Witt Groups of Varieties and the Purity Problem

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Summary We provide a general algorithm used to prove purity for functors with transfers. As a basic example we consider the Witt group of an algebraic variety.

1 The Witt Ring of a Field

Symmetric bilinear spaces. Let k be a field of characteristic $\neq 2$. A *symmetric bilinear space* over k is a pair (V,b) consisting of a vector space V and a symmetric isomorphism $b\colon V \to V^{\#}$ into its dual $V^{\#} = \mathrm{Hom}_k(V,k)$.

Observe that the map b defines a symmetric bilinear form $B\colon V \times V \to k$ via $B(x,y) := b(x)(y)$. Given a basis $\{e_1, e_2, \dots, e_n\}$ of V and a symmetric bilinear space there is a symmetric matrix

$$M_b := \Big(b(e_i)(e_j)\Big)_{i,j=1,\dots,n} = \Big(B(e_i,e_j)\Big)_{i,j=1,\dots,n}.$$

By definition, the map $b\colon V \to V^{\#}$ is an isomorphism if and only if M_b is invertible.

Isometries. Assume we are given two symmetric bilinear spaces (V,b) and (V',b'). An *isometry* $\varphi\colon (V,b) \to (V',b')$ between them is an isomorphism $\varphi\colon V \to V'$ of vector spaces such that the square

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$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b'} \\ V^{\#} & \xleftarrow{\varphi^{\#}} & V'^{\#} \end{array}$$

commutes. For the respective matrices M_b and $M_{b'}$, it means that

$$M_{b'} = C_\varphi^t \cdot M_b \cdot C_\varphi,$$

where C_φ is the transformation matrix corresponding to φ.

Hyperbolic spaces. We define the hyperbolic space of V as

$$H(V) := \left(V \oplus V^{\#}, \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix} \right).$$

A symmetric bilinear space isometric to $H(V)$ is called *hyperbolic*.

The Witt ring. We define the *orthogonal sum* and the *tensor product* of symmetric bilinear spaces as

$$(V,b) \perp (V',b') := (V \oplus V', b \oplus b'),$$
$$(V,b) \otimes (V',b') := (V \otimes V', b \otimes b').$$

The orthogonal sum and the tensor product respect isometries.

Let $KO(k)$ denote the *Grothendieck group* of isometry classes of symmetric bilinear spaces with respect to the orthogonal sum. Let H be the subgroup of $KO(k)$ generated by the classes of hyperbolic spaces. The *Witt group* of k is defined to be the quotient

$$W(k) := KO(k)/H.$$

The tensor product of spaces turns $W(k)$ into a commutative ring with a unit $1 = (k, id)$, where k is a vector space of rank 1.

Properties:

- Let l/k be a field extension. The base change induces a ring homomorphism $i^*\colon W(k) \to W(l)$. Hence, the assignment $l/k \mapsto W(l)$ is a covariant functor from the category of field extensions to the category of commutative rings.
- Let l/k be a finite field extension and $s\colon l \to k$ be a nontrivial k-linear map. Then the composite $s \circ B$, where B is a bilinear form over l, defines a bilinear form over k and, moreover, induces a well-defined group homomorphism $s_*\colon W(l) \to W(k)$ called a Scharlau *transfer*.
- There is the *projection formula*:

$$s_*(i^*(\alpha) \cdot \beta) = \alpha \cdot s_*(\beta) \quad \text{for } \alpha \in W(k) \text{ and } \beta \in W(l).$$

  In particular, the composite $s_* \circ i^*\colon W(k) \to W(k)$ is given by multiplication by $s_*(1)$.

Relations with quadratic forms. By a *quadratic form* q over k, we mean a homogeneous polynomial of degree 2 over k

$$q(x) = \sum_{i,j=1,\dots,n} a_{ij}x_ix_j \quad \text{with } a_{ij} \in k.$$

Assume we are given a symmetric bilinear space (V,b). Let M_b be the symmetric matrix corresponding to the space (V,b) and the chosen basis $\{e_1,e_2,\dots,e_n\}$. We can associate to M_b a quadratic form

$$q(x) = x^t \cdot M_b \cdot x.$$

In the opposite direction, let $q(x)$ be a quadratic form over k. Then we may define a symmetric matrix M_q as

$$M_q := \left(\frac{1}{2}(a_{ij}+a_{ji})\right)_{ij}$$

and, hence, a map $b\colon V \to V^{\#}$ by $b(e_i)(e_j) := (M_q)_{ij}$. If b is an isomorphism, then q is called *nonsingular*.

