Inversion and Invariance of Characteristic Terms: Part I

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Fondly dedicated to the 104th birthday of my father
Professor S.K. Abhyankar who was born on 4/4/1904
Also dedicated to the memory of
Professor Alladi Ramakrishnan in great admiration

Summary In my 1967 paper with almost the same title which appeared in volume 89 of the American Journal of Mathematics, I proved the invariance of the characteristic terms in the fractional power series expansion of a branch of an algebraic plane curve over fields of characteristic zero. Now I extend the results by a more generous interpretation of the characteristic terms, and by relaxing the characteristic zero hypothesis.

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1 Introduction

A branch of an algebraic or analytic plane curve can be parametrized by expressing both the variables as power series in a parameter; we call this the MT (= Maclaurin–Taylor) expansion. In case of zero characteristic, by Hensel's Lemma or by Newton's Theorem on fractional power series expansion, one of the variables can be arranged to be a power of the parameter, and then certain divisibility properties of the exponents in the expansion of the other variable lead to the characteristic terms whose importance was first pointed out by Smith [34] and Halphen [23] as noted in Zariski's famous book [35].

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In my 1967 paper [4] I showed that, as long as the variable which is a power of the parameter is nontangential, the characteristic terms remain invariant. This I did by first showing that if I flip the variables, then the characteristic terms change by a definite inversion formula whose proof essentially depends on the binomial theorem. This will be reviewed in Sect. 4. In Sect. 5, I shall relate this to quadratic transformations and establish the invariance of another type of characteristic term, namely, the first exponent whose coefficient is transcendental over a certain subfield of the ground field. While doing this, I shall reorganize the NT (= newtonian) expansion into the ED (= euclidean) expansion, which is a generalized form of the so called HN (= Hamburger-Noether) expansion. The reorganization will partly make things work even in the mixed characteristic meromorphic case.

As basic references for this paper, the reader may profitably consult my Rambling Article [5], Tata Notes [6], Engineering Book [9], and Algebra Book [12].

After fixing the notation in Sect. 2, a host of Remarks and Lemmas will be collected in Sect. 3. These deal with Euclidean Sequences (3.1), Characteristic Sequences (3.2), Binomial Lemmas (3.3) and (3.4), Special Subfields (3.5), Gap Lemmas (3.6) and (3.7), Valuation Expansions (3.8) and (3.9), and Uniqueness of Power Series Rings (3.10).

In Sect. 6, I shall show how the above mentioned first transcendental coefficient is related to a generator of the residue field of the branch. Moreover, the generator can be chosen so that the said coefficient is a polynomial in it. This leads to an algebraic incarnation of the topological theory of dicritical divisors which I shall describe. In Sect. 7, I shall relate field generators to dicritical divisors.

In Sect. 8, I shall preview Part II which will include various topics from algebraic curve theory such as the conductor and genus formulas of Dedekind and Noether, and the automorphism theorems of Jung and Kulk. In Part II, I shall also relate all this to the Jacobian problem which conjectures that if the Jacobian of n polynomials in n variables over a characteristic zero field equals a nonzero constant, then the variables can be expressed as polynomials in the given polynomials; see [13–15].

As hinted in the Note following Lemma (3.4) of Sect. 3, Newton's Binomial Theorem For Fractional Exponents is the real heart of this paper. I was very lucky in having studied this in the hand-written manuscript of my father's book [1] two years before it was published when I was 11 years old. Very relevant is the following comment which he makes on page 235 of his book:

From

$$(a + b)^n = a^n + \cdots$$

we get the standard form

$$(1 + x)^n = 1 + \cdots$$

by writing 1 for a and x for b; the standard form is simpler and is more convenient to use; all problems regarding binomial expansions can be solved by using the standard form.

Coming to the idea of Inversion in the title of this paper, let me repeat from page 194 of my Engineering Book [9] the following quotation from page 323 of the chapter on Abel in Bell's Men of Mathematics [17]:

Instead of assuming that people are depraved because they drink to excess, Galton inverted this hypothesis … For the moment we need note only that Galton, like Abel, inverted his problem – turned it upside-down, and inside-out, back-end-to and foremost-end-backward … 'you must always invert,' as Jacobi said when asked the secret of his mathematical discoveries. He was recalling what Abel and he had done.

On page 309 of the chapter on Abel, Bell says: One of his (= Abel's) classics in this direction is the first proof of the general binomial theorem, special cases of which had been stated by Newton and Euler.

In other words, Bell disagrees with my viewpoint that Newton stated and proved the most general form of the Binomial Theorem.

In this connection, let me repeat what I said on page 417 of my Ramblings Article [5]: Generally speaking, from Newton to Cauchy, mathematicians used power series without regard to convergence. They were criticized for this and the matter was rectified by the analysts Cauchy and Abel who developed a rigorous theory of convergence. After another hundred years or so we were taught, say by Hensel, Krull, and Chevalley, that it really didn't matter, i.e., we may disregard convergence after all! So the algebraist was freed from the shackles of analysis, or rather (as in Vedanta philosophy) he was told that he always was free but had only forgotten it temporarily.

Now one good way to study the rest of this paper is to INVERT it by first reading the last section called EPILOGUE, which is sort of an extended Introduction or a Birds Eye View of the entire paper. Another idea is to start with Sect. 4 and refer to Sects. 2 and 3 as necessary. More precisely, start by reading definition (••) of a valuation sequence given at the beginning of Sect. 4. Our goal in Sect. 4 is to show that the newtonian expansion of the first two terms of that sequence partly determines the newtonian expansion of any two consecutive terms.

2 Notation

We shall mostly follow the notation and terminology of my Kyoto paper [7] and my books [9, 12]. In particular: $\mathbb{N}$ = the set of all nonnegative integers, $\mathbb{N}_+$ = the set of all positive integers, $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and $R^\times$ = the set of all nonzero elements in a ring R. The GCD of a set of integers S is the unique nonnegative generator of the ideal $S\mathbb{Z}$ in the ring of integers $\mathbb{Z}$ generated by S; if the set S contains a noninteger then $\mathrm{GCD}(S) = \infty$. A set of integers J is bounded from below means for some integer e we have $e \le j$ for all $j \in J$, and we write $\min J$ for the smallest element of such a set, with the convention that if J is the empty set $\emptyset$ then $\min J = \infty$.

To fix some more notation: Recall that a quasilocal ring is a (commutative with identity) ring R having a unique maximal ideal $M(R)$; we let $H(R)$ stand for its residue field $R/M(R)$, and by $H_R : R \to R/M(R)$ we denote the residue class epimorphism; note that then $H(R) = H_R(R)$. By a coefficient set of R, we mean a subset k of R with $0 \in k$ and $1 \in k$ such that H_R maps k bijectively onto

$H(R)$. By a coefficient field of R, we mean a coefficient set k of R such that k is a subfield of R. For any subfield K of R, we note that H_R maps K isomorphically onto the subfield $H_R(K)$ of $H(R)$ and we let $\mathrm{trdeg}_K H(R)$ and $[H(R) : K]$ stand for $\mathrm{trdeg}_{H_R(K)} H(R)$ and $[H(R) : H_R(K)]$, respectively. Given an element z in an overring of R, we say that z is residually transcendental over K at R to mean that $z \in R$ and $H_R(z)$ is transcendental over $H_R(K)$.

Recall that a field extension L/K is algebraic (resp: finite algebraic, transcendental, simple transcendental, pure transcendental) means $[K(w) : K] < \infty$ for all $w \in L$ (resp: $[L : K] < \infty$, $\mathrm{trdeg}_K L > 0$, $\mathrm{trdeg}_K L = 1$ and $L = K(t)$ for some $t \in L$, $\mathrm{trdeg}_K L = \nu \in \mathbb{N}$ and $L = K(t_1, \dots, t_\nu)$ for some $t_1, \dots, t_\nu$ in L). Recall that an affine domain over a field is a domain which is a finitely generated ring extension of that field. The characteristic of a field K is denoted by $\mathrm{ch}(K)$. The dimension $\dim(R)$ of a ring R is the maximum length n of a chain of prime ideals

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

in R.

A noetherian quasilocal ring R is called a local ring. The smallest number of generators of $M(R)$ is called the embedding dimension of R and is denoted by $\mathrm{emdim}(R)$. We always have $\mathrm{emdim}(R) \geq \dim(R)$ and R is regular means equality holds; a regular local ring is always a domain. A DVR is a one-dimensional regular local domain; Alternatively, a DVR is the valuation ring of a real discrete valuation in the following sense. A valuation is a map $W : L \to G \cup \{\infty\}$, where L is a field and G is an ordered abelian group, such that for all u, u' in L we have $W(uu') = W(u) + W(u')$ and $W(u + u') \geq \min(W(u), W(u'))$ and for any u in L we have: $W(u) = \infty \Leftrightarrow u = 0$. We put $G_W = W(K^\times)$ and $R_W = \{u \in K : W(u) \geq 0\}$ and call these the value group and the valuation ring of W. Now R_W is a ring with the unique maximal ideal $M(R_W) = \{u \in K : W(u) > 0\}$. Thus R_W is a quasilocal ring. If $G_W = \mathbb{Z}$ then W is said to be real discrete.

A quasilocal ring V dominates a quasilocal ring S means S is a subring of V with $M(S) \subset M(V)$, and then: $\mathrm{restrdeg}_S V$ denotes the residual transcendence degree of V over S, i.e., the transcendence degree of $H(V)$ over $H_V(S)$; we say that V is residually rational over S to mean that $H(V) = H_V(S)$; we say that V is residually algebraic (resp: residually finite algebraic, residually transcendental, residually simple transcendental, residually pure transcendental) over S to mean that the field extension $H(V)/H_V(S)$ is algebraic (resp: finite algebraic, transcendental, simple transcendental, pure transcendental). Given any subring A of a quasilocal ring V, upon letting S to be the localization of A at the prime ideal $A \cap M(V)$, we put $\mathrm{restrdeg}_A V = \mathrm{restrdeg}_S V$ and call it the residual transcendence degree of V over A, and we say that V is residually rational (resp: residually algebraic, residually finite algebraic, residually transcendental, residually simple transcendental, residually pure transcendental) over A to mean that V is residually rational (resp: residually algebraic, residually finite algebraic, residually transcendental, residually simple transcendental, residually pure transcendental) over S.

For any local domain R and any $z \in R^{\times}$, we define $\mathrm{ord}_R z$ to be the largest nonnegative integer e such that $z \in M(R)^e$; if $z = 0$ then we put $\mathrm{ord}_R z = \infty$. If R is regular then we extend this to the quotient field QF(R) of R by putting

$$\mathrm{ord}_R(x/y) = \mathrm{ord}_R x - \mathrm{ord}_R y$$

for all x, y in $R^{\times}$; if $\dim(R) > 0$ then this gives a real discrete valuation of QF(R) whose valuation ring V dominates R and is residually pure transcendental over R of residual transcendence degree $\dim(R) - 1$. See (Q35.5) on pages 559–577 of [12].

Given any subring K of a domain L, by the transcendence degree of L over K we mean the transcendence degree of QF(L) over QF(K), and we continue to denote it by $\mathrm{trdeg}_K L$; note that by convention, if $\mathrm{trdeg}_K L = \infty$ then $(\mathrm{trdeg}_K L) - 1 = \infty$. Given any subring K of a field L, by $\overline{D}(L/K)$ we denote the set of all valuation rings V with $\mathrm{QF}(V) = L$ such that $K \subset V$, and by $D(L/K)$ we denote the set of all $V \in \overline{D}(L/K)$ such that $\mathrm{trdeg}_{H_V(K)} H(V) = (\mathrm{trdeg}_K L) - 1$; we call these V the *valuation rings* and *prime divisors* of L/K respectively. Note that if L is a finitely generated field extension of a field K then every member of $D(L/K)$ is a DVR; moreover if $\mathrm{trdeg}_K L = 1$ then L is the only member of $\overline{D}(L/K)$ which does not belong to $D(L/K)$.

Given any affine domain A over a field K with $\mathrm{QF}(A) = L$, by $\overline{I}(A/K)$ and $I(A/K)$ we denote the set of all $V \in \overline{D}(L/K)$ and $V \in D(L/K)$, respectively, such that $A \not\subset V$; we call these V the *infinity valuation rings* and *infinity divisors* of A/K respectively. Note that all members of $D(L/K)$, and hence all members of $I(A/K)$, are DVRs. Also note that if $\mathrm{trdeg}_K L = 1$ then $I(A/K)$ is a nonempty finite set, and for every $V \in D(L/K)$ we have $[H(V) : K] \in \mathbb{N}_+$. Let us recall that DD = Dedekind Domain = normal noetherian domain of dimension at most one. Note that the localizations of a DD at the various nonzero prime ideals in it are DVRs whose intersection is the given DD. Note that a domain is a PID iff it is a DD as well as a UFD. Also note that a domain is a PID iff it is a notherian UFD of dimension at most one. Let us say that a domain is proper to mean that it is not a field. In particular, a proper PID is a PID which is not a field.

Given any local domain R, by $\overline{D}(R)^{\Delta}$ we denote the set of all $V \in \overline{D}(\mathrm{QF}(R)/R)$ such that V dominate R, and we let $D(R)^{\Delta}$ denote the set of all $V \in \overline{D}(R)^{\Delta}$ such that $\mathrm{restrdeg}_R V = \dim(R) - 1$; we call these V the *valuation rings* of QF(R) dominating R and *prime divisors* of R respectively; note that then for every $V \in \overline{D}(R)^{\Delta}$ we have $\mathrm{restrdeg}_R V \le \dim(R)$, and for every $V \in D(R)^{\Delta}$ we have that V is a DVR.

The habitat for most of the Remarks and Lemmas of the next section will be a DVR V with its quotient field $\mathrm{QF}(V) = L$, its completion $\widehat{V}$, a coefficient field K, and a uniformizing parameter T, i.e., an element of V of order 1. Note that $\widehat{V}$ can be identified with the power series ring $K[[T]]$ and L with a subfield of the meromorphic series field $K((T))$. For any

$$y = y(T) = \sum_{i \in \mathbb{Z}} A_i T^i \in K((T)) \quad \text{with} \quad A_i \in K$$

we define the T-support $\mathrm{Supp}_T\, y(T)$ of $y(T)$ to be the set of all $i \in \mathbb{Z}$ with $A_i \neq 0$, and then we define the T-order and T-initial-coefficient of $y(T)$ by putting

$$\mathrm{ord}_T\, y(T) = \min \mathrm{Supp}_T\, y(T)$$

and

$$\mathrm{inco}_T\, y(T) = A_e \quad \text{where} \quad e = \mathrm{ord}_T\, y(T)$$

with the understanding that if $y(T) = 0$ then $\mathrm{ord}_T\, y(T) = \infty$ and $\mathrm{inco}_T\, y(T) = 0$; note that in case of $\widehat{V} = K[[T]]$ we have $\mathrm{ord}_V\, y = \mathrm{ord}_T\, y(T)$ for all $y \in L$. By a *special subfield* S of $K((T))$ we mean either the null ring $S = \{0\} \subset K$ or a subfield S of $K((T))$ such that: if $a \in S \cap K^\times$ and $b \in K^\times$ with $b^q = a^p$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}_+$ then $b \in S$; if $S \subset K$ then we may call S a *special subfield* of K. Observe that if k is any special subfield of K then k as well as $k((T))$ are special subfields of $K((T))$; by convention, if $k = \{0\}$ then $k((T)) = \{0\}$. We put

$$\mathrm{sub}_T\, y(T) = \begin{cases} \text{the smallest special subfield } k \text{ of } K \\ \text{such that } A_i \in S \text{ for all } i \in \mathbb{Z} \\ \text{(with the note that } k = \{0\} \Leftrightarrow y(T) = 0) \end{cases}$$

We call $\mathrm{sub}_T\, y(T)$ the *T-subfield* of $y(T)$.

As a weaker version of algebraic closedness, we say that a field K is *root-closed* to mean that for every $a \in K$ and $n \in \mathbb{N}_+$ we have $X^n - a = (X - a_1)\dots(X - a_n)$ for some $a_1, \dots, a_n$ in K. Both the notions of a special subfield and a root-closed field are inspired by root extraction, i.e., the finding of square-roots, cube-roots, and so on. The process of root extraction also inspires the concept of a *quasiroot-closed* domain which we shall introduce in Remark (3.10).

3 Remarks and Lemmas

We start off by codifying the euclidean algorithm (= method of long division) of finding the GCD of a pair of integers.

Remark on Euclidean algorithm (3.1). By a *euclidean sequence pair*, we mean a pair $\big((e_j)_{0\le j\le l}, (p_j)_{0\le j<l}\big)$ of sequences of integers $e_j \in \mathbb{Z}$ and $p_j \in \mathbb{Z}$ with $l \in \mathbb{N}_+$ such that:

$$e_1 \neq 0 = p_0 = 0 \neq p_j \text{ for } 2 \le j < l \text{ with } p_j > 0 \text{ for } 3 \le j < l, \tag{1}$$

$$e_{j-1} = p_j e_j + e_{j+1} \text{ with } 0 < e_{j+1} < |e_j| \text{ for } 1 \le j \le l-1, \tag{2}$$

$$|e_j| > |e_l| = \mathrm{GCD}(e_0, e_1) = \mathrm{GCD}(e_0, \dots, e_l) \text{ for } 1 \le j \le l-1, \tag{3}$$

$$l = 1 \Leftrightarrow e_0 \equiv 0 \mod (e_1). \tag{4}$$

The usual euclidean algorithm implies that any pair of integers (e_0, e_1) with $e_1 \neq 0$ can be embedded in a unique euclidean sequence pair $\big((e_j)_{0\le j\le l},\ (p_j)_{0\le j<l}\big)$ which we call the *euclidean extension* of (e_0, e_1).

To apply this construction to orders of elements, let V be a DVR with

$$V \subset \widehat{V} = \text{the completion of } V \quad \text{and} \quad \mathrm{QF}(V) = L \subset \widehat{L} = \mathrm{QF}(\widehat{V})$$

and let K be a coefficient set of V.

By a (V, K)-*protosequence* we mean a sequence

$$(z_j, e_j, p_j, A_l^*(\nu), e_l^*, z_l^*)_{\nu\in\mathbb{Z}, 0\le j\le l+1}$$

where

$$\big((e_j)_{0\le j\le l}, (p_j)_{0\le j<l}\big)$$

is a euclidean sequence pair and

$$\begin{cases} z_j \in L^\times \text{ with } \mathrm{ord}_V z_j = e_j \text{ for } 0 \le j \le l, \\ \text{and } z_{j-1} = z_j^{p_j} z_{j+1} \text{ for } 1 \le j \le l-1 \end{cases} \tag{5}$$

and

$$\begin{cases} z_{l+1} \in L \text{ with } \mathrm{ord}_V z_{l+1} = e_{l+1} \text{ and } p_l = p_{l+1} \in \mathbb{Z} \cup \{\infty\} \\ \text{and } z_l^* \in L \text{ with } \mathrm{ord}_V z_l^* = e_l^* \\ \text{such that } z_{l+1} = 0 \Leftrightarrow p_l = \infty \Leftrightarrow z_l^* = 0 \\ \text{and } z_{l+1} \neq 0 \Rightarrow (e_{l-1}/e_l) \le p_l(e_l/|e_l|) \text{ with } 0 < e_{l+1} < |e_l| \end{cases} \tag{6}$$

and

$$\begin{cases} A_l^*(\nu) \in K \text{ for all } \nu \in \mathbb{Z} \text{ such that} \\ A_l^*(\nu) \begin{cases} = 0 & \text{if } \nu < (e_{l-1}/|e_l|) \\ \neq 0 & \text{if } \nu = (e_{l-1}/|e_l|) \\ = 0 & \text{if } \nu > p_l(e_l/|e_l|) \text{ and } z_{l+1} \neq 0 \end{cases} \end{cases} \tag{7}$$

such that in $\widehat{L}$ we have

$$z_l^* = z_{l-1} - \sum_{(e_{l-1}/|e_l|)\le\nu<\infty} A_l^*(\nu) z_l^{\nu(|e_l|/e_l)} = \begin{cases} 0 & \text{if } z_{l+1} = 0 \\ z_l^{p_l} z_{l+1} & \text{if } z_{l+1} \neq 0. \end{cases} \tag{8}$$

Any pair of elements (z_0, z_1) in $L^\times$ with $\mathrm{ord}_V z_1 \neq 0$ can clearly be embedded in a unique (V, K)-protosequence

$$(z_j, e_j, p_j, A_l^*(\nu), e_l^*, z_l^*)_{\nu\in\mathbb{Z}, 0\le j\le l+1}$$

which we call the (V, K)-*protoexpansion* of (z_0, z_1).

To contrast the above expansion (8) with the usual expansions in terms of a uniformizing parameter T of $\widehat{V}$, we note that

$$\text{for } 0 \le j \le l+1$$

there exist unique

$$\begin{cases} A_j(\nu) \in K \text{ for all } \nu \in \mathbb{Z} \\ \text{with } A_j(\nu) = 0 \text{ for } \nu < e_j \\ \text{and if } e_j \in \mathbb{Z} \text{ then } A_j(e_j) \neq 0 \end{cases} \tag{9}$$

such that

$$z_j = z_j(T) = \sum_{e_j \le \nu < \infty} A_j(\nu) T^{\nu}. \tag{10}$$

In case z_0, z_1 belong to V, we may visualize $x = z_1(T), y = z_0(T)$ as giving a parametrization of a branch of a curve in the (x, y)-plane centered at the point $(z_1(0), z_0(0))$.

To continue our construction by a (V, K)-*presequence* we mean a sequence

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)})_{\nu \in \mathbb{Z}, 0 \le j \le l(i)+1, 0 \le i \le \kappa} \quad \text{with } \kappa \in \mathbb{N}$$

where

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)})_{\nu \in \mathbb{Z}, 0 \le j \le l(i)+1}$$

is a (V, K)-protosequence for $0 \le i \le \kappa$ with

$$(z^*_{il(i)}, z_{il(i)}) = (z_{i+1,0}, z_{i+1,1}) \text{ for } 0 \le i < \kappa \tag{11}$$

and

$$z_{\kappa, l(\kappa)+1} = 0. \tag{12}$$

Now given any pair of elements (z_0, z_1) in $L^{\times}$ with $\mathrm{ord}_V z_1 \neq 0$, clearly there exists a unique (V, K)-presequence

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)})_{\nu \in \mathbb{Z}, 0 \le j \le l(i)+1, 0 \le i \le \kappa}$$

with $(z_{00}, z_{01}) = (z_0, z_1)$, and we call this the (V, K)-*preexpansion* of (z_0, z_1).

In a moment we shall relate the above expansion with the characteristic terms coming out of Newton's fractional power series expansion. To do this we start off with a string of definitions in the following Remark.

Remark on GCD dropping sequence (3.2). A GCD sequence is a system d consisting of its length $h(d) \in \mathbb{N}$ and its sequence $(d_i)_{0 \le i \le h(d)+2}$ where $d_0 = 0$, $d_i \in \mathbb{N}_+$ for $1 \le i \le h(d)+1$, $d_i \in d_{i+1}\mathbb{Z}$ for $0 \le i \le h(d)$, and $d_{h(d)+2} \in \widehat{\mathbb{R}}$. A *charseq* (= characteristic sequence) is a system m consisting of its length $h(m) \in \mathbb{N}$ and its sequence $(m_i)_{0 \le i \le h(m)+1}$ where $m_0 \in \mathbb{Z}^{\times}$, $m_i \in \mathbb{Z}$ for $1 \le i \le h(m)$, and

$m_{h(m)+1} \in \widehat{\mathbb{R}}$. Given any charseq m with $h = h(m)$, its GCD *sequence* is the GCD sequence $d = d(m)$ obtained by putting $h(d) = h$, and $d_i = \mathrm{GCD}(m_0, \ldots, m_{i-1})$ for $0 \le i \le h+2$; its *reciprocal sequence* $n(m)$ is the sequence $n = (n_i)_{1 \le i \le h+1}$ obtained by putting $n_i = d_1/d_i$ for $1 \le i \le h+1$; its *difference sequence* is the charseq $q = q(m)$ obtained by putting $h(q) = h$ with $q_i = m_i$ for $0 \le i \le 1$ and $q_i = m_i - m_{i-1}$ for $2 \le i \le h+1$; note that clearly $d(q) = d(m)$. Given any charseq q with $h = h(q)$ and $d = d(q)$, its *inner product sequence* is the charseq $s = s(q)$ obtained by putting $h(s) = h$ with $s_0 = q_0$ and $s_i = \sum_{1 \le j \le i} q_j d_j$ for $1 \le i \le h+1$, and its *normalized inner product sequence* is the charseq $r = r(q)$ obtained by putting $h(r) = h$ with $r_0 = s_0$ and $r_i = s_i/d_i$ for $1 \le i \le h+1$. Note that then $d(r) = d(q)$.

Let us also note that if $m_{h+1} = \infty$ then $q_{h+1} = s_{h+1} = r_{h+1} = d_{h+2} = \infty$ by the *infinity convention* according to which: for all $c \in \mathbb{R}$ we have $\infty \pm c = \infty$ and $-\infty \pm c = -\infty$, for all $c \in \mathbb{R}_+ =$ the set of all positive real numbers we have $\infty c = \infty/c = \infty$ and $-\infty c = -\infty/c = -\infty$, and we have $\infty + \infty = \infty$.

It is worth observing that any one of the four sequences $m, q(m), s(q(m))$, $r(q(m))$ determines the other three.

Given any charseq m, by the *characteristic pair sequence* of m we mean the sequence $(\widehat{m}_i(m), \widehat{n}_i(m))_{1 \le i \le h(m)}$ defined by putting $\widehat{m}_i(m) = m_i/d_{i+1}(m)$ and $\widehat{n}_i(m) = d_i(m)/d_{i+1}(m)$ for $1 \le i \le h(m)$; we call $\widehat{m}(m) = \widehat{m}_i(m)_{1 \le i \le h(m)}$ the *derived numerator sequence* of m, and we call $\widehat{n}(m) = \widehat{n}_i(m)_{1 \le i \le h(m)}$ the *derived denominator sequence* of m.

A charseq m is *upper-unbounded* means $m_{h(m)+1} = \infty$.

For any set of integers J which is bounded from below and for any nonzero integer l, we define the GCD-*dropping sequence* $m = m(J, l)$ of J relative to l by saying that m is the unique upper-unbounded charseq with $m_0 = l$ and $m_1 = \min J$ such that for $2 \le i \le h(m) + 1$ we have

$$m_i = \min\{j \in J : j \text{ is nondivisible by } \mathrm{GCD}(m_0, \ldots, m_{i-1})\}.$$

If $F(X, Y)$ is a monic polynomial of positive degree N in Y with coefficients in the univariate meromorphic series field $K((X))$ over an algebraically closed field K with $N \not\equiv 0 \mod (\mathrm{ch}(K))$ such that F is a power of a monic irreducible member of $K((X))[Y]$, then by Newton's Theorem on fractional meromorphic series expansion, we can factor

$$F(T^N, Y) = \prod_{1 \le i \le N} [Y - \eta_i(T)] \quad \text{with} \quad \eta_i(T) \in K((T)).$$

Clearly $\mathrm{Supp}_T \eta_i(T)$ is independent of i; we denote this common support by $\mathrm{Supt}(F)$ and call it the *newtonian support* of F. We define the *newtonian charseq* $m(F, l)$ of F relative to a nonzero integer l by putting $m(F, l) = m(\mathrm{Supt}(F), l)$.

In [4], we had assumed $F(X, Y) \in K[[X]][Y]$ and $l = N$. In Sect. 4, we shall now reprove the assertions of [4] for the somewhat more general case of $F(X, Y) \in K((X))[Y]$ and $l = \pm N$.

