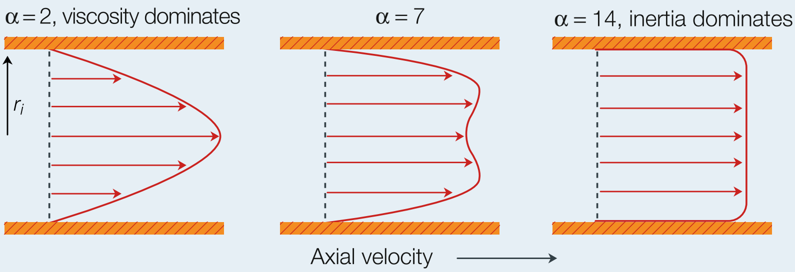
Chapter 8
Oscillatory Flow Theory



Oscillatory flow velocity profiles in a stiff tube during laminar sinusoidal flow. Flow is driven by a sinusoidal, oscillatory, pressure gradient. The velocity profile varies over the sinusoidal cycle and only the profile at the moment of maximal forward flow is depicted here. The velocity profile depends on the Womersley parameter, $\alpha^2 = r_i^2\,\omega\rho/\eta$, where r_i is the internal radius, ω the circular frequency ($2\pi f$, f frequency), ρ the blood density, and η the blood viscosity. Womersley's parameter α expresses the relative importance of inertial effects over viscous (frictional) effects. For low values of the Womersley parameter ($\alpha < 3$, low frequency, small radius), viscous effects dominate and the profile becomes parabolic, as in Poiseuille flow (left figure). For a medium range of α values the velocity profile becomes flatter and the maximum velocity does not occur at the tube's center (middle figure). For large values of α, $\alpha > 10$, i.e., high frequency and/or large vessels the profile becomes flat, because inertial effects dominate. The theory is based on sinusoidal oscillations of pressure and flow. This implies that application of the theory to hemodynamics requires Fourier analysis (Appendix 1). From work on comprehensive models we conclude that the contribution of oscillatory flow theory only gives a small correction over and above the use of a resistance with inertance in series (Appendix 3). However, in the calculation of wall shear and in the calculations and measurements of local flow profiles, the theory is of great importance.

Description

The pressure-flow relation for steady flow, where only frictional losses are considered (resistance, law of Poiseuille), and the relation between oscillatory or pulsatile pressure and flow when only blood mass (inertance) is taken into consideration, are simplifications of reality.

N. Westerhof et al., *Snapshots of Hemodynamics: An Aid for Clinical Research and Graduate Education*, DOI 10.1007/978-1-4419-6363-5_8,

The relation between oscillatory, sinusoidal, pressure drop and flow through a blood vessel can be derived from the Navier-Stokes equations. The assumptions are to a large extent similar to the derivation of Poiseuille's law: uniform and straight blood vessel, rigid wall, Newtonian viscosity, etc. The result is that flow is still laminar but pulsatile, i.e., not constant in time, and the flow profile is no longer parabolic. The theory is based on sinusoidal pressure-flow relations, and therefore called oscillatory flow theory.

The flow profile depends on the, circular, frequency of oscillation, ω, with $\omega = 2\pi f$, with f the frequency; the tube radius, r_i, the viscosity, η, and density, ρ, of the blood. These variables were taken together in a single dimensionless (no units) parameter called Womersley's alpha parameter [1]:

$$\alpha^2 = r_i^2 \omega \cdot \rho / \eta$$

If the local pressure gradient, $\Delta P/l$, is a sinusoidal wave with amplitude A^* and circular frequency, ω, then the corresponding velocity profile is given by the formula [1]:

$$v(r,t) = \text{Real}\left[(\text{A}^*/\text{i}\omega\rho) \cdot \{1 - \text{J}_0(\alpha \cdot \text{y} \cdot \text{i}^{3/2})/\text{J}_0(\alpha \cdot \text{i}^{3/2})\} \cdot \text{e}^{\text{i}\omega t}\right]$$

where y is the relative radial position, $y = r/r_i$, and $\text{i} = \sqrt{-1}$. Flow is given as:

