aligned} \xi_j(k+1) &= \sum_{i=1}^{N} \alpha_i(k) \Big[\mathbf{A}_{ji} \xi_k + \mathbf{D}_{ji} x_{k-d_k} + \widehat{\Gamma}_{ji} \omega_k \\ &\quad + \widehat{g}_j(k, x(k), x(k-d(k))) \Big] \\ z_j(k) &= \sum_{i=1}^{N} \alpha_i(k) \widehat{G}_{ji} \xi_k + \sum_{i=1}^{N} \alpha_i(k) \widehat{H}_{ji} \xi_{k-d_k} + \sum_{i=1}^{N} \alpha_i(k) \Phi_{ji} \omega_k \end{aligned} \tag{10.36}$$

where

$$\xi_j(k) = \begin{bmatrix} x_j(k) \\ \hat{x}_j(k) \end{bmatrix}, \ \mathbf{A}_{ji} = \begin{bmatrix} A_{ji} & B_{ji} K_{cji} \\ K_{oji} C_{ji} & L_{ji} \end{bmatrix}, \ \mathbf{D}_{ji} = \begin{bmatrix} D_{ji} & 0 \\ K_{oji} F_{ji} & 0 \end{bmatrix}$$

$$\widehat{\Gamma}_{ji} = \begin{bmatrix} \Gamma_{ji} \\ K_{oji}\Psi_{ji} \end{bmatrix}, \ \widehat{G}_{ji} = \begin{bmatrix} G_{ji} \ 0 \end{bmatrix}, \ \widehat{H}_{ji} = \begin{bmatrix} H_{ji} \ 0 \end{bmatrix}$$
$$\widehat{g}_j(k, x(k), x(k-d(k))) = \begin{bmatrix} g_j(k, x(k), x(k-d(k))) \\ 0 \end{bmatrix} \tag{10.37}$$

The objective now is to determine the controller

$$\mathcal{K}_{ji} = \begin{bmatrix} L_{ji} & K_{oji} \\ K_{cji} & 0 \end{bmatrix} \in \Re^{(s_j+m_j)\times(s_j+p_j)}$$

such that the feedback controlled system (10.36) is delay-dependent asymptotically stable with an $\mathcal{L}_2-$ gain $< \gamma_j$. It is worth mentioning that the determination of the controller $\mathcal{K}_i$ by several methods [27, 295, 354, 432] using suitable transformations. In the sequel, we use a systematic convex procedure to determine the gains $L_{ji}, \ K_{oji}, \ K_{cji}$ by elaborating on the results of **Theorem 10.4**. For this purpose, we introduce the following block matrices

$$\widetilde{\mathcal{P}}_{ji} = \begin{bmatrix} \mathcal{P}_{aji} & 0 \\ 0 & \mathcal{P}_{cji} \end{bmatrix}, \ \widetilde{\mathcal{Q}}_{ji} = \begin{bmatrix} \mathcal{Q}_{aji} & 0 \\ 0 & \mathcal{Q}_{cji} \end{bmatrix}, \ \widetilde{\mathcal{R}}_{ji} = \begin{bmatrix} \mathcal{R}_{aji} & 0 \\ 0 & \mathcal{R}_{cji} \end{bmatrix}$$
$$\widetilde{\mathcal{W}}_{aji} = \begin{bmatrix} \mathcal{W}_{vji} & 0 \\ 0 & \mathcal{W}_{wji} \end{bmatrix}, \ \widetilde{\mathcal{W}}_{cji} = \begin{bmatrix} \mathcal{W}_{rji} & 0 \\ 0 & \mathcal{W}_{sji} \end{bmatrix}, \ \widetilde{\mathcal{M}}_j = \begin{bmatrix} \mathcal{M}_{aj} & 0 \\ 0 & \mathcal{M}_{cj} \end{bmatrix}$$
$$\widetilde{\mathcal{S}}_j = \begin{bmatrix} \mathcal{S}_{aj} & 0 \\ 0 & \mathcal{S}_{cj} \end{bmatrix}, \ \widetilde{\mathcal{Z}}_j = \begin{bmatrix} \mathcal{Z}_{aj} & 0 \\ 0 & \mathcal{Z}_{cj} \end{bmatrix}$$
$$\mathbf{X}_{js} = \widetilde{\mathcal{P}}_{js}^{-1} = \begin{bmatrix} \mathbf{X}_{ajs} & 0 \\ 0 & \mathcal{X}_{cjs} \end{bmatrix} \tag{10.38}$$

Remark 10.8 Note that the block form of the weighting matrices is used, without loss of generality, for the purpose of preserving LMIs as a basis for computing the unknown gain matrices. Our experience has indicated that nonblock diagonal matrices tend generally to yield nonlinear matrix inequalities requiring iterative algebraic equations that are computationally demanding.

Now, in line with the preceding section, it follows from **Theorem 10.11** and in the manner of the foregoing section that augmented switched system (10.36) is delay-dependent asymptotically stable if there exist weighting matrices $\widetilde{\mathcal{P}}_{ji}, \ \widetilde{\mathcal{P}}_{js}, \ \widetilde{\mathcal{Q}}_{ji}, \ \widetilde{\mathcal{R}}_{ji}, \ \widetilde{\mathcal{S}}_{ji}, \ \widetilde{\mathcal{W}}_{aji}, \ \widetilde{\mathcal{W}}_{cji},$ parameter matrices $\widetilde{L}_{ja}, \ \widetilde{L}_{jc}, \ \widetilde{M}_{ja}, \ \widetilde{M}_{jc}, \ \widetilde{N}_{ja}, \ \widetilde{N}_{jc}, \forall (i,s) \in \mathcal{S}, \ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs

$$\Sigma_{jsi} = \begin{bmatrix} \hat{\Sigma}_{jsi} & \hat{\Sigma}_{jwi} & \hat{\Sigma}_{jxi} & \check{\Pi}_{jsi} \\ \bullet & -\hat{\Sigma}_{jzi} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\check{\Pi}_{jti} \end{bmatrix} < 0 \tag{10.39}$$

where

$$\hat{\Sigma}_{jsi} = \begin{bmatrix} \tilde{\Sigma}_{jio} & \tilde{\Sigma}_{jia} & \widehat{M}_{ja} & -\widehat{L}_{ja} & \mathbf{A}^t_{ji} & 0 \\ \bullet & -\tilde{\Sigma}_{jie} & \widehat{M}_{jc} & -\widehat{L}_{jc} & \mathbf{D}^t_{ji} & 0 \\ \bullet & \bullet & -\widetilde{\mathcal{R}}_{ji} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\widetilde{\mathcal{S}}_{ji} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \widetilde{\mathcal{P}}^{-1}_{js} - I_j & \widehat{\Gamma}_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix}$$

$$\check{\Pi}_{jsi} = \begin{bmatrix} \mathbf{A}^t_{ji} & \phi_j \widetilde{E}^t_j & 0 & \sqrt{d_{jM} - d_{jm}}\left(\mathbf{A}^t_{ji} - I_j\right) & \sqrt{d_{jM}}\left(\mathbf{A}^t_{ji} - I_j\right) \\ \mathbf{D}^t_{ji} & 0 & \psi_j \widetilde{E}^t_{dj} & \sqrt{d_{jM} - d_{jm}} D^t_{ji} & \sqrt{d_{jM}}\mathbf{D}^t_{ji} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \widehat{\Gamma}^t_{ji} & 0 & 0 & \sqrt{d_{jM} - d_{jm}}\widehat{\Gamma}^t_{ji} & \sqrt{d_{jM}}\widehat{\Gamma}^t_{ji} \end{bmatrix}$$

$$\check{\Pi}_{jti} = \text{diag}\left[\widetilde{\mathcal{P}}^{-1}_{js} \; I_j \; I_j \; \widetilde{\mathcal{W}}^{-1}_{jai} \; \widetilde{\mathcal{W}}^{-1}_{jci} \right]$$

$$\tilde{\Sigma}_{jio} = d_{js}\widetilde{\mathcal{Q}}_{ji} - \widetilde{\mathcal{P}}_{ji} + \widetilde{\mathcal{R}}_{ji} + \widetilde{\mathcal{S}}_{ji} + \widehat{N}_{ja} + \widehat{N}^t_{ja}$$

$$\tilde{\Sigma}_{jia} = \widehat{L}_{ja} - \widehat{M}_{ja} - \widehat{N}_{ja} + \widehat{N}^t_{jc}$$

$$\tilde{\Sigma}_{jie} = \widetilde{\mathcal{Q}}_{ji} - \widehat{L}_{jc} - \widehat{L}^t_{jc} + \widehat{M}_{jc} + \widehat{M}^t_{jc} + \widehat{N}_{jc} + \widehat{N}^t_{jc}$$

$$\hat{\Sigma}_{jwi} = \begin{bmatrix} \sqrt{d_{jM} - d_{jm}}\widehat{L}_{jai} & \sqrt{d_{jM} - d_{jm}}\widehat{M}_{jai} & \sqrt{d_{jM}}\widehat{N}_{jai} \\ \sqrt{d_{jM} - d_{jm}}\widehat{L}_{jci} & \sqrt{d_{jM} - d_{jm}}\widehat{M}_{jci} & \sqrt{d_{jM}}\widehat{N}_{jci} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\Sigma}_{jxi} = \text{diag}\left[\widetilde{\mathcal{W}}_{jai} + \widetilde{\mathcal{W}}_{jci} \; \widetilde{\mathcal{W}}_{jai} \; \widetilde{\mathcal{W}}_{jci} \right]$$

$$\hat{\Sigma}_{jzi} = \left[\widehat{G}_{ji} \; \widehat{H}_{ji} \; 0 \; 0 \; 0 \; \Phi_{ji} \right]^t \tag{10.40}$$

and

$$\widehat{L}_{jai} = \begin{bmatrix} L_{jai1} \\ L_{jai2} \end{bmatrix}, \; \widehat{L}_{jci} = \begin{bmatrix} L_{jci1} \\ L_{jci2} \end{bmatrix}, \; \widehat{M}_{jai} = \begin{bmatrix} M_{jai1} \\ M_{jai2} \end{bmatrix}$$

$$\widehat{M}_{jci} = \begin{bmatrix} M_{jci1} \\ M_{jci2} \end{bmatrix}, \; \widehat{N}_{jai} = \begin{bmatrix} N_{jai1} \\ N_{jai2} \end{bmatrix}, \; \widehat{N}_{jci} = \begin{bmatrix} N_{jai1} \\ N_{jai2} \end{bmatrix} \tag{10.41}$$

The following theorem states the main design result

Theorem 10.9 *Given the bounds $d_{jM} > 0$, $d_{jm} > 0$, $j \in \mathcal{N}$. Then system (10.3) with dynamic output-feedback controller (10.35). Then system (10.3) with dynamic output-feedback controller (10.35) is delay-dependent asymptotically stable with ℓ_2-performance bound γ if there exist parameter matrices $\mathbf{X}_{aji}$, $\mathbf{X}_{cji}$, $\mathbf{Y}_{eji}$, $\mathbf{Y}_{fji}$,*

$\mathbf{Y}_{sji}, \{\widehat{\Theta}_{ji1}, ..., \widehat{\Theta}_{ji22}\}$ *and scalars* $\gamma_j > 0, \ \forall (j,s) \in \mathcal{N}, \ \forall (i) \in \mathcal{S}$ *satisfying the following LMIs*

$$\overline{\Sigma}_{jsi} = \begin{bmatrix} \widehat{\Sigma}_{joi} & \widehat{\Sigma}_{jwi} & \widehat{\Sigma}_{jxi} & \widehat{\Sigma}_{jpi} \\ \bullet & -\widehat{\Sigma}_{jzi} & 0 & 0 \\ \bullet & \bullet & -I_j & 0 \\ \bullet & \bullet & \bullet & -\widehat{\Sigma}_{jqi} \end{bmatrix} < 0 \tag{10.42}$$

where

$$\widetilde{\Sigma}_{joi} = \begin{bmatrix} \widetilde{\Theta}_{ji1} & \widetilde{\Theta}_{ji2} & \widehat{\Theta}_{jmai} & -\widehat{\Theta}_{jlai} & \widehat{\Sigma}_{jaxi} & 0 \\ \bullet & -\widetilde{\Theta}_{ji3} & \widehat{\Theta}_{jmci} & -\widehat{\Theta}_{jlci} & \widehat{\Sigma}_{jdxi} & 0 \\ \bullet & \bullet & -\widehat{\Theta}_{jri} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\widehat{\Theta}_{jsi} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \mathbf{X}_{js} - I_j & \widehat{\Gamma}_{ji} \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma_j^2 I_j \end{bmatrix}$$

$$\widehat{\Sigma}_{jpi} = \begin{bmatrix} \widehat{\Sigma}_{jpi1} & \widehat{\Sigma}_{jpi2} \end{bmatrix}$$

$$\widehat{\Sigma}_{jpi1} = \begin{bmatrix} \widehat{\Sigma}_{jaxi} & \phi_j \widehat{\Sigma}_{jmxi} & 0 \\ \widehat{\Sigma}_{jdxi} & 0 & \psi_j \widehat{\Sigma}_{jnxi} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \widehat{\Gamma}^t_{ji} & 0 & 0 \end{bmatrix}$$

$$\widehat{\Sigma}_{jpi2} = \begin{bmatrix} \sqrt{d_{jM} - d_{jm}}(\widehat{\Sigma}_{jaxi} - \mathbf{X}_{js}) & \sqrt{d_{jM}}(\widehat{\Sigma}_{jaxi} - \mathbf{X}_{js}) \\ \sqrt{d_{jM} - d_{jm}}\widehat{\Sigma}_{jdxi} & \sqrt{d_{jM}}\widehat{\Sigma}_{jdxi} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sqrt{d_{jM} - d_{jm}}\widehat{\Gamma}^t_{ji} & \sqrt{d_{jM}}\widehat{\Gamma}^t_{ji} \end{bmatrix}$$

$$\widehat{\Sigma}_{jqi} = \text{diag}\begin{bmatrix} \mathbf{X}_{js} & I_j & I_j & 2\mathbf{X}_{js} - \widehat{\Theta}_{jwai} & 2\mathbf{X}_{js} - \widehat{\Theta}_{jwci} \end{bmatrix}$$

$$\widetilde{\Theta}_{ji1} = \text{diag}\begin{bmatrix} \widetilde{\Theta}_{ji1a} & \widetilde{\Theta}_{ji1c} \end{bmatrix}$$

$$\widetilde{\Theta}_{ji1a} = d_{js}\widehat{\Theta}_{ji1} - \mathbf{X}_{aji} + \widehat{\Theta}_{ji3} + \widehat{\Theta}_{ji5} + \widehat{\Theta}_{ji19} + \widehat{\Theta}^t_{ji19}$$

$$\widetilde{\Theta}_{ji1c} = d_{js}\widehat{\Theta}_{ji2} - \mathbf{X}_{cji} + \widehat{\Theta}_{ji4} + \widehat{\Theta}_{ji6} + \widehat{\Theta}_{ji20} + \widehat{\Theta}^t_{ji20}$$

$$\widetilde{\Theta}_{ji2} = \text{diag}\begin{bmatrix} \widetilde{\Theta}_{ji2a} & \widetilde{\Theta}_{ji2c} \end{bmatrix}$$

$$\widetilde{\Theta}_{ji2a} = \widehat{\Theta}^t_{ji11} - \widehat{\Theta}^t_{ji15} - \widehat{\Theta}^t_{ji19} + \widehat{\Theta}^t_{ji21}$$

$$\widetilde{\Theta}_{ji2c} = \widehat{\Theta}^t_{ji12} - \widehat{\Theta}^t_{ji16} - \widehat{\Theta}^t_{ji20} + \widehat{\Theta}^t_{ji22}$$

$$\widetilde{\Theta}_{ji3} = \text{diag}\begin{bmatrix} \widetilde{\Theta}_{ji3a} & \widetilde{\Theta}_{ji3c} \end{bmatrix}$$

$$\widetilde{\Theta}_{ji3a} = \widehat{\Theta}_{ji1} - \widehat{\Theta}_{ji13} - \widehat{\Theta}^t_{ji13} + \widehat{\Theta}_{ji17} + \widehat{\Theta}^t_{ji17} + \widehat{\Theta}_{ji21} + \widehat{\Theta}^t_{ji21}$$

$$\begin{aligned}
\widetilde{\Theta}_{ji3c} &= \widehat{\Theta}_{ji2} - \widehat{\Theta}_{ji14} - \widehat{\Theta}^t_{ji14} + \widehat{\Theta}_{ji18} + \widehat{\Theta}^t_{ji18} + \widehat{\Theta}_{ji22} + \widehat{\Theta}^t_{ji22} \\
\widehat{\Sigma}_{jaxi} &= \begin{bmatrix} \mathbf{X}_{ajs} A^t_{ji} & \mathbf{Y}^t_{eji} \\ \mathbf{Y}^t_{cji} B^t_{ji} & \mathbf{Y}^t_{eji} \end{bmatrix}, \; \widehat{\Sigma}_{jdxi} = \begin{bmatrix} \mathbf{X}_{ajs} D^t_{ji} & \mathbf{Y}^t_{fji} \\ 0 & 0 \end{bmatrix} \\
\widehat{\Sigma}_{jmxi} &= \begin{bmatrix} \mathbf{X}_{ajs} E_j \\ 0 \end{bmatrix}, \; \widehat{\Sigma}_{jnxi} = \begin{bmatrix} \mathbf{X}_{ajs} E_{dj} \\ 0 \end{bmatrix} \\
\widehat{\Theta}_{jlai} &= \text{diag}\left[\widehat{\Theta}_{ji11} \; \widehat{\Theta}_{ji12} \right], \; \widehat{\Theta}_{jmai} = \text{diag}\left[\widehat{\Theta}_{ji15} \; \widehat{\Theta}_{ji16} \right] \\
\widehat{\Theta}_{jlci} &= \text{diag}\left[\widehat{\Theta}_{ji13} \; \widehat{\Theta}_{ji14} \right], \; \widehat{\Theta}_{jmci} = \text{diag}\left[\widehat{\Theta}_{ji17} \; \widehat{\Theta}_{ji18} \right] \\
\widehat{\Theta}_{jri} &= \text{diag}\left[\widehat{\Theta}_{ji3} \; \widehat{\Theta}_{ji4} \right], \qquad \widehat{\Theta}_{jsi} = \text{diag}\left[\widehat{\Theta}_{ji5} \; \widehat{\Theta}_{ji6} \right] \\
\widehat{\Theta}_{jwai} &= \text{diag}\left[\widehat{\Theta}_{ji7} \; \widehat{\Theta}_{ji8} \right], \qquad \widehat{\Theta}_{jwci} = \text{diag}\left[\widehat{\Theta}_{ji9} \; \widehat{\Theta}_{ji10} \right]
\end{aligned} \tag{10.43}$$

where $\mathbf{X}_{js}$ *and* $\widehat{\Gamma}_{ji}$ *are given by (10.38). Moreover, the feedback gain matrix is given by*

$$L_{ji} = \mathbf{Y}_{sji} \mathbf{X}^{-1}_{cji}, \; K_{oji}, \; K_{cji} = \mathbf{Y}_{cji} \mathbf{X}^{-1}_{cji}$$

Proof Using the congruent transformation

$$\begin{aligned}
\mathbf{T}_{js} &= \text{diag}[\mathbf{T}_{js1}, \; \mathbf{T}_{js2}], \\
\mathbf{T}_{js1} &= \text{diag}[\mathbf{X}_{js}, \; \mathbf{X}_{js}, \; \mathbf{X}_{js}, \; \mathbf{X}_{js}, \; I_j, \; I_j], \\
\mathbf{T}_{js2} &= \text{diag}[\mathbf{X}_{js}, \; \mathbf{X}_{js}, \; \mathbf{X}_{js}, \; I_j, \; \mathbf{X}_{js}, \; I_j, \; I_j, \; I_j, \; I_j]
\end{aligned}$$

into LMI (10.39) using (10.40) and (10.41) along with the linearizations

$$\begin{aligned}
\widehat{\Theta}_{jqi} &= \mathcal{X}_{js} \widetilde{\mathcal{Q}}_{ji} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} \mathcal{Q}_{aji} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} \mathcal{Q}_{cji} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji1} & 0 \\ 0 & \widehat{\Theta}_{ji2} \end{bmatrix} \\
\widehat{\Theta}_{jri} &= \mathcal{X}_{js} \widetilde{\mathcal{R}}_{ji} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} \mathcal{R}_{aji} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} \mathcal{R}_{cji} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji3} & 0 \\ 0 & \widehat{\Theta}_{ji4} \end{bmatrix} \\
\widehat{\Theta}_{jsi} &= \mathcal{X}_{js} \widetilde{\mathcal{S}}_{ji} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} \mathcal{S}_{aji} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} \mathcal{S}_{cji} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji5} & 0 \\ 0 & \widehat{\Theta}_{ji6} \end{bmatrix} \\
\widehat{\Theta}_{jwai} &= \mathcal{X}_{js} \widetilde{\mathcal{W}}_{jai} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} \mathcal{W}_{vji} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} \mathcal{W}_{wji} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji7} & 0 \\ 0 & \widehat{\Theta}_{ji8} \end{bmatrix} \\
\widehat{\Theta}_{jwci} &= \mathcal{X}_{js} \widetilde{\mathcal{W}}_{jci} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} \mathcal{W}_{rji} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} \mathcal{W}_{sji} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji9} & 0 \\ 0 & \widehat{\Theta}_{ji10} \end{bmatrix} \\
\widehat{\Theta}_{jlai} &= \mathcal{X}_{js} \widehat{L}_{jai} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} L_{jai1} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} L_{jai2} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji11} & 0 \\ 0 & \widehat{\Theta}_{ji12} \end{bmatrix} \\
\widehat{\Theta}_{jlci} &= \mathcal{X}_{js} \widehat{L}_{jci} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} L_{jci1} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} L_{jci2} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji13} & 0 \\ 0 & \widehat{\Theta}_{ji14} \end{bmatrix} \\
\widehat{\Theta}_{jmai} &= \mathcal{X}_{js} \widehat{M}_{jai} \mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs} M_{jai1} \mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs} M_{jai2} \mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji15} & 0 \\ 0 & \widehat{\Theta}_{ji16} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\widehat{\Theta}_{jmci} &= \mathcal{X}_{js}\widehat{L}_{jai}\mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs}M_{jai1}\mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs}M_{jai2}\mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji17} & 0 \\ 0 & \widehat{\Theta}_{ji18} \end{bmatrix} \\
\widehat{\Theta}_{jlnai} &= \mathcal{X}_{js}\widehat{L}_{jai}\mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs}N_{jai1}\mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs}N_{jai2}\mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji19} & 0 \\ 0 & \widehat{\Theta}_{ji20} \end{bmatrix} \\
\widehat{\Theta}_{jnci} &= \mathcal{X}_{js}\widehat{L}_{jai}\mathcal{X}_{js} = \begin{bmatrix} \mathbf{X}_{ajs}N_{jai1}\mathbf{X}_{ajs} & 0 \\ 0 & \mathbf{X}_{cjs}N_{jai2}\mathbf{X}_{cjs} \end{bmatrix} = \begin{bmatrix} \widehat{\Theta}_{ji21} & 0 \\ 0 & \widehat{\Theta}_{ji22} \end{bmatrix} \\
\mathbf{Y}_{cji} &= K_{cji}\mathbf{X}_{cjs}, \;\; \mathbf{Y}_{sji} = L_{ji}\mathbf{X}_{cjs}
\end{aligned} \tag{10.44}$$

We then deploy the algebraic inequalities

$$(\mathbf{X}_{js} - \mathbf{X}_{js}\widetilde{\mathcal{W}}_{jai}\mathbf{X}_{js})^t(\mathbf{X}_{js}\widetilde{\mathcal{W}}_{jai}\mathbf{X}_{js})^{-1}(\mathbf{X}_{js} - \mathbf{X}_{js}\widetilde{\mathcal{W}}_{jai}\mathbf{X}_{js}) \geq 0$$

which leads to

$$\widetilde{\mathcal{W}}_{jai}^{-1} \geq 2\mathbf{X}_{js} - \widehat{\Theta}_{jwai}$$

and similarly

$$\widetilde{\mathcal{W}}_{jci}^{-1} \geq 2\mathbf{X}_{js} - \widehat{\Theta}_{jwci}$$

We finally cast $\mathbf{T}_{js}^t \Sigma_{jsi}\mathbf{T}_{js}$ into the LMI (10.42) with (10.43) as desired. ■

Finally, we provide a numerical simulation example.

10.1.5 Simulation Example A

A discrete model for water pollution control of the type (10.3) simulating three consecutive reaches ($N = 3$, a reach is of length 6–10 km) and having three operating points (modes) is considered ($\mathcal{S} = 3$) which reflects seasonal water plans. The first mode corresponds to light dumped effluents where the water control policy would be proportional to pretreated waste water, the second mode reflects the moderate dumped effluents where the water control policy would be proportional to change in stream velocity, and the third indicates heavy dumped effluents where the water control policy would be a combined action of both proportional to changes in pretreated waste water and stream velocity. The water pollution model represents two aggregate bio-strata, the first one is for algae (first order) and the other is of fourth-order simulating the concentration levels of ammonia products, phosphate products, BOD (biochemical oxygen demand), and DO (dissolved oxygen). The data values are taken from [117, 254]. In this study, we wish to design switched feedback controllers for this system based on **Theorems 10.7** and **10.9**. Switching occurs between three modes $(1, 2, 3)$ described by the following coefficients:

Mode 1:

$$A_{1i} = \begin{bmatrix} 0.608+\alpha(i) & 0.01 & 0.01 & 0.11 & 0.13 \\ -0.01 & 0.457+\alpha(i) & 0 & 0.22 & 0.24 \\ -0.01 & 0 & -0.332+\alpha(i) & 0.24 & 0.29 \\ -0.12 & -0.21 & -0.25 & -0.617+\alpha(i) & 0.35 \\ -0.13 & -0.21 & -0.31 & -0.25 & 0.536+\alpha(i) \end{bmatrix}$$

$$D_{1i} = \begin{bmatrix} 0.4+\beta(i) & 0 & 0 & 0 & 0 \\ 0 & 0.29+\beta(i) & 0 & -0.02 & 0.04 \\ 0 & 0 & -0.33+\beta(i) & 0 & 0.16 \\ 0 & 0 & -0.05 & -0.18+\beta(i) & 0.03 \\ 0 & 0 & -0.03 & 0 & 0.13+\beta(i) \end{bmatrix}$$

$$B_{1i} = \begin{bmatrix} 0.3 & 0 \\ 0 & \beta(i) \\ 0 & \beta(i) \\ 0 & 0.35 \\ 0 & 0.45 \end{bmatrix}, \ C_{1i}^t = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \\ 0 & 0.25 \\ 0 & 0.25 \\ 0 & 0.25 \end{bmatrix}, \ F_{1i}^t = \begin{bmatrix} -0.61 & 0 \\ -0.01 & 0 \\ 0 & -0.05 \\ 0 & -0.03 \\ 0.01 & -0.01 \end{bmatrix}$$

$$G_{1i} = \begin{bmatrix} 0.1 & 0.3+\beta(i) & 0 & 0.2+\beta(i) & 0.1 \end{bmatrix}, \ \Psi_{1i}^t = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \ \Phi_{1i} = [0.6]$$
$$\Gamma_{1i}^t = \begin{bmatrix} 0.5 & 0.1 & 0.2 & 0.1 & 0.2 \end{bmatrix}, \ H_{1i} = \begin{bmatrix} 0.1 & 0.01 & 0.01 & 0.03 & 0.04 \end{bmatrix}$$

Mode 2:

$$A_{2i} = \begin{bmatrix} 0.51+\kappa(i) & 0.01 & 0.01 & 0.11 & 0.13 \\ -0.01 & 0.412+\kappa(i) & 0 & 0.22 & 0.24 \\ -0.01 & 0 & -0.362+\kappa(i) & 0.24 & 0.29 \\ -0.12 & -0.21 & -0.25 & -0.568+\kappa(i) & 0.35 \\ -0.13 & -0.21 & -0.31 & -0.25 & 0.304+\kappa(i) \end{bmatrix}$$

$$D_{2i} = \begin{bmatrix} 0.21+\beta(i) & 0 & 0 & 0 & 0 \\ 0 & 0.20+\beta(i) & 0 & -0.02 & 0.04 \\ 0 & 0 & -0.15+\beta(i) & 0 & 0.16 \\ 0 & 0 & -0.05 & -0.14+\beta(i) & 0.03 \\ 0 & 0 & -0.03 & 0 & 0.26+\beta(i) \end{bmatrix}$$

$$B_{2i} = \begin{bmatrix} 0.4 & 0 \\ 0 & \beta(i) \\ 0 & \beta(i) \\ 0 & 0.45 \\ 0 & 0.35 \end{bmatrix}, \ C_{2i}^t = \begin{bmatrix} 1 & 0 \\ 0 & 0.15 \\ 0 & 0.15 \\ 0 & 0.15 \\ 0 & 0.15 \end{bmatrix}, \ F_{2i}^t = \begin{bmatrix} -0.48 & 0 \\ -0.01 & 0 \\ 0 & -0.05 \\ 0 & -0.03 \\ 0.01 & -0.01 \end{bmatrix}$$

$$G_{2i} = \begin{bmatrix} 0.2 & 0.2+\beta(i) & 0 & 0.1+\beta(i) & 0.2 \end{bmatrix}, \ \Psi_{2i}^t = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \ \Phi_{2i} = [0.5]$$
$$\Gamma_{2i}^t = \begin{bmatrix} 0.4 & 0.1 & 0.3 & 0.1 & 0.3 \end{bmatrix}, \ H_{2i} = \begin{bmatrix} 0.1 & 0.02 & 0.01 & 0.04 & 0.03 \end{bmatrix}$$

Mode 3:

$$A_{3i} = \begin{bmatrix} 0.324+\delta(i) & 0.01 & 0.02 & 0.12 & 0.11 \\ -0.01 & 0.503+\delta(i) & 0 & 0.18 & 0.14 \\ -0.01 & 0 & -0.401+\delta(i) & 0.21 & 0.28 \\ -0.11 & -0.22 & -0.24 & -0.582+\delta(i) & 0.25 \\ -0.11 & -0.20 & -0.28 & -0.23 & 0.61+\delta(i) \end{bmatrix}$$

$$D_{3i} = \begin{bmatrix} 0.34+\beta(i) & 0 & 0 & 0 & 0 \\ 0 & 0.277+\beta(i) & 0 & -0.02 & 0.02 \\ 0 & 0 & -0.38+\beta(i) & 0 & 0.16 \\ 0 & 0 & -0.05 & -0.17+\beta(i) & 0.02 \\ 0 & 0 & -0.03 & 0 & 0.14+\beta(i) \end{bmatrix}$$

$$B_{3i} = \begin{bmatrix} 0.6 & 0 \\ 0 & \beta(i) \\ 0 & \beta(i) \\ 0 & 0.15 \\ 0 & 0.55 \end{bmatrix}, \; C_{3i}^t = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \\ 0 & 0.35 \\ 0 & 0.25 \\ 0 & 0.35 \end{bmatrix}, \; F_{3i}^t = \begin{bmatrix} -0.44 & 0 \\ -0.01 & 0 \\ 0 & -0.05 \\ 0 & -0.03 \\ 0.01 & -0.01 \end{bmatrix}$$

$$G_{3i} = \begin{bmatrix} 0.1 & 0.4+\beta(i) & 0 & 0.3+\beta(i) & 0.1 \end{bmatrix}, \; \Psi_{3i}^t = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \; \Phi_{3i} = [0.7]$$
$$\Gamma_{3i}^t = \begin{bmatrix} 0.4 & 0.1 & 0.3 & 0.1 & 0.3 \end{bmatrix}, \; H_{3i} = \begin{bmatrix} 0.1 & 0.02 & 0.02 & 0.04 & 0.03 \end{bmatrix}$$

Coupling:

$$E_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \; E_3 = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.3 \end{bmatrix}$$
$$E_{d1} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \; E_{d2} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \; E_{d3} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

where the respective values of the parameters $\alpha(i), \; \beta(i), \; \delta(i), \; \kappa(i)$ are given in Table 10.1.

Table 10.1 Ranges of model parameters

Mode	1	2	3
$\alpha(i)$	0.1	0.2	0.15
$\beta(i)$	−0.2	−0.3	−0.15
$\delta(i)$	0.2	0.1	0.3
$\kappa(i)$	−0.1	0.1	0.2

Choosing $d_{jm} = 2, \; d_{jM} = 9, \; j = 1, \ldots, 3$ and invoking the software environment [74], the feasible solution of LMIs (10.32) and (10.33) yields the state-feedback gains:

$$\gamma_1 = 2.9411$$
$$K_{11} = \begin{bmatrix} -0.1754 & -0.8234 & -0.0274 & 0.0005 & -0.0068 \\ -0.4567 & -0.5754 & -0.0312 & 0.0004 & -0.0075 \end{bmatrix}$$

$$K_{12} = \begin{bmatrix} -0.2633 & -0.9012 & 0.0004 & 0.0105 & -0.0132 \\ -0.3219 & -0.4105 & -0.0002 & -0.0114 & -0.1125cc \end{bmatrix}$$

$$K_{13} = \begin{bmatrix} -0.2318 & -0.8241 & 0.0001 & 0.0114 & -0.0162 \\ -0.3144 & -0.5003 & -0.0001 & -0.0127 & -0.1231cc \end{bmatrix}$$

$$\gamma_2 = 3.4022$$

$$K_{21} = \begin{bmatrix} -0.5469 & -0.8775 & -0.0014 & 0.0003 & -0.0138 \\ -0.6114 & -0.7004 & -0.0332 & 0.0004 & -0.0039 \end{bmatrix}$$

$$K_{22} = \begin{bmatrix} -0.6229 & -0.9347 & -0.0012 & 0.0002 & -0.0141 \\ -0.4328 & -0.6803 & -0.0292 & 0.0003 & -0.0028 \end{bmatrix}$$

$$K_{23} = \begin{bmatrix} -0.7111 & -0.7813 & -0.0105 & 0.0011 & -0.0008 \\ -0.8121 & -0.6163 & -0.0372 & 0.0123 & -0.0008 \end{bmatrix}$$

$$\gamma_3 = 3.3856$$

$$K_{31} = \begin{bmatrix} -0.4213 & -0.2815 & -0.0104 & 0.0001 & -0.0098 \\ -0.6114 & -0.7004 & -0.0332 & -0.0004 & -0.0009 \end{bmatrix}$$

$$K_{32} = \begin{bmatrix} -0.5464 & -0.3781 & -0.0202 & 0.0001 & -0.0144 \\ -0.2897 & -0.6121 & -0.0087 & 0.0011 & -0.0022 \end{bmatrix}$$

$$K_{33} = \begin{bmatrix} -0.6842 & -0.7453 & -0.0115 & 0.00101 & -0.0017 \\ -0.8002 & -0.5972 & -0.0122 & -0.0146 & -0.0012 \end{bmatrix}$$

By contrast, using second-order dynamic feedback controller ($s_1 = s_2 = s_3 = 2$) the feasible solution of LMIs (10.42) and (10.43) yields the output-feedback gains:

$$L_{11} = \begin{bmatrix} -0.4172 & -0.5715 \\ 0.3332 & -0.2908 \end{bmatrix}, \; L_{12} = \begin{bmatrix} -0.3986 & -0.3335 \\ 0.4242 & -0.3127 \end{bmatrix}$$

$$L_{13} = \begin{bmatrix} -0.2983 & -0.6121 \\ 0.3542 & -0.3128 \end{bmatrix}, \; K_{o11} = \begin{bmatrix} -0.5155 & 0.0126 \\ 0.3125 & -0.3978 \end{bmatrix}$$

$$K_{o12} = \begin{bmatrix} -0.4875 & 0.0134 \\ 0.2985 & -0.3865 \end{bmatrix}, \; K_{o13} = \begin{bmatrix} -0.5155 & 0.0516 \\ 0.0325 & -0.3252 \end{bmatrix}$$

$$K_{c11} = \begin{bmatrix} -0.3107 & 0.0131 \\ 0.0025 & 0.0942 \end{bmatrix}, \quad K_{c12} = \begin{bmatrix} -0.3107 & 0.0091 \\ 0.0165 & 0.0888 \end{bmatrix}$$

$$K_{c13} = \begin{bmatrix} -0.3107 & 0.3201 \\ 0.0425 & 0.1126 \end{bmatrix}, \qquad \gamma_1 = 1.5337$$

$$L_{21} = \begin{bmatrix} -0.5021 & -0.6602 \\ 0.3401 & -0.2877 \end{bmatrix}, \; L_{22} = \begin{bmatrix} -0.0894 & -0.6716 \\ -0.3514 & 0.4306 \end{bmatrix}$$

$$L_{23} = \begin{bmatrix} -0.7181 & -0.9001 \\ 0.0533 & -0.1418 \end{bmatrix}, \; K_{o21} = \begin{bmatrix} -0.4933 & 0.2125 \\ -0.3008 & -0.4243 \end{bmatrix}$$

$$K_{c22} = \begin{bmatrix} -0.4123 & -0.2525 \\ 0.0178 & 0.1223 \end{bmatrix}, \; K_{o23} = \begin{bmatrix} -0.0045 & -0.0716 \\ -0.0514 & -0.4415 \end{bmatrix}$$

$$K_{c21} = \begin{bmatrix} -0.8354 & -0.0894 \\ 0.4306 & 0.2157 \end{bmatrix}, \; K_{c23} = \begin{bmatrix} -0.8354 & -0.7105 \\ -0.5222 & 0.2157 \end{bmatrix}$$

$$K_{c23} = \begin{bmatrix} -0.8354 & -0.1077 \\ 0.4081 & 0.2157 \end{bmatrix}, \; \gamma_2 = 1.6265,$$

$$L_{31} = \begin{bmatrix} -0.1897 & -0.7101 \\ -0.1928 & -0.2677 \end{bmatrix}, \; L_{32} = \begin{bmatrix} -0.2093 & -0.7555 \\ -0.2108 & -0.2797 \end{bmatrix}$$

$$L_{33} = \begin{bmatrix} -0.1024 & -0.4521 \\ -0.2108 & -0.2797 \end{bmatrix}, \; K_{o31} = \begin{bmatrix} -0.5977 & -0.7956 \\ 0.3254 - 0.4705 & \end{bmatrix}$$

$$K_{o32} = \begin{bmatrix} -0.6328 & -0.1234 \\ 0.0508 & -0.5121 \end{bmatrix}, \; K_{o33} = \begin{bmatrix} -0.6328 & -0.0005 \\ 0.3017 & -0.4877 \end{bmatrix}$$

$$K_{c31} = \begin{bmatrix} -0.6645 & -0.0055 \\ 0.0116 & 0.3448 \end{bmatrix}, \; K_{c32} = \begin{bmatrix} -0.4833 & -0.0024 \\ 0.4107 & -0.7487 \end{bmatrix}$$

$$K_{c33} = \begin{bmatrix} -0.5937 & -0.0245 \\ 0.1118 & 0.2948 \end{bmatrix}, \; \gamma_3 = 1.7355$$

The ensuing results show that the developed switched feedback control policies have been quite effective in clearing out the impact of sudden dumped pollutant disturbance, and the closed-loop water-quality system settles to regular levels of water-quality constituents. This is in agreement with our theoretical developments. To further illustrate the validity of our design method, we simulate the closed-loop water-quality system in both state-feedback and dynamic output-feedback cases. The corresponding state trajectories are plotted in Figs. 10.1, 10.2, 10.3, 10.4, 10.5 under state-feedback and in Figs. 10.6, 10.7, 10.8, 10.9, and 10.10 using dynamic output feedback.

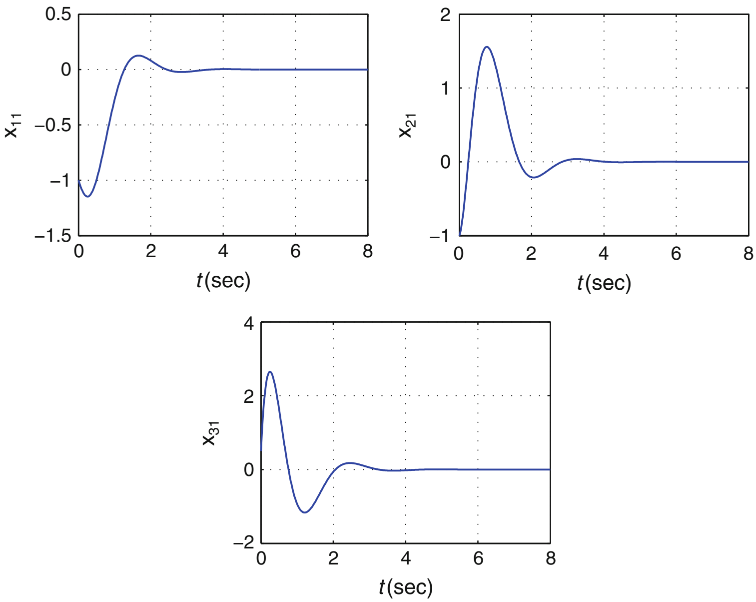

Fig. 10.1 Algae trajectories under switched state feedback

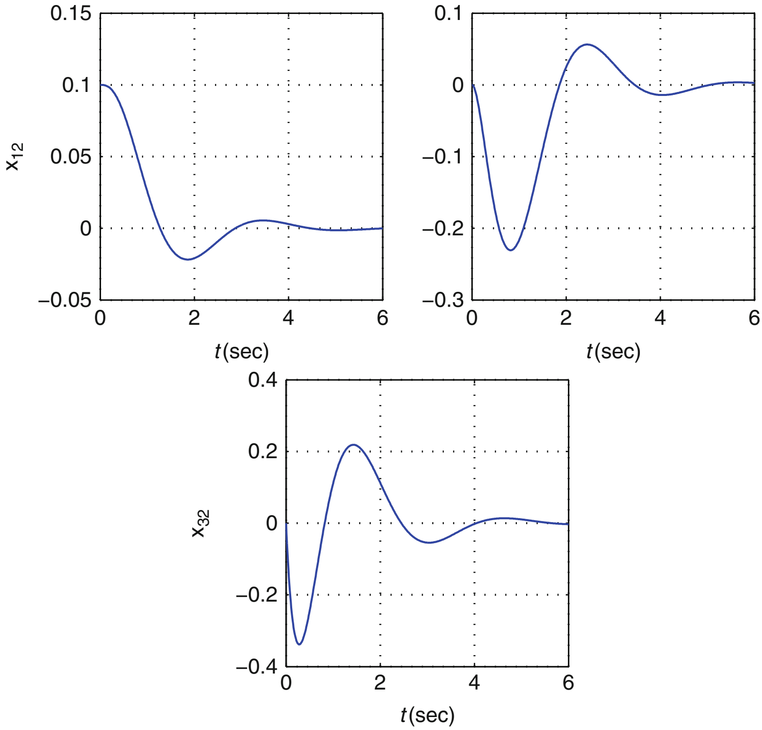

Fig. 10.2 Ammonia nitrogen trajectories under switched state feedback

It is readily seen from the computational results that the switched dynamic output-feedback controller is more responsive than the switched state-feedback controller since the closed-loop system settles rather quickly with a better performance index. Hence, the switched dynamic output-feedback controller is more effective in clearing the disturbed water system.

10.2 Interconnected Continuous-Time Systems

This section develops a decentralized approach to the robust stability and stabilization problems of class of interconnected continuous-time switched systems with cone-bounded uncertainties and nonlinearities. This class consists of coupled nominally linear subsystems with unknown-but-bounded time-varying state-delay. We showed that multi-controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems with large parameter variations. We developed a delay-dependent decentralized structure that guarantees global asymptotic stability with local disturbance attenuation on the subsystem level. Then, we constructed decentralized switched control schemes based on state feedback to ensure stabilizability of the global system with $\mathcal{L}_2$-performance bound.