First binomial lemma (3.3). Let us consider the univariate meromorphic series field $K((T))$ over a field K. Let $y \in K((T))^\times$ and $z \in K((T))$ be such that

$$\mathrm{ord}_T y = v < w = \mathrm{ord}_T z$$

with

$$\mathrm{inco}_T y = \eta \quad \text{and} \quad \mathrm{inco}_T z = \zeta.$$

Then for any $n \in \mathbb{Z}$ we have

$$\begin{cases} (y+z)^n = y' + z' \text{ where } y' = y^n \in K((T))^\times \text{ and } z' \in K((T)) \\ \text{with } \mathrm{inco}_T y' = \eta' \text{ and } \mathrm{inco}_T z' = \zeta' \\ \text{are such that } \mathrm{sub}_T y' \subset \mathrm{sub}_T y \\ \text{with } \mathrm{ord}_T y' = nv = v' < w' = (n-1)v + w \le \mathrm{ord}_T z' \\ \text{and } w' - v' = w - v \text{ with } \eta' = \eta^n \\ \text{and if } n \not\equiv 0 \mod (\mathrm{ch}(K)) \text{ then } \mathrm{ord}_T z' = w' \text{ with } \zeta' = n\zeta\eta^{n-1}. \end{cases} \tag{$\bullet$}$$

Moreover, assuming $n \not\equiv 0 \mod (\mathrm{ch}(K))$, we have that:

$$\begin{cases} \text{if } \mathrm{ord}_T(y - \eta T^v) = w \\ \text{then } \mathrm{ord}_T(y' - \eta' T^{v'}) = w' \end{cases} \tag{1*}$$

whereas

$$\begin{cases} \text{if } y \in K((T^d)) \text{ and } w/d \notin \mathbb{Z} \text{ for some } d \in \mathbb{N}_+ \\ \text{then for that } d \text{ we have } y' \in K((T^d)) \text{ and } w'/d \notin \mathbb{Z} \end{cases} \tag{2*}$$

while

$$\begin{cases} \text{if } y \in k((T)) \text{ and } \zeta \notin k \text{ for some subfield } k \text{ of } K \\ \text{then for that } k \text{ we have } y' \in k((T)) \text{ and } \mathrm{inco}_T z' \notin k. \end{cases} \tag{3*}$$

Proof. (1*)–(3*) follow from ($\bullet$). So it suffices to prove ($\bullet$). If $n \in \mathbb{N}$ then we are done because by the binomial theorem we have

$$z' = \sum_{1 \le i \le n} \binom{n}{i} z^i y^{n-i} = nzy^{n-1} + \cdots + z^n.$$

If $-n \in \mathbb{N}_+$ then we are done by applying the geometric series identity to the previous case. In greater detail, if $-n \in \mathbb{N}_+$ then by the previous case we can write

$$(y+z)^{-n} = \overline{y} + \overline{z} \quad \text{where} \quad \overline{y} = y^{-n} \in K((T))^\times \quad \text{and} \quad \overline{z} \in K((T))$$

with

$$\mathrm{inco}_T \overline{y} = \overline{\eta} \quad \text{and} \quad \mathrm{inco}_T \overline{z} = \overline{\zeta}$$

are such that $\text{sub}_T\overline{y} \subset \text{sub}_T y$ with

$$\text{ord}_T\overline{y} = -nv = \overline{v} < \overline{w} = (-n-1)v + w \le \text{ord}_T\overline{z}$$

and

$$\overline{w} - \overline{v} = w - v \text{ with } \overline{\eta} = \eta^{-n}$$

and

$$\text{if } n \not\equiv 0 \mod (\text{ch}(K)) \text{ then } \text{ord}_T\overline{z} = \overline{w} \text{ with } \overline{\zeta} = -n\zeta\eta^{-n-1}.$$

By the geometric series identity $(1+X)^{-1} = 1 - X + X^2 - \ldots$ we get

$$(\overline{y}+\overline{z})^{-1} = \overline{y}^{-1}(1+(\overline{z}/\overline{y}))^{-1} = \overline{y}^{-1} - \overline{z}\,\overline{y}^{-2} + \overline{z}^2\overline{y}^{-3} - \ldots$$

and

$$\overline{y}^{-1} = \left(\overline{\eta}^{-1}T^{-\overline{v}}\right)\left(1 - (\overline{y}\,\overline{\eta}^{-1}T^{-\overline{v}} - 1) + (\overline{y}\,\overline{\eta}^{-1}T^{-\overline{v}} - 1)^2 - \ldots\right)$$

with $\text{sub}_T\overline{y}^{-1} \subset \text{sub}_T\overline{y}$, and therefore the desired result follows by taking

$$y' = \overline{y}^{-1} \quad \text{and} \quad z' = -\overline{z}\,\overline{y}^{-2} + \overline{z}^2\overline{y}^{-3} - \ldots.$$

Second Binomial Lemma (3.4). Let K be a root-closed field and let us consider the univariate meromorphic series field $K((T))$. Let $y \in K((T))^\times$ and $z \in K((T))$ be such that

$$\text{ord}_T y = v < w = \text{ord}_T z$$

with

$$\text{inco}_T y = \eta \quad \text{and} \quad \text{inco}_T z = \zeta.$$

Let $n = p/q$ where p and q are integers with $q > 0$ and $\text{GCD}(p,q) = 1$ such that $q \not\equiv 0 \mod (\text{ch}(K))$ and $pv \equiv 0 \mod (q)$. Then *mantrawise* (= briefly suggestivewise) we have (•) of (3.3), i.e., *bhashyawise* (= precisely detailwise) we have that:

$$\begin{cases} \text{there exists } y' \in K((T))^\times \text{ with } (y')^q = y^p \text{ and } \text{sub}_T y' \subset \text{sub}_T y \\ \text{and for any such } y' \text{ there exists } z' \in K((T))^\times \text{ such that} \\ \text{upon letting } x = y' + z' \text{ with } \text{inco}_T y' = \eta' \text{ and } \text{inco}_T z' = \zeta' \\ \text{we have } x^q = (y+z)^p \\ \text{with } \text{ord}_T y' = nv = v' < w' = (n-1)v + w \le \text{ord}_T z' \\ \text{and } w' - v' = w - v \text{ with } (\eta')^q = \eta^p \\ \text{and if } p \not\equiv 0 \mod (\text{ch}(K)) \text{ then } \text{ord}_T z' = w' \text{ with } q\zeta' = p\zeta\eta'/\eta. \end{cases} \qquad (\dagger)$$

Moreover, assuming $p \not\equiv 0 \mod (\text{ch}(K))$, we have (1*)–(3*) of (3.3).

Proof. (1*)–(3*) follow from (†). So it suffices to prove (†). Since the field K is root-closed, we have $\xi^q = \eta^p$ for some $\xi \in K$. Applying Hensel's Lemma to $Y^q - (1+X)^p$ we get an identity in $K[[X]]$ saying that

$$(1 + b_1 X + b_2 X^2 + \dots)^q = (1+X)^p \quad \text{with} \quad b_1, b_2, \dots \text{ in } K. \tag{1}$$

Differentiating both sides with respect to X and then putting $X = 0$ we get

$$q b_1 = p. \tag{2}$$

Substituting $X = y\eta^{-1}T^{-v} - 1$ in (1) and letting

$$y' = \xi \left(1 + b_1(y\eta^{-1}T^{-v} - 1) + b_2(y\eta^{-1}T^{-v} - 1)^2 + \dots\right) T^{(pv)/q}$$

we get $y' \in K((T))^\times$ with $(y')^q = y^p$ and $\mathrm{sub}_T y' \subset \mathrm{sub}_T y$.

Now let y' be any element of $K((T))^\times$ such that

$$(y')^q = y^p \quad \text{and} \quad \mathrm{sub}_T y' \subset \mathrm{sub}_T y. \tag{3}$$

Then letting $\mathrm{inco}_T y' = \eta'$ we clearly get

$$(\eta')^q = \eta^p. \tag{4}$$

For any $x \in K((T))$ we have

$$x^q = (y+z)^p \Leftrightarrow (x/y')^q = (1 + (z/y))^p$$

which follows by dividing the LHS by (3), and hence substituting $X = z/y$ in (1) and letting

$$x = y'(1 + b_1(z/y) + b_2(z/y)^2 + \dots)$$

we get $x \in K((T))^\times$ such that $x^q = (y+z)^p$ and

$$x - y' = y'(b_1(z/y) + b_2(z/y)^2 + \dots). \tag{5}$$

Now letting $z' = x - y'$ and $\mathrm{inco}_T z' = \zeta'$, by (3)–(5) we see that $z' \in K((T))$ and $x = y' + z'$ with

$$\mathrm{ord}_T y' = nv = v' < w' = (n-1)v + w \le \mathrm{ord}_T z' \;\text{ and }\; w' - v' = w - v$$

and

$$\text{if } \; p \not\equiv 0 \mod (\mathrm{ch}(K)) \; \text{ then } \; \mathrm{ord}_T z' = w' \; \text{ with } \; q\zeta' = p\zeta\eta'/\eta.$$

Note. Lemma (3.3) was not used in Lemma (3.4). So the former was reproved in the latter. The former used the Binomial Theorem for integer exponents, while the latter used the Binomial Theorem for fractional exponents in disguise. Removing the disguise, Mantrawise, Lemma (3.4) follows by saying that by the Binomial Theorem for fractional exponents we have

$$(y+z)^n = y^n\left(1 + n(y/z) + (n(n-1)/2)(y/z)^2 + \dots\right)$$

and so we are done by taking $y' = y^n$ and $z' = (y+z)^n - y'$; but care has to be taken when $\mathrm{ch}(K) \neq 0$. In spite of what was said in the Introduction, we shall not directly use (3.4), i.e., we shall not explicitly use the Binomial Theorem for fractional exponents, but really it is lurking everywhere!!

Remark on special subfields (3.5). The essence of the above two Binomial Lemmas (3.3) and (3.4) is the Invariance of the Gap, i.e., the equation $w' - v' = w - v$, which underlies all the claims of [4] as well as their generalization in the present paper.

Now consider the univariate meromorphic series field $K((T))$ over a field K.

The two cases (2*) and (3*) of (3.3) and (3.4) can be unified by introducing the notion of the (T,S)-gap v of $y(T) = T^e \sum_{0\le i<\infty} A_i T^i$ with $A_i \in K$ and $A_0 \neq 0$, where S is any subfield of $K((T))$, by putting $v = \min\{i \in \mathbb{N} : A_i T^i \notin S\}$, in case (2*) we take $S = K((T^d))$ and in case (3*) we take $S = k((T))$. To include the ordinary gaps as in case (1*), like the gap of length 4 between T and T^5 in $T + T^5 + T^6 + \dots$, we have to allow S to be the null ring, which is not a subfield of $K((T))$ under the usual convention. This is why we introduced the notion of a special subfield.

More generally, we define a *quasispecial subfield* S of $K((T))$ to be either the nullring $S = \{0\} \subset K((T))$ or a subfield S of $K((T))$; if $S \subset K$ then we may call S a *quasispecial subfield* of K. Now let S be a quasispecial subfield of $K((T))$. Given any $y = y(T) \in K((T))^\times$ let

$$y(T) = T^e \sum_{0\le i<\infty} A_i T^i \quad \text{with} \quad \mathrm{ord}_T y(T) = e \quad \text{and} \quad A_i \in K \quad \text{with} \quad A_0 \neq 0$$

and

$$v = \begin{cases} \min\{i \in \mathbb{N}_+ : A_i T^i \notin S\} & \text{if } S = \{0\} \\ \min\{i \in \mathbb{N} : A_i T^i \notin S\} & \text{if } S \neq \{0\} \end{cases}$$

with the convention that the minimum of the empty set of integers is ∞. We define the (T,S)-*gap* and the (T,S)-*coefficient* of $y(T)$ by putting

$$\mathrm{gap}_{(T,S)} y(T) = v \quad \text{and} \quad \mathrm{coef}_{(T,S)} y(T) = \begin{cases} A_v & \text{if } v \neq \infty \\ 0 & \text{if } v = \infty. \end{cases}$$

We are particularly interested in the following cases ($1^\sharp$), ($2^\sharp$), ($3^\sharp$) of a quasispecial subfield S of $K((T))$; note that in each of these cases S is a special subfield of $K((T))$.

$$\begin{cases} S = \{0\}; \\ \text{note that then } v = \operatorname{ord}_T(y(T)T^{-e} - A_0). \end{cases} \tag{$1^\sharp$}$$

$$\begin{cases} S = K((T^d)) \text{ where } d \in \mathbb{N}_+; \\ \text{note that then } v = \min(\operatorname{Supp}_T(y(T)T^{-e} - A_0) \setminus d\mathbb{Z}). \end{cases} \tag{$2^\sharp$}$$

$$\begin{cases} S = k((T)) \text{ where } k \text{ is a nonnull special subfield of } K; \\ \text{note that then } v = \begin{cases} \min\{i \in \mathbb{N}_+ : A_i \notin k\} & \text{if } A_0 \in k \\ 0 & \text{if } A_0 \notin k. \end{cases} \end{cases} \tag{$3^\sharp$}$$

To prepare for proving the next Lemma (3.6), *let S be a quasispecial subfield of $K((T))$ and let $y(T), z(T), x(T)$ in $K[[T]]^\times$ be such that*

$$y(T) = T \sum_{0 \le i < \infty} A_i T^i \quad \text{with} \quad \operatorname{ord}_T y(T) = 1 \quad \text{and} \quad \operatorname{gap}_{(T,S)} y(T) = v$$

and

$$z(T) = T \sum_{0 \le j < \infty} B_j T^j \quad \text{with} \quad \operatorname{ord}_T z(T) = 1 \quad \text{and} \quad \operatorname{gap}_{(T,S)} z(T) = w$$

and

$$x(T) = y(z(T)) = T \sum_{0 \le l < \infty} C_l T^l \quad \text{with} \quad \operatorname{ord}_T x(T) = 1 \quad \text{and} \quad \operatorname{gap}_{(T,S)} x(T) = \pi$$

where A_i, B_j, C_l are in K with

$$A_0 \neq 0 \neq B_0 \neq 0 \neq C_0$$

and where we note that now $e = 1$. For $0 \le l < \infty$ we clearly have

$$C_l T^l = \sum_{0 \le i \le l} \left(A_i T^i \times \text{the term of } T\text{-degree } l - i \text{ in } \left(\sum_{0 \le j \le l-i} B_j T^j \right)^{i+1} \right)$$

and hence

$$C_l T^l = \begin{cases} A_0 B_l T^l + B_0 A_l T^l + \sum_{0<i<l} A_i T^i D_{il} & \text{if } l \neq 0 \\ A_0 B_0 & \text{if } l = 0 \end{cases} \tag{I}$$

with

$$D_{il} = \sum\nolimits^{*} M_\lambda \prod_{0 \le j \le l-i} (B_j T^j)^{\lambda_j} \tag{II}$$

where $\sum^*$ indicates summation over all $(\lambda_0, \ldots, \lambda_{l-i}) \in \mathbb{N}^{l-i+1}$ for which

$$\sum_{0 \le j \le l-i} j\lambda_j = l - i \tag{III}$$

and M_λ is the multinomial coefficient

$$M_\lambda = \frac{(i+1)!}{\lambda_0! \ldots \lambda_{l-i}!}.$$

We shall now prove the following assertions:

$$\begin{cases} \text{(1) If } 0 < i < l < \infty \text{ with } l \le \min(v, w) \text{ then } A_i T^i D_{il} \in S. \\ \text{(2) } \pi \ge \min(v, w). \\ \text{(3) If } v < w \text{ then } \pi = v \text{ and } C_v T^v - B_0 A_v T^v \in S. \\ \text{(4) If } w < v \text{ then } \pi = w \text{ and } C_w T^w - A_0 B_w T^w \in S. \\ \text{(5) } C_0 - A_0 B_0 = 0 \in S. \\ \text{(6) If } 0 \ne v = w \ne \infty \text{ then } C_v T^v - (A_0 B_v T^v + B_0 A_v T^v) \in S. \\ \text{(7) If } x(T) = T \text{ then } v = w \text{ and } A_0 B_0 = 1. \\ \text{(8) If } x(T) = T \text{ and } 0 \ne v = w \ne \infty \text{ then } A_0 B_v T^v + B_0 A_v T^v \in S. \end{cases} \tag{IV}$$

To prove (1) let $0 < i < l < \infty$ with $l \le \min(v, w)$. Since $0 < i < l \le v$, we get $A_i T^i \in S$. If $S = \{0\}$ then $A_i T^i = 0$ and hence $A_i T^i D_{il} = 0 \in S$. If $S \ne \{0\}$ then $1 \in S$ and because $i < l \le w$, every term in each product involved in (II) belongs to S, and hence again $A_i T^i D_{il} \in S$.

To prove (2) let $0 \le l < \min(v, w)$. Then $A_0 B_l T^l \in S$ and $B_0 A_l T^l \in S$, and hence by (I) and (1) we get $C_l T^l \in S$. It follows that $\pi \ge \min(v, w)$.

To prove (3) let $v < w$. If $v \ne 0$ then $A_0 B_v T^v \in S$ and hence by (I) and (1) we get $C_v T^v - B_0 A_v T^v \in S$; but $B_0 \in S^\times$ with $A_v T^v \notin S$ and therefore $C_v T^v \notin S$; consequently by (2) we see that $\pi = v$. If $v = 0$ then $A_0 \notin S$ with $B_0 \in S$ and hence by (I) we get $C_0 - B_0 A_0 = 0 \in S$ with $C_0 \notin S$ and therefore $\pi = 0 = v$.

To prove (4) let $w < v$. If $w \ne 0$ then $B_0 A_w T^w \in S$ and hence by (I) and (1) we get $C_w T^w - A_0 B_w T^w \in S$; but $A_0 \in S^\times$ with $B_w T^w \notin S$ and therefore $C_w T^w \notin S$; consequently by (2) we see that $\pi = w$. If $w = 0$ then $B_0 \notin S$ with $A_0 \in S$ and hence by (I) we get $C_0 - A_0 B_0 = 0 \in S$ with $C_0 \notin S$ and therefore $\pi = 0 = w$.

By (I) we obviously get (5). By (I) and (1) we see that if $0 \ne v = w \ne \infty$ then $C_v T^v - (A_0 B_v T^v + B_0 A_v T^v) \in S$, which proves (6).

If $x(T) = T$ then $\pi = \infty$ with $C_0 = 1$, and hence by (3) and (4) we get $v = w$ and by (5) we get $A_0 B_0 = 1$, which proves (7).

If $x(T) = T$ and $0 \neq v = w \neq \infty$ then $\pi = \infty$, and hence by (6) we get $A_0 B_v T^v + B_0 A_v T^v \in S$, which proves (8).

Gap Lemma (3.6). Consider the univariate meromorphic series field $K((T))$ over a root-closed field K. Let $y(T)$ and $z(T)$ in $K((T))^\times$ with

$$\mathrm{ord}_T\, y(T) = e \neq 0 \neq \epsilon = \mathrm{ord}_T z(T)$$

be such that

$$y(T) = T^e \sum_{0 \leq i < \infty} A_i T^i \quad \text{and} \quad z(T) = T^\epsilon \sum_{0 \leq j < \infty} B_j T^j$$

where

$$A_i \quad \text{and} \quad B_j \quad \text{are in} \quad K \quad \text{with} \quad A_0 \neq 0 \neq B_0.$$

Assume that $e \not\equiv 0 \mod (\mathrm{ch}(K))$ and $\epsilon \not\equiv 0 \mod (\mathrm{ch}(K))$. Then by Hensel's Lemma, there exist $\widehat{y}(T)$ and $\widehat{z}(T)$ in $K[[T]]^\times$ with

$$\mathrm{ord}_T \widehat{y}(T) = 1 = \mathrm{ord}_T \widehat{z}(T)$$

such that

$$\widehat{y}(T)^e = y(T) \quad \text{and} \quad \widehat{z}(T)^\epsilon = z(T)$$

and

$$\widehat{y}(T) = T \sum_{0 \leq i < \infty} \widehat{A}_i T^i \quad \text{and} \quad \widehat{z}(T) = T \sum_{0 \leq j < \infty} \widehat{B}_j T^j$$

where

$$\widehat{A}_i \quad \text{and} \quad \widehat{B}_j \quad \text{are in} \quad K \quad \text{with} \quad \widehat{A}_0^e = A_0 \neq 0 \neq B_0 = \widehat{B}_0^\epsilon.$$

Given any *special subfield* S of $K((T))$ let

$$\mathrm{gap}_{(T,S)} y(T) = v \quad \text{with} \quad \mathrm{gap}_{(T,S)} \widehat{y}(T) = \widehat{v}$$

and

$$\mathrm{gap}_{(T,S)} z(T) = w \quad \text{with} \quad \mathrm{gap}_{(T,S)} \widehat{z}(T) = \widehat{w}.$$

Assume that

$$v \neq 0 \neq w.$$

Then

$$v = \widehat{v} \quad \text{with} \quad \mathrm{coef}_{(T,S)} y(T) = \left(\mathrm{coef}_{(T,S)} \widehat{y}(T)\right) e \widehat{A}_0^{e-1} \tag{1}$$

and

$$w = \widehat{w} \quad \text{with} \quad \mathrm{coef}_{(T,S)} z(T) = \left(\mathrm{coef}_{(T,S)} \widehat{z}(T)\right) \epsilon \widehat{B}_0^{\epsilon-1}. \tag{2}$$

Moreover,

$$\text{if}\quad \widehat{y}(\widehat{z}(T)) = T \quad \text{then} \quad v = w \quad \text{and} \quad \widehat{A}_0\widehat{B}_0 = 1 \tag{3}$$

and

$$\begin{cases} \text{if}\quad \widehat{y}(\widehat{z}(T)) = T \quad \text{and} \quad \infty \neq v = w \neq \infty \quad \text{then} \\ \left(\text{coef}_{(T,S)}z(T)\right) e\widehat{A}_0^e T^v + \left(\text{coef}_{(T,S)}y(T)\right) \epsilon\widehat{B}_0^\epsilon T^v \in S. \end{cases} \tag{4}$$

Proof. (1) follows from (3.3) by noting that $\widehat{v} > 0$ and taking

$$(n, v, w, y, z) = \left(e, 1, \widehat{v}+1, T \sum_{0\le i<\widehat{v}} \widehat{A}_i T^i, T \sum_{\widehat{v}\le i<\infty} \widehat{A}_i T^i\right)$$

and (2) follows from (3.3) by noting that $\widehat{w} > 0$ and taking

$$(n, v, w, y, z) = \left(\epsilon, 1, \widehat{w}+1, T \sum_{0\le i<\widehat{w}} \widehat{B}_i T^i, T \sum_{\widehat{w}\le i<\infty} \widehat{B}_i T^i\right).$$

By (3.5)(IV)(7), we see that

$$\text{if}\quad \widehat{y}(\widehat{z}(T)) = T \quad \text{then} \quad \widehat{v} = \widehat{w} \quad \text{and} \quad \widehat{A}_0\widehat{B}_0 = 1 \tag{$'$}$$

and by (3.5)(IV)(8) we see that

$$\begin{cases} \text{if}\quad \widehat{y}(\widehat{z}(T)) = T \quad \text{and} \quad \widehat{v} = \widehat{w} \neq \infty \quad \text{then} \\ \left(\text{coef}_{(T,S)}\widehat{z}(T)\right) \widehat{A}_0 T^{\widehat{v}} + \left(\text{coef}_{(T,S)}\widehat{y}(T)\right) \widehat{B}_0 T^{\widehat{v}} \in S. \end{cases} \tag{$''$}$$

Now, in view of (1) and (2), by ($'$) we get (3), and by ($''$) we get (4).

Remark on gap lemma (3.7). We shall now paraphrase (3.6) by using the language of DVRs.

So let V be a DVR with

$$V \subset \widehat{V} = \text{the completion of } V \quad \text{and} \quad \text{QF}(V) = L \subset \widehat{L} = \text{QF}(\widehat{V}).$$

Let T be a uniformizing parameter of $\widehat{V}$. Assume that $\text{ch}(L) = \text{ch}(H(V))$ and let K be a coefficient field of $\widehat{V}$. Note that then $\widehat{V} = K((T))$. Assume that $H(V)$, and hence K, is root-closed.

Given any $y = y(T) \in K((T))^\times$ and $z = z(T) \in K((T))^\times$ let

$$\text{ord}_T y = e \;\text{ with }\; \text{inco}_T y = A \quad \text{and} \quad \text{ord}_T z = \epsilon \;\text{ with }\; \text{inco}_T z = B.$$

Since K is root-closed, we can choose

$$\widehat{A} \in K^\times \;\text{ with }\; (\widehat{A})^e = A \quad \text{and} \quad \widehat{B} \in K^\times \;\text{ with }\; (\widehat{B})^\epsilon = B.$$

Assuming $e \not\equiv 0 \mod (\mathrm{ch}(K))$, with the chosen $\widehat{A}$, by Hensel's Lemma there exists a unique $\widehat{y} = \widehat{y}(T) \in K((T))^\times$ such that

$$(\widehat{y})^e = y \quad \text{and} \quad \mathrm{ord}_T\widehat{y} = 1 \ \text{ with } \ \mathrm{inco}_T\widehat{y} = \widehat{A}.$$

Clearly $\theta(T) \mapsto \theta(\widehat{y}(T))$ gives an automorphism $K((T)) \to K((T))$ and hence there exists a unique $\tilde{z} = \tilde{z}(T) \in K((T))^\times$ such that

$$\tilde{z}(\widehat{y}(T)) = z(T).$$

We call $\tilde{z} = \tilde{z}(T)$ the (V, K, T)-*expansion* of z in terms of y relative to $\widehat{A}$, or briefly we call $\tilde{z} = \tilde{z}(T)$ the (V, K, T)-*expansion* of $(z, y, \widehat{A})$. Concerning the dependence of this expansion on $\widehat{A}$, let us note that

$$\begin{cases} \text{if } \widehat{A}^* \text{ is any other member of } K \text{ with } (\widehat{A}^*)^e = A \\ \text{then } \omega = \widehat{A}^*/\widehat{A} \text{ is an } e\text{-th root of 1 in } K \\ \text{and for the } (V, K, T)\text{-expansion } \tilde{z}^* \text{ of } (z, y, \widehat{A}^*) \\ \text{we have } \tilde{z}^*(T) = \tilde{z}(\omega T) \\ \text{and hence } \mathrm{Supp}_T\tilde{z}^*(T) = \mathrm{Supp}_T\tilde{z}(T). \end{cases} \tag{$\flat$}$$

Assuming $e \not\equiv 0 \mod (\mathrm{ch}(K))$ but without assuming any condition on ϵ, with the chosen $\widehat{A}$, in view of ($\flat$) we may put

$$m(z, y, V, K) = m(\mathrm{Supp}_T\tilde{z}(T), e)$$

(because $\mathrm{Supp}_T\tilde{z}(T)$ is independent of $\widehat{A}$) and call it the (V, K)-*charseq* of (z, y).

Also *assuming* $\epsilon \not\equiv 0 \mod (\mathrm{ch}(K))$, with the chosen $\widehat{B}$, by Hensel's Lemma there exists a unique $\widehat{z} = \widehat{z}(T) \in K((T))^\times$ such that

$$(\widehat{z})^\epsilon = z \quad \text{and} \quad \mathrm{ord}_T\widehat{z} = 1 \ \text{ with } \ \mathrm{inco}_T\widehat{z} = \widehat{B}.$$

Clearly $\theta(T) \mapsto \theta(\widehat{z}(T))$ gives an automorphism $K((T)) \to K((T))$ and hence there exists a unique $\widetilde{y} = \widetilde{y}(T) \in K((T))^\times$ such that

$$\widetilde{y}(\widehat{z}(T)) = y(T).$$

Note that now $\widetilde{y} = \widetilde{y}(T)$ is the (V, K, T)-expansion of $(y, z, \widehat{B})$.

Again clearly there exist unique $z^\dagger(T)$ and $y^\dagger(T)$ in $K((T))$ such that

$$z^\dagger(\widehat{y}(T)) = \widehat{z}(T) \quad \text{and} \quad y^\dagger(\widehat{z}(T)) = \widehat{y}(T). \tag{$\bullet$}$$

Substituting the first equation of (•) in its second, we get

$$y^\dagger(z^\dagger(\widehat{y}(T))) = \widehat{y}(T)$$

and hence

$$y^\dagger(z^\dagger(T)) = T \tag{1}$$

Raising the second equation of (•) to the e-th power and the first to the ϵ-th power we get

$$y^\dagger(T)^e = \widetilde{y}(T) \quad \text{and} \quad z^\dagger(T)^\epsilon = \tilde{z}(T). \tag{2}$$

By the first equation of (•) we get

$$\mathrm{ord}_T z^\dagger(T) = 1 \quad \text{with} \quad \mathrm{inco}_T z^\dagger(T) = \widehat{B}/\widehat{A} \tag{3}$$

and by the second equation of (•) we get

$$\mathrm{ord}_T y^\dagger(T) = 1 \quad \text{with} \quad \mathrm{inco}_T y^\dagger(T) = \widehat{A}/\widehat{B}. \tag{4}$$

Now we claim the FIRST INVERSION THEOREM which says that:

$$\begin{cases} \text{Given any special subfield } S \text{ of } K((T)), \\ \text{upon letting } \mathrm{gap}_{(T,S)}\widetilde{y}(T) = v \text{ and } \mathrm{gap}_{(T,S)}\tilde{z}(T) = w, \\ \text{we have the following.} \\ \text{(1*) If } v \neq 0 \neq w \text{ then } v = w. \\ \text{(2*)} \begin{cases} \text{If } \infty \neq v \neq 0 \neq w \neq \infty \text{ then} \\ \left(\mathrm{coef}_{(T,S)}\tilde{z}(T)\right) e\widehat{A}^{e+\epsilon}T^v + \left(\mathrm{coef}_{(T,S)}\widetilde{y}(T)\right) \epsilon\widehat{B}^{\epsilon+e}T^v \in S. \end{cases} \\ \text{(3*) If } S = K((T^d)) \text{ for some } d \in \mathbb{N}_+ \text{ then } 0 \neq v = w \neq 0. \end{cases} \tag{I}$$

Namely, in view of (1)–(4), by (3.6)(3) and (3.6)(4) we obtain (1*) and (2*) respectively. If $S \neq \{0\}$ then by definition $v \neq 0 \neq w$ and hence by (1*) we get (3*).

Next we claim the SECOND INVERSION THEOREM which says that:

$$\begin{cases} \text{Upon letting } m = m(z, y, V, K) \text{ and } m' = m(y, z, V, K) \\ \text{we have } 0 \neq h(m) = h(m') \neq 0 \\ \text{and } e = m_0 = m'_1 \text{ with } \epsilon = m'_0 = m_1 \\ \text{and } m_\mu - \epsilon = m'_\mu - e \text{ for } 2 \leq \mu \leq h(m) + 1 \\ \text{and } d_1(m) = |e| \text{ with } d_1(m') = |\epsilon| \\ \text{and } d_2(m) = d_2(m') = \mathrm{GCD}(e, \epsilon) \\ \text{and } d_\mu(m) = d_\mu(m') \text{ for } 2 \leq \mu \leq h(m) + 2. \end{cases} \tag{II}$$

Namely, everything is obvious except the assertion $h(m) = h(m')$ together with the assertions that for $2 \le \mu \le h(m) + 1$ we have

$$m_\mu - \epsilon = m'_\mu - e \quad \text{and} \quad d_{\mu+1}(m) = d_{\mu+1}(m').$$

Clearly the assertions about $h(m)$ and $d_{\mu+1}(m)$ follow from the assertion about $m_\mu - \epsilon$. By induction on μ let us prove that for $1 \le \mu \le h(m) + 1$ we have

$$m_\mu - \epsilon = m'_\mu - e.$$

For $\mu = 1$ this is line 3 of (II). To go from μ to $\mu + 1$ can be achieved by taking

$$d = d_{\mu+1}(m)$$

in (I)(3*). This completes the proof of (II).

Remark on valuation protoexpansions (3.8). To merge Remarks (3.1), (3.2), and (3.7), let V be a DVR with

$$V \subset \widehat{V} = \text{the completion of } V \quad \text{and} \quad \text{QF}(V) = L \subset \widehat{L} = \text{QF}(\widehat{V}).$$

Let T be a uniformizing parameter of $\widehat{V}$. Assume that $\text{ch}(L) = \text{ch}(H(V)) = 0$ and let K be a coefficient field of $\widehat{V}$. Note that then $\widehat{V} = K((T))$. Assume that $H(V)$, and hence K, is root-closed.