We have just provided a bijection

isometry classes of nonsingular quadratic forms	$\leftrightarrow$	*isometry classes of symmetric bilinear spaces*

which is compatible with the orthogonal sum and the tensor product of spaces. Hence, to compute the Witt group we can use the language of quadratic forms. The following two properties turn out to be very important for computations:

- *Diagonalization*. Any quadratic form q over k can be diagonalized. Namely, there exists an isometry such that q transforms into a diagonal form

  $$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \quad \text{where } a_i \in k.$$

  We denote such a form as $\langle a_1,a_2,\dots,a_n\rangle$. In particular, a hyperbolic space corresponds to the form $\langle 1,-1,1,-1,\dots,1,-1\rangle$.
- *Witt decomposition*. Any nonsingular quadratic form q can be written uniquely (up to an isometry) as an orthogonal sum of a maximal hyperbolic subspace H and the so called anisotropic part of q

  $$q = q_{an} \perp H.$$

Examples.

- $W(\mathbb{C}) = \mathbb{Z}/2$ or, more generally, $W(k) = \mathbb{Z}/2$ whenever k is quadratically closed, that is, any element of k is a square. Indeed, in this case any form can be diagonalized to $\langle 1,1,\dots,1\rangle$
- $W(\mathbb{R}) = \mathbb{Z}$ (use the signature)

- $W(\mathbb{F}_p) = \begin{cases} \mathbb{Z}/2[k^*/k^{*2}] & \text{if } p \equiv 1 \mod 4 \\ \mathbb{Z}/4 & \text{if } p \equiv 3 \mod 4 \end{cases}$

2 The Witt Ring of a Variety

The affine case. Let R be a commutative ring with unit, $1/2 \in R$. The previous definition of the Witt ring perfectly works if one replaces

a field k	*by*	*a ring R*
a k-vector space V	*by*	*a finitely generated projective R-module P*
an isomorphism b	*by*	*an isomorphism of R-modules* $b\colon P \to P^\#$

In this way, we obtain the definition of the Witt ring of a commutative ring R. Let $X = \mathrm{Spec}(R)$ be the associated affine scheme. Then we define

$$W(X) := W(R).$$

The Euler trace. We extend the notion of the Scharlau transfer to the affine case as follows.

Let T/S be a finite flat extension of commutative rings. Let $s\colon T \to S$ be an S-linear map such that the induced map

$$\lambda\colon T \to \mathrm{Hom}_S(T,S)$$

defined by $\lambda(x)(y) := s(xy)$ is an isomorphism.

Then to every symmetric bilinear space (P,B) over T, we associate the bilinear form $(P_S, s \circ B)$, where P_S denotes P considered as an S-module. This bilinear form gives rise to a symmetric bilinear space over S and, moreover, induces a generalized Scharlau transfer

$$s_*\colon W(T) \to W(S).$$

Example. Let k be a field and let T/S be a finite extension of smooth, purely d-dimensional k-algebras. Let Ω_S and Ω_T be the modules of Kähler differentials of S and T over k and let $\omega_S = \bigwedge^d \Omega_S$ and $\omega_T = \bigwedge^d \Omega_T$.

Assume ω_S and ω_T are trivial. Then there exists an isomorphism of T-modules

$$\lambda\colon T \xrightarrow{\cong} \mathrm{Hom}_S(T,S)$$

which induces an S-linear map $\varepsilon\colon T \to S$ via $\varepsilon(x) := \lambda(1)(x)$, called an *Euler trace*. Observe that from ε we get back λ as $\lambda(x)(y) := \varepsilon(xy)$.

The general case. Let X be a scheme with structure sheaf $\mathscr{O}_X$, $1/2 \in \Gamma(\mathscr{O}_X)$. A symmetric bilinear space over X is a pair $(\mathscr{E}, b)$, consisting of a vector bundle $\mathscr{E}$ and an isomorphism $b\colon \mathscr{E} \to \mathscr{E}^\#$, where $\mathscr{E}^\# = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$ such that $b^\# = b$ (we identify $\mathscr{E}$ with its double dual $\mathscr{E}^{\#\#}$).

An isometry $\varphi\colon (\mathscr{E},b) \to (\mathscr{E}',b')$ of symmetric bilinear spaces is an $\mathscr{O}_X$-linear isomorphism such that $\varphi^{\#} \circ b' \circ \varphi = b$. The orthogonal sum and the tensor product are defined in an obvious way.

If we now introduce hyperbolic spaces $H(\mathscr{E})$ in the same manner as before and then take the quotient, the result will *not* be the Witt group of X but a finer version of it. To obtain the correct definition we have to use the notion of a *metabolic* space instead of hyperbolic. This was first observed by Knebusch [11]. Roughly speaking, the reason is that

$$\textit{metabolic} = \textit{locally hyperbolic}.$$

Metabolic spaces. Let $(\mathscr{E},b)$ be a symmetric bilinear space over X and let $\mathscr{F}$ be a subbundle of $\mathscr{E}$, that is, a locally direct summand of $\mathscr{E}$.