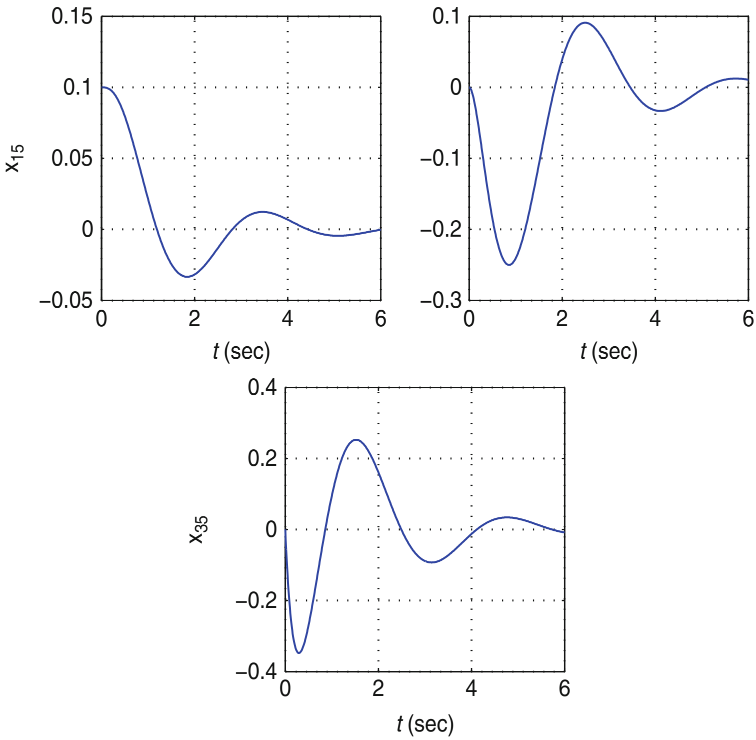

Fig. 10.3 Phosphate phosphorous trajectories under switched state feedback

10.2.1 Introduction

There are real-world systems consisting of coupled units or subsystems which directly interact with each other in a simple and predictable fashion to serve a common pool of objectives. When viewed as a whole, the resulting overall system often displays rich and complex behavior. Typical examples are found in electric power systems with strong interactions, water networks which are widely distributed in space, traffic systems with many external signal or large-space flexible structures, to name a few, which are often termed *large-scale* or *interconnected* systems. It becomes increasingly evident that the underlying notions of interconnected systems manifest the complexity as an essential and dominating problem in systems theory and practice and that several associated problems cannot be tackled using one-shot approaches. Recent research investigations have revealed [8] that the crucial need for improved methodologies relies on: (1) dividing the analysis and synthesis of the overall system into independent or almost independent subproblems, (2) searching for new ideas of coping with the incomplete information about the system, and (3) seeking appropriate methods of handling the uncertainties and for dealing with delays. System complexity frequently leads to severe difficulties that are encountered in the tasks of analyzing, designing, and implementing appropriate control

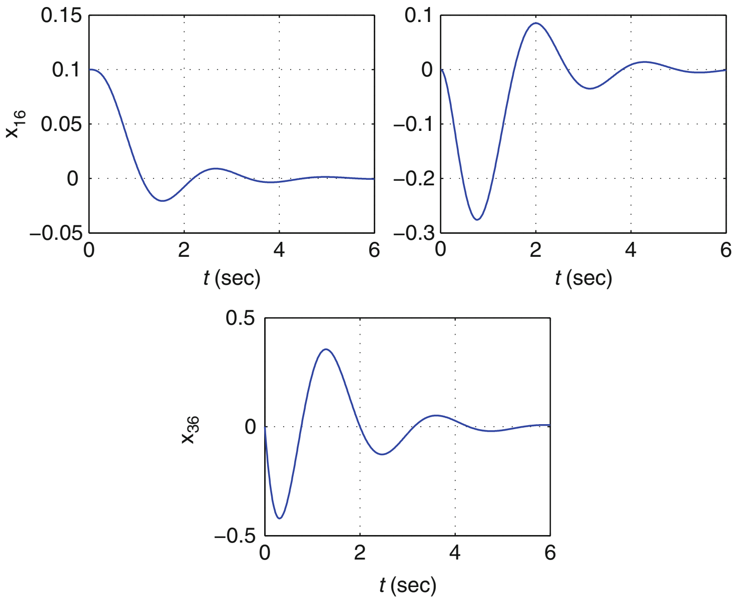

Fig. 10.4 BOD trajectories under switched state feedback

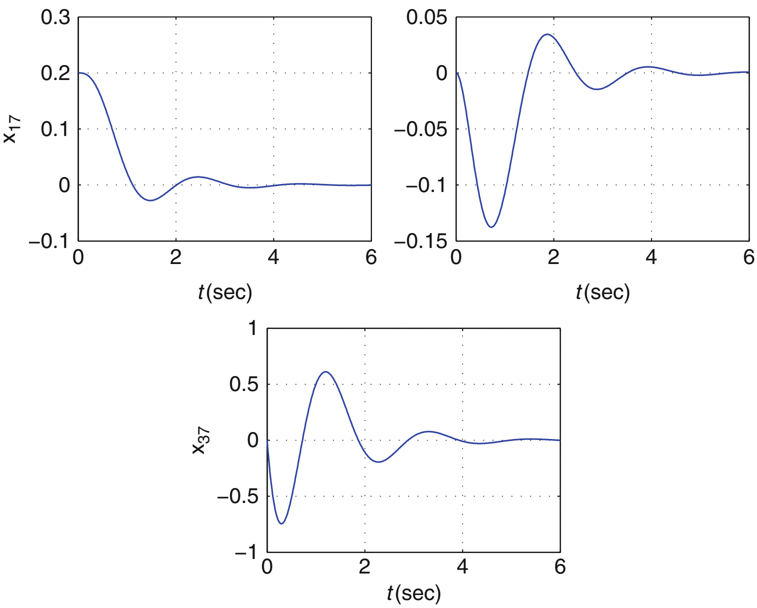

Fig. 10.5 DO trajectories under switched state feedback

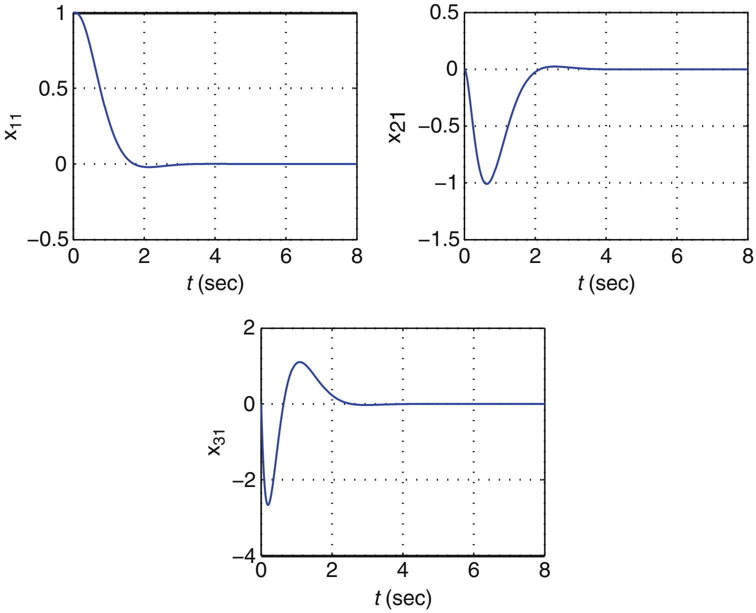

Fig. 10.6 Algae trajectories under switched output feedback

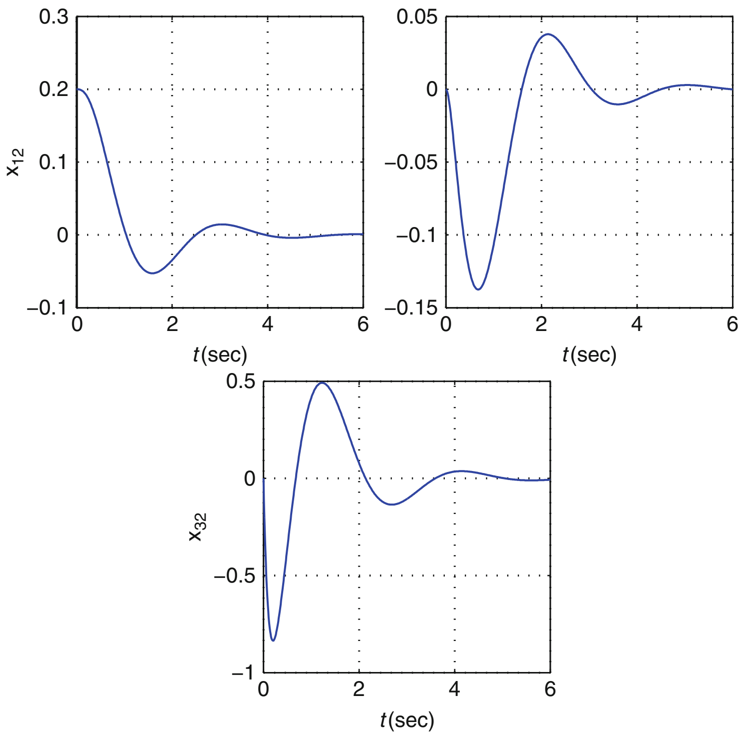

Fig. 10.7 Ammonia nitrogen trajectories under switched output feedback

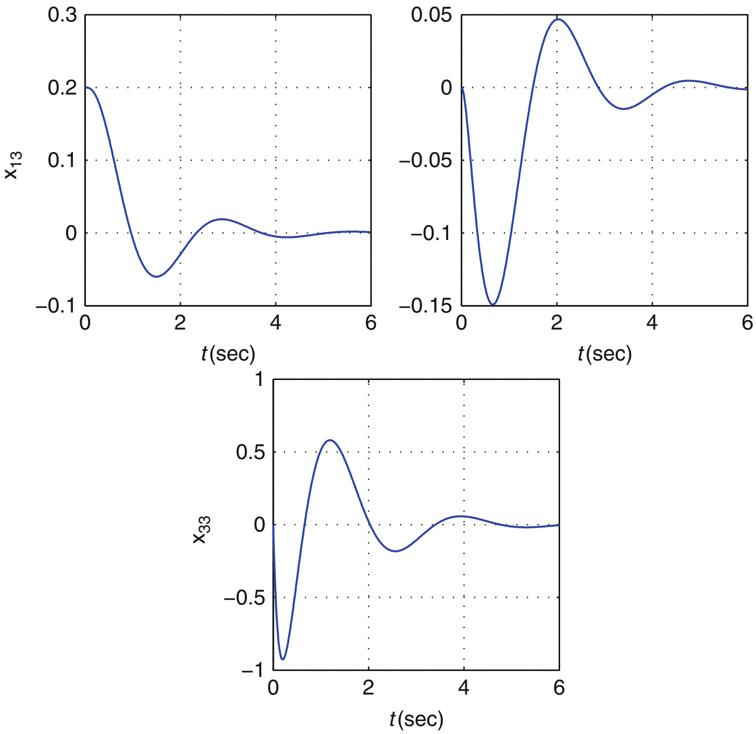

Fig. 10.8 Phosphate phosphorous trajectories under switched output feedback

methods. These difficulties arise mainly from the following well-known reasons: *dimensionality; information structure constraints; uncertainty; and delays*. Pertinent results can be found in [141, 201, 276, 348].

On another research front, switched systems are composed of a finite family of continuous- or discrete-time subsystems (called modes) and a rule that governs the switching among them. Switched systems have received growing attention in the control literature [15, 28, 42, 47] and have been intensively studied in recent years due to their widespread applications, including power systems and chemical processes and mechanical systems [47]. Each mode is regarded as a state of a finite-state machine whose evolution dynamics determines the switching. Research investigations into problems pertaining to switched dynamical systems have received a great deal of attention because of the fast development in computing technologies, see [174, 193, 327, 366] and the references cited therein. In [351] a design procedure based on proportional plus delay control is presented for a class of flexible structures possessing multiple modes.

From the published results, we conclude that the study of switched linear systems provides additional insights into some long-standing problems, such as robust, adaptive, and intelligent control, gain scheduling, or multi-rate digital control. The recent results in switched systems have benefited many real-world systems such

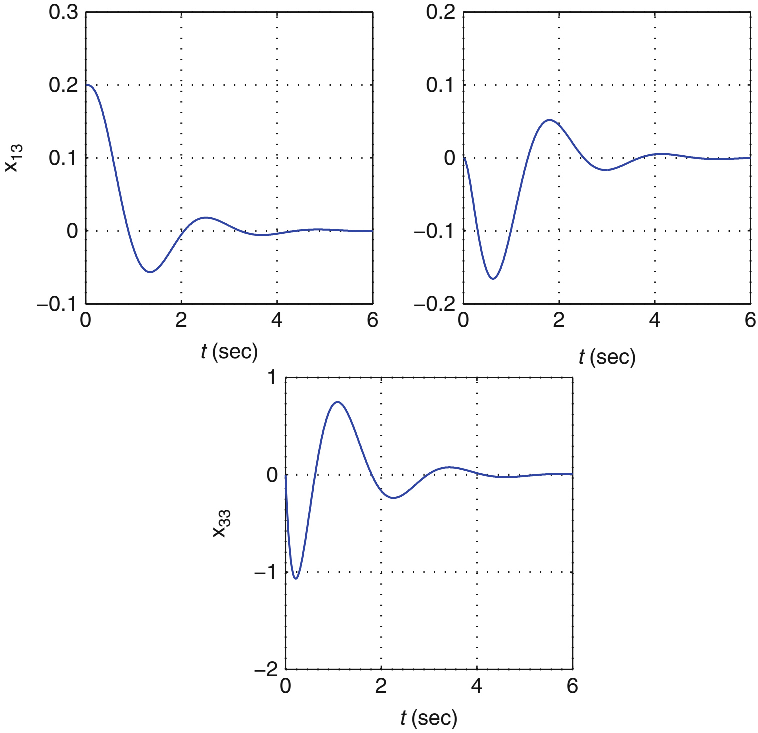

Fig. 10.9 BOD trajectories under switched output feedback

as power systems, automotive control, air traffic control, network and congestion control. One important problem in uncertain switched systems is the design of switching rules which guarantee quadratic stability and performance and such switching rules must be independent of uncertainties. The min-projection strategy introduced in [331] as a simple stabilization method for systems composed of several subsystems. This motivates need of multi-controller switched schemes for large-scale complex systems when implementing low-order local controllers. However, all these references deal with a centralized switching rule. In addition, it appears that the problems of stability analysis and control design interconnected switched systems with time-varying delays have not been fully resolved thus far. Therefore, in this paper, both problems are addressed where we consider the subsystems representing the lower-level local dynamics governed by delayed differential equations, while the supervisor is the high-level coordinator producing the switches among the local dynamics. The dynamics of the global system is therefore determined by both the subsystem and the switching signal, which may depend on the time, its own past value, the state/output, and/or possibly an external signal. We deal with the problem of low-order $\mathcal{H}_\infty$ state-feedback or output-feedback controller design with a decentralized switching rule for a class of switching

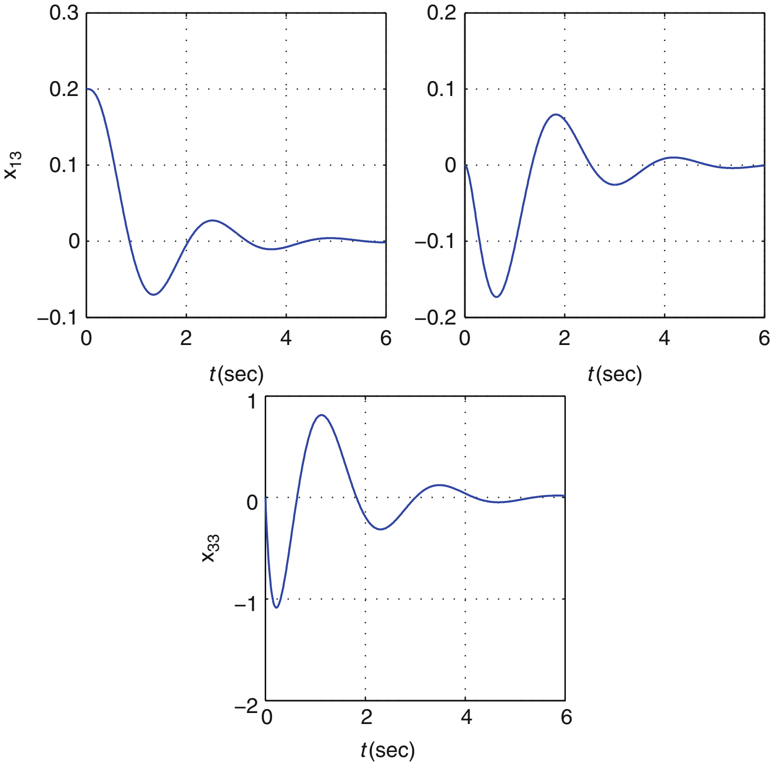

Fig. 10.10 DO trajectories under switched output feedback

discrete-time interconnected systems, where we extend further the results of [26, 224–227, 232, 239–241, 243, 247, 248, 252, 260, 270–272, 274, 275, 279–281, 283–285] to the class of interconnected switched continuous-time systems with unknown-but-bounded state delay. In our work, we showed that multi-controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems and/or systems with large uncertainties. We developed a delay-dependent decentralized structure that guarantees asymptotic stability with local disturbance attenuation. Then, we constructed a decentralized switched control scheme based on state feedback to ensure stabilizability of the global system with $\mathcal{L}_2$-performance bound.

10.2.2 Problem Statement and Preliminaries

A class of nonlinear interconnected discrete-time systems with state-delay Σ composed of N coupled subsystems $\Sigma_j, \quad j \in \mathcal{N} = \{1, \ldots, N\}$, which is represented by

$$\begin{aligned}
\Sigma_j: \quad \dot{x}_j(t) &= A_{j\xi} x_j(t) + D_{j\xi} x_j(t-\tau_j(t)) + B_{j\xi} u_j(t) + \Gamma_{j\xi}\omega_j(t) \\
&\quad + g_j(t, x(t), x(t-\tau(t))) \\
y_j(t) &= C_{j\xi} x_j(t) + F_{j\xi} x_j(t-\tau_j(t)) + \Psi_{j\xi}\omega_j(t) \\
z_j(t) &= G_{j\xi} x_j(t) + H_{j\xi} x_j(t-\tau_j(t)) + \Phi_{j\xi}\omega_j(t)
\end{aligned} \tag{10.45}$$

where the function $\xi = \xi(x_j, t) : \Re^n \times \Re_+ \to \mathcal{S} = \{1, 2, \ldots, S\}$ is a switching rule within subsystem Σ_j which takes its values in the finite set of modes $\mathcal{S}$. This rule is selected for all j such that $\xi(x_j, t) = s$ implies that the sth switching mode is activated for the jth subsystem of the interconnected system. In general, the switching rule is a piecewise constant function depending on the subsystem state in each time. It is seen that system (10.45) can be viewed as autonomous switched system in which the effective system changes when the state $x_j(t)$ hits predefined boundaries, that is, the switching rule is dependent on the system trajectories.

The factors $\tau_j, \; j \in \{1, ..., N\}$ are unknown time-delay factors satisfying

$$0 \;\le\; \varphi_j \;\le\; \tau_j(t) \;\le\; \varrho_j, \quad \dot{\tau}_j(t) \;\le\; \mu_j \tag{10.46}$$

where the bounds $\varrho_j, \; \le \; \varphi_j, \; \mu_j$ are known constants in order to guarantee smooth growth of the state trajectories.

The system matrices $\{A_\alpha, D_\alpha, ..., \Phi_\alpha\}$ take values, at arbitrary discrete instants, in the finite set of

$$\{(A_{j1}, D_{j1}, ..., \Phi_{j1}), (A_{j2}, D_{j2}, ..., \Phi_{j2}), ..., (A_{jN}, D_{jN}, ..., \Phi_{jN})\}$$

Thus the matrices $(A_{js}, D_{js}, ..., \Phi_{js})$ denotes the sth model of local subsystem j corresponding to operational mode s and hence (10.45) represents a time-controlled switched system [42]. Typically, the switching rule ξ is not known a priori but we assume its instantaneous value is available in real time for practical implementations. Define the indicator function

$$\begin{aligned}
\alpha(t) &= [\alpha_1(t), ..., \alpha_N(t)]^t, \quad \forall j \;\in\; \mathcal{N} \\
\alpha_s(t) &= \begin{cases} 1 & \text{when the } j\text{th subsystem (10.45) is in the sth mode} \\ & \qquad (A_{js}, D_{js}, ..., \Phi_{js}) \\ 0 & \qquad \text{otherwise} \end{cases}
\end{aligned} \tag{10.47}$$

It is obvious that $\alpha_i(t) : \Re_+ \to \{0, 1\}, \; \sum_{j=1}^{N} \alpha_j(t) = 1, \; t \in \Re_+, \; i \in \mathcal{N}$. Now we cast system (10.45) into the form

$$\begin{aligned}
\Sigma_j: \quad \dot{x}_j(t) = \sum_{i=1}^{N} \alpha_i(t) \Big[& A_{ji} x_j(t) + D_{ji} x_j(t-\tau_j(t)) + B_{ji} u_j(t) + \Gamma_{ji}\omega_j(t) \\
& + g_j(t, x(t), x(t-\tau(t))) \Big]
\end{aligned}$$

$$y_j(t) = \sum_{i=1}^{N} \alpha_i(t) \Big[C_{ji} x_j(t) + F_{ji} x_j(t - \tau_j(t)) + \Psi_{ji} \omega_j(t) \Big]$$
$$z_j(t) = \sum_{i=1}^{N} \alpha_i(t) \Big[G_{ji} x_j(t) + H_{ji} x_j(t - \tau_j(t)) + \Phi_{ji} \omega_j(t) \Big] \qquad (10.48)$$

where, relating the local subsystems to the global system,

$$x(t) = \left(x_1^t(t), ..., x_N^t(t) \right)^t \in \Re^n, \; n = \sum_{j=1}^{N} n_j$$
$$x(t - \tau(t)) = \left(x_1^t\left(t - \tau_1(t)\right), ..., x_N^t\left(t - \tau_N(t)\right) \right)^t \in \Re^n, \; n = \sum_{j=1}^{N} n_j$$
$$u(t) = \left(u_1^t(t), ..., u_N^t(t) \right)^t \in \Re^p, \; p = \sum_{j=1}^{N} p_j$$
$$y(t) = \left(y_1^t(t), ..., y_N^t(t) \right)^t \in \Re^m, \; m = \sum_{j=1}^{N} m_j$$
$$z(t) = \left(z_1^t(t), ..., z_N^t(t) \right)^t \in \Re^q, \; q = \sum_{j=1}^{N} q_j$$

being the state, delayed state, control input, measured output, and performance output vectors of interconnected (global) system Σ and $\omega(t) = \left(\omega_1^t(t), ..., \omega_N^t(t) \right)^t \in \Re^q$ is the disturbance input, which is assumed to belong to $\mathcal{L}_2[0, \infty)$. It is significant to observe in the foregoing setup that there are N distinct switching rules where each subsystem has been assigned one local state-dependent switching rule that operates independently from other rules.

The associated matrices are real constants and modeled as

$$\mathcal{A}_s = \text{diag}\{A_{1s}, \ldots, A_{Ns}\}, \; A_{js} \in \Re^{n_j \times n_j}$$
$$\mathcal{B}_s = \text{diag}\{B_{1s}, \ldots, B_{Ns}\}, \; B_{js} \in \Re^{n_j \times p_j}$$
$$\mathcal{D}_s = \text{diag}\{D_{1s}, \ldots, D_{Ns}\}, \; D_{js} \in \Re^{n_j \times n_j}$$
$$\mathcal{C}_s = \text{diag}\{C_{1s}, \ldots, C_{Ns}\}, \; C_{js} \in \Re^{q_j \times n_j}$$
$$\mathcal{H}_s = \text{diag}\{H_{1s}, \ldots, H_{Ns}\}, \; H_{js} \in \Re^{q_j \times p_j}$$

The function $g_j : \Re_+ \times \Re^n \times \Re^n \rightarrow \Re^{n_j}$ is a piecewise-continuous vector function in its arguments and it satisfies thequadratic inequality

$$g_j^t(t, x(t), x(t-\tau(t)))\ g_j(t, x(t), x(t-\tau(t))) \leq \phi_j^2\ x^t(t)\widetilde{E}_j^t \widetilde{E}_j x(t) \\ +\ \psi_j^2\ x^t(t-\tau_j(t))\widetilde{E}_{dj}^t \widetilde{E}_{dj} x(t-\tau_j(t)) \tag{10.49}$$

where $\phi_j > 0,\ \psi_j > 0$ are bounding parameters such that

$$\widetilde{\Phi} = \text{diag}\left\{\phi_j^2 I_{r_1}, \ldots, \phi_j^2 I_{r_N}\right\},\ \widetilde{\Psi} = \text{diag}\left\{\psi_j^2 I_{s_1}, \ldots, \psi_j^2 I_{s_N}\right\}$$

with I_{m_j} being the $m_j \times m_j$ identity matrix. From (10.49) and the notation

$$g(t, x(t), x(t-\tau(t))) = \left[g_1^t(t, x(t), x(t-\tau_1(t))), \ldots, g_N^t(t, x(t), x(t-\tau_N(t)))\right]^t$$

it is always possible to find matrices $\Phi,\ \Psi$ such that

$$g_j^t(t, x(t), x(t-\tau(t)))\ g_j(t, x(t), x(t-\tau(t))) \leq x^t(t)E^t\Phi^{-1}Ex(t) \\ +\ x^t(t-\tau_j(t))E_d^t\Psi^{-1}E_d x(t-\tau_j(t)) \tag{10.50}$$

where $E = \text{diag}\{E_1, \ldots, E_N\},\ E_d = \text{diag}\{E_{d1}, \ldots, E_{dN}\},\ \delta_j = \phi_j^{-2},\ \nu_j = \psi_j^{-2},\ \Phi = \text{diag}\{\delta_1 I_{r_1}, \ldots, \delta_N I_{r_N}\}$
$\Psi = \text{diag}\{\nu_{d1} I_{s_1}, \ldots, \nu_{dN} I_{s_N}\}$ with $E_j \in \Re^{r_j \times n_j},\ E_{dj} \in \Re^{s_j \times n_j}$.

Letting $\zeta(t) = \left[x^t(t)\ x^t(t-\tau_j(t))\ g_j^t(t, x(t), x(t-\tau_j(t)))\right]^t \stackrel{\Delta}{=} \left[\zeta_1^t, \ldots, \zeta_N^t\right]^t$, then (10.50) can be conveniently written as

$$\zeta^t \text{diag}\left[-E^t\Phi^{-1}E\ -\ E_d^t\Psi^{-1}E_d\ \ I\right]\zeta \leq\ 0 \tag{10.51}$$

and in view of the block structure of matrices, it turns out for Σ_j that

$$\zeta_j^t \text{diag}\left[-\delta_j^{-1}E_j^t E_j\ -\nu_j^{-1}E_{dj}^t E_{dj}\ \ I_j\right]\zeta_j \leq\ 0 \tag{10.52}$$

Remark 10.10 This paper essentially develops a flexible conceptual framework for multi-controller state-dependent switching structure among smooth controllers with all the effort and computations being performed on the subsystem level thereby providing an efficient decentralized feedback control design guaranteeing the level of disturbance attenuation for the overall interconnected systems. In this work, we consider that the modes are represented by discrete-time linear systems with unknown-but-bounded delays. The subsystems thus represent the lower-level local dynamics governed by difference equations, while the supervisor is the high-level coordinator producing the switches among the local dynamics. The dynamics of the system is determined by both the subsystem and the switching signal. In general, a switching signal may depend on the time, its own past value, the state/output, and/or possibly an external signal.

10.2.3 Delay-Dependent $\mathcal{L}_2$ Gain Analysis

In this section, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and ℓ_2 gain analysis. Toward our goal, we consider the Lyapunov–Krasovskii functional (LKF):

$$
\begin{aligned}
V(t) &= \sum_{j=1}^{n_s} V_j(t), \quad V_j(t) = V_{oj}(t) + V_{aj}(t) + V_{cj}(t) + V_{ej}(t) \\
&\quad + V_{mj}(t) + V_{nj}(t)), \\
V_{oj}(t) &= x_j^t(t)\mathcal{P}_{ji}x_j(t), \ V_{aj}(t) = \int_{-\varrho_j}^{0}\int_{t+s}^{t} \dot{x}_j^t(\alpha)\mathcal{Q}_{ji}\dot{x}_j(\alpha)\mathrm{d}\alpha \ \mathrm{d}s \\
V_{mj}(t) &= \varphi_j \int_{-\varphi_j}^{0}\int_{t+s}^{t} \dot{x}_j^t(\alpha)\Theta_{ji}\dot{x}_j(\alpha)\mathrm{d}\alpha \ \mathrm{d}s, \\
V_{nj}(t) &= (\varrho_j - \varphi_j)\int_{-\varrho_j}^{-\varphi_j}\int_{t+s}^{t} \dot{x}_j^t(\alpha)\Lambda_{ji}\dot{x}_j(\alpha)\mathrm{d}\alpha \ \mathrm{d}s, \\
V_{cj}(t) &= \int_{t-\tau(t)}^{t} x_j^t(s)\mathcal{Z}_{ji}x_j(s) \ \mathrm{d}s, \\
V_{ej}(t) &= \int_{t-\varrho_j}^{t} x_j^t(s)\Upsilon_{ji}x_j(s) \ \mathrm{d}s
\end{aligned} \tag{10.53}
$$

where $0 < \mathcal{P}_{ji}, \ 0 < \Theta_{ji}, \ 0 < \mathcal{Q}_{ji}, \ 0 < \mathcal{Z}_{ji}, \ 0 < \Lambda_{ji}, \ 0 < \Upsilon_{ji}, \ j \in \{1, ..., N\}, \ i \in \{1, ..., S\}$ are weighting matrices of appropriate dimensions. The first term in (5.51) is standard to nominal systems without delay while the second and third terms correspond to the delay-dependent conditions since they provide measures of the individual and derivative signal energies during the delay-period (recall that $\int_{t-\varrho_j}^{t} \dot{x}_j(\alpha)\mathrm{d}\alpha = x_j(t) - x_j(t-\varrho_j)$ by the Leibniz–Newton formula) and the fourth term corresponds to the intra-connection delays. The main result is provided by the following theorem:

Theorem 10.11 *Given the bounds $\varphi_j > 0, \ \varrho_j > 0, \ \mu_j > 0, \ j = 1, ..., N$. The global system Σ with subsystem Σ_j given by (10.48) is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist weighting matrices $\mathcal{X}_{ji}, \ \mathcal{Y}_{ji}, \ \mathcal{M}_{ji}, \ \mathcal{S}_{ji}, \ \mathcal{W}_{ji}, \ \mathcal{R}_{ji}, \ \forall i \in \mathcal{S}, \ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs*

$$
\widetilde{\Omega}_{ji} = \begin{bmatrix}
\Omega_{ji1} & \Omega_{ji2} & \Omega_{ji4} & \mathcal{X}_{ji}E_j^t & 0 & \Omega_{ji7} \\
\bullet & -\Omega_{ji6} & \Omega_{ji5} & 0 & \mathcal{X}_{ji}E_{dj}^t & 0 \\
\bullet & \bullet & -\Omega_{ji8} & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -\delta_j I & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\nu_j I & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -I
\end{bmatrix} < 0 \tag{10.54}
$$

$$\Omega_{ji1} = \begin{bmatrix} \Omega_{oji} & \mathcal{W}_{ji} & D_{ji}\mathcal{X}_{ji} & \mathcal{W}_{ji} \\ \bullet & -\Omega_{aji} & 0 & 0 \\ \bullet & \bullet & -\Omega_{cji} & \mathcal{S}_{ji} \\ \bullet & \bullet & \bullet & -\mathcal{R}_{ji} - \mathcal{S}_{ji} \end{bmatrix}, \ \Omega_{ji7} = \begin{bmatrix} G^t_{ji} \\ 0 \\ H^t_{ji} \\ 0 \\ 0 \\ \Phi^t_{ji} \end{bmatrix} \tag{10.55}$$

$$\Omega_{ji2} = \begin{bmatrix} \mathcal{X}_{ji} & \Gamma_{ji} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \Omega_{ji4} = \begin{bmatrix} \varrho_j \mathcal{X}_{ji} A^t_{ji} & (\varrho_j - \varphi_j)\mathcal{X}_{ji} A^t_{ji} \\ 0 & 0 \\ \varrho_j \mathcal{X}_{ji} D^t_{ji} & (\varrho_j - \varphi_j)\mathcal{X}_{ji} D^t_{ji} \\ 0 & 0 \end{bmatrix},$$

$$\Omega_{ji6} = \begin{bmatrix} I_j & 0 \\ \bullet & \gamma_j^2 I_j \end{bmatrix}$$

$$\Omega_{ji5} = \begin{bmatrix} \varrho_j I_j & (\varrho_j - \varphi_j) I_j \\ \varrho_j \Gamma_{ji} & (\varrho_j - \varphi_j)\Gamma_{ji} \end{bmatrix}, \ \Omega_{ji8} = \begin{bmatrix} 2I_j - \mathcal{W}_{ji} & 0 \\ \bullet & 2I_j - \mathcal{W}_{ji} \end{bmatrix} \tag{10.56}$$

$$\begin{aligned} \Omega_{oji} &= A_{ji}\mathcal{X}_{ji} + \mathcal{X}_{ji}A^t_{ji} + \mathcal{M}_{ji} + \mathcal{R}_{ji} + \mathcal{Y}_{ji} - \mathcal{W}_{ji}, \\ \Omega_{aji} &= \mathcal{M}_{ji} + \mathcal{W}_{ji} + \mathcal{S}_{ji}, \quad \Omega_{cji} = (1-\mu_j)\mathcal{M}_{ji} + 2\mathcal{S}_{ji} \end{aligned} \tag{10.57}$$

Proof A straightforward computation gives the time derivative of $V(t)$ along the solutions of (10.45) with $u(t) \equiv 0$ as

$$\begin{aligned} \dot{V}_{oj}(t) &= 2x^t_j \mathcal{P}_{ji}\dot{x}_j \\ &= 2x^t_j \mathcal{P}_{ji} \sum_{i=1}^N \alpha_i(t) \Big[A_{ji}x_j(t) + D_{ji}x_j(t - \tau_j(t)) + \Gamma_{ji}\omega_j(t) + g_j \Big] \\ \dot{V}_{aj}(t) &= \Big[x^t_j(t)\mathcal{Q}_{ji}x(t) - x^t_j(t-\varphi_j)\mathcal{Q}_{ji}x_j(t-\varphi_j) \Big] \\ \dot{V}_{cj}(t) &\le \Big[x^t_j(t)\mathcal{Z}_{ji}x_j(t) - (1-\mu_j)\, x^t_j(t-\tau_j)\mathcal{Z}_{ji}x_j(t-\tau_j) \Big] \\ \dot{V}_{ej}(t) &= \Big[x^t_j(t)\Upsilon_{ji}x_j(t) - x^t_j(t-\varrho_j)\Upsilon_{ji}x_j(t-\varrho_j) \Big] \\ \dot{V}_{mj}(t) &= \varphi_j^2 \dot{x}^t_j(t)\Theta_{ji}\dot{x}_j(t) - \varphi_j \int_{t-\varphi_j}^t \dot{x}^t_j(s)\theta_{ji}\dot{x}_j(s)\mathrm{d}s \\ &\le \varphi_j^2 \dot{x}^t_j(t)\Theta_{ji}\dot{x}_j(t) - \varphi_j \int_{t-\varphi_j}^t \dot{x}^t(s)\Theta_{ji}\dot{x}(s)\mathrm{d}s \\ \dot{V}_{nj}(t) &= (\varrho_j - \varphi_j)^2 \dot{x}^t_j(t)\Lambda_{ji}\dot{x}_j(t) \\ &\quad - (\varrho_j - \varphi_j)\int_{t-\varrho_j}^{t-\varphi_j} \dot{x}^t_j(s)\Lambda_{ji}\dot{x}_j(s)\mathrm{d}s \\ &\le (\varrho_j - \varphi_j)^2 \dot{x}^t_j(t)\Lambda_{ji}\dot{x}_j(t) \end{aligned} \tag{10.58}$$

$$- (\varrho_j - \varphi_j) \int_{t-\varrho_j}^{t-\varphi_j} \dot{x}_j^t(s) \Lambda_{ji} \dot{x}_j(s) \mathrm{d}s \tag{10.59}$$

Applying **Lemma A.13**, we get

$$\begin{aligned} &- \varphi_j \int_{t-\varphi_j}^{t} \dot{x}_j^t(\alpha) \Theta_{ji} \dot{x}(\alpha) \mathrm{d}\alpha \le \\ &\begin{bmatrix} x_j(t) \\ x_j(t-\varphi_j) \end{bmatrix}^t \begin{bmatrix} -\Theta_{ji} & \Theta_{ji} \\ \bullet & -\Theta_{ji} \end{bmatrix} \begin{bmatrix} x_j(t) \\ x_j(t-\varphi_j) \end{bmatrix} \end{aligned} \tag{10.60}$$

Similarly,

$$\begin{aligned} &-(\varrho_j - \varphi_j) \int_{t-\varrho_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \Lambda_{ji} \dot{x}_j(\alpha) \mathrm{d}\alpha \\ &= -(\varrho_j - \varphi_j) \left[\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \Lambda_{ji} \dot{x}_j(\alpha) \mathrm{d}\alpha + \int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha) \Lambda_{ji} \dot{x}_j(\alpha) \mathrm{d}\alpha \right] \\ &\le -(\tau_j - \varphi_j) \left[\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \Lambda_{ji} \dot{x}_j(\alpha) \mathrm{d}\alpha \right] \\ &\quad - (\varrho - \tau) \left[\int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha) \Lambda_{ji} \dot{x}_j(\alpha) \mathrm{d}\alpha \right] \\ &\le - \left(\int_{t-\tau_j}^{t-\varphi_j} \dot{x}_j^t(\alpha) \mathrm{d}\alpha \right) \Lambda_{ji} \left(\int_{t-\tau}^{t-\varphi} \dot{x}_j^t(\alpha) \mathrm{d}\alpha \right) \\ &\quad - \left(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}_j^t(\alpha) \mathrm{d}\alpha \right) \Lambda_{ji} \left(\int_{t-\varrho_j}^{t-\tau_j} \dot{x}^t(\alpha) \mathrm{d}\alpha \right) \\ &= -[x(t-\varphi_j) - x(t-\tau_j)]^t \Lambda_{ji} [x(t-\varphi_j) - x(t-\tau_j)] \\ &\quad - [x(t-\tau_j) - x(t-\varrho_j)]^t \Lambda_{ji} [x(t-\tau_j) - x(t-\varrho_j)] \end{aligned} \tag{10.61}$$

Since (10.58) has to be satisfied under arbitrary switching, it follows that this holds for the particular case $\alpha_i(k) = 1, \ \alpha_{m \neq i}(k) = 0$. This implies by combining (10.53), (10.54), (10.55), (10.56), (10.57), (10.58), (10.59), (10.60), and (10.61) and using Schur complements with $u(t) \equiv 0$ that

$$\begin{aligned} \dot{V}_j(t)|_{(10.48)} \le &\ \chi_j^t(t) \ \Xi_{ji} \ \chi_j(t) \\ &+ \varphi_j^2 \dot{x}_j^t(t) \Theta_{ji} \dot{x}_j(t) + (\varrho_j - \varphi_j)^2 \dot{x}_j^t(t) \Lambda_{ji} \dot{x}_j(t) \end{aligned} \tag{10.62}$$

$$\begin{aligned} \chi_j(t) &= \left[x_j^t(t) \ x_j^t(t-\varphi_j) \ x_j^t(t-\tau_j) \ x_j^t(t-\varrho_j) \ g_j \ \omega_j \right]^t \\ \Xi_{ji} &= \begin{bmatrix} \Xi_{ji1} & \Xi_{ji2} \\ \bullet & \Xi_{ji3} \end{bmatrix}, \ \Xi_{ji3} = \begin{bmatrix} 0 & 0 \\ \bullet & 0 \end{bmatrix} \end{aligned} \tag{10.63}$$

$$\Xi_{ji1} = \begin{bmatrix} \Xi_{oji} & \Theta_{ji} & \mathcal{P}_{ji} D_{ji} & \Theta_{ji} \\ \bullet & -\Xi_{aji} & 0 & 0 \\ \bullet & \bullet & -\Xi_{cji} & \Lambda_{ji} \\ \bullet & \bullet & \bullet & -\Upsilon_{ji} - \Lambda_{ji} \end{bmatrix}$$

$$\Xi_{ji2} = \begin{bmatrix} \mathcal{P}_{ji} & \mathcal{P}_{ji} \Gamma_{ji} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{10.64}$$

$$\Xi_{oji} = \mathcal{P}_{ji} A_{ji} + A^t_{ji} \mathcal{P}_{ji} + \mathcal{Q}_{ji} + \Upsilon_{ji} + \mathcal{Z}_{ji} - \Theta_{ji},$$
$$\Xi_{aji} = \mathcal{Q}_{ji} + \Theta_{ji} + \Lambda_{ji}, \;\; \Xi_{cj} = (1 - \mu_j) \mathcal{Z}_{ji} + 2\Lambda_{ji} \tag{10.65}$$

where $\dot{V}_j(t)|_{(10.48)}$ defines the Lyapunov derivative along the solutions of system (10.45). By Schur complements we express (10.62) into the form $\pi^t(t)\widehat{\Xi}_{ji}\pi(t)$ for some $\pi(t) \neq 0$ where

$$\widehat{\Xi}_{ji} = \begin{bmatrix} \Xi_{ji1} & \Xi_{ji2} & \Xi_{ji4} \\ \bullet & -\Xi_{ji3} & \Xi_{ji5} \\ \bullet & \bullet & -\Xi_{ji6} \end{bmatrix}, \;\; \Xi_{ji4} = \begin{bmatrix} \varrho_j A^t_{ji} \Theta_{ji} & (\varrho_j - \varphi_j) A^t_{ji} \Lambda_{ji} \\ 0 & 0 \\ \varrho_j D^t_{ji} \Theta_{ji} & (\varrho_j - \varphi_j) D^t_{ji} \Lambda_{ji} \\ 0 & 0 \end{bmatrix}$$

$$\Xi_{ji5} = \begin{bmatrix} \varrho_j \Theta_{ji} & (\varrho_j - \varphi_j) \Lambda_{ji} \\ \varrho_j \Gamma_{ji} \Theta_{ji} & (\varrho_j - \varphi_j) \Gamma_{ji} \Lambda_{ji} \end{bmatrix}, \;\; \Xi_{ji6} = \begin{bmatrix} \Theta_{ji} & 0 \\ \bullet & \Theta_{ji} \end{bmatrix} \tag{10.66}$$

By resorting to the **S**-procedure [27], inequalities (10.52) and (10.66) can be rewritten together for some $\sigma_j \geq 0$ as

$$\widehat{\Xi}_{ji} = \begin{bmatrix} \hat{\Xi}_{ji1} & \Xi_{ji2} & \Xi_{ji4} \\ \bullet & -\hat{\Xi}_{ji3} & \Xi_{ji5} \\ \bullet & \bullet & -\Xi_{ji6} \end{bmatrix}, \;\; \hat{\Xi}_{ji3} = \begin{bmatrix} I_j & 0 \\ \bullet & 0 \end{bmatrix}$$

$$\hat{\Xi}_{ji1} = \begin{bmatrix} \hat{\Xi}_{oji} & \Theta_{ji} & \mathcal{P}_{ji} D_{ji} & \Theta_{ji} \\ \bullet & -\Xi_{aji} & 0 & 0 \\ \bullet & \bullet & -\hat{\Xi}_{cji} & \Lambda_{ji} \\ \bullet & \bullet & \bullet & -\Upsilon_{ji} - \Lambda_{ji} \end{bmatrix} \tag{10.67}$$

$$\hat{\Xi}_{oji} = \Xi_{oji} + \sigma_j \phi_j^2 E^t_j E_j, \;\; \hat{\Xi}_{cji} = \Xi_{cji} + \sigma_j \psi_j^2 E^t_{dj} E_{dj} \tag{10.68}$$

which describe nonstrict LMIs since $\sigma_j \geq 0$. Recalling from [31] that minimization under nonstrict LMIs corresponds to the same result as minimization under strict LMIs when both strict and nonstrict LMI constraints are feasible. Moreover, if there is a solution for (10.67) for $\sigma_j = 0$, there will also be a solution for some $\sigma_j > 0$ and sufficiently small ϕ_j, ψ_j. Therefore, we safely replace $\sigma_j \geq 0$ by $\sigma_j > 0$. Equivalently, we may further rewrite (10.67) with some manipulations in the form

$$\tilde{\Xi}_{ji} = \begin{bmatrix} \tilde{\Xi}_{ji1} & \Xi_{ji2} & \Xi_{ji4} \\ \bullet & -\hat{\Xi}_{ji3} & \Xi_{ji5} \\ \bullet & \bullet & -\Xi_{ji6} \end{bmatrix}$$

$$\tilde{\Xi}_{ji1} = \begin{bmatrix} \tilde{\Xi}_{oji} & \Theta_{ji} & \mathcal{P}_{ji} D_{ji} & \Theta_{ji} \\ \bullet & -\Xi_{aji} & 0 & 0 \\ \bullet & \bullet & -\tilde{\Xi}_{cji} & \Lambda_{ji} \\ \bullet & \bullet & \bullet & -\Upsilon_{ji} - \Lambda_{ji} \end{bmatrix} \tag{10.69}$$

$$\tilde{\Xi}_{oji} = \bar{\Xi}_{oji} + \phi_j^2 E_j^t E_j, \ \tilde{\Xi}_{cji} = \bar{\Xi}_{cji} + \psi_j^2 E_{dj}^t E_{dj} \tag{10.70}$$

where $\bar{\Xi}_{oji}, \bar{\Xi}_{cji}$ correspond to Ξ_{oji}, Ξ_{cji} with bar values defined by $\bar{\mathcal{P}}_{ji} = \sigma_j^{-1}\mathcal{P}_{ji}$, $\bar{\mathcal{Q}}_{ji} = \sigma_j^{-1}\mathcal{Q}_{ji}$,$\bar{\Theta}_{ji} = \sigma_j^{-1}\Theta_{ji}$, $\bar{\Upsilon}_{ji} = \sigma_j^{-1}\Upsilon_{ji}$, $\bar{\Lambda}_{ji} = \sigma_j^{-1}\Lambda_{ji}$, $\bar{\mathcal{Z}}_{ji} = \sigma_j^{-1}\mathcal{Z}_{ji}$. Using the linearizations

$$\mathcal{X}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}, \ \mathcal{M}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}\bar{\mathcal{Q}}_{ji}\bar{\mathcal{P}}_{ji}^{-1}, \ \mathcal{Y}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}\bar{\mathcal{Z}}_{ji}\bar{\mathcal{P}}_{ji}^{-1}, \ \mathcal{W}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}\bar{\Theta}_{ji}\bar{\mathcal{P}}_{ji}^{-1}$$
$$\mathcal{R}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}\bar{\Upsilon}_{ji}\bar{\mathcal{P}}_{ji}^{-1}, \ \mathcal{S}_{ji} = \bar{\mathcal{P}}_{ji}^{-1}\bar{\Lambda}_{ji}\bar{\mathcal{P}}_{ji}^{-1}$$

with $\delta_j = \phi_j^{-2}$ and $\nu_j = \psi_j^{-2}$. Applying the congruent transformation

$$T = \mathrm{diag}[\mathcal{X}_{ji}, \ \mathcal{X}_{ji}, \ \mathcal{X}_{ji}, \ \mathcal{X}_{ji}, \ I, \ I, \ I, \ I]$$

with some arrangement, we can express (10.69) in the form (10.55) with $G_{ji} \equiv 0$, $H_{ji} \equiv 0$, $\Phi_{ji} \equiv 0$, $\Gamma_{ji} \equiv 0$. This establishes robust stability of the nonlinear interconnected system (10.45) under the constraint (10.47) with maximal $\phi_j, \ \psi_j$.