Given any pair of elements (z_0, z_1) in $L^\times$ with $\text{ord}_V z_1 \ne 0$, by (3.1) and (3.7) there exists a system

$$(z_j, e_j, p_j, A_l^*(\nu), e_l^*, z_l^*, A_j(\nu), \widehat{A}_j, \tilde{A}_j(\nu), m^{(j)}, \tilde{z}_j)_{\nu\in\mathbb{Z}, 0\le j\le l+1}$$

where

$$(z_j, e_j, p_j, A_l^*(\nu), e_l^*, z_l^*)_{\nu\in\mathbb{Z}, 0\le j\le l+1}$$

is the (V, K)-protoexpansion of (z_0, z_1) as described in (3.1)(1)–(3.1)(8) with $A_j(\nu)$ as in (3.1)(9) and (3.1)(10), and

$$\text{for } 0 \le j \le l + 1$$

we have

$$\begin{cases} \widehat{A}_j \in K^\times \text{ with } (\widehat{A}_j)^{e_j} = \text{inco}_T z_j & \text{if } j \ne l + 1, \\ \widehat{A}_j \in K^\times \text{ with } (\widehat{A}_j)^{e_j} = \text{inco}_T z_j & \text{if } j = l + 1 \text{ and } z_{l+1} \ne 0, \\ \widehat{A}_j = 0 \in K & \text{if } j = l + 1 \text{ and } z_{l+1} = 0, \end{cases} \tag{1}$$

and

$$m^{(j)} = \begin{cases} m(z_j, z_{j+1}, V, K) & \text{if } l \neq j \neq l+1 \\ m(z_j, z_{j+1}, V, K) & \text{if } j = l \text{ and } z_{l+1} \neq 0 \\ m(z^*_{j-1}, z_{j-1}, V, K) & \text{if } j = l+1 \text{ and } z_{l+1} \neq 0 \\ m(\emptyset, 1) & \text{if } l \leq j \leq l+1 \text{ and } z_{l+1} = 0 \end{cases} \tag{2}$$

and

$$\tilde{z}_j = \tilde{z}_j(T) = \sum_{\nu \in \mathbb{Z}} \tilde{A}_j(\nu) T^\nu \ \text{ with } \ \tilde{A}_j(\nu) \in K \tag{3}$$

is the (V, K, T)-expansion of $(z_j, z_{j+1}, \widehat{A}_{j+1})$ in case $l \neq j \neq l+1$, and in the remaining cases:

$$\begin{cases} \text{if } j = l \text{ and } z_{l+1} \neq 0 \\ \text{then (3) is the } (V, K, T)\text{-expansion of } (z_j, z_{j+1}, \widehat{A}_{j+1}) \end{cases} \tag{4}$$

and

$$\begin{cases} \text{if } j = l+1 \text{ and } z_{l+1} \neq 0 \\ \text{then (3) is the } (V, K, T)\text{-expansion of } (z^*_{j-1}, z_{j-1}, \widehat{A}_{j-1}) \end{cases} \tag{5}$$

and

$$\begin{cases} \text{if } l \leq j \leq l+1 \text{ and } z_{l+1} = 0 \text{ then in (3) we take} \\ \tilde{z}_j = \tilde{z}_j(T) = 0 = \tilde{A}_j(\nu) \text{ for all } \nu \in \mathbb{Z}; \end{cases} \tag{6}$$

we call such a system a *mixed* (V, K, T)-*protoexpansion* of (z_0, z_1). It follows that

$$\begin{cases} \text{if } z_{l+1} \neq 0 \text{ then } e^*_l = p_l e_l + e_{l+1} = m_2^{(l-1)} \\ \text{and } \tilde{A}_{l-1}(\nu) = \begin{cases} 0 & \text{if } e^*_l > \nu \not\equiv 0 \mod (e_l) \\ A^*_l(\nu/|e_l|) & \text{if } e^*_l > \nu \equiv 0 \mod (e_l) \\ \tilde{A}_{l+1}(\nu - p_l e_l) & \text{if } e^*_l \leq \nu; \end{cases} \end{cases} \tag{7}$$

In view of (3.1)(5),

$$\text{for } \ 0 \leq j \leq l-2$$

upon letting

$$\begin{cases} \breve{A}_{j+2}(\nu) \in K \text{ for all } \nu \in \mathbb{Z} \text{ such that} \\ \breve{A}_{j+2}(\nu) = 0 \text{ for } \nu < e_{j+2} \text{ and } \breve{A}_{j+2}(e_{j+2}) \neq 0 \\ \text{and } \breve{z}_{j+2} = \breve{z}_{j+2}(T) = \sum_{e_{j+2} \leq \nu < \infty} \breve{A}_{j+2}(\nu) T^\nu \\ \text{is the } (V, K, T)\text{-expansion of } (z_{j+2}, z_{j+1}, \widehat{A}_{j+1}) \end{cases} \tag{1_j}$$

we have

$$\breve{A}_{j+2}(\nu + e_{j+2}) = \tilde{A}_j(\nu + e_j) \text{ for all } \nu \in \mathbb{Z} \tag{2_j}$$

and hence

$$\begin{cases} \text{upon letting } \breve{m}^{(j+2)} = m(z_{j+2}, z_{j+1}, V, K) \\ \text{we have } 0 \neq h(m^{(j)}) = h(\breve{m}^{(j+2)}) \neq 0 \\ \text{and } e_{j+1} = m_0^{(j)} = \breve{m}_0^{(j+2)} \text{ and } e_j = m_1^{(j)} \text{ with } e_{j+2} = \breve{m}_1^{(j+2)} \\ \text{and } m_\mu^{(j)} - e_j = \breve{m}_\mu^{(j+2)} - e_{j+2} \text{ for } 1 \leq \mu \leq h(m^{(j)}) + 1 \\ \text{and } d_1(m^{(j)}) = d_1(\breve{m}^{(j+2)}) = |e_{j+1}| \\ \text{and } d_2(m^{(j)}) = d_2(\breve{m}^{(j+2)}) = \mathrm{GCD}(e_j, e_{j+1}) = \mathrm{GCD}(e_{j+1}, e_{j+2}) \\ \text{and } d_\mu(m^{(j)}) = d_\mu(\breve{m}^{(j+2)}) \text{ for } 1 \leq \mu \leq h(m^{(j)}) + 2. \end{cases} \tag{3_j}$$

and, in view of (3_j), by taking $(z, y) = (z_{j+1}, z_{j+2})$ in (2.7)(II) we see that

$$\begin{cases} 0 \neq h(m^{(j)}) = h(m^{(j+1)}) \neq 0 \\ \text{and } e_{j+1} = m_0^{(j)} = m_1^{(j+1)} \text{ and } e_j = m_1^{(j)} \text{ with } e_{j+2} = m_0^{(j+1)} \\ \text{and } m_\mu^{(j)} - e_j = m_\mu^{(j+1)} - e_{j+1} \text{ for } 2 \leq \mu \leq h(m^{(j)}) + 1 \\ \text{and } d_1(m^{(j)}) = |e_{j+1}| \text{ with } d_1(m^{(j+1)}) = |e_{j+2}| \\ \text{and } d_2(m^{(j)}) = d_2(m^{(j+1)}) = \mathrm{GCD}(e_j, e_{j+1}) = \mathrm{GCD}(e_{j+1}, e_{j+2}) \\ \text{and } d_\mu(m^{(j)}) = d_\mu(m^{(j+1)}) \text{ for } 2 \leq \mu \leq h(m^{(j)}) + 2. \end{cases} \tag{4_j}$$

Now $(4_0) + \cdots + (4_{j-1}) \Rightarrow$

$$\begin{cases} \text{for } 0 \leq j \leq l-1 \text{ we have } 0 \neq h(m^{(0)}) = h(m^{(j)}) \neq 0 \\ \text{and } e_1 = m_0^{(0)} \text{ and } e_0 = m_1^{(0)} \text{ with } e_{j+1} = m_0^{(j)} \text{ and } e_j = m_1^{(j)} \\ \text{and } m_\mu^{(0)} - e_0 = m_\mu^{(j)} - e_j \text{ for } 2 \leq \mu \leq h(m^{(0)}) + 1 \\ \text{and } d_1(m^{(0)}) = |e_1| \text{ with } d_1(m^{(j)}) = |e_{j+1}| \\ \text{and } d_2(m^{(0)}) = d_2(m^{(j)}) = \mathrm{GCD}(e_0, e_1) = \mathrm{GCD}(e_j, e_{j+1}) \\ \text{and } d_\mu(m^{(0)}) = d_\mu(m^{(j)}) \text{ for } 2 \leq \mu \leq h(m^{(0)}) + 2. \end{cases} \tag{I}$$

Moreover, in view of (2)–(7) we see that

$$\begin{cases} \text{if } z_{l+1} \neq 0 \text{ then } h(m^{(l+1)}) = h(m^{(l-1)}) - 1 \\ \text{and } e_l = m_0^{(l+1)} = m_0^{(l-1)} \\ \text{and } p_l e_l + e_{l+1} = m_1^{(l+1)} = m_2^{(l-1)} \text{ with } e_{l-1} = m_1^{(l-1)} \\ \text{and } m_\mu^{(l+1)} = m_{\mu+1}^{(l-1)} \text{ for } 2 \leq \mu \leq h(m^{(l+1)}) + 1 \\ \text{and } d_1(m^{(l+1)}) = |e_l| \\ \text{and } d_1(m^{(l-1)}) = |e_l| = d_2(m^{(l-1)}) = \mathrm{GCD}(e_l, e_{l-1}) \\ \text{and } d_2(m^{(l+1)}) = d_3(m^{(l-1)}) = \mathrm{GCD}(e_l, e_{l+1}) \\ \text{and } d_\mu(m^{(l+1)}) = d_{\mu+1}(m^{(l-1)}) \text{ for } 2 \leq \mu \leq h(m^{(l+1)}) + 2. \end{cases} \tag{II}$$

Preamble for next lemma. Having dealt with case (3.5)($2^\sharp$), turning to case (3.5)($3^\sharp$), let

$$\begin{cases} S = k((T)) \text{ where } k \text{ is a nonnull special subfield of } K, \\ \theta = \text{an unspecified member of } k^\times \\ (\theta \text{ is called Abhyankar's nonzero and may be read as } \theta), \\ \theta' = \text{an unspecified member of } k \\ (\theta' \text{ is called Abhyankar's constant and may be read as } \theta'), \\ \text{gap}_{(T,S)}\tilde{z}_j(T) = v_j \text{ with coef}_{(T,S)}\tilde{z}_j(T) = \bar{A}_j \text{ for } 0 \le j \le l \\ \text{with the understanding that if } z_{l+1} = 0 \text{ then } v_l = \infty \text{ and } \bar{A}_l = 0, \end{cases} \tag{8}$$

and let

$$z_l^\dagger = \sum_{(e_{l-1}/|e_l|) \le \nu < (v_{l-1}+e_{l-1})|e_l|^{-1}} A_l^*(\nu) z_l^{\nu(|e_l|/e_l)} \in K[z_l, z_l^{-1}] \tag{9}$$

and

$$z_l^\flat = z_{l-1} - z_l^\dagger \in L \text{ with } \text{ord}_V z_l^\flat = e_l^\flat \tag{10}$$

and let

$$z_l^\flat = z_l^\flat(T) = \sum_{\nu \in \mathbb{Z}} A_l^\flat(\nu) T^\nu \text{ with } A_l^\flat(\nu) \in K \tag{11}$$

be the usual expansion in $K((T))$ and

$$\begin{cases} \text{if } z_l^\flat \neq 0 \\ \text{then let } \tilde{z}_l^\flat = \tilde{z}_l^\flat(T) \text{ be the } (V,K,T)\text{-expansion of } (z_l^\flat, z_l, \widehat{A}_l) \\ \text{and let gap}_{(T,S)}\tilde{z}_l^\flat(T) = v_l^\flat \text{ with coef}_{(T,S)}\tilde{z}_l^\flat(T) = \bar{A}_l^\flat. \end{cases} \tag{12}$$

Finally let

$$z_l^{\flat\flat} = z_{l-1}/z_l^{(e_{l-1}-e_l)/e_l} \in L^\times \text{ with } \text{ord}_V z_l^{\flat\flat} = e_l^{\flat\flat} \tag{13}$$

and note that then

$$e_l^{\flat\flat} = e_l. \tag{14}$$

With the above notation at hand, we shall now prove the:

Coefficient lemma (III). We have the following.

(1*) If $\widehat{A}_1 \in k$ and $v_0 > 0$ then for $0 \le j \le l$ we have $\widehat{A}_j \in k$, and for $0 \le j \le l-1$ we have $v_j = v_0$ with $\bar{A}_j = \theta \bar{A}_0 + \theta'$.

(2*) $z_{l+1} \neq 0 \Leftrightarrow m_2^{(l-1)} \neq \infty \Rightarrow m_2^{(l-1)} = p_l e_l + e_{l+1}$.

(3*) If $\widehat{A}_l \in k$ and $v_{l-1} = \infty$ then $v_l = \infty$ and $\widehat{A}_{l+1} \in k$ with $\tilde{z}_{l-1}(T) \in k((T))$ and $A_l^*(\nu) \in k$ for all $\nu \in \mathbb{Z}$.

(4*) If $\widehat{A}_l \in k$ and $v_{l-1} + m_1^{(l-1)} > m_2^{(l-1)}$ then $z_{l+1} \neq 0$ with $\widehat{A}_{l+1} \in k$ and $v_l + m_2^{(l-1)} = v_{l-1} + m_1^{(l-1)}$ with $\bar{A}_l = \theta \bar{A}_{l-1} + \theta'$.

(5*) If $\widehat{A}_l \in k$ and $v_{l-1}+m_1^{(l-1)} < m_2^{(l-1)}$ then $z_l^\flat \neq 0$ and $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \equiv 0 \mod (e_l)$.

(6*) If $\widehat{A}_l \in k$ and $v_{l-1} \neq \infty$ with $v_{l-1} + m_1^{(l-1)} = m_2^{(l-1)}$ then $z_{l+1} \neq 0 \neq z_l^\flat$ and $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l)$.

(7*) If $\widehat{A}_1 \in k$ and $v_0 = 0$ then $\text{inco}_T z_0 = \theta\!\!\!- \bar{A}_0 \in K \setminus k$ and

$$\frac{\text{inco}_T z_l^{\flat\flat}}{\text{inco}_T z_l} = \theta\!\!\!- \bar{A}_0^E \quad \text{with} \quad E = (-1)^{l+1}(e_1/e_l) \in \mathbb{Z}^\times.$$

Prenote. In the statements as well as proofs of (1*)–(7*), some quantity such as v_{l-1} may take the value ∞, and then the reader is advised to follow the infinity convention described in the second paragraph of (3.2).

Note. In the following proofs of (1*)–(7*), we shall frequently invoke two obvious but very useful principles which in the context of (8) may be stated thus. The MP = MULTIPLICATIVE PRINCIPLE says that if $\widehat{A}_{j+1} \in k$ and $z^\sharp \in L$ is such that $z^\sharp = \theta\!\!\!- z_j z_{j+1}^p$ with $p \in \mathbb{Z}$ then: $z^\sharp \neq 0$ and upon letting $\tilde{z}^\sharp(T)$ be the (V,K,T)-expansion of $(z^\sharp, z_{j+1}, \widehat{A}_{j+1})$ and putting

$$\text{gap}_{(T,S)}\tilde{z}^\sharp(T) = v^\sharp \quad \text{with} \quad \text{coef}_{(T,S)}\tilde{z}^\sharp(T) = \bar{A}^\sharp$$

we have

$$v^\sharp = v_j \quad \text{with} \quad \bar{A}^\sharp = \theta\!\!\!- \bar{A}_j \quad \text{and: if } v_j > 0 \text{ then } \{\text{inco}_T z^\sharp, \text{inco}_T z_j\} \subset k.$$

The AP = ADDITIVE PRINCIPLE says that if $\widehat{A}_{j+1} \in k$ and $z^{\sharp\sharp} \in L$ is such that

$$z^{\sharp\sharp} = z_j - \sum_{\nu \le \theta} \tilde{A}_j(\nu)\widehat{z}_{j+1}^{\nu} \quad \text{where} \quad \theta \in \mathbb{Z} \quad \text{with} \quad \theta < v_j + e_j$$

and where $\widehat{z}_{j+1} = \widehat{z}_{j+1}(T) \in K((T))$ is such that

$$\widehat{z}_{j+1}^{e_{j+1}} = z_{j+1} \quad \text{and} \quad \text{inco}_T \widehat{z}_{j+1} = \widehat{A}_{j+1}$$

then: $z^{\sharp\sharp} \neq 0$ and upon letting $\text{ord}_V z^{\sharp\sharp} = e^{\sharp\sharp}$ and upon letting $\tilde{z}^{\sharp\sharp}(T)$ be the (V,K,T)-expansion of $(z^{\sharp\sharp}, z_{j+1}, \widehat{A}_{j+1})$ and putting

$$\text{gap}_{(T,S)}\tilde{z}^{\sharp\sharp}(T) = v^{\sharp\sharp} \quad \text{with} \quad \text{coef}_{(T,S)}\tilde{z}^{\sharp\sharp}(T) = \bar{A}^{\sharp\sharp}$$

we have

$$v^{\sharp\sharp} + e^{\sharp\sharp} = v_j + e_j \quad \text{with} \quad \bar{A}^{\sharp\sharp} = \bar{A}_j.$$

Proof of (1*). In view of an obvious induction, it suffices to show that, given any integer $j \in \{0, \ldots, l-2\}$ with $\widehat{A}_{j+1} \in k$ and $v_j > 0$, we have $\{\widehat{A}_j, \widehat{A}_{j+2}\} \subset k$ and $v_j = v_{j+1}$ with $\bar{A}_j = \theta \bar{A}_{j+1} + \theta'$. Now for any $j \in \{0, \ldots, l-2\}$ with $\widehat{A}_{j+1} \in k$ and $v_j > 0$, by (2_j) and MP we see that

$$\{\widehat{A}_j, \widehat{A}_{j+2}\} \subset k \text{ and } \mathrm{gap}_{(T,S)} \breve{z}_{j+2}(T) = v_j \text{ with } \mathrm{coef}_{(T,S)} \breve{z}_{j+2}(T) = \theta \bar{A}_j$$

and by taking $(z, y) = (z_{j+1}, z_{j+2})$ in (3.7)(I) we see that

$$\mathrm{gap}_{(T,S)} \breve{z}_{j+2}(T) = v_{j+1} \text{ with } \mathrm{coef}_{(T,S)} \breve{z}_{j+2}(T) = \theta \bar{A}_{j+1} + \theta'$$

and by combining the above two displays we get $\{\widehat{A}_j, \widehat{A}_{j+2}\} \subset k$ and $v_j = v_{j+1}$ with $\bar{A}_j = \theta \bar{A}_{j+1} + \theta'$.

Proof of (2*). In view of (2), this follows from (3.1)(6) to (3.1)(8).

Proof of (3*). In view of (1)–(7), this follows from (3.1)(6) to (3.1)(8) together with (3.7)(I).

Proof of (4*). Assuming $\widehat{A}_l \in k$ and $v_{l-1} + m_1^{(l-1)} > m_2^{(l-1)}$, by (2*) we have

$$\widehat{A}_l \in k \text{ with } z_{l+1} \neq 0 \text{ and } m_2^{(l-1)} = p_l e_l + e_{l+1} \neq \infty. \qquad (0^\sharp)$$

To prove that

$$\widehat{A}_{l+1} \in k \text{ and } v_l + m_2^{(l-1)} = v_{l-1} + m_1^{(l-1)} \text{ with } \bar{A}_l = \theta \bar{A}_{l-1} \qquad (\sharp)$$

we proceed thus. In view of (3.1)(8), by $(0^\sharp)$ we have

$$\widehat{A}_l \in k \text{ with } z_{l+1} \neq 0 \text{ and } z_l^* = z_l^{p_l} z_{l+1}. \qquad (1^\sharp)$$

Let $\tilde{z}_l^* = \tilde{z}_l^*(T)$ be the (V, K, T)-expansion of $(z_l^*, z_l, \widehat{A}_l)$ and let

$$\mathrm{gap}_{(T,S)} \tilde{z}_l^*(T) = v_l^* \text{ with } \mathrm{coef}_{(T,S)} \tilde{z}_l^*(T) = \bar{A}_l^*. \qquad (2^\sharp)$$

Clearly

$$\mathrm{ord}_V z_{l-1} = m_1^{(l-1)} \qquad (3^\sharp)$$

and by $(0^\sharp)$ and $(1^\sharp)$ we see that

$$\mathrm{ord}_V z_l^* = m_2^{(l-1)}. \qquad (4^\sharp)$$

In view of (3.1)(8), by ($2^\sharp$)–($4^\sharp$) and AP with $z_j = z_{l-1}$ and $z^{\sharp\sharp} = z_l^*$ it follows that

$$v_l^* + m_2^{(l-1)} = v_{l-1} + m_1^{(l-1)} \text{ with } \bar{A}_l^* = \bar{A}_{l-1}. \tag{$5^\sharp$}$$

Let $\breve{z}_{l+1}(T)$ be the (V, K, T)-expansion of $(z_{l+1}, z_l, \widehat{A}_l)$ and let

$$\text{gap}_{(T,S)}\breve{z}_{l+1}(T) = \breve{v}_{l+1} \text{ with } \text{coef}_{(T,S)}\breve{z}_{l+1}(T) = \breve{A}_{l+1}. \tag{$6^\sharp$}$$

By ($5^\sharp$) we see that $v_l^* > 0$, and hence by ($1^\sharp$), ($6^\sharp$) and MP with $(z^\sharp, z_j, z_{j+1}) = (z_l^*, z_{l+1}, z_l)$ we get

$$\widehat{A}_{l+1} \in k \text{ and } \breve{v}_{l+1} = v_l^* \text{ with } \breve{A}_{l+1} = \theta\, \bar{A}_l^*. \tag{$7^\sharp$}$$

In view of ($6^\sharp$) and ($7^\sharp$), by taking $(y, z) = (z_l, z_{l+1})$ in (3.7)(I) we see that

$$\breve{v}_{l+1} = v_l \text{ with } \breve{A}_{l+1} = \theta\, \bar{A}_l + \theta'. \tag{$8^\sharp$}$$

Combining ($5^\sharp$), ($7^\sharp$) and ($8^\sharp$) we get ($\sharp$).

Proof of (5*). Assuming $\widehat{A}_l \in k$ and $v_{l-1} + m_1^{(l-1)} < m_2^{(l-1)}$, in view of (9)–(12) and (3.1)(6)–(3.1)(8), by (2*) we see that

$$z_l^\flat \neq 0 \text{ with } e_l^\flat = v_{l-1} + e_{l-1} \equiv 0 \mod (e_l)$$

and hence, in view of (9)–(12) and (3.1)(6)–(3.1)(8), by AP with $z_j = z_{l-1}$ and $z^{\sharp\sharp} = z_l^\flat$ we see that

$$v_l^\flat = 0 \text{ with } \bar{A}_l^\flat = \bar{A}_{l-1}.$$

Proof of (6*). Assuming $\widehat{A}_l \in k$ and $v_{l-1} \neq \infty$ with $v_{l-1} + m_1^{(l-1)} = m_2^{(l-1)}$, in view of (9)–(12) and (3.1)(6)–(3.1))(8), by (2*) we see that

$$z_{l+1} \neq 0 \neq z_l^\flat \text{ with } e_l^\flat = v_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l)$$

and hence, in view of (9)–(12) and (3.2)(6)–(3.2)(8), by AP with $z_j = z_{l-1}$ and $z^{\sharp\sharp} = z_l^\flat$ we see that

$$v_l^\flat = 0 \text{ with } \bar{A}_l^\flat = \bar{A}_{l-1}.$$

Proof of (7*). Assuming $\widehat{A}_1 \in k$ and $v_0 = 0$, we clearly have

$$\text{inco}_T z_0 = \theta\, \bar{A}_0 \in K \setminus k. \tag{$0'$}$$

To prove the equation

$$\frac{\mathrm{inco}_T z_l^{\flat\flat}}{\mathrm{inco}_T z_l} =_\theta \bar{A}_0^E \quad \text{with} \quad E = (-1)^{l+1}(e_1/e_l) \in \mathbb{Z}^\times \tag{$'$}$$

we define the euclidean postextension of the integer pair (e_0, e_1) with $e_1 \neq 0$ to be the sequence pair $\left((e'_j)_{0 \le j \le l+1}, (p'_j)_{0 \le j \le l}\right)$ obtained by putting $e'_j = e_j$ or 0 accordings as $0 \le j \le l$ or $j = l + 1$, and $p'_j = p_j$ or e_{l-1}/e_l accordings as $0 \le j \le l - 1$ or $j = l$. Note that now

$$e'_{j-1} = p'_j e'_j + e'_{j+1} \quad \text{for } 1 \le j \le l. \tag{1$'$}$$

Given any integers e''_0, e''_1, let us define integers $e''_2, \ldots, e''_{l+1}$ by requiring that

$$e''_{j-1} = p'_j e''_j + e''_{j+1} \quad \text{for } 1 \le j \le l. \tag{2$'$}$$

Let $M_j = \begin{pmatrix} e'_{j-1} & e'_j \\ e''_{j-1} & e''_j \end{pmatrix}$ for $1 \le j \le l + 1$, and $N_j = \begin{pmatrix} 0 & 1 \\ 1 & -p'_j \end{pmatrix}$ for $1 \le j \le l$. Then

$$M_j N_j = M_{j+1} \quad \text{with} \quad \det(N_j) = -1 \quad \text{for } 1 \le j \le l \tag{3$'$}$$

and hence

$$\det(M_{l+1}) = (-1)^l \det(M_1). \tag{4$'$}$$

Clearly $\det(M_{l+1}) = e''_{l+1} e'_l$ and if $(e''_0, e''_1) = (1, 0)$ then $\det(M_1) = -e'_1$. Therefore

$$\text{if} \quad (e''_0, e''_1) = (1, 0) \quad \text{then} \quad e''_{l+1} = (-1)^{l+1}(e_1/e_l). \tag{5$'$}$$

Let the sequence $(z'_j)_{0 \le j \le l+1}$ be defined by putting $z'_j = z_j$ or $z_{l-1}/z_l^{p'_l}$ according as $0 \le j \le l$ or $j = l + 1$. Then

$$z'_{j-1} = (z'_j)^{p'_j} z'_{j+1} \quad \text{for } 1 \le j \le l. \tag{6$'$}$$

Assuming $A \in K^\times$ to be such that $\mathrm{inco}_T z'_j =_\theta A^{e''_j}$ for $0 \le j \le 1$, by (1$'$), (2$'$), and (6$'$) we see that

$$\mathrm{inco}_T z'_j =_\theta A^{e''_j}$$

for $0 \le j \le l + 1$; consequently by (5$'$) we conclude that

$$\begin{cases} \text{if } (e''_0, e''_1) = (1, 0) \text{ and } A \in K^\times \text{ is such that} \\ \mathrm{inco}_T z'_j =_\theta A^{e''_j} \text{ for } 0 \le j \le 1, \\ \text{then } \mathrm{inco}_T z'_{l+1} =_\theta A^E \text{ with } E = (-1)^{l+1}(e_1/e_l) \in \mathbb{Z}^\times. \end{cases} \tag{7$'$}$$

By (13) and (14) we have

$$\mathrm{inco}_T z'_{l+1} = \frac{\mathrm{inco}_T z_l^{\flat\flat}}{\mathrm{inco}_T z_l}$$

and hence by taking $A = \bar{A}_0$ in (7′) we get (′).

Remark on valuation preexpansions (3.9). For further merging of Remarks (3.1), (3.2), and (3.7), let V be a DVR with

$$V \subset \widehat{V} = \text{the completion of } V \quad \text{and} \quad \mathrm{QF}(V) = L \subset \widehat{L} = \mathrm{QF}(\widehat{V}).$$

Let T be a uniformizing parameter of $\widehat{V}$. Assume that $\mathrm{ch}(L) = \mathrm{ch}(H(V)) = 0$ and let K be a coefficient field of $\widehat{V}$. Note that then $\widehat{V} = K((T))$. Assume that $H(V)$, and hence K, is root-closed.

Given any pair of elements (z_0, z_1) in $L^\times$ with $\mathrm{ord}_V z_1 \neq 0$, by (3.1) and (3.8) there exists a system

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)}, A_{ij}(\nu), \widehat{A}_{ij}, \tilde{A}_{ij}(\nu), m^{(ij)}, \tilde{z}_{ij})_{\nu\in\mathbb{Z}, 0\le j\le l(i)+1, 0\le i\le\kappa}$$

such that

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)})_{\nu\in\mathbb{Z}, 0\le j\le l(i)+1, 0\le i\le\kappa}$$

is the (V, K)-preexpansion of (z_0, z_1) and

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)}, A_{ij}(\nu), \widehat{A}_{ij}, \tilde{A}_{ij}(\nu), m^{(ij)}, \tilde{z}_{ij})_{\nu\in\mathbb{Z}, 0\le j\le l(i)+1}$$

is a mixed (V, K, T)-protoexpansion of (z_{i0}, z_{i1}) for $0 \le i \le \kappa$; in analogy with a mixed protoexpansion, we call such a system a *mixed* (V, K, T)*-preexpansion* of (z_0, z_1).

Let us record that, for $0 \le i \le \kappa \in \mathbb{N}$, by (3.1)(1)–(3.1)(5) we now have a pair of sequences

$$\big((e_{ij})_{0\le j\le l(i)}, (p_{ij})_{0\le j< l(i)}\big)$$

of integers $e_{ij} \in \mathbb{Z}$ and $p_{ij} \in \mathbb{Z}$ with $l(i) \in \mathbb{N}_+$ such that:

(1) $p_{i0} = 0 \neq e_{il(i)}$,
(2) $e_{i,j-1} = p_{ij}e_{ij} + e_{i,j+1}$ with $p_{ij} \neq 0 < e_{i,j+1} < |e_{ij}|$ for $1 \le j \le l(i) - 1$,
(3) $|e_{ij}| > |e_{il(i)}| = \mathrm{GCD}(e_{i0}, e_{i1}) = \mathrm{GCD}(e_{i0}, \dots, e_{il(i)})$ for $1 \le j \le l(i) - 1$,
(4) $l(i) = 1 \Leftrightarrow e_{i0} \equiv 0 \mod (e_{i1})$,
(5) $z_{i,j-1} = z_{ij}^{p_{ij}} z_{i,j+1}$ for $1 \le j \le l(i) - 1$.

Let us also record that (3.1)(6)–(3.1)(12) and (3.8)(1)–(3.8)(7) hold with obvious modifications. Note that (3.1)(11) *is used in proving* (II) below.