For a subbundle $\mathscr{F}$ of $\mathscr{E}$ we define its *orthogonal complement* $\mathscr{F}^{\perp}$ as the kernel of the composite $i^{\#} \circ b$, where $i^{\#}\colon \mathscr{E}^{\#} \to \mathscr{F}^{\#}$ is the dual of the inclusion $i\colon \mathscr{F} \to \mathscr{E}$. Clearly, $i^{\#}$ is an epimorphism and b induces an isomorphism $\mathscr{E}/\mathscr{F}^{\perp} \to \mathscr{F}^{\#}$. In particular, $\mathscr{F}^{\perp}$ is again a subbundle of $\mathscr{E}$.

A subbundle $\mathscr{F}$ of $\mathscr{E}$ is called a *lagrangian* of $\mathscr{E}$ if $\mathscr{F} = \mathscr{F}^{\perp}$. A symmetric bilinear space is called *metabolic* if it contains a lagrangian.

Observe that in the field case, that is, $X = \mathrm{Spec}(k)$, the orthogonal complement of a subspace U of (V,b) coincides with

$$U^{\perp} = \{x \in V \mid B(x,y) = 0 \text{ for all } y \in U\},$$

where B is the corresponding bilinear form.

The Witt group. It is defined to be the quotient of the Grothedieck group $KO(X)$ of isometry classes of symmetric bilinear spaces over X modulo the subgroup M generated by metabolic spaces, that is,

$$W(X) := KO(X)/M.$$

As before, the tensor product turns $W(X)$ into a ring. Note that in the affine case this definition gives the previously defined Witt ring.

Examples.

- $W(\mathbb{A}^n_X) = W(X)$, where X is affine – Karoubi [10]
- $W(\mathbb{P}^n_k) = W(k)$ for $n = 1$ follows from the description of vector bundles over the projective line and for $n \geq 2$ – Arason [1]
- For an irreducible smooth quasiprojective complex curve C

  $$W(C) = \mathbb{Z}/2 \oplus \mathrm{Disc}(C),$$

  where $\mathrm{Disc}(C)$ is the group of isometry classes of symmetric bilinear spaces of rank 1. (This follows from [9].) In particular, if C is projective of genus g, then

  $$W(C) = (\mathbb{Z}/2)^{1+2g}.$$

  For conic, elliptic, and hyperelliptic curves, see Parimala [18].

- For an irreducible smooth quasiprojective complex surface X

$$W(X) = \mathbb{Z}/2 \oplus \mathrm{Disc}(X) \oplus {}_2\mathrm{Br}(X).$$

For real projective surfaces, see Sujatha [21]. For affine threefolds, see Parimala [19].

One of the main technical tools is the exact sequence

$$W(X) \to W(k(X)) \xrightarrow{\partial_x} \bigoplus_{x \in X^{(1)}} W(k(x)),$$

where X is an integral regular scheme over k, $k(X)$ is its quotient field, $k(x)$ is the residue field at a point x of codimension 1, and ∂_x is a second residue homomorphism which depends on the choice of a local parameter. The exactness of this sequence at the $W(k(X))$-term is called *purity* and is the main subject of these lectures.

Purity for the Witt groups is known for the following cases:

- X is a regular integral noetherian scheme of dimension at most 2 by Colliot-Thélène and Sansuc [8].
- X is a regular integral noetherian affine scheme of dimension 3 by Ojanguren, Parimala, Sridharan, Suresh [15].
- $X = \mathrm{Spec}(R)$, where R is a local regular ring containing a field by Ojanguren and Panin [14].

Triangular Witt groups. A vast generalization of the notion of the Witt group of a scheme was provided by Balmer [2]. He introduced and studied the notion of the Witt group $\mathscr{W}$ of a *triangulated category with duality*. For triangular Witt groups the following computations were obtained:

- $\mathscr{W}(\mathbb{P}_X(\mathscr{E}))$, $\mathscr{W}(\text{quadric})$ by Nenashev [12], Walter
- $\mathscr{W}(\text{Grassmannian})$ by Balmer and Calmès [4]

3 Purity

Let A be a smooth algebra over an infinite field k and let K be its quotient field. Let $R = A_{\mathfrak{q}}$ be the localization of A at a prime ideal $\mathfrak{q} \in \mathrm{Spec}(A)$. Such a ring R is called a *local regular ring of geometric type*.

Let $F\colon A\text{-Alg} \to Ab$ be a covariant functor from the category of A-algebras to the category of Abelian groups. From now on

- the local regular ring of geometric type R
- the covariant functor F

will be the main objects of our discussion.

Definition 1. (cf. [6, 2.1.1]) For every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, consider the group homomorphism $F(R_\mathfrak{p}) \to F(K)$ induced by the canonical inclusion of $R_\mathfrak{p}$ into its quotient field K.

We say that an element $\alpha \in F(K)$ is *unramified at a prime ideal* $\mathfrak{p}$ if it is in the image of $F(R_\mathfrak{p})$. We say that an element $\alpha \in F(K)$ is *unramified over* R if it is unramified at every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ of height 1.

Definition 2. (cf. [6, 2.1.4.(b)]) We say that *purity* holds for the functor F over R if every unramified element