Consider the $\mathcal{L}_2 - gain$ performance measure

$$J = \sum_{j=1}^{n_s} \int_0^\infty \left(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) \right) \mathrm{d}s$$

For any $w_j(t) \in \mathcal{L}_2(0, \infty) \neq 0$ with zero initial condition $x_j(0) = 0$, hence $V(0) = 0$, we have

$$J = \sum_{j=1}^{n_s} \int_0^\infty \left(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) \ + \dot{V}_j(t)|_{(10.48)} - \ V_j(\infty) \right) \mathrm{d}s$$
$$\leq \sum_{j=1}^{n_s} \int_0^\infty \left(z_j^t(s) z_j(s) - \gamma_j^2 w_j^t(s) w_j(s) \ + \dot{V}_j(t)|_{(10.48)} \right) \mathrm{d}s$$

Proceeding as before, we make use of (10.62) to get

$$\sum_{j=1}^{n_s}\Big(z_j^t(s)z_j(s) - \gamma_j^2 w_j^t(s)w_j(s) \ + \dot{V}_j(t)|_{(10.48)}\Big)$$
$$= \sum_{j=1}^{n_s} \chi_j^t(t,s)\ \widetilde{\Omega}_{ji}\ \chi_j(t,s) \tag{10.71}$$

where $\widetilde{\Omega}_{ji}$ is given by (10.54). It is readily seen that when $\widehat{\Xi}_j < 0$ the condition

$$\sum_{j=1}^{n_s}\Big(z_j^t(s)z_j(s) - \gamma_j^2 w_j^t(s)w_j(s) \ + \dot{V}_j(t)|_{(10.48)}\Big) < 0$$

for arbitrary $s \in [t,\infty)$, which implies for any $w_j(t) \in \mathcal{L}_2(0,\infty) \neq 0$ that $J < 0$ leading to $\sum_{j=1}^{n_s} ||z_j(t)||_2 < \sum_{j=1}^{n_s} \gamma_j\ ||w(t)_j||_2$, which assures the desired performance. ■

The following corollaries provide some relevant special cases:

Corollary 10.12 *Given the bounds $\varphi_j > 0,\ \varrho_j > 0,\ \mu_j > 0,\ j = 1,...,N$. The linear system Σ_n with subsystem Σ_{nj} given by*

$$\begin{aligned}
\Sigma_{nj}:\quad \dot{x}_j(t) &= \sum_{i=1}^{N}\alpha_i(t)\Big[A_{ji}x_j(t) + D_{ji}x_j(t-\tau_j(t)) + B_{ji}u_j(t) + \Gamma_{ji}\omega_j(t)\Big] \\
y_j(t) &= \sum_{i=1}^{N}\alpha_i(t)\Big[C_{ji}x_j(t) + F_{ji}x_j(t-\tau_j(t)) + \Psi_{ji}\omega_j(t)\Big] \\
z_j(t) &= \sum_{i=1}^{N}\alpha_i(t)\Big[G_{ji}x_j(t) + H_{ji}x_j(t-\tau_j(t)) + \Phi_{ji}\omega_j(t)\Big]
\end{aligned} \tag{10.72}$$

is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist weighting matrices $\mathcal{X}_{ji},\ \mathcal{Y}_{ji},\ \mathcal{M}_{ji},\ \mathcal{S}_{ji},\ \mathcal{W}_{ji},\ \mathcal{R}_{ji},\ \forall i \in \mathcal{S},\ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs

$$\check{\Omega}_{ji} = \begin{bmatrix} \Omega_{ji1} & \check{\Omega}_{ji2} & \Omega_{ji4} & \Omega_{ji7} \\ \bullet & -\check{\Omega}_{ji6} & \check{\Omega}_{ji5} & 0 \\ \bullet & \bullet & -\Omega_{ji8} & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{10.73}$$

$$\begin{aligned}
\check{\Omega}_{ji2}^t &= \big[\,\Gamma_{ji}\ 0\ 0\ 0\,\big],\ \check{\Omega}_{ji6} = \Big[\,0\ \gamma_j^2 I_j\,\Big], \\
\check{\Omega}_{ji5} &= \big[\,\varrho_j\Gamma_{ji}\ (\varrho_j-\varphi_j)\Gamma_{ji}\,\big]
\end{aligned} \tag{10.74}$$

Proof Follows from **Theorem 10.11** by deleting the contributions of $g_j(.)$ and the remaining entries are given in (10.55), (10.56), and (10.57).

Corollary 10.13 *Given the bounds $\varphi_j > 0,\ \varrho_j > 0,\ \mu_j > 0,\ j = 1, ..., N$. The linear system Σ_m with subsystem Σ_{mj} given by*

$$\begin{aligned}\Sigma_{mj}:\quad \dot{x}_j(t) &= \sum_{i=1}^{N} \alpha_i(t)\Big[A_{ji}x_j(t) + B_{ji}u_j(t) + \Gamma_{ji}\omega_j(t)\\ &\quad + g_j(t, x(t), x(t-\tau(t)))\Big]\\ y_j(t) &= \sum_{i=1}^{N} \alpha_i(t)\Big[C_{ji}x_j(t) + F_{ji}x_j(t-\tau_j(t)) + \Psi_{ji}\omega_j(t)\Big]\\ z_j(t) &= \sum_{i=1}^{N} \alpha_i(t)\Big[G_{ji}x_j(t) + H_{ji}x_j(t-\tau_j(t)) + \Phi_{ji}\omega_j(t)\Big]\end{aligned} \tag{10.75}$$

is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist weighting matrices $\mathcal{X}_{ji},\ \mathcal{Y}_{ji},\ \mathcal{M}_{ji},\ \mathcal{S}_{ji},\ \mathcal{W}_{ji},\ \mathcal{R}_{ji},\ \forall i \in \mathcal{S},\ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs

$$\widetilde{\Omega}_{ji} = \begin{bmatrix} A_{ji}\mathcal{X}_{ji} + \mathcal{X}_{ji}A^t_{ji} & \mathcal{X}_{ji} & \Gamma_{ji} & \mathcal{X}_{ji}E^t_j & 0 & G^t_{ji} \\ \bullet & -I_j & 0 & 0 & \mathcal{X}_{ji}E^t_{dj} & 0 \\ \bullet & \bullet & -\gamma_j^2 I_j & 0 & 0 & \Phi^t_{ji} \\ \bullet & \bullet & \bullet & -\delta_j I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\nu_j I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{10.76}$$

Proof Follows from **Theorem 10.11** by deleting the contributions of $D_j(.)$ and the effectively considering $V(t) = \sum_{j=1}^{n_s} x^t_j(t)\mathcal{P}_{ji}x_j(t)$. ■

10.2.4 Switched State-Feedback Design

Next, we address the feedback control problem for the interconnected continuous-time systems Σ by focusing the design effort on the subsystem Σ_j as given by (10.48). The goal is to find global decentralized feedback switching controllers and a decentralized switching rule asymptotically stabilizing the system formed by subsystems $\Sigma_j,\ j = 1, \ldots, N$. This decentralized feedback controllers are composed of N local feedback controllers and each equipped with the corresponding local switching rule. In the sequel, we start with decentralized switched-state-feedback scheme. We seek to design a switched state feedback

$$u_j(k) = \sum_{i=1}^{N} \alpha_i(k) K_{ji}\, x_j(k), \qquad i \in \mathcal{S},\ j \in \mathcal{N}$$

which guarantees that the controlled switched system achieves a prescribed performance level, where $K_{ji} \in \Re^{p_j \times n_j}$ is the local state-feedback gain matrix at the mode i. Letting $\mathcal{A}_{ji} = A_{ji} + B_{ji} K_{ji}$, it is readily seen from **Theorem 10.11** that the closed-loop switched system

$$\begin{aligned} \Sigma_{cj}: \quad \dot{x}_j(t) &= \sum_{i=1}^{N} \alpha_i(t) \Big[\mathcal{A}_{ji} x_j(t) + D_{ji} x_j(t - \tau_j(t)) + \Gamma_{ji} \omega_j(k) \\ &\quad + g_j(t, x(t), x(t - \tau(t))) \Big] \\ z_j(t) &= \sum_{i=1}^{N} \alpha_i(t) \Big[G_{ji} x_j(t) + H_{ji} x_j(t - \tau_j(t)) + \Phi_{ji} \omega_j(t) \Big] \end{aligned} \tag{10.77}$$

is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma_j$ if there exist weighting matrices $\mathcal{X}_{ji},\ \mathcal{Y}_{ji},\ \mathcal{M}_{ji}, \mathcal{S}_{ji},\ \mathcal{W}_j,\ \mathcal{R}_j,\ \forall i \in \mathcal{S},\ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs

$$\overline{\Omega}_{ji} = \begin{bmatrix} \overline{\Omega}_{ji1} & \Omega_{ji2} & \overline{\Omega}_{ji4} & \mathcal{X}_{ji} E_j^t & 0 & \Omega_{ji7} \\ \bullet & -\Omega_{ji6} & \Omega_{ji5} & 0 & \mathcal{X}_{ji} E_{dj}^t & 0 \\ \bullet & \bullet & -\Omega_{ji8} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\delta_j I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\nu_j I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{10.78}$$

$$\begin{aligned} \overline{\Omega}_{ji1} &= \begin{bmatrix} \overline{\Omega}_{oji} & \mathcal{W}_{ji} & D_{ji} \mathcal{X}_{ji} & \mathcal{W}_{ji} \\ \bullet & -\Omega_{aji} & 0 & 0 \\ \bullet & \bullet & -\Omega_{cji} & \mathcal{S}_{ji} \\ \bullet & \bullet & \bullet & -\mathcal{R}_{ji} - \mathcal{S}_{ji} \end{bmatrix} \\ \overline{\Omega}_{ji4} &= \begin{bmatrix} \varrho_j \mathcal{A}_{ji} & (\varrho_j - \varphi_j) \mathcal{A}_{ji} \\ 0 & 0 \\ \varrho_j D_{ji} & (\varrho_j - \varphi_j) D_{ji} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{10.79}$$

$$\begin{aligned} \overline{\Omega}_{oji} &= \mathcal{A}_{ji} \mathcal{X}_{ji} + \mathcal{X}_{ji} \mathcal{A}_{ji}^t + \mathcal{M}_{ji} + \mathcal{R}_{ji} + \mathcal{Y}_{ji} - \mathcal{W}_{ji} \\ \mathcal{A}_{ji} &= A_{ji} + B_{ji} K_{ji} \end{aligned} \tag{10.80}$$

where $\Omega_{ji2},\ \Omega_{ji3},\ \Omega_{ji5},\ \Omega_{ji6},\ \Omega_{ji8}$ are given by (10.56) and (10.57). The following theorem establishes the main design result:

Theorem 10.14 *Given the bounds $\varphi_j > 0,\ \varrho_j > 0,\ \mu_j > 0,\ j = 1, ..., N$. The global system Σ_c with subsystem Σ_{cj} given by (10.77) is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ_j if there exist weighting matrices $\mathcal{X}_{ji},\ \mathcal{G}_{ji},\ \mathcal{Y}_{ji},\ \mathcal{M}_{ji},\ \mathcal{S}_{ji},\ \mathcal{W}_{ji},\ \mathcal{R}_{ji},\ \forall i \in \mathcal{S},\ \forall j \in \mathcal{N}$ and scalars $\gamma_j > 0$ satisfying the following LMIs*

$$\widehat{\Omega}_{ji} = \begin{bmatrix} \widehat{\Omega}_{ji1} & \Omega_{ji2} & \widehat{\Omega}_{ji4} & \mathcal{X}_{ji} E_j^t & 0 & \Omega_{ji7} \\ \bullet & -\Omega_{ji6} & \Omega_{ji5} & 0 & \mathcal{X}_{ji} E_{dj}^t & 0 \\ \bullet & \bullet & -\Omega_{ji8} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\delta_j I & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\nu_j I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{10.81}$$

$$\widehat{\Omega}_{ji1} = \begin{bmatrix} \widehat{\Omega}_{oji} & \mathcal{W}_{ji} & D_{ji}\mathcal{X}_{ji} & \mathcal{W}_{ji} \\ \bullet & -\Omega_{aji} & 0 & 0 \\ \bullet & \bullet & -\Omega_{cji} & \mathcal{S}_{ji} \\ \bullet & \bullet & \bullet & -\mathcal{R}_{ji} - \mathcal{S}_{ji} \end{bmatrix} \tag{10.82}$$

$$\widehat{\Omega}_{ji4} = \begin{bmatrix} \varrho_j \left(\mathcal{X}_{ji} A_{ji}^t + \mathcal{G}_{ji}^t \mathcal{B}_{ji}^t\right) & (\varrho_j - \varphi_j)\left(\mathcal{X}_{ji} A_{ji}^t + \mathcal{G}_{ji}^t \mathcal{B}_{ji}^t\right) \\ 0 & 0 \\ \varrho_j D_{ji}^t & (\varrho_j - \varphi_j) D_{ji}^t \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \widehat{\Omega}_{oji} &= A_{ji}\mathcal{X}_{ji} + \mathcal{X}_{ji} A_{ji}^t + B_{ji}\mathcal{G}_{ji} + \mathcal{G}_{ji}^t B_{ji}^t \\ &\quad + \mathcal{M}_{ji} + \mathcal{R}_{ji} + \mathcal{Y}_{ji} - \mathcal{W}_{ji} \end{aligned} \tag{10.83}$$

where $\Omega_{ji2},\ \Omega_{ji3},\ \Omega_{ji5},\ \Omega_{ji6},\ \Omega_{ji7},\ \Omega_{ji8},\ \Omega_{aji},\ \Omega_{cji}$ are given by (10.55), (10.56), and (10.57). Moreover, the local state-feedback gain matrices are given by $\mathcal{K}_{ji} = \mathcal{G}_{ji}\ \mathcal{X}_{ji}^{-1}$

Proof The results immediately follow by substituting $\mathcal{G}_{ji} = \mathcal{K}_{ji}\mathcal{X}_{ji}$ and arranging the terms. ■

10.2.5 Simulation Example B

To demonstrate the theoretical developments, we consider an aggregate model representing physiochemical changes of three consecutive identical reaches of the River Nile with each reach (about 6 km length) being subject to sewage-dump from pollution station. The subsystem model is based on the variation of concentrations of three bio-strata of water-quality constituents: algae, nitrogen group, and phosphate-BOD-DO group. Two types of water-quality control are enforced: one through effluent discharge and the other is through water velocity. There are two modes of operation corresponding to high-pollutionlevel (H) and low-pollution level (L).

The dynamical equations in the form (10.48), (10.49), (10.50), (10.51), and (10.52) are given for $j = 1, 2, 3$ by

Mode H:

$$A_j = \begin{bmatrix} -0.893 & 0.003 & 0 \\ -0.003 & -1.146 & -0.0026 \\ -0.006 & -0.561 & -1.257 \end{bmatrix}, \; D_j = \begin{bmatrix} -0.225 & 0 & 0.001 \\ -0.001 & -0.136 & -0.001 \\ -0.002 & -0.041 & -0.233 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.1 \\ 0 & 0.2 \end{bmatrix}, G_j^t = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}, H_j^t = \begin{bmatrix} 0.05 \\ 0.05 \\ 0.05 \end{bmatrix}, \Gamma_j = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \Phi_j = 0.4$$

$$E_j = \begin{bmatrix} 0.3 \; 0.1 \; 0.1 \end{bmatrix}, \; E_{dj} = \begin{bmatrix} 0.05 \; 0.02 \; 0.02 \end{bmatrix}$$

Mode L:

$$A_j = \begin{bmatrix} -1.341 & 0.007 & 0 \\ -0.003 & -1.477 & -0.005 \\ -0.007 & -0.888 & -1.745 \end{bmatrix}, \; D_j = \begin{bmatrix} -0.235 & 0 & 0.001 \\ -0.001 & -0.156 & -0.001 \\ -0.002 & -0.041 & -0.363 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \\ 0 & 0.4 \end{bmatrix}, G_j^t = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}, H_j^t = \begin{bmatrix} 0.09 \\ 0.09 \\ 0.09 \end{bmatrix}, \Gamma_j = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \Phi_j = 0.6$$

$$E_j = \begin{bmatrix} 0.5 \; 0.3 \; 0.2 \end{bmatrix}, \; E_{dj} = \begin{bmatrix} 0.07 \; 0.04 \; 0.04 \end{bmatrix}$$

The feasible solution of **Theorem 10.11** is summarized in Table 10.2.

Table 10.2 Performance results

Mode	Method	φ_j	ϱ_j	μ_j	γ_j
High	**Theorem 10.11**	0.425	2.975	1.284	2.693
		0.613	2.987	1.337	2.584
		0.762	3.553	1.414	2.752
		0.884	3.973	1.454	2.788
Low	**Theorem 10.11**	0.425	3.225	1.254	2.773
		0.613	3.557	1.297	2.635
		0.762	3.986	1.311	2.826
		0.884	4.113	1.335	2.895

We observe that the water-system stability is preserved for both modes of operation. Application of **Theorem 10.14** yields the following feasible set of results

Mode H:

$$\gamma_j = 2.1057, \; \varphi_j = 0.585, \; \varrho_j = 3.166, \; \mu_j = 1.366$$

$$\mathcal{K}_{jH} = \begin{bmatrix} -0.2781 & -0.5715 & -0.4219 \\ 0.1132 & -0.6125 & -0.7326 \end{bmatrix}$$

Mode L:

$$\gamma_j = 2.6342,\ \varphi_j = 0.615,\ \varrho_j = 3.245,\ \mu_j = 1.297$$
$$\mathcal{K}_{jL} = \begin{bmatrix} -0.5571 & -0.6865 & -0.6673 \\ 0.1665 & -0.8636 & -0.5908 \end{bmatrix}$$

Typical closed-loop state trajectories are depicted in Figs. 10.11, 10.12, and 10.13 and the associated water-control trajectories are plotted in Fig. 10.14. From the ensuing results, it is evident that the developed switched state-feedback control has been able to smooth the effects of dumped pollutants. Of particular interest is that most of the control effort are exerted during 1/3 the planning period.

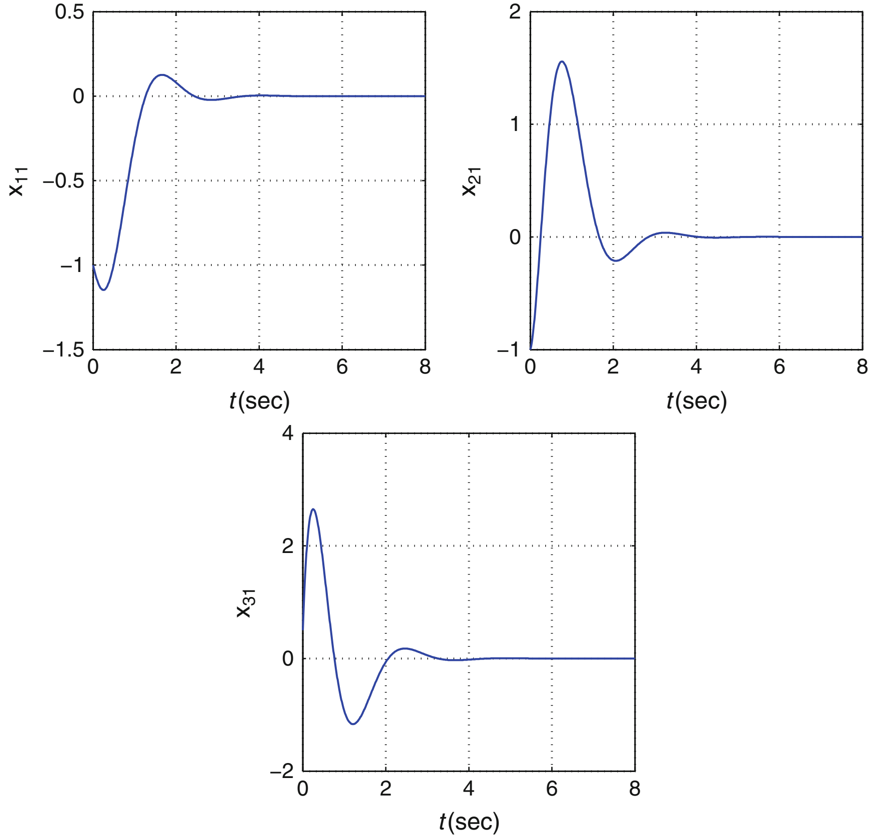

Fig. 10.11 Algae trajectories under switched-state feedback

10.2.6 Simulation Example C

A switched system of the type 10.45 is composed of three identical subsystems ($j = 1, 2, 3$) subject to two modes of operation with the following data

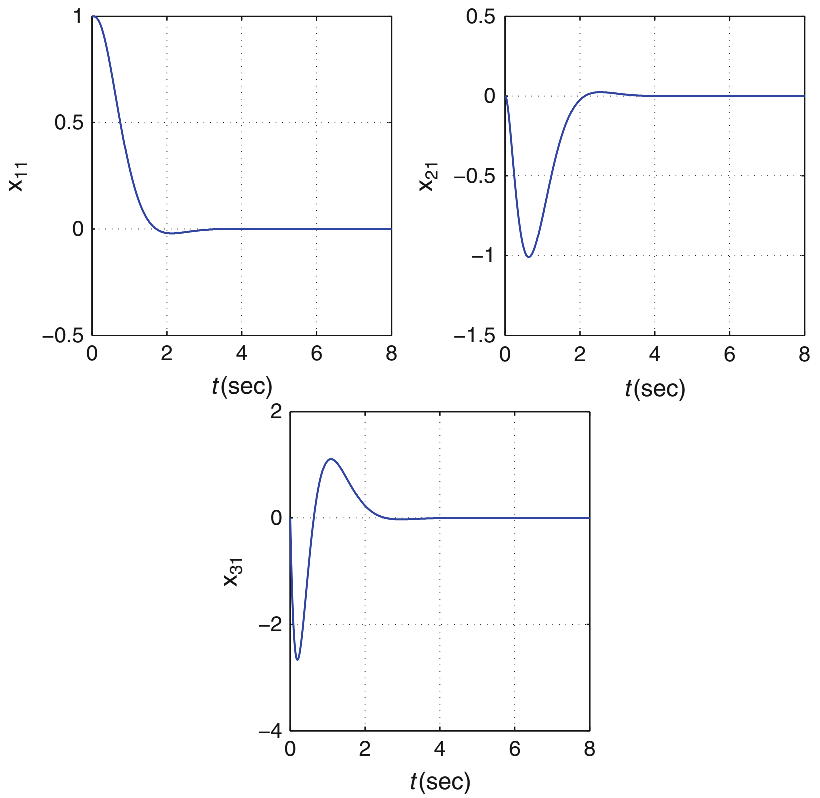

Fig. 10.12 Nitrogen group trajectories under switched state feedback

Mode 1:

$$A_j = \begin{bmatrix} -24.3 & 1.2 \\ -4.95 & -2.1 \end{bmatrix}, \ D_j = \begin{bmatrix} -1.2 & 0 \\ 0 & -0.83 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ G_j^t = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}, \ H_j^t = \begin{bmatrix} 0.04 \\ 0.03 \end{bmatrix}, \ \Gamma_j = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}, \ \Phi_j = 0.7$$

$$E_j = \begin{bmatrix} 0.75 & 0.1 \end{bmatrix}, \ E_{dj} = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}$$

Mode 2:

$$A_j = \begin{bmatrix} -12.0 & 6.9 \\ -3.1 & -14.1 \end{bmatrix}, \ D_j = \begin{bmatrix} -1.3 & 0 \\ 0 & -0.55 \end{bmatrix}$$

$$B_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ G_j^t = \begin{bmatrix} 0.03 \\ 0.03 \end{bmatrix}, \ H_j^t = \begin{bmatrix} 0.06 \\ 0.04 \end{bmatrix}, \ \Gamma_j = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, \ \Phi_j = 0.8$$

$$E_j = \begin{bmatrix} 0.5 & 0.4 \end{bmatrix}, \ E_{dj} = \begin{bmatrix} 0.32 & 0.3 \end{bmatrix}$$

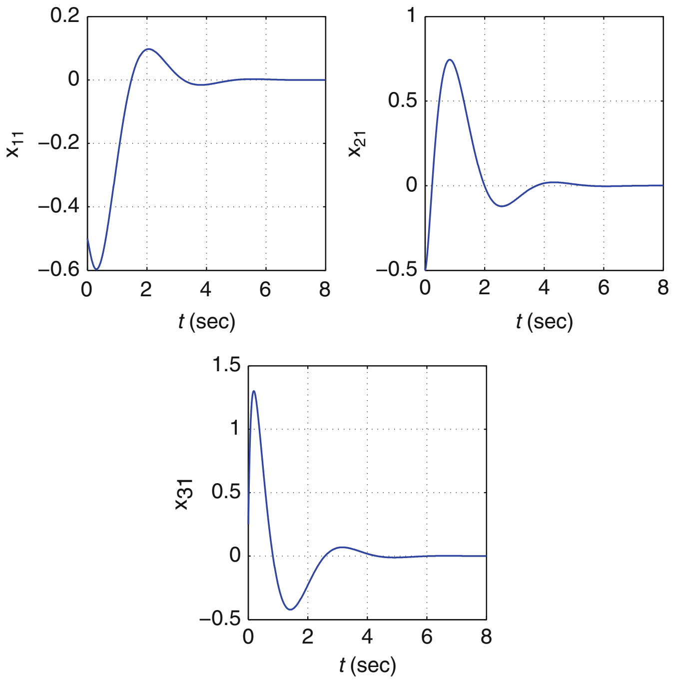

Fig. 10.13 Phosphate-BOD-DO group trajectories under switched state feedback

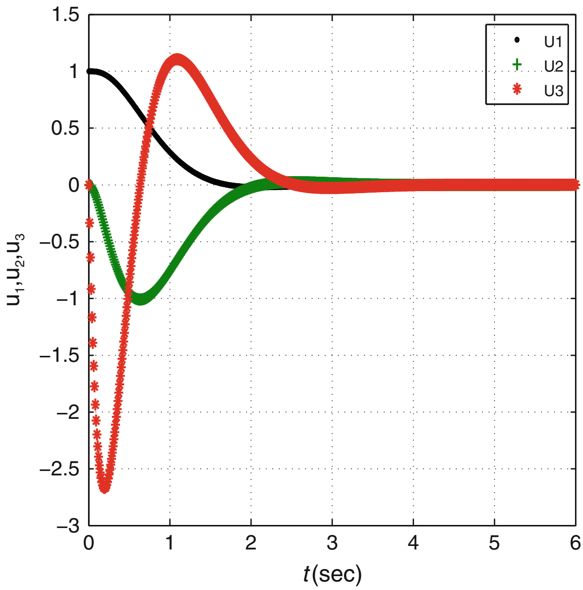

Fig. 10.14 Water-control trajectories under switched state feedback

Application of **Theorem 10.14** yields the following feasible set of results

Mode 1:

$$\gamma_j = 0.808,\ \varphi_j = 0.375,\ \varrho_j = 2.776,\ \mu_j = 1.212$$
$$\mathcal{K}_{j1} = \begin{bmatrix} -1.44 & -0.02 \end{bmatrix}$$

Mode 2:

$$\gamma_j = 0.911,\ \varphi_j = 0.425,\ \varrho_j = 2.855,\ \mu_j = 1.311$$
$$\mathcal{K}_{j2} = \begin{bmatrix} 0.01 & -0.74 \end{bmatrix}$$

10.3 Notes and References

A decentralized approach has been developed to the robust stability and stabilization problems of class of nonlinear interconnected discrete-time systems with arbitrary switched rules. This class of interconnected systems has been considered to consist of coupled nominally linear subsystems with unknown-but-bounded state delay. We have shown that multi-controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems with large uncertainties. We have developed a delay-dependent decentralized structure that guarantees asymptotic stability with local disturbance attenuation on the subsystem level. Then, we have constructed decentralized switched control schemes based on state feedback and dynamic output feedback to ensure stabilizability of the global system with ℓ_2-performance bound. The theoretical developments have been demonstrated by numerical simulation of a multi-reach water-pollution control problem. The analysis pursued in this chapter represents a new and preliminary avenue in the theory of switched time-delay systems, which should attract the attention of researchers and readers to attempt to move further and attain subsequent results. The reader is advised to look at [8] for more useful avenues.

Part VI
Applications

Chapter 11
Applications to Water-Quality Control

Switched systems have been intensively studied in recent years due to their widespread applications, including power systems and chemical processes [41] and mechanical systems [47]. A wide class of switched systems composed of a finite family of continuous- or discrete-time subsystems (called modes) and a rule that governs the switching between these subsystems. Each mode is regarded as a state of a finite-state machine whose evolution dynamics determines the switching. In this work, we consider that the modes are represented by discrete-time linear systems with unknown-but-bounded delays. Research investigations into problems pertaining to switched dynamical systems have received a great deal of attention because of the fast development in computing technologies [15, 28, 42, 47, 174, 193] and the references cited therein.

The phenomena of delays, on the contrary, is often encountered in several physical and man-made systems due to finite capabilities of information processing, inherent phenomena like mass transport flow and recycling and/or by product of computational delays [216]. The available results can be categorized into two broad categories: *delay-independent* and *delay-dependent*. In the former category, the stability and stabilization results are feasible irrespective of the size of the delay. In the latter category, methods are developed to take the information about time delay into consideration in the process of controller design. By and large, *delay-dependent* methods are regarded as more practical and yield less conservative designs. Results on discrete-time systems with state delay are found in [25, 159, 213] and recent developments of delay-dependent stability and control are presented in [34, 71].

11.1 Application I: Water-Quality Control

In this section, the results of [26, 44, 224–234, 239–243, 246–248, 251, 252, 259–261, 268, 270–272, 274, 275, 279–281, 283–286, 288–325, 327–329, 331–334, 421] are extended further to the class of switched discrete-time systems with unknown-but-bounded state delay. The analytical development is inspired by the contemporary research activities in water resources development, planning, and management [119, 294, 363]. Through a multi-model representation of preserving

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water-quality constituents to standard in multi-reach fresh water streams, the problem of water-quality control is cast as the problem of delay-dependent $\mathcal{L}_2$ gain analysis and is fully investigated. New delay-dependent asymptotic stability criteria are developed under arbitrary switching based on appropriately constructed switched Lyapunov–Krasovskii functionals and cast as feasibility testing of linear matrix inequalities (LMIs). Switched control synthesis is performed to design switched feedback schemes, based on state- or output-measurements, to guarantee that the corresponding closed-loop system enjoys the delay-dependent asymptotic stability with an $\mathcal{L}_2$ gain smaller than a prescribed constant level. All the developed results are expressed in terms of convex optimization over LMIs and tested on a representative water-quality example of the River Nile.

11.1.1 Motivating Example

It has been widely recognized that water and related resources systems play a major role in the socio-economic development due to their long-range economic effects [119]. Perhaps the most significant environmental problems nowadays is that of excessive pollution levels in streams of fresh water as these are usually the main source for potable supply. The problems arise from the discharge of industrial wastes and sewage into the water basin, which in turn disturb the balance of the ecological system. There has been increased activity by different environmental boards toward preserving water-quality standards. This calls for integrated approaches [363]. It turns out that one effective method for studying water-quality standards in streams using mathematical modeling and computer control is by simulating the in-stream interaction of the chemical and biochemical constituents in steady-flow rivers. It turns out under reasonable simplifying assumptions [107, 117, 253] that a comprehensive picture about the growth of water-quality constituents can be described by a linearized model evaluated about several operating points (multiple modes). By considering a representative reach of the River Nile subjected to dumped wastewater and applying computer control methods [254, 277], the water-quality model can then be expressed in multi-model form as

$$
\begin{aligned}
x_{k+1} &= A_{\xi} x_k + A_{d\xi} x_{k-d_k} + \Gamma_{\xi} \omega_k \\
y_k &= C_{\xi} x_k + C_{d\xi} x_{k-d_k} + \Psi_{\xi} \omega_k \\
z_k &= G_{\xi} x_k + G_{d\xi} x_{k-d_k} + \Phi_{\xi} \omega_k
\end{aligned} \tag{11.1}
$$

where $d_k > 0$ a time-varying delay reflecting the mixing effect of biochemical constituents in the reach at time k,[1] $x_k \in \Re^n$ is the state vector of water-quality constituents (like algae, ammonia nitrogen, dissolved oxygen, biochemical oxygen demand), $y_k \in \Re^p$ is the measured output vector, $z_k \in \Re^q$ is the performance

[1] In systems terminology, the delay factor represents an average time to clear up the water-stream and through control effects, water-quality constituents are brought back to their standard levels.

vector, $\omega_k \in \Re^q$ is the disturbance input (due to irregular discharge of effluents) which is assumed to belong to $\ell_2[0, \infty)$, and ξ is a switching rule which takes its values in the finite set modes $\mathbf{N} = \{1, 2, ..., N\}$. This means that the matrices $(A_\alpha, A_{d\alpha}, ..., \Phi_\alpha)$ take values, at arbitrary discrete instants, in the finite set of

$$\{(A_1, A_{d1}, ..., \Phi_1), ..., (A_N, A_{dN}, ..., \Phi_N)\}$$

Thus the matrices $(A_j, A_{dj}, ..., \Phi_j)$ denotes the jth water-quality model corresponding to operational mode j and hence (11.1) represents a time-controlled switched system [42]. Typically, the switching rule ξ is not known a priori but we assume its instantaneous value is available in real time for practical implementations by water pollution management. Define the indicator function

$$\begin{aligned}
\alpha(k) &= [\alpha_1(k), ..., \alpha_N(k)]^t, \quad \forall\, i \in \mathbf{N} \\
\alpha_i(k) &= 1 \quad \text{when system (11.1) is in the } i\text{th mode,} \\
&\quad (A_N, A_{dN}, ..., \Phi_N) \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

Thus we cast system (11.1) into the form

$$\begin{aligned}
x_{k+1} &= \sum_{i=1}^{N} \alpha_i(k) A_i x_k + \sum_{i=1}^{N} \alpha_i(k) A_{di} x_{k-d_k} + \sum_{i=1}^{N} \alpha_i(k) \Gamma_i \omega_k \\
y_k &= \sum_{i=1}^{N} \alpha_i(k) C_i x_k + \sum_{i=1}^{N} \alpha_i(k) C_{di} x_{k-d_k} + \sum_{i=1}^{N} \alpha_i(k) \Psi_i \omega_k \\
z_k &= \sum_{i=1}^{N} \alpha_i(k) G_i x_k + \sum_{i=1}^{N} \xi_i(k) G_{di} x_{k-d_k} \\
&\quad + \sum_{i=1}^{N} \alpha_i(k) \Phi_i \omega_k
\end{aligned} \tag{11.2}$$

It is significant that system (11.2) is quite general in the context of switched time-delay systems as in includes state, measurement, and output delays. In this paper, we are interested in the stability analysis and control synthesis for this class of discrete-time switched systems with time-varying delays. In the sequel, it is assumed that the delay d_k is a time-varying function satisfying

$$d_m \le d_k \le d_M \tag{11.3}$$

where the lower bound $d_m > 0$ and the upper bound $d_M > 0$ are known constant scalars.

11.1.2 Delay-Dependent $\mathcal{L}_2$ Gain Analysis

In this section, we develop new criteria for LMI-based characterization of delay-dependent asymptotic stability and $\mathcal{L}_2$ gain analysis. Introduce

$$\begin{aligned} \delta x_k &= x_{k+1} - x_k,\ x_{k-d_k} = x_k - \sum_{j=k-d_k}^{k-1} \delta x_j \\ \delta x_k &= (A_j - I)x_k + A_{dj}\ x_{k-d_k} + \Gamma_j w_k \\ \zeta_k &= \begin{bmatrix} x_k^t & x_{k-d_k}^t & x_{k-d_M}^t \end{bmatrix}^t,\ \overline{d} = (d_M - d_m + 1) \end{aligned} \tag{11.4}$$

The following theorem establishes the main LMI-based stability result for switched system (11.2):

Theorem 11.1 *Given the bounds $d_M > 0$ and $d_m > 0$. System (11.2) is delay-dependent asymptotically stable with ℓ_2-performance bound γ_i if there exist weighting matrices $\mathcal{P}_i,\ \mathcal{P}_j$,*
$\mathcal{Q}_s,\ \mathcal{R}_i,\ \mathcal{W}_{ai},\ \mathcal{W}_{ci}$, parameter matrices $\mathcal{M}_i,\ \mathcal{S}_i,\ \mathcal{Z}_i,\ \forall (i,j,s) \in \boldsymbol{N}$ and scalars $\gamma_i > 0$ satisfying the following LMIs for $(i,j,s) \in \boldsymbol{N}$

$$\Omega_i = \begin{bmatrix} \bar{\Omega} + \Omega_{ai} + \Omega_{ai}^t + \Omega_{ci} & \Omega_{zi} \\ \bullet & -\Omega_{wi} \end{bmatrix} < 0 \tag{11.5}$$

where

$$\begin{aligned} \bar{\Omega} &= \begin{bmatrix} \Omega_{oj} & \Omega_{mj} & 0 & \Omega_{nj} \\ \bullet & \Omega_{qs} & 0 & \Omega_{sj} \\ \bullet & \bullet & -\mathcal{R}_i & 0 \\ \bullet & \bullet & \bullet & -\Omega_{vj} \end{bmatrix} \\ \Omega_{oj} &= A_i^t \mathcal{P}_j A_i - \mathcal{P}_i + \overline{d}\mathcal{Q}_i + \mathcal{R}_i + G_i^t G_i \\ \Omega_{mj} &= A_i^t \mathcal{P}_j A_{di} + G_i^t G_{di},\ \Omega_{nj} = A_i^t \mathcal{P}_j \Gamma_i + G_i^t \Phi_i \\ \Omega_{ai} &= \begin{bmatrix} \mathcal{M}_i + \mathcal{Z}_i & \mathcal{S}_i - \mathcal{M}_i & -\mathcal{S}_i - \mathcal{Z}_i & 0 \end{bmatrix} \\ \Omega_{qs} &= A_{di}^t \mathcal{P}_j A_{di} - \mathcal{Q}_s + G_{di}^t G_{di} \\ \Omega_{sj} &= A_{di}^t \mathcal{P}_j \Gamma_i + G_{di}^t \Phi_i,\ \Omega_{ci} = d_M \Omega_{cci}^t\ (\mathcal{W}_{ai} + \mathcal{W}_{ci})\ \Omega_{cci} \\ \Omega_{cci} &= \begin{bmatrix} A_i - I & A_{di} & 0 & I \end{bmatrix} \\ \Omega_{zi} &= \begin{bmatrix} \sqrt{d_M}\mathcal{M}_i & \sqrt{d_M - d_m}\mathcal{S}_i & \sqrt{d_M}\mathcal{Z}_i \end{bmatrix} \\ \Omega_{vj} &= \gamma_i^2 I - \Gamma_i^t \mathcal{P}_j \Gamma_i - \Phi_i^t \Phi_i \\ \Omega_{wi} &= \text{diag}\begin{bmatrix} \mathcal{W}_{ai} & \mathcal{W}_{ai} & \mathcal{W}_{ci} \end{bmatrix} \end{aligned} \tag{11.6}$$

Proof See the Appendix

Remark 11.2 It should be noted that the lower bound d_m and the upper bound d_M account for extreme cases of light and heavy waste dump loadings, respectively.

System stability and stabilization of water and related resources systems are generally expressed in terms of algebraic Riccati inequalities (ARIs). Seeking computational convenience and effectiveness, the solutions to the problems of stability analysis and control synthesis are cast into convex optimization that are handled using interior-point minimization algorithms that have been recently coded into efficient numerical software [74]. It is remarked that LMIs and ARIs are equivalent [27]; however, parameter tuning intrinsic to the ARIs can be avoided by using the framework of feasibility testing of linear matrix inequalities (LMIs). It is crucial to observe that **Theorem 11.1** provides least-conservative stability criteria since it employs reduced number of LMI variables and does not rely on overbounding relations and inequalities.

Remark 11.3 In comparison with the published results [25, 34, 71, 82, 159, 213, 262, 354], it is crucial to observe that **Theorem 11.1** provides least-conservative stability criteria since it employs reduced number of LMI variables and does not rely on overbounding relations and inequalities.

11.1.3 Switched Feedback Control

Next, we address the feedback control problem for discrete-time switched system

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{N} \alpha_i(k) A_i x_k + \sum_{i=1}^{N} \alpha_i(k) A_{di} x_{k-d_k} \\ &\quad + \sum_{i=1}^{N} \alpha_i(k) B_i u_k + \sum_{i=1}^{N} \alpha_i(k) \Gamma_i \omega_k \\ y_k &= \sum_{i=1}^{N} \alpha_i(k) C_i x_k + \sum_{i=1}^{N} \alpha_i(k) C_{di} x_{k-d_k} \\ &\quad + \sum_{i=1}^{N} \alpha_i(k) \Psi_i \omega_k \\ z_k &= \sum_{i=1}^{N} \alpha_i(k) G_i x_k + \sum_{i=1}^{N} \xi_i(k) G_{di} x_{k-d_k} \\ &\quad + \sum_{i=1}^{N} \alpha_i(k) \Phi_i \omega_k \end{aligned} \tag{11.7}$$

where $u_k \in \Re^m$ is the control input. In case of water-quality system, the control inputs correspond to command signals proportional to change in stream velocity and after-treated discharges from waste water facilities. In the sequel, two switched feedback schemes are considered: one utilizes state feedback and the other is based on output feedback.

11.1.4 Switched State Feedback

With reference to system (11.7), we seek to design a switched state feedback

$$u_k = \sum_{i=1}^{N} \alpha_i(k) K_i \, x_k, \; i \in \mathbf{N}$$

that guarantees the controlled switched system achieves a prescribed performance level. This corresponds to regulating the water-quality constituents to practical standards. Letting $\mathcal{A}_i = A_i + B_i K_i$, it is readily seen from Theorem 11.1 that the closed-loop switched system is delay-dependent asymptotically stable with an $\mathcal{L}_2 - gain < \gamma$ if there exist weighting matrices $\mathcal{Q}_i, \; \mathcal{Q}_s, \; \mathcal{P}_i, \; \mathcal{P}_j, \; \mathcal{R}_i, \; \mathcal{W}_{ai}, \; \mathcal{W}_{ci},$ parameter matrices $\mathcal{M}_i, \; \mathcal{S}_i, \; \mathcal{Z}_i, \; \forall(i, j, s) \in \mathbf{N}$ and scalars $\gamma_i > 0$ satisfying the following LMIs for $(i, j, s) \in \mathbf{N}$

$$\begin{bmatrix} \Theta_{oj} & \Theta_{mj} & \Theta_{xi} & \Theta_{nj} & d_M \Theta_{cci}^t \mathcal{W}_i & \Omega_{zi} \\ \bullet & \Theta_{qs} & 0 & \Omega_{sj} & 0 & 0 \\ \bullet & \bullet & -\mathcal{R}_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\Omega_{vv} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -d_M \mathcal{W}_i & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Omega_{wi} \end{bmatrix} < 0$$

$$\begin{aligned} \Theta_{oj} &= \mathcal{A}_i^t \mathcal{P}_j \mathcal{A}_i - \mathcal{P}_i + \overline{d} \mathcal{Q}_i + \mathcal{R}_i + G_i^t G_i \\ &\quad + \mathcal{M}_i + \mathcal{Z}_i + \mathcal{M}_i^t + \mathcal{Z}_i^t, \; \mathcal{W}_i = (\mathcal{W}_{ai} + \mathcal{W}_{ci}) \\ \Theta_{qs} &= A_{di}^t \mathcal{P}_j A_{di} - \mathcal{Q}_s + G_{di}^t G_{di} \\ \Theta_{cci} &= \left[\mathcal{A}_i - I \;\; A_{di} \;\; 0 \;\; I \right] \\ \Theta_{mj} &= \mathcal{A}_i^t \mathcal{P}_j A_{di} + G_i^t G_{di} + \mathcal{S}_i - \mathcal{M}_i \\ \Theta_{nj} &= \mathcal{A}_i^t \mathcal{P}_j \Gamma_i + G_i^t \Phi_i, \; \Theta_{xi} = -\mathcal{S}_i - \mathcal{Z}_i \end{aligned} \tag{11.8}$$

The following theorem states the main design result

Theorem 11.4 *Let the bounds $d_M > 0$ and $d_m > 0$ and the matrices $\mathcal{W}_{ai} > 0, \; \mathcal{W}_{ci} > 0$ be given. Then system (11.2) with $u_k = \sum_{i=1}^{N} \alpha_i(k) K_i \, x_k, \; i \in \boldsymbol{N}$ is delay-dependent asymptotically stable with ℓ_2-performance bound γ if there exist parameter matrices $\mathcal{X}_i, \; \mathcal{R}_i, \; \mathcal{X}_j, \; \mathcal{Q}_s,$*
$\mathcal{Y}_i, \; \{\widehat{\Theta}\}_1^8$ and scalars $\gamma_i > 0$ satisfying the following LMIs for $(i, j, s) \in \boldsymbol{N}$

$$\begin{bmatrix} -\widehat{\Theta}_{si} & \widehat{\Theta}_4 - \widehat{\Theta}_3 & -\widehat{\Theta}_4 - \widehat{\Theta}_5 & 0 & d_M \widehat{\Theta}_{vi} & \widehat{\Theta}_w & \widehat{\Theta}_{cj} & \mathcal{X}_i G_i^t \\ \bullet & -\widehat{\Theta}_8 & 0 & 0 & 0 & 0 & A_{di}^t & G_{di}^t \\ \bullet & \bullet & -\mathcal{R}_i & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma_i^2 I & 0 & 0 & \Gamma_i & \Phi_i^t \\ \bullet & \bullet & \bullet & \bullet & -d_M \mathcal{W}_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\widehat{\Theta}_k & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \quad (11.9)$$

$$\begin{aligned} \widehat{\Theta}_{si} &= \mathcal{X}_i - \overline{d}\widehat{\Theta}_1 - \widehat{\Theta}_2 - \widehat{\Theta}_4 - \widehat{\Theta}_4^t - \widehat{\Theta}_5 - \widehat{\Theta}_5^t, \\ \widehat{\Theta}_k &= \mathrm{diag}\left[\, \widehat{\Theta}_6 \;\; \widehat{\Theta}_6 \;\; \widehat{\Theta}_7 \,\right] \\ \widehat{\Theta}_{cj} &= \mathcal{X}_j A_i^t + \mathcal{Y}_j^t B_i^t, \;\; \widehat{\Theta}_{vi} = \left(\mathcal{X}_i A_i^t - \mathcal{X}_i + \mathcal{Y}_i^t B_i^t\right) \mathcal{W}_i \\ \widehat{\Theta}_w &= \left[\, \sqrt{d_M}\widehat{\Theta}_3 \;\; \sqrt{d_M - d_m}\widehat{\Theta}_4 \;\; \sqrt{d_M}\widehat{\Theta}_5 \,\right] \end{aligned} \quad (11.10)$$

Moreover, the feedback gain matrix is given by $K_i = \mathcal{Y}_i \mathcal{X}_i^{-1}$

Proof See the Appendix

11.1.5 Switched Static Output Feedback

Now, the objective is to design a switched output feedback $u_k = \sum_{i=1}^{N} \alpha_i(k) H_i \; y_k$, $\forall i \in \mathbf{N}$, where the set of output matrices $\{C_i, \; i \in \mathbf{N}\}$ are assumed to be of full-row rank, such that the switched closed-loop system

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{N} \alpha_i(k) \widehat{A}_i x_k + \sum_{i=1}^{N} \alpha_i(k) \widehat{A}_{di} x_{k-d_k} \\ &\quad + \sum_{i=1}^{N} \alpha_i(k) \widehat{\Gamma}_i \omega_k \\ z_k &= \sum_{i=1}^{N} \alpha_i(k) \widehat{G}_i x_k + \sum_{i=1}^{N} \alpha_i(k) G_{di} x_{k-d_k} \\ &\quad + \sum_{i=1}^{N} \alpha_i(k) \Phi_i \omega_k \\ \widehat{A}_i &= A_i + B_i H_i C_i, \;\; \widehat{A}_{di} = A_{di} + B_i H_i C_{di} \\ \widehat{\Gamma}_i &= \Gamma_i + B_i H_i \Psi_i, \;\; \widehat{G}_i = G_i + D_i H_i C_i \end{aligned} \quad (11.11)$$

is delay-dependent asymptotically stable with an $\mathcal{L}_2-$ gain $< \gamma$. Therefore, in line with the preceding section, it follows from Theorem 11.1 that switched system (11.2) is delay-dependent asymptotically stable if there exist weighting matrices

$\mathcal{P}_i,\ \mathcal{P}_j,\ \mathcal{Q}_s,\ \mathcal{R}_i,\ \mathcal{W}_{ai},\ \mathcal{W}_{ci}$, parameter matrices $\mathcal{M}_i,\ \mathcal{S}_i,\ \mathcal{Z}_i,\ \forall(i,j,s)\ \in\ \mathbf{N}$ and scalars $\gamma_i > 0$ satisfying the following LMIs for $(i,j,s)\ \in\ \mathbf{N}$

$$\widetilde{\Theta} = \begin{bmatrix} -\hat{\Theta}_i & \hat{\Theta}_{mi} & \Theta_{xi} & \Theta_{nj} & d_M \Omega_{cc}^t \mathcal{W} & \Omega_{zi} & \widehat{A}_i^t \mathcal{P}_j & \widehat{G}_i^t \\ \bullet & -\mathcal{Q}_s & 0 & 0 & 0 & 0 & \widehat{A}_{di}^t \mathcal{P}_j & G_{di}^t \\ \bullet & \bullet & -\mathcal{R}_i & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma_i^2 I & 0 & 0 & \widehat{\Gamma}_i \mathcal{P}_j & \Phi_i^t \\ \bullet & \bullet & \bullet & \bullet & -d_M \mathcal{W}_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Omega_{wi} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\mathcal{P}_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

$$-\hat{\Theta}_i = -\mathcal{P}_i + \overline{d}\mathcal{Q}_i + \mathcal{R}_i + \mathcal{M}_i + \mathcal{Z}_i + \mathcal{M}_i^t + \mathcal{Z}_i^t,$$
$$\hat{\Theta}_{mi} = \mathcal{S}_i - \mathcal{M}_i \tag{11.12}$$

where the matrices $\Theta_{xi},\ \Theta_{nj},\ \Omega_{zi},\ \Omega_{wi}$ are given in (11.8). The following theorem states the main design result

Theorem 11.5 *Let the bounds $d_M > 0$ and $d_m > 0$ and the matrices $\mathcal{W}_{ai} > 0,\ \mathcal{W}_{ci} > 0$ be given. Then system (11.2) with $u_k = \sum_{i=1}^N \alpha_i(k) H_i\ y_k,\ \forall i \in \boldsymbol{N}$ is delay-dependent asymptotically stable with ℓ_2-performance bound γ if there exist parameter matrices $\widehat{\mathcal{X}}_i,\ \widehat{\mathcal{X}}_j,\ \mathcal{R}_i,\ \mathcal{G}_j,\ \mathcal{F}_j,\ \{\widehat{\Theta}\}_1^8$ and scalars $\gamma_i > 0$ satisfying the following LMIs for $(i,j,s)\ \in\ \boldsymbol{N}$*

$$\begin{bmatrix} -\widehat{\Theta}_{si} & \widehat{\Theta}_4 - \widehat{\Theta}_3 & -\widehat{\Theta}_4 - \widehat{\Theta}_5 & 0 & \mathrm{d}_M \widehat{\Theta}_{vi} & \widehat{\Theta}_w & \widetilde{\Theta}_{cj}^t & \widehat{\Theta}_{ei} \\ \bullet & -\widehat{\Theta}_8 & 0 & 0 & 0 & 0 & \widehat{\Theta}_{dj}^t & G_{di}^t \\ \bullet & \bullet & -\mathcal{R}_i & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma_i^2 I & 0 & 0 & \Gamma_i & \Phi_i^t \\ \bullet & \bullet & \bullet & \bullet & -\mathrm{d}_M \mathcal{W}_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\widehat{\Theta}_k & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\widehat{\mathcal{X}}_j & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{11.13}$$

$$\widetilde{\Theta}_{cj}^t = \widehat{\mathcal{X}}_j A_i^t + C_i^t \mathcal{G}_j^t B_i^t,\ \widetilde{\Theta}_{dj} = \widehat{\mathcal{X}}_j A_{di}^t + \mathcal{F}_j^t B_i^t \tag{11.14}$$

Moreover, the feedback gain matrix is given by $H_i = R_i C_i \widehat{X}_i C_i^{\dagger}$[2]

Proof See the Appendix.