Now rewriting (3.8)(I) and (3.8)(II) in terms of the difference sequence $q(m)$ defined in (3.2) we respectively see that

$$\begin{cases} \text{for } 0 \le j \le l(i)-1 \text{ and } 0 \le i \le \kappa \\ \text{we have } 0 \ne h(m^{(i0)}) = h(m^{(ij)}) \ne 0 \\ \text{and } e_{i1} = m_0^{(i0)} \text{ and } e_{i0} = m_1^{(i0)} \\ \text{with } e_{i,j+1} = m_0^{(ij)} \text{ and } e_{ij} = m_1^{(ij)} \\ \text{and } q_\mu(m^{(i0)}) = q_\mu(m^{(ij)}) \text{ for } 2 \le \mu \le h(m^{(i0)})+1 \\ \text{and } d_1(m^{(i0)}) = |e_{i1}| \text{ with } d_1(m^{(ij)}) = |e_{i,j+1}| \\ \text{and } d_2(m^{(i0)}) = d_2(m^{(ij)}) = \mathrm{GCD}(e_{i0}, e_{i1}) = \mathrm{GCD}(e_{ij}, e_{i,j+1}) \\ \text{and } d_\mu(m^{(i0)}) = d_\mu(m^{(ij)}) \text{ for } 2 \le \mu \le h(m^{(i0)})+2 \end{cases} \tag{I}$$

and

$$\begin{cases} \text{for } 0 \le i < \kappa \\ \text{we have } h(m^{(i+1,0)}) = h(m^{(i,l(i)-1)}) - 1 \\ \text{and } e_{i+1,1} = m_0^{(i+1,0)} = m_0^{(i,l(i)-1)} = e_{il(i)} \\ \text{and } e_{i+1,0} = m_1^{(i+1,0)} = m_2^{(i,l(i)-1)} = p_{il(i)}e_{il(i)} + e_{i,l(i)+1} \\ \text{and } e_{i,l(i)-1} = m_1^{(i,l(i)-1)} \\ \text{and } q_\mu(m^{(i+1,0)}) = q_{\mu+1}(m^{(i,l(i)-1)}) \text{ for } 2 \le \mu \le h(m^{(i+1,0)})+1 \\ \text{and } d_1(m^{(i+1,0)}) = |e_{i+1,1}| \\ \text{and } d_1(m^{(i,l(i)-1)}) = |e_{il(i)}| = d_2(m^{(i,l(i)-1)}) = \mathrm{GCD}(e_{il(i)}, e_{i,l(i)-1}) \\ \text{and } d_2(m^{(i+1,0)}) = d_3(m^{(i,l(i)-1)}) = \mathrm{GCD}(e_{i+1,0}, e_{i+1,1}) \\ \text{and } d_\mu(m^{(i+1,0)}) = d_{\mu+1}(m^{(i,l(i)-1)}) \text{ for } 2 \le \mu \le h(m^{(i+1,0)})+2. \end{cases} \tag{II}$$

Combining (I) and (II) we get the concise THIRD INVERSION THEOREM which shows the power of the difference sequence and which says that:

$$\begin{cases} \text{for } 0 \le j \le l(i)-1 \text{ and } 0 \le i \le \kappa \\ \text{we have } h(m^{(ij)}) = h(m^{(00)}) - i \\ \text{and } q_0(m^{(ij)}) = e_{i,j+1} = \mathrm{ord}_V z_{i,j+1} \text{ with } z_{i,j+1} \in L^\times \\ \text{and } q_1(m^{(ij)}) = e_{ij} = \mathrm{ord}_V z_{ij} \text{ with } z_{ij} \in L^\times \\ \text{and } q_\mu(m^{(ij)}) = q_{\mu+i}(m^{(00)}) \text{ for } 2 \le \mu \le h(m^{(ij)})+1 \\ \text{and } d_\mu(m^{(ij)}) = d_{\mu+i}(m^{(00)}) \text{ for } 2 \le \mu \le h(m^{(ij)})+2. \end{cases} \tag{III}$$

Remark on root-closed fields (3.10). The concepts of root-closed fields and special subfields, as well as Newton's Binomial Theorem for fractional exponents, all lead to the idea of root extraction, which in turn inspires the following generalization (I) of a 1936 result of F. K. Schmidt, where we use the terminology according to which: By a *quasiroot-closed pair* we mean a pair (R, I) consisting of a domain R and a nonzero ideal I in it such that

$$\begin{cases} \text{for every } a \in I \text{ we have } b_n^n = (1+a) \text{ for some } b_n \in R \\ \text{for infinitely many } n \in \mathbb{N}_+. \end{cases}$$

By a *quasiroot-closed domain* we mean a domain R such that (R, I) is a quasiroot-closed pair for some nonzero ideal I in R. By $\mathcal{N}(R)$ we denote the *normalization* of a domain R, i.e., the integral closure of R in $\mathrm{QF}(R)$.

(I) Let (R, I) be any quasiroot-closed pair.

(1) Then for every DVR V with $\mathrm{QF}(R) =$ a subfield of $\mathrm{QF}(V)$ we have $R \subset V$.
(2) More generally, for every noetherian domain W with $\mathrm{QF}(R) =$ a subfield of $\mathrm{QF}(W)$ we have $R \subset \mathcal{N}(W)$.
(3) Moreover, if R is noetherian and W is any quasiroot-closed noetherian domain with $\mathrm{QF}(R) = \mathrm{QF}(W)$ then $\mathcal{N}(R) = \mathcal{N}(W)$.
(4) Finally, if R is a DVR then for every normal noetherian domain W with $\mathrm{QF}(R) = \mathrm{QF}(W) \neq W$ we have $R = W$.

Proof of (1). If $R \not\subset V$ then for some $x \in R$ we will have $\mathrm{ord}_V(x) = -q$ with $q \in \mathbb{N}_+$. Since $I \neq \{0\}$, we can take $0 \neq y \in I$. Upon letting $a = yx^m$ for large $m \in \mathbb{N}_+$ we get $a \in I$ and $\mathrm{ord}_V a = -p$ with $p \in \mathbb{N}_+$. Clearly $\mathrm{ord}_V(1+a) = -p$. Now taking $n > p$, the equation $b_n^n = (1+a)$ implies $\mathrm{ord}_V b_n = p/n \notin \mathbb{Z}$ which is a contradiction. Therefore, $R \subset V$.

Proof of (2). Follows from (1) by noting that by Theorem (4.10) on page 118 of Nagata [28] $\mathcal{N}(W)$ is the intersection of all DVRs V with $\mathrm{QF}(W) = \mathrm{QF}(V)$ and $W \subset V$.

Proof of (3). By (2) we get $\mathcal{N}(R) \subset \mathcal{N}(W)$ with $\mathcal{N}(W) \subset \mathcal{N}(R)$ and hence $\mathcal{N}(R) = \mathcal{N}(W)$.

Proof of (4). Follows from (2) by noting that there is no subring strictly between a DVR and its quotient field.

Recall that a quasilocal domain R is *henselian* means it satisfies the following condition: If $f(Y)$ is any monic polynomial of degree $n > 0$ with coefficients in R such that, letting $\bar{f}(Y)$ denote the polynomial obtained by applying H_R to the coefficients of $f(Y)$, we have $\bar{f}(Y) = g^*(Y)h^*(Y)$ where $g^*(Y)$ and $h^*(Y)$ are monic coprime polynomials in $H(R)[Y]$, then there exits unique monic $g(Y)$ in $h(Y)$ in $R[Y]$ such that $f(Y) = g(Y)h(Y)$ with $\overline{g}(Y) = g^*(Y)$ and $\overline{h}(Y) = h^*(Y)$. In order to apply (I) to this case, by taking

$$f(Y) = Y^n - (1+a) \text{ and } n \not\equiv 0 \mod (\mathrm{ch}(H(R))$$

we see that:

(II) If R is a henselian quasilocal domain which is not a field then $(R, M(R))$ is a quasiroot-closed pair.

By (I) and (II) we get the following:

(III) If R and S are henselian local domains with $R \neq \mathrm{QF}(R) = \mathrm{QF}(S) \neq S$ then $R = S$.

In this connection, referring to [12], we note that:

(IV) Every complete local domain is henselian. The r-variable power series ring $K[[X_1, \ldots, X_r]]$ over a field K with $r \in \mathbb{N}_+$ is an r-dimensional complete local domain which is normal and unequal to its quotient field $K((X_1, \ldots, X_r))$.

By (III) and (IV) we see that:

(V) If r and s are positive integers and K and L are fields for which we have $K((X_1, \ldots, X_r)) = L((Y_1, \ldots, Y_s))$, then we have $K[[X_1, \ldots, X_r]] = L[[Y_1, \ldots, Y_s]]$ and $r = s$.

4 Newtonian Expansion

In Remarks (3.1) and (3.9), we organized the valuation data in $\kappa + 1$ blocks of sizes $l(0), l(1), \ldots, l(\kappa)$. Now we shall reorganize it in a single sequence of length $l(0) + l(1) + \cdots + l(\kappa)$. To be more precise, the blocks were of sizes $l(0) + 2, \ldots, l(\kappa) + 2$ where the last two members of a block essentially coincided with the second and third members of the next block. Likewise the reorganized single sequence will more precisely be of length $l(0)+\cdots+l(\kappa)-\kappa+1$. In Sect. 5, we shall give a brief review of quadratic transformations and discuss invariance properties of newtonian characteristic sequences. In PART II, we shall revisit Newton's polygonal method and thereby deduce certain integral dependence properties of the coefficients of fractional power series expansions.

Let V be a DVR with

$$V \subset \widehat{V} = \text{the completion of } V \quad \text{and} \quad \mathrm{QF}(V) = L \subset \widehat{L} = \mathrm{QF}(\widehat{V})$$

and let K be a coefficient set of V. In (3.1) we have defined what we mean by a (V, K)-presequence

$$(z_{ij}, e_{ij}, p_{ij}, A^*_{il(i)}(\nu), e^*_{il(i)}, z^*_{il(i)})_{\nu \in \mathbb{Z}, 0 \leq j \leq l(i)+1, 0 \leq i \leq \kappa}. \qquad (\bullet)$$

Note that then

$$\begin{cases} \kappa \in \mathbb{N} \text{ with } l(\kappa) \in \mathbb{N}_+, \\ 2 \le l(i) \in \mathbb{N}_+ \text{ for } 0 \le i < \kappa, \\ z_{ij} \in L \text{ with } \mathrm{ord}_V z_{ij} = e_{ij}, \\ z^*_{il(i)} \in L \text{ with } \mathrm{ord}_V z^*_{il(i)} = e^*_{il(i)}, \\ p_{ij} \in \mathbb{Z} \cup \{\infty\} \text{ with } A^*_{il(i)}(\nu) \in K, \end{cases} \tag{$1^\dagger$}$$

where the quantities $z_{ij}, z^*_{il(i)}, p_{ij}, A^*_{il(i)}(\nu)$ satisfy the conditions described in (3.1). In particular we have $\mathrm{ord}_V z_{00} = e_{00} \in \mathbb{Z}$ with $\mathrm{ord}_V z_{01} = e_{01} \in \mathbb{Z}^\times$. Moreover, having noted that the pair (z_{00}, z_{01}) uniquely determines (•), we have called (•) the (V, K)-preexpansion of (z_{00}, z_{01}).

Now we define a (V, K)-*sequence* to be a sequence

$$(z_j, e_j, p_j, B_j(\nu), \epsilon(i), \iota(j))_{\nu\in\mathbb{Z}, 0\le j\le\lambda, 0\le i\le\kappa} \tag{••}$$

where

$$\begin{cases} \kappa \in \mathbb{N} \text{ with } \lambda = \epsilon(\kappa) \in \mathbb{N}_+, \\ \epsilon(i) \in \mathbb{N}_+ \text{ for } 0 \le i \le \kappa, \\ \epsilon(i) < \epsilon(i+1) \text{ for } 0 \le i < \kappa, \end{cases} \tag{$1^\ddagger$}$$

and

$$\iota(j) = \begin{cases} \max\{i : 1 \le i \le \kappa + 1 \text{ with } \epsilon(i-1) \le j\} & \text{if } j \ge \epsilon(0) \\ 0 & \text{if } j < \epsilon(0) \end{cases} \tag{$2^\ddagger$}$$

and

$$\begin{cases} z_j \in L^\times \text{ with } \mathrm{ord}_V z_j = e_j \in \mathbb{Z} \text{ for } 0 \le j \le \lambda, \\ e_1 \ne 0 < e_{j+1} < |e_j| \text{ for } 1 \le j < \lambda, \\ p_j \in \mathbb{Z} \text{ for } 0 \le j < \lambda \text{ with } p_\lambda = \infty, \\ p_0 = 0 \ne p_j \text{ for } 2 \le j < \lambda \text{ with } p_j > 0 \text{ for } 3 \le j < \lambda, \\ B_j(\nu) \in K \text{ for } 0 \le j \le \lambda \text{ and } \nu \in \mathbb{Z}, \\ B_0(\nu) = 0 \text{ for all } \nu \in \mathbb{Z}, \end{cases} \tag{$3^\ddagger$}$$

with

$$z_{j-1} - \sum_{(e_{j-1}/|e_j|)\le\nu<\infty} B_j(\nu) z_j^{\nu(|e_j|/e_j)} = \begin{cases} 0 \text{ in } \widehat{L} & \text{if } j = \lambda \\ z_j^{p_j} z_{j+1} & \text{if } 1 \le j < \lambda \end{cases} \tag{$4^\ddagger$}$$

are such that

$$\begin{cases} \text{if } j \in \{1,\ldots,\lambda\} \setminus \{\epsilon(0),\ldots,\epsilon(\kappa)\} \\ \text{then } B_j(\nu) = 0 \text{ for all } \nu \in \mathbb{Z} \\ \text{and } e_{j-1}/e_j \notin \mathbb{Z} \text{ with } e_{j-1} = p_j e_j + e_{j+1}, \\ \text{and } z_{j-1} = z_j^{p_j} z_{j+1}, \end{cases} \tag{5‡}$$

and

$$\begin{cases} \text{if } j \in \{\epsilon(0),\ldots,\epsilon(\kappa)\} \\ \text{then } e_{j-1}/e_j \in \mathbb{Z} \text{ with } e_{j-1}/e_j \le p_j(e_j/|e_j|), \\ \text{and } B_j(\nu) \begin{cases} = 0 & \text{if } \nu < (e_{j-1}/|e_j|) \\ \ne 0 & \text{if } \nu = (e_{j-1}/|e_j|) \\ = 0 & \text{if } \nu > p_j(e_j/|e_j|) \text{ and } j \ne \lambda, \end{cases} \end{cases} \tag{6‡}$$

and we make the *convention* that

$$\epsilon(-1) = 0 \quad \text{and} \quad \epsilon(\kappa+1) = \infty. \tag{7‡}$$

Any pair of elements (z_0, z_1) in $L^\times$ with $\operatorname{ord}_V z_1 \ne 0$ can clearly be embedded in a unique (V, K)-sequence (••) which we call the (V, K)-*expansion* of (z_0, z_1).

Note that if $(z_{00}, z_{01}) = (z_0, z_1)$ then (•) and (••) determine each other by the relations

$$\begin{cases} \lambda = l(0) + \cdots + l(\kappa) - \kappa, \\ \epsilon(i) = l(0) + \cdots + l(i) - i \text{ for } 0 \le i \le \kappa, \\ z_j = z_{0j} \text{ for } 0 \le j \le \epsilon(0), \\ z_j = z_{i,j+1-\epsilon(i-1)} \text{ for } 1 \le i \le \kappa \text{ and } \epsilon(i-1) \le j \le \epsilon(i), \\ p_j = p_{0j} \text{ for } 0 \le j \le \epsilon(0), \\ p_j = p_{i,j+1-\epsilon(i-1)} \text{ for } 1 \le i \le \kappa \text{ and } \epsilon(i-1) \le j \le \epsilon(i), \end{cases} \tag{2†}$$

and

$$\begin{cases} B_j(\nu) = A^*_{il(i)}(\nu) \text{ for } 1 \le i \le \kappa \text{ and } j = \epsilon(i), \\ z_j^{p_j} z_{j+1} = z^*_{il(i)} \text{ for } 1 \le i < \kappa \text{ and } j = \epsilon(i). \end{cases} \tag{3†}$$

Descriptive Note (8†). In a more descriptive manner, the i-th row of (•) as a “matrix” looks like

$$z_{i0}, z_{i1}, \ldots, z_{i,l(i)+1}$$

and a slight trimming converts it into the i-th $= \iota$-th subsequence of (••) which looks like

$$z_{\epsilon(i-1)}, z_{\epsilon(i-1)+1}, \ldots, z_{\epsilon(i)-1}$$

with the convention ($7^\ddagger$) that $\epsilon(-1) = 0$; namely, for $i = 0$, delete the last two terms of the i-th row whereas, for $i > 0$, delete the first and the last two terms of the i-th row. Moreover, at the $\epsilon(i)$-th spot of (••) with $0 \le i < \kappa$ we put the following expansion with nonempty support:

$$z_{\epsilon(i)-1} = \left(\sum_{\nu} B_{\epsilon(i)}(\nu) z_{\epsilon(i)}^{\nu}\right) + z_{\epsilon(i)}^{p_{\epsilon(i)}} z_{\epsilon(i)+1}.$$

In (••), the basic sequence is $(z_j, e_j, p_j, B_j(\nu))_{\nu\in\mathbb{Z},0\le j\le\lambda}$. The remaining two quantities $\epsilon(i)$ and $\iota(j)$ are determined by the basic sequence thus. The $\epsilon(i)$ are those values of j at which the support of the function $\nu \mapsto B_j(\nu)$ is nonempty; we label the $\epsilon(i)$ so that they increase with i. The $\iota(j)$ are the counters to locate $\epsilon(i)$. In other words, if $j = 0, 1, 2, \ldots, \lambda$ are the markers of the train stations as we march along the basic sequence, then $\epsilon(i)$ is the label of a crowded station (say, a junction), and for $0 \le j \le \lambda$ we have $\iota(j) = i \Leftrightarrow \epsilon(i-1) \le j < \epsilon(i)$, i.e., we have

$$\epsilon(\iota(j) - 1) \le j < \epsilon(\iota(j))$$

with the convention ($7^\ddagger$) that $\epsilon(-1) = 0$ and $\epsilon(\kappa + 1) = \infty$. With this convention we can write

$$\epsilon(-1) = 0 < \epsilon(0) < \epsilon(1) < \cdots < \epsilon(\kappa) = \lambda < \infty = \epsilon(\kappa + 1).$$

Definition. Let T be a uniformizing parameter of $\widehat{V}$. *Assume that* $\operatorname{ch}(L) = \operatorname{ch}(H(V)) = 0$ and K is a coefficient field of $\widehat{V}$. Note that then $\widehat{V} = K((T))$. *Assume that* $H(V)$, and hence K, is root-closed. Given any pair (z_0, z_1) in $L^\times$ with $\operatorname{ord}_V z_1 \neq 0$, in view of (3.7) and what we have said above, there exists a system

$$(z_j, e_j, p_j, B_j(\nu), \epsilon(i), \iota(j), A_j(\nu), \widehat{A}_j, \tilde{A}_j(\nu), m^{(j)}, \tilde{z}_j)_{\nu\in\mathbb{Z},0\le j\le\lambda,0\le i\le\kappa} \quad (\bullet\bullet\bullet)$$

such that (••) is the (V, K)-expansion of (z_0, z_1) and

$$\text{for } 0 \le j \le \lambda$$

we have

$$\widehat{A}_j \in K^\times \text{ with } (\widehat{A}_j)^{e_j} = \operatorname{inco}_T z_j \tag{1}$$

and

$$m^{(j)} = m(z_j, z_{j+1}, V, K) \tag{2}$$

and

$$\begin{cases} A_j(\nu) \in K \text{ for all } \nu \in \mathbb{Z} \\ \text{with } A_j(\nu) = 0 \text{ for } \nu < e_j \text{ and } A_j(e_j) \neq 0 \end{cases} \tag{3}$$

and

$$\begin{cases} \tilde{A}_j(\nu) \in K \text{ for all } \nu \in \mathbb{Z} \\ \text{with } \tilde{A}_j(\nu) = 0 \text{ for } \nu < e_j \text{ and } \tilde{A}_j(e_j) \neq 0 \end{cases} \tag{4}$$

such that

$$z_j = z_j(T) = \sum_{e_j \le \nu < \infty} A_j(\nu) T^\nu \tag{5}$$

is the usual expansion of z_j in $K((T))$ and

$$\tilde{z}_j = \tilde{z}_j(T) = \sum_{e_j \le \nu < \infty} \tilde{A}_j(\nu) T^\nu \tag{6}$$

is the (V, K, T)-expansion of $(z_j, z_{j+1}, \widehat{A}_{j+1})$ with the proviso that

$$\begin{cases} \text{if } j = \lambda \text{ then } m^{(j)} = m(\emptyset, 1) \\ \text{and } \tilde{z}_j = \tilde{z}_j(T) = 0 = \tilde{A}_j(\nu) \text{ for all } \nu \in \mathbb{Z}; \end{cases} \tag{7}$$

we call such a system a *mixed* (V, K, T)-*expansion* of (z_0, z_1).

Since (•) and (••) determine each other, referring to (3.2) for notation, (3.9)(III) may be paraphrased as the:

First invariance theorem (I). For $0 \le j \le \lambda - 1$ we have

$$\begin{cases} h(m^{(j)}) = h(m^{(0)}) - \iota(j) \\ \text{and } q_0(m^{(j)}) = e_{j+1} = \text{ord}_V z_{j+1} \text{ with } z_{j+1} \in L^\times \\ \text{and } q_1(m^{(j)}) = e_j = \text{ord}_V z_j \text{ with } z_j \in L^\times \\ \text{and } q_\mu(m^{(j)}) = q_{\mu+\iota(j)}(m^{(0)}) \text{ for } 2 \le \mu \le h(m^{(j)}) + 1 \\ \text{and } d_\mu(m^{(j)}) = d_{\mu+\iota(j)}(m^{(0)}) \text{ for } 2 \le \mu \le h(m^{(j)}) + 2. \end{cases}$$

Moreover, we have

$$h(m^{(\lambda)}) = h(m^{(0)}) - \iota(\lambda) = 0 \quad \text{with} \quad \iota(\lambda) = \kappa + 1.$$

Preamble for next theorem. Referring to (3.5) for notation, having just dealt with case (3.5)($2^\sharp$), turning to case (3.5)($3^\sharp$) let

$$\begin{cases} S = k((T)) \text{ where } k \text{ is a nonnull special subfield of } K, \\ \boldsymbol{\theta} = \text{an unspecified member of } k^\times \\ (\boldsymbol{\theta} \text{ is called Abhyankar's nonzero and may be read as } \theta), \\ \boldsymbol{\theta}' = \text{an unspecified member of } k \\ (\boldsymbol{\theta}' \text{ is called Abhyankar's constant and may be read as } \theta'), \\ \text{gap}_{(T,S)} \tilde{z}_j(T) = \nu_j \text{ with } \text{coef}_{(T,S)} \tilde{z}_j(T) = \bar{A}_j \text{ for } 0 \le j \le \lambda \\ \text{with the understanding that } \nu_\lambda = \infty \text{ and } \bar{A}_\lambda = 0 \end{cases} \tag{8}$$

and

$$\text{for } 1 \le l \le \lambda$$

let

$$z_l^{\dagger} = \sum_{(e_{l-1}/|e_l|) \le \nu < (\nu_{l-1}+e_{l-1})|e_l|^{-1}} B_l(\nu) z_l^{\nu(|e_l|/e_l)} \in K[z_l, z_l^{-1}] \tag{9}$$

and

$$z_l^{\flat} = z_{l-1} - z_l^{\dagger} \in L \quad \text{with} \quad \text{ord}_V z_l^{\flat} = e_l^{\flat} \tag{10}$$

and let

$$z_l^{\flat} = z_l^{\flat}(T) = \sum_{\nu \in \mathbb{Z}} A_l^{\flat}(\nu) T^{\nu} \quad \text{with } A_l^{\flat}(\nu) \in K \tag{11}$$

be the usual expansion in $K((T))$ and

$$\begin{cases} \text{if } z_l^{\flat} \neq 0 \\ \text{then let } \tilde{z}_l^{\flat} = \tilde{z}_l^{\flat}(T) \text{ be the } (V, K, T)\text{-expansion of } (z_l^{\flat}, z_l, \widehat{A}_l) \\ \text{and let } \text{gap}_{(T,S)} \tilde{z}_l^{\flat}(T) = \nu_l^{\flat} \text{ with } \text{coef}_{(T,S)} \tilde{z}_l^{\flat}(T) = \bar{A}_l^{\flat} \end{cases} \tag{12}$$

and finally let

$$z_l^{\flat\flat} \in L \quad \text{with} \quad \text{ord}_V z_l^{\flat\flat} = e_l^{\flat\flat} \tag{13}$$

and

$$z_l^{\flat\flat\flat} \in L \quad \text{with} \quad \text{ord}_V z_l^{\flat\flat\flat} = e_l^{\flat\flat\flat} \tag{14}$$

be defined by putting

$$z_l^{\flat\flat} = \begin{cases} 0 & \text{if } e_{l-1}/e_l \notin \mathbb{Z} \\ z_{l-1}/z_l^{(e_{l-1}-e_l)/e_l} & \text{if } e_{l-1}/e_l \in \mathbb{Z} \end{cases} \tag{15}$$

and

$$z_l^{\flat\flat\flat} = \begin{cases} 0 & \text{if } z_l^{\flat} = 0 \\ 0 & \text{if } z_l^{\flat} \neq 0 \text{ and } e_l^{\flat}/e_l \notin \mathbb{Z} \\ z_l^{\flat}/z_l^{(e_l^{\flat}-e_l)/e_l} & \text{if } z_l^{\flat} \neq 0 \text{ and } e_l^{\flat}/e_l \in \mathbb{Z} \end{cases} \tag{16}$$

and let

$$z_l^{\flat\flat} = z_l^{\flat\flat}(T) = \sum_{\nu \in \mathbb{Z}} A_l^{\flat\flat}(\nu) T^{\nu} \quad \text{with } A_l^{\flat\flat}(\nu) \in K \tag{17}$$

and

$$z_l^{\flat\flat\flat} = z_l^{\flat\flat\flat}(T) = \sum_{\nu \in \mathbb{Z}} A_l^{\flat\flat\flat}(\nu) T^{\nu} \quad \text{with } A_l^{\flat\flat\flat}(\nu) \in K \tag{18}$$

be the usual expansion in $K((T))$.

With the above notation at hand, we shall now prove the:

Second invariance theorem (II). For $0 \le j \le \lambda - 1$ we have the following.

(1*) If $\widehat{A}_{j+1} \in k$ with $v_j > 0$ and $l = \epsilon(\iota(j))$ with $v_j + m_1^{(l-1)} > m_2^{(l-1)}$ then: $\widehat{A}_l \in k$ and $v_{l-1} = v_j$ with $\bar{A}_{l-1} = \not\theta\, \bar{A}_j + \not\theta\,'$ and we have $l < \lambda$ with $\widehat{A}_{l+1} \in k$ and $v_l = v_j + m_1^{(l-1)} - m_2^{(l-1)} > 0$ with $\bar{A}_l = \not\theta\, \bar{A}_j + \not\theta\,'$.

(2*) If $\widehat{A}_{j+1} \in k$ with $v_j > 0$ and $l = \epsilon(\iota(j))$ with $v_j + m_1^{(l-1)} < m_2^{(l-1)}$ then: $\widehat{A}_l \in k$ and $v_{l-1} = v_j$ with $\bar{A}_{l-1} = \not\theta\, \bar{A}_j + \not\theta\,'$ and we have $z_l^\flat \neq 0$ and $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \equiv 0 \mod (e_l)$.

(3*) If $\widehat{A}_{j+1} \in k$ with $\infty \neq v_j > 0$ and $l = \epsilon(\iota(j))$ with $v_j + m_1^{(l-1)} = m_2^{(l-1)}$ then: $\widehat{A}_l \in k$ and $v_{l-1} = v_j$ with $\bar{A}_{l-1} = \not\theta\, \bar{A}_j + \not\theta\,'$ and $l - \lambda \neq 0 \neq z_l^\flat$ and we have $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l)$.

(3**) Notation. For stating the following generalization (4*)–(6*) of (1*)–(3*) we introduce the quantities $\mu(j)$, $\mu^*(j)$, and $\mu(j, j')$ thus. We put

$$\mu(j) = \max\{\mu \in \{1, \dots, h(m^{(j)}) + 1\} : v_j + m_1^{(j)} \ge m_\mu^{(j)}\}$$

and we note that if $v_j = \infty$ then $\mu(j) = h(m^{(j)}) + 1$, whereas if $v_j \neq \infty$ then $\mu(j)$ is the unique integer with $1 \le \mu(j) \le h(m^{(j)})$ such that

$$m_{\mu(j)}^{(j)} \le v_j + m_1^{(j)} < m_{\mu(j)+1}^{(j)}.$$

If $v_j = \infty$ then we put $\mu^*(j) = \infty$, whereas if $v_j \neq \infty$ then we put

$$\mu^*(j) = v_j + m_1^{(j)} - m_{\mu(j)}^{(j)}.$$

For $j \le j' \le \lambda - 1$ we put

$$\mu(j, j') = \iota(j') - \iota(j) + 1$$

and we note that then $\mu(j, j) = 1$ and hence $\mu^*(j) = v_j + m_{\mu(j,j)}^{(j)} - m_{\mu(j)}^{(j)}$. The proofs of (4*)–(6*) will be by induction on $\mu(j, j')$ starting with the

ground case of $\mu(j, j') = 1$,

i.e., the case when

$$\iota(j') = \iota(j) \quad \text{and} \quad \epsilon(\iota(j) - 1) \le j \le j' < \epsilon(\iota(j)).$$

(4*) If $\widehat{A}_{j+1} \in k$ with $\mu^*(j) = \infty$ then for $j \le j' \le \lambda - 1$ we have $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ with $v_{j'+1} = v_{j'} = \infty$ and $\tilde{z}_{j'}(T) \in k((T))$ with $B_{j'+1}(\nu) \in k$ for all $\nu \in \mathbb{Z}$.

(5*) If $\widehat{A}_{j+1} \in k$ with $\infty \ne \mu^*(j) > 0$ then, letting $l = \epsilon(\iota(j) + \mu(j) - 1)$, for $j \le j' \le l-1$ we have $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ and $1 \le \mu(j, j') \le \mu(j, l-1) = \mu(j)$ with $\mu(j, j') + \mu(j') = 1 + \mu(j)$ and $\infty \ne \mu^*(j') = \mu^*(j) > 0$ with $\bar{A}_{j'} = \theta\!\!\!- \bar{A}_j + \theta\!\!\!-'$, and moreover: $z_l^\flat \ne 0$ and $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \equiv 0 \mod (e_l)$, and finally: $\bar{A}_j \in K \setminus k$ and

$$\frac{\mathrm{inco}_T z_l^{\flat\flat\flat}}{\mathrm{inco}_T z_l} = \theta\!\!\!- \bar{A}_l^\flat = \theta\!\!\!- \bar{A}_j + \theta\!\!\!-'.$$

(6*) If $\widehat{A}_{j+1} \in k$ with $\mu^*(j) = 0$ and $\mu(j) \ne 1$ then, letting $l = \epsilon(\iota(j) + \mu(j) - 2)$, for $j \le j' \le l-1$ we have $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ and $1 \le \mu(j, j') \le \mu(j, l-1) = \mu(j) - 1$ with $\mu(j, j') + \mu(j') = 1 + \mu(j)$ and $\mu^*(j') = \mu^*(j) = 0$ with $\bar{A}_{j'} = \theta\!\!\!- \bar{A}_j + \theta\!\!\!-'$, and moreover: $\widehat{A}_l \in k$ and $v_{l-1} = v_j$ with $\bar{A}_{l-1} = \theta\!\!\!- \bar{A}_j + \theta\!\!\!-'$ and $l - \lambda \ne 0 \ne z_l^\flat$ and $v_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = v_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l)$.