$$Q(t) = \text{Real}\left[(\pi \text{r}_\text{i}^2 \text{A}^*/\text{i}\omega\rho) \cdot \{1 - 2\text{J}_1(\alpha \cdot \text{i}^{3/2})/\alpha \cdot \text{i}^{3/2}\text{J}_0(\alpha \cdot \text{i}^{3/2})\} \cdot \text{e}^{\text{i}\omega t}\right]$$

with $\text{i} = \sqrt{-1}$, and J_0 and J_1 are Bessel functions of order 0 and 1, respectively. The Real means that only the real part of the mathematically complex formula is taken.

Since the heart does not generate a single sine wave but a sum of sine waves (see Appendix 1) the flow profile *in vivo* is found by addition of the velocity profiles of the various harmonics, and is very complex. The relation between pressure drop and flow as given above is the so-called longitudinal impedance of a vessel segment (see Appendix 3). Experiments have shown that the theory is accurate.

On the basis of the oscillatory flow theory Womersley predicted that for $\alpha > 0.5$, the area ratio of two equal daughters and a mother vessel should be between 1.33 and 1.15 to minimize local wave reflection (Chap. 21). For large α, i.e., $\alpha > 10$, where inertia dominates the viscous effects, this area ratio is 1.15, while Murray's law predicts 1.25 for the area ratio (Chap. 2).

Physiological and Clinical Relevance

Womersley's oscillatory flow theory [1] reduces to Poiseuille's law for very low α. This means that in the periphery with small blood vessels (small r) and little oscillation, there is no need for the oscillatory flow theory and we can describe the pressure-flow relation with Poiseuille's law. For the very large conduit arteries, where $\alpha > 10$, friction does not play a significant role and the pressure-flow relation can be described with inertance alone. For α values in between, the combination of the resistance plus the inductance approximates the oscillatory pressure-flow relations (see Appendix 3).

Models of the entire arterial system have indicated that, even in intermediate size arteries, the oscillatory effects on the velocity profiles are not large. The main factors contributing to pressure and flow wave shapes in the arterial tree are due to branching, non-uniformity and bending of the blood vessels etc. Thus for global hemodynamics, i.e., wave travel, input impedance, Windkessel models etc., the longitudinal impedance of a segment of artery (Appendix 3) can be described, in a sufficiently accurate way, by an inertance only in the aorta and major arteries, an inertance in series with a resistance in medium sized conduit vessels, and a resistance only in peripheral arteries.

The oscillatory flow theory is, however, of importance when local phenomena are studied. For instance, detailed flow profiles and calculation of shear stress at the vascular wall require the use of the oscillatory flow theory.

There is another dimensionless parameter of importance in unsteady, oscillatory flow problems: the Strouhal number. The Strouhal number can be written as $St = \omega D/v$, where D is diameter, ω circular frequency, $2\pi f$, with f frequency in cycles per second, and v velocity. It represents a measure of the ratio of inertial forces due to oscillatory flow and the inertial forces due to convective acceleration. It can be shown that the Strouhal number relates in combination with the Reynolds number (Re, Chap. 4) to the square of Womersley's parameter α:

$$\mathrm{St} \cdot \mathrm{Re} = (\omega D / v) \cdot (\rho v D / \eta) = D^2 \omega \rho / \eta = 4\alpha^2$$

The Strouhal number has been used in experimental studies of arterial flows in the past, such as vortex shedding phenomena distal of a cardiac (mitral) valve. It is widely accepted, however, that the most relevant parameter expressing the relative significance of inertial effects due to oscillatory flow is the Womersley parameter α.

Before Ultrasound Doppler velocity and other flow measurement techniques became available, attempts were made to derive blood flow wave shapes and Cardiac Output from the measurement of two pressures in the aorta a few centimeters apart. The pressure drop and aortic size allowed for the calculation of flow using the oscillatory flow theory.

Reference

1. Womersley JR. The mathematical analysis of the arterial circulation in a state of oscillatory motion. 1957, Wright Air Dev. Center, Tech Report WADC-TR-56-614.