Finally, we provide a numerical simulation example.

---

[2] $C_i^{\dagger}$ is the right Moore–Penrose inverse of $C_i = C_i^t \left(C_i C_i^t\right)^{-1}$ since $\mathrm{rank}[C_i] = p$.

Illustrative Example

A discrete water-pollution model of the type (11.2) with multiple operating points is considered. The model represents two aggregate bio-strata, the one for algae and the other for ammonia products. We wish to design switched feedback controllers for this system based on **Theorems 11.4** and **11.5**. Switching occurs between three modes described by the following coefficients:

Mode 1

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.6 & 0 \\ 0.2 & 0.3 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

$$C_{d1} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix},\ G_{d1} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

$$\Phi_1 = [0.6],\ \Psi_1 = [0.01]$$

Mode 2

$$A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -0.5 & 0.1 \\ 0 & -0.4 \end{bmatrix},\ \Gamma_2 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$$

$$C_{d2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},\ C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix},\ G_{d2} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$$

$$\Phi_2 = [0.3],\ \Psi_2 = [0.02]$$

Mode 3

$$A_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.6 & 0.3 \end{bmatrix},\ A_{d3} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix},\ \Gamma_3 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

$$C_{d3} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},\ C_3 = \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix},\ G_{d3} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

$$\Psi_3 = [0.1],\ \Psi_2 = [0.02]$$

Note that single control action (either proportional to change in stream velocity or pretreated waste water) is used in modes 1 and 2 and both control actions are used in mode 3. Choosing $\underline{d}_m = 2,\ d_M = 6$ and invoking the software environment [74], the feasible solution of LMIs (11.9)-(11.10) yields the state feedback gains:

$$\gamma_1 = 2.3411, \; K_1 = \begin{bmatrix} -0.1144 & -0.6754 \end{bmatrix}$$
$$\gamma_2 = 3.5448, \; K_2 = \begin{bmatrix} -0.5539 & -0.8685 \end{bmatrix}$$
$$\gamma_3 = 1.8694, \; K_3 = \begin{bmatrix} -0.3313 & -0.2306 \end{bmatrix}$$

On the contrary, the feasible solution of LMIs (11.13)-(11.14) yields the output-feedback gains:

$$\gamma_1 = 1.5421, \; H_1 = \begin{bmatrix} -0.7184 & -0.6716 \end{bmatrix}$$
$$\gamma_2 = 2.1367, \; H_2 = \begin{bmatrix} -1.0345 & -0.4415 \end{bmatrix}$$
$$\gamma_3 = 1.3742, \; H_3 = \begin{bmatrix} -0.5977 & -0.4705 \end{bmatrix}$$

These results validate our theoretical developments. To further show the validity of our design method, we simulate the closed-loop water-quality system using the disturbance $\omega_k = 0.04\exp(0.04k)\sin(0.05\pi k)$, a randomly generated switching signal from a uniform distribution in the interval $(0, 1)$. The obtained state and control trajectories from 400 samples over a time horizon of 10 s are plotted in Figs. 11.1, 11.2, 11.3, and 11.4. It is readily seen from the computational results that the switched dynamic controller is more effective in clearing the disturbed water system than the switched static controller. This takes place on the expense of being more involved to realize.

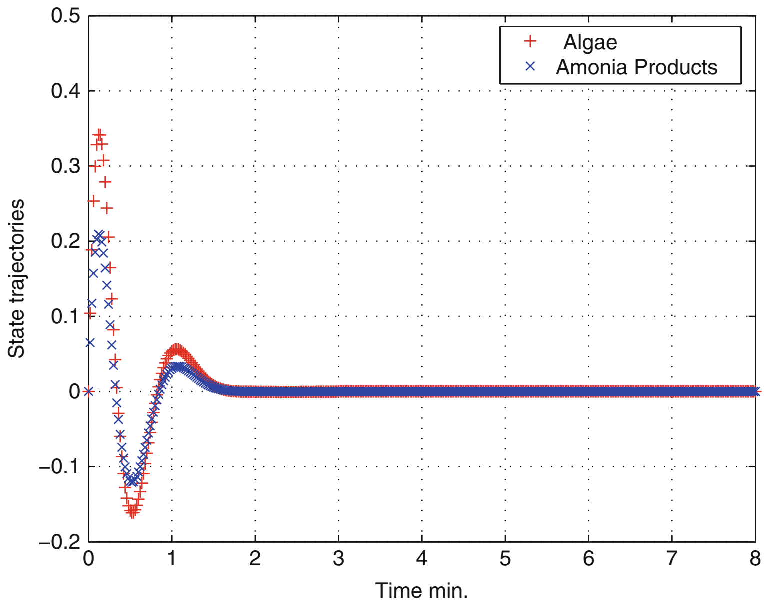

Fig. 11.1 Algae and ammonia trajectories under switched state feedback

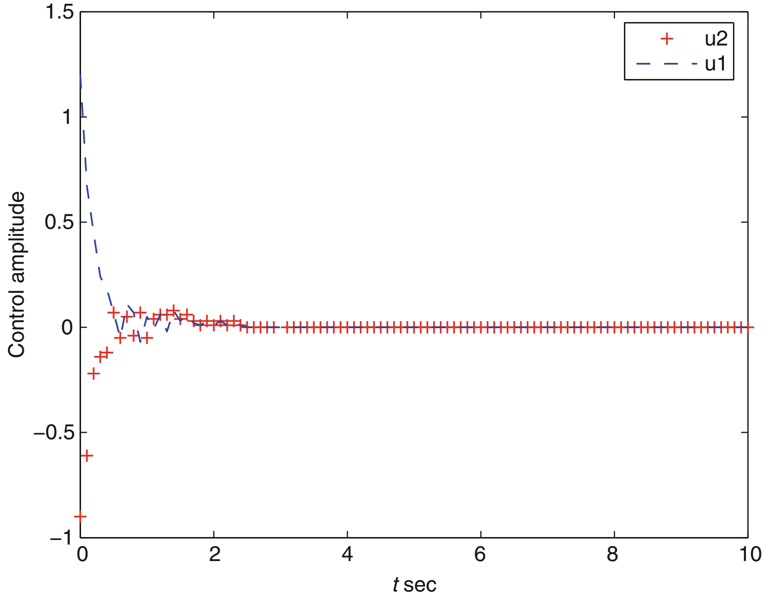

Fig. 11.2 Trajectories of switched state feedback control

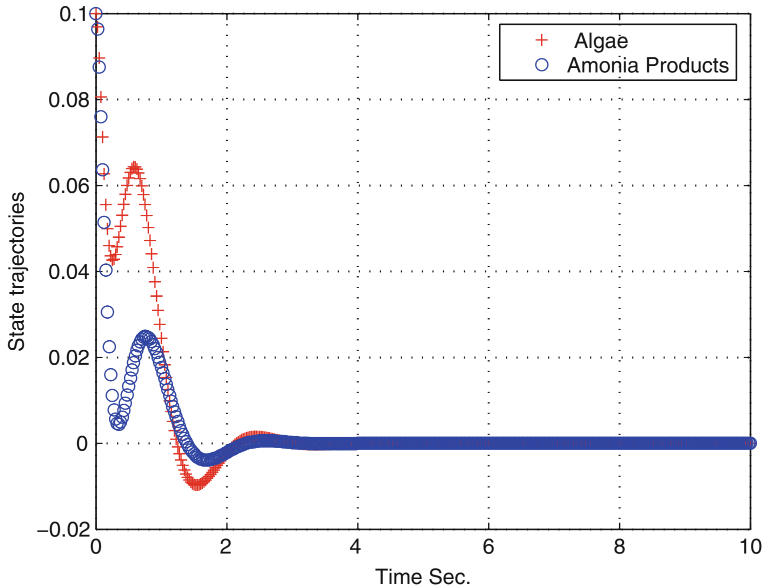

Fig. 11.3 Algae and ammonia trajectories under switched output feedback

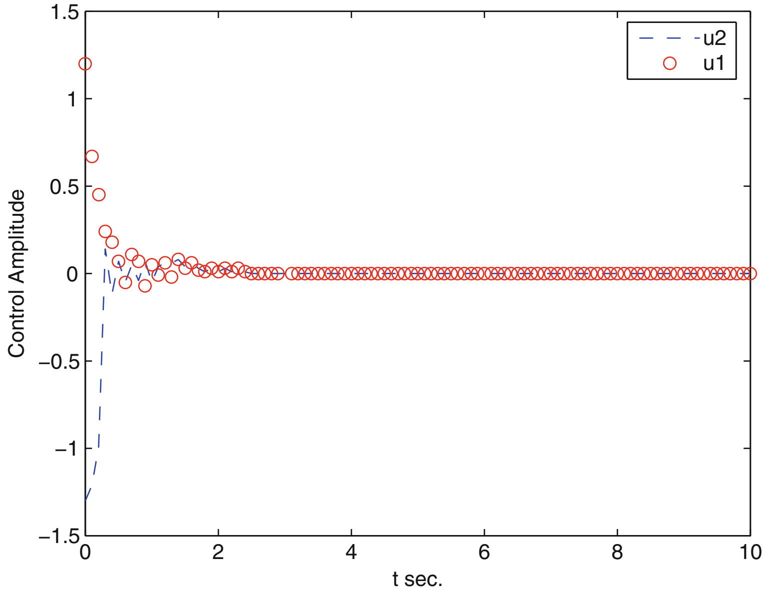

Fig. 11.4 Trajectories of switched output feedback control

11.2 Appendix

11.2.1 Proof of Theorem 11.1

Consider the switched Lyapunov–Krasovskii functional (SLKF):

$$\begin{aligned}
V_k &= V_{ok} + V_{ak} + V_{ck} + V_{mk} + V_{nk} \\
V_{ok} &= x_k^t \sum_{i=1}^{N} \alpha_i(k) \mathcal{P}_i x_k, \;\; V_{ak} = \sum_{j=k-d_k}^{k-1} x_j^t \sum_{i=1}^{N} \alpha_i(k) \mathcal{Q}_i x_j \\
V_{ck} &= \sum_{k-d_M}^{k-1} x_j^t \sum_{i=1}^{N} \alpha_i(k) \mathcal{R}_i x_j \\
V_{nk} &= \sum_{m=-d_M}^{-1} \sum_{j=k+m}^{k-1} \delta x_j^t \sum_{i=1}^{N} \alpha_i(k) (\mathcal{W}_{ai} + \mathcal{W}_{ci}) \delta x_j \\
V_{mk} &= \sum_{m=-d_M+1}^{-d_m} \sum_{j=k+m}^{k-1} x_j^t \sum_{i=1}^{N} \alpha_i(k) \mathcal{Q}_i x_j
\end{aligned} \tag{11.15}$$

where $\mathcal{P}_i, \; \mathcal{Q}_i, \; \mathcal{R}_i, \; \mathcal{W}_{ai}, \; \mathcal{W}_{ci}$ are weighting matrices of appropriate dimensions. The first term in (8.8) is standard to the delayless nominal system while the second and fifth correspond to the delay-dependent conditions. The third and fourth terms are added to compensate for the enlargement in the time interval from $(k-1 \rightarrow d-d_k)$ to $(k-1 \rightarrow d-d_M)$. A straightforward computation gives the first-difference of $\Delta V_k = V_{k+1} - V_k$ along the solutions of (11.2) with $w_k \equiv 0$ as

$$
\begin{aligned}
&\Delta V_k \le x_{k+1}^t \sum_{i=1}^{N} \alpha_i(k+1)\mathcal{P}_i x_{k+1} - x_k^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{P}_i x_k + \\
&x_k^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_i x_k - x_{k-d_k}^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_i x_{k-d_k} + \\
&\sum_{j=k-d_M+1}^{k-d_m} x_j^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_i x_j + x_k^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_i x_k - \\
&x_{k-d_M}^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{R}_i x_{k-d_M} + \\
&(d_M - d_m) x_k^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_i x_k - \sum_{j=k-d_M+1}^{k-d_m} x_j^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{Q}_i x_j \\
&+ d_M \delta x_k^t \sum_{i=1}^{N} \alpha_i(k)(\mathcal{W}_{ai} + \mathcal{W}_{ci})\delta x_k - \\
&\sum_{s=k-d_k}^{k-1} \delta x_s^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{ai}\delta x_s - \sum_{s=k-d_M}^{k-d_k-1} \delta x_s^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{ai}\delta x_s \\
&- \sum_{s=k-d_M}^{k-1} \delta x_s^t \sum_{i=1}^{N} \alpha_i(k)\mathcal{W}_{ci}\delta x_s < 0
\end{aligned}
\tag{11.16}
$$

Since (11.16) has to be satisfied under arbitrary switching, it follows that this holds for the particular case $\alpha_i(k) = 1,\ \alpha_{m\neq i}(k) = 0,\ \alpha_j(k+1) = 1,\ \alpha_{m\neq j}(k+1) = 0,$ $\alpha_s(k-d_k) = 1$ and $\alpha_{m\neq s}(k-d_k) = 0$. Together with the following identities:

$$
\begin{aligned}
2\zeta_k^t \mathcal{M}[x_k - x_{k-d_k} - \sum_{j=k-d_k}^{k-1} \delta x_j] = 0 \\
2\zeta_k^t \mathcal{S}[x_{k-d_k} - x_{k-d_M} - \sum_{j=k-d_M}^{k-d_k-1} \delta x_j] = 0 \\
2\zeta_k^t \mathcal{Z}[x_k - x_{k-d_M} - \sum_{j=k-d_M}^{k-1} \delta x_j] = 0
\end{aligned}
\tag{11.17}
$$

for arbitrary parameter matrices $\mathcal{M},\ \mathcal{S},\ \mathcal{Z}$, and ζ_k from (11.4), algebraic manipulation of (11.16) using (11.2) yields

$$\begin{aligned}
\Delta V_k &\le \zeta_k^t\ \widehat{\Omega}\ \zeta_k \\
&- \sum_{j=k-d_k}^{k-1} \left[\xi_k^t \mathcal{M} + \delta x_j^t \mathcal{W}_a\right] \mathcal{W}_a^{-1} [\mathcal{M}^t \xi_k + \mathcal{W}_a \delta x_j] \\
&- \sum_{j=k-d_M}^{k-d_k-1} \left[\xi_k^t \mathcal{S} + \delta x_j^t \mathcal{W}_{ai}\right] \mathcal{W}_{ai}^{-1} [\mathcal{S}^t \xi_k + \mathcal{W}_{ai} \delta x_j] \\
&- \sum_{j=k-d_M}^{k-1} \left[\xi_k^t \mathcal{Z} + \delta x_j^t \mathcal{W}_{ci}\right] \mathcal{W}_{ci}^{-1} [\mathcal{Z}^t \xi_k + \mathcal{W}_{ci} \delta x_j]
\end{aligned} \tag{11.18}$$

$$\begin{aligned}
\widehat{\Omega} &= \hat{\Omega}_s + \hat{\Omega}_a + \hat{\Omega}_a^t + \hat{\Omega}_c + d_M \mathcal{M} \mathcal{W}_{ai}^{-1} \mathcal{M}^t + \\
&(d_M - d_m) \mathcal{S} \mathcal{W}_{ai}^{-1} \mathcal{S}^t + d_M \mathcal{N} \mathcal{W}_{ci}^{-1} \mathcal{N}^t
\end{aligned} \tag{11.19}$$

where $\hat{\Omega}_s,\ \hat{\Omega}_a,\ \hat{\Omega}_c$ correspond to $\Omega_s,\ \Omega_a,\ \Omega_c$ in (11.6) when $G_i \equiv 0,\ G_{di} \equiv 0,\ \Gamma_i \equiv 0,\ \Phi_i \equiv 0$. Since $\mathcal{W}_{ai} > 0,\ \mathcal{W}_{ci} > 0$, it follows from (11.5) by combining (11.8)–(11.18) and Schur complements that $\widehat{\Omega} < 0$. In turn, this implies that $\Delta V_k < -\varrho ||x_k||^2$ for a sufficiently small $\varrho > 0$, which establishes the internal asymptotic stability. Next, consider the performance measure

$$J_K = \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j \right)$$

For any $w_k \in \ell_2(0, \infty) \ne 0$ and zero initial condition $x_o = 0$, we have

$$J_K \le \sum_{j=0}^{K} \left(z_j^t z_j - \gamma^2 w_j^t w_j \ + \Delta V(x)|_{(11.2)} \right)$$

where $\Delta V(x)|_{(11.2)}$ defines the Lyapunov difference along with the solutions of system (11.2). Proceeding as before, we get

$$z_j^t z_j - \gamma^2 w_j^t w_j \ + \Delta V_k|_{(11.2)} = \left[\zeta_k^t\ w_j^t\right]\ \widetilde{\Omega}\ \left[\zeta_k^t\ w_j^t\right]^t \tag{11.20}$$

where $\widetilde{\Omega}$ corresponds to Ω_i in (11.5) by Schur complements. It is readily seen that

$$z_j^t z_j - \gamma^2 w_j^t w_j \ + \Delta V_k|_{(11.1)} < 0$$

for arbitrary $j \in [0, K)$, which implies for any $w_k \in \ell_2(0, \infty) \ne 0$ that $J < 0$ leading to $||z_k||_2 < \gamma\ ||w_k||_2$ and the proof is completed. ■

11.2.2 Proof of Theorem 11.4

Applying Schur complements, we express inequalities (11.8) conveniently in the form

$$\tilde{\Theta} = \begin{bmatrix} -\hat{\Theta}_i & \hat{\Theta}_{mi} & \Theta_{xi} & \Theta_{nj} & d_M \Omega^t_{cci} \mathcal{W}_i & \Omega_{zi} & \mathcal{A}^t_i \mathcal{P}_j & G^t_i \\ \bullet & -\mathcal{Q}_s & 0 & 0 & A^t_{di} \mathcal{P}_j & G^t_{di} & & \\ \bullet & \bullet & -\mathcal{R}_i & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I & 0 & 0 & \Gamma_j \mathcal{P}_j & \Phi^t_i \\ \bullet & \bullet & \bullet & \bullet & -d_M \mathcal{W}_i & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Omega_{wi} & 0 & 0 \\ \bullet & \bullet\bullet & \bullet & \bullet\bullet & & -\mathcal{P}_i & 0 & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix} < 0$$

$$-\hat{\Theta}_i = -\mathcal{P}_i + \overline{d}\mathcal{Q}_i + \mathcal{R}_i + \mathcal{M}_i + \mathcal{Z}_i + \mathcal{M}^t_i + \mathcal{Z}^t_i,$$
$$\hat{\Theta}_{mi} = \mathcal{S}_i - \mathcal{M}_i \tag{11.21}$$

Define $\mathcal{X}_i = \mathcal{P}_i^{-1}$ and using the congruent transformation

$$T = \text{diag}[\mathcal{X}_i,\ \mathcal{X}_s,\ \mathcal{X}_i,\ I,\ I,\ \mathcal{X}_i,\ \mathcal{X}_j,\ I]$$

into LMI (11.21) along with the linearizations

$$\begin{aligned} \widehat{\Theta}_1 &= \mathcal{X}_i \mathcal{Q}_i \mathcal{X}_i,\ \widehat{\Theta}_5 = \mathcal{X}_i \mathcal{Z}_i \mathcal{X}_i,\ \widehat{\Theta}_8 = \mathcal{X}_i \mathcal{Q}_s \mathcal{X}_i, \\ \widehat{\Theta}_2 &= \mathcal{X}_i \mathcal{R}_i \mathcal{X}_i,\ \widehat{\Theta}_6 = \mathcal{X}_i \mathcal{W}_{ai} \mathcal{X}_i \\ \mathcal{Y}_i &= K_i \mathcal{X}_i,\ \widehat{\Theta}_3 = \mathcal{X}_i \mathcal{M}_i \mathcal{X}_i,\ \widehat{\Theta}_7 = \mathcal{X}_i \mathcal{W}_{ci} \mathcal{X}_i \end{aligned} \tag{11.22}$$

we cast $T^t \tilde{\Theta} T$ into the LMI (11.9) as desired.

11.2.3 Proof of Theorem 11.5

Starting from LMI (11.12), we apply the congruent transformation

$$[\widehat{X}_i,\ I,\ I,\ I,\ I,\ I,\ \widehat{X}_j,\ I]$$

with $\widehat{X}_i = \mathcal{P}_i^{-1},\ C_i \widehat{X}_i = \widehat{T}_i C_i,\ \mathcal{G}_i = H_i \widehat{T}_i$ along with the linearizations (10.34) in addition to $\mathcal{F}_j = H_i C_{di} \widehat{X}_j,$ we obtain LMI (11.13) as desired. The output gain is computed by $H_i = R_i \widehat{T}_i^{\dagger}$ and $\widehat{T}_i^{\dagger} = C_i \widehat{X}_i C_i^{\dagger}$. ■

11.3 Notes and References

We have studied delay-dependent analysis and control synthesis for a class of linear discrete-time switched state-delay systems under arbitrary switching using an appropriate switched Lyapunov functional. LMI-based feasibility conditions have been developed to ensure that the linear switched discrete delay system is delay-dependent asymptotically stable with an $\mathcal{L}_2-$ gain smaller than a prescribed constant level. Subsequently, switched feedback schemes have been designed using state measurements and output measurements, to guarantee that the corresponding closed-loop system enjoys the same delay-dependent asymptotic stability with an $\mathcal{L}_2-$ gain smaller than a prescribed constant level. All the developed results have been expressed in terms of convex optimization over LMIs and have been tested by Matlab simulation on a representative water-quality example.

Chapter 12
Applications to Mutli-Rate Control

A class of hybrid multi-rate control models with time-delay and switching controllers are formulated based on combined remote control and local control strategies. The problem of robust dissipative control for this discrete system is investigated. An improved Lyapunov–Krasovskii functional is constructed and the subsequent analysis provides some new sufficient conditions in the form of linear matrix inequalities (LMIs) for both nominal and uncertain representations. Several special cases of practical interests are derived. A numerical simulation example is given to illustrate the effectiveness of the theoretical result.

12.1 Introduction

In the past decade, the most successful network developed has been the Internet that has proved a powerful tool for distributed collaborative work. The emerging Internet technologies offer unprecedented interconnection capability and ways of distributing collaborative work, and these have great potential to bring the advantages of these ways of working to the high-level control of process plants. These advantages include: (1) enabling remote monitoring and adjustment of plants; (2) enabling collaboration between skilled plant managers situated in geographically diverse locations; and (3) enabling the business to relocate the physical location of plant management staff easily in response to business needs.

In the areas of control theory and application, researchers began to exploit the advantages of the Internet for control systems, namely Internet-based control systems, and its controllability and stability. These new types of control systems are characterized as globally remote monitoring and adjustment of plants over the Internet. In recent years, Internet-based control systems have gained considerable attention in science and engineering [299, 300, 308], since they provide a new and convenient unified framework for system control and practical applications. Examples include intelligent home environments, wind-mill and solar power stations, small-scale hydroelectric power stations, and other highly geographically distributed devices, as well as tele-manufacturing, tele-surgery, and tele-control of spacecraft.

M.S. Mahmoud, *Switched Time-Delay Systems*,
DOI 10.1007/978-1-4419-6394-9_12,

With the explosive dominance of Internet services, physical systems stand to benefit from the patterns of retrieving data and reacting to system fluctuations from anywhere around the globe at any time. On the educational level, web-based virtual control laboratories for distance learning purposes have been realized [315, 360, 380, 415]. Essentially, Internet-based control systems have been developed by means of extending discrete control systems, which do not explicitly consider Internet transmission features [315]. It is further elaborated in [415] to address the Internet transmission issue as characterized by unpredictable time delay and data loss. One of the major challenges in Internet-based control systems is how to deal with the Internet transmission delay, and this has demonstrated the need for new control structure and relative elements as framework for Internet-based control systems.

The existing approaches of overcoming network transmission delay mainly focus on designing a model based on time-delay compensator or a state observer to reduce the effect of the transmission delay. In [416], the overcoming of the Internet time delay has been investigated from the control system architecture angle, including introducing a tolerant time to the fixed sampling interval to potentially maximize the possibility of succeeding the transmission on time. A dual-rate control scheme for Internet-based control systems has been proposed in [410] where a two-level hierarchy was used in the dual-rate control scheme. At the lower level, a local controller that is implemented to control the plant at a higher frequency to stabilize the plant and guarantee the plant being under control even when the network communication is lost for a long time. At the higher level, a remote controller is employed to remotely regulate the desirable reference at a lower frequency to reduce the communication load and increase the possibility of receiving data over the Internet on time. The local and the remote controller are composed of some modes, which is enabled due to the time and state of the network. Since the time delay is variable and the uncertainty of the process parameters is unavoidable, a dual-rate Internet-based control system may be unstable for certain control intervals.

The interest in the stability of networked control systems has grown in recent years due to its theoretical and practical significance [83, 113, 235, 343, 380, 430, 433]. To the best our knowledge, there is virtually no result on robust dissipative Internet-based control systems despite the significance of dissipativity analysis and synthesis in dynamic systems [139, 203]. Results on the robust dissipative control problem for time-delay systems were reported in [236, 237]. This motivates the present dissipativity investigation of multi-rate Internet-based switching control systems with time-varying delays and uncertainties.

In this chapter, we study the modeling and robust dissipative control for Internet-based switching control systems with multi-rate scheme, time-varying delays, and uncertainties from a general perspective. The controller switches between some modes due to the time and state of the network, either different time or the state changing may cause the controller to change its mode and the mode may change at each instant time. Based on remote control and local control strategy, a new class of multi-rate switching control models with time delay is formulated. Some new robust dissipative properties of such systems under arbitrary switching are investigated. All subsequent stability analysis and synthesis provide for both nominal and uncertain representations some new sufficient conditions in the form of linear matrix inequalities (LMIs), which can be conveniently explained by efficient interior-point minimization methods [27]. Several special cases of practical interests are derived.

12.2 Problem Statement

The Internet provides a great potential for the high-level control of process plants and remote dynamical systems. Internet-based (or remote network) control is a new concept which has received much attention in the previous years. However, little work has so far been done which aimed at developing systematic design methods for the design of such Internet-based process control systems. In this work, we focus on one approach: that is, dual-rate network control of linear plants. We consider a class of discrete-time dynamical systems under the action of a dual-rate control structure with remote controller (RC) and local controller (LC), a block diagram of which is depicted in Fig. 12.1.

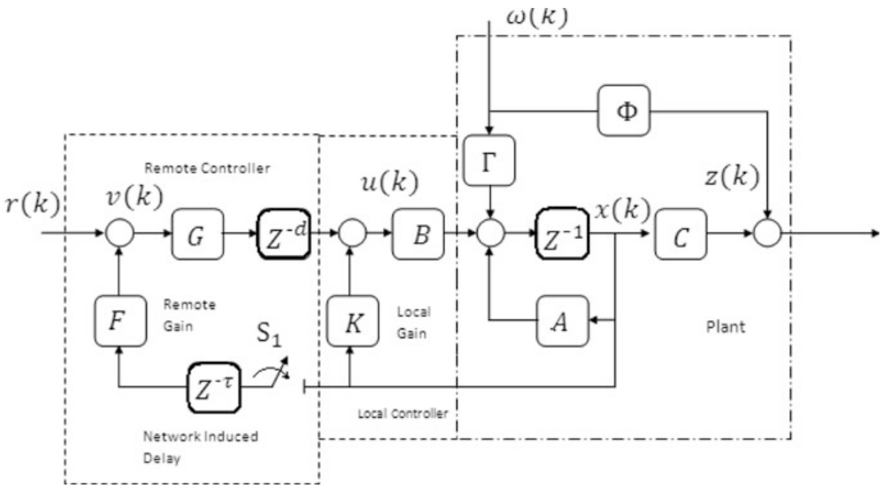

Fig. 12.1 Network control model

In the sequel, it is assumed that the sampling interval of the RC is s multiple of the LC, with s being a positive integer, and the switching device $S1$ closes only at the time instant $k = ns,\ n \in \mathbb{N}$ and otherwise, it switches off. Correspondingly, RC updates its state at $k = ns,\ n \in \mathbb{N}$ only, and otherwise, it keeps intact.

12.2.1 Dual-Rate Network Control Model

In this diagram, there are three building blocks: the plant is a linear shift-invariant system represented by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \Gamma\omega(k) \\ z(k) &= Cx(k) + \Lambda\omega(k) \end{aligned} \tag{12.1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^m$ is the local control vector, $z(k) \in \Re^q$ is the output observation vector, and $\omega(k) \in \Re^q$ is the exogenous vector which is assumed to belong to $\ell_2[0, \infty)$, r(k) is the input, and for the dissipativity analysis one can let r(k) = 0. The matrices $A \in \Re^{n\times n}, B \in \Re^{n\times m}, C \in \Re^{q\times n}, \Gamma \in \Re^{n\times q}, \Lambda \in \Re^{q\times q}$ are constants.

The second building block is a local controller (LC) of state-feedback type and described by

$$u(k) = Gv(k - d(k)) - K_j x(k) \tag{12.2}$$

where $v(k) \in \Re^m$ is the remote control vector, $G \in \Re^{m\times m}$, and $K_j \in \Re^{m\times n}$, $j \in \{1, 2, ..., N_a\}$ are mode gain switching matrices where the switching rules are designated by

$$\begin{aligned} &j(k) = \phi(k, x(k)), \quad j \in \{1, 2, ..., N_a\}, \quad m(k) = \sigma(k, x(k)) \\ &m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N} \end{aligned}$$

implying that the switching controllers have N_a and N_o modes, respectively, and $k, \ s \ \in \mathsf{N}$. The third building block is an RC implemented via a network and given by

$$\begin{aligned} v(k - d(k))(k) &= r(k - d(k)) - F_m x(k - d(k) - \tau(k)), \quad k = n\ s \\ v(k - d(k))(k) &= r(ns - d(k)) - F_m x(ns - d(k) - \tau(k)) \\ k \ &\in \{ns + 1, ..., ns + s - 1\} \end{aligned} \tag{12.3}$$

where $d(k), \tau(k)$ are communication- and network-induced delays and k, $n \in I_+ \stackrel{\Delta}{=} \{0, 1, 2, 3, ...\}$. For all practical purposes, we consider that these delays are not exactly known and satisfy

$$0 < d_m \le d(k) \le d_M, \quad 0 < \tau_m \le \tau(k) \le \tau_M \tag{12.4}$$

where d_m, d_M, τ_m, τ_M are constants designating delay bounds with d_m, τ_m reflecting a finite delay irrespective of the technology level of the communication links whereas d_M, τ_M reflecting the maximum allowable bound beyond which the networked system will become unstable.

With $j \in \{1, 2, ..., N_a, \; m \in \{1, 2, ..., N_o\}$, it follows from (12.1, 12.2, and 12.3) that for $k = n\ s$,

$$
\begin{aligned}
x(k+1) &= (A - BK_j)x(k) - BGF_m x(k - d(k) - \tau(k)) + BGr(k - d(k)) \\
&\quad + \Gamma\omega(k) \\
z(k) &= Cx(k) + \Lambda\omega(k)
\end{aligned} \tag{12.5}
$$

and for $k \in \{ns + 1, ..., ns + s - 1\}$

$$
\begin{aligned}
x(k+1) &= (A - BK_j)x(k) - BGF_m x(ns - d(k) - \tau(k)) + BGr(ns - d(k)) \\
&\quad + \Gamma\omega(k) \\
z(k) &= Cx(k) + \Lambda\omega(k)
\end{aligned} \tag{12.6}
$$

Remark 12.1 When undertaking dissipativity analysis, we let $r(k) \equiv 0$, in which case system (12.5) and (12.6) reduce to

$$
\begin{aligned}
x(k+1) &= (A - BK_j)x(k) - BGF_m x(k - \eta(k)) + \Gamma\omega(k), \quad k = n\ s \\
z(k) &= Cx(k) + \Lambda\omega(k)
\end{aligned} \tag{12.7}
$$

and

$$
\begin{aligned}
x(k+1) &= (A - BK_j)x(k) - BGF_m x(k - \eta(k)) + \Gamma\omega(k) \\
& k \in \{ns + 1, ..., ns + s - 1\} \\
z(k) &= Cx(k) + \Lambda\omega(k)
\end{aligned} \tag{12.8}
$$

where $\eta(k) = d(k) + \tau(k), \; k, \; n \in I_+, \; s > 0$

12.2.2 Hybrid Control Model

To study the behavior of dual-control models under dissipativity, we use the change of variables

$$
\mathsf{A}_j = (A - BK_j), \quad \mathsf{B}_m = -BGF_m \tag{12.9}
$$

into (12.7) and (12.8) to obtain

$$
\begin{aligned}
x(k+1) &= \mathsf{A}_j x(k) + \mathsf{B}_m x(k - \eta(k)) + \Gamma\omega(k), \quad k = n\ s \\
x(k+1) &= \mathsf{A}_j x(k) + \mathsf{B}_m x(ns - \eta(k)) + \Gamma\omega(k) \\
& k \in \{ns + 1, ..., ns + s - 1\} \\
z(k) &= Cx(k) + \Lambda\omega(k)
\end{aligned} \tag{12.10}
$$

Observe that at each time instant k, there will be only one mode of each controller that enabled with $\eta(k) > 0, k, s \in \mathsf{N}$. Moreover, noting that as $k = ns + r, r = 0, 1, ..., s-1$ and $ns - \eta(k) = k - (\eta(k) + r)$, then system (12.10) can be expressed as

$$\begin{aligned} x(k+1) &= \mathsf{A}_j x(k) + \mathsf{B}_m x(k - \theta(k)) + \Gamma \omega(k) \\ 0 &< \eta(k) \leq \theta(k) \leq \eta(k) + s - 1 \\ z(k) &= C x(k) + \Lambda \omega(k) \end{aligned} \tag{12.11}$$

In the case the system matrices undergo structured parametric uncertainties, we have the uncertain model

$$\begin{aligned} x(k+1) &= (\mathsf{A}_j + \Delta A(k))x(k) + (\mathsf{B}_m + \Delta B(k))x(k - \theta(k)) \\ &\quad + \Gamma\ \omega(k) \\ z(k) &= (C + \Delta C(k))x(k) + \Lambda\ \omega(k) \end{aligned} \tag{12.12}$$

and the associated uncertain matrices are given by

$$\begin{aligned} [\Delta A(k)\ \Delta B(k)] &= M_o \Delta(k)[N_a,\ N_b],\ \ \Delta^t(k)\Delta(k) \leq I\ \forall k \\ \Delta C(k) &= M_c \Delta(k) N_a \end{aligned} \tag{12.13}$$

where in view of (12.4) for both models (12.12) and (12.13), we have

$$0 \leq \theta_m \leq \theta(k) \leq \theta_M \tag{12.14}$$

12.3 Dissipativity Analysis

In the sequel, we study the dissipativity analysis of the following discrete-time switching models in the nominal case:

$$\begin{aligned} x(k+1) &= \mathsf{A}_j x(k) + \mathsf{B}_m x(k - \theta(k)) + \Gamma \omega(k) \\ z(k) &= C x(k) + \Lambda \omega(k) \\ j(k) &= \phi(k, x(k)),\ \ j \in \{1, 2, ..., N_a\},\ \ m(k) = \sigma(k, x(k)) \\ 0 &< \theta_m \leq \theta(k) \leq \theta_M,\ \ m \in \{1, 2, ..., N_o\} \end{aligned} \tag{12.15}$$

and in the uncertain case

$$\begin{aligned} x(k+1) &= (\mathsf{A}_j + \Delta A(k))x(k) + (\mathsf{B}_m + \Delta B(k))x(k - \theta(k)) + \Gamma\ \omega(k) \\ z(k) &= (C + \Delta C(k))x(k) + \Lambda\ \omega(k) \\ j(k) &= \phi(k, x(k)),\ \ j \in \{1, 2, ..., N_a\},\ \ m(k) = \sigma(k, x(k)) \\ 0 &< \theta_m \leq \theta(k) \leq \theta_M,\ \ m \in \{1, 2, ..., N_o\} \end{aligned}$$

$$\begin{aligned}\Delta A(k) &= M_o \Delta(k) N_a,\ \Delta B(k) = M_o \Delta(k) N_b \\ \Delta C(k) &= M_c \Delta(k) N_a,\ \Delta^t(k)\Delta(k) \le I\ \forall k\end{aligned} \tag{12.16}$$

The following dissipativity definition is adopted

Definition 12.2 Hill and Moylan [139] Given matrices $0 < \mathcal{Q}^t = \mathcal{Q} \in \Re^{q\times q},\ 0 < \mathcal{R}^t = \mathcal{R} \in \Re^{q\times q},\ \mathcal{S}$ system (12.11) is called (**Q**,**S**,**R**)-dissipative if for some real function $\eta(.),\ \eta(0) = 0$,

$$\sum_{r=0}^{K}\Big[z^t(j)\mathbf{Q}z(j) + 2\omega^t(j)\mathbf{S}z(j) + \omega^t(r)\mathbf{R}\omega(r)\Big] + \eta(x_o) \ge 0,\ \forall K \ge 0 \tag{12.17}$$

Furthermore, if for some scalar $\alpha > 0$,

$$\sum_{r=0}^{K}\Big[z^t(r)\mathbf{Q}z(r) + 2w^t(r)\mathbf{S}z(r) + w^t(r)\mathbf{R}w(r)\Big] + \eta(x_o) \ge \sum_{r=0}^{K}\alpha\omega^t(r)\omega(r),\ \forall\ K\ \ge 0 \tag{12.18}$$

System (12.11) is called strictly (**Q**,**S**,**R**)-dissipative

Remark 12.3 It is significant to observe that **Definition 12.2** paves the way to a general performance objective

$$\Delta V(k) + \delta||z||^2 + \varepsilon||\omega||^2 - \beta z^t\omega\ \le\ 0 \tag{12.19}$$

for an energy function $V(k) = x^t(k)\Xi x(k)$, $\Xi > 0$. The significance of the objective (12.18) is readily interpreted upon summing up inequality (12.19) over the period $[0,\ K]$ to yield

$$x^t(K)\Xi x(K) \le x^t(0)\Xi x(0) - \sum_{j=0}^{K}\Big[\delta||z(j)||^2 + \varepsilon||\omega(j)||^2 - \beta z^t(j)\omega(j)\Big] \tag{12.20}$$

By Rayleigh's inequality [27], $\lambda_m(\Xi)||x||^2 \le x^t(k)\Xi x(k) \le \lambda_M(\Xi)||x||^2$, we get

$$\lambda_m(\Xi)||x||^2 \le \lambda_M(\Xi)||x||^2 - \Big[\sum_{j=0}^{K}\delta||z(j)||^2 + \varepsilon||\omega(j)||^2 - \beta z^t(j)\omega(j)\Big] \tag{12.21}$$

As will be shown in the sequel, this allows several optimization possibilities in a unified eigenvalue problem framework.

Our objective hereafter is to establish conditions for system (12.15) or (12.16) with the switching controllers to be strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-dissipative.

12.3.1 Dissipative Stability of the Nominal System

In the sequel, we use $\theta_s = (\theta_M - \theta_m)$ which accounts for the discrete span, and define the state increment $\delta x(k) = x(k+1) - x(k)$. The following theorem provides the desired stability result:

Theorem 12.4 *Suppose that the gains K_j, F_m are specified. Given the bounds $\theta_M > 0, \theta_m > 0$ and a scalar convergence rate $\sigma > 0$. System (12.15) is delay-dependent asymptotically stable and strictly $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-dissipative under arbitrary switching rules ϕ and σ if there exist weighting matrices $\mathcal{P}, \mathcal{Q}, \mathcal{Z}, \mathcal{S}, \mathcal{R}_{aj}, \mathcal{R}_{cj}$, parameter matrices $\Theta_{aj}, \Theta_{cj}, \Psi_{aj}, \Psi_{cj}, \Phi_{aj}, \Phi_{cj}$ satisfying the following LMIs for $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$*

$$\widetilde{\Pi} = \begin{bmatrix} \Omega_{jm} & \Upsilon_{jm} & -C^t\mathbf{Q}\Lambda - C^t\mathbf{S}^t & \mathsf{A}_j^t\mathcal{P} \\ \bullet & -\Xi_j & 0 & \mathsf{B}_m^t\mathcal{P} \\ \bullet & \bullet & -\Sigma & \Gamma^t \\ \bullet & \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{12.22}$$

$$\Upsilon_{jm} = \begin{bmatrix} \sqrt{\theta_s}\Phi_{aj} & \sqrt{\theta_s}\Psi_{aj} & \sqrt{\theta_M}\Theta_{aj} & \sqrt{\theta_s}(\mathsf{A}_j - I)^t\mathcal{R}_{aj} & \sqrt{\theta_M}(\mathsf{A}_j - I)^t\mathcal{R}_{cj} \\ \sqrt{\theta_s}\Phi_{cj} & \sqrt{\theta_s}\Psi_{cj} & \sqrt{\theta_M}\Theta_{cj} & \sqrt{\theta_s}\mathsf{B}_m^t\mathcal{R}_{aj} & \sqrt{\theta_M}\mathsf{B}_m^t\mathcal{R}_{cj} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_{jm} = \begin{bmatrix} \widetilde{\Pi}_o & \Pi_{aj} & \Psi_{aj} & -\Phi_{aj} \\ \bullet & -\Pi_{cj} & \Psi_{cj} & -\Phi_{cj} \\ \bullet & \bullet & -\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S} \end{bmatrix}$$

$$\Xi_j = \mathrm{diag}\left[\mathcal{R}_{aj} + \mathcal{R}_{cj} \;\; \mathcal{R}_{aj} \;\; \mathcal{R}_{cj} \;\; \mathcal{R}_{aj} \;\; \mathcal{R}_{cj} \right] \tag{12.23}$$

where

$$\widetilde{\Pi}_o = -\mathcal{P} + (\theta_s + 1)\mathcal{Q} + \mathcal{Z} + \mathcal{S} + \Theta_{aj} + \Theta_{aj}^t - C^t\mathbf{Q}C$$

$$\Pi_{aj} = -\Theta_{aj} + \Theta_{cj}^t + \Phi_{aj} - \Psi_{aj}$$

$$\Pi_{cj} = \mathcal{Q} + \Theta_{cj} + \Theta_{cj}^t - \Phi_{cj} - \Phi_{cj}^t + \Psi_{cj} + \Psi_{cj}^t$$

$$\Sigma = (\mathbf{R} - \alpha I) + \Lambda^t \mathbf{Q} \Lambda + \mathbf{S} \Lambda + \Lambda^t \mathbf{S}^t \tag{12.24}$$

Proof In terms of the state increment $\delta x(k) = x(k+1) - x(k)$, consider the Lyapunov–Krasovskii functional (LKF):

$$\begin{aligned}
V(k) &= V_o(k) + V_a(k) + V_c(k) + V_m(k) + V_n(k) \\
V_o(k) &= x^t(k)\mathcal{P}x(k), \ \ V_a(k) = \sum_{j=k-\theta(k)}^{k-1} x^t(j)\mathcal{Q}x(j) \\
V_c(k) &= \sum_{j=k-\theta_m}^{k-1} x^t(j)\mathcal{Z}x(j) + \sum_{j=k-\theta_M}^{k-1} x^t(j)\mathcal{S}x(j) \\
V_m(k) &= \sum_{j=-\theta_M+1}^{-\theta_m} \sum_{m=k+j}^{k-1} x^t(m)\mathcal{Q}x(m) \\
V_n(k) &= \sum_{j=-\theta_M}^{-\theta_m-1} \sum_{m=k+j}^{k-1} \delta x^t(m)\mathcal{R}_a\delta x(m) \\
&\quad + \sum_{j=-\theta_M}^{-1} \sum_{m=k+j}^{k-1} \delta x^t(m)\mathcal{R}_c\delta x(m)
\end{aligned} \tag{12.25}$$

where $\mathcal{P} > 0, \mathcal{Q} > 0, \mathcal{Z} > 0, \mathcal{S} > 0, \mathcal{R}_a > 0, \mathcal{R}_c > 0$ are weighting matrices of appropriate dimensions. A straightforward computation gives the first difference of $\Delta V(k) = V(k+1) - V(k)$ along the solutions of (12.15) as

$$\begin{aligned}
\Delta V_o(k) &= x^t(k+1)\mathcal{P}x(k+1) - x^t(k)\mathcal{P}x(k) \\
&= [\mathsf{A}_j x(k) + \mathsf{B}_m x(k-\theta(k)) + \Gamma\omega(k)]^t \mathcal{P} \\
&\qquad [\mathsf{A}_j x(k) + \mathsf{B}_m x(k-\theta(k)) + \Gamma\omega(k)] \\
&\quad - x^t(k)\mathcal{P}x(k) \\
\Delta V_a(k) &\le x^t(k)\mathcal{Q}x(k) - x^t(k-\theta(k))\mathcal{Q}x(k-\theta(k)) \\
&\quad + \sum_{j=k-\theta_M+1}^{k-\theta_m} x^t(j)\mathcal{Q}x(j) \\
\Delta V_c(k) &= x^t(k)\mathcal{Z}x(k) - x^t(k-\theta_m)\mathcal{Z}x(k-\theta_m) \\
&\quad + x^t(k)\mathcal{S}x(k) - x^t(k-\theta_M)\mathcal{S}x(k-\theta_M) \\
\Delta V_m(k) &= (\theta_M - \theta_m)x^t(k)\mathcal{Q}x(k) - \sum_{j=k-\theta_M+1}^{k-\theta_m} x^t(k)\mathcal{Q}x(k)
\end{aligned}$$

$$\Delta V_n(k) = (\theta_M - \theta_m)\delta x^t(k)\mathcal{R}_a\delta x(k) + \theta_M\delta x^t(k)\mathcal{R}_c\delta x(k) - \sum_{j=k-\theta_M}^{k-d_m-1}\delta x^t(j)\mathcal{R}_a\delta x(j) - \sum_{j=k-\theta_M}^{k-1}\delta x^t(j)\mathcal{R}_c\delta x(j) \quad (12.26)$$

To facilitate the delay-dependence analysis, we invoke the following identities

$$\Big[2x^t(k)\Theta_{aj} + 2x^t(k-\theta(k))\Theta_{cj}\Big]\Big[x(k) - x(k-\theta(k)) - \sum_{j=k-\theta(k)}^{k-1}\delta x(j)\Big] = 0$$

$$\Big[2x^t(k)\Phi_{aj} + 2x^t(k-\theta(k))\Phi_{cj}\Big]\Big[x(k-\theta(k)) - x(k-\theta_M) - \sum_{j=k-\theta_M}^{k-\theta(k)-1}\delta x(j)\Big] = 0$$

$$\Big[2x^t(k)\Psi_{aj} + 2x^t(k-\theta(k))\Psi_{cj}\Big]\Big[x(k-\theta_m) - x(k-\theta(k)) - \sum_{j=k-\theta(k)}^{k-\theta_m-1}\delta x(j)\Big] = 0 \quad (12.27)$$

for some matrices $\Theta_a, \Phi_a, \Psi_a, \Theta_c, \Phi_c, \Psi_c$. By algebraic manipulations, we set $\omega(k) \equiv 0$ and proceed to get from (12.26) and (12.27)

$$\begin{aligned}
\Delta V_k &= \Delta V_o(k) + \Delta V_a(k) + \Delta V_c(k) + \Delta V_m(k) + \Delta V_n(k) \\
&\quad + \Big[2x^t(k)\Theta_{aj} + 2x^t(k-\theta(k))\Theta_{cj}\Big]\Big[x(k) - x(k-\theta(k)) - \sum_{j=k-d(k)}^{k-1}\delta x(j)\Big] \\
&\quad + [2x^t(k)\Phi_{aj} + 2x^t(k-\theta(k))\Phi_{cj}] \\
&\quad \Big[x(k-\theta(k)) - x(k-d_M) - \sum_{j=k-d_M}^{k-d(k)-1}\delta x(j)\Big] \\
&\quad + [2x^t(k)\Psi_{aj} + 2x^t(k-\theta(k))\Psi_{cj}] \\
&\quad \Big[x(k-\theta_m) - x(k-\theta(k)) - \sum_{j=k-d(k)}^{k-d_m-1}\delta x(j)\Big] \\
&= \mu^t(k)\ \widetilde{\Upsilon}\ \mu(k), \quad \mu = \big[\,x^t(k)\ x^t(k-\theta(k))\ x^t(k-\theta_m)\ x^t(k-\theta_M)\,\big]^t \\
\widetilde{\Upsilon} &= \Omega_{jm} + \theta_s\Phi_j(\mathcal{R}_{aj} + \mathcal{R}_{cj})^{-1}\Phi_j^t + \theta_s\Psi_j\mathcal{R}_{aj}^{-1}\Psi_j^t + \theta_M\Theta_j\mathcal{R}_{cj}^{-1}\Theta_j^t
\end{aligned}$$

$$\Psi_j = \begin{bmatrix} \Psi_{aj} \\ \Psi_{cj} \\ 0 \\ 0 \end{bmatrix}, \ \Phi_j = \begin{bmatrix} \Phi_{aj} \\ \Phi_{cj} \\ 0 \\ 0 \end{bmatrix}, \ \Theta_j = \begin{bmatrix} \Theta_{aj} \\ \Theta_{cj} \\ 0 \\ 0 \end{bmatrix} \tag{12.28}$$

Schur complement operations to (12.22) using (12.23) with $C \equiv 0, \ \Lambda \equiv 0$ leads to $\widetilde{\Upsilon} < 0$. In view of this and (12.28), it follows that there exists a small scalar $\sigma > 0$ guaranteeing that $\Delta V_k \le -\sigma||x(k)||^2$. This proves that the system (12.15) is asymptotically stable for $\theta_m \le \theta(k) \le \theta_M$.

Considering the performance of system (12.15), we get from (12.26) and (12.27)

$$\begin{aligned} J_K &= \sum_{r=0}^{K} \Big[\Delta V_k - z^t(j)\mathbf{Q}z(j) - 2\,\omega^t(j)\mathbf{S}z(j) - \omega^t(r)(\mathbf{R} - \alpha I)\omega(r) \Big] \\ &\le [\mu^t(r) \ \ \omega^t(r)] \ \ \widetilde{\Pi} \ \ [\mu^t(r) \ \ \omega^t(r)]^t \end{aligned} \tag{12.29}$$

In view of (12.22) using (12.23), we get $J_K < 0$ implying that

$$\begin{aligned} &\sum_{r=0}^{K} \Big[z^t(j)\mathbf{Q}z(j) + 2\,\omega^t(j)\mathbf{S}z(j) + \omega^t(r)(\mathbf{R} - \alpha I)\omega(r) \Big] \\ &\qquad > \sum_{r=0}^{K} \Delta V_r = V_{K+1} - V(0) \end{aligned}$$

In addition, since $V_k = V(x_k) \ge 0$, it follows that

$$\begin{aligned} &\sum_{r=0}^{K} \Big[z^t(j)\mathbf{Q}z(j) + 2\,\omega^t(j)\mathbf{S}z(j) + \omega^t(r)(\mathbf{R} - \alpha I)\omega(r) \Big] \\ &\qquad \ge -V(0) \stackrel{\Delta}{=} \eta, \ \forall K \in \mathbf{N}, \ \forall \omega \in \ell_2[0, \infty) \end{aligned}$$

which implies from **Definition 12.2** that system (12.15) is delay-dependent asymptotically stable and *strictly* $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-*dissipative* under arbitrary switching rules ϕ and σ and therefore completes the proof.

12.3.2 Dissipative Synthesis of the Nominal System

The following corollary establishes a method to determine the gains:

Corollary 12.5 *Given the bounds $\theta_M > 0, \theta_m > 0$ and matrices $\mathcal{R}_{aj}, \mathcal{R}_{cj}$. System (12.15) is delay-dependent asymptotically stable and strictly* $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-*dissipative under arbitrary switching rules ϕ and σ if there exist weighting matrices*

$\mathcal{X}, \mathcal{Y}_a, \mathcal{Y}_c, \mathcal{Y}_e, \mathcal{Y}_{rj}, \mathcal{Y}_{tj}, \mathcal{Y}_{ej}, \mathcal{Y}_{sj}, \mathcal{Y}_{vj}, \mathcal{Y}_{wj}, \mathcal{X}_{aj}, \mathcal{X}_{cj}$ *satisfying the following LMIs for* $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$

$$\widetilde{\Pi}_x = \begin{bmatrix} \widehat{\Omega}_{jm} & \widehat{\Upsilon}_{jm} & -C^t\mathbf{Q}\Lambda - C^t\mathbf{S}^t & \mathcal{X}A_j^t - \mathcal{Y}_{kj}^t B^t \\ \bullet & -\widehat{\Xi}_j & 0 & -\mathcal{Y}_m^t G^t B^t \\ \bullet & \bullet & -\Sigma & \Gamma^t \\ \bullet & \bullet & \bullet & -\mathcal{X} \end{bmatrix} < 0 \tag{12.30}$$

$$\widehat{\Upsilon}_{jm} = \begin{bmatrix} \sqrt{\theta_s}\mathcal{Y}_{ej} & \sqrt{\theta_s}\mathcal{Y}_{vj} & \sqrt{\theta_M}\mathcal{Y}_{rj} & \sqrt{\theta_s}\mathcal{X}_{sj}\mathcal{R}_{aj} & \sqrt{\theta_M}(\mathcal{X}_{sj} - I)\mathcal{R}_{cj} \\ \sqrt{\theta_s}\mathcal{Y}_{sj} & \sqrt{\theta_s}\mathcal{Y}_{wj} & \sqrt{\theta_M}\mathcal{Y}_{tj} & -\sqrt{\theta_s}\mathcal{Y}_m^t G^t B^t \mathcal{R}_{aj} & -\sqrt{\theta_M}\mathcal{Y}_m^t G^t B^t \mathcal{R}_{cj} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\Omega}_{jm} = \begin{bmatrix} \widehat{\Pi}_o & \hat{\Pi}_{aj} & \mathcal{Y}_{vj} & -\mathcal{Y}_{ej} \\ \bullet & -\hat{\Pi}_{cj} & \mathcal{Y}_{wj} & -\mathcal{Y}_{sj} \\ \bullet & \bullet & -\mathcal{Y}_c & 0 \\ \bullet & \bullet & \bullet & -\mathcal{Y}_e \end{bmatrix}$$

$$\widehat{\Xi}_j = \mathrm{diag}\left[\, \mathcal{X}_{aj} + \mathcal{X}_{cj} \;\; \mathcal{X}_{aj} \;\; \mathcal{X}_{cj} \;\; \mathcal{X}_{aj} \;\; \mathcal{X}_{cj} \,\right] \tag{12.31}$$

where

$$\begin{aligned} \widehat{\Pi}_o &= -\mathcal{X} + (\theta_s + 1)\mathcal{Y}_a + \mathcal{Y}_c + \mathcal{Y}_e + \mathcal{Y}_{rj} + \mathcal{Y}_{rj}^t - \mathcal{X}C^t\mathbf{Q}C\mathcal{X}, \\ \widehat{\Pi}_{aj} &= -\mathcal{Y}_{rj} + \mathcal{Y}_{tj}^t + \mathcal{Y}_{ej} - \mathcal{Y}_{vj} \\ \widehat{\Pi}_{cj} &= \mathcal{Y}_a + \mathcal{Y}_{tj} + \mathcal{Y}_{tj}^t - \mathcal{Y}_{sj} - \mathcal{Y}_{sj}^t + \mathcal{Y}_{wj} + \mathcal{Y}_{wj}^t, \;\; \mathcal{X}_{sj} = \mathcal{X}A_j^t - \mathcal{X} - \mathcal{Y}_{kj}^t B^t \\ \Sigma &= (\mathbf{R} - \alpha I) + \Lambda^t\mathbf{Q}\Lambda + \mathbf{S}\Lambda + \Lambda^t\mathbf{S}^t \end{aligned} \tag{12.32}$$

Moreover, the feedback gains are given by $K_j = \mathcal{Y}_{kj}\mathcal{X}^{-1}$ *and* $F_m = \mathcal{Y}_{fm}\mathcal{X}^{-1}$.

Proof Introducing the congruent transformation

$$\begin{aligned} \mathcal{T} &= \mathrm{diag}[\mathcal{T}_a,\ \mathcal{T}_e,\ I],\ \mathcal{T}_a = \mathrm{diag}[\mathcal{X},\ \mathcal{X},\ \mathcal{X},\ \mathcal{X}] \\ \mathcal{T}_e &= \mathrm{diag}[\mathcal{X},\ \mathcal{X},\ \mathcal{X},\ I,\ I,\ I],\ \mathcal{X} = \mathcal{P}^{-1} \end{aligned}$$

and making the change of variables

$$\begin{aligned} &\mathcal{Y}_a = \mathcal{X}\mathcal{Q}\mathcal{X},\ \mathcal{Y}_c = \mathcal{X}\mathcal{Z}\mathcal{X},\ \mathcal{Y}_e = \mathcal{X}\mathcal{S}\mathcal{X},\ \mathcal{Y}_{rj} = \mathcal{X}\Theta_{aj}\mathcal{X} \\ &\mathcal{Y}_{tj} = \mathcal{X}\Theta_{cj}\mathcal{X},\ \mathcal{Y}_{ej} = \mathcal{X}\Phi_{aj}\mathcal{X} \\ &\mathcal{Y}_{sj} = \mathcal{X}\Phi_{cj}\mathcal{X},\ \mathcal{Y}_{vj} = \mathcal{X}\Psi_{aj}\mathcal{X},\ \mathcal{Y}_{wj} = \mathcal{X}\Psi_{cj}\mathcal{X},\ \mathcal{Y}_{kj} = \mathcal{X}K_j \\ &\mathcal{X}_{aj} = \mathcal{X}R_{aj}\mathcal{X},\ \mathcal{X}_{cj} = \mathcal{X}R_{cj}\mathcal{X},\ \mathcal{Y}_{fm} = \mathcal{X}F_m \end{aligned}$$

Simple manipulations show that $\mathcal{T}\ \widetilde{\Pi}\ \mathcal{T} = \widetilde{\Pi}_x$.

12.3.3 Dissipativity Stability of the Uncertain System

In preparation to develop robust dissipative stability of system (12.16), we introduce the following block matrices:

$$\widehat{M} = \begin{bmatrix} M_o & 0 \\ 0 & M_o \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \widehat{N}^t = \begin{bmatrix} 0 & 0 \\ \sqrt{\theta_s}(N_a - N_b K_j)\mathcal{R}_{aj} & -\sqrt{\theta_s} N_b G F_m \mathcal{R}_{aj} \\ \sqrt{\theta_M}(N_a - N_b K_j)\mathcal{R}_{cj} & -\sqrt{\theta_M} N_b G F_m \mathcal{R}_{cj} \\ 0 & 0 \\ (N_a - N_b K_j)\mathcal{P} & N_b G F_m \mathcal{P} \end{bmatrix}$$

$$\widetilde{M} = \begin{bmatrix} -N_a^t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \widetilde{N}^t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (\mathbf{Q}\Lambda - \mathbf{S}^t)^t M_c \\ 0 \end{bmatrix}, \ \mathcal{U} = \begin{bmatrix} N_a^t & \pi C^t \mathbf{Q} & \widehat{M} & \varepsilon \widehat{N}^t & \widetilde{M} & \nu \widetilde{N}^t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{V} = \mathrm{diag}\left[\pi I \ \left(\mathbf{Q} - \pi \mathbf{Q} M_c M_c^t \mathbf{Q}\right) \ \varepsilon I \ \varepsilon I \ \nu I \ \nu I \right]$$

$$\bar{\Xi}_j = \mathrm{diag}\left[\mathcal{R}_{aj}^{-1} + \mathcal{R}_{cj}^{-1} \ \mathcal{R}_{aj}^{-1} \ \mathcal{R}_{cj}^{-1} \ \mathcal{R}_{aj}^{-1} \ \mathcal{R}_{cj}^{-1} \right] \tag{12.33}$$

for some scalars $\pi > 0, \ \varepsilon > 0, \ \nu > 0$. The corresponding robust results are provided by the following theorem and corollary:

Theorem 12.6 *Suppose that the gains K_j, F_m are specified. Given the bounds $\theta_M > 0, \theta_m > 0$ and a scalar convergence rate $\sigma > 0$. System (12.16) is delay-dependent asymptotically stable and strictly $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-dissipative under arbitrary switching rules ϕ and σ if there exist weighting matrices $\mathcal{P}, \mathcal{Q}, \mathcal{Z}, \mathcal{S}, \mathcal{R}_{aj}, \mathcal{R}_{cj}$, parameter matrices $\Theta_{aj}, \Theta_{cj}, \Psi_{aj}, \Psi_{cj}, \Phi_{aj}, \Phi_{cj}$ and scalars $\pi > 0, \varepsilon > 0, \nu > 0$ satisfying the following LMIs for $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$*

$$\overline{\Pi} = \begin{bmatrix} \overline{\Omega}_{jm} & \Upsilon_{jm} & -C^t \mathbf{Q}\Lambda - C^t \mathbf{S}^t & \mathsf{A}_j^t \mathcal{P} & \mathcal{U} \\ \bullet & -\Xi_j & 0 & \mathsf{B}_m^t \mathcal{P} & 0 \\ \bullet & \bullet & -\Sigma & \Gamma^t & 0 \\ \bullet & \bullet & \bullet & -\mathcal{P} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{V} \end{bmatrix} < 0 \tag{12.34}$$

$$\overline{\Omega}_{jm} = \begin{bmatrix} \overline{\Pi}_o & \Pi_{aj} & \Psi_{aj} & -\Phi_{aj} \\ \bullet & -\Pi_{cj} & \Psi_{cj} & -\Phi_{cj} \\ \bullet & \bullet & -\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S} \end{bmatrix} \tag{12.35}$$

where $\overline{\Pi}_o = -\mathcal{P} + (\theta_s + 1)\mathcal{Q} + \mathcal{Z} + \mathcal{S} + \Theta_{aj} + \Theta_{aj}^t$ and the matrices Ξ_j, Υ_{jm} are given in (12.22 and 12.23).

Proof Incorporating the system matrices of (12.16) into LMI (12.30) of **Theorem 12.4** and manipulating with the help of inequalities A.1.1 and A.1.2 in the Appendix and (12.33), we obtain LMI (12.34).

12.3.4 Dissipative Synthesis of the Uncertain System

By a parallel development to **Corollary 12.5**, we have the following result for system (12.16):

Corollary 12.7 *Given the bounds $\theta_M > 0,\ \theta_m > 0$ and matrices $\mathcal{R}_{aj},\ \mathcal{R}_{cj}$. System (12.16) is delay-dependent asymptotically stable and strictly* $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$*-dissipative under arbitrary switching rules ϕ and σ if there exist weighting matrices $\mathcal{X},\ \mathcal{Y}_a,\ \mathcal{Y}_c,\ \mathcal{Y}_e,\ \mathcal{Y}_q,\ \mathcal{X}_{xj},\ \mathcal{X}_{ym},\ \mathcal{Y}_{rj},\ \mathcal{Y}_{tj},\ \mathcal{Y}_{ej},\ \mathcal{Y}_{sj},\ \mathcal{Y}_{vj},\ \mathcal{Y}_{wj},$ $\widehat{\mathcal{X}}_{aj},\ \widehat{\mathcal{X}}_{cj}$ and scalars $\pi > 0,\ \varepsilon > 0,\ \nu > 0$ satisfying the following LMIs for $j \in \{1, 2, ..., N_a\},\ m \in \{1, 2, ..., N_o\},\ N_a,\ N_o \in \mathbf{N}$*

$$\overline{\Pi}_x = \begin{bmatrix} \overline{\Omega}_{jm} & \overline{\Upsilon}_{jm} & -C^t\mathbf{Q}\Lambda - C^t\mathbf{S}^t & \mathcal{X}A_j^t - \mathcal{Y}_{kj}^t B^t & \overline{\mathcal{U}} \\ \bullet & -\widehat{\Xi}_j & 0 & -\mathcal{Y}_m^t G^t B^t & 0 \\ \bullet & \bullet & -\Sigma & \Gamma^t & 0 \\ \bullet & \bullet & \bullet & -\mathcal{X} & 0 \\ \bullet & \bullet & \bullet & \bullet & -\mathcal{V} \end{bmatrix} < 0 \tag{12.36}$$

$$\overline{\Upsilon}_{jm} = \begin{bmatrix} \sqrt{\theta_s}\mathcal{Y}_{ej} & \sqrt{\theta_s}\mathcal{Y}_{vj} & \sqrt{\theta_M}\mathcal{Y}_{rj} & \sqrt{\theta_s}\mathcal{X}_{sj} & \sqrt{\theta_M}(\mathcal{X}_{sj} - I) \\ \sqrt{\theta_s}\mathcal{Y}_{sj} & \sqrt{\theta_s}\mathcal{Y}_{wj} & \sqrt{\theta_M}\mathcal{Y}_{tj} & -\sqrt{\theta_s}\mathcal{Y}_m^t G^t B^t & -\sqrt{\theta_M}\mathcal{Y}_m^t G^t B^t \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\Omega}_{jm} = \begin{bmatrix} \widehat{\Pi}_o & \widehat{\Pi}_{aj} & \mathcal{Y}_{vj} & -\mathcal{Y}_{ej} \\ \bullet & -\widehat{\Pi}_{cj} & \mathcal{Y}_{wj} & -\mathcal{Y}_{sj} \\ \bullet & \bullet & -\mathcal{Y}_c & 0 \\ \bullet & \bullet & \bullet & -\mathcal{Y}_e \end{bmatrix},\ \overline{\Xi}_j = \mathrm{diag}\left[\widehat{\mathcal{X}}_{aj} + \widehat{\mathcal{X}}_{cj}\ \widehat{\mathcal{X}}_{aj}\ \widehat{\mathcal{X}}_{cj}\ \widehat{\mathcal{X}}_{aj}\ \widehat{\mathcal{X}}_{cj}\right]$$

$$\overline{\mathcal{U}} = \begin{bmatrix} \mathcal{X}N_a^t & \mathcal{Y}_q & \mathcal{X}\widehat{M} & \widehat{\mathcal{Y}} & \mathcal{X}\widetilde{M} & \nu\mathcal{X}\widetilde{N}^t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\mathcal{Y}} = \begin{bmatrix} 0 & 0 \\ \sqrt{\theta_s}\mathcal{X}_{xj} & -\sqrt{\theta_s}\mathcal{X}_{ym} \\ \sqrt{\theta_M}\mathcal{X}_{xj} & -\sqrt{\theta_M}\mathcal{X}_{ym} \\ 0 & 0 \\ \mathcal{X}_{xj} & \mathcal{X}_{ym} \end{bmatrix} \tag{12.37}$$

where

$$\widehat{\Pi}_o = -\mathcal{X} + (\theta_s + 1)\mathcal{Y}_a + \mathcal{Y}_c + \mathcal{Y}_e + \mathcal{Y}_{rj} + \mathcal{Y}_{rj}^t,\ \widehat{\Pi}_{aj} = -\mathcal{Y}_{rj} + \mathcal{Y}_{tj}^t + \mathcal{Y}_{ej} - \mathcal{Y}_{vj}$$
$$\widehat{\Pi}_{cj} = \mathcal{Y}_a + \mathcal{Y}_{tj} + \mathcal{Y}_{tj}^t - \mathcal{Y}_{sj} - \mathcal{Y}_{sj}^t + \mathcal{Y}_{wj} + \mathcal{Y}_{wj}^t,\ \mathcal{X}_{sj} = \mathcal{X} A_j^t - \mathcal{X} - \mathcal{Y}_{kj}^t B^t$$
$$\Sigma = (\mathbf{R} - \alpha I) + \Lambda^t \mathbf{Q} \Lambda + \mathbf{S}\Lambda + \Lambda^t \mathbf{S}^t \tag{12.38}$$

Moreover, the feedback gains are given by $K_j = \mathcal{Y}_{kj}\mathcal{X}^{-1}$ *and* $F_m = \mathcal{Y}_{fm}\mathcal{X}^{-1}$.

Proof Introducing the congruent transformation

$$\mathcal{T} = \text{diag}[\mathcal{T}_a,\ \mathcal{T}_e,\ \mathcal{T}_o],\ \mathcal{T}_a = \text{diag}[\mathcal{X},\ \mathcal{X},\ \mathcal{X},\ \mathcal{X}]$$
$$\mathcal{T}_e = \text{diag}[\mathcal{X},\ \mathcal{X},\ \mathcal{X},\ I,\ I,\ I],\ \mathcal{X} = \mathcal{P}^{-1}$$
$$\mathcal{T}_o = \text{diag}[I,\ I,\ I,\ I,\ \mathcal{X},\ \mathcal{X},\ I,\ I]$$

and making the change of variables

$$\mathcal{Y}_a = \mathcal{X}\mathcal{Q}\mathcal{X},\ \mathcal{Y}_c = \mathcal{X}\mathcal{Z}\mathcal{X},\ \mathcal{Y}_e = \mathcal{X}\mathcal{S}\mathcal{X},\ \mathcal{Y}_{rj} = \mathcal{X}\Theta_{aj}\mathcal{X}$$
$$\mathcal{Y}_{tj} = \mathcal{X}\Theta_{cj}\mathcal{X},\ \mathcal{Y}_{ej} = \mathcal{X}\Phi_{aj}\mathcal{X}$$
$$\mathcal{Y}_{sj} = \mathcal{X}\Phi_{cj}\mathcal{X},\ \mathcal{Y}_{vj} = \mathcal{X}\Psi_{aj}\mathcal{X},\ \mathcal{Y}_{wj} = \mathcal{X}\Psi_{cj}\mathcal{X},\ \mathcal{Y}_{kj} = \mathcal{X}K_j$$
$$\widehat{\mathcal{X}}_{aj} = \mathcal{X}R_{aj},\ \widehat{\mathcal{X}}_{cj} = \mathcal{X}R_{cj},\ \mathcal{Y}_{fm} = \mathcal{X}F_m,\ \mathcal{Y}_q = \pi\mathcal{X}C^t\mathbf{Q}$$
$$\mathcal{X}_{xj} = \mathcal{X}(N_a\mathcal{X} - N_b\mathcal{Y}_{kj}),\ \mathcal{X}_{ym} = \mathcal{X}N_bG\mathcal{Y}_{fm}$$

Simple manipulations show that $\mathcal{T}\ \overline{\Pi}\ \mathcal{T} = \overline{\Pi}_x$.

12.4 Special Cases

In order to illustrate the generality of our analysis and synthesis approach, hereafter we provide results pertaining to some relevant interesting cases in system theory. These results are summarized by the following corollaries:

Corollary 12.8 *Suppose that the gains* $K_j,\ F_m$ *are specified. Given the bounds* $\theta_M > 0,\ \theta_m > 0$ *and a scalar convergence rate* $\sigma > 0$. *System (12.15) is delay-dependent asymptotically stable with disturbance attenuation* γ *under arbitrary switching rules* ϕ *and* σ *if there exist weighting matrices* $\mathcal{P}, \mathcal{Q}, \mathcal{Z}, \mathcal{S}, \mathcal{R}_{aj}, \mathcal{R}_{cj}$, *parameter matrices* $\Theta_{aj}, \Theta_{cj}, \Psi_{aj}, \Psi_{cj}, \Phi_{aj}, \Phi_{cj}$ *satisfying the following LMIs for* $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$

$$\widetilde{\Pi}_\infty = \begin{bmatrix} \Omega_{jmd} & \Upsilon_{jm} & C^t\Lambda & \mathsf{A}_j^t\mathcal{P} \\ \bullet & -\Xi_j & 0 & \mathsf{B}_m^t\mathcal{P} \\ \bullet & \bullet & -\widehat{\Sigma} & \Gamma^t \\ \bullet & \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{12.39}$$

$$\Omega_{jmd} = \begin{bmatrix} \widetilde{\Pi}_o & \Pi_{aj} & \Psi_{aj} & -\Phi_{aj} \\ \bullet & -\Pi_{cj} & \Psi_{cj} & -\Phi_{cj} \\ \bullet & \bullet & -\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S} \end{bmatrix} \tag{12.40}$$

where

$$\begin{aligned} \widetilde{\Pi}_o &= -\mathcal{P} + (\theta_s + 1)\mathcal{Q} + \mathcal{Z} + \mathcal{S} + \Theta_{aj} + \Theta_{aj}^t + C^t C \\ \widehat{\Sigma} &= \gamma_o^2 I - \Lambda^t \Lambda \end{aligned} \tag{12.41}$$

Proof Follows from **Theorem 12.4** by setting $\mathbf{Q} = -I,\ \mathbf{S} = 0,\ (\mathbf{R} - \alpha I) = \gamma^2 I.$

Corollary 12.9 *Suppose that the gains $K_j,\ \ F_m$ are specified. Given the bounds $\theta_M > 0, \theta_m > 0$ and a scalar convergence rate $\sigma > 0$. System (12.15) is delay-dependent asymptotically stable and strictly positive real under arbitrary switching rules ϕ and σ if there exist weighting matrices $\mathcal{P}, \mathcal{Q}, \mathcal{Z}, \mathcal{S}, \mathcal{R}_{aj}, \mathcal{R}_{cj}$, parameter matrices $\Theta_{aj}, \Theta_{cj}, \Psi_{aj}, \Psi_{cj}, \Phi_{aj}, \Phi_{cj}$ satisfying the following LMIs for $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$*

$$\widetilde{\Pi}_s = \begin{bmatrix} \Omega_{jms} & \Upsilon_{jm} & -C^t & \mathsf{A}_j^t \mathcal{P} \\ \bullet & -\Xi_j & 0 & \mathsf{B}_m^t \mathcal{P} \\ \bullet & \bullet & -\widetilde{\Sigma} & \Gamma^t \\ \bullet & \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{12.42}$$

$$\Omega_{jms} = \begin{bmatrix} \widetilde{\Pi}_s & \Pi_{aj} & \Psi_{aj} & -\Phi_{aj} \\ \bullet & -\Pi_{cj} & \Psi_{cj} & -\Phi_{cj} \\ \bullet & \bullet & -\mathcal{Z} & 0 \\ \bullet & \bullet & \bullet & -\mathcal{S} \end{bmatrix} \tag{12.43}$$

where

$$\begin{aligned} \widetilde{\Pi}_s &= -\mathcal{P} + (\theta_s + 1)\mathcal{Q} + \mathcal{Z} + \mathcal{S} + \Theta_{aj} + \Theta_{aj}^t \\ \widetilde{\Sigma} &= \Lambda + \Lambda^t \end{aligned} \tag{12.44}$$

Proof Follows from **Theorem 12.4** by setting $\mathbf{Q} = 0,\ \mathbf{S} = I,\ (\mathbf{R} - \alpha I) = 0.$

Corollary 12.10 *Suppose that the gains K_j, F_m are specified. Given the bounds $\theta_M > 0, \theta_m > 0$ and a scalar convergence rate $\sigma > 0$. System (12.15) is delay-dependent asymptotically stable and passive under arbitrary switching rules ϕ and σ if there exist weighting matrices $\mathcal{P}, \mathcal{Q}, \mathcal{Z}, \mathcal{S}, \mathcal{R}_{aj}, \mathcal{R}_{cj}$, parameter matrices $\Theta_{aj}, \Theta_{cj}, \Psi_{aj}, \Psi_{cj}, \Phi_{aj}, \Phi_{cj}$ satisfying the following LMIs for $j \in \{1, 2, ..., N_a\}, m \in \{1, 2, ..., N_o\}, N_a, N_o \in \mathbf{N}$*

$$\tilde{\Pi} = \begin{bmatrix} \Omega_{jms} & \Upsilon_{jm} & -C^t & \mathsf{A}_j^t \mathcal{P} \\ \bullet & -\Xi_j & 0 & \mathsf{B}_m^t \mathcal{P} \\ \bullet & \bullet & -\Sigma_p & \Gamma^t \\ \bullet & \bullet & \bullet & -\mathcal{P} \end{bmatrix} < 0 \tag{12.45}$$

where $\Sigma_p = \beta I + \Lambda + \Lambda^t$.

Proof Follows from **Theorem 12.4** by setting $\mathbf{Q} = 0$, $\mathbf{S} = I$, $(\mathbf{R} - \alpha I) = \beta I$.

12.5 Numerical Simulation

In order to demonstrate the applicability and effectiveness of the developed approach, we present a numerical simulation example. Consider system (12.16) with the following data ($N_a = 2,\ N_o = 3$)

$$A_1 = \begin{bmatrix} 0 & 0.1 & -0.05 \\ 0 & 0.01 & 0.3 \\ 0.1 & 0.4 & 0.6 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -0.3 & -0.01 \\ 0.1 & 0.02 & 0.1 \\ 0.2 & -0.3 & 0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 1.2 & 1.1 & -1.8 \\ 0.7 & 0.2 & 0.5 \\ 0.2 & -0.3 & 1.4 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0.7 & 0.4 & -1.2 \\ -0.5 & 1.1 & -2.3 \\ 0.3 & -1.4 & 1.9 \end{bmatrix},\ B_3 = \begin{bmatrix} -1.1 & 0.4 & 0.5 \\ 1.3 & 0.5 & -1.3 \\ -0.1 & 0.8 & -1.5 \end{bmatrix},\ C^t = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix},\ M_o = \begin{bmatrix} 0.4 \\ 0 \\ 0.2 \end{bmatrix},\ N_a^t = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},\ N_b^t = \begin{bmatrix} 0.2 \\ 0.3 \\ 0 \end{bmatrix},\ M_c = 0.5,\ \Lambda = 0.1$$

Observe that the system under consideration is unstable. Using the Matlab LMI Control Toolbox with $\theta_m = 4,\quad \theta_M = 8$, it is found that the feasible solution of LMI (12.36) is given by

$$\mathcal{X} = \begin{bmatrix} 0.8737 & 0.4545 & 0.2458 \\ 0.4545 & 0.7785 & 0.1658 \\ 0.2458 & 0.1658 & 0.5756 \end{bmatrix},\ \mathcal{Y}_{k1} = \begin{bmatrix} 0.7624 & 0.1103 & 0.0005 \\ 0.1205 & 0.0144 & 0.2367 \end{bmatrix}$$

$$\mathcal{Y}_{k2} = \begin{bmatrix} 0.6543 & 0.2103 & 0.0101 \\ 0.0145 & 0.5282 & 0.2175 \end{bmatrix},\ \mathcal{Y}_{f1} = \begin{bmatrix} 0.9012 & 0.6334 & 0.0008 \\ 0.1546 & 0.1024 & 0.1055 \end{bmatrix}$$

$$\mathcal{Y}_{f2} = \begin{bmatrix} 0.7172 & 0.5481 & 0.0003 \\ 0.4323 & 0.1455 & 0.0019 \end{bmatrix},\ \mathcal{Y}_{f3} = \begin{bmatrix} 0.8344 & 0.5756 & 0.6362 \\ 0.4143 & 0.2872 & 0.0005 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0.0077 & 0.1467 & -0.0447 \\ 0.0050 & 0.0026 & 0.0609 \\ 0.0033 & 0.0033 & 0.0714 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.0813 & -0.0326 & -0.0078 \\ -0.1063 & 0.0087 & 0.4208 \\ -0.0653 & -0.0446 & -0.0218 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -0.0292 & 0.0607 & -0.0036 \\ 0.0914 & -0.0469 & -0.0160 \\ 0.0024 & 0.0061 & 0.0823 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.0143 & 0.0732 & -0.0144 \\ -0.0787 & 0.2397 & -0.0321 \\ -0.0051 & 0.0032 & 0.0723 \end{bmatrix}$$

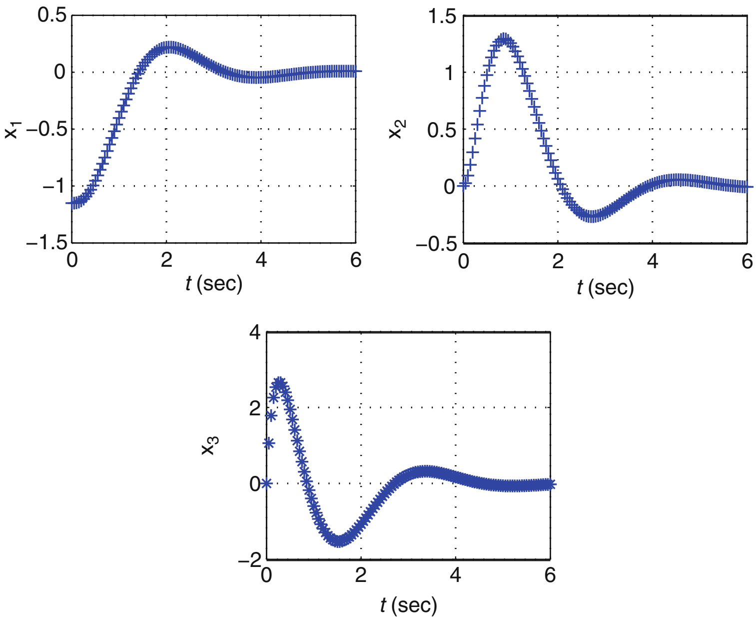

Fig. 12.2 State trajectories

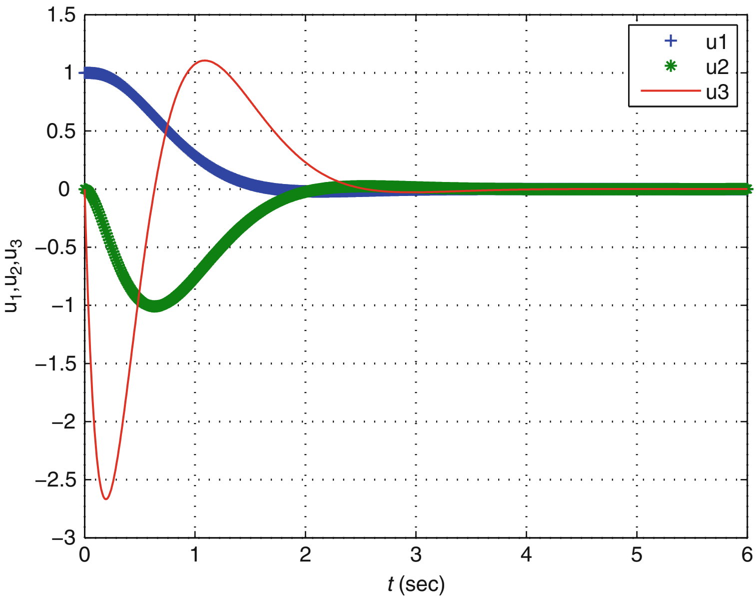

Fig. 12.3 Control trajectories: input 1

$$F_3 = \begin{bmatrix} -0.0269 & 0.1033 & 0.0446 \\ -0.2409 & 0.5193 & -0.0459 \\ 0.0043 & -0.0056 & -0.0663 \end{bmatrix}$$

State trajectories simulation is depicted in Fig. 12.2 and the corresponding controls are plotted in Figs. 12.3 and 12.4. It can be easily verified that the closed-loop system now is stable, which emphasizes the effectiveness of the robust dissipative approach in stabilizing the system by remote and local controllers.

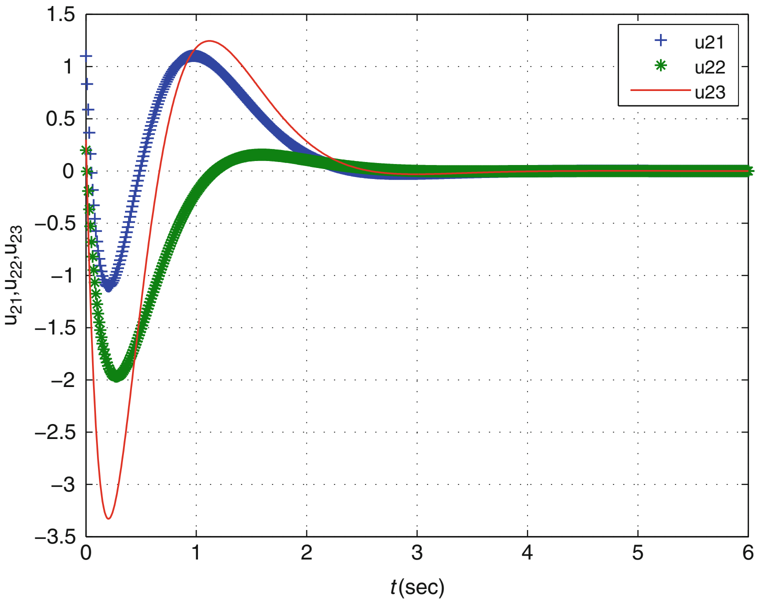

Fig. 12.4 Control trajectories: input 2

12.6 Conclusions

In this chapter, we have formulated a class of hybrid multi-rate control models with time-delay and switching controllers based on combined remote control and local control strategies. We have addressed the problem of designing robust dissipative controllers based on generalized criteria. An improved Lyapunov–Krasovskii functional has been constructed and the subsequent analysis has established some new sufficient conditions in the form of LMIs for both nominal and uncertain representations. Several special cases of practical interests have been derived. A numerical simulation example has been given to illustrate the effectiveness of the theoretical result.

Appendix

In this appendix, we collect some useful mathematical inequalities and lemmas, which have been extensively used throughout the book.

A.1 Basic Inequalities

All mathematical inequalities are proved for completeness. They are termed facts due to their high frequency of usage in the analytical developments.

A.1.1 Inequality 1

For any real matrices Σ_1, Σ_2, and Σ_3 with appropriate dimensions, it follows that

$$\Sigma_1 \Sigma_2 + \Sigma_2^t \Sigma_1^t \leq \alpha \Sigma_1 \Sigma_1^t + \alpha^{-1} \Sigma_2^t \Sigma_2, \forall \alpha > 0$$

Proof This inequality can be proved as follows. Since $\Phi^t \Phi \geq 0$ holds for any matrix Φ, then take Φ as

$$\Phi = [\alpha^{1/2} \Sigma_1 - \alpha^{-1/2} \Sigma_2]$$

Expansion of $\Phi^t \Phi \geq 0$ gives $\forall \alpha > 0$

$$\alpha \Sigma_1 \Sigma_1^t + \alpha^{-1} \Sigma_2^t \Sigma_2 - \Sigma_1^t \Sigma_2 - \Sigma_2^t \Sigma_1 \geq 0$$

which by simple arrangement yields the desired result.

A.1.2 Inequality 2

Let $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ and $0 < R = R^t$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \leq I$. Then for any

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$\rho > 0$ satisfying $\rho\Sigma_2^t\Sigma_2 < R$, the following matrix inequality holds:

$$(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}\left(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t\right) \le \rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3\left(R - \rho\Sigma_2^t\Sigma_2\right)^{-1}\Sigma_3^t$$

Proof The proof of this inequality proceeds like the previous one by considering that

$$\Phi = \left[\left(\rho^{-1}\Sigma_2\Sigma_2^t\right)^{-1/2}\Sigma_2 R^{-1}\Sigma_3^t - \left(\rho^{-1}\Sigma_2\Sigma_2^t\right)^{-1/2} H^t(t)\Sigma_1^t\right]$$

Recall the following results

$$\rho\Sigma_2^t\Sigma_2 < R,$$
$$\left[R - \rho\Sigma_2^t\Sigma_2\right]^{-1} = \left[R^{-1} + R^{-1}\Sigma_2^t\left[\rho^{-1}I - \Sigma_2 R^{-1}\Sigma_2^t\right]^{-1}\Sigma_2 R^{-1}\Sigma_2\right]$$

and

$$H^t(t)H(t) \le I \Longrightarrow H(t)H^t(t) \le I$$

Expansion of $\Phi^t\Phi \ge 0$ under the condition $\rho\Sigma_2^t\Sigma_2 < R$ with standard matrix manipulations gives

$$\begin{aligned}
&\Sigma_3 R^{-1}\Sigma_2^t H^t(t)\Sigma_1^t + \Sigma_1 H(t)\Sigma_2 R^{-1}\Sigma_3^t + \Sigma_1 H(t)\Sigma_2\Sigma_2^t H^t(t)\Sigma_1^t \le \\
&\rho^{-1}\Sigma_1 H(t)H^t(t)\Sigma_1^t + \Sigma_3^t R^{-1}\Sigma_2\left[\rho^{-1}I\Sigma_2\Sigma_2^t\right]^{-1}\Sigma_2 R^{-1}\Sigma_3^t \Longrightarrow \\
&(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}\left(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t\right) - \Sigma_3 R^{-1}\Sigma_3^t \le \\
&\rho^{-1}\Sigma_1 H(t)H^t(t)\Sigma_1^t + \Sigma_3^t R^{-1}\Sigma_2\left[\rho^{-1}I - \Sigma_2\Sigma_2^t\right]^{-1}\Sigma_2 R^{-1}\Sigma_3^t \Longrightarrow \\
&(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}\left(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t\right) \le \\
&\Sigma_3\left[R^{-1} + \Sigma_2\left[\rho^{-1}I - \Sigma_2\Sigma_2^t\right]^{-1}\Sigma_2 R^{-1}\right]\Sigma_3^t + \\
&\rho^{-1}\Sigma_1 H(t)H^t(t)\Sigma_1^t = \\
&\rho^{-1}\Sigma_1 H(t)H^t(t)\Sigma_1^t + \Sigma_3\left(R - \rho\Sigma_2^t\Sigma_2\right)^{-1}\Sigma_3^t
\end{aligned}$$

which completes the proof.

A.1.3 Inequality 3

For any real vectors β, ρ and any matrix $Q^t = Q > 0$ with appropriate dimensions, it follows that

$$-2\rho^t\beta \le \rho^t Q\rho + \beta^t Q^{-1}\beta$$

Proof Starting from the fact that

$$[\rho + Q^{-1}\beta]^t Q[\rho + Q^{-1}\beta] \ge 0,\ Q > 0$$

which, when expanded and arranged, yields the desired result.

A.1.4 Inequality 4 (Schur Complements)

Given a matrix Ω composed of constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ as follows

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_3 \\ \Omega_3^t & \Omega_2 \end{bmatrix}$$

We have the following results

(A) $\Omega \ge 0$ if and only if either

$$\begin{cases} \Omega_2 \ge 0 \\ \Pi = \Upsilon\Omega_2 \\ \Omega_1 - \Upsilon\Omega_2\Upsilon^t \ge 0 \end{cases} \tag{A.1}$$

or

$$\begin{cases} \Omega_1 \ge 0 \\ \Pi = \Omega_1\Lambda \\ \Omega_2 - \Lambda^t\Omega_1\Lambda \ge 0 \end{cases} \tag{A.2}$$

hold where Λ, Υ are some matrices of compatible dimensions.

(B) $\Omega > 0$ if and only if either

$$\begin{cases} \Omega_2 > 0 \\ \Omega_1 - \Omega_3\Omega_2^{-1}\Omega_3^t > 0 \end{cases}$$

or

$$\begin{cases} \Omega_1 \ge 0 \\ \Omega_2 - \Omega_3^t\Omega_1^{-1}\Omega_3 > 0 \end{cases}$$

hold where Λ, Υ are some matrices of compatible dimensions.

In this regard, matrix $\Omega_3\Omega_2^{-1}\Omega_3^t$ is often called the Schur complement $\Omega_1(\Omega_2)$ in Ω.

Proof (A) To prove (A.1), we first note that $\Omega_2 \ge 0$ is necessary. Let $z^t = \begin{bmatrix} z_1^t & z_2^t \end{bmatrix}$ be a vector partitioned in accordance with Ω. Thus we have

$$z^t \Omega z = z_1^t \Omega_1 z_1 + 2 z_1^t \Omega_3 z_2 + z_2^t \Omega_2 z_2 \tag{A.3}$$

Select z_2 such that $\Omega_2 z_2 = 0$. If $\Omega_3 z_2 \neq 0$, let $z_1 = -\pi \Omega_3 z_2, \pi > 0$. Then it follows that

$$z^t \Omega z = \pi^2 z_2^t \Omega_3^t \Omega_1 \Omega_3 z_2 - 2\pi z_2^t \Omega_3^t \Omega_3 z_2$$

which is negative for a sufficiently small $\pi > 0$. We thus conclude $\Omega_1 z_2 = 0$, which then leads to $\Omega_3 z_2 = 0, \forall z_2$ and consequently

$$\Omega_3 = \Upsilon \Omega_2 \tag{A.4}$$

for some Υ.

Since $\Omega \geq 0$, the quadratic term $z^t \Omega z$ possesses a minimum over z_2 for any z_1. By differentiating $z^t \Omega z$ from (A.3) wrt z_2^t, we get

$$\frac{\partial (z^t \Omega z)}{\partial z_2^t} = 2\Omega_3^t z_1 + 2\Omega_2 z_2 = 2\Omega_2 \Upsilon^t z_1 + 2\Omega_2 z_2$$

Setting the derivative to zero yields

$$\Omega_2 \Upsilon z_1 = -\Omega_2 z_2 \tag{A.5}$$

Using (A.4) and (A.5) in (A.3), it follows that the minimum of $z^t \Omega z$ over z_2 for any z_1 is given by

$$\min_{z_2} z^t \Omega z = z_1^t [\Omega_1 - \Upsilon \Omega_2 \Upsilon^t] z_1$$

which proves the necessity of $\Omega_1 - \Upsilon \Omega_2 \Upsilon^t \geq 0$.

On the contrary, we note that the conditions (A.1) are necessary for $\Omega \geq 0$ and since together they imply that the minimum of $z^t \Omega z$ over z_2 for any z_1 is nonnegative, they are also sufficient.

Using similar argument, conditions (A.2) can be derived as those of (A.1) by starting with Ω_1.

The proof of (B) follows as direct corollary of (A).

A.1.5 Inequality 5

For any quantities u and v of equal dimensions and for all $\eta_t = i \in \mathcal{S}$, it follows that the following inequality holds

$$||u + v||^2 \leq [1 + \beta^{-1}]||u||^2 + [1 + \beta]||v||^2 \tag{A.6}$$

for any scalar $\beta > 0, i \in \mathcal{S}$

Proof Since

$$[u + v]^t [u + v] =$$
$$u^t u + v^t v + 2u^t v \tag{A.7}$$

It follows by taking norm of both sides of (A.7) for all $i \in \mathcal{S}$ that

$$||u + v||^2 \leq ||u||^2 + ||v||^2 + 2||u^t v|| \tag{A.8}$$

We know from the triangle inequality that

$$2||u^t v|| \leq \beta^{-1}||u||^2 + \beta||v||^2 \tag{A.9}$$

On substituting (A.9) into (A.8), it yields (A.6).

A.1.6 Inequality 6

Given matrices $0 < \mathcal{Q}^t = \mathcal{Q}, \mathcal{P} = \mathcal{P}^t$, then it follows that

$$-\mathcal{P}\mathcal{Q}^{-1}\mathcal{P} \leq -2\mathcal{P} + \mathcal{Q} \tag{A.10}$$

This can be easily established by considering the algebraic inequality

$$(\mathcal{P} - \mathcal{Q})^t \mathcal{Q}^{-1}(\mathcal{P} - \mathcal{Q}) \geq 0$$

and expanding to get

$$\mathcal{P}\mathcal{Q}^{-1}\mathcal{P} - 2\mathcal{P} + \mathcal{Q} \geq 0 \tag{A.11}$$

which, when manipulating, yields (A.10). An important special case is obtained when $\mathcal{P} \equiv I$, that is

$$-\mathcal{Q}^{-1} \leq -2I + \mathcal{Q} \tag{A.12}$$

This inequality proves useful when using Schur complements to eliminate the quantity $\mathcal{Q}^{-1}$ from the diagonal of an LMI without alleviating additional math operations.

A.2 Lemmas

The basic tools and standard results that are utilized in robustness analysis and resilience design in the different chapters are collected hereafter.

Lemma A.1 *Consider the functional*

$$H = \int_{\alpha(t)}^{\beta(t)} \int_{t-s}^{t} g(r)\mathrm{d}r\mathrm{d}s$$

with $g(r) > 0$*, then the time-derivative* $\dot{H}$ *is given by*

$$\dot{H} = [\beta(t) - \alpha(t)]g(t) - [1 - \dot{\beta}] \int_{t-\beta(t)}^{t-\alpha(t)} g(s)\mathrm{d}s$$
$$+[\dot{\beta} - \dot{\alpha}] \int_{t-\alpha}^{t} g(s)\mathrm{d}s$$

Proof Defining $G(s) = \int g(s)\mathrm{d}s$ and applying the classical Leibniz rule of calculus

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{q(x)}^{p(x)} f(x,t)\mathrm{d}t = \frac{\mathrm{d}p}{\mathrm{d}x} f(x, p(x))$$
$$-\frac{\mathrm{d}q}{\mathrm{d}x} f(x, q(x)) + \int_{q(x)}^{p(x)} \frac{\partial f}{\partial x}(x,t)\mathrm{d}t$$

the desired result readily follows. ■

Lemma A.2 *The matrix inequality*

$$-\Lambda + S\Omega^{-1}S^t < 0 \tag{A.13}$$

holds for some $0 < \Omega = \Omega^t \in \Re^{n\times n}$*, if and only if*

$$\begin{bmatrix} -\Lambda & S\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} < 0 \tag{A.14}$$

holds for some matrices $\mathcal{X} \in \Re^{n\times n}$ *and* $\mathcal{Z} \in \Re^{n\times n}$.

Proof ($\Longrightarrow$) By Schur complements, inequality (A.13) is equivalent to

$$\begin{bmatrix} -\Lambda & S\Omega^{-1} \\ \bullet & -\Omega^{-1} \end{bmatrix} < 0 \tag{A.15}$$

Setting $\mathcal{X} = \mathcal{X}^t = \mathcal{Z} = \Omega^{-1}$, we readily obtain inequality (A.14).
($\Longleftarrow$) Since the matrix $[I \;\; S]$ is of full rank, we obtain

$$\begin{bmatrix} I \\ S^t \end{bmatrix}^t \begin{bmatrix} -\Lambda & S\mathcal{X} \\ \bullet & -\mathcal{X} - \mathcal{X}^t + \mathcal{Z} \end{bmatrix} \begin{bmatrix} I \\ S^t \end{bmatrix} < 0 \Longleftrightarrow$$
$$-\Lambda + S\mathcal{Z}S^t < 0 \Longleftrightarrow$$
$$-\Lambda + S\Omega^{-1}S^t < 0, \mathcal{Z} = \Omega^{-1} \tag{A.16}$$

which completes the proof. ■

Lemma A.3 *The matrix inequality*

$$A\mathcal{P} + \mathcal{P}A^t + D^t\mathcal{R}D + \mathcal{M} < 0 \tag{A.17}$$

holds for some $0 < \mathcal{P} = \mathcal{P}^t \in \Re^{n\times n}$, *if and only if*

$$\begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + \mathcal{M} & \mathcal{P} + A\mathcal{W} - \mathcal{V} & D^t\mathcal{R} \\ \bullet & -\mathcal{W} - \mathcal{W}^t & 0 \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix} < 0 \tag{A.18}$$

holds for some $\mathcal{V} \in \Re^{n\times n}$ *and* $\mathcal{W} \in \Re^{n\times n}$.

Proof ($\Longrightarrow$) By Schur complements, inequality (A.17) is equivalent to

$$\begin{bmatrix} A\mathcal{P} + \mathcal{P}A^t + \mathcal{M} & D^t\mathcal{R} \\ \bullet & -\mathcal{R} \end{bmatrix} < 0 \tag{A.19}$$

Setting $\mathcal{V} = \mathcal{V}^t = \mathcal{P}, \mathcal{W} = \mathcal{W}^t = \mathcal{R}$, it follows from **Lemma (A.2)** with Schur complements that there exists $\mathcal{P} > 0$, $\mathcal{V}$, $\mathcal{W}$ such that inequality (A.18) holds.
($\Longleftarrow$) In a similar way, Schur complements to inequality (A.18) imply that

$$\begin{aligned} &\begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + \mathcal{M} & \mathcal{P} + A\mathcal{W} - \mathcal{V} & D^t\mathcal{R} \\ \bullet & -\mathcal{W} - \mathcal{W}^t & 0 \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix} < 0 \\ &\Longleftrightarrow \begin{bmatrix} I \\ A \end{bmatrix} \begin{bmatrix} A\mathcal{V} + \mathcal{V}^t A^t + \mathcal{M} + D^t\mathcal{P}^{-1}D & \mathcal{P} + A\mathcal{W} - \mathcal{V} \\ \bullet & -\mathcal{W} - \mathcal{W}^t \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix}^t < 0 \\ &\Longleftrightarrow A\mathcal{P} + \mathcal{P}A^t + D^t\mathcal{P}D + \mathcal{M} < 0, \mathcal{V} = \mathcal{V}^t \end{aligned} \tag{A.20}$$

which completes the proof. ■

The following lemmas are found in [334].

Lemma A.4 *Given any* $x \in \Re^n$*:*

$$\max\{[x^t R H \Delta G x]^2 : \Delta \in \Re\} = x^t R H H^t R x x^t G^t G x$$

Lemma A.5 *Given matrices* $0 \le X = X^t \in \Re^{p\times p}, Y = Y^t < 0 \in \Re^{p\times p}, 0 \le Z = Z^t \in \Re^{p\times p}$, *such that*

$$[\xi^t Y \xi]^2 - 4[\xi^t X \xi \xi^t Z \xi]^2 > 0$$

for all $0 \ne \xi \in \Re^p$ *is satisfied. Then there exists a constant* $\alpha > 0$ *such that*

$$\alpha^2 X + \alpha Y + Z < 0$$

The following lemma can be found in [301].

Lemma A.6 *For given two vectors* $\alpha \in \Re^n$, $\beta \in \Re^m$ *and matrix* $\mathcal{N} \in \Re^{n\times m}$ *defined over a prescribed interval* Ω*, it follows for any matrices* $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, *and* $Z \in \Re^{m\times m}$*, the following inequality holds*

$$-2\int_{\Omega} \alpha^t(s)\mathcal{N}\beta(s)\mathrm{d}s \le \int_{\Omega} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix}^t \begin{bmatrix} X & Y-\mathcal{N} \\ Y^t-\mathcal{N}^t & Z \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} \mathrm{d}s$$

where

$$\begin{bmatrix} X & Y \\ Y^t & Z \end{bmatrix} \ge 0$$

An algebraic version of **Lemma A.6** is stated below

Lemma A.7 *For given two vectors* $\alpha \in \Re^n$, $\beta \in \Re^m$ *and matrix* $\mathcal{N} \in \Re^{n\times m}$ *defined over a prescribed interval* Ω*, it follows for any matrices* $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, *and* $Z \in \Re^{m\times m}$*, the following inequality holds*

$$\begin{aligned} -2\alpha^t\mathcal{N}\beta &\le \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^t \begin{bmatrix} X & Y-\mathcal{N} \\ Y^t-\mathcal{N}^t & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \alpha^t X\alpha + \beta^t(Y^t-\mathcal{N}^t)\alpha + \alpha^t(Y-\mathcal{N})\beta + \beta^t Z\beta \end{aligned}$$

subject to

$$\begin{bmatrix} X & Y \\ Y^t & Z \end{bmatrix} \ge 0$$

The following lemma can be found in [216]

Lemma A.8 *Let* $0 < Y = Y^t$ *and* M, N *be given matrices with appropriate dimensions. Then it follows that*

$$Y + M\Delta N + N^t\Delta^t M^t < 0, \forall \Delta^t\Delta \le I$$

holds if and only if there exists a scalar $\varepsilon > 0$ *such that*

$$Y + \varepsilon MM^t + \varepsilon^{-1}N^tN < 0$$

In the following lemma, we let $X(z) \in \Re^{n\times p}$ be a matrix function of the variable z. A matrix $X_*(z)$ is called the orthogonal complement of $X(z)$ if $X^t(z)X_*(z) = 0$ and $X(z)X_*(z)$ is nonsingular (of maximum rank).

Lemma A.9 *Let* $0 < L = L^t$ *and* X, Y *be given matrices with appropriate dimensions. Then it follows that the inequality*

$$L(z) + X(z)PY(z) + Y^t(z)P^tX^t(z) > 0 \tag{A.21}$$

holds for some P *and* $z = z_o$ *if and only if the following inequalities*

$$X_*^t(z)L(z)X_*(z) > 0,\ Y_*^t(z)L(z)Y_*(z) > 0 \tag{A.22}$$

hold with $z = z_o$.

It is significant to observe that feasibility of matrix inequality (A.21) with variables P and z is equivalent to the feasibility of (A.22) with variable z and thus the matrix variable P has been eliminated from (A.21) to form (A.22). Using Finsler's lemma [27], we can express (A.22) in the form

$$L(z) - \beta X(z)X^t(z) > 0,\ L(z) - \beta Y(z)Y^t(z) > 0 \tag{A.23}$$

for some $\beta \in \mathbf{R}$.

The following is a statement of the reciprocal projection Lemma [4]

Lemma A.10 *Let* $P > 0$ *be a given matrix. The following statements are equivalent:*

(i) $\mathcal{M} + Z + Z^t < 0$

(ii) the LMI problem

$$\begin{bmatrix} \mathcal{M} + P - (V + V^t) & V^t + Z^t & array \\ V + Z & -P & \end{bmatrix} < 0$$

is feasible with respect to the general matrix V.

A useful lemma that is frequently used in overbounding given inequalities is presented.

Lemma A.11 *For matrices* X *and* Y *and* $K^t = K \geq 0$ *of appropriate dimensions, with* $K^\dagger$ *being the Moore–Penrose generalized inverse of matrix* K*, the following lemma is proved:*

$$X^tKK^\dagger Y + Y^tKK^\dagger X \leq X^tKX + Y^tK^\dagger Y$$

In particular, if x *and* y *are vectors and* $K^t = K > 0$*, then*

$$x^ty \leq (1/2)x^tKx + (1/2)y^tK^{-1}y$$

Proof Lrt the real Schur decomposition of K be $K = U^tVU$ where $U = U^{-t}$ is orthogonal and $V = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is the diagonal matrix of eigenvalues. The lemma then follows from

$$
\begin{aligned}
0 &\le \left[\sqrt{V}UX - \sqrt{V^{\dagger}}U^{-t}Y\right]^t \left[\sqrt{V}UX - \sqrt{V^{\dagger}}U^{-t}Y\right] \\
&\le X^tU^tVUX + Y^tU^{-1}V^{\dagger}UY - Y^tU^{-1}\sqrt{V^{\dagger}}\sqrt{V}UX \\
&- X^tU^t\sqrt{V}\sqrt{V^{\dagger}}U^{-t}X
\end{aligned}
$$

as $U^{-1}\sqrt{V^{\dagger}}\sqrt{V}U = U^t\sqrt{V}\sqrt{V^{\dagger}}U^{-t} = KK^{\dagger}$ and $K^{\dagger} = U^{-1}V^{\dagger}U$

Lemma A.12 *Let* $x \in \Re^n$, $\mathcal{P}^t = \mathcal{P} \in \Re^{n\times n}$, $\mathcal{Z} \in \Re^{m\times n}$, $rank(\mathcal{Z}) = r < n$. *The following statements are equivalent:*

(I) $x^t\mathcal{P}x < 0, \forall x \neq 0 : \mathcal{Z}x = 0, thatis, \forall x(\neq 0) \in \mathbf{Ker}\mathcal{Z}$,

(II) $\exists \mathcal{B} \in \Re^{n\times m} : \mathcal{P} + \mathcal{B}\mathcal{Z} + \mathcal{Z}^t\mathcal{B}^t < 0$.

This is frequently called *Finsler's Lemma* and can be found in [27].

A.3 Linear Matrix Inequalities

It has been shown in [27] that a wide variety of problems arising in system and control theory can conveniently reduced to a few standard convex or quasi-convex optimization problems involving linear matrix inequalities (LMIs). The resulting optimization problems can then be solved numerically very efficiently using commercially available interior-point methods.

A.3.1 Basics

One of the earliest LMIs arises in Lyapunov theory. It is well known that the differential equation

$$\dot{x}(t) = Ax(t) \tag{A.24}$$

has all of its trajectories converge to zero (stable) if and only if there exists a matrix $P > 0$ such that

$$A^tP + AP < 0 \tag{A.25}$$

This leads to the LMI formulation of stability, that is, *a linear time-invariant system is asymptotically stable if and only if there exists a matrix* $0 < P = P^t$ *satisfying the LMIs*

$$A^tP + AP < 0, P > 0$$

Given a vector variable $x \in \mathbf{R}^n$ and a set of matrices $0 < G_j = G_j^t \in \mathbf{R}^{n\times n}$, $j = 0, ..., p$, then a basic compact formulation of a linear matrix inequality is

$$G(x) \stackrel{\Delta}{=} G_0 + \sum_{j=1}^{p} x_j G_j > 0 \tag{A.26}$$

Notice that (A.26) implies that $v^t G(x) v > 0 \forall 0 \neq v \in \Re^n$. More importantly, the set $\{x | G(x) > 0$ is convex. Nonlinear (convex) inequalities are converted to LMI form using Schur complements in the sense that

$$\begin{bmatrix} Q(x) & S(x) \\ \bullet & R(x) \end{bmatrix} > 0 \tag{A.27}$$

where $Q(x) = Q^t(x)$, $R(x) = R^t(x)$, $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0,\ Q(x) - S(x) R^{-1}(x) S^t(x) > 0 \tag{A.28}$$

More generally, the constraint

$$Tr[S^t(x) P^{-1}(x) S(x)] < 1,\ P(x) > 0$$

where $P(x) = P^t(x) \in \Re^{n\times n}$, $S(x) \in \Re^{n\times p}$ depend affinely on x, is handled by introducing a new (slack) matrix variable $Y(x) = Y^t(x) \in\in \Re^{p\times p}$ and the LMI (in x and Y):

$$Tr Y < 1, \begin{bmatrix} Y & S(x) \\ \bullet & P(x) \end{bmatrix} > 0 \tag{A.29}$$

Most of the time, our LMI variables are matrices. It should be clear from the foregoing discussions that a quadratic matrix inequality (QMI) in the variable P can be readily expressed as a linear matrix inequality (LMI) in the same variable.

A.3.2 Some Standard Problems

Here we provide some common convex problems that we encountered throughout the monograph. Given an LMI $G(x) > 0$, the corresponding LMI problem (LMIP) is to

find a feasible $x \equiv x^f$ *such that* $G(x^f) > 0$,
or *determine that the LMI is infeasible*.

It is obvious that this is a convex feasibility problem.

The generalized eigenvalue problem (GEVP) is to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on a variable, subject to an LMI constraint. GEVP has the general form

$$\begin{aligned} &\text{minimize } \lambda \\ &\text{subject to } \lambda B(x) - A(x) > 0,\, B(x) > 0,\, C(x) > 0 \end{aligned} \quad \text{(A.30)}$$

where A, B, C are symmetric matrices that are affine functions of x. Equivalently stated

$$\begin{aligned} &\text{minimize } \ \lambda_M[A(x), B(x)] \\ &\text{subject } \ \text{to } B(x) > 0, C(x) > 0 \end{aligned} \quad \text{(A.31)}$$

where $\lambda_M[X, Y]$ denotes the largest generalized eigenvalue of the pencil $\lambda Y - X$ with $Y > 0$. This is a quasi-convex optimization problem since the constraint is convex and the objective $\lambda_M[A(x), B(x)]$ is quasi-convex.

The eigenvalue problem (EVP) is to minimize the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an LMI constraint. EVP has the general form

$$\begin{aligned} &\text{minimize } \lambda \\ &\text{subject } \ \text{to } \lambda I - A(x) > 0,\, B(x) > 0 \end{aligned} \quad \text{(A.32)}$$

where A, B are symmetric matrices that are affine functions of the optimization variable x. This is a convex optimization problem.
EVPs can appear in the equivalent form of minimizing a linear function subject to an LMI, that is

$$\begin{aligned} &\text{minimize } c^t x \\ &\text{subject } \ \text{to } G(x) > 0 \end{aligned} \quad \text{(A.33)}$$

where $G(x)$ is an affine function of x. Examples of $G(x)$ include

$$PA + A^t P + C^t C + \gamma^{-1} PBB^t P < 0, \;\; P > 0$$

It should be stressed that the standard problems (LMIPs, GEVPs, EVPs) are tractable, from both theoretical and practical viewpoints:

They can be solved in polynomial time.

They can solved in practice very efficiently using commercial software [74].

A.3.3 The S-Procedure

In some design applications, we faced the constraint that some quadratic functions be negative whenever some other quadratic function is negative. In such cases, this constraint can be expressed as an LMI in the data variables defining the quadratic functions.

Let $G_o, ..., G_p$ be quadratic functions of the variable $\xi \in \Re^n$:

$$G_j(\xi) \stackrel{\Delta}{=} \xi^t R_j \xi + 2u_j^t \xi + v_j, \;\; j = 0, ..., p, R_j = R_j^t$$

We consider the following condition on $G_o, ..., G_p$:

$$G_o(\xi) \leq 0 \forall \xi \;\text{ such that }\; G_j(\xi) \geq 0, \;\; j = 0, ..., p \tag{A.34}$$

It is readily evident that if there exist scalars $\omega_1 \geq 0, ..., \omega_p \geq 0$ such that

$$\forall \xi, \;\; G_o(\xi) - \sum_{j=1}^{p} \omega_j G_j(\xi) \geq 0 \tag{A.35}$$

then inequality (A.34) holds. Observe that if the functions $G_o, ..., G_p$ are affine, then Farkas lemma [27] state that (A.34) and (A.35) are equivalent. Interestingly enough, inequality (A.35) can written as

$$\begin{bmatrix} R_o & u_o \\ \bullet & v_o \end{bmatrix} - \sum_{j=1}^{p} \omega_j \begin{bmatrix} R_j & u_j \\ \bullet & v_j \end{bmatrix} \geq 0 \tag{A.36}$$

The foregoing discussions were stated for nonstrict inequalities. In case of strict inequality, we let $R_o, ..., R_p \in \Re^{n \times n}$ be symmetric matrices with the following qualifications

$$\xi^t \; R_o \, \xi > 0 \forall \xi \;\text{ such that }\; \xi^t G_j \xi \geq 0, \;\; j = 0, ..., p \tag{A.37}$$

Once again, it is obvious that there exist scalars $\omega_1 \geq 0, ..., \omega_p \geq 0$ such that

$$\forall \xi, G_o(\xi) - \sum_{j=1}^{p} \omega_j G_j(\xi) > 0 \tag{A.38}$$

then inequality (A.37) holds. Observe that (A.38) is an LMI in the variables $R_o, \omega_1, ..., \omega_p$.

It should be remarked that the **S**-procedure deals with nonstrict inequalities and allows the inclusion of constant and linear terms. In the strict version, only quadratic functions can be used.

A.4 Some Continuous Lyapunov–Krasovskii Functionals

In this section, we provide some Lyapunov–Krasovskii functionals and their time derivatives which are of common use in stability studies throughout the text.

$$V_1(x) = x^t P x + \int_{-\tau}^{0} x^t(t+\theta) Q x(t+\theta) \mathrm{d}\theta \tag{A.39}$$

$$V_2(x) = \int_{-\tau}^{0} \left[\int_{t+\theta}^{t} x^t(\alpha) R x(\alpha) d\alpha \right] \mathrm{d}\theta \tag{A.40}$$

$$V_3(x) = \int_{-\tau}^{0} \left[\int_{t+\theta}^{t} \dot{x}^t(\alpha) W \dot{x}(\alpha) d\alpha \right] \mathrm{d}\theta \tag{A.41}$$

where x is the state vector, τ is a constant delay factor, and the matrices $0 < P^t = P$, $0 < Q^t = Q$, $0 < R^t = R$, $0 < W^t = W$ are appropriate weighting factors.

Standard matrix manipulations lead to

$$\dot{V}_1(x) = \dot{x}^t P x + x^t P \dot{x} + x^t(t) Q x(t) - x^t(t-\tau) Q x(t-\tau) \tag{A.42}$$

$$\begin{aligned} \dot{V}_2(x) &= \int_{-\tau}^{0} \left[x^t(t) R x(t) - x^t(t+\alpha) R x(t+\alpha) \right] \mathrm{d}\theta \\ &= \tau x^t(t) R x(t) - \int_{-\tau}^{0} x^t(t+\theta) R x(t+\theta) \Big] \mathrm{d}\theta \end{aligned} \tag{A.43}$$

$$\dot{V}_3(x) = \tau \dot{x}^t(t) W x(t) - \int_{t-\tau}^{t} \dot{x}^t(\alpha) W \dot{x}(\alpha) \mathrm{d}\alpha \tag{A.44}$$

A.5 Some Formulas on Matrix Inverses

This concerns some useful formulas for inverting matrix expressions in terms of the inverses of its constituents.

A.5.1 Inverse of Block Matrices

Let A be a square matrix of appropriate dimension and partitioned in the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{A.45}$$

where both A_1 and A_4 are square matrices. If A_1 is invertible, then

$$\varDelta_1 = A_4 - A_3 A_1^{-1} A_2$$

is called the Schur complement of A_1. Alternatively, if A_4 is invertible, then

$$\Delta_4 = A_1 - A_2 A_4^{-1} A_3$$

is called the Schur complement of A_4.

It is well known [157] that matrix A is invertible if and only if either

$$A_1 \text{ and } \Delta_1 \text{ are invertible,}$$

or

$$A_4 \text{ and } \Delta_4 \text{ are invertible.}$$

Specifically, we have the following equivalent expressions

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} \Upsilon_1 & -A_1^{-1} A_2 \Delta_1^{-1} \\ -\Delta_1^{-1} A_3 A_1^{-1} & \Delta_1^{-1} \end{bmatrix} \tag{A.46}$$

or

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} \Delta_4^{-1} & -\Delta_4^{-1} A_2 A_4^{-1} \\ -A_4^{-1} A_3 \Delta_4^{-1} & \Upsilon_4 \end{bmatrix} \tag{A.47}$$

where

$$\begin{aligned} \Upsilon_1 &= A_1^{-1} + A_1^{-1} A_2 \Delta_1^{-1} A_3 A_1^{-1} \\ \Upsilon_4 &= A_4^{-1} + A_4^{-1} A_3 \Delta_4^{-1} A_2 A_4^{-1} \end{aligned} \tag{A.48}$$

Important special cases are

$$\begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_4^{-1} A_3 A_1^{-1} & A_4^{-1} \end{bmatrix} \tag{A.49}$$

and

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} & -A_1^{-1} A_2 A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} \tag{A.50}$$

A.5.2 Matrix Inversion Lemma

Let $A \in \Re^{n \times n}$ and $C \in \Re^{m \times m}$ be nonsingular matrices. By using the definition of matrix inverse, it can be easily verified that

$$[A + BCD]^{-1} = A^{-1} - A^{-1} B [D A^{-1} B + C^{-1}]^{-1} D A^{-1} \tag{A.51}$$

A.6 Some Discrete Lyapunov–Krasovskii Functionals

In this section, we provide some general form of discrete Lyapunov–Krasovskii functionals and their first difference, which can be used in stability studies of discrete time throughout the text.

$$
\begin{aligned}
V(k) &= V_o(k) + V_a(k) + V_c(k) + V_m(k) + V_n(k) \\
V_o(k) &= x^t(k)\mathcal{P}_\sigma x(k),\; V_a(k) = \sum_{j=k-d(k)}^{k-1} x^t(j)\mathcal{Q}_\sigma x(j) \\
V_c(k) &= \sum_{j=k-d_m}^{k-1} x^t(j)\mathcal{Z}_\sigma x(j) + \sum_{j=k-d_M}^{k-1} x^t(j)\mathcal{S}_\sigma x(j) \\
V_m(k) &= \sum_{j=-d_M+1}^{-d_m} \sum_{m=k+j}^{k-1} x^t(m)\mathcal{Q}_\sigma x(m) \\
V_n(k) &= \sum_{j=-d_M}^{-d_m-1} \sum_{m=k+j}^{k-1} \delta x^t(m)\mathcal{R}_{a\sigma}\delta x(m) \\
&\quad + \sum_{j=-d_M}^{-1} \sum_{m=k+j}^{k-1} \delta x^t(m)\mathcal{R}_{c\sigma}\delta x(m)
\end{aligned}
\tag{A.52}
$$

where

$$
\begin{aligned}
&0 < \mathcal{P}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{P}_j,\; 0 < \mathcal{Q}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{Q}_j,\; 0 < \mathcal{S}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{S}_j \\
&0 < \mathcal{Z}_\sigma = \sum_{j=1}^{N} \lambda_j \mathcal{Z}_j,\; 0 < \mathcal{R}_{a\sigma} = \sum_{j=1}^{N} \lambda_j \mathcal{R}_{aj},\; 0 < \mathcal{R}_{c\sigma} = \sum_{j=1}^{N} \lambda_j \mathcal{R}_{cj}
\end{aligned}
\tag{A.53}
$$

are weighting matrices of appropriate dimensions. Consider now a class of discrete-time systems, with interval-like time delays can be described by

$$
\begin{aligned}
x(k+1) &= A_\sigma x(k) + D_\sigma x(k-d_k) + \Gamma_\sigma \omega(k) \\
z(k) &= C_\sigma x(k) + G_\sigma x(k-d_k) + \Sigma_\sigma \omega(k)
\end{aligned}
\tag{A.54}
$$

where $x(k) \in \Re^n$ is the state, $z(k) \in \Re^q$ is the controlled output, and $\omega(k) \in \Re^p$ is the external disturbance which is assumed to belong to $\ell_2[0, \infty)$. In the sequel, it is assumed that d_k is time varying and satisfying

$$
d_m \le d_k \le d_M \tag{A.55}
$$

where the bounds $d_m > 0$ and $d_M > 0$ are constant scalars. The system matrices containing uncertainties which belong to a real convex bounded polytopic model of the type

$$[A_\sigma, D_\sigma, ..., \Sigma_\sigma] \in \widehat{\Xi}_\lambda := \Big\{[A_\lambda, D_\lambda, ..., \Sigma_\lambda] \\ = \sum_{j=1}^{N} \lambda_j [A_j, D_j, ..., \Sigma_j], \lambda \in \Lambda \Big\} \tag{A.56}$$

where Λ is the unit simplex

$$\Lambda \stackrel{\Delta}{=} \Big\{(\lambda_1, \ldots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \geq 0 \Big\} \tag{A.57}$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A, ..., \Sigma\}$ to imply generic system matrices and $\{A_j, ..., \Sigma_j, j \in \mathcal{N}\}$ to represent the respective values at the vertices. In what follows, we provide a definition of exponential stability of system (A.54):

A straightforward computation gives the first difference of $\Delta V(k) = V(k+1) - V(k)$ along the solutions of (A.54) with $\omega(k) \equiv 0$ as

$$\begin{aligned}
\Delta V_o(k) &= x^t(k+1)\mathcal{P}_\sigma x(k+1) - x^t(k)\mathcal{P}_\sigma x(k) \\
&= [A_\sigma x(k) + D_\sigma x(k-d_k)]^t \mathcal{P}_\sigma [A_\sigma x(k) + D_\sigma x(k-d_k)] \\
&\quad - x^t(k)\mathcal{P}_\sigma x(k) \\
\Delta V_a(k) &\leq x^t(k)\mathcal{Q}x(k) - x^t(k-d(k))\mathcal{Q}x(k-d(k)) + \sum_{j=k-d_M+1}^{k-d_m} x^t(j)\mathcal{Q}x(j) \\
\Delta V_c(k) &= x^t(k)\mathcal{Z}x(k) - x^t(k-d_m)\mathcal{Z}x(k-d_m) + x^t(k)\mathcal{S}x(k) \\
&\quad - x^t(k-d_M)\mathcal{S}x(k-d_M) \\
\Delta V_m(k) &= (d_M - d_m)x^t(k)\mathcal{Q}x(k) - \sum_{j=k-d_M+1}^{k-d_m} x^t(k)\mathcal{Q}x(k) \\
\Delta V_n(k) &= (d_M - d_m)\delta x^t(k)\mathcal{R}_a\delta x(k) + d_M \delta x^t(k)\mathcal{R}_c \delta x(k) \\
&\quad - \sum_{j=k-d_M}^{k-d_m-1} \delta x^t(j)\mathcal{R}_a\delta x(j) - \sum_{j=k-d_M}^{k-1} \delta x^t(j)\mathcal{R}_c\delta x(j)
\end{aligned} \tag{A.58}$$

A.7 Additional Inequalities

A basic inequality that has been frequently used in the stability analysis of time-delay systems is called *Jensen's Inequality* or *the Integral Inequality*, a detailed account of which is available in [105]:

Lemma A.13 *For any constant matrix* $0 < \Sigma \in \Re^{n \times n}$*, scalar* $\tau_* < \tau(t) < \tau^+$ *and vector function* $\dot{x} : [-\tau^+, -\tau_*] \to \Re^n$ *such that the following integration is well defined, then it holds that*

$$-(\tau^+ - \tau_*) \int_{t-\tau^+}^{t-\tau_*} \dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s \leq \begin{bmatrix} x(t-\tau_*) \\ x(t-\tau^+) \end{bmatrix}^t \begin{bmatrix} -\Sigma & \Sigma \\ \bullet & -\Sigma \end{bmatrix} \begin{bmatrix} x(t-\tau_*) \\ x(t-\tau^+) \end{bmatrix}$$

Building on **Lemma A.13**, the following lemma specifies a particular inequality for quadratic terms:

Lemma A.14 *: For any constant matrix* $0 < \Sigma \in \Re^{n \times n}$*, scalar* $\tau_* < \tau(t) < \tau^+$ *and vector function* $\dot{x} : [-\tau^+, -\tau_*] \to \Re^n$ *such that the following integration is well defined, then it holds that*

$$-(\tau^+ - \tau_*) \int_{t-\tau^+}^{t-\tau_*} \dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s \leq \xi^t(t) \Upsilon \xi(t)$$

$$\xi(t) = \begin{bmatrix} x(t-\tau_*) \\ x(t-\tau(t)) \\ x(t-\tau^+) \end{bmatrix}^t, \quad \Upsilon = \begin{bmatrix} -\Sigma & \Sigma & 0 \\ \bullet & -2\Sigma & \Sigma \\ \bullet & \bullet & -\Sigma \end{bmatrix}$$

Proof Considering the case $\tau_* < \tau(t) < \tau^+$ and applying the Leibniz–Newton formula, it follows that

$$-(\tau^+ - \tau_*) \int_{t-\tau^+}^{t-\tau_*} \dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s - (\tau^+ - \tau_*) \int_{t-\tau(t)}^{t-\tau_*} \Big[\dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s$$

$$+ \int_{t-\tau^+}^{t-\tau(t)} \dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s \Big]$$

$$\leq -(\tau(t) - \tau_*) \int_{t-\tau(t)}^{t-\tau_*} \Big[\dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s$$

$$-(\tau^+ - \tau(t)) \int_{t-\tau^+}^{t-\tau(t)} \dot{x}^t(s) \Sigma \dot{x}(s) \mathrm{d}s \Big]$$

$$\leq - \int_{t-\tau(t)}^{t-\tau_*} \dot{x}^t(s) \mathrm{d}s \, \Sigma \int_{t-\tau(t)}^{t-\tau_*} \dot{x}^t(s) \mathrm{d}s$$

$$- \int_{t-\tau_+}^{t-\tau(t)} \dot{x}^t(s) \mathrm{d}s \, \Sigma \int_{t-\tau_+}^{t-\tau(t)} \dot{x}^t(s) \mathrm{d}s$$

$$= [x(t-\tau_*) - x(t-\tau(t))]^t \Sigma [x(t-\tau_*) - x(t-\tau(t))] \\ -[x(t-\tau(t)) - x(t-\tau^+)]^t \Sigma [x(t-\tau(t)) - x(t-\tau^+)]$$

which completes the proof.

Lemma A.15 *For any constant matrix* $0 < \Sigma \in \Re^{n\times n}$*, scalar* η*, any* $t \in [0, \infty)$*, and vector function* $g : [t-\eta, t] \to \Re^n$ *such that the following integration is well defined, then it holds that*

$$\left(\int_{t-\eta}^{t} g(s)\mathrm{d}s\right)^t \Sigma \int_{t-\eta}^{t} g(s)\mathrm{d}s \leq \eta \int_{t-\eta}^{t} g^t(s)\Sigma g(s)\mathrm{d}s \tag{A.59}$$

Proof It is simple to show that for any $s \in [t-\eta, t]$, $t \in [0, \infty)$, and Schur complements

$$\begin{bmatrix} g^t(s)\Sigma g(s) & g^t(s) \\ \bullet & \Sigma^{-1} \end{bmatrix} \geq 0$$

Upon integration, we have

$$\begin{bmatrix} \int_{t-\eta}^{t} g^t(s)\Sigma g(s)\mathrm{d}s & \int_{t-\eta}^{t} g^t(s)\mathrm{d}s \\ \bullet & \eta\Sigma \end{bmatrix} \geq 0$$

By Schur complements, we obtain inequality (A.59).

The following lemmas show how to produce equivalent LMIs by elimination procedure.

Lemma A.16 *There exists* $\mathcal{X}$ *such that*

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} & \mathcal{X} \\ \bullet & \mathcal{R} & \mathcal{Z} \\ \bullet & \bullet & \mathcal{S} \end{bmatrix} > 0 \tag{A.60}$$

if and only if

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \bullet & \mathcal{R} \end{bmatrix} > 0, \begin{bmatrix} \mathcal{R} & \mathcal{Z} \\ \bullet & \mathcal{S} \end{bmatrix} > 0 \tag{A.61}$$

Proof Since LMIs (A.61) form sub-blocks on the principal diagonal of LMI (A.60), necessity is established. To show sufficiency, apply the congruence transformation

$$\begin{bmatrix} I & 0 & 0 \\ \bullet & I & 0 \\ 0 & -V^t R^{-1} & I \end{bmatrix}$$

to LMI (A.60), it is evident that (A.60) is equivalent to

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} & \mathcal{X} - \mathcal{Q}\mathcal{R}^{-1}\mathcal{Z} \\ \bullet & \mathcal{R} & 0 \\ \bullet & \bullet & \mathcal{S} - \mathcal{Z}^t\mathcal{R}^{-1}\mathcal{Z} \end{bmatrix} > 0 \tag{A.62}$$

Clearly (A.61) is satisfied for $\mathcal{X} = \mathcal{Q}\mathcal{R}^{-1}\mathcal{Z}$ if (A.61) is satisfied in view of Schur complements.

Lemma A.17 *There exists* $\mathcal{X}$ *such that*

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} + \mathcal{X}\mathcal{G} & \mathcal{X} \\ \bullet & \mathcal{R} & \mathcal{Z} \\ \bullet & \bullet & \mathcal{S} \end{bmatrix} > 0 \tag{A.63}$$

if and only if

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \bullet & \mathcal{R} - \mathcal{V}\mathcal{G} - \mathcal{G}^t\mathcal{V}^t + \mathcal{G}^t\mathcal{Z}\mathcal{G} \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathcal{R} - \mathcal{V}\mathcal{G} - \mathcal{G}^t\mathcal{V}^t + \mathcal{G}^t\mathcal{Z}\mathcal{G} & \mathcal{V} - \mathcal{G}^t\mathcal{Z} \\ \bullet & \mathcal{Z} \end{bmatrix} > 0 \tag{A.64}$$

Proof Applying the congruence transformation

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\mathcal{G} & I \end{bmatrix}$$

to LMI (A.63) and using **Lemma A.16**, we readily obtain the results.

Lemma A.18 *There exists* $0 < \mathcal{X}^t = \mathcal{X}$ *such that*

$$\begin{bmatrix} \mathcal{P}_a + \mathcal{X} & \mathcal{Q}_a \\ \bullet & \mathcal{R}_a \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathcal{P}_c - \mathcal{X} & \mathcal{Q}_c \\ \bullet & \mathcal{R}_c \end{bmatrix} > 0 \tag{A.65}$$

if and only if

$$\begin{bmatrix} \mathcal{P}_a + \mathcal{P}_c & \mathcal{Q}_a & \mathcal{Q}_c \\ \bullet & \mathcal{R}_a & 0 \\ \bullet & \bullet & \mathcal{R}_c \end{bmatrix} > 0 \tag{A.66}$$

Proof It is obvious from Schur complements that LMI (A.66) is equivalent to

$$\mathcal{R}_a > 0, \mathcal{R}_c > 0$$
$$\Xi = \mathcal{P}_a + \mathcal{P}_c - \mathcal{Q}_a\mathcal{R}_a^{-1}\mathcal{Q}_a^t - \mathcal{Q}_c\mathcal{R}_c^{-1}\mathcal{Q}_c^t > 0 \tag{A.67}$$

On the other hand, LMI (A.65) is equivalent to

$$\begin{aligned}
&\mathcal{R}_a > 0, \mathcal{R}_c > 0\\
&\Xi_a = \mathcal{P}_a + \mathcal{X} - \mathcal{Q}_a \mathcal{R}_a^{-1} \mathcal{Q}_a^t > 0,\\
&\Xi_c = \mathcal{P}_c - \mathcal{X} - \mathcal{Q}_c \mathcal{R}_c^{-1} \mathcal{Q}_c^t > 0
\end{aligned} \tag{A.68}$$

It is readily evident from (A.67) and (A.68) that $\Xi = \Xi_a + \Xi_c$ and hence the existence of $\mathcal{X}$ satisfying (A.68) implies (A.67). By the same token, if (A.67) is satisfied, $\mathcal{X} = \mathcal{Q}_a \mathcal{R}_a^{-1} \mathcal{Q}_a^t - \mathcal{P}_a - \frac{1}{2}\Xi$ yields $\Xi_a = \Xi_c = \Xi_a = \frac{1}{2}\Xi$ and (A.68) is satisfied.

References

1. Abdelaziz, T. H. S. and M. Valas'ek, Pole-Placement for SISO Linear Systems by State-Derivative Feedback, *IEE Proc. Contr. Theory Appl.*, vol. 151, 2004, pp. 377–385.
2. Alessandri, A., M. Baglietto and G. Battistelli, Leunberger Observers for Switching Discrete-Time Linear Systems, *Proceedings of the 44th IEEE CDC*, Seville, Spain, 2005, pp. 7014–7019.
3. Anderson, B. D. O. and J. B. Moore, *Optimal Filtering*, Prentice Hall, New York, NY, 1979.
4. Apkarian, P. and H. D. Tuan, Parameterized LMIs in Control Theory, *SIAM J. Contr. Optim.*, vol. 38, 1998, pp. 1241–1264.
5. Apkarian, P. and H. D. Tuan, Continuous-Time Analysis, Eigenstructure Assignment, and $\mathcal{H}_2$ Synthesis with Enhanced Linear Matrix Inequalities (LMI) Characterizations, *IEEE Trans. Automat. Contr.*, vol. 46, 2001, pp. 1941–1946.
6. Assuncao, E., M. C. M. Teixeira, F. A. Faria, N. A. P. Da Silva and R. Cardim, Robust State-Derivative Feedback LMI-Based Designs for Multivariable Linear Systems, *Int. J. Contr.*, vol. 80, 2007, pp. 1260–1270.
7. Bakule, L., Complexity-Reduced Guaranteed Cost Control Design for Delayed Uncertain Symmetrically Connected Systems, *Proc. Am. Contr. Conf.*, 2005, pp. 2590–2595.
8. Bakule, L., Decentralized Control: An Overview, *Annu. Rev. Contr.*, vol. 32, 2008, pp. 87–98.
9. Bakule, L. and M. de la Sen, Decentralized Stabilization of Uncertain Symmetric Composite Systems, *Proc. IFAC/IFORS/IMACS Symp. Large Scale Syst.: Theory Appl.*, vol. 1, 1998, pp. 265–270.
10. Bakule, L. and J. Lunze, Decentralized Design of Feedback Control for Large-Scale Systems, *Kybernetika*, vol. 24, 1988, pp. 1–100.
11. Bakule, L. and J. Rodellar, Decentralized Control Design of Uncertain Nominally Linear Symmetric Composite Systems, *IEE Proc. Contr. Theory Appl.*, vol. 143, 1996, pp. 530–536.
12. Bauer, P., K. Premaratne and J. Duran, A Necessary and Sufficient Condition for Robust Asymptotic Stability of Time-Variant Discrete Systems, *IEEE Trans. Automat. Contr.*, vol. 38, no. 9, 1993, pp. 1427–1430.
13. Bemporad, A. and M. Morari, .Control of Systems Integrating Logic, Dynamics, and Constraints, *Automatica*, vol. 35, 1999, pp. 407–427.
14. Bemporad, A., G. Ferrari-Trecate and M. Morari, Observability and Controllability of Piecewise Affine and Hybrid Systems, *IEEE Transactions on Automatic Control*, vol. 45 no. 10, 2000, pp. 1864–1876.
15. Benzaouia, A., E. De Santis, P. Caravani and N. Daraoui, Constrained Control of Switching Systems: A Positive Invariant Approach, *Int. J. Contr.*, vol. 80, 2007, 1379–1387.
16. Bernstein, D. S. and W. M. Haddad, Steady-State Kalman Filtering with an $\mathcal{H}_\infty$ Error Bound, *Syst. Contr. Lett.*, vol. 16, 1991, pp. 309–317.
17. Bingulac, S. and H. F. VanLandingham, *Algorithms for Computer-Aided Design of Multivariable Control Systems*, Marcel-Dekker, New York, NY, 1993.
18. Blanchini, F., Nonquadratic Lyapunov Functions for Robust Control, *Automatica*, vol. 31, no. 3, 1995, pp. 451–461.

19. Blanchini, F., S. Miani and F. Mesquine, A Separation Principle for Linear Switching Systems and Parametrization of All Stabilizing Controllers, *IEEE Trans. Automat. Contr.*, vol. 54, 2009, pp. 279–292.
20. Blondel, V. and J. Tsitsiklis, NP-hardness of Some Linear Control Design Problems, *SIAM J. Contr. Optim.*, vol. 35, no. 6, 2007, pp. 2118–2127.
21. Boker, G. and J. Lunze, Stability and Performance of Switching Kalman Filters, *Int. J. Contr.*, vol. 75, 2002, pp. 1269–1281.
22. Boukas, E. K., Static Output Feedback Control for Stochastic Hybrid Systems: LMI Approach, *Automatica*, vol. 42, 2006, pp. 183–188.
23. Boukas, E. K., Discrete-Time Systems with Time-Varying Time Delay: Stability and Stabilizability, *Math Probl. Eng.*, vol. 2006, Article ID 42489, 2006.
24. Boukas, E. K. and Z. K. Liu, *Deterministic and Stochastic Time-delay Systems*, Birkhauser, Boston, MA, 2002.
25. Boukas, E. K. and M. S. Mahmoud, A Practical Approach to Control of Nonlinear Discrete-Time State-Delay Systems, *Optimal Contr. Appl. Methods*, vol. 10, 2007, pp. 1–21.
26. Saif, A. W., M. S. Mahmoud and Y. Shi, A Parameterized Delay-Dependent Approach to Control of Uncertain Switched Discrete-Time Delay Systems, *Int. J. Innovative Comput. Inf. Contr.*, vol. 5, no. 10, 2009, pp. 1071–1082.
27. Boyd, S., L. El. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM Stud. Appl. Math., Philadelphia, 1994.
28. Branicky, M., Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems, *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, 1998, pp. 475–582.
29. Brogan, W. L., *Modern Control Theory*, 3rd edn., Prentice-Hall, Englewood Cliffs, NJ, 1991.
30. Cao, Y. and P. M. Frank, Robust $\mathcal{H}_\infty$ Disturbance Attenuation for a Class of Uncertain Discrete-Time Fuzzy Systems, *IEEE Trans. Fuzzy Syst.*, vol. 8, 2000, pp. 406–415.
31. Cao, Y. Y. and P. M. Frank, Analysis and Synthesis of Nonlinear Time-Delay Systems via Fuzzy Control Approach, *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 2, 2000, pp. 200–211.
32. Cao, Y. Y., Y. X. Sum and C. W. Cheng, Delay-Dependent Robust Stabilization of Uncertain Systems with Multiple State-Delays, *IEEE Trans. Automat. Contr.*, vol. 43, no. 11, 1998, pp. 1608–1612.
33. Cao, Y. Y., Y. X. Sum and J. Lam, Delay-Dependent Robust $\mathcal{H}_\infty$ Control for Uncertain Systems with Time-Varying Delays, *IEE Proc.-Contr. Theory Appl.*, vol. 145, 1998, pp. 338–344.
34. Chen, W. H., Z. H. Guan and X. Lu, Delay-Dependent Guaranteed Cost Control for Uncertain Discrete-Time Systems with Delay, *IEE Proc.-Contr. Theory Appl.*, vol. 150, 2003, pp. 412–416.
35. Cheng, C. J., J. J. Yan and C. C. Hwang, Globally Asymptotic Stability of a Class of Neutral-Type Neural Networks with Delays, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 1191–1195.
36. Cheng, D., L. Guo, Y. Lin and Y. Wang, Stabilization of Switched Linear Systems, *IEEE Trans. Automat. Contr.*, vol. 50, no. 4, 2005, pp. 661–666.
37. Cheng, C. J., T. L. Liao, J. J. Yan and C. C. Hwang, Exponential Synchronization of a Class of Neural Networks with Time-Varying Delays, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 209–215.
38. Choi, H. L. and J. T. Lim, Global Asymptotic Stabilization of a Class of Nonlinear Time-Delay Systems by Output Feedback, *IEICE Trans.-Fundamentals*, vol. E88-A, 2005, pp. 3804–3809.
39. Chowdhury, F. N., B. Jiang and C. M. Belcastro, Reduction of False Alarms in Fault Detection Problems, *Int. J. Innovative Comput. Inf. Contr.*, vol. 2, no. 3, 2006, pp. 481–490.
40. Christofides, P. D. and N. H. El-Ferra, *Control of Nonlinear and Hybrid Process Systems*, Springer, Berlin, 2005.
41. Cuzzola, F. A. and M. Morari, A Generalized Approach for Analysis and Control of Discrete-Time Piecewise Affine and Hybrid Systems, in *Hybrid Systems: Computation and Control*, Di Benedetto, M. D., San Giovanni-Vincentelli, A. L. (Editors), Springer, Berlin, 2001, pp. 135–164.

42. Daafouz, J., P. Riedinger and C. Lung, Stability Analysis and Control Synthesis for Switched Systems: A Hybrid Lyapunov Function Approach, *IEEE Trans. Automat. Contr.*, vol. 47, 2002, pp. 1883–1887.
43. Dayawansa, J. M. and C. F. Martin, A Converse Lyapunov Theorem for a Class of Dynamical Systems which undergo Switching, *IEEE Trans. Automat. Contr.*, vol. 44, no. 4, 1999, pp. 751–760.
44. De Oliveita, M. C., J. C. Geromel and J. Bernussou, Extended $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norm Characterizations and Controller Parameterizations for Discrete-Time Systems, *Int. J. Contr.*, vol. 75, 2002, pp. 666–679.
45. de Souza, C. E., M. Fu and L. Xie, $\mathcal{H}_\infty$ Analysis and Synthesis of Discrete-Time Systems with Time-Varying Uncertainty, *IEEE Trans. Automat. Contr.*, vol. 8, 1993, pp. 459–462.
46. DeCarlo, R. A., *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1989.
47. DeCarlo, R., M. Branicky, S. Pettersson and B. Lennartson, Perspectives and Results on the Stability and Stabilizability of Hybrid Systems, *Proc. IEEE*, vol. 88, 2000, pp. 1069–1082.
48. Dorato, P., Non-Fragile Controllers Design: An Overview, *Proceedings of the American Control Conference*, Philadelphia, PA, 1998, pp. 2829–2831.
49. Doyle, J. C. and G. Stein, Robustness with Observers, *IEEE Trans. Automat. Contr.*, vol. 24, 1979, pp. 607–611.
50. Du, D., B. Jiang and S. Zhou, Delay-Dependent Robust Stabilization of Uncertain Discrete-Time Switched Linear Systems with Time-Varying State Delays, *Int. J. Syst. Sci.*, vol. 39, 2008, pp. 305–313.
51. Du, D., B. Jiang, P. Shi and S. Zhou, $\mathcal{H}_\infty$ Filtering of Discrete-Time hybrid Switched Systems with State-Delays via Switched Lyapunov Functions Approach, *IEEE Trans. Automat. Contr.*, vol. 52, no. 8, 2007, pp. 1520–1525.
52. Du, D., S. Zhou and B. Zhang, Oliveita, Generalized H2 Output Feedback Controller Design for Uncertain Discrete-Time hybrid Systems via Switched Lyapunov Functions, *Nonlinear Anal.*, vol. 65, 2006, pp. 2135–2146.
53. Duan, G. R. and X. Zhang, Regularizability of Linear Descriptor Systems via Output plus Partial State-Derivative Feedback, *Asian J. Contr.*, vol. 5, 2003, pp. 334–340.
54. Dugard, L. and E. I. Verriest, (Editors), *Stability and Control of Time-Delay Systems*, Springer, Berlin, 1997.
55. El Ghaoui, L. F. Oustry and M. Ait Rami, A Cone Complementarity Linearization Algorithm for Static Output Feedback and Related Problems, *IEEE Trans. Automat. Contr.*, vol. 42, 1997, pp. 1171–1176.
56. El-Fara, N. H., P. Mhaskar and P. D. Christofides, Output Feedback Control of Switched Nonlinear Systems using Multiple Lyapunov Functions, *Syst. Contr. Lett.*, vol. 54, 2005, pp. 1163–1162.
57. Elsayed, A. and M. Grimble, A new approach to $\mathcal{H}_\infty$ design for optimal digital linear filters, *IMA J. Math. Contr. Inf.*, vol. 6, no. 3, 1989, pp. 233–251.
58. El-Sayed, M. and P. S. Krishnaprasad, Homogeneous Interconnected Systems: An Example, *IEEE Trans. Automat. Contr.*, vol. 26, 1981, pp. 894–901.
59. Ergen, W. K., Kinetics of the Circulating-Fuel Nuclear Reactor, *J. Appl. Phys.*, vol. 25, 1954, pp. 702–711.
60. Fattoh, A., O. Sename and J. M. Dion, An Unknown Input Observer Design for Linear Time-Delay Systems, *Proceedings of the 38th CDC*, Sydney, 1999, pp. 4222–4226.
61. Feng, G., Observer-Based Output Feedback Controller Design of Piecewise Discrete-Time Linear Systems, *IEEE Trans. Circuits Syst.-I*, vol. 50, 2003, pp. 448–451.
62. Feng, G., Robust Filtering Design of Piecewise Discrete-Time Linear Systems, *IEEE Trans. Signal Process.*, vol. 53, 2005, pp. 599–605.
63. Feng, G., Stability Analysis of Piecewise Discrete-Time Linear Systems, *IEEE Trans. Automat. Contr.*, vol. 47, 2002, pp. 1108–1112.
64. Fofana, M. S., Dimensional Reduction of Nonlinear Time-Delay Systems, *Int. J. Math. Math. Sci.*, vol. 2, 2005, pp. 311–328.

65. Fridman E. and U. Shaked, An Improved Stabilization Method for Linear Systems with Time-Delay, *IEEE Trans. Automat. Contr.*, vol. 47, 2002, pp. 1931–1937.
66. Fridman E. and U. Shaked, Delay-Dependent Stability and $\mathcal{H}_\infty$ Control: Constant and Time-Varying Delays, *Int. J. Contr.*, vol. 76, 2003, pp. 48–60.
67. Fridman E. and U. Shaked, Stability and Guaranteed-Cost Control of Uncertain Discrete-Delay Systems, *Int. J. Contr.*, vol. 78, 2005, pp. 235–246.
68. Fridman, E. and U. Shaked, A Descriptor System Approach to $\mathcal{H}_\infty$ Control of Linear Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 47, no. 2, 2002, pp. 253–274.
69. Fridman, E. and U. Shaked, An Improved Delay-Dependent $\mathcal{H}_\infty$ Filtering of Linear Neutral Systems, *IEEE Trans. Signal Process.*, vol. 42, 2004, pp. 668–673.
70. Fridman, E. and U. Shaked, New Bounded Real Lemma Representations for Time-Delay Systems and their Application, *IEEE Trans. Automat. Contr.*, vol. 46, 2001, pp. 1973–1979.
71. Fridman, E. and U. Shaked, Delay-Dependent $\mathcal{H}_\infty$ Control of Uncertain Discrete Delay Systems, *Eur. J. Contr.*, vol. 11, 2005, pp. 29–37.
72. Fu, M., C. E. de Souza and L. Xie, $\mathcal{H}_\infty$-Estimation for Uncertain Systems, *Int. J. Robust Nonlinear Contr.*, vol. 2, 1992, pp. 87–105.
73. Gahinet, P. and P. Apkarian, A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control, *Int. J. Robust Nonlinear Contr.*, vol. 4, 1994, pp. 421–448.
74. Gahinet, P., A. Nemirovski, A. L. Laub and M. Chilali, *LMI Control Toolbox*, The Math Works, Boston, MA, 1995.
75. Gao, H., Stability and Synchronization of Discrete-Time Markovian Jumping Neural Networks with Mixed Mode-Dependent Time-Delays, *IEEE Trans. Neural Netw.*, vol. 20, no. 7, 2009, pp. 1102–1116.
76. Gao, F., S. Zhong and X. Gao, Delay-Dependent Stability of a Type of Linear Switching Systems with Discrete and Distributed Time-Delays, *Appl. Math Comput.*, vol. 196, 2008, pp. 24–39.
77. Gao, H. and C. Wang, A Delay-Dependent Approach to Robust $\mathcal{H}_\infty$ Filtering for Uncertain Discrete-Time State-Delayed Systems, *IEEE Trans. Signal Process.*, vol. 52, no. 6, 2004, pp. 1631–1640.
78. Gao, H. and C. Wang, New Approaches to Robust $\ell_1 - \ell_2$ and $\mathcal{H}_\infty$ Filtering for Uncertain Discrete Time Systems, *Sci. China (Series E)*, vol. 33, 2003, pp. 695–706.
79. Gao, H. and C. Wang, A Delay-Dependent Approach to Robust $\mathcal{H}_\infty$ and $L_2 - L_\infty$ Filtering for a Class of Uncertain Nonlinear Time-Delayed Systems, *IEEE Trans. Automat. Contr.*, vol. 48, no. 9, 2003, pp. 1661–1666.
80. Gao, H. and C. Wang, Comments and Further Results on 'A Descriptor System Approach to $\mathcal{H}_\infty$ Control of Linear Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 48, no. 3, 2003, pp. 520–525.
81. Gao, H., Robust Filtering for Bilinear Uncertain Stochastic Discrete-Time Systems, *IEEE Trans. Signal Process.*, vol. 50, no. 3, 2002, pp. 560–567.
82. Gao, H. and T. Chen, New Results on Stability of Discrete-Time Systems with Time-Varying State-Delay, *IEEE Trans. Automat. Contr.*, vol. 52, no. 2, 2007, pp. 328–334.
83. Gao, H., T. Chen and J. Lam, A New Delay System Approach to Network-based Control, *Automatica*, vol. 44, 2008, pp. 39–52.
84. Gao, H., X. Meng and T. Chen, Stabilization of Networked Control Systems with New Delay Characterization, *IEEE Trans. Automat. Contr.*, vol. 53, no. 9, 2008, pp. 2142–2148.
85. Gao, H., J. Lam, C. Wang and L. Wang, Delay-Dependent Output-Feedback Stabilization of Discrete-Time Systems with Time-varying State Delay, *IEE Proc. Contr. Theory Appl.*, vol. 151, 2004, pp. 691–698.
86. Germani, A., C. Manes and P. Palumbo, Filtering of Switching Systems via a Singular Minimax Approach, *Proceeding of the 41th IEEE Conference on Decision and Control*, Las Vegas, NV, December 2002.
87. Geromel, J. C., Optimal Linear Filtering with Parameter Uncertainty, *IEEE Trans. Signal Process.*, vol. 47, 1999, pp. 168–175.

88. Geromel, J. C. and P. Colaneri, Stability and Stabilization of Continuous-Time Switched Linear Systems, *SIAM J. Contr. Optim.*, vol. 45, no. 5, 2006, pp. 1915–1930.
89. Geromel, J. C. and G. S. Deaecto, Switched State Feedback Control for Continuous-Time Uncertain Systems, *Automatica*, vol. 45, 2009, pp. 593–597.
90. Geromel, J. C. and P. Colaneri, Stability and Stabilization of Discrete-Time Switched Linear Systems, *Int. J. Contr.*, vol. 79, no. 7, 2006, pp. 719–728.
91. Geromel, J. C., J. Bernussou and M. C. de Oliveira, $\mathcal{H}_2$ Norm Optimization with Constrained Dynamic Output Feedback Controllers: Decentralized and Reliable Control, *IEEE Trans. Automat. Contr.*, vol. 44, 1999, pp. 1449–1454.
92. Geromel, J. C., P. Colaneri and P. Bolzern, Dynamic Output Feedback Control of Switched Linear Systems, *IEEE Trans. Automat. Contr.*, vol. 53, no. 4, 2008, pp. 720–733.
93. Geromel, J. C., M. C. D. Olivera and J. Bernussou, Robust Filtering of Discrete-Time Linear Systems with Parameter Dependent Lyapounov Functions, *SIAM J. Contr. Optim.*, vol. 41, 2002, pp. 700–711.
94. Geromel, J. C., P. L. D. Peres and S. R. de Souza, A Convex Approach to the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem for Discrete-Time Uncertain Systems, *SIAM J. Contr. Optim.*, vol. 33, 1995, pp. 1816–1833.
95. Goebel, R., R. G. Sanfelice and A. R. Teel, Hybrid Dynamical Systems, *IEEE Contr. Syst. Mag.*, vol. 9, 2009, pp. 28–93.
96. Gorecki, H., S. Fuska, P. Garbowski and A. Korytowski, *Analysis and Synthesis of Time-Delay Systems*, Wiley, New York, NY, 1989.
97. Gouaisbaut, F. and D. Peaucelle, A Note on Stability of Time Delay Systems, *Proceedings of the 5th IFAC Symposium on Robust Control Design*, Toulouse, France, vol. 2, July 57, 2006.
98. Gouaisbaut, F. and D. Peaucelle, Delay-Dependent Robust Stability of Time Delay Systems, *Proceedings of the 5th IFAC Symposium on Robust Control Design*, Toulouse, France, vol. 2, July 57, 2006.
99. Grossman, R. L., A. P. Ravn and H. Rischel, (Editors), *Hybrid Systems*, Springer, New York, NY, 1993.
100. Gu, K. Q., Discretized Lyapunov Functional for Uncertain Systems with Multiple Time-Delay, *Int. J. Contr.*, vol. 72, no. 16, 1999, pp. 1436–1445.
101. Gu, K., An Integral Inequality in the Stability Problem of Time-Delay Systems, *Proceedings of 39th IEEE Conference on Decision and Control*, Sydney, 2000, pp. 2805–2810.
102. Gu, K., Survey on Recent Results in the Stability and Control of Time Delay Systems, *J. Dyn. Syst., Meas. Contr.*, vol. 125, 2005, pp. 158–165.
103. Gu, K. Q. and S. I. Niculescu, Additional Dynamics in Transformed Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 45, 2000, pp. 572–575.
104. Gu, K. Q. and S. I. Niculescu, Further Remarks on Additional Dynamics in Various Model Transformations of Linear Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 46, 2001, pp. 497–500.
105. Gu, K., V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*, Birkhauser, Boston, MA, 2003.
106. Guo, L., $\mathcal{H}_\infty$ Output Feedback Control for Delay Systems with Nonlinear and Parametric Uncertainties, *IEE Proc. Contr. Theory Appl.*, vol. 49, 2002, pp. 226–236.
107. Haimes, Y. Y., *Hierarchical Analyses of Water Resources Systems*, McGraw-Hill, New York, NY, 1977.
108. Hale, J. K. and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, Berlin, 1993.
109. Hale, J., *Theory of Functional Differential Equations*, Springer, New York, NY, 1977.
110. Han, Q.-L., A Descriptor System Approach to Robust Stability of Uncertain Neutral Systems with Discrete and Distributed Delays, *Proceedings of the American Control Conference*, Denver, vol. 2, June 2003, pp. 5098–5103.
111. Han, Q.-L., A Descriptor System Approach to Robust Stability of Uncertain Neutral Systems with Discrete and Distributed Delays, *Automatica*, vol. 40, 2004, 1791–1796.

112. Han, Q.-L., New Results for Delay-Dependent Stability of Linear Systems with Timevarying Delay, *Int. J. Syst. Sci.*, vol. 33, 2003, pp. 213–228.
113. Han, K. H., S. Kim, Y. J. Kim and J. H. Kim, Internet Control Architecture for Internetbased Personal Robot, *Autonomous Robots*, vol. 10, 2001, pp. 135–147.
114. Han, Q. L., On Robust Stability of Neutral Systems with Time-Varying Discrete Delay and Norm-Bounded Uncertainty, *Automatica*, vol. 40, no. 6, 2004, pp. 1087–1092.
115. Han, Q.-L., Stability Criteria for a Class of Linear Neutral Systems with Time-varying Discrete and Distributed Delays, *IMA J. Math Contr. Inf.*, vol. 20, 2003, pp. 371–386.
116. Han, W., *Stability of Motion*, Springer, New York, NY, 1967.
117. Hassan, M. F., M. S. Mahmoud and M. I. Younis, A Dynamic Leontief Modeling Approach to Management for Optimal Utilization in Water Resources Systems, *IEEE Trans. Syst. Man Cybern.*, vol. SMC-11, 1981, pp. 552–558.
118. Hassibi, A. and S. Boyd, Quadratic Stabilization and Control of Piecewise-Linear Systems, *Proceedings of American Control Conference (ACC)*, Philadelphia, PA, 1998, pp. 3659–3664.
119. Hattingh, J. and M. Claassen, SecuringWater Quality for Life, *Int. J. Water Resour. Dev.*, vol. 24, 2008, pp. 401–405.
120. He, Y., M. Wu and Q. L. Han, Delay-Dependent $\mathcal{H}_\infty$ Control of Linear Discrete-Time Systems with an Interval-Like Time-Varying Delay, *Int. J. Syst. Sci.*, vol. 39, no. 3, 2008, pp. 427–436.
121. He, Y., M. Wu and J. H. She, Delay-Dependent Stability Criteria for Linear Systems with Multiple Time-Delays, *IEE Proc.- Contr. Theory Appl.*, vol. 153, no. 4, 2006, pp. 447–452.
122. He, Y., G. P. Liu, D. Rees and M. Wu, $\mathcal{H}_\infty$ Filtering for Discrete-Time Systems with Time-Varying Delay, *Signal Process.*, vol. 89, no. 3, 2009, pp. 275–282.
123. He, Y., Q. G. Wang, C. Lin and M. Wu, Augmented Lyapunov Functional and Delay-Dependent Stability Criteria for Neutral Systems, *Int. J. Robust Nonlinear Contr.*, vol. 15, no. 8, 2005, pp. 923–933.
124. He, Y., Q. G. Wang, L. H. Xie and C. Lin, Further Improvement of Free-Weighting Matrices Technique for Systems with Time-Varying Delay, *IEEE Trans. Automat. Contr.*, vol. 52, no. 2, 2007, pp. 293–299.
125. He, Y., M. Wu, G. P. Liu and J. H. She, Output Feedback Stabilization for Discrete-Time Systems with a Time-Varying Delay, *Proceedings of the 26th Chinese Control Conference*, China, 2007, pp. 64–70.
126. He, Y., M. Wu, G. P. Liu and J. H. She, Output Feedback Stabilization for a Discrete-Time System with a Time-Varying Delay, *IEEE Trans. Automat. Contr.*, vol. 53, no. 10, 2008, pp. 2372–2377.
127. He, Y., M. Wu, J. H. She and G. P. Liu, Delay-Dependent Robust Stability Criteria for Uncertain Neutral Systems with Mixed Delays, *Syst. Contr. Lett.*, vol. 51, no. 1, 2004, pp. 57–65.
128. He, Y., M. Wu, J. H. She and Y. C. Wang, Parameter-Dependent Lyapunov Functional for Stability of Time-Delay Systems with Polytopic Uncertainties, *IEEE Trans. Automat. Contr.*, vol. 49, no. 5, 2004, pp. 828–832.
129. Heemels, W. P. M. H., De Schutter, B. and A. Bemporad, On The Equivalence of Classes of Hybrid Systems: Mixed Logical Dynamical and Complimentarily Systems, *Automatica*, vol. 37, no. 7, 2001, pp. 1085–1091.
130. Hein, Y., Q. P. Ha and V. N. Phat, Stability and Stabilization of Switched Linear Dynamic Systems with Time Delay and Uncertainties, *Appl. Math Comput.*, vol. 210, no. 1, 2009, pp. 223–231.
131. Henzinger, T. and S. Sastry, (Editors), *Hybrid Systems: Computation and Control*, Springer, New York, NY, 1990.
132. Hespanha, J. P., Modeling and Analysis of Stochastic Hybrid Systems, *IEE Proc.-Contr. Theory Appl.*, vol. 153, 2006, pp. 520–535.
133. Hespanha, J. P., Uniform Stability of Switched Linear Systems: Extensions of Lasalles Invariance Principle, *IEEE Trans. Automat. Contr.*, vol. 49, no. 4, 2004, pp. 470–482.

134. Hespanha, J. P. and A. S. Morse, Stability of Switched Systems with Average Dwell-Time, *Proceedings of the 38th IEEE Conference on Decision Control*, Sydney, 1999, pp. 2655–2660.
135. Hespanha, J. P. and A. S. Morse, Switching Between Stabilizing Controllers, *Automatica*, vol. 38, 2002, pp. 1905–1917.
136. Hespanha, J. P., D. Liberzon, D. Angeli and E. D. Sontag, Nonlinear Norm-Observability Notions and Stability of Switched Systems, *IEEE Trans. Automat. Contr.*, vol. 52, no. 2, 2007, pp. 154–168.
137. Hespanha, J. P., P. Naghshtabrizi and Y. Xu, A Survey of Recent Results in Networked Control Systems, *Proc. IEEE*, vol. 95, no. 1, 2007, pp. 138–162.
138. Hetel, L., J. Daafouz and C. Iung, Stabilization of Arbitrary Switched Linear Systems With Unknown Time-Varying Delays, *IEEE Trans. Automat. Contr.*, vol. 51, no. 10, 2006, pp. 1668–1674.
139. Hill, D. J. and P. J. Moylan, Dissipative Dynamical Systems: Basic Input-Output and State Properties, *J. Franklin Inst.*, vol. 309, 1980, pp. 327–357.
140. Hlava, J., Time Delay Systems Applications in Textile Industry – Modeling of Sliver Drafting Process, *Time Delay Systems 2003 – A Proceedings Volume from the 4th IFAC Workshop*, Elsevier, Oxford, 2003, pp. 287–292.
141. Hovd, M. and S. Skogestad, Control of Symmetrically Interconnected Plants, *Automatica*, vol. 30, 1994, pp. 957–973.
142. Hu, Z., Decentralized Stabilization of Large-Scale Interconnected Systems with Delays, *IEEE Trans. Automat. Contr.*, vol. 39, no. 5, 1994, pp. 180–182.
143. Huang, S., J. Lam, G. H. Yang and S. Zhang, Fault Tolerant Decentralized $\mathcal{H}_\infty$ Control for Symmetric Composite Systems, *IEEE Trans. Automat. Contr.*, vol. 44, 1999, pp. 2108–2114.
144. Imura, J. I. and A. Van der Schaft, Characterization of Well-Posedness of Piecewise-Linear Systems, *IEEE Trans. Automat. Contr.*, vol. 45, 2000, pp. 1600–1619.
145. Ismail, A. and M. S. Mahmoud, A Descriptor Approach to Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Jumping Time-delay Systems., IMA J. Math Control and Info., vol. 21, 2004, 95–114.
146. Ivanescu, D., J. M. Dion, L. Dugard and S. I. Niculescu, .Dynamical Compensation for Time-Delay Systems: An LMI Approach, *Int. J. Robust Nonlinear Contr.*, vol. 10, 2000, pp. 611–628.
147. Ji, Z. and L. Wang, Disturbance Attenuation of Uncertain Switched Linear Systems, *Proceedings of the Conference on Decision and Control*, Sydney, Australia, 2004, pp. 3708–3713.
148. Ji, Z. and L. Wang, Robust $\mathcal{H}_\infty$ Control and Quadratic Stabilization of Uncertain Switched Linear Systems, *Proceedings of the American Control Conference*, Portland, OR, 2004, pp. 4543–4548.
149. Ji, Z. and L. Wang, Robust Stability and Stabilization of a Class of Nonlinear Switched Systems, *Proceedings of the IASTED Conference on Modeling, Identification, and Control*, Lanzarote, Spain, 2006, pp. 37–42.
150. Ji, Z., L. Wang and G. Xie, Quadratic Stabilization of Switched Systems, *Int. J. Syst. Sci.*, vol. 36, no. 6, 2005, pp. 395–404.
151. Jiang, B. and F. N. Chowdhury, Fault Estimation and Accommodation for Linear MIMO Discrete-Time Systems, *IEEE Trans. Contr. Syst. Technol.*, vol. 13, no. 3, 2005, pp. 493–499.
152. Jiang, X. and Q. L. Han, $\mathcal{H}_\infty$ Control for Linear Systems with Interval Time-Varying Delay, *Automatica*, vol. 41, no. 12, 2005, pp. 2099–2106.
153. Jiang, X., Q.-L. Han and X. Yu, Stability Criteria for Linear Discrete-Time Systems with Interval-Like Time-varying Delay, *Proceedings of the American Control Conference*, Portland, OR, 2005, pp. 2817–2822.
154. Jiang, B., M. Staroswiecki and V. Cocquempot, $\mathcal{H}_\infty$ Fault Detection Filter Design for Linear Discrete-Time Systems with Multiple Time-Delays, *Int. J. Syst. Sci.*, vol. 34, no. 5, 2003, pp. 365–373.
155. Jing, X. J., D. L. Tan and Y. C. Wang, An LMI Approach to Stability of Systems with Severe Time-Delay, *IEEE Trans. Automat. Contr.*, vol. 49, no. 7, 2004, pp. 1192–1195.

156. Johansson, M. and A. Rantzer, Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems', *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, 1998, pp. 555–559.
157. Kailath T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
158. Kalman, R. E., A New Approach to Linear Filtering and Prediction Problems, *Trans. ASME-J. Basic Eng.*, vol. 82, 1960, pp. 35–45.
159. Kapila, V. and W. M. Haddad, Memoryless $\mathcal{H}_\infty$ Controllers for Discrete-Time Systems with Time-Delay, *Automatica*, vol. 34, no. 5, 1998, pp. 1141–1144.
160. Khalil, H. K., *Nonlinear Systems*, 3rd edn., Prentice Hall, Upper Saddle River, NJ, 2002.
161. Khargonekar P. P. and M. A. Rotea, Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control: A Convex Optimization Approach, *IEEE Trans. Automat. Contr.*, vol. 36, 1991, pp. 824–837.
162. Kharitonov, V. L. and D. Melchor-Aguilar, Delay-Dependent Stability Conditions, *Syst. Contr. Lett.*, vol. 40, 2000, pp. 71–76.
163. Kharitonov, V. L. and D. Melchor-Aguilar, On Delay-Dependent Stability Conditions for Time-varying Systems, *Syst. Contr. Lett.*, vol. 46, 2000, pp. 173–180.
164. Kharitonov, V. L. and D. Melchor-Aguilar, Additional Dynamics for General Class of Time-delay Systems, *IEEE Trans. on Automat. Contr.*, vol. 48, 2003, pp. 1060–1064.
165. Kharitonov, V. L. and D. Melchor-Aguilar, Lyapunov-Krasovskii Functionals for Additional Dynamics, *Int. J. Robust Nonlinear Contr.*, vol. 13, 2003, pp. 793–804.
166. Kharitonov, V. L. and S. I. Niculescu, On the Stability of Linear Systems with Uncertain Delay, *IEEE Trans. Automat. Contr.*, vol. 48, no. 1, 2003, pp. 127–132.
167. Kharitonov, V. L. and A. P. Zhabko, Lyapunov-Krasovskii Approach to the Robust Stability Analysis of Time-Delay Systems, *Automatica*, vol. 39, 2003, pp. 15–20.
168. Kim, J. H., Delay and its Time-Derivative Dependent Robust Stability of Time-Delayed Linear Systems with Uncertainty, *IEEE Trans. Automat. Contr.*, vol. 46, no. 5, 2001, pp. 789–792.
169. Kim, S., S. A. Campbell and X. Z. Liu, Stability of a Class of Linear Switching Systems with Time-Delay, *IEEE Trans. Circuits Syst.*, vol. 53, no. 2, 2006, pp. 384–393.
170. Kim, D. K., P. G. Park and J. W. Ko, Output Feedback $\mathcal{H}_\infty$ Control of Systems Over Communication Networks using a Deterministic Switching Systems Approach, *Automatica*, vol. 40, no. 3, 2004, pp. 1205–1212.
171. Kolomanovskii, V. and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer, New York, NY, 1992.
172. Kolmanovskii, V. B. and J. P. Richard, Stability of Some Linear Systems with Delays, *IEEE Trans. Automat. Contr.*, vol. 44, 1999, pp. 984–989.
173. Kolmanovskii, V. B., S.-I. Niculescu and J. P. Richard, On the Lyapunov-Krasovskii Functionals for Stability Analysis of Linear Delay Systems, *Int. J. Contr.*, vol. 72, 1999, pp. 374–384.
174. Koutsoukos, X. D. and P. J. Antsaklis, Design of Stabilizing Switching Control Laws for Discrete- and Continuous-Time Linear Systems Using Piecewise-Linear Lyapunov Functions, *Int. J. Contr.*, vol. 75, 2002, pp. 932–945.
175. Krogh, B. and N. Lynch, (Editors), *Hybrid Systems: Computation and Control*, Springer, New York, NY, 2000.
176. Kulkarni, V., M. Jun and J. Hespanha, Piecewise Quadratic Lyapunov Functions for Piecewise Affine Time-Delay Systems, *Proceedings of the ACCC*, Boston, MA, 2004, pp. 2975–2984.
177. Kwon, W. H., J. W. Kang, Y. S. Lee and Y. S. Moon, A Simple Receding Horizon Control for State Delayed Systems and its Stability Criterion, *J. Process. Contr.*, vol. 13, 2003, pp. 539–551.
178. Lee, K. H., Robust Decentralized Stabilization of a Class of Linear Discrete-Time Systems with Nonlinear Interactions, *Int. J. Contr.*, vol. 80, 2007, pp. 1544–1551.
179. Lee, C. S. and G. Leitmann, Continuous Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Linear Delay Systems: An Application to River Pollution Control, *Comput. Math. Model.*, vol. 16, 1988, pp. 929–938.

180. Lee, T. N. and U. L. Radovic, Decentralized Stabilization of Linear Continuous or Discrete-Time Systems with Delays in the Interconnection, *IEEE Trans. Automat. Contr.*, vol. 33, no. 5, 1989, pp. 757–760.
181. Lee, Y. S., Y. S. Moon, W. H. Kwon and P. G. Park, Delay-Dependent $\mathcal{H}_\infty$ Control for Uncertain Systems with a State-Delay, *Automatica*, vol. 40, 2004, pp. 65–72.
182. Lehman, B., J. Bentsman, S. V. Lunel and E. I. Verriest, Vibrational Control of Nonlinear Time-Lag Systems with Bounded Delay: Averaging Theory, Stability and Transient Behavior, *IEEE Trans. Automat. Contr.*, vol. 39, no. 5, 1994, pp. 898–912.
183. Leite, V. J. and M. F. Miranda, Robust Stabilization of Discrete-Time Systems with Time-Varying Delay: An LMI Approach, *Math Probl. Eng.*, 2008, vol. 2008, pp. 1–15.
184. Leite, V. J. S. and P. L. D. Peres, Robust Control Through Piecewise Lyapunov Functions for Discrete Time-Varying Uncertain Systems, *Int. J. Contr.*, vol. 77, 2004, pp. 230–238.
185. Leite, V. J., S. Tarbouriech and P. L. D. Peres, A Convex Approach for Robust State=Feedback Control of Discrete-Time Systems with State Delay, *Proceedings of the American Control Conference*, Boston, MA, June 30 – July 2, 2004, pp. 2870–2875.
186. Leith, D. J., R. N. Shorten, W. E. Leithead, O. Mason and P. Curran, Issues in the Design of Switched Linear Control Systems: A Benchmark Study, *Int. J. Adaptive Contr. Signal Process.*, vol. 17, no. 2, 2003, pp. 103–118.
187. Leonessa, A., W. M. Haddad and V.-S. Chellaboina, *Hierarchical Nonlinear Switching Control Design with Applications to Propulsion Systems*, Springer, London, 2000.
188. Li, X. and C. de Souza, Criteria for Robust Stability and Stabilization of Test for Uncertain Linear Systems with State-Delay, *Automatica*, vol. 33, 1997, pp. 1657–1662.
189. Li, H. and M. Fu, A Linear Matrix Inequality Approach to Robust $\mathcal{H}_\infty$ Filtering, *IEEE Trans. Signal Process.*, vol. 47, 1997, pp. 2338–2350.
190. Li, C., H. Wang and X. Liao, Delay-Dependent Robust Stability of Uncertain Fuzzy Systems with Time-Varying Delays, *IEE Proc. Contr. Theory Appl.*, vol. 151, 2004, pp. 417–421.
191. Li, Z., C. Wen, Y. C. Soh and W. Xie, Generalized Matrix Measure of Switched Nonlinear Systems, *IEEE Trans. Automat. Contr.*, vol. 47, 2002, pp. 178–183.
192. Liberzon, D., *Switching in Systems and Control*, Birkhauser, Boston, MA, 2003.
193. Liberzon, D. and A. Morse, Basic Problems in Stability and Design of Switched Systems, *IEEE Contr. Syst. Mag.*, vol. 19, no. 5, 1999, pp. 59–70.
194. Lien, C. H., K. W. Yu, Y. F. Lin, Y. J. Chung and L. Y. Chung, Global Exponential Stability for Uncertain Delayed Neural Networks of Neural Type with Mixed Time Delays, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 38, 2008, pp. 709–720.
195. Lin, H. and P. J. Antsaklis, Hybrid State Feedback Stabilization with ℓ_2 Performance for Discrete-Time Switched Linear Systems, *Int. J. Contr.*, vol. 81, 2008, pp. 1114–1124.
196. Lin, H. and P. J. Antsaklis, Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results, *IEEE Trans. Automat. Contr.*, vol. 54, 2009, pp. 308–322.
197. Lin, D., X. Liu and S. Zhong, Delay-Dependent Robust Stability and Control Synthesis for Uncertain Switched Neutral Systems with Mixed Delays, *Appl. Math. Comput.*, vol. 202, 2008, pp. 828–839.
198. Lin, Q., G. Wang and T. H. Lee, A Less Conservative Robust Stability Test for Linear Uncertain Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 51, 2006, pp. 87–91.
199. Liu, X. P., Output Regulation of Strongly Coupled Symmetric Composite Systems, *Automatica*, vol. 28, 1992, pp. 1037–1041.
200. Liu, P.-L. and T.-J. Su, Robust Stability of Interval Time-delay Systems with Delay-Dependence, *Syst. Contr. Lett.*, vol. 33, 1998, pp. 231–239.
201. Liu, F. and S. Y. Zhang, Robust Decentralized Output Feedback Control of Similar Composite System with Uncertainties Unknown, *Proceedings of the American Control Conference*, vol. 6, Boston, MA, 1999, pp. 3838–3842.
202. Liu, X. G., R. R. Martin, M. Wu and M. L. Tang, Delay-Dependent Robust Stabilization of Discrete-Time Systems with Time-Varying Delays, *IEE Proc. Contr. Theory Appl.*, vol. 153, no. 6, 2006, pp. 689–702.

203. Lozano, R., B. Brogliato, O. Egeland and B. Maschke, *Dissipative Systems Analysis and Control: Theory and Applications*, Springer, London, 2000.
204. Mahmoud, M. S., Output Feedback stabilization of Uncertain Systems with State Delay, in *Analysis and Synthesis Techniques in Complex Control and Dynamic Systems*, vol. 63, Leondes, C. T. (Editor), Academic, New York, NY, 1994, pp. 197–257.
205. Mahmoud, M. S., Stabilizing Control for a Class of Uncertain Interconnected Systems, *IEEE Trans. Automat. Contr.*, vol. AC-39, 1994, pp. 2484–2488.
206. Mahmoud, M. S., Output Feedback Stabilization of Uncertain Systems with State Delay. in *Control and Dynamic Systems Series*, vol. 63, Leondes, C. T. (Editor), Academic, New York, NY, 1994, pp. 197–257.
207. Mahmoud, M. S., Guaranteed Stabilization of Interconnected Discrete-Time Systems, *Int. J. Syst. Sci.*, vol. 26, no. 1, 1995, pp. 337–358.
208. Mahmoud, M. S., Design of Stabilizing Controllers for Uncertain Discrete-Time Systems with State-Delay, *Syst. Anal. Model. Simulat.*, vol. 21, 1995, pp. 13–27.
209. Mahmoud, M. S., Dynamic Control of Systems with Variable State-Delays, *Int. J. Robust Nonlinear Contr.*, vol. 6, 1996, pp. 123–146.
210. Mahmoud, M. S., Robust Performance Results for Discrete-Time Systems, *J. Math. Probl. Eng. Syst.*, vol. 4, 1997, pp. 17–38.
211. Mahmoud, M. S., Robust Stability and Stabilization of a Class of Uncertain Nonlinear Systems with Delays, *J. Math. Probl. Eng.*, vol. 3, 1997, pp. 1–21.
212. Mahmoud, M. S., Robust Stability and Stabilization of a Class of Nonlinear Systems with Delays, *J. Math. Probl. Eng.*, vol. 4, no. 2, 1998, pp. 165–185.
213. Mahmoud, M. S., Robust $\mathcal{H}_\infty$ Control of Discrete Systems with Uncertain Parameters and Unknown Delays, *Automatica*, vol. 36, 2000, pp. 627–635.
214. Mahmoud, M. S., Delay-Dependent Robust Stability and Stabilization for Systems with Mismatched Uncertainties, *IMA J. Math Contr. Inf.*, vol. 17, 2000, pp. 309–323.
215. Mahmoud, M. S., Robust $\mathcal{H}_\infty$ Control of Discrete Systems with Uncertain Parameters and Unknown Delays, *Automatica*, vol. 36, 2000, pp. 627–635.
216. Mahmoud, M. S., *Robust Control and Filtering for Time-Delay Systems*, Marcel-Dekker, New-York, NY, 2000.
217. Mahmoud, M. S., Robust Stability and $\mathcal{H}_\infty$ Estimation for Uncertain Discrete Systems with State-Delay, *J. Math. Probl. Eng.*, vol. 7, no. 5, 2001, pp. 393–412.
218. Mahmoud, M. S., Resilient Linear Filtering of Uncertain Systems, *Automatica*, vol. 40, no. 10, 2004, pp. 1797–1802.
219. Mahmoud, M. S., Robust Linear Filtering of Uncertain Systems, *Automatica*, vol. 40, 2004, pp. 1797–1802.
220. Mahmoud, M. S., Uncertain Jumping Systems with Strong and Weak Functional Delays, *Automatica*, vol. 40, 2004, pp. 501–510.
221. Mahmoud, M. S., *Resilient Control of Uncertain Dynamical Systems*, Springer, Berlin, 2004.
222. Mahmoud, M. S., New Results on Robust Control Design of Discrete-Time Uncertain Systems, *IEE Proc. Contr. Theory Appl.*, vol. 152, 2005, pp. 453–459.
223. Mahmoud, M. S., Resilient $\mathcal{L}_2/\mathcal{L}_\infty$ Filtering of Polytopic Systems with State-Delays, *Proc. IET Contr. Theory Appl.*, vol. 1, 2007, pp. 141–154.
224. Mahmoud, M. S., Linear Hybrid Systems with Time-Varying Delays: $\mathcal{H}_\infty$ Stabilization Schemes, *Int. J. Syst. Contr. Commun.*, vol. 1, no. 2, 2008, pp. 147–169.
225. Mahmoud, M. S., Delay-Dependent $\mathcal{H}_2$ Filtering of a Class of Switched Discrete-Time State-Delay Systems, *J. Signal Process.*, vol. 88, no. 11, 2008, pp. 2709–2719.
226. Mahmoud, M. S., Switched Discrete-Time with Time-Varying Delays: A Generalized $\mathcal{H}_2$ Approach, *Comput. Math Appl.*, vol. 57, no. 1, 2009, pp. 79–95.
227. Mahmoud, M. S., Generalized $\mathcal{H}_2$ Control of Switched Discrete-Time with Unknown Delays, *Appl. Math Comput.*, vol. 311, no. 1, 2009, pp. 33–44.
228. Mahmoud, M. S., State-Feedback Stabilization of Linear Systems with Input and State Delays, *Proc. IME Part I- J. Syst. Contr. Eng.*, vol. 233, no. 14, June 2009, pp. 557–565. DOI: 10.1243/09596518JSCE723.

229. Mahmoud, M. S., New Exponentially-Convergent State Estimation Method for Delayed Neural Networks, *Neurocomputing*, vol. 72, no. 16–18, 2009, pp. 3935–3942.
230. Mahmoud, M. S., Decentralized Reliable Control of Interconnected Systems with Time-Varying Delays, *J. Optim. Theory Appl.*, vol. 143, no. 5, 2009, pp. 497–518.
231. Mahmoud, M. S., Stabilization of Linear Time-Delay Systems by Three-Term Controllers, *Proceedings of the 3rd International Multi-Conference on Systems and Control (MSC 2009)*, Saint Petersburg, Russia, July 8–10, 2009, pp. 296–301.
232. Mahmoud, M. S., Delay-Dependent Dissipativity of Singular Time-Delay Systems, *IMA J. Math. Contr. Inf.*, vol. 26, no. 1, 2009, pp. 45–58.
233. Mahmoud, M. S., Improved Stability and Stabilization Approach to Linear Interconnected Time-Delay Systems, *Optimal Contr. Appl. Methods*, vol. 31, no. 3, 2010, pp. 81–92.
234. Mahmoud, M. S., Stabilization of Time-Delay Systems by PID Controllers, *Proceedings of the 6th International Multi-Conference on Systems, Signals and Devices (SSD'09)*, Djerba-Tunisia, March 23–26, 2009, Paper # 1569171107.
235. Mahmoud, M. S. and Abdulla Ismail, Role of Delay in Networked Control Systems, *Proceedings of the 10th IEEE International Conference on Electronics, Circuits and Systems*, University of Sharaja, UAE, December 15–17, 2003, pp. 40–43.
236. Mahmoud, M. S. and Abdulla Ismail, Passivity Analysis and Synthesis of Discrete-Time Delay Systems, *J. Dyn. Continuous, Disc. Impulsive Syst., Series A*, vol. 11, no. 4, 2004, pp. 525–544.
237. Mahmoud, M. S. and Abdulla Ismail, New Results on Delay-Dependent Control of Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 50, 2005, pp. 95–100.
238. Mahmoud, M. S. and Abdulla Ismail, Delay-Dependent Robust Stability and Stabilization of Neutral Systems, *J. Appl. Math. Sci.*, vol. 1, 2007, pp. 491–505.
239. Mahmoud, M. S. and Abdulla Ismail, Resilient Control of Nonlinear Discrete-Time State-Delay Systems, *Appl. Math Comput.*, vol. 206, no. 2, 2008, pp. 561–569.
240. Mahmoud, M. S. and N. B. Almutairi, Improved Resilient Filtering for Linear Discrete-Time Systems, *J. Math Contr. Sci. Appl.*, vol. 1, no. 2, 2007, pp. 351–364.
241. Mahmoud, M. S. and N. B. Almutairi, Improved Resilient Filtering for Linear Discrete-Time Systems, *J. Math. Contr. Sci. Appl.*, vol. 1, no. 2, 2007, pp. 351–364.
242. Mahmoud, M. S. and N. B. Almutairi, Decentralized Stabilization of Interconnected Systems with Time-Varying Delays, *Eur. J. Contr.*, vol. 15, no. 6, 2009, pp. 624–633.
243. Mahmoud, M. S. and N. B. Almutairi, Stability and Implementable $\mathcal{H}_\infty$ Filters for Singular Systems with Nonlinear Perturbations, *Nonlinear Dyn.*, vol. 57, no. 6, 2009, pp. 401–410.
244. Mahmoud, M. S. and N. F. Al-Muthairi, Design of Robust Controllers for Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. AC-39, 1994, pp. 995–999.
245. Mahmoud, M. S. and N. F. Al-Muthairi, Quadratic Stabilization of Continuous-Time Systems with State Delay and Norm-Bounded Time Varying Uncertainties, *IEEE Trans. Automat. Contr.*, vol. AC-39, 1994, pp. 2135–2139.
246. Mahmoud, M. S. and A. Y. AlRayyah, Adaptive Control of Systems with Mismatched Nonlinearities and Time-Varying Delays using State-Measurements, *IET Contr. Theory Appl.*, vol. 4, no. 1, 2010, pp. 27–36.
247. Mahmoud, M. S. and A. Y. AlRayyah, Efficient Parameterization to Stability and Feedback Synthesis of Linear Time-Delay Systems, *IET Contr. Theory Appl.*, vol. 3, no. 8, 2009, pp. 1107–1118.
248. Mahmoud, M. S. and Awatef Kamal M Ali, A Robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Approach to Uncertain Systems with Time-varying Delays, *Proc. IME Part I- J Syst. Contr. Eng.*, vol. 223, no. 14, 2009, pp. 507–520.
249. Mahmoud, M. S. and S. Bingulac, Robust Design of Stabilizing Controllers for Interconnected Time-Delay Systems, *Automatica*, vol. 34, 1998, pp. 795–800.
250. Mahmoud, M. S. and A. A. Boujarwah, Robust $\mathcal{H}_\infty$ Filtering for a Class of Linear Parameter-Varying Systems, *IEEE Trans. Circuits Syst.-I*, vol. 48, no. 9, 2001, pp. 1131–1138.

251. Mahmoud, M. S. and S. El-Ferik, New Stability and Stabilization Methods for Nonlinear Systems with Time-Varying Delays, *Optimal Contr. Appl. Methods*, vol. 31, no. 2, 2010, pp. 273–287.
252. Mahmoud, M. S. and H. E. Emara-Shabaik, New $\mathcal{H}_2$ Filter for Uncertain Singular Systems using Strict LMIs, *J. Circuits, Syst. Signal Process.*, vol. 28, no. 5, 2009, pp. 665–677.
253. Mahmoud, M. S. and M. F. Hassan, A Decentralized Water Quality Control Scheme, *IEEE Trans. Syst. Man Cybern.*, vol. SMC-16, 1986, pp. 694–702.
254. Mahmoud, M. S. and S. J. Saleh, Regulation of Water Quality Standards in Streams by Decentralized Control, *Int. J. Contr.*, vol. 41, 1985, pp. 525–540.
255. Mahmoud, M. S. and S. J. Saleh, Regulation of Water Quality Standards in Streams by Decentralized Control, *Int. J. Contr.*, vol. 41, 1985, pp. 525–540.
256. Mahmoud, M. S. and P. Shi, Robust Kalman Filtering for Continuous Time-Lag Systems with Jump Parameters, *IEEE Trans. Circuits Syst.-I*, vol. 50, no. 1, 2003, pp. 98–105.
257. Mahmoud, M. S. and P. Shi, Robust Stability, Stabilization and $\mathcal{H}_\infty$ Control of Time-Delay Systems with Markovian Jump Parameters, *Int. J. Robust Nonlinear Contr.*, vol. 13, 2003, pp. 755–784.
258. Mahmoud, M. S. and P. Shi, Optimal Guaranteed Cost Filtering for Markovian Jump Discrete-Time Systems, *Math. Probl. Eng.*, vol. 12, no. 1, 2004, pp. 33–48.
259. Mahmoud, M. S. and Yuanqing Xia, Design of Reduced-Order $\mathcal{L}_2 - \mathcal{L}_\infty$ Filter for Singular for Discrete-Time Systems Using Strict LMIs, *IET Contr. Theory Appl.*, vol. 4, no. 3, 2010, pp 509–519.
260. Mahmoud, M. S. and Yuanqing Xia, Robust Filter Design for Piecewise Discrete-Time Systems with Time-Varying Delays, *Int. J. Robust Nonlinear Contr.*, vol. 20, no. 3, 2010, pp. 540–560.
261. Mahmoud, M. S. and Yuanqing Xia, Robust Stability and Stabilization of a Class of Nonlinear Switched Discrete-Time Systems with Time-Varying Delays, *J. Optim. Theory Appl.*, vol. 143, no. 2, 2009, pp. 329–355.
262. Mahmoud, M. S. and L. Xie, Guaranteed Cost Control of Uncertain Discrete Systems with Delays, *Int. J. Contr.*, vol. 73, no. 2, 2000, pp. 105–114.
263. Mahmoud, M. S. and L. H. Xie, Guaranteed Cost Control of Uncertain Discrete Systems with Delays, *Int. J. Contr.*, vol. 73, 2000, pp. 105–114.
264. Mahmoud, M. S. and M. Zribi, Robust and $\mathcal{H}_\infty$ Stabilization of Interconnected Systems with Delays, *IEE Proc.-Contr. Theory Appl.*, vol. 145, 1998, pp. 558–567.
265. Mahmoud, M. S. and M. Zribi, $\mathcal{H}_\infty$ Controllers for Time-Delay Systems using Linear Matrix Inequalities, *J. Optim. Theory Appl.*, vol. 100, 1999, pp. 89–123.
266. Mahmoud, M. S. and M. Zribi, Output Feedback Design for Uncertain Systems with Delays, *IMA J. Math Contr. Inf.*, vol. 19, 2002, pp. 297–312.
267. Mahmoud, M. S. and M. Zribi, Robust Output Feedback Control for Nonlinear Systems Including Actuators, *J. Syst. Anal. Model. Simulat.*, vol. 43, no. 6, June 2003, pp. 771–792.
268. Mahmoud, M. S., Abdulla Ismail and F. M. AL-Sunni, A Parameterization Approach to Stability and Feedback Stabilization of Linear Time-Delay Systems, *Proc. IME Part I- J. Syst. Contr. Eng.*, vol. 223, no. 17, 2009, pp. 929–939.
269. Mahmoud, M. S., N. F. Al-Muthairi and S. Bingulac, Robust Kalman Filtering for Continuous Time-Lag Systems, *Syst. Contr. Lett.*, vol. 38, no. 4, 1999, pp. 309–319.
270. Mahmoud, M. S., F. M. AL-Sunni and Y. Shi, Dissipativity Results for Linear Singular Time-Delay Systems, *Int. J. Innovative Comput. Inf. Contr.*, vol. 4, no. 11, 2008, pp. 2833–2845.
271. Mahmoud, M. S., F. M. AL-Sunni and Y. Shi, Mixed $\mathcal{H}_2 - \mathcal{H}_\infty$ Control of Uncertain Jumping Time-Delay Systems, *J. Franklin Inst.*, vol. 345, no. 5, 2008, pp. 536–552.
272. Mahmoud, M. S., F. M. AL-Sunni and Y. Shi, Switched Discrete-Time Delay Systems: Delay-Dependent Analysis and Synthesis, *J. Circuits, Syst. Signal Process.*, vol. 28, no. 5, 2009, pp. 735–761.
273. Mahmoud, M. S., E. K. Boukas and P. Shi, Resilient Feedback Stabilization of Discrete-Time Systems with Delays, *IMA J. Math. Contr. Inf.*, vol. 24, no. 4, 2007, pp. 1–16.

274. Mahmoud, M. S., E. K. Boukas and P. Shi, .Resilient Feedback Stabilization of Discrete-Time Systems with Delays, *IMA J. Math. Contr. Inf.*, vol. 25, no. 2, 2008, pp. 141–156.
275. Mahmoud, M. S., E. K. Boukas and P. Shi, Robust Adaptive Control of Uncertain Discrete-Time State-Delay Systems, *Comput. Math Appl.*, vol. 55, no. 11, 2008, pp. 2887–2902.
276. Mahmoud, M. S., M. F. Hassan and M. G. Darwish, *Large Scale Control Systems: Theories and Techniques*, Marcel Dekker, New York, NY, 1985.
277. Mahmoud, M. S., M. F. Hassan and S. J. Saleh, Decentralized Structures for Stream Water Quality Control Problems, *Optimal Contr. Appl. Methods*, vol. 6, 1987, pp. 167–186.
278. Mahmoud, M. S., M. N. Nounou and H. N. Nounou, Analysis and Synthesis of Uncertain Switching Discrete-Time Systems, *IMA J. Math. Contr. Inf.*, vol. 23, no. 6, 2006, pp. 245–257.
279. Mahmoud, M. S., M. N. Nounou and H. N. Nounou, Analysis and Synthesis of Uncertain Switched Discrete-Time Systems, *IMA J. Math. Contr. Inf.*, vol. 24, 2007, pp. 245–257.
280. Mahmoud, M. S., A. W. Saif and P. Shi, Stabilization of Linear Switched Delay Systems: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Methods, *J. Optim. Theory Appl.*, vol. 142, no. 3, 2009, pp. 583–607.
281. Mahmoud, M. S., A. W. Saif and P. Shi, Stabilization of Linear Switched Delay Systems: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Methods, *J. Optim. Theory Appl.*, vol. 142, no. 3, July 2009, pp. 583–607.
282. Mahmoud, M. S., P. Shi and A. Ismail, Robust Kalman Filtering for Discrete-Time Markovian Jump systems with Parameter Uncertainty, *J. Comput. Appl. Math.*, vol. 169, no. 1, 2004, pp. 53–69.
283. Mahmoud, M. S., P. Shi and Y. Shi, Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Uncertain Jumping Systems with Functional Time-Delays, *Int. J. Robust Nonlinear Contr.*, vol. 18, no. 3, 2008, pp. 296–318.
284. Mahmoud, M. S., P. Shi and Y. Shi, Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Uncertain Jumping Systems with Functional Time-Delays, *Int. J. Robust Nonlinear Contr.*, vol. 18, no. 3, 2008, pp. 296–318.
285. Mahmoud, M. S., Y. Shi and F. M. AL-Sunni, Robust $\mathcal{H}_2$ Output Feedback Control for a Class of Time-Delay Systems, *Int. J. Innovative Comput. Inf. Contr.*, vol. 4, no. 12, 2008, pp. 3247–3257.
286. Mahmoud, M. S., Y. Shi and F. M. AL-Sunni, Dissipativity Analysis and Synthesis of a Class of Nonlinear Systems with Time-Varying Delays, *J. Franklin Inst.*, vol. 346, 2009, pp. 570–592.
287. Mahmoud, M. S., L. Xie and Y. C. Soh, Robust Kalman Filtering for Discrete State-Delay Systems, *IEE Proc. – Contr. Theory Appl.*, vol. 147, 2000, pp. 613–618.
288. Mancilla-Aguilar, J. L. and R. A. Garcia, A Converse Lyapunov Theorem for Nonlinear Switched Systems, *Syst. Contr. Lett.*, vol. 41, no. 1, 2000, pp. 67–71.
289. Martinez, L. A. M. and C. H. Moog, New Insights on the Analysis of Nonlinear Time-Delay Systems: Application to The Triangular Equivalence, *Syst. Contr. Lett.*, vol. 56, 2007, pp. 133–140.
290. McClamroch, N. H. and I. Kolmanvsky, Performance Benefits of Hybrid Control Design for Linear and Nonlinear Systems, *Proc. IEEE*, vol. 88, 2000, pp. 1083–1096.
291. Meyer, C., S. Schroder and R. W. De Doncker, Solid-State Circuits Breakers and Current Limiters for Medium-Voltage Systems having Distributed Power Systems, *IEEE Trans. Power Electron.*, vol. 19, 2004, pp. 1333–1340.
292. Michel, A., Recent Trends in the Stability Analysis of Hybrid Dynamical Systems, *IEEE Trans. Circuits Syst.-Part I*, 46, 1999, pp. 120–134.
293. Mignone, D. G. F. Trecate and M. Morari, Stability and Stabilization of Piecewise Affine and Hybrid Systems: An LMI Approach, *Proceedings of the 39th IEEE CDC*, Sydney, Australia, 2000, pp. 504–509.
294. Milot, J., M. J. Rodriguez and J. B. Serodes Song, Contribution of Neural Networks for Modeling Trihalomethanes Occurrence in Drinking Water, *J. Water Resour. Plann. Manage.*, vol. 128, 2002, pp. 370–376.

295. Miranda, M. F. and V. J. S. Leite, Convex Analysis and Synthesis for Uncertain Discrete-Time Systems with Time-Varying State-Delay, *Proceedings of American Control Conference*, Seattle, Washington, June 11–13, 2008, pp. 4910–4915.
296. Molchanov A. P. and E. Pyatnitskiy, Criteria of Asymptotic Stability of Differential and Difference Inclusions Encountered in Control Theory, *Syst. Contr. Lett.*, vol. 13, 1989, pp. 59–64.
297. Montagner, V. F., V. J. S. Leite, R. C. L. F. Oliveira and P. L. D. Peres, State-Feedback Control of Switched Linear Systems: An LMI Approach, *J. Comput. Appl. Math.*, vol. 194, 2006, pp. 192–206.
298. Montagner, V. F., V. J. S. Leite, S. Tarbouriech and P. L. D. Peres, Stability and Stabilizability of Discrete-Time Switched Linear Systems with State-Delay, *Proceedings of American Control Conference*, Portland, OR, 2005, pp. 3806–3811.
299. Montestruque, L. A. and P. J. Antsaklis, On the Model-based Control of Networked Systems, *Automatica*, vol. 39, 2003, pp. 1837–1843.
300. Montestruque, L. A. and P. J. Antsaklis, Stability of Model-based Networked Control Systems with Time-Varying Transmission Times. *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1562–1573.
301. Moon, Y. S., P. Park, W. H. Kwon and Y. S. Lee, Delay-Dependent Robust Stabilization of Uncertain State-Delayed Systems, *Int. J. Contr.*, vol. 74, no. 14, 2001, pp. 1447–1455.
302. Morse, A. S., (Editor), *Control using Logic-Based Switching*, Springer, New York, NY, 1996.
303. Mukaidani, H., An LMI Approach to Guaranteed Cost Control for Uncertain Time-Delay Systems, *IEEE Trans. Circuits Syst.-I*, vol. 50, 2003, pp. 795–800.
304. Myshkis, A., *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer, The Netherlands, 1999.
305. Nagpal, K. M. and P. P. Khargonekar, Filtering and Smoothing in $\mathcal{H}_\infty$ Setting, *IEEE Trans. Automat. Contr.*, vol. 36, 1991, pp. 152–166.
306. Narendra, K. S. and J. Balakrishnan, A common Lyapunov Function for Stable LTI Systems with Commuting A-Matrices, *IEEE Trans. Automat. Contr.*, vol. 39, no. 12, 1994, pp. 2469–2471.
307. Nesic, D. and D. Liberzon, A Unified Framework for Design and Analysis of Networked and Quantized Control Systems, *IEEE Trans. Automat. Contr.*, vol. 54, 2009, pp. 732–747.
308. Nesic, D. and A. R. Teel, Input-to-State Stability of Networked Control Systems, *Automatica*, vol. 40, 2004, pp. 2121–2128.
309. Nian, X. and R. Li, Robust Stability of Uncertain Large-Scale Systems with Time-Delay, *Int. J. Syst. Sci.*, vol. 32, 2001, pp. 541–544.
310. Niculescu, S.-I., On Delay-Dependent Stability Under Model Transformations of Some Neutral Linear Systems, *Int. J. Contr.*, vol. 74, pp. 609–617.
311. Niculescu, S.-I., *Delay Effects on Stability: A Robust Control Approach*, Springer, Berlin, 2001.
312. Niculescu, S.-I., J.-M. Dion and L. Dugard, Robust Stabilization for Uncertain Time delay Systems Containing Saturating Actuators, *IEEE Trans. Automat. Contr.*, vol. 41, 1996, pp. 742–747.
313. Ohtake, H., K. Tanaka and H. O. Wang, Switching Fuzzy Controller Design Based on Switching Lyapunov Function for a Class of Nonlinear Systems, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 13–23.
314. Oucheriah, S., Decentralized Stabilization of Large-Scale Systems with Multiple Delays in the Interconnection, *Int. J. Contr.*, vol. 73, 2000, pp. 1213–1223.
315. Overstreet, J. W. and A. Tzes, An Internet-Based Real-Time Control Engineering Laboratory, *IEEE Contr. Syst. Mag.*, vol. 19, 1999, pp. 320–326.
316. Pagilla, P. R., Robust Decentralized Control of Large-Scale Interconnected Systems: General Interconnections, *Proceedings of American Control Conference*, San Diego, CA, 1999, pp. 4527–4531.
317. Palhares, R. M. and P. L. D. Peres, Robust $\mathcal{H}_\infty$ Filtering Design with Pole Placement Constraint via LMIs, *J. Optim. Theory Appl.*, vol. 102, 1999, pp. 239–261.

318. Palhares, R. M. and P. L. D. Peres, Robust Filtering with Guaranteed Energy-to-Peak Performance- An LMI Approach, *Automatica*, vol. 36, 2000, pp. 851–855.
319. Palhares, R. M., C. D. Campos, P. Y. Ekel, M. C. R. Leles and M. F. S. V. DAngelo, Delay-Dependent Robust $\mathcal{H}_\infty$ Control of Uncertain Linear Systems with Time-Varying Delays, *Comput. Math. Appl.*, vol. 50, 2005, pp. 13–32.
320. Palhares, R. M., C. D. Campos, P. Y. Ekel, M. C. R. Leles and M. F. S. V. DAngelo, Delay-Dependent Robust $\mathcal{H}_\infty$ Control of Uncertain Linear Systems with Lumped Delays, *IEE Proc. Contr. Theory Appl.*, vol. 153, 2005, pp. 27–33.
321. Palhares, R. M., P. Peres and C. de Souza, Robust $\mathcal{H}_\infty$ Filtering for Linear Continuous-Time Uncertain Systems with Multiple Delays: An LMI Approach, *Proceedings of the 3rd IFAC Conference Robust Control Design*, Prague, Check Republic, 2000.
322. Park, P., A Delay-Dependent Stability Criterion for Systems with Uncertain Time-Invariant Systems, *IEEE Trans. Automat. Contr*., vol. 44, no. 4, 1999, pp. 876–879.
323. Park, J.-H., A New Delay-dependent Criterion for Neutral Systems with Multiple Delays, *J. Comput. Appl. Math.*, vol. 136, 2001, pp. 177–184.
324. Park, P. G. and J.W. Ko, Stability and Robust Stability for Systems with a Time-Varying Delay, *Automatica*, vol. 43, 2007, pp. 1855–1858.
325. Parlakci, M. N. A., Robust Stability of Uncertain Systems with Time-Varying State-Delayed Systems, *IEE Proc.- Contr. Theory Appl.*, vol. 153, no. 4, 2006, pp. 469–477.
326. Pepyne, D. L. and C. G. Cassandras, Optimal Control of Hybrid Systems in Manufacturing, *Proc. IEEE*, vol. 88, 2000, pp. 1108–1123.
327. Persis, C. D., R. D. Santis and A. S. Morse, Supervisory Control with State-Dependent Dwell-Time Logic and Constraints, *Automatica*, vol. 40, 2004, pp. 269–275.
328. Petersen, I. R., A Stabilization Algorithm for a Class of Uncertain Linear Systems, *Syst. Contr. Lett*., vol. 8, 1987, pp. 351–357.
329. Petersen, I. R. and D. C. McFarlane, Optimal Guaranteed Cost Control and Filtering for Uncertain Linear Systems, *IEEE Trans. Automat. Contr.*, vol. 39, 1994, pp. 1971–1977.
330. Pettersson, S., Synthesis of Switched Linear Systems, *Proceedings of the 42nd IEEE Conference on Decision and Control*, Boston, MA, 2003, pp. 5283–5288.
331. Pettersson, S. and B. Lennartson, *Controller Design of Hybrid Systems, in Lecture Notes in Computer Science*, vol. 1201, Springer, Berlin, 1997, pp. 240–254.
332. Pettersson, S. and B. Lennartson, Stabilization of Hybrid Systems Using a Min-Projection Strategy, *Proceedings of the 2001 American Control Conference*, Arlington, VA, 2001, pp. 223–228.
333. Pettersson, S. and B. Lennartson, Hybrid System Stability and Robustness Verification Using Linear Matrix Inequalities, *Int. J. Contr.*, vol. 75, 2002, pp. 1335–1355.
334. Pettit, N. B. L. and P. E. Wellstead, Analysing Piecewise Linear Dynamical Systems, *IEEE Contr. Syst. Mag.*, vol. 15, 1995, pp. 43–50.
335. Qiu, J., G. Feng and J. Yang, New Results on Robust Energy-to-Peak Filtering for Discrete-Time Switched Polytopic Linear Systems with Time-Varying Delay, *IET Contr. Theory Appl.*, vol. 2, no. 9, 2008, pp. 795–806.
336. Rantzer, A. and M. Johansson, Piecewise Linear Quadratic Optimal Control, *IEEE Trans. Automat. Contr*., vol. 45, 2000, pp. 629–637.
337. Reithmeier, E. and G. Leitmann, Robust Vibration Control of Dynamical Systems Based on the Derivative of the State, *Arc. Appl. Mech.*, vol. 72, 2003, pp. 856–864.
338. Richard, J. P., Time-Delay Systems: An Overview of Some Recent Advances and Open Problems, *Automatica*, vol. 39, no. 10, 2003, pp. 1667–1694.
339. Rotea, M. A., The Generalized $\mathcal{H}_2$ Control Problem, *Automatica*, vol. 29, 1993, pp. 373–383.
340. Savkin, A. V. and R. J. Evans, *Hybrid Dynamical Systems*, Birkhauser, Boston, MA, 2002.
341. Savkin, A. V., I. R. Petersen, E. Skafidas and R. J. Evans, Hybrid Dynamical Systems: Robust Control Synthesis Problems, *Syst. Contr. Lett.*, vol. 29, 1996, pp. 81–90.
342. Savkin, A. V., E. Skafidas and R. J. Evans, Robust Output Feedback Stabilizability via Controller Switching, *Automatica*, vol. 35, 1999, pp. 69–74.

343. Seiler, P. and R. Sengupta, An $\mathcal{H}_\infty$ Approach to Networked Control, *IEEE Trans. Automat. Contr.*, vol. 50, 2005, pp. 356–364.
344. Shaked, U. and C. E. de Souza, Robust Minimum Variance Filtering, *IEEE Trans. Signal Process.*, vol. 43, 1995, pp. 2474–2483.
345. Shi, P., E. K. Boukas and R. K. Agarwal, Optimal Guaranteed Cost Control of Uncertain Discrete Time-Delay Systems, *J. Comput. Appl. Math.*, vol. 157, 2003, pp. 435–451.
346. Shi, P., M. Karan and Y. Kaya, Robust Kalman Filter Design for Markovian Jump Linear Systems with Norm-Bounded Unknown Nonlinearities, *Circuits Syst. Signal Process.*, vol. 24, no. 2, 2005, pp. 135–150.
347. Shi, P., M. Mahmoud and S. Nguang, Robust Filtering for Jumping Systems with Mode-Dependent Delays, *Signal Process.*, vol. 86, 2006, pp. 140–152.
348. Siljak, D. D., *Decentralized Control of Complex Systems*, Academic, New York, NY, 1991.
349. Siljak, D. D. and A. I. Zecevic, Control of Large-Scale Systems: Beyond Decentralized Feedback, *Ann. Rev. Contr.*, vol. 29, 2005, pp. 169–179.
350. Siljak, D. D., D. M. Stipanovic and A. I. Zecevic, Robust Decentralized Turbine/Governor Using Linear Matrix Inequalities, *IEEE Trans. Power Syst*, vol. 17, 2002, pp. 715–722.
351. Singh, T. and S. R. Vadali, Robust Time-Delay Control of Multimode Systems, *Int. J. Contr.*, vol. 62, no. 2, 1995, pp. 1319–1339.
352. Siouris, G., *An Engineering Approach to Optimal Control and Estimation Theory*, Wiley, New York, NY, 1996.
353. Skafidas, E., R. J. Evans, A. V. Savkin and I. R. Petersen, Stability Results for Switched Controller Systems, *Automatica*, vol. 35, 1999, pp. 553–564.
354. Song, S. H. and J. K. Kim, $\mathcal{H}_\infty$ Control of Discrete-Time Linear Systems with Norm-Bounded Uncertainties and Time-Delay in State, *Automatica*, vol. 34, 1998, pp. 137–139.
355. Sontag, E. D., Interconnected Automata and Linear Systems: A Theoretical Framework in Discrete-Time, in *Hybrid systems III–Verification and Control*, Alur, R., Henzinger, T. A., Sontag, E. D. (Editors), Springer-Verlag, Berlin, vol. 1066, 1996, pp. 436-448.
356. Sorenson, H., (Editor), *Kalman Filtering: Theory and Applications*, IEEE Press, New York, NY, 1985.
357. Antsaklis, P. J. and M. Lemon (Editors), Special Issue on Hybrid Systems, *IEEE Trans. Automat. Contr.*, vol. AC-43, 1998.
358. Morse, A. S., C. C. Pantelides, S. S. Sastry, J. M. Schumacher (Editors), Special Issue on Hybrid Systems, *Automatica*, vol. 34, 1999.
359. Antsaklis, P. J., (Editor), Special Issue on Hybrid Systems, *Proc. IEEE*, vol. 88, 2000.
360. Srinivasagupta, D. and B. Joseph, Internet-mediated Process Control Laboratory, *IEEE Contr. Syst. Mag.*, vol. 23, 2003, pp. 110–118.
361. Stipanovic, D. M. and D. D. Siljak, Robust Stability and Stabilization of Discrete-Time Nonlinear Systems: The LMI Approach, *Int. J. Contr.*, vol. 74, 2001, pp. 873–879.
362. Sturm, J. F., Using SeDuMi 1.02, aMATLAB Toolbox for Optimization over Symmetric Cones, *Optim. Methods Softw.*, vol. 11, no. 14, 1999, pp. 625–653.
363. Subbaroo Vemula, V. R., P. P. Mujumdar and S. Ghosh, Risk Evaluation in Water Quality Management of a River System, *J. Water Resour. Plann. Manage.*, vol. 130, 2004, pp. 411–423.
364. Sun, Z. and S. S. Ge, Analysis and Synthesis of Switched Linear Control Systems, *Automatica*, vol. 41, 2005, pp. 181–195.
365. Sun, Z. and S. S. Ge, Switched Stabilization of High-Order Switched Linear Systems, *Proceedings of the 44th IEEE CDC Conference on Decision and Control*, Boston, MA, 2005, pp. 4873–4878.
366. Sun, Z. and S. S. Ge, *Switched Linear Systems*, Springer, London, 2005.
367. Sun, T. J. and C. G. Huang, Robust Stability of Delay Dependence for Linear Uncertain Systems, *IEEE Trans. Automat. Contr.*, vol. 37, no. 10, 1992, pp. 1656–1659.
368. Sun, Y. G., L. Wang and G. Xie, Delay-Dependent Robust Stability and $\mathcal{H}_\infty$ Control for Uncertain Discrete-Time Switched Systems with Mode-Dependent Time-Delays, *Appl. Math. Comput.*, vol. 180, 2006, pp. 428–435.

369. Sun, Y. G., L. Wang and G. Xie, Delay-Dependent Robust Stability and $\mathcal{H}_\infty$ Control for Uncertain Discrete-Time Switched Systems with Mode-Dependent Time-Delays, *Appl. Math. Comput.*, vol. 187, 2007, pp. 1228–1237.
370. Sun, X., J. Zhao and D. J. Hill, Stability and $\mathcal{L}_2$ Gain Analysis for Switched Delay Systems: A Delay-Dependent Method, *Automatica*, vol. 42, 2006, pp. 1769–1774.
371. Sun, X. M., J. Zhao and W. Wang, Two Design Schemes for Robust Adaptive Control of a Class of Linear Uncertain Delay Systems, *Int. J. Innovative Comput., Inf. Contr.*, vol. 3, 2007, pp. 385–396.
372. Sun, X. M., W. Wang, G. P. Liu and J. Zhao, Stability Analysis for Linear Switched Systems with Time-Varying Delays, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 38, 2008, pp. 528–533.
373. Suplin, V., E. Fridman and U. Shaked, $\mathcal{H}_\infty$ Control of Linear Uncertain Time-Delay Systems-A Projection Approach, *IEEE Trans. Automat. Contr.*, vol. 51, 2006, pp. 680–685.
374. Suplin, V., E. Fridman and U. Shaked, $\mathcal{H}_\infty$ Control of Linear Uncertain Time-Delay Systems-A Projection Approach, *IEEE Trans. Automat. Contr.*, vol. 51, 2006, pp. 680–685.
375. Tarbouriech, S., P. L. D. Peres, G. Garcia and I. Queinnec, Delay-Dependent Stabilization and Disturbance Tolerance for Time-delay Systems Subject to Actuator Saturation, *IEE Proc. Contr. Theory Appl.*, vol. 149, 2002, pp. 387–393.
376. Trecate, G. F., F. A. Cuzzola, D. Mignone and M. Morari, Analysis of Discrete-Time Piecewise Affine and Hybrid Systems, *Automatica*, vol. 38, 2002, pp. 2139–2146.
377. Tshii, H. and B. A. Francis, Stabilizing a Linear System by Switching Control with Dwell Time, *IEEE Trans. Automat. Contr.*, vol. 47, no. 2, 2002, pp. 1962–1973.
378. Tsubone, T. and T. Saito, Stabilizing and Destabilizing Control for a Piecewise-Linear Circuit, *IEEE Trans. Circuits Syst.-I*, vol. 45, 1998, pp. 172–177.
379. Vaandrager, F. and J. van Schuppen, (Editors), *Hybrid Systems: Computation and Control*, Springer, New York, NY, 1999.
380. Wang, F. Y. and D. Liu, (Editors), *Networked Control Systems: Theory and Applications*, Springer, London, 2008.
381. Wang, Z. and F. Yang, Robust Filtering for Uncertain Linear Systems with Delayed States and Outputs, *IEEE Trans. Circuits Syst. I*, vol. 49, 2002, pp. 125–130.
382. Wang, R. and J. Zhao, Exponential Stability Analysis for Discrete-Time Switched Linear Systems with Time-Delay, *Int. J. Innovative Comput. Inf. Contr.*, vol. 3, 2007, pp. 1557–1564.
383. Wang, D., D. H. Zhou and Y. H. Jin, Nonlinear State Predictor for a Class of Nonlinear Time-Delay Systems, *Int. J. Syst. Sci.*, vol. 35, no. 3, 2004, pp. 197–210.
384. Wang, Y., C. de-Souze and L. H. Xie, Decentralized Output Feedback Control of Interconnected Uncertain Delay Systems, *Proceedings of the 12th IFAC Congress*, Australia, vol. 6, 1993, pp. 38–42.
385. Wang, M., B. Chen and P. Shi, Adaptive Neural Control for a Class of Perturbed Strict-Feedback Nonlinear Time-Delay Systems, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 38, 2008, pp. 721–726.
386. Wang, Z. W., R. Zhou and C. Wen, Load-Frequency Controller Design for Power Systems, *Proc. IEE Part C*, vol. 140, 1993, pp. 11–17.
387. Weisheng, C. and L. Junmin, Backstepping Tracking Control for Nonlinear Time-Delay Systems, *J. Syst. Eng. Electron.*, vol. 17, no. 4, 2006, pp. 846–852.
388. Wicks, M. A., P. Peleties and R. A. De Carlo, Construction of Piecewise Lyapunov Functions for Stabilizing Switched Systems, *Proceedings of the 33rd Conference on Decision and Control*, December 1994, pp. 3492–3497.
389. Williams, S. M. and R. G. Hoft, Adaptive Frequency Domain Control of PPM Switched Power Line Conditioner, *IEEE Trans. Power Electron.*, vol. 6, 1991, pp. 665–670.
390. Wu, H. N., Delay-Dependent Stability Analysis and Stabilization for Discrete-Time Fuzzy Systems with State-Delay: A Fuzzy Lyapunov-Krasovskii Functional Approach, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 954–962.
391. Wu, M., Y. He and J. H. She, New Delay-Dependent Stability Criteria and Stabilizing Method for Neutral Systems, *IEEE Trans. Automat. Contr.*, vol. 49, no. 12, 2004, pp. 2266–2271.

392. Wu, M., Y. He, J. H. She and G. P. Liu, Delay-Dependent Criteria for Robust Stability of Time-Varying Delay Systems, *Automatica*, vol. 40, no. 8, 2004, pp. 1435–1439.
393. Wu, L., P. Shi, C. Wang and H. Gao, Delay-Dependent Robust $\mathcal{H}_\infty$ and $\mathcal{L}_2 - \mathcal{H}_\infty$ Filtering for LPV Systems with Both Discrete and Distributed Delays, *IEE Proc. Contr. Theory Appl.*, vol. 153, 2006, pp. 483–492.
394. Xie, L. H., Output Feedback $\mathcal{H}_\infty$ Control of Systems with Parameter Uncertainty'. *Int. J. Contr.*, vol. 63, 1996, pp. 741–750.
395. Xie, G. and L. Wang, Quadratic Stability and Stabilization of Discrete-Time Switched Systems with State-Delay, *Proceedings of the 43rd Conference on Decision Control*, Atlantis, Paradise Island, Bahamas, December 14–17, 2004, pp. 3235–3240.
396. Xie, G. M. and L. Wang, Stability and Stabilization of Switched Linear Systems with State Delay, *Proceedings of the 16th Mathematical Theory of Networks and Systems Conference*, Leuven, Belgium, 2004.
397. Xie, G. M. and L. Wang, Stabilization of Switched Linear Systems with Time Delay in Detection of Switching Signal, *J. Math Anal. Appl.*, vol. 305, no. 1, 2005, pp. 277–290.
398. Xie, L. and Y. C. Soh, Robust Kalman Filtering for Uncertain Systems, *Syst. Contr. Lett.*, vol. 22, 1994, pp. 123–129.
399. Xie, L., C. E. de souza and M. Fu, $\mathcal{H}_\infty$ Estimation for Discrete-Time Linear Uncertain Systems, *Int. J. Robust Nonlinear Contr.*, vol. 1, 1991, pp. 111–123.
400. Xie, L., C. E. de souza and Y. Wang, Robust Filtering for a Class of Discrete-Time Uncertain Nonlinear Systems: An $\mathcal{H}_\infty$ Approach, *Int. J. Robust Nonlinear Contr.*, vol. 6, 1996, pp. 297–312.
401. Xu, S. and J. Lam, Positive Real Control for Uncertain Singular Time-Delay Systems via Output Feedback Controllers, *Eur. J. Contr.*, vol. 292, 2004, pp. 122–127.
402. Xu, S. and J. Lam, Improved Delay-Dependent Stability Criteria for Time-Delay Systems, *IEEE Trans. Automat. Contr*, vol. 50, 2005, pp. 384–387.
403. Xu, S, J. Lam and Y. Zou, Delay Dependent Guaranteed Cost Control for Uncertain Systems with State and Input Delays, *IEE Proc.-Contr. Theory Appl.*, vol. 153, 2006, pp. 307–313.
404. Xu, S. and J. Lam, On Equivalence and Efficiency of Certain Stability Criteria for Improved Delay-Dependent Stability Criteria for Time-Delay Systems, *IEEE Trans. Automat. Contr.*, vol. 52, 2007, 95–101.
405. Xu, S. and J. Lam, A Survey of Linear Matrix Inequality Techniques in Stability of Delay Systems, *Int. J. Syst. Sci.*, vol. 39, no. 12, 2008, pp. 1095–1113.
406. Xu, S., J. Lam and Y. Zou, Simplified Descriptor System Approach to Delay-Dependent Stability and Performance Analyses for Time-Delay Systems, *IEE Proc.-Contr. Theory Appl.*, vol. 152, no. 2, 2005, pp. 147–151.
407. Xu, S., J. Lam and Y. Zou, Improved Conditions for Delay-Dependent Robust Stability and Stabilization of Uncertain Discrete Time-Delay Systems, *Asian J. Contr.*, vol. 7, no. 3, 2005, pp. 344–348.
408. Xu, H., Y. Zou and S. Xu, Robust $\mathcal{H}_\infty$ Control for a Class of Uncertain Nonlinear Two-Dimensional Systems, *Int. J. Innovative Comput. Inf. Contr.*, vol. 1, no. 2, 2005, pp. 181–191.
409. Yakubovich, V. A., Σ-Procedure in Nonlinear Control Theory, *Vestink Leningradskogo Universiteta, Ser. Matematika*, vol. 233, 1971, pp. 62–77.
410. Yang, S. H. and C. Dai, Multi-Rate Control in Internet-based Control Systems, in Proceedings of UK Control, Sahinkaya, H., Edge, R. (Editors), Bath, UK, 2004, ID.053.
411. Yang, G. H. and J. L. Wang, Robust resilient Kalman Filtering for Uncertain Linear Systems with estimator Gain Uncertainty, *IEEE Trans. Automat. Contr.*, vol. 46, 2001, pp. 343–348.
412. Yang, G. H. and S. Y. Zhang, Stabilizing Controllers for Uncertain Symmetric Composite Systems, *Automatica*, vol. 31, 1995, pp. 337–340.
413. Yang, G. H. and S. Y. Zhang, Robust Stability and Stabilization of Uncertain Symmetric Composite Systems with Circulant Structures, *Proceedings of the 13th Triennial World Congress*, San Francisco, CA, 1996, pp. 67–72.
414. Yang, G. H. and S. Y. Zhang, Robust Control for Nonlinear Similar Composite Systems with Uncertain Parameters, *IEE Proc. Contr. Theory Appl.*, vol. 147, 2000, pp. 80–84.

415. Yang, S. H., X. Chen and J. L. Alty, Design Issues and Implementation of Internet based Process Control Systems, *Contr. Eng. Pract.*, vol. 11, 2003, pp. 709–720.
416. Yang, S. H., X. Chen, L. Tan and L. Yang, Time-Delay and Data Loss Compensation for Internet-based Process Control Systems, *Trans. Inst. Meas. Contr.*, vol. 27, 2005, pp. 103–108.
417. Yanushevsky, R. T., A Class of Nonlinear Time-Delay Systems and Related Optimal Problems, *Comput. Math. Appl.*, vol. 37, no. 6, 1999, pp. 73–78.
418. Yong H., Q. G. Wang, C. Lin and M. Wu, Delay-Range-Dependent Stability for Systems with Time-Varying Delay, *Automatica*, vol. 43, 2007, pp. 371–376.
419. Yu, L., Q.-L. Han, S. Yu and J. Gao, Delay-Dependent Conditions for Robust Absolute Stability of Uncertain Time-delay Systems, *Proceedings of the 42nd Conference Decision and Control*, Maui, Hawaii, vol. 2, 2003, pp. 6033–6037.
420. Yuan, K., J. Cao and H. X. Li, Robust Stability of Switched Cohen-Grossberg Neural Networks with Mixed Time-Varying Delays, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 1356–1363.
421. Yuanqing Xia, Y., Zheng Zhu and M. S. Mahmoud, H2 control for Networked Control Systems with Markovian Data Losses and Delay, *Proceedings of the 2nd International Symposium on Intelligent Informatics (ISII 2009)*, QinHuangDao, China, September 13–15, 2009, and in ICIC Express Letters (ICIC-EL).
422. Zhai, G. and H. Lin, Control Failure Time-Analysis for Symmetric $\mathcal{H}_\infty$ Control Systems, *Int. J. Contr.*, vol. 77, 2004, pp. 598–605.
423. Zhai, G. and A. N. Michel, On Practical Stability of Switched Systems, *Proceedings of the Conference on Decision and Control*, Las Vegas, NV, 2003, pp. 3488–3493.
424. Zhai, G., H. Lin and P. J. Antsaklis, Quadratic Stabilizability of Switched Linear Systems with Polytopic Uncertainties, *Int. J. Contr.*, vol. 76, 2003, pp. 747–753.
425. Zhai, G. S., Y. Sun, X. K. Chen and N. M. Anthony, Stability and $\mathcal{L}_2$ Gain Analysis for Switched Symmetric Systems with Time Delay, *Proceedings of the American Control Conference*, Portland, OR, 2003, pp. 2682–2687.
426. Zhai, G., B. Shu, K. Yasuda and A. N. Michel, Disturbance Attenuation Properties of Time-Controlled Switched Systems, *J. Franklin Inst.*, vol. 338, 2001, pp. 765–779.
427. Zhai, G., Y. Sun, X. Chen and A. N. Michel, Stability and $\mathcal{L}_2$ Gain Analysis for Switched Symmetric Systems with Time-Delay, *Proceedings of the American Control Conference*, Denver, CO, June 2003, pp. 4–6.
428. Zhang, B. and S. Xu, Robust $\mathcal{H}_\infty$ Filtering for Uncertain Discrete Piecewise Time-Delay Systems, *Int. J. Contr.*, vol. 80, no. 4, 2007, pp. 636–645.
429. Zhang, L., E. K. Boukas and P. Shi, Exponential $\mathcal{H}_\infty$ Filtering for Uncertain Discrete-Time Switched Linear Systems with Average Dwell Time: A μ- Dependent Approach, *Int. J. Robust Nonlinear Contr.*, vol. 18, no. 11, 2008, pp. 1188–1207.
430. Zhang W., M. S. Branicky and S. M. Phillips, Stability of Networked Control Systems, *IEEE Contr. Syst. Mag.*, vol. 21, 2001, pp. 84–99.
431. Zhang, H., C. Li and X. Liao, Stability Analysis and $\mathcal{H}_\infty$ Controller Design of Fuzzy Large-Scale Systems based on Piecewise Lyapunov Functions, *IEEE Trans. Syst. Man Cybern.-Part B*, vol. 36, 2006, pp. 685–698.
432. Zhang, L., P. Shi and E. K. Boukas, $\mathcal{H}_\infty$ Output-Feedback Control for Switched Linear Discrete-Time Systems with Time-Varying Delays, *Int. J. Contr.*, vol. 80, 2007, pp. 1354–1365.
433. Zhang L., Y. Shi, T. Chen and B. Huang, A New Method for Stabilization of Networked Control Systems with Random Delays, *IEEE Trans. Automat. Contr.*, vol. 50, 2005, pp. 1177–1181.
434. Zhang, J., Y. Xia and M. S. Mahmoud, Robust Generalized $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Static Output Feedback Control for Uncertain Discrete-Time Fuzzy Systems, *IEE Contr. Theory Appl.*, vol. 3, no. 7, 2009, pp. 865–876.
435. Zhang, B., S. Xu and Y. Zou, Improved Stability Criterion and Its Applications in Delayed Controller Design for Discrete-Time Systems, *Automatica*, vol. 44, 2008, pp. 2963–2967.

436. Zhang, L., P. Shi, E. Boukas and C. Wang, $\mathcal{H}_\infty$ Control of Switched Linear Discrete-Time Systems with Polytopic Uncertainties, *Optimal Contr. Appl. Methods*, vol. 27, 2006, pp. 273–291.
437. Zhang, L., P. Shi, E. K. Boukas and C.Wang, $\mathcal{H}_\infty$ Control of Switched Linear Discrete-Time Systems with Polytopic Uncertainties, *Optimal Contr. Appl. Methods*, vol. 27, 2006, pp. 273–291.
438. Zhang, L., P. Shi, E.-K. Boukas and C. Wang, Robust $\ell_2 - \ell_\infty$ Filtering for Switched Linear Discrete Time-Delay Systems with Polytopic Uncertainties, *IET Contr. Theory Appl.*, vol. 1, no. 3, 2007, pp. 722–730.
439. Zhang, L., P. Shi, C. Wang and H. J. Gao, Robust $\mathcal{H}_\infty$ Filtering for Switched Linear Discrete-Time Systems with Polytopic Uncertainties, *Int. J. Adapt. Contr. Signal Process.*, vol. 20, 2006, pp. 291–304.
440. Zhang, X. M., M. Wu, J. H. She and Y. He, Delay-Dependent Stabilization of Linear Systems with Time-Varying State and Input-Delays, *Automatica*, vol. 41, no. 8, 2005, pp. 1405–1412.
441. Zhao, J. and G. M. Dimirovski, Quadratic Stability of a Class of Switched Nonlinear Systems, *IEEE Trans. on Automat. Contr.*, vol. 49, 2004, pp. 574–578.
442. Zheng, F. and P. M. Frank, Robust Control of Uncertain Distributed Delay Systems with Application to the Stabilization of Combustion in Rocket Motor Chambers, *Automatica*, vol. 38, 2002, pp. 487–497.
443. Zhou, K. and J. C. Doyle, *Essentials of Robust Control*, Prentice-Hall, Englewood Cliffs, NJ, 1998.
444. Zhou, S. and J. Lam, Robust Stabilization of Delayed Singular Systems with Linear Fractional Parametric Uncertainties, *Circuits Syst. Signal Process.*, vol. 22, no. 6, 2003, pp. 579–588.
445. Zhou, S. and T. Li, Stabilization of Delayed Discrete-Time Fuzzy Systems via Basis-Dependent Lyapunov-Karsovskii Function, *Fuzzy Sets Syst.*, vol. 151, 2005, pp. 139–153.
446. Zhu, X., C. Hua and S. Wang, State Feedback Controller Design of Networked Control Systems with Time Delay in the Plant, *Int. J. Innovative Comput. Inf. Contr.*, vol. 4, 2008, pp. 283–290.
447. Zhu, Y., D. Q. Li and G. Feng, $\mathcal{H}_\infty$ Controller Synthesis of Uncertain Piecewise Continuous-Time Linear Systems, *IEE Proc.-Contr. Theory Appl.*, vol. 152, 2005, pp. 513–519.
448. Zribi, M., M. S. Mahmoud, M. Karkoub and T. Li, $\mathcal{H}_\infty$-Controllers for Linearized Time-Delay Power Systems, *Proc. IEE Generation, Trans. Distr.*, vol. 147, no. 6, 2000, pp. 401–408.

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About the Author

Magdi Sadek Mahmoud obtained Ph.D. degree in systems engineering from Cairo University, 1974. He has been a professor of systems engineering since 1984. He is now on the faculty of KFUPM, Saudi Arabia. He worked at different universities worldwide in Egypt, Kuwait, UAE, UK, USA, Singapore, Saudi Arabia, and Australia. He has given invited lectures in Venezuela, Germany, China, UK, and USA. He has been actively engaged in teaching and research in the development of modern methodologies to computer control, systems engineering, and information technology. He is the principal author of 23 books inclusive book chapters, and the author/co-author of more than 450 peer-reviewed papers. He is the recipient of two national, one regional, and a university prize for outstanding research in engineering. He is a fellow of the IEE, a senior member of the IEEE, the CEI (UK), and a registered consultant engineer of information engineering and systems (Egypt).