(6**) Notation. To facilitate stating claim (7*), we supplement the definition of the derived denominator sequence $\widehat{n}_i(m)_{1 \le i \le h(m)}$ of a charseq m with $h(m) > 0$ given in (3.2) by introducing its signed version

$$n_i^{\flat\flat}(m) = (-1)^{n_i^\flat(m)} \widehat{n}_i(m)$$

where the positive integer $n_i^\flat(m)$ is defined thus. Let $\left((e_j^{(i)})_{0 \le j \le l^{(i)}}, (p_j^{(i)})_{0 \le j < l^{(i)}}\right)$ be the euclidean extension of $(e_0^{(i)}, e_1^{(i)})$ where

$$(e_0^{(i)}, e_1^{(i)}) = \begin{cases} (q_i(m), d_i(m)) & \text{if } 2 \le i \le h(m) \\ (q_1(m), q_0(m)) & \text{if } i = 1. \end{cases}$$

Now (paying special attention to the $j = 0$ case) we put

$$n_i^\flat(m) = \begin{cases} l^{(i)} + 1 & \text{if } e_1^{(i)} > 0 \\ l^{(i)} & \text{if } e_1^{(i)} \le 0. \end{cases}$$

(7*) If $\widehat{A}_{j+1} \in k$ with $\mu^*(j) = 0$ then, letting $l = \epsilon(\iota(j) + \mu(j) - 1)$, we have $\bar{A}_j \in K \setminus k$ and

$$\frac{\mathrm{inco}_T z_l^{\flat\flat}}{\mathrm{inco}_T z_l} = \theta\!\!\!- (\bar{A}_j + \theta\!\!\!-')^E \quad \text{with} \quad E = n_{\mu(j)}^{\flat\flat}(m^{(j)}) \in \mathbb{Z}^\times.$$

Note. In proving Theorem (II), we shall be using the following Reincarnated Version of Lemma (3.8)(III). The said Reincarnated Version says that the Original Version remains valid when for $0 \le j \le \lambda - 1$, upon letting $l = \epsilon(\iota(j))$, we substitute the subsequence $(z_j, z_{j+1}, \ldots, z_l)$ and its associated quantities $(e_j, \ldots, e_l), \ldots$ for the sequence $(z_0, z_1, \ldots, z_l)$ together with its associated quantities considered in (3.8). Note that in the said substitution we put $A_l^*(\nu) = B_l(\nu)$.

Reincarnated coefficient lemma (III). For $0 \le j \le \lambda - 1$, upon letting $l = \epsilon(\iota(j))$, we have the following.

(1*) If $\widehat{A}_{j+1} \in k$ with $\nu_j > 0$ then for $j \le j' \le l$ we have $\widehat{A}_{j'} \in k$, and for $j \le j' \le l-1$ we have $\nu_{j'} = \nu_j$ with $\bar{A}_{j'} = \theta\, \bar{A}_j + \theta'$.

(2*) $l < \lambda \Leftrightarrow l - \lambda \neq 0 \Leftrightarrow m_2^{(l-1)} \neq \infty \Rightarrow m_2^{(l-1)} = p_l e_l + e_{l+1}$.

(3*) If $\widehat{A}_l \in k$ and $\nu_{l-1} = \infty$ then $\nu_l = \infty$ and $\widehat{A}_{l+1} \in k$ with $\tilde{z}_{l-1}(T) \in k((T))$ and $B_l(\nu) \in k$ for all $\nu \in \mathbb{Z}$.

(4*) If $\widehat{A}_l \in k$ and $\nu_{l-1} + m_1^{(l-1)} > m_2^{(l-1)}$ then $l < \lambda$ with $\widehat{A}_{l+1} \in k$ and $\nu_l + m_2^{(l-1)} = \nu_{l-1} + m_1^{(l-1)}$ with $\bar{A}_l = \theta\, \bar{A}_{l-1} + \theta'$.

(5*) If $\widehat{A}_l \in k$ and $\nu_{l-1} + m_1^{(l-1)} < m_2^{(l-1)}$ then $z_l^\flat \neq 0$ and $\nu_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = \nu_{l-1} + e_{l-1} \equiv 0 \mod (e_l)$.

(6*) If $\widehat{A}_l \in k$ and $\nu_{l-1} \neq \infty$ with $\nu_{l-1} + m_1^{(l-1)} = m_2^{(l-1)}$ then $l - \lambda \neq 0 \neq z_l^\flat$ and $\nu_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and $e_l^\flat = \nu_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l)$.

(7*) If $\widehat{A}_{j+1} \in k$ and $\nu_j = 0$ then $\mathrm{inco}_T z_j = \theta\, \bar{A}_j \in K \setminus k$ and

$$\frac{\mathrm{inco}_T z_l^{\flat\flat}}{\mathrm{inco}_T z_l} = \theta\, \bar{A}_j^E \quad \text{with} \quad E = (-1)^{l+1-j}(e_{j+1}/e_l) \in \mathbb{Z}^\times.$$

Proof of (II)(1).* Now if $\widehat{A}_{j+1} \in k$ with $\nu_j > 0$ and $l = \epsilon(\iota(j))$ with $\nu_j + m_1^{(l-1)} > m_2^{(l-1)}$ then by (III)(1*) we get $\widehat{A}_l \in k$ with $\nu_{l-1} = \nu_j$ and also $\nu_{l-1} + m_1^{(l-1)} > m_2^{(l-1)}$ with $\bar{A}_{l-1} = \theta\, \bar{A}_j + \theta'$ and hence by (III)(4*) we conclude that $l < \lambda$ with $\widehat{A}_{l+1} \in k$ and $\nu_l = \nu_j + m_1^{(l-1)} - m_2^{(l-1)} > 0$ with $\bar{A}_l = \theta\, \bar{A}_{l-1} + \theta'$.

Proof of (II)(2).* Now if $\widehat{A}_{j+1} \in k$ with $\nu_j > 0$ and $l = \epsilon(\iota(j))$ with $\nu_j + m_1^{(l-1)} < m_2^{(l-1)}$ then by (III)(1*) we get $\widehat{A}_l \in k$ with $\nu_{l-1} + m_1^{(l-1)} < m_2^{(l-1)}$ and $\nu_{l-1} = \nu_j$ with $\bar{A}_{l-1} = \theta\, \bar{A}_j + \theta'$ and hence by (III)(5*) we conclude that $z_l^\flat \neq 0$ and $\nu_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and

$$e_l^\flat = \nu_{l-1} + e_{l-1} \equiv 0 \mod (e_l).$$

Proof of (II)(3).* Now if $\widehat{A}_{j+1} \in k$ with $\infty \neq \nu_j > 0$ and $l = \epsilon(\iota(j))$ with $\nu_j + m_1^{(l-1)} = m_2^{(l-1)}$ then by (III)(1*) we get $\widehat{A}_l \in k$ with $\nu_{l-1} + m_1^{(l-1)} = m_2^{(l-1)}$ and $\nu_{l-1} = \nu_j$ with $\bar{A}_{l-1} = \theta\, \bar{A}_j + \theta'$ and hence by (III)(6*) we conclude that $l - \lambda \neq 0 \neq z_l^\flat$ and $\nu_l^\flat = 0$ with $\bar{A}_l^\flat = \bar{A}_{l-1}$ and

$$e_l^\flat = \nu_{l-1} + e_{l-1} \not\equiv 0 \mod (e_l).$$

Proof of (II)(4)*. Assuming $\widehat{A}_{j+1} \in k$ with $\mu^*(j) = \infty$, and given any j' with $j \le j' \le \lambda - 1$, by induction on $\mu(j, j')$ we shall show that $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ with $v_{j'+1} = v_{j'} = \infty$ and $\tilde{z}_{j'}(T) \in k((T))$ with $B_{j'+1}(\nu) \in k$ for all $\nu \in \mathbb{Z}$. In the ground case we are done by (III)(1*) and (III)(3*). So let $\mu(j, j') > 1$ and assume true for all smaller values of $\mu(j, j')$. Now letting $j_1 = \epsilon(\iota(j))$ and $j'' = j_1 - 1$ we have $j \le j'' < j'' + 1 = j_1 \le j' \le \lambda - 1$ with (i) $\mu(j, j'') = 1$ and (ii) $\mu(j_1, j') = \mu(j, j') - 1$. In view of (i), by (III)(1*) and (III)(3*) we get (iii) $\widehat{A}_{j_1+1} \in k$ and (iv) $\mu^*(j_1) = \infty$. In view of (ii) to (iv) we are done by the induction hypothesis.

Note on proofs of (II)(5*)–(II)(7*). In the following arguments we may tacitly use (I) together with the fact that for $1 \le j \le \lambda - 1$ we have $m_0^{(j)} = q_0(m^{(j)})$ and $m_\mu^{(j)} = q_1(m^{(j)}) + \cdots + q_\mu(m^{(j)})$ for $1 \le \mu \le h(m^{j)}) + 1$. This is particularly relevant for comparing $\mu(j)$ and $\mu(j')$ with $j \ne j'$. Similarly for $\mu^*(j)$ and $\mu^*(j')$.

Proof of (II)(5)*. Assume that $\widehat{A}_{j+1} \in k$ with $\infty \ne \mu^*(j) > 0$, and let us put $l = \epsilon(\iota(j) + \mu(j) - 1)$.

In case of $\mu(j) = 1$ everything follows from (III)(1*) and (III)(5*).

In the general case, given any j' with $j \le j' \le l - 1$, by induction on $\mu(j, j')$ we shall show that $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ and $1 \le \mu(j, j') \le \mu(j, l - 1) = \mu(j)$ with $\mu(j, j') + \mu(j') = 1 + \mu(j)$ and $\infty \ne \mu^*(j') = \mu^*(j) > 0$ with $\bar{A}_{j'} = \theta \bar{A}_j + \theta'$. In the ground case we are done by (III)(1*). So let $\mu(j, j') > 1$ and assume true for all smaller values of $\mu(j, j')$. Now upon letting $j_1 = \epsilon(\iota(j))$ and $j'' = j_1 - 1$ we see that $j \le j'' < j'' + 1 = j_1 \le j' \le \lambda - 1$ with (i) $\mu(j, j'') = 1$ and (ii) $\mu(j_1, j') = \mu(j, j') - 1$. Assuming $\mu(j) > 1$, in view of (i), by (III)(1*) and (III)(4*) we also conclude that (iii) $\widehat{A}_{j_1+1} \in k$ and (iv) $\infty \ne \mu^*(j_1) > 0$ and (v) $\iota(j_1) + \mu(j_1) = \iota(j) + \mu(j)$. In view of (ii) to (v) we are done by the induction hypothesis.

In view of what we have proved in the above paragraph, by (III)(5*) we get the "moreover" and the "finally."

Proof of (II)(6)*. Assume that $\widehat{A}_{j+1} \in k$ with $\mu^*(j) = 0$ and $\mu(j) \ne 1$, and let us put $l = \epsilon(\iota(j) + \mu(j) - 2)$.

In case of $\mu(j) = 2$ everything follows from (III)(1*) and (III)(6*).

In the general case, given any j' with $j \le j' \le l - 1$, by induction on $\mu(j, j')$ we shall show that $\{\widehat{A}_{j'}, \widehat{A}_{j'+1}\} \subset k$ and $1 \le \mu(j, j') \le \mu(j, l - 1) = \mu(j) - 1$ with $\mu(j, j') + \mu(j') = 1 + \mu(j)$ and $\mu^*(j') = \mu^*(j) = 0$ with $\bar{A}_{j'} = \theta \bar{A}_j + \theta'$. In the ground case we are done by (III)(1*). So let $\mu(j, j') > 1$ and assume true for all smaller values of $\mu(j, j')$. Now upon letting $j_1 = \epsilon(\iota(j))$ and $j'' = j_1 - 1$ we see that $j \le j'' < j'' + 1 = j_1 \le j' \le \lambda - 1$ with (i) $\mu(j, j'') = 1$ and (ii) $\mu(j_1, j') = \mu(j, j') - 1$. Assuming $\mu(j) > 2$, in view of (i), by (III)(1*) and (III)(4*) we also conclude that (iii) $\widehat{A}_{j_1+1} \in k$ and (iv) $\mu^*(j_1) = 0$ and (v) $\iota(j_1) + \mu(j_1) = \iota(j) + \mu(j)$. In view of (ii) to (v) we are done by the induction hypothesis.

In view of what we have proved in the above paragraph, by (III)(6*) we get the "moreover."

Proof of (II)(7).* This follows from (II)(6*) and (III)(7*). In greater detail, the case of $\mu(j) = 1$ is done by (III)(7*). So assume that $\mu(j) \neq 1$ and let

$$(z'_0, z'_1) = (z^\flat_L, z_L) \quad \text{where} \quad L = \epsilon(\iota(j) + \mu(j) - 2) \tag{$'$}$$

and let

$$(z'_J, e'_J, p'_J, B'_J(\nu), \epsilon'(i), \iota'(J))_{\nu\in\mathbb{Z}, 0\le J\le\lambda', 0\le i\le\kappa'} \tag{$\bullet\bullet'$}$$

be the (V, K)-expansion of (z'_0, z'_1). Also let

$$(z'_J, e'_J, p'_J, B'_J(\nu), \epsilon'(i), \iota'(J), A'_J(\nu), \widehat{A}'_J, \dots)_{\nu\in\mathbb{Z}, 0\le J\le\lambda', 0\le i\le\kappa'} \tag{$\bullet\bullet\bullet'$}$$

be the mixed (V, K, T)-expansion of (z'_0, z'_1), and let

$$v'_J, \bar{A}'_J, (z'_J)^{\flat\flat}, \dots$$

have the corresponding meanings. Then assuming

$$\widehat{A}_{j+1} \in k \quad \text{with} \quad \mu^*(j) = 0$$

by (II)(6*) we see that

$$\bar{A}'_0 = \theta\, \bar{A}_j + \theta\,' \quad \text{with} \quad e'_0 \not\equiv 0 \mod (e'_1) \tag{i}$$

and

$$\widehat{A}'_1 \in k \quad \text{with} \quad v'_0 = 0. \tag{ii}$$

In view of (i) and (ii), upon letting

$$l' = \epsilon'(\iota'(0)) \tag{iii}$$

and applying (III)(7*) with $j = 0$ to the "primed" system we see that

$$\text{inco}_T z'_0 = \theta\, \bar{A}'_0 \in K \setminus k \tag{iv}$$

and

$$\frac{\text{inco}_T (z'_{l'})^{\flat\flat}}{\text{inco}_T z'_{l'}} = \theta\, (\bar{A}'_0)^{E'} \text{ with } E' = (-1)^{l'+1}(e'_1/e'_{l'}) \in \mathbb{Z}^\times. \tag{v}$$

Now clearly

$$z^\flat_L = z_L^{p_L} z_{L+1}. \tag{vi}$$

In view of (vi), upon letting

$$l = \epsilon(\iota(j) + \mu(j) - 1) \tag{vii}$$

we see that

$$z'_{l'} = z_l \quad \text{and} \quad (z'_{l'})^{\flat\flat} = z_l^{\flat\flat} \quad \text{with} \quad E' = n^{\flat\flat}_{\mu(j)}(m^{(j)}).$$

By (i)–(vii) we conclude that

$$\bar{A}_j \in K \setminus k \tag{i*}$$

and

$$\frac{\text{inco}_T z_l^{\flat\flat}}{\text{inco}_T z_l} = \theta(\bar{A}_j + \theta')^E \text{ with } E = n^{\flat\flat}_{\mu(j)}(m^{(j)}) \in \mathbb{Z}^\times. \tag{ii*}$$

Note on the proof of (II)(7*). To get a clearer picture of the above proof remember that, as explained in the Descriptive Note (8‡), the (V, K)-sequence (••) is obtained by straightening the (V, K)-presequence (•), and while doing this we drop the first element of each row, except the first; the dropped element is reinstated by the concept of $z_l^\flat$ where we observe that $z_l^\flat = z_l^{p_l} z_{l+1}$. Also remembering (3.1)(8) and (3.1)(11) we observe that

$$\frac{\text{inco}_T z_l^{\flat\flat}}{\text{inco}_T z_l} = A_l^*(e_{l-1}/|e_l|) = \text{the first coefficient of the summation in (3.1)(8).}$$

At any rate, (••′) is obtained by chopping off the initial $0 \le j \le L-1$ piece of (••) and replacing the chopped off piece by $z_L^\flat = z_L^{p_L} z_{L+1}$. Finally observe that the $j = 0$ case of (II)(7*) requires special treatment which is taken care of in (II)(6**).

5 Quadratic Transformations

For details referring to [2, 3, 9–11] in general, and specifically to (Q35.8) on pages 569–577 of [12], let us recall some basic facts about QDTs = Quadratic Transformations.

Recall that, spec(S) denotes the set of all prime ideals in a ring S. If S is a domain then the modelic spec $\mathfrak{V}(S)$ = the set of all localizations of S at various prime ideals in S, and if J is an ideal in S then the modelic blowup

$$\mathfrak{W}(S, J) = \bigcup_{0 \ne x \in J} \mathfrak{V}(S[Jx^{-1}])$$

where $Jx^{-1} = \{yx^{-1} : y \in J\}$; if S is quasilocal then the dominating modelic blowup $\mathfrak{W}(S, J)^\Delta$ = the set of all those members of $\mathfrak{W}(S, J)$ which dominate S.

Let R be a positive dimensional local domain. By a QDT of R we mean a member of $\mathfrak{W}(R, M(R))^\Delta$. For any QDT S of R we have $0 < \dim(S) \le \dim(R)$ with $\dim(R) - \dim(S) = \text{restrdeg}_R S$, and $S/M(S)$ is a finitely generated field extension of $R/M(R)$. We have $\dim(S) = 1$ for at least one and at most a finite number of QDTs S of R. If R is regular then every QDT S of R is regular, and $\dim(S) = 1$

for exactly one S which then coincides with the valuation ring of the real discrete valuation ord_R mentioned in Sect. 2, and hence in particular it is residually pure transcendental over R. Some QDT of R coincides with R iff R is a DVR. If V is any valuation ring dominating R then V dominates exactly one QDT S of R, and we call S the QDT of R along V.

A QDT of a positive dimensional local domain R may also be called a first QDT of R; by a second QDT of R we mean a first QDT of a first QDT of $R, \ldots$, by a j-th QDT of R we mean a first QDT of a $(j-1)$-th QDT of R. We declare R to be the only zeroeth QDT of R. By a QDT *sequence* of R we mean a sequence $(R_j)_{0\le j<\infty}$ with $R_0 = R$ such that R_j is a first QDT of R_{j-1} for $0 < j < \infty$.

If V is any valuation ring dominating a positive dimensional local domain R then, for any nonnegative integer j, there is a unique j-th QDT R_j of R which is dominated by V and we call it the j-th QDT of R along V; we call $(R_j)_{0\le j<\infty}$ the QDT *sequence* of R along V. To get a concrete set of generators of $M(R_j)$ for all j, we proceed thus.

Definition (♮). Let V be the valuation ring of a valuation $W : L \to G \cup \{\infty\}$ of a field L and let K be a coefficient set of V. Let

$$\overline{L} = \{z \in L : W(z) = 0 \text{ or } \infty\}.$$

Given any $(z_0, \ldots, z_\tau) \in L^{\tau+1} \setminus \overline{L}^{\tau+1}$ where τ is a positive integer, we shall define its QDT sequence $(0^\sharp)$ along (V, K). The *reader may prefer* to first study the $\tau = 1$ case starting in Note (III*). Now clearly there exists a unique sequence

$$(z_{0j}, \ldots, z_{\tau j}, c_{0j}, \ldots, c_{\tau j}, t(j))_{0\le j<\infty} \tag{$0^\sharp$}$$

with $(z_{00}, \ldots, z_{\tau 0}) = (z_0, \ldots, z_\tau)$ and

$$(z_{0j}, \ldots, z_{\tau j}, c_{0j}, \ldots, c_{\tau j}, t(j)) \in (L^{\tau+1} \setminus \overline{L}^{\tau+1}) \times K^{\tau+1} \times \{0, \ldots, \tau\}$$

for $0 \le j < \infty$ such that for $0 \le j < \infty$ and $0 \le t \le \tau$ we have

$$z_{t,j+1} = \begin{cases} z_{tj} \text{ with } c_{tj} = 1 & \text{if } t = t(j) \\ \overline{z}_{tj} \text{ with } c_{tj} = 0 & \text{if } t \ne t(j) \text{ and } W(\overline{z}_{tj}) \ne 0 \\ \overline{z}_{tj} - c_{tj} \in M(V) \text{ with } c_{tj} \ne 0 & \text{if } t \ne t(j) \text{ and } W(\overline{z}_{tj}) = 0 \end{cases} \tag{$1^\sharp$}$$

where

$$\overline{z}_{tj} = \begin{cases} \frac{z_{tj}}{z_{t(j)j}} & \text{if } 0 < W(z_{t(j)j}) \le W(z_{tj}) \\ \frac{z_{tj}}{z_{t(j)j}} & \text{if } W(z_{t(j)j}) < 0 > W(z_{tj}) \\ \frac{z_{tj}}{1/z_{t(j)j}} & \text{if } W(z_{tj}) < 0 < W(z_{t(j)j}) \\ \frac{z_{tj}}{1/z_{t(j)j}} & \text{if } W(z_{t(j)j}) < 0 < |W(z_{t(j)j})| \le W(z_{tj}) \\ z_{tj} & \text{if } 0 = W(z_{tj}) < |W(z_{t(j)j})| \end{cases} \tag{$2^\sharp$}$$

and where, upon letting

$$\begin{cases} t^*(j) = \{0 \le t \le \tau : 0 \ne W(z_{tj}) \ne \infty\} \\ t^{**}(j) = \max(t^*(j)) \\ t_+^*(j) = \{t \in t^*(j) : 0 < W(z_{tj}) < \infty\} \\ t_+^{**}(j) = \max\{t \in t_+^*(j) : W(z_{tj}) \le W(z_{t'j}) \ \forall\, t' \in t_+^*(j)\} \\ t_-^*(j) = \{t \in t^*(j) : t \ge t_+^{**}(j)\} \\ t_-^{**}(j) = \max\{t \in t_-^*(j) : |W(z_{tj})| \le |W(z_{t'j})| \ \forall\, t' \in t_-^*(j)\} \end{cases} \tag{$3^\sharp$}$$

with the understanding that if $t_+^*(j) = \emptyset$ then $t_+^{**}(j) = 0 = t_-^{**}(j)$, we put

$$t(j) = \begin{cases} t_-^{**}(j) & \text{if } t_+^*(j) \ne \emptyset \\ t^{**}(j) & \text{if } t_+^*(j) = \emptyset. \end{cases} \tag{$4^\sharp$}$$

Noting that for all $j \in \mathbb{N}$ we have $0 \ne W(z_{t(j)j}) < \infty$, for $0 \le t \le \tau$ we put

$$\pi(t,j) = \begin{cases} 1 & \text{if } 0 < W(z_{t(j)j}) \le W(z_{tj}) \\ 1 & \text{if } W(z_{(j)j}) < 0 > W(z_{tj}) \\ -1 & \text{if } W(z_{tj}) < 0 < W(z_{t(j)j}) \\ -1 & \text{if } W(z_{t(j)j}) < 0 < |W(z_{t(j)j})| \le W(z_{tj}) \\ 0 & \text{if } 0 = W(z_{tj}) < |W(z_{t(j)j})| \end{cases} \tag{$5^\sharp$}$$

and we observe that $z_{t(j)j}^{\pi(t,j)}$ is the denominator in each line of ($2^\sharp$).

Let us define the flipping set $\Phi^\sharp$ of ($0^\sharp$) by putting

$$\Phi^\sharp = \text{the set of all } j \in \mathbb{N}_+ \text{ such that } t(j-1) \ne t(j). \tag{$6^\sharp$}$$

Let $p(u)_{1 \le u < \widehat{\lambda}}$ be the unique sequence such that $\{p(u) : 1 \le u < \widehat{\lambda}\} = \Phi^\sharp$ with $p(u) < p(u+1)$ whenever $1 \le u < u+1 < \widehat{\lambda}$ where $\widehat{\lambda} = \infty$ or $\operatorname{card}(\Phi^\sharp) + 1$ according as the cardinality $\operatorname{card}(\Phi^\sharp)$ is infinite or finite.

Let us define the translation set $\Psi^\sharp$ of ($0^\sharp$) by putting

$$\Psi^\sharp = \begin{cases} \text{the set of all } j \in \mathbb{N} \text{ such that} \\ \text{for every } t \in \{0, \dots, \tau\} \text{ with } z_{tj} \ne 0 \text{ we have} \\ \dfrac{z_{tj}}{z_{t(j)j}^{n(t,j)}} \in V \setminus M(V) \text{ for some } n(t,j) \in \mathbb{Z} \end{cases} \tag{$7^\sharp$}$$

and let us note that this defines $n(t,j)$ uniquely. Let $u(i)_{0 \le i < \widehat{\kappa}}$ be the unique sequence such that $\{p(u(i)) : 0 \le i < \widehat{\kappa}\} = \Phi^\sharp \cap \Psi^\sharp$ with $u(i) < u(i+1)$ whenever

$0 \leq i < i+1 < \widehat{\kappa}$ where $\widehat{\kappa} = \infty$ or $\mathrm{card}(\Phi^\sharp \cap \Psi^\sharp)$ according as the cardinality $\mathrm{card}(\Phi^\sharp \cap \Psi^\sharp)$ is infinite or finite.

We call $(0^\sharp)$ the QDT *sequence* of $(z_0, \ldots, z_\tau)$ along (V, K) and we call

$$(\pi(0, j), \ldots, \pi(\tau, j), p(u), u(i))_{0 \leq j < \infty, 1 \leq u < \widehat{\lambda}, 0 \leq i < \widehat{\kappa}} \tag{$8^\sharp$}$$

the *supplement* of the QDT sequence.

Note (I^*). The proofs of the following Lemmas (I) and (II) are straightforward. Lemma (II) deals with a situation when $(z_{0j}, \ldots, z_{\tau j})$ are generators of the maximal ideal $M(R_j)$ of a local domain R_j dominated by V; in that situation clearly j belongs to $N^\sharp$ where $N^\sharp = \{j \in \mathbb{N} : W(z_{tj}) > 0 \text{ for } 0 \leq t \leq \tau\}$. Note that if $j \in N^\sharp$ then only the first line of $(2^\sharp)$ is relevant. Also note that:

(i) $j \in N^\sharp$ for a certain value of j implies $j \in N^\sharp$ for all bigger values of j.
(ii) $t_+^*(j) \neq \emptyset$ for a certain value of j implies $t_+^*(j) \neq \emptyset$ for all bigger values of j.
(iii) If W is real, i.e., if the value group G_W is order isomorphic to an additive subgroup of $\mathbb{R}$ then $j \in N^\sharp$ for all sufficiently large values of j.
(iv) If $j < j^*$ in $\mathbb{N} \cup \{\infty\}$ are such that $t(j) = t(j')$ for all $j \leq j' < j^*$ then $z_{t(j)j} = z_{t(j')j'}$ whenever $j \leq j' < j^*$.

Finally note that by definition

$$|W(z)| = W(z) \text{ or } -W(z) \text{ according as } W(z) \geq 0 \text{ or } W(z) < 0.$$

Lemma (I). *Let $j \in \Phi^\sharp$ and $j < j^* \in \mathbb{N}_+ \cup \{\infty\}$ be such that for all $j' \in \mathbb{N}$ with $j < j' < j^*$ we have $j' \notin \Phi^\sharp$, and if $j^* \neq \infty$ then we have $j^* \in \Phi^\sharp$. Then we have the following.*

(I.1) For all $j' \in \mathbb{N}$ with $j \leq j' < j^*$ we have $t(j') = t(j)$ and $z_{t(j')j'} = z_{t(j)j}$. If $j^* \neq \infty$ then we have $t(j^*) \neq t(j)$.
(I.2) If $j^* = \infty$ then we have $1 < \widehat{\lambda} < \infty$ and $p(\widehat{\lambda} - 1) = j$. If $j^* \neq \infty$ then for a unique integer u with $1 \leq u < u+1 < \widehat{\lambda}$ we have $p(u) = j < j^* = p(u+1)$.
(I.3) Assume $j \notin \Psi^\sharp$. Then for $0 \leq i < \widehat{\kappa}$ we have $j \neq p(u(i))$. Moreover, either: for all $t \in \{0, \ldots, \tau\} \setminus \{t(j)\}$ we have $z_{tj} = 0$, or: for some $t \in \{0, \ldots, \tau\} \setminus \{t(j)\}$ we have $z_{tj} \neq 0$ with $z_{tj}/z_{t(j)j}^n \notin V \setminus M(V)$ for all $n \in \mathbb{Z}$. In the "either" case, for all $t \in \{0, \ldots, \tau\} \setminus \{t(j)\}$ and for all $j' \in \mathbb{N}$ with $j \leq j' < j^*$, we have $c_{tj'} = 0 = z_{tj'}$. Furthermore, for every t of the "or" case and for all $j' \in \mathbb{N}$ with $j \leq j' < j^*$, we have

$$c_{tj'} = 0 \quad \text{with} \quad \pi(t, j') = \pi(t, j) \quad \text{and} \quad z_{tj} = z_{t(j)j}^{\pi(t,j)(j'-j)} z_{tj'}.$$

(I.4) Assume $j \in \Psi^\sharp$. Then $j = p(u(i))$ for a unique i with $0 \leq i < \widehat{\kappa}$. Moreover, if $j^* = \infty$ then for any $t \in \{0, \ldots, \tau\} \setminus \{t(j)\}$, whereas if $j^* \neq \infty$ then for $t = t(j^*)$, for all $j' \in \mathbb{N}$ with $j \leq j' < j^*$ we have

$$z_{tj} - \sum_{j \le \nu < j'} c_{t\nu} z_{t(j)j}^{\pi(t,j)(\nu-j+1)} = z_{t(j)j}^{\pi(t,j)(j'-j)} z_{tj'} \text{ with } \pi(t,j') = \pi(t,j) \quad \text{(i)}$$

which may be viewed as a Taylor Expansion with Remainder discussed in (9.5). If $j^* = \infty$ then (i) gives rise to the equation

$$z_{tj} = \sum_{j \le \nu < j^*} c_{t\nu} z_{t(j)j}^{\pi(t,j)(\nu-j+1)} \quad \text{(ii)}$$

which may be thought of as an infinite Taylor Expansion discussed in (9.2), with a suitable interpretation of the equality; see (9.3) for the case when V is a DVR.

(I.5) Assuming $j \in \Psi^\sharp$ and letting $\overline{\nu} = \{\nu \in \mathbb{N} : j \le \nu < j^* \text{ with } c_{t\nu} \ne 0\}$ we have the following. If $j^* = \infty$ and $t \in \{0, \dots, \tau\} \setminus \{t(j^*)\}$ then letting $\nu_1 < \cdots < \nu_w$ or $\nu_1 < \nu_2 < \dots$ be the finitely many or infinitely many values of $\nu \in \overline{\nu}$ and putting $\nu_0 = j - 1$ we have

$$n(t, \nu_\nu + 1) = \nu_{\nu+1} - \nu_\nu$$

for $0 \le \nu < w$ or $0 \le \nu < \infty$ respectively. If $j^* \ne \infty$ and $t = t(j^*)$ then letting $\nu_1 < \cdots < \nu_w$ be the values of $\nu \in \overline{\nu}$ and putting $\nu_0 = j - 1$ we have

$$n(t, \nu_\nu + 1) = \nu_{\nu+1} - \nu_\nu$$

for $0 \le \nu < w$.

Sketch proof of (I.5). Letting

$$X = z_{t(j)j} \text{ and } (N(q), Z(q)) = (n(t,q), z_{tq})$$

for all $q \in \mathbb{N}$, we have

$$Z(q) = C(q)X^{N(q)} + X^{N(q)}Z(q + N(q)) \text{ where } 0 \ne C(q) \in K. \quad \text{[q]}$$

Comparing $[\nu_0 + 1]$ and (i) with $j' = \nu_1 + 1$ we see that

$$N(\nu_0 + 1) = \nu_1 - \nu_0 \text{ and } C(\nu_0 + 1) = c_{t\nu_1}$$

with

$$Z(\nu_0 + 1) = c_{t\nu_1} X^{\nu_1 - \nu_0} + X^{\nu_1 - \nu_0} Z(\nu_1 + 1).$$

Substituting $[\nu_1 + 1]$ in the last equation and comparing the resulting equation and (i) with $j' = \nu_2 + 1$ we see that

$$N(\nu_1 + 1) = \nu_2 - \nu_1 \text{ and } C(\nu_1 + 1) = c_{t\nu_2}$$

with

$$Z(\nu_0 + 1) = c_{t\nu_1} X^{\nu_1 - \nu_0} + c_{t\nu_2} X^{\nu_2 - \nu_1} + X^{\nu_2 - \nu_1} Z(\nu_2 + 1).$$

And so on. Thus by induction on v we get

$$N(\nu_v + 1) = \nu_{v+1} - \nu_v \text{ and } C(\nu_v + 1) = c_{\nu_{v+1}}$$

with

$$Z(\nu_0{+}1){=}c_{t\nu_1} X^{\nu_1-\nu_0}{+}c_{t\nu_2} X^{\nu_2-\nu_1}{+}\cdots{+}c_{t\nu_{v+1}} X^{\nu_{v+1}-\nu_v}{+}X^{\nu_{v+1}-\nu_v} Z(\nu_{v+1}{+}1)$$

for all relevant values of v.

Lemma (II). *Assume that V dominates a positive dimensional local domain R for which $M(R) = (z_0, \ldots, z_\tau)R$. Let $(R_j)_{0 \le j < \infty}$ be the QDT sequence of R along V. Then we have the following.*

(II.1) If $n \in \mathbb{N}$ is such that K contains a coefficient set K_j of R_n for $0 \le j \le n$, then for $0 \le j \le n$ we have

$$\{c_{0j}, \ldots, c_{\tau j}\} \in K_j = K \cap R_j \text{ and } M(R_j) = (z_{0j}, \ldots, z_{\tau j})R_j.$$

(II.2) If V is a DVR then $\widehat{\lambda} \in \mathbb{N}_+$ and for all integers $n \ge \widehat{\lambda}$ we have $t(n) = t(\widehat{\lambda})$. If V is a DVR and $\mathrm{QF}(R) = \mathrm{QF}(V)$ then $\widehat{\lambda} \in \mathbb{N}_+$ and for all integers $n \ge \widehat{\lambda}$ we have $t(n) = t(\widehat{\lambda})$ and $W(z_{t(n)n}) = 1$.

(II.3) If R is regular of dimension $\tau + 1$ and V is a prime divisor of R then there exists a unique positive integer n such that for all integers $0 \le j < n \le \mu$ we have $R_j \ne R_n = R_\mu = V$ and $\dim(R_j) > \dim(R_n) = \dim(R_\mu) = 1$. Moreover, R_n is residually pure transcendental over R_{n-1} of residual transcendence degree $\dim(R_{n-1}) - 1$. Finally n is the essential length of the QDT sequence $(R_j)_{0 \le j < \infty}$ in the sense of Note (II**) below.

(II.4) If R is one dimensional and V is a prime divisor of R then V is residually finite algebraic over R and there exists $n \in \mathbb{N}$ such that for all integers $\mu \ge n$ we have $M(V) = M(R_\mu)V$ with $V/M(V) = R_\mu/M(\mu)$.

Note (II).* For (II.3) see Proposition 3 of [2] and its proof. The first part of (II.4) is proved in Theorem 1(4) of [2], and the rest of (II.4) follows from it by (II.2). It may be tempting to think that (II.4) implies $V = R_\mu$ for large μ, but Example (E3.2) on page 206 of Nagata [28] shows this to be untrue.

Note (II).* Given any positive dimensional local domain R and any QDT sequence $(R_j)_{0 \le j < \infty}$ of R, by the *essential length* of the QDT sequence we mean the unique $n \in \mathbb{N} \cup \{\infty\}$ such that if $n = \infty$ then for all $j \in \mathbb{N}$ we have $R_j \ne R_{j+1}$, whereas if $n \in \mathbb{N}$ then for all $j \in \mathbb{N}$ with $j < n$ we have $R_j \ne R_{j+1}$ and for all $j \in \mathbb{N}$ with $j \ge n$ we have $R_j = R_{j+1}$. Note that $R_j = R_{j+1}$ iff R_j is a DVR.

Lemma (III). *Assume that $\tau = 1$ and V dominates a two dimensional regular local domain R with quotient field L and $M(R) = (z_0, z_1)R$. Let $(R_j)_{0\le j<\infty}$ be the QDT sequence of R along V. Then we have the following.*

(III.1) The essential length of the QDT sequence $(R_j)_{0\le 0<\infty}$ is finite or infinite according as V is residually transcendental or algebraic over R.
(III.2) If V is residually transcendental over R then V is a prime divisor of R.
(III.3) Assume that V is residually algebraic over R. Then the value group G_W is order isomorphic to either (i) the set of all lexicographically ordered pairs of integers or (ii) the additive group of all integers or (iii) a noncyclic additive subgroup of $\mathbb{Q}$ or (iv) an additive subgroup of $\mathbb{R}$ of the form $\{a_1\pi_1 + a_2\pi_2 : (a_1, a_2) \in \mathbb{Z}^2\}$ for some positive real numbers π_1, π_2 which are linearly independent over $\mathbb{Q}$. In these cases, we shall respectively say that V is nonreal discrete or real discrete or rational nondiscrete or irrational. Now assume that K contains a coefficient set K_j of R_j for all $j \in \mathbb{N}$. Then:

(i*) $\mathrm{card}(\Phi^\sharp) \neq \infty \neq \mathrm{card}(\Psi^\sharp)$ iff V is nonreal discrete;
(ii*) $\mathrm{card}(\Phi^\sharp) \neq \infty = \mathrm{card}(\Psi^\sharp)$ iff V is real discrete;
(iii*) $\mathrm{card}(\Phi^\sharp) = \infty = \mathrm{card}(\Psi^\sharp)$ iff V is rational nondiscrete;
(iv*) $\mathrm{card}(\Phi^\sharp) = \infty \neq \mathrm{card}(\Psi^\sharp)$ iff V is irrational.

Proof. In view of Lemma (II) this follows from [2, 3].

Note (III).* In the next two Lemmas we continue to give special attention to the $\tau = 1$ case. Here we make some definitions for that case. For any nonnegative integer j we let $t'(j)$ be the unique member of $\{0, 1\}$ different from $t(j)$. By the *quadratic expansion* of any $(z_0, z_1) \in L^2 \setminus \overline{L}^2$ along (V, K) we mean the sequence

$$(z_{0j}, z_{1j}, c_{0j}, c_{1j}, t(j), t'(j))_{0\le j<\infty} \tag{9$^\sharp$}$$

where $(z_{0j}, z_{1j}, c_{0j}, c_{1j}, t(j))_{0\le j<\infty}$ is the $\tau = 1$ version of (0$^\sharp$); moreover, by the *supplement* of the quadratic expansion we mean the $\tau = 1$ version of (8$^\sharp$), i.e.,

$$(\pi(0, j), \pi(1, j), p(u), u(i))_{0\le j<\infty, 1\le u<\widehat{\lambda}, 0\le i<\widehat{\kappa}}. \tag{10$^\sharp$}$$

Since the euclidean algorithm played a crucial role in it, the (V, K)-expansion

$$(z_j, e_j, p_j, B_j(\nu), \epsilon(i), \iota(j))_{\nu\in\mathbb{Z}, 0\le j\le\lambda, 0\le i\le\kappa} \tag{$\bullet\bullet$}$$

introduced in (4.1) is called the *euclidean expansion* of (z_0, z_1) along (V, K), and ($\bullet\bullet\bullet$) is called the *mixed euclidean expansion* of (z_0, z_1) along (V, K, T). In Lemma (IV) we shall give a *stand alone description* of the quadratic expansion. In Lemma (V) we shall restate the $\tau = 1$ case of Lemma (I). In Part II, we shall compare the quadratic expansion with the euclidean expansion.

Lemma (IV). *Assuming $\tau = 1$, for the quadratic expansion* (9$^\sharp$) *of* (z_0, z_1) *along* (V, K) *with* $(z_{0j}, z_{1j}) \in L^2 \setminus \overline{L}^2$ *for* $0 \le j < \infty$, *we have the following.*

(IV.1) Recalling that for every $j \in \mathbb{N}$ we have $(z_{0j}, z_{1j}) \in L^2 \setminus \overline{L}^2$, we can paraphrase the characterizations $(3^\sharp)$–$(5^\sharp)$ of $t(j)$ and $\pi(t, j)$ by saying that $t(j) \in \{0, 1\}$ with $z_{t(j)j} \notin \overline{L}$ and with $\pi(t(j), j) = 1$ with $\pi(t'(j), j) \in \{0, 1, -1\}$ satisfy (1)–(8) stated below.

(1) If $0 < W(z_{1j}) \leq W(z_{0j})$ then $t(j) = 1$ and $\pi(t'(j), j) = 1$.
(2) If $0 < W(z_{0j}) < W(z_{1j})$ then $t(j) = 0$ and $\pi(t'(j), j) = 1$.
(3) If $W(z_{1j}) < 0 > W(z_{0j})$ then $t(j) = 1$ and $\pi(t'(j), j) = 1$.
(4) If $W(z_{1j}) > 0 > W(z_{0j})$ then $t(j) = 1$ and $\pi(t'(j), j) = -1$.
(5) If $W(z_{1j}) < 0 < -W(z_{1j}) \leq W(z_{0j})$ then $t(j) = 1$ and $\pi(t'(j), j) = -1$.
(6) If $W(z_{1j}) < 0 < W(z_{0j}) < -W(z_{1j})$ then $t(j) = 0$ and $\pi(t'(j), j) = -1$.
(7) If $W(z_{1j}) \neq 0 = W(z_{0j})$ then $t(j) = 1$ and $\pi(t'(j), j) = 0$.
(8) If $W(z_{1j}) = 0 \neq W(z_{0j})$ then $t(j) = 0$ and $\pi(t'(j), j) = 0$.

(IV.2) Next the definitions $(1^\sharp)$ and $(2^\sharp)$ can be paraphrased by saying that for $0 \leq j < \infty$ and $0 \leq t \leq 1$ we have

$$z_{t,j+1} = \begin{cases} z_{tj} \text{ with } c_{tj} = 1 & \text{if } t = t(j) \\ \overline{z}_{tj} \text{ with } c_{tj} = 0 & \text{if } t = t'(j) \text{ and } W(\overline{z}_{tj}) \neq 0 \\ \overline{z}_{tj} - c_{tj} \in M(V) \text{ with } c_{tj} \neq 0 & \text{if } t = t'(j) \text{ and } W(\overline{z}_{tj}) = 0 \end{cases}$$

where

$$\overline{z}_{tj} = \frac{z_{tj}}{z_{t(j)j}^{\pi(t,j)}}.$$

(IV.3) To paraphrase definition $(6^\sharp)$ of the flipping set $\Phi^\sharp$, recalling that

$$\Phi^\sharp = \text{the set of all } j \in \mathbb{N}_+ \text{ such that } t(j-1) \neq t(j).$$

and

$$\widehat{\lambda} = \text{card}(\Phi^\sharp) + 1 \in \mathbb{N}_+ \cup \{\infty\}$$

we supplement the definition of $p(u)_{1 \leq u < \widehat{\lambda}}$ by the *convention*

$$p(-1) = p(0) = 0$$

and we note that now the members of $\Phi^\sharp \cup \{0\}$ are labelled as

$$p(-1) = p(0) = 0 < p(1) < p(2) < \dots \qquad \text{if } \widehat{\lambda} = \infty$$

and

$$p(-1) = p(0) = 0 < p(1) < \dots < p(\widehat{\lambda} - 1) \qquad \text{if } \widehat{\lambda} \in \mathbb{N}_+.$$

(IV.4) To paraphrase definition ($7^\sharp$) of the translation set $\Psi^\sharp$, recalling that

$$\Psi^\sharp = \begin{cases} \text{the set of all } j \in \mathbb{N} \text{ such that} \\ \text{for every } t \in \{0,1\} \text{ with } z_{tj} \neq 0 \text{ we have} \\ \frac{z_{tj}}{z_{t(j)j}^{n(t,j)}} \in V \setminus M(V) \text{ for a (unique) } n(t,j) \in \mathbb{Z} \end{cases}$$

and

$$\widehat{\kappa} = \mathrm{card}(\Phi^\sharp \cap \Psi^\sharp) \in \mathbb{N} \cup \{\infty\}$$

we supplement the definition of $u(i)_{0 \le i < \widehat{\kappa}}$ by the *convention*

$$u(-1) = 0$$

and we note that now we have the integer sequences

$$u(-1) = 0 < u(0) < u(1) < \dots \qquad \text{if } \widehat{\kappa} = \infty$$

and

$$u(-1) = 0 < u(0) < \dots < u(\widehat{\kappa} - 1) \qquad \text{if } \widehat{\kappa} \in \mathbb{N}$$

while the members of $(\Phi^\sharp \cap \Psi^\sharp) \cup \{0\}$ are labelled as

$$p(u(-1)) = 0 < p(u(0)) < p(u(1)) < \dots \qquad \text{if } \widehat{\kappa} = \infty$$

and

$$p(u(-1)) = 0 < p(u(0)) < \dots < p(u(\widehat{\kappa} - 1)) \qquad \text{if } \widehat{\kappa} \in \mathbb{N}.$$

Lemma (V). *Assume $\tau = 1$. Let $j \in \Phi^\sharp$ and $j < j^* \in \mathbb{N}_+ \cup \{\infty\}$ be such that for all $j' \in \mathbb{N}$ with $j < j' < j^*$ we have $j' \notin \Phi^\sharp$, and if $j^* \neq \infty$ then we have $j^* \in \Phi^\sharp$. Then we have the following.*

(V.1) For all $j' \in \mathbb{N}$ with $j \le j' < j^*$ we have $t(j') = t(j)$ and $z_{t(j')j'} = z_{t(j)j}$. If $j^* \neq \infty$ then we have $t(j^*) = t'(j)$.

(V.2) If $j^* = \infty$ then we have $1 < \widehat{\lambda} < \infty$ and $p(\widehat{\lambda} - 1) = j$. If $j^* \neq \infty$ then for a unique integer u with $1 \le u < u + 1 < \widehat{\lambda}$ we have $p(u) = j < j^* = p(u+1)$.

(V.3) Assume $j \notin \Psi^\sharp$. Then for $0 \le i < \widehat{\kappa}$ we have $j \neq p(u(i))$. Moreover, either: $z_{t'(j)j} = 0$, or: $z_{t'()j} \neq 0$ with $z_{t'(j)j}/z^n_{t(j)j} \notin V \setminus M(V)$ for all $n \in \mathbb{Z}$. In the "either" case, for $t = t'(j)$ and for all $j' \in \mathbb{N}$ with $j \le j' < j^*$, we have $c_{tj'} = 0 = z_{tj'}$. In the "or" case, for $t = t'(j)$ and for all $j' \in \mathbb{N}$ with $j \le j' < j^*$, we have

$$c_{tj'} = 0 \quad \text{with} \quad \pi(t, j') = \pi(t, j) \quad \text{and} \quad z_{tj} = z_{t(j)j}^{\pi(t,j)(j'-j)} z_{tj'}.$$

(V.4) Assume $j \in \Psi^\sharp$. Then $j = p(u(i))$ for a unique i with $0 \le i < \widehat{\kappa}$. Moreover, if $j^* = \infty$ then for $t = t'(j)$, whereas if $j^* \ne \infty$ then for $t = t(j^*)$, for all $j' \in \mathbb{N}$ with $j \le j' < j^*$ we have

$$z_{tj} - \sum_{j \le \nu < j'} c_{t\nu} z_{t(j)j}^{\pi(t,j)(\nu-j+1)} = z_{t(j)j}^{\pi(t,j)(j'-j)} z_{tj'} \text{ with } \pi(t,j') = \pi(t,j) \tag{i}$$

which may be viewed as a Taylor Expansion with Remainder discussed in (9.5). If $j^* = \infty$ then (i) gives rise to the equation

$$z_{tj} = \sum_{j \le \nu < j^*} c_{t\nu} z_{t(j)j}^{\pi(t,j)(\nu-j+1)} \tag{ii}$$

which may be thought of as an infinite Taylor Expansion discussed in (9.2), with a suitable interpretation of the equality; see (9.3) for the case when V is a DVR.

(V.5) Assuming $j \in \Psi^\sharp$ and letting $\overline{\nu} = \{\nu \in \mathbb{N} : j \le \nu < j^* \text{ with } c_{t\nu} \ne 0\}$ we have the following. If $j^* = \infty$ and $t = t'(j^*)$ then letting $\nu_1 < \cdots < \nu_w$ or $\nu_1 < \nu_2 < \ldots$ be the finitely many or infinitely many values of $\nu \in \overline{\nu}$ and putting $\nu_0 = j - 1$ we have

$$n(t, \nu_v + 1) = \nu_{v+1} - \nu_v$$

for $0 \le v < w$ or $0 \le v < \infty$ respectively. If $j^* \ne \infty$ and $t = t(j^*)$ then letting $\nu_1 < \cdots < \nu_w$ be the values of $\nu \in \overline{\nu}$ and putting $\nu_0 = j - 1$ we have

$$n(t, \nu_v + 1) = \nu_{v+1} - \nu_v$$

for $0 \le v < w$.

Note on inversion and invariance (VI). The three Inversion Theorems of Sects. (3.7) and (3.9), the two Invariance Theorems of Sect. 4, and the above quadratic transformation Lemmas (I)–(V) of this section are refinements of the results of my papers [2, 4]. More about all this in Part II.

6 Dicritical Divisors

The concept of dicritical divisors arose in the topological study of a map $\mathbb{C}^2 \to \mathbb{C}$ given by a polynomial $f \in k[X,Y] \setminus k$ when k is the field of complex numbers. The term dicritical divisor seems to have been introduced by Mattei and Moussu [27], and was then used by Artal-Bartolo [16], Eisenbud–Neumann [21], Fourrier [22], Le–Weber [26], Neumann [29], Rudolph [31], and others. On the other hand, Pierrette Cassou-Noguès [18, 19] and Neumann–Norbury [30] use the alternative term horizontal divisors.

In Definition (6.1) we introduce the algebraic incarnation of dicritical divisors. In Note (6.2) we pay a heuristic visit to the original topological version.

The dicritical divisors may be viewed as a nonempty finite set of univariate polynomials strategically (and quite algebraically) located inside the belly of a randomly chosen bivariate polynomial. It is certainly amazing that, until 1980, no endoscopic examination of bivariate polynomial bellies (=affine plane curve bellies) revealed their existence. We have stressed "and quite algebraically" to indicate that in our treatment we do not use any topology or analysis which, under the pretext of geometric viewpoint, only muddies the water. Of course, it may be admitted that one person's clarity can be another person's muddying of waters and vice versa. Positively speaking, muddying may amount to stirring!!

In Note (6.6) I shall introduce the dicritical divisor theory of local rings and compare it to the analogous theory of quasirational and nonquasirational surface singularities coming out of my papers [2, 8].

Preamble for (6.1)–(6.4). Let us consider the bivariate polynomial ring $B = k[X, Y]$ over a field k and let $L = k(X, Y) = \mathrm{QF}(B)$ where $\mathrm{QF}(B)$ denotes the quotient field of B. Given any

$$f = f(X, Y) \in B \setminus k$$

of (total) degree N, by B_f we denote the localization of B at the multiplicative set $k[f]^\times$, and we note that then B_f is the affine domain $k(f)[X, Y]$ over the field $k(f)$ with $\mathrm{QF}(B_f) = k(X, Y) = L$ and we have $\mathrm{trdeg}_{k(f)} L = 1$. Now a localization of a UFD is a UFD, and irreducibles in the localization are essentially the same as irreducibles in the original UFD except that the localization has more units. Consequently B_f is a one-dimensional UFD and hence it is a DD as well as a PID. It follows that B_f is the affine coordinate ring of an irreducible nonsingular affine plane curve over $k(f)$.

Note that $D(L/k)$ is the set of all valuation rings V with $\mathrm{QF}(V) = L$ and $k \subset V$ such that $\mathrm{trdeg}_k H(V) = 1$ where $H_V : V \to H(V) = V/M(V)$ is the residue class epimorphisms; moreover, every member of $D(L/k)$ is a DVR, and $I(B/k)$ is the set of all $V \in D(L/k)$ with $B \not\subset V$. Also note that $D(L/k(f))$ is the set of all valuation rings V with $\mathrm{QF}(V) = L$ and $k(f) \subset V \neq L$; moreover, every member of $D(L/k(f))$ is a DVR, and $I(B_f/k(f))$ is the set of all $V \in D(L/k(f))$ with $B_f \not\subset V$.

Definition (6.1). For every $V \in D(L/k(f))$ we put

$$\deg(V) = \deg_f V = [H(V) : k(f)] \in \mathbb{N}_+$$

and we call this the f-degree of V, or briefly the degree of V. Moreover, for every $V \in I(B_f/k(f))$ we put

$$\mathrm{ind}(V) = \mathrm{ind}_f V = -\min(\mathrm{ord}_V X, \mathrm{ord}_V Y) \in \mathbb{N}_+$$

and we call this the f-index of V, or briefly the index of V. Finally we put

$$I(B/k, f) = \begin{cases} \text{the set of all } V \in I(B/k) \text{ at which} \\ f \text{ is residually transcendental over } k \end{cases}$$

and we observe that

$$I(B/k, f) = I(B_f/k(f)) = \text{a nonempty finite set.} \tag{†}$$

Now labelling the distinct members of $I(B/k, f)$ as $V_1, \ldots, V_m$, we call them the *dicritical divisors* of f (in B). In Part II we shall show that by the "sigma-eeye-feye" formula from extension theory of DVRs we have

$$\sum_{1 \le i \le m} \operatorname{ind}(V_i)\deg(V_i) = N \tag{•}$$

and, if k is algebraically closed and is of characteristic zero, then for $1 \le i \le m$

$$\begin{cases} H(V_i) = H_{V_i}(k(t_i)) \text{ for some } t_i \in V_i \text{ so that } H_{V_i}(f) = H_{V_i}(P_i(t_i)) \\ \text{where } P_i(Z) \in k[Z] \setminus k \text{ is a univariate polynomial} \\ \text{whose } Z\text{-degree equals } \deg(V_i) \end{cases} \tag{••}$$

In the proof we shall use Newton's fractional power series expansion. In Part II we shall also show that the characteristic zero hypothesis can be removed by replacing Newton expansion by Hamburger-Noether expansion. Note that the integers m and $\deg(V_1), \ldots, \deg(V_m)$ depend only on f as a element of the ring B and not on the particular generators X, Y of that ring, but the integers $\operatorname{ind}(V_1), \ldots, \operatorname{ind}(V_m)$ do depend on X, Y, as will be shown in Example (6.5).

Note (6.2). Momentarily assuming k to be the complex number field $\mathbb{C}$, the dicritical divisors may be "heuristically explained" thus. The polynomial map $\mathbb{C}^2 \to \mathbb{C}^1$ which is given by $(a, b) \mapsto f(a, b)$ can be extended to a rational map $P^2 \to P^1$ of the complex projective plane to the complex projective line. But as a "rational map" it may have points of indeterminacy. We get rid of these by "blowing up" P^2 to get a compact complex nonsingular surface W on which the map f extends to a well defined map $\phi : W \to P^1$. Just as P^2 is obtained by adding one projective line (called the line at infinity) to $\mathbb{C}^2$, the surface W is obtained by adding a finite number of projective lines $P_1^1, \ldots, P_n^1$ to $\mathbb{C}^2$. Consideration of connectivity tells us that, depending on the particular line P_i^1, the restriction of the map ϕ to P_i^1 maps it either onto the entire target line P^1 or to a single point of it, i.e., it is either surjective or constant. Those P_i^1 for which it is surjective are called dicritical divisors. By suitably relabelling, we may assume that $P_1^1, \ldots, P_m^1$ are dicritical while $P_{m+1}^1, \ldots, P_n^1$ are not. It can be shown that m is positive. Moreover, it can also be shown that by deleting a suitable point from a dicritical P_i^1 and also deleting a suitable point from the target P^1, the resulting map $\mathbb{C}_i^1 \to \mathbb{C}^1$ is given by a univariate polynomial $P_i(Z)$ of some degree d_i; note that d_i is the degree of the ramified covering $P_i^1 \to P^1$. By rotating the axes, i.e., by making a homogeneous linear transformation, we may assume that f is monic of degree N in Y. It turns out that then

$$\sum_{1 \le i \le m} e_i d_i = N$$

where the positive integer e_i is the ramification index coming out of the Dedekind Domain theory which is the same thing as the Riemann Surface theory.

As a side remark recall that f is a field generator means $k(f, g) = k(X, Y)$ for some rational function $g \in k(X, Y)$; it turns out that if the polynomial f is a field generator then the complementary generator g can be chosen to be a polynomial iff $d_i = 1$ for some dicritical P_i^1. Without assuming f to be a field generator, how do we show that the dicritical divisors are independent of the particular blow up W and how do we algebracize them?

To consider the independence, let $\overline{\phi} : \overline{W} \to P^1$ be any other blow up, and label the projective lines in $\overline{W} \setminus \mathbb{C}^2$ as $\overline{P}_1^1, \ldots, \overline{P}_{\overline{m}}^1, \overline{P}_{\overline{m}+1}^1, \ldots, \overline{P}_{\overline{n}}^1$ so that the first $\overline{m}$ are dicritical while the remaining ones are not. It can be shown that there exists a blow up $\widetilde{\phi} : \widetilde{W} \to P^1$ together with maps $\theta : \widetilde{W} \to W$ and $\overline{\theta} : \widetilde{W} \to \overline{W}$ such that $\phi\,\theta = \widetilde{\phi} = \overline{\phi}\,\overline{\theta}$. Label the projective lines in $\widetilde{W} \setminus \mathbb{C}^2$ as $\widetilde{P}_1^1, \ldots, \widetilde{P}_{\widetilde{m}}^1, \widetilde{P}_{\widetilde{m}+1}^1, \ldots, \widetilde{P}_{\widetilde{n}}^1$ so that the first $\widetilde{m}$ are dicritical while the remaining ones are not. It can be shown that $\widetilde{m} = \overline{m} = m$ and after suitable labelling, for $1 \le i \le m$, we have $\theta(\widehat{P}_i^1) = P_i^1$ and $\overline{\theta}(\widehat{P}_i^1) = \overline{P}_i^1$ with induced bijections $\widetilde{P}_i^1 \to P_i^1$ and $\widetilde{P}_i^1 \to \overline{P}_i^1$.

Now let us proceed to the algebraization which will actually reprove the independence. Recall that: for any finitely generated field extension L of a field K we have put $D(L/K) =$ the set of all prime divisors of L/K, i.e., the set of all DVRs V with quotient field QF$(V) = L$ such that $K \subset V$ and $\text{trdeg}_K H(V) = (\text{trdeg}_K L) - 1$ where $H(V) = V/M(V) =$ the residue field of V; for any affine domain A over K with QF$(A) = L$ we have put $I(A/K) =$ the set of all infinity divisors of A/K, i.e., the set of all $V \in D(L/K)$ such that $A \not\subset V$. Henceforth, we consider the bivariate polynomial ring $B = k[X, Y]$ over a field k and we let QF$(B) = L = k(X, Y)$ and we put $I(B/k, f) =$ the set of all those members V of $I(B/k)$ for which f is residually transcendental over k. Let V_i be the local ring of P_i^1 on W. Then clearly $V_i \in I(B/k)$ for $1 \le i \le n$, and we have: $V_i \in I(B/k, f) \Leftrightarrow 1 \le i \le m$.

It can also be shown that $I(B/k) =$ the totality of the local rings of the projective lines on various blow ups of P^2 which are in the complements of $\mathbb{C}^2$. At any rate, $I(B/k, f)$ is a nonempty finite set which we have defined without any aid of blowing ups, and this is our algebraic definition of dicritical divisors of f. Since $I(B/k, f)$ does correspond to the geometrically defined dicritical divisors on any blow up of P^2 on which the rational map $P^2 \to P^1$ becomes well-defined, this reproves the independence in a more succinct manner; the geometric proof sketched in the paragraph before last was rather fuzzy at best. This is the beauty of the approach by "models" which are collections of local rings and so on; for details see the Algebra and Geometry books [9, 12].

Now the $I(B/k, f)$ from surface theory coincides with the $I(B_f/k(f))$ from curve theory, where we have put $B_f = k(f)[X,Y]$. Note that B_f can be identified with the affine coordinate ring of the generic curve $f^\sharp = 0$ where we take an indeterminate u over k and put $f^\sharp = f - u$. Substituting f for u, this generic curve acquires the confusing equation $f = f$. The confusion (like the Maya covering the Brahma) can be removed by using two sets of variables giving $f(\overline{X}, \overline{Y}) = f(X,Y)$. Indeed, experience shows that such $f = f$ arguments provide exceptionally powerful tools! Although the curve $f = 0$ may be reducible and may even have multiple components and may be full of singularities, but miraculously the curve $f^\sharp = 0$ is irreducible and nonsingular. The best way to see this is to realize B_f as the localization of B at the multiplicative subset $k[f]^\times$ = the set of all nonzero elements in $k[f]$. Of course, the nonsingularity of $f^\sharp$ is only at finite distance, i.e., in general it will have singularities at infinity.

In any case, $I(B_f/k(f))$ is nothing but the set of all branches of $f^\sharp$ at infinity. To deal with them we put $F(X,Y) = f(X^{-1}, Y)$ and $F^\sharp(X,Y) = F(X,Y) - u$. Now

$$F(X,Y) = Y^N + \sum_{1\le j\le N} A_j(X)Y^{N-j} \quad \text{where} \quad A_j(X) \in k(X) \subset k((X)).$$

The branches of $f^\sharp$ at infinity are the branches of $F^\sharp$ which in turn are the irreducible factors in $k(u)((X))[Y]$ written as

$$F^\sharp(X,Y) = \prod_{1\le i\le m} F_i^\sharp(X,Y) \quad \text{with} \quad F_i^\sharp(X,Y) = Y^{N_i} + \sum_{1\le j\le N_i} A_{ij}^\sharp(X)Y^{N_i-j}$$

where $A_{ij}^\sharp(X) \in k(u)((X))$. Yes, it is not an accident that this is the same m as the number of dicritical divisors $V_1, \dots, V_m$. Indeed, after suitable labelling, there is a natural isomorphism σ_i of V_i onto the DVR $V_i^\dagger$ given by the branch $F_i^\sharp$.

Basically, assuming k to be an algebraically closed field of characteristic zero, we shall end up finding $t_i^\dagger$ in an algebraic closure of $k(u)$ such that $H(V_i^\dagger) = k(t_i^\dagger)$ and $u = P_i(t_i^\dagger)$ where $P_i(Z) \in k[Z]$ is the univariate polynomial of degree d_i we spoke of in the first paragraph of this Note. Upon letting $t_i = \sigma^{-1}(t^\dagger)$ we would then get $t_i \in V_i$ such that $H(V_i) = k(t_i)$ and $f = P_i(t_i)$.

To find $t_i^\dagger$ we use Newton's polygonal method to solve the equation $F_i^\sharp(X,Y) = 0$ and thereby expand Y as a fractional meromorphic series $\widetilde{Y}$ in X, and also to expand X as a fractional meromorphic series $\widetilde{X}$ in Y. Now we use the inversion formula given in [4] to compare these two expansions. Details in Part II.

Philosophy (6.3). The importance of polynomials derives from the fact that they can be viewed as functions in two different ways. To the algebraist, a bivariate polynomial

$$f = f(X,Y) = \sum_{i+j\le N} a_{ij}X^iY^j \in k[X,Y] \setminus k \quad \text{with} \quad a_{ij} \in k$$

of (total) degree N is a function $\mathbb{N}^2 \to k$ given by $(i, j) \mapsto a_{ij}$. To the analyst, who prefers his field to be the complex number field $\mathbb{C}$, it is a map $\mathbb{C}^2 \to \mathbb{C}$ given by $(\alpha, \beta) \mapsto f(\alpha, \beta)$. Finally, to the geometer, who is an animal linking the analyst with the algebraist, it defines a plane curve $C : f(X, Y) = 0$; if k is algebraically closed then the points of C belong to k^2; if k is not algebraically closed then it is better to let the points of C live in spec$(k[X, Y])$.

Before he proceeds to "compactify" $\mathbb{C}^2$ and $\mathbb{C}$, the analyst thinks of the "fibers" of the map $\mathbb{C}^2 \to \mathbb{C}$ above various values c of f, and then he may perform catastrophic tortuous surgery, and so on.

In place of this, as algebraists (or algebraic-geometers) we take an indeterminate u over $k(X, Y)$ and think of the "generic curve" $f^\sharp = 0$ where

$$f^\sharp = f^\sharp(X, Y) = f(X, Y) - u \in k(u)[X, Y].$$

By "identifying" u with f, i.e., by the shocking (= absurd sounding but surprisingly correct and extremely useful) equation $f = f$, we can take B_f to be the affine coordinate ring of $f^\sharp$. As noted above, B_f is a PID and hence $f^\sharp$ is an irreducible nonsingular affine plane curve. Instead of saying that we can take B_f to be the affine coordinate ring of $f^\sharp$, let us be more pedantic and set up an isomorphism between the two. Now the affine coordinate ring $B_f^\sharp$ of $f^\sharp$ is given by

$$H_f : B^\sharp = k(u)[X, Y] \to k(u)[X^\sharp, Y^\sharp] = B_f^\sharp$$

where H_f is a $k(u)$-epimorphism which sends (X, Y) to $(X^\sharp, Y^\sharp)$ and for whose kernel we have

$$\ker(H_f) = f^\sharp B^\sharp.$$

Taking indeterminates $(\overline{X}, \overline{Y})$ over $k(X, Y)$, we view B_f as an affine coordinate ring by considering the $k(f)$-epimorphism

$$\overline{H}_f : \overline{B}_f = k(f)[\overline{X}, \overline{Y}] \to k(f)[X, Y] = B_f$$

which sends $(\overline{X}, \overline{Y})$ to (X, Y) and for whose kernel we have

$$\ker(\overline{H}_f) = (f(\overline{X}, \overline{Y}) - f(X, Y))\overline{B}_f.$$

Also we have an obvious k-isomorphism

$$\widehat{H}_f : B^\sharp = k(u)[X, Y] \to k(f)[\overline{X}, \overline{Y}] = \overline{B}_f$$

which sends (u, X, Y) to $(f, \overline{X}, \overline{Y})$. Now the said isomorphism

$$\widetilde{H}_f (= \text{restriction of } H_f^\sharp) : B_f \to B_f^\sharp$$

is the unique isomorphism such that $\widetilde{H}_f \overline{H}_f \widehat{H}_f = H_f$, i.e., such that the obvious rectangle

$$\begin{array}{ccc}
B_f = k(f)[X,Y] & \xrightarrow{\widetilde{H}_f\,(=\text{restriction of } H_f^\sharp)} & B_f^\sharp = k(u)[X^\sharp, Y^\sharp] \\
\overline{H}_f \big\uparrow & & H_f \big\uparrow \\
\overline{B}_f = k(f)[\overline{X}, \overline{Y}] & \xleftarrow{\qquad \widehat{H}_f \qquad} & B^\sharp = k(u)[X,Y].
\end{array}$$

commutes. Moreover, the said isomorphism extends to an isomorphism

$$H_f^\sharp : L = \mathrm{QF}(B_f) = k(X,Y) \to k(u)(X^\sharp, Y^\sharp) = \mathrm{QF}(B_f^\sharp) = L_f^\sharp$$

of the function fields.

To distinguish between $B_f/k(f)$ (resp: $L/k(f)$) and $B_f^\sharp/k(u)$ (resp: $L_f^\sharp/k(u)$) we may call them the affine coordinate ring (resp: function field) of the intrinsic generic curve and the extrinsic generic curve, respectively.

The affine coordinate ring $B_{f,k}$ of f is given by the k-epimorphism

$$H_{f,k} : B = k[X,Y] \to B_{f,k} = k[x,y] = B_{f,k} \subset k(x,y) = L_{f,k}$$

which sends (X,Y) to (x,y) and for whose kernel we have

$$\ker(H_{f,k}) = fB$$

where $L_{f,k}$ is the total quotient ring of $B_{f,k}$, which means the quotient field if f is irreducible (in B).

Assuming f to be irreducible, $I(B_{f,k}/k)$ is a nonempty finite subset of $D(L_{f,k}/k)$ which is a set of DVRs; for every $V \in D(L_{f,k}/k)$ we put

$$\deg_{f,k}(V) = [H(V) : k] \in \mathbb{N}_+$$

and we call this is the (f,k)-degree of V; for every $V \in I(B_{f,k}/k)$ we put

$$\mathrm{ind}_{f,k} V = -\min(\mathrm{ord}_V x, \mathrm{ord}_V y) \in \mathbb{N}_+$$

and we call this the (f,k)-index of V.

Note that, without assuming f to be irreducible, for every $V \in D(L/k(f))$, upon letting $V^\sharp = H_f^\sharp(V)$, we have

$$V^\sharp \in D(L_f^\sharp/k(u)) \quad \text{with} \quad \deg(V) = \deg_{(f^\sharp, k(u))} V^\sharp$$

and if $V \in I(B_f/k(f))$ then we have

$$V^\sharp \in I(B_f^\sharp/k(u)) \quad \text{with} \quad \text{ind}(V) = \text{ind}_{(f^\sharp,k(u))} V^\sharp.$$

Remark on infinity (6.4). Continuing the discussion of (6.3), without assuming f to be irreducible, to take care of points at infinity, we introduce two different incarnations $\dot{f} = \dot{f}(\dot{X}, \dot{Y})$ and $\ddot{f} = \ddot{f}(\ddot{X}, \ddot{Y})$ of f thus.

We write

$$f(X,Y) = \sum_{0 \le l \le N} f_l(X,Y) \quad \text{with} \quad f_l(X,Y) = \sum_{i+j=l} a_{ij} X^i Y^j$$

where f_l is either zero or is homogeneous of degree l. We call $f_N = f_N(X,Y)$ the degree form of f which we denote by defo(f) or f^+. Now we let

$$(\dot{X}, \dot{Y}) = (1/X, Y/X) \quad \text{and} \quad \dot{B} = k[\dot{X}, \dot{Y}]$$

with

$$\dot{f}(\dot{X}, \dot{Y}) = \dot{X}^N f(1/\dot{X}, \dot{Y}/\dot{X}) = \sum_{0 \le l \le N} \dot{X}^{N-l} f_l(1, \dot{Y}) \in k[\dot{X}, \dot{Y}]$$

and

$$(\ddot{X}, \ddot{Y}) = (X/Y, 1/Y) \quad \text{and} \quad \ddot{B} = k[\ddot{X}, \ddot{Y}]$$

with

$$\ddot{f}(\ddot{X}, \ddot{Y}) = \dot{Y}^N f(\dot{X}/\dot{Y}, 1/\dot{Y}) = \sum_{0 \le l \le N} \dot{Y}^{N-l} f_l(\dot{X}, 1) \in k[\ddot{X}, \ddot{Y}]$$

and we note that $\dot{f}$ and $\ddot{f}$ are polynomials of degree N.

Let L_∞ consist of X together with all irreducible homogeneous polynomials in $k[X,Y] \setminus k$ which are monic in Y. We call L_∞ the line at infinity (over k). If $Q \in L_\infty \setminus \{X\}$ is of degree 1 then $Q = Y - \beta X$ where $\beta \in k$ and with Q we associate the triple $(1, \beta, 0) \in k^3$ by putting $Q(1, \beta, 0) = Q$. With X associate the triple $(0, 1, 0)$ by putting $Q(0, 1, 0) = X$; note that $Q(1, 0, 0) = Y$. Thinking of the usual projective line (over k) as consisting of all triples $(\alpha, \beta, 0) \in k^3$ such that if $\alpha \neq 0$ then $\alpha = 1$ and if $\alpha = 0$ then $\beta = 1$, the mapping which sends $(\alpha, \beta, 0)$ to $Q(\alpha, \beta, 0)$ gives a bijection of the said line onto the set of degree 1 points of L_∞. For any $Q \in L_\infty$, we let $e(f, Q)$ be the largest nonnegative integer such that $Q^{e(f,Q)}$ divides f^+ in B; we call $e(f, Q)$ the exponent of Q in f. Clearly $\{Q \in L_\infty : e(f, Q) > 0\}$ is a nonempty finite set and labelling its distinct members which are different from X as $\{Q_1, \ldots, Q_p\}$ and letting $Q_0 = X$ we have

$$f^+ = \theta \prod_{0 \le i \le p} Q_i^{e_i} \quad \text{with} \quad e_i = e(f, Q_i)$$

and hence, as a case of Bézout's theorem, we get the obvious equation

$$\sum_{0\le i\le p} e_i d_i = N \quad \text{with} \quad d_i = \deg(Q_i)$$

which says that f and L_∞ meet in N points counted properly.

Recall that for any finite number of elements $x_1, \dots, x_r$ in an overfield of k we have defined

$$\mathfrak{W}(k; x_1, \dots, x_r) = \bigcup_{1\le j\le r \text{ with } x_j\neq 0} \mathfrak{V}(k[x_1/x_j, \dots, x_r/x_j])$$

and for any subset J of a domain S let us put

$$\mathfrak{V}(S, J) = \{R \in \mathfrak{V}(S) : JR \neq R\}.$$

Also recall that any $V \in \overline{D}(L/k)$ dominates a unique member of $\mathfrak{W}(k; x_1, \dots, x_r)$ which is called the *center* of V on $\mathfrak{W}(k; x_1, \dots, x_r)$.

We define the projective plane and the projective line over k by putting

$$\mathcal{P}_k^2 = \mathfrak{W}(k; X, Y, 1) \quad \text{with} \quad \mathcal{P}_k^1 = \mathfrak{W}(k; X, 1)$$

and we define the affine plane and the affine line over k by putting

$$\mathcal{A}_k^2 = \mathfrak{V}(B) \quad \text{with} \quad \mathcal{A}_k^1 = \mathfrak{V}(k[X])$$

and we define the projective point and the affine point over k by putting

$$\mathcal{P}_k^0 = \mathcal{A}_k^0 = \{k\}$$

and we note that then

$$\mathcal{P}_k^2 = \mathfrak{V}(B) \cup \mathfrak{V}(\dot{B}) \cup \mathfrak{V}(\ddot{B})$$

and by putting

$$\dot{\mathcal{A}}_k^1 = \mathfrak{V}(\dot{B}, (\dot{X}\dot{B})) \quad \text{with} \quad \ddot{\mathcal{A}}_k^0 = \mathfrak{V}(\ddot{B}, (\ddot{X}, \ddot{Y})\ddot{B})$$

we have the disjoint unions

$$\mathcal{P}_k^2 = \mathcal{A}_k^2 \coprod \dot{\mathcal{A}}_k^1 \coprod \ddot{\mathcal{A}}_k^0 \quad \text{with} \quad \mathcal{P}_k^1 = \mathcal{A}_k^1 \coprod \mathcal{A}_k^0.$$

Informally speaking, $\ddot{\mathcal{A}}_k^0$ is the set consisting only of the local ring of the origin in the $(\ddot{X}, \ddot{Y})$-plane, and so we may identify $\ddot{\mathcal{A}}_k^0$ with $\mathcal{A}_k^0$. Again informally speaking, $\dot{\mathcal{A}}_k^1$ is the line $\dot{X} = 0$ in the plane $\mathfrak{V}(\dot{B})$; formally speaking, to identify $\dot{\mathcal{A}}_k^1$ with the X-line $\mathcal{A}_k^1 = \mathfrak{V}(k[X])$, considering the k-epimorphism $\dot{B} \to k[X]$ given by $(\dot{X}, \dot{Y}) \mapsto (0, X)$, and remembering the commutativity of epimorphism and localization, we note that $R \mapsto R/(\dot{X}R)$ gives a bijection $\dot{\mathcal{A}}_k^1 \to \mathcal{A}_k^1$. Thus $\dot{B}$ is the preferred chart to study the line at infinity in $\mathcal{P}_k^2$, i.e.,

$$\mathcal{P}_k^2 \setminus \mathcal{A}_k^2 = \dot{\mathcal{A}}_k^1 \coprod \ddot{\mathcal{A}}_k^0.$$

To match this line at infinity with L_∞, first we define the local ring $R(L_\infty)$ of L_∞ by putting

$$R(L_\infty) = \dot{B}_{\dot{X}\dot{B}}$$

and noting that this is the unique one-dimensional member of $\dot{\mathcal{A}}_k^1$; it can also be characterized as the DVR R_∞ of L/k for which

$$\mathrm{ord}_{R_\infty} g = -\deg(g) \text{ for all } g \in B.$$

Next we define the local ring $R(Q)$ of $Q \in L_\infty$ by putting

$$R(Q) = \begin{cases} \ddot{B}_{(\ddot{X},\ddot{Y})\ddot{B}} & \text{if } Q = X \\ \dot{B}_M \text{ where } M = (\dot{X}, Q/X^{\deg(Q)})\dot{B} & \text{if } Q \neq X \end{cases}$$

and we note that $Q \mapsto R(Q)$ gives bijections $\{X\} \to \ddot{\mathcal{A}}_k^0$ and $L_\infty \setminus \{X\} \to \dot{\mathcal{A}}_k^1$. To complete the picture, we define the local ring $R(Q)$ of any $Q \in \mathrm{spec}(B)$ by putting

$$R(Q) = B_Q$$

so that $Q \mapsto R(Q)$ gives a bijection $\mathrm{spec}(B) \to \mathcal{A}_k^2$. Thus,

$$Q \mapsto R(Q) \quad \text{gives a bijection} \quad SP_k^2 \to \mathcal{P}_k^2$$

where by definition

$$\text{the spectral projective plane } SP_k^2 = \mathrm{spec}(B) \coprod L_\infty \coprod \{L_\infty\}.$$

Moreover, for any $(\alpha, \beta, 1) \in k^3$ we put

$$Q(\alpha, \beta, 1) = (X - \alpha, Y - \beta)B \in \mathrm{spec}(B)$$

and we note that then

$$(\alpha, \beta, \gamma) \mapsto R(Q(\alpha, \beta, \gamma)) \quad \text{gives a bijection} \quad UP_k^2 \to RP_k^2$$

where by definition

$$\text{the usual projective plane } UP_k^2 = \begin{cases} \text{the set of all } (\alpha, \beta, \gamma) \in k^3 \\ \text{such that: if } \gamma \neq 0 \text{ then } \gamma = 1, \\ \text{if } \gamma = 0 \neq \alpha \text{ then } \alpha = 1, \\ \text{if } \gamma = 0 = \alpha \text{ then } \beta = 1, \end{cases}$$

and

$$\text{the rational projective plane } RP_k^2 = \begin{cases} \text{the set of rational points of } \mathcal{P}_k^2, \\ \text{i.e., 2-dimensional members of } \mathcal{P}_k^2 \\ \text{which are residually rational over } k. \end{cases}$$

To summarize, we have maps

$$UP_k^2 \xrightarrow{Q} SP_k^2 \xrightarrow{R} \mathcal{P}_k^2 \quad \text{with} \quad \text{im}(QR) = RP_k^2$$

where the first injective map is $(\alpha, \beta, \gamma) \mapsto Q(\alpha, \beta, \gamma)$ and the second bijective map is $Q \mapsto R(Q)$.

Let us observe that $I(B/k, f) \subset I(B/k) \setminus \{R_\infty\}$, and moreover the center of any $V \in I(B/k) \setminus \{R_\infty\}$ on $\mathcal{P}_k^2$ is the two dimension regular local domain R, with quotient field L and $[H(R) : k] < \infty$, described thus:

(‡) $R = k[x, y]_J$ with $x \in M(R) \setminus M(R)^2$ where

$$(x, y) = (1/X, Y/X) \text{ or } (x, y) = (1/Y, X/Y) \text{ according as } X \notin V \text{ or } x \in V$$

and J is the maximal ideal in $k[x, y]$ generated by x and a nonconstant irreducible monic polynomial $\zeta(y) \in k[y]$. Furthermore, if $V \in I(B/k, f)$ then V is a dicritical divisor of f in R with $fx^N \in R$ and we have

$$F_N(1, y) \in \zeta(y)k[y] \text{ or } F_N(y, 1) \in \zeta(y)k[y] \text{ according as } X \notin V \text{ or } x \in V.$$

By Lemma (II) of Sect. 5, it follows that if $V \in I(B/k, f)$ then the relative algebraic closure k' of k in $H(V)$ is a finite algebraic extension of k and $H(V)$ is a simple transcendental extension of k'; we say that f is *residually a polynomial* over B relative to V to mean that $f \in V$ and $H_V(f) \in k'[t] \setminus k'$ for some $t \in H(V)$ with $H(V) = k'(t)$.

Further discussion in Part II.

Example (6.5). To indicate the dependence of N and m on f, let us write N_f and m_f for them. Then clearly m_f and $\deg_f(V_1), \dots, \deg_f(V_m)$ depend only on f as an element of B and not on the particular generators X, Y of B. This can be paraphrased by letting $\text{Aut}_k(B)$ be the group of all k-automorphisms of B and saying that for every τ in $\mathrm{Aut}_k(B)$ we have that: (i) $m_{\tau(f)} = m_f$; (ii) $\tau(V_i)_{1\le i\le m}$ are the dicritical divisors of $\tau(f)$; and (iii) $\deg_{\tau(f)}(\tau(V_i)) = \deg_f(V_i)$ for $1 \le i \le m$. Let us call f a ring generator to mean that $B = k[f, g]$ for some g in B. Then it is clear that f is a ring generator iff $N_{\tau(f)} = 1$ for some τ in $\mathrm{Aut}_k(B)$. Therefore by (6.1)(•), it follows that:

$$f \text{ is a ring generator} \Leftrightarrow m_f = 1 = \deg_f(V_1) \Rightarrow \mathrm{ind}_f(V_1) = N_f.$$

Now to exhibit the dependence of $\mathrm{ind}_f(V_i)$ on X, Y, it suffices to take f to be the ring generator $Y - X^N$ with any $N \in \mathbb{N}_+$ and noting that $\mathrm{ind}_f(V_1) = N_f = N$ but $\mathrm{ind}_{\tau(f)}(\tau(V_1)) = N_{\tau(f)} = 1$ where τ in $\mathrm{Aut}_k(B)$ is given by $(X, Y) \mapsto (X, Y + X^N)$.

Note (6.6). Let R be a two dimensional regular local domain. Now given any $z \in \mathrm{QF}(R)^\times$, by a *dicritical divisor* of z in R we mean a prime divisor V of R such that z is residually transcendental over R relative to V. By Lemma (II) of Sect. 5, we know that the residue field $K^* = H(V)$ of any prime divisor V of R is of the form $K^* = K'(t)$ where the finite algebraic field extension K' of $K = H_V(R)$ is the relative algebraic closure of K in K^* and the element t is not algebraic over K'. Assuming $z \in \mathrm{QF}(R)$ to be residually transcendental over R relative to V, after writing

$$H_V(z) = \frac{P(t)}{Q(t)}$$

where $P(t), Q(t)$ are nonzero members of $K'[t]$ having no nonconstant common factor in $K'[t]$, we define the *relative polar degree* $\mathrm{rpdeg}_{(V,t)}z$ of z relative to (V, t) to be the number of distinct nonconstant irreducible monic factors of $Q(t)$ in $K'[t]$. Note that

$$\max(\deg_t P(t), \deg_t Q(t))$$

is a positive integer which is independent of t as long as $K^* = K'(t)$; we denote this positive integer by $\mathrm{resdeg}_{(V,R)}z$ and call it the *residue degree* of z relative to (V, R). We also define the *polar degree* $\mathrm{pdeg}_V z$ of z relative to V to be the minimum of $\mathrm{rpdeg}_{(V,t)}z$ taken over all $t \in K^*$ with $K^* = K'(t)$. We say that z is *residually a polynomial* over R relative to V to mean that $\mathrm{pdeg}_V z = 0$, i.e., to mean that $H_V(z) \in K'[t] \setminus K'$ for some $t \in K^*$ with $K^* = K'(t)$; note that for any such t we have $\mathrm{resdeg}_{(V,R)}z = \deg_t P(t)$; moreover if t' and $P'(t')$ are any other such values of t and $P(t)$ then $P'(t') = aP(bt + c)$ for some a, b, c in K' with $a \ne 0 \ne b$.

(†*) As an analogue of (6.1)(†) we note that any $z \in \mathrm{QF}(R)^\times$ has at most a finite number of dicritical divisors in R. Moreover, this number is zero iff either $z \in R$ or $1/z \in R$. [To see this, first observe that if z has a dicritical divisor in R then obviously $z \notin R$ and $1/z \notin R$. So henceforth assume that $z \notin R$ and $1/z \notin R$. Now R is normal because it is regular, and hence by the bracketed proof on pages 75–76 of [3] we find an epimorphism $h : R[z] \to H(R)[Z]$ with indeterminate Z such that $h(z) = Z$ and $h(x) = H_R(x)$ for all $x \in R$. It follows that $M(R)R[z]$ is a prime ideal in $R[z]$ with $(M(R)R[z]) \cap R = M(R)$. Let S be the localization of $R[z]$ at $M(R)R[z]$ and let T be the integral closure of S in $\mathrm{QF}(R)$. By Lemma (T54) on page 268 of [12] we have $\dim(S) = 1$ and hence by Theorem (4.10) on page 118 of Nagata [28] we see that

$$T = V_1 \cap \cdots \cap V_e$$

where e is a positive integer and $V_1, \ldots, V_e$ are pairwise distinct DVRs with quotient field QF(R). Clearly $V_1, \ldots, V_e$ are exactly all the dicritical divisors of z in R.]

Given any F, G in $R^\times$, by a *dicritical divisor* of (F, G) in R we mean a dicritical divisor of F/G in R. The above terms relative polar degree rpdeg, residue degree resdeg, polar degree pdeg, and residually a polynomial, are now applicable with z replaced by (F, G).

Geometrically speaking, we may visualize R to be the local ring of a simple point of an algebraic or arithmetical surface, and think of z as a *rational function* at that simple point, and (F, G) as the *pencil* of curves $F = uG$ at that point. Let us call the pencil *special* to mean that G equals a unit times a power of a regular parameter, i.e., $GR = x^m R$ for some $x \in M(R) \setminus M(R)^2$ and $m \in \mathbb{N}$.

By (6.4)(‡) we see that a bivariate polynomial $f \in B \setminus k$ gives rise to a special pencil in each relevant R, and hence the following Local Ring Proposition LRP would imply the following Polynomial Ring Proposition PRP.

LRP says that if (F, G) is any special pencil in a two dimensional regular local ring R then F/G is residually a polynomial over R relative to any dicritical divisor V of F/G in R.

PRP says that if f is any nonconstant member of a bivariate polynomial ring $B = k[X, Y]$ then f is residually a polynomial over B relative to any dicritical divisor of f in R.

Let A be a two-dimensional affine domain over an algebraically closed field and let R be the localization of A at a maximal ideal. Now (†*) says that if R is regular then, for any rational function

$$z = F/G$$

with $F \neq 0 \neq G$ in R, z has only a finite number of dicritical divisors in R; morover, if the pencil (F, G) is special then z is residually a polynomial over R relative to every dicritical divisor of z in R. In view of the results of [2, 8], it can be shown that all except a finite number of prime divisors V of R are residually simple transcendental over R; moreover, if R is regular then the said finite number is zero. This is the analogue from the theory of quasirational singularities we spoke of in the preamble of this section. Thus a (possibly singular) point of a surface in the quasirational theory is replaced by a rational function at a simple point of a surface in the dicritical theory.

Needless to say that a simple point in the former theory is replaced by a special pencil in the latter theory. Likewise, residually simple transcendental in the former theory is replaced by residually a polynomial in the latter theory.

As a final philosophical comment, I wish to observe that the LHS $I(B/k, f)$ of the equation (6.1)(†) represents points at infinity of the projective plane while its RHS $I(B_f, /k(f))$ represents the branches at infinity of a generic plane curve. Thus the LHS stands for the projective viewpoint while the RHS stands for the meromorphic viewpoint. Although, in [6, 7, 13–15], I have been beating the drums of the meromorphic viewpoint, it has suddenly dawned on me that the difference between these two methods is merely a matter of semantics!!

More discussion in Part II.

7 Field Generators

Consider the bivariate polynomial ring $k[X,Y]$ over a field k. A polynomial $f(X,Y) \in k[X,Y]$ is a field generator means for some $g = g(X,Y) \in k(X,Y)$ we have $k(X,Y) = k(f,g)$; here the complementary generator g may or may not be a polynomial. In his 1974 Purdue Ph.D. Thesis [25], Jan gave an example of a field generator which has no complementary polynomial field generator. In Theorem (7.6) I shall give a criterion for the existence of a complementary polynomial field generator. Recently, Pierrette Cassou-Noguès [18, 19] ascribed this criterion to Russell [32, 33], and she used it to revisit Jan's example. However, I shall give a short, almost obvious, proof of (7.6) which is completely independent of the rest of this paper. The criterion (7.6) can be paraphrased by saying that a field generator f has a complementary polynomial field generator iff f has a dicritical divisor of degree 1.

Note that if a polynomial f is a field generator then the generic curve $f = u$, where u is an indeterminate, is a curve of genus zero having a rational place over $k(u)$, and conversely. In Example (7.7), I shall discuss the circle to illustrate this fact. It was conjectured by me and proved by my student Jan in his Thesis [25] that a field generator has at most two points at infinity. Without assuming f to be a field generator, in Part II I shall generalize this by giving a bound on the number of points at infinity of f in terms of the genus of $f = u$.

Preamble for (7.1)–(7.5). Let L be a finitely generated field extension of a field K with $\text{trdeg}_K L = \epsilon$. Let A be an affine domain over K with $\text{QF}(A) = L$ where $QF(A)$ denotes the quotient field of A. Note that $D(L/K)$ is the set of all valuation rings V with $\text{QF}(V) = L$ and $K \subset V$ such that $\text{trdeg}_K H(V) = \epsilon - 1$ where

$$H_V : V \to H(V) = V/M(V)$$

is the residue class epimorphisms and we are identifying $H(K)$ with K; moreover, every member of $D(L/K)$ is a DVR, and $I(A/K)$ is the set of all $V \in D(L/K)$ with $A \not\subset V$.

Lemma (7.1). *Assume that $L = K(x)$ where x is transcendental over K. Let V be the $(1/x)$-adic valuation, i.e., let V be the localization of $K[x]$ at the prime ideal generated by $1/x$. Then $V \in D(L/K)$ with $H(V) = K$.*

Proof. Obvious.

Lemma (7.2). *Assume that $L = K(y)$ where y is transcendental over K. Let $V \in D(L/K)$ be such that $H(V) = K$. Then $L = K(x)$ for some $x \in L$ such that V is the $(1/x)$-adic valuation. Moreover, if K is infinite and $V_2, \dots, V_m$ are any finite number of members of $D(L/K) \setminus \{V\}$, then x can be chosen so that we also have $x \notin M(V_2) \cup \dots \cup M(V_m)$.*

Proof. If V is the $(1/y)$-adic valuation then taking $z = y$ we see that $L = K(z)$ and V is the $(1/z)$-adic valuation. If not then V must be the localization of $K[y]$ at the prime ideal generated by $y - a$ for some $a \in K$, and taking $z = 1/(y - a)$ we see that $L = K(z)$ and V is the $(1/z)$-adic valuation. Now without the "Moreover" it suffices to take $x = z$. With the "Moreover" we clearly have $z \in V_2 \cup \dots \cup V_m$ and, since K is infinite, for all except a finite number of $c \in K$ we must have

$$z + c \notin M(V_2) \cup \dots \cup M(V_m)$$

and it suffices to take $x = z + c$.

Lemma (7.3). *Assume that $L = K(x)$ where x is transcendental over K. Let V be the $(1/x)$-adic valuation and assume that $V \in I(A/K)$. Let $\{V_2, \dots, V_m\}$ be the distinct elements of $I(A/K) \setminus \{V\}$, and note that for $2 \le i \le m$ we clearly have $K[x] \subset V_i$ and V_i is the localization of $K[x]$ at the prime ideal generated by an irreducible element x_i in $k[x]$. Now assume that A is a UFD. Then A is a proper PID, and A is the localization of $K[x]$ at the multiplicative set consisting of all monomials in $x_2, \dots, x_m$. Moreover, if $x \notin M(V_2) \cup \dots \cup M(V_m)$ then clearly x is an irreducible element in A.*

Proof. To see that A equals the said localization, note that A is normal because it is a UFD, and hence A is the intersection of all the members of $D(A/K) \setminus I(A/K)$, but this intersection clearly equals the said localization.

Lemma (7.4). *Assume that $\epsilon = 1$ and $L = K(x)$ for some $x \in A$. Then $H(V) = K$ for some $V \in I(A/K)$.*

Proof. Take V to be the $(1/x)$-adic valuation and apply (7.1).

Lemma (7.5). *Assume that $\epsilon = 1$ and $L = K(y)$ for some $y \in L$. Also assume that, K is infinite, A is a UFD, and $H(V) = K$ for some $V \in I(A/K)$. Then A is a proper PID and $L = K(x)$ for some irreducible $x \in A$.*

Proof. Take $\{V_2, \dots, V_m\} = I(A/K) \setminus \{V\}$ and apply (7.2) and (7.3).

Preamble for (7.6). Consider the bivariate polynomial ring $B = k[X, Y]$ over a field k and let $L = k(X, Y) = \mathrm{QF}(B) =$ the quotient field of B. Given any

$$f = f(X, Y) \in B \setminus k,$$

by B_f we denote the localization of B at the multiplicative set $k[f]^\times$, and we note that then B_f is the affine domain $k(f)[X, Y]$ over the field $k(f)$ with $\mathrm{QF}(B_f) = k(X, Y) = L$ and we have $\mathrm{trdeg}_{k(f)} L = 1$. Note that a localization of a UFD is a

UFD, and irreducibles in the localization are essentially the same as irreducibles in the original UFD except that the localization has more units. Hence we get:

Theorem (7.6). *In the above setup we have the following.*

(1) If $L = k(f,g)$ *for some* $g \in B$, *then* $H(V) = k(f)$ *for some* $V \in I(B_f/k(f))$.
(2) If $L = k(f,l)$ *for some* $l \in L$ *and* $H(V) = k(f)$ *for some* $V \in I(B_f/k(f)$, *then* $L = k(f,g)$ *for some* $g \in B$.

Proof. Taking $(K, L, A) = (k(f), k(X,Y), B_f)$, (1) follows from (7.4). Likewise (2) follows from (7.5) after noting that the irreducible $x \in B_f$ when multiplied by a suitable $b \in k[f]^{\times}$ produces an irreducible $bx \in B$ and we obviously have $k(f,bx) = k(f,x) = L$.

Example (7.7). We illustrate the above theorem by showing that the circle is a field generator over $\mathbb{C}$ but not over $\mathbb{R}$. The underlying obvious fact behind this is that f is a field generator of $L = k(X,Y)$ iff the general curve $f^{\sharp} = f(X,Y) - u$, where u is an indeterminate, is of genus zero and has a rational place over $k(u)$, i.e., a $V \in D(L^{\sharp}_f/k(u))$ which is residually rational over $k(u)$; here $L^{\sharp}_f$ is the function field of $f^{\sharp}$, i.e., the quotient field of the residue class ring of $k(u)[X,Y]$ modulo the ideal generated by $f^{\sharp}$. For the circle $f = X^2 + Y^2 - 1$ with $k = \mathbb{R}$, if $f^{\sharp}$ had a rational place then we can find a nonzero triple $(a(u), b(u), c(u))$ in $k[u]$ such that

$$a(u)^2 + b(u)^2 = c(u)^2 + uc(u)^2.$$

Since the equation $x^2 + y^2 = 0$ has no solution in $\mathbb{R}$ other than $(0,0)$, it follows that if $(a(u), b(u)) \neq 0$ then the LHS of the above equation is a nonzero polynomial of even degree. But if $c(u) \neq 0$ then the RHS of the equation is a nonzero polynomial of odd degree. Therefore, the circle is not a field generator over $\mathbb{R}$. Over $k = \mathbb{C}$ it is a field generator because $k(f, X + iY) = k(X,Y)$.

8 Preview of Part II

As said in the Introduction, Part II will include various topics from algebraic curve theory such as the conductor and genus formulas of Dedekind and Noether, and the automorphism theorems of Jung and Kulk. In Part II, I shall also relate all this to the Jacobian problem which conjectures that if the Jacobian of n polynomials in n variables over a characteristic zero field equals a nonzero constant then the variables can be expressed as polynomials in the given polynomials; see [13–15]. As indicated in the preamble of Sect. 4, in Part II, I shall revisit Newton's polygonal method. As said at the end of Sect. 5, in Part II, I shall say more about the Inversion and Invariance Theorems and about quadratic transformations. As said in Sect. 6, in Part II, I shall discuss Dicritical Divisors some more. Finally, as said in the beginning of Sect. 7, in Part II, I shall give a bound on the number of points at infinity of an algebraic plane curve.

9 Epilogue

Let me close with a chatty survey of the paper which can also serve as an alternative Introduction.

9.1 Trigonometry

In high-school we learn the expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x \sum_{0 \le i < \infty} a_i x^i$$

where $a_i = 0$ or $\frac{(-1)^{i/2}}{(i+1)!}$ according as i is odd or even. The fact that in the expansion of $\sin x$ there is no x^2 term but there is an x^3 term, may be codified by saying that $\sin x$ has a gap of size 2, i.e., 2 is the smallest positive value of i for which $a_i \neq 0$. Now

$$\sin^{-1} x = x + \frac{x^3}{3!} +$$

and so the inverse function has a gap of the same size 2.

It was around 1665 that Newton gave the above two expansions and Gregory gave the expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

and from this it follows that

$$\tan x = x + \frac{x^3}{3} + \ldots$$

but the full expansion of $\tan x$ is rather complicated and was obtained by Bernoulli only in the next century. At any rate the size of the gap in $\tan x$ as well as $\tan^{-1} x$ is again 2. All these formulas can be found in Chrystal's Algebra [20] published in 1886 and Hobson's Trigonometry [24] published in 1891. I was lucky to have studied these two excellent books towards the end of my high-school years at the suggestion of my father. After hundred years, they are still being reprinted and I highly recommend them to all students of mathematics.

Renaming the above type of gap as absolute gap, given any positive integer d, let us define the d-gap to be the smallest value of i which is nondivisible by d and for which $a_i \neq 0$. Then in all the above examples, the value of the d-gap is 2 for every $d > 2$. As an example of a function with 3-gap 7, we can consider the power series

$$x + x^4 + x^7 + x^8 + x^9 + \cdots = x(1 + x^3 + x^6 + x^7 + x^8 + \dots).$$

To illustrate yet another type of gap, consider the power series

$$x(1 + x^2 + x^3 + \theta x^5 + \theta^2 x^6 + x^7 + \dots)$$

where θ is a transcendental number. This has a transcendentality gap of size 5, i.e., after factoring out x, the smallest power with transcendental coefficient is x^5.

Formalizing all this, in (3.5) we were led to the definition of the (T, S)-gap v of a nonzero meromorphic series

$$y(T) = T^e \sum_{0 \le i < \infty} A_i T^i \quad \text{with} \quad \text{ord}_T y(T) = e \quad \text{and} \quad A_i \in K \quad \text{with} \quad A_0 \neq 0$$

over a field K, where S is a subfield of the meromorphic series field $K((T))$ and $v = \min\{i \in \mathbb{N} : A_i T^i \notin S\}$. For the definitions of meromorphic series, ord, field, etc., see pages 25–32 and 67–88 of [9], or pages 1–39 of [12]. In particular see the first paragraph of Sect. 2 for the symbols $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}$, and so on.

In the above examples we wrote x for T, and let $e = 1$. In the d-gap case we take $S = K((T^d))$, and in the transcendentality gap case we take $S = k((T))$ where k is an algebraically closed subfield of K. In the absolute gap case we take S to be the null ring $\{0\}$ although technically speaking it is not a subfield.

Assuming $e = 1$, let $z(T) \in K((T))$ be the inverse of $y(T)$, i.e., $\text{ord}_T z(T) = 1$ with $y(z(T)) = T$; note that if $y(T) = \sin T$ then $z(T) = \sin^{-1} T$, and if $y(T) = \tan^{-1} T$ then $z(T) = \tan T$. In (3.5)(IV)(7) we show that the (T, S)-gap of $z(T)$ equals the (T, S)-gap of $y(T)$. We prove this gap invariance by relating the coefficients of $y(T)$ and $z(T)$. Applying the said relating of coefficients to $\tan^{-1} x$ we can recover the Bernoulli expansion of $\tan x$.

Actually, in (3.5) we prove something which is more general than gap invariance. Namely, for any $z(T) \in K((T))$ with $\text{ord}_T z(T) = 1$, without assuming $y(z(T)) = T$ but considering the composition $x(T) = y(z(T))$, by using the multinomial theorem

$$(X_1 + \cdots + X_r)^n = \sum \frac{n!}{t_1! \dots t_r!} X_1^{t_1} \dots X_r^{t_r} \quad \text{with} \quad r \text{ and } n \text{ in } \mathbb{N} \qquad (1)$$

where the summation is over all $t = (t_1, \ldots, t_r) \in \mathbb{N}^r$ with $t_1 + \cdots + t_r = n$, we express the coefficients of x as polynomials in the coefficients of y and z. As a consequence we show that the (T, S)-gaps v, w, π of x, y, z satisfy the relations

$$\begin{cases} \pi \geq \min(v, w) \\ v < w \Rightarrow \pi = v \\ w < v \Rightarrow \pi = w. \end{cases} \tag{2}$$

The $r = 2$ case of (1) is Newton's Binomial Theorem for positive integer exponents which he obtained around 1665. Soon after he generalized it to fractional exponents which led him to his famous theorem on fractional meromorphic series expansion of algebraic functions. For Newton's Theorem and the related result called Hensel's Lemma see pages 89–108 of [9].

In (3.6)(1) and (3.6)(2) we prove some properties of the (T, S)-gap by using the Binomial Lemma (3.3). It should be stressed that in this usage the full force of (3.3) has to be brought into play including the information about the relationship between the initial coefficients of the various meromorphic series.

9.2 Taylor Expansion and Valuations

A power series

$$f(T) = \sum_{0 \leq i < \infty} \alpha_i T^i \in K[[T]] \text{ with } \alpha_i \in K \tag{1}$$

over a field K is a meromorphic series without negative degree terms, i.e., with $\operatorname{ord}_T f(T) \geq 0$. Differentiating both sides i-times and then putting $T = 0$ we get

$$\alpha_i = \frac{f^{(i)}(0)}{i!} \tag{2}$$

where $f^{(i)}(T)$ denotes the i-th T-derivative of $f(T)$. Formula (1) with the value of α_i as in Formula (2), is called the Taylor expansion of $f(T)$. Sometimes it is called the Maclaurin expansion. Maclaurin and Taylor were disciples of Newton. We can use this to deduce the expansions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$$

from the identities

$$\frac{\mathrm{d} \sin x}{\mathrm{d}x} = \cos x \text{ with } \frac{\mathrm{d} \cos x}{\mathrm{d}x} = -\sin x \quad \text{and} \quad \sin 0 = 0 \text{ with } \cos 0 = 1.$$

The definitions of $\sin x$ and $\cos x$ give the last two identities while the first two follow from the equation $\sin^2 x + \cos^2 x = 1$ by implicit differentiation.

For further commentary on Taylor Expansion see pages 104–105 of [9]. There, and on pages 39–43 of [12], you will also find the definition of a (real) discrete valuation of a field L as a surjective (= onto) map $W : L \to \mathbb{Z} \cup \{\infty\}$ such that for all u, u' in L we have $W(uu') = W(u) + W(u')$ and

$$W(u + u') \geq \min(W(u), W(u')) \tag{1}$$

and for any u in L we have: $W(u) = \infty \Leftrightarrow u = 0$. Replacing $\mathbb{Z}$ by any ordered abelian group and deleting the adjective "surjective" we get the definition of a (general) valuation. Note that

$$\begin{cases} W(u) < W(u') \Rightarrow W(u + u') = W(u) \\ W(u') < W(u) \Rightarrow W(u + u') = W(u') \end{cases} \tag{2}$$

Writing v, w, π for $W(u), W(u'), W(u + u')$ and then comparing (1) and (2) with (9.1)(2) we observe an analogy between valuations under sums and gaps under compositions. See pages 65–70 of [9] for the fact that, in case G is subgroup of $\mathbb{R}$, (1) and (2) may be reformulated by saying that sometimes the usual triangle inequality can be replaced by a stronger inequality which requires all triangles to be isosceles.

For any W we put $G_W = W(K^\times)$ and $R_W = \{u \in K : W(u) \geq 0\}$ and call these the value group and the valuation ring of W. Now R_W is a ring with the unique maximal ideal $M(R_W) = \{u \in K : W(u) > 0\}$. Thus R_W is a quasilocal ring to which the second paragraph of Sect. 2 is applicable. More generally, by a valuation ring of a field L we mean the valuation ring of some valuation of L. Finally, by a valuation ring we mean a valuation ring of some field. It can be shown that a ring V is a valuation ring iff V is domain such that: $x \neq 0 \neq y$ in $V \Rightarrow$ either $x/y \in V$ or $y/x \in V$.

This would be a good time to read the rest of Sect. 2. An ambitious reader may also gradually look up the material on pages 43–201 of [12].

9.3 *Discrete Valuation Rings or DVRs*

As a supplement to the reading of Sect. 2, let us add some details about DVRs = discrete valuation rings.

We defined a DVR to be a one-dimensional regular local domain. If V is any DVR then $u \mapsto \mathrm{ord}_V u$ gives a discrete valuation of the field $\mathrm{QF}(V)$ whose valuation ring coincide with V. Conversely, the valuation ring R_W of any discrete valuation W of a field L is a DVR and for all $u \in L$ we have $\mathrm{ord}_{R_W} u = W(u)$. As another characterization of a DVR we note that a domain V is a DVR iff V is a PID such that V has exactly a nonzero prime ideal P and $P^0, P^1, P^2, P^3, \ldots$ are exactly all the distinct nonzero ideals in V. As yet another characterization of a DVR we note that a domain V is a DVR iff V is a DD with exactly one nonzero prime ideal, where DD = Dedekind Domain = a normal noetherian domain of dimension at most one. Here noetherian ring means a ring in which every ideal is finitely generated. Normal domain means a domain which is integrally closed in its quotient field, i.e., every element of its quotient field which is integral over it (i.e., satisfies a monic polynomial equation over the domain) over the domain belongs to the domain. We note that the valuation ring of any valuation is normal.

Recall that a multiplicative set in a domain E is subset M of $E^\times$ with $1 \in M$ such that the product of any two elements in M belongs to M, and the localization E_M of E at M is defined by putting $E_M = \{u/v : u \in E \text{ and } v \in M\}$; note that E_M is a subdomain of QF(E), and if E is noetherian (resp: UFD) then E_M is noetherian (resp: UFD). In case $M = E \setminus P$ for a prime ideal P in E, we may write E_P in place of $E_{E\setminus P}\}$; note that E_P is a quasilocal domain with $M(E_P) = PE_P$.

A typical example of a DVR V is provided by taking a UFD E and letting $V = E_{pE}$ where p is a nonzero nonunit irreducible element in E. For instance, take $E = \mathbb{Z}$ and let p = a prime number, or take E to be the polynomial ring $K[X_1, \ldots, X_n]$ in a finite number of variables over a field K and $p = p(X_1, \ldots, X_n)$ = a nonconstant irreducible polynomial, or take E to be the power series ring $K[[X_1, \ldots, X_n]]$ in a finite number of variables over a field K and $p = p(X_1, \ldots, X_n)$ = a nonzero nonunit irreducible power series.

In the one variable power series case, $K[[X]]$ is itself a DVR. In the one variable polynomial case of $E = K[X]$, for every $a \in K$, the localization $E_a = E_{(X-a)E}$ is a DVR. Moreover,

$$E_\infty = K[1/X]_{(1/X)K[1/X]}$$

is also a DVR; this is the valuation ring of the discrete valuation W of $K(X)$ with $W(X) = -1$ which we call the $(1/X)$-adic valuation of $K(X)$. If K is algebraically closed, then E_∞ together with $(E_a)_{a\in K}$ are exactly all the distinct DVRs with $K \subset V$ and QF(V) $= K(X)$. In case K is not algebraically closed, we have to replace $(E_a)_{a\in K}$ by (E_{pE}) with p varying over all nonconstant monic irreducible polynomials in X over K.

Let V be a DVR with quotient field L, let $H_V : V \to H(V)$ be the residue class epimorphism, let T be a uniformizing parameter of V, and let k be a coefficient set of V. The passage from $\mathbb{Q}$ to $\mathbb{R}$ suggests the definition of the completion $\widehat{V}$ of V together with the quotient field $\widehat{L}$ of $\widehat{V}$ thus. A sequence $y = (y_i)_{1\le i<\infty}$ in L is Cauchy means for every $\epsilon \in \mathbb{N}_+$ there exists $N_\epsilon \in \mathbb{N}_+$ such that for all $i > N_\epsilon$ and $j > N_\epsilon$ we have $\text{ord}_V(y_i - y_j) > \epsilon$. This is equivalent to the Cauchy sequence $y' = (y'_i)_{1\le i<\infty}$ if for every $\epsilon \in \mathbb{N}_+$ there exists $M_\epsilon \in \mathbb{N}_+$ such that for all $i > M_\epsilon$ we have $\text{ord}_V(y_i - y'_i) > \epsilon$. Now $\widehat{L}$ may be defined to be the set of all equivalence classes of Cauchy sequences. Moreover $\widehat{V}$ may be defined to be the set of those members of $\widehat{L}$ which contain a Cauchy sequence consisting of elements of V. Sums and products in $\widehat{L}$ in an obvious manner. This makes $\widehat{L}$ an overfield of L and $\widehat{V}$ an overdomain of V in such a manner that $\widehat{L}$ is the quotient field of $\widehat{V}$. Now $\widehat{V}$ is a DVR and for all $x \in L$ we have $\text{ord}_V x = \text{ord}_{\widehat{V}} x$. Given a sequence $z_1, z_2, \ldots$

and an element z in $\widehat{L}$ we say that z_i tend to z, in symbols $z_i \to z$, to mean that $\mathrm{ord}_{\widehat{V}}(z - z_i) \to \infty$, and we put $\sum_{1 \le i < \infty} z_i = z$ to mean that $\sum_{1 \le j \le i} z_j \to z$. Taking any uniformizing parameter T and coefficient set k of $\widehat{V}$, by mimicking the idea of Taylor expansion, we can show that any $z \in \widehat{L}^\times$ with $\mathrm{ord}_{\widehat{V}} z = e$ can uniquely be expressed as

$$z = \sum_{e \le i < \infty} a_i T^i$$

where $a_i \in k$ with $a_e \neq 0$; we may call this the Taylor expansion of z in $k((T))$; we can extend the sum to the left of e by putting $a_i = 0$ for all $i < e$; if $z = 0$ then we can take $a_i = 0$ for all $i \in \mathbb{Z}$. If k is a coefficient field then $k((T))$ is the usual power series ring.

Let us sketch a proof of the observation made in Sect. 2 to the effect that if A is an affine domain over a field K such that the transcendence degree of the quotient field L of A over K is 1, then $I(A/K)$ is a nonempty finite set where $I(A/K)$ is defined to be the set of all DVRs V with $\mathrm{QF}(V) = L$ such that $A \not\subset V$. For any $x \in A$, let $J(x)$ be the set of all DVRs V with $\mathrm{QF}(V) = L$ such that $x \notin V$. If x is algebraic over K then clearly $J(x)$ is empty. If x is transcendental over K then $J(x)$ is a nonempty finite set because now $L/K(x)$ is a finite algebraic field extension and the members of $J(x)$ are the valuation rings of the extensions to L of the $(1/x)$-adic valuation of $K(x)$. We can write $A = K[x_1, \dots, x_n]$ where $x_1, \dots, x_n$ is a finite set of elements in A at least one of which is transcendental over K. It only remains to note that $I(A/K) = \cup_{1 \le i \le n} J(x_i)$. Geometrically speaking, A represents the affine coordinate ring of a curve C in $\mathbb{A}^n_K$ = the affine n-space over K, and $I(A/K)$ represents the set of branches of C at infinity. Recall that

$$I(A/K) \subset D(L/K) = \begin{cases} \text{the set of all DVRs } V \\ \text{with } K \subset V \text{ and } \mathrm{QF}(V) = L. \end{cases}$$

$D(L/K)$ represents the set of all branches of C, and $D(L/K) \setminus I(A/K)$ represents the set of all branches of C at finite distance.

To talk more about the branches of C in case $n = 2$ and K is algebraically closed, let $f(X, Y)$ be the bivariate irreducible polynomial in $K[X, Y]$ such that $f(x, y) = 0$ where $(x, y) = (x_1, x_2)$. Note that $f(X, Y)$ is unique up to multiplication by a nonzero element of K, and $f(X, Y) = 0$ is an affine equation C. To use homogeneous coordinates, let $F(X, Y, Z) = Z^d f(X/Z, Y/Z)$ where d is the degree of f. Now a point of C at finite distance is of the form $(a, b, 1)$ where a, b in K with $f(a, b) = 0$, and at infinity it is either of the form $(a, 1, 0)$ where $a \in K$ with $F(a, 1, 0) = 0$ or of the form $(1, 0, 0)$ with $F(1, 0, 0) = 0$. Let I_y be the set of all those members V of $I(A/K)$ for which $\mathrm{ord}_V y \le \mathrm{ord}_V x$ and let I_x be the set of all the remaining members of $I(A/K)$. We define the center of any $V \in D(L/K)$ on C thus: if $V \notin I(A/K)$ then it is the point $(a, b, 1)$ of C such that $\mathrm{ord}_V(x - a) > 0 < \mathrm{ord}_V(y - b)$; if $V \in I_y$ then it is the point $(a, 1, 0)$ of C such that $\mathrm{ord}_V((x/y) - a) > 0$; if $V \in I_x$ then it is the point $(1, 0, 0)$ of C. It can be shown that every point of C is the center of at least one and at most a finite number of branches of C. For $V \in D(L/K) \setminus I(A/K)$ and its center $(a, b, 1)$ on C, taking a uniformizing parameter T of $\widehat{V}$, we get the Taylor expansions

$$x = z_1(T) \in K[[T]] \quad \text{and} \quad y = z_0(T) \in K[[T]]$$

with $z_1(0) = a$ and $z_0(0) = b$. We call this a parametrization of C at the point $(a, b, 1)$. It elucidates the material in the short paragraph of (3.1) just before the definition of (V, K)-presequence.

9.4 Newton Expansion and Hamburger-Noether Expansion

Having elucidated a part of (3.1), let us elucidate parts of (3.2) and (3.7). So consider

$$x = z_1(T) \in K((T)) \quad \text{and} \quad y = z_0(T) \in K((T))$$

with

$$\text{ord}_T z_1(T) = \epsilon \in \mathbb{Z}^\times \quad \text{and} \quad \text{ord}_T z_0(T) = e \in \mathbb{Z}^\times$$

where K is an algebraically closed field of characteristic zero. Following Newton, we can expand y in terms of x by first taking an ϵ-th root $\delta(T)$ of x, i.e.,

$$\delta(T) \in K((T)) \text{ with } \delta(T)^\epsilon = z_1(T)$$

and then rewriting y in terms of it as

$$y = \eta(T) \in K((T)) \quad \text{with} \quad \eta(\delta(T)) = z_0(T)$$

Let J be the T-support of $\eta(T)$. The charseq (= characteristic sequence) $m(J, \epsilon)$ is, roughly speaking, a record of the members of J where the GCD with ϵ drops. This is introduced in (3.2) and studied in (3.7). Here the main tool is the concept of d-gap mentioned in (9.1).

We call $\eta(T)$ the Newton expansion of z_0 in terms of z_1. In (3.1) we replicate this without taking roots, and call it the (V, K)-preexpansion which we develop further in (3.8), (3.9), and (4.1) where it culminates into the Valuation Theoretic expansion, i.e., the (V, K)-expansion; here V is a certain DVR. The avoidance of roots motivates items (6)–(8) of (3.1).

The Valuation Theoretic expansion is a generalized version of the so called Hamburger-Noether expansion. The Mixed Valuation Theoretic expansion, i.e., the (V, K, T)-expansion of (4.1) is a mixture of the Newton expansion and the Valuation Theoretic expansion.

Let us now further describe the organization of these numerous expansions.

In (3.1) we introduce the (V, K)-protoexpansion as a simple sequence, and the (V, K)-preexpansion as a double sequence consisting of several sequences each of which is a (V, K)-protoexpansion. In (3.1) we reorganize the (V, K)-preexpansion as a simple sequence which we call the (V, K)-expansion. This reorganization is something like reorganizing an m by n matrix (a_{ij}) as the simple sequence

$$a_{11}, \ldots, a_{1m}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}$$

of length mn. Actually, the rows of the (V, K)-preexpansion may have different lengths. Namely, the i-th row looks like $z_{i0}, \ldots, z_{i,l(i)+1}$ and has length $l(i) + 2$. We chop off its first term and then the first two terms of the chopped off version coincide with the last two terms of the previous row, i.e., $(z_{i-1.l(i-1)}, z_{i-1,l(i-1)+1}) = (z_{i1}, z_{i2})$, and so we glue the two rows at the coincidental terms. Doing this for all except the first row, the (V, K)-preexpansion converts into a single sequence which we call the (V, K)-expansion.

In (3.8), (3.9), and (4.1), we inject some newtonian expansions into the (V, K)-protoexpansion, the (V, K)-preexpansion, and the (V, K)-expansion, and then we call the resulting object the mixed (V, K, T)-protoexpansion, the mixed (V, K, T)-preexpansion, and the mixed (V, K, T)-expansion, respectively.

9.5 Taylor Series with Remainder

The Taylor formula (9.2)(1) may be truncated at some value of i, say $i = j$, and then the last term α_j need not equal $\frac{f^{(j)}(0)}{j!}$. The resulting formula is called Taylor series with remainder. This is illustrated by the crucial formula (3.1)(8) which explains the avoidance of roots mentioned in (9.4). Note that in (3.1), the quantity p_l is not defined until items (6)–(8), and in case of $z_{l+1} \neq 0$, the summation in (8) terminates at $\nu = p_l(e_l/|e_l|)$, i.e., (8) is reduced to the equation

$$z_{l-1} = \left(\sum_{(e_{l-1}/|e_l|) \leq \nu \leq p_l(e_l/|e_l|)} A_l^*(\nu) z_l^{\nu(|e_l|/e_l)} \right) + z_l^* \quad \text{with} \quad z_l^* = z_l^{p_l} z_{l+1}.$$

Also note that in (3.1) we have $e_j > 0$ and $p_j > 0$ for all $j > 1$ and hence, in case of $l \neq 1$, items (6)–(8) become more transparent by putting $|e_l| = e_l$. Finally note that formula (4.1)($4^{\ddagger}$) is another incarnation of (3.1)(8).

To illustrate (3.1)(8) by an example, consider the DVR $V = K[[T]]$ having uniformizing parameter T with coefficient field K, and let

$$z_l = T^3 \text{ and } z_{l-1} = T^6 + T^{9+u} \text{ with } 0 \leq u < 3.$$

Then

$$z_{l-1} = \begin{cases} z_l^2 + z_l^3 + z_l^* \text{ with } z_l^* = z_{l+1} = 0 \;\&\; p_l = \infty & \text{if } u = 0 \\ z_l^2 + z_l^* \text{ with } z_l^* = z_l^3 z_{l+1} \;\&\; z_{l+1} = T^u \;\&\; (p_l, e_{l+1}) = (3, 2) & \text{if } u \neq 0. \end{cases}$$

Let us now further comment on the formation of the mixed (V, K, T)-expansion we talked about in (9.4) above. In (3.8) we consider the sequence

$$(z_0, z_1, \dots, z_l, z_{l+1}, z_l)$$

of meromorphic series in $K((T))$, and we expand each term of the sequence relative to the next term in the newtonian manner, i.e., as a (V, K, T)-expansion. For the last two pairs, this is possible only if $z_{l+1} \neq 0$. The flipping of z_{l+1} and z_l in the end of the sequence is meant for connecting it smoothly to the next sequence of the presequence as achieved in (3.9). Think of two wagons of a railway train being connected at the smooth round buffers. Thus in (3.8), we are constructing a perfect wagon which in (3.9) gets joined to other wagon to form a whole train. In (4.1), the whole train is thought of as a single very long wagon which is called the mixed (V, K, T)-expansion.

9.6 Polynomials and Power Series

The field $K(X_1, \dots, X_n)$ of rational functions over s field K does not determine the polynomial ring $K[X_1, \dots, X_n]$ as can be seen by noting that clearly we have $K[1/X_1, \dots, 1/X_n] \neq K[X_1, \dots, X_n]$ but $K(1/X_1, \dots, 1/X_n) = K(X_1, \dots, X_n)$. However, the quotient field $K((X_1, \dots, X_n))$ of the power series ring $K[[X_1, \dots, X_n]]$ does determine the said ring. See (3.10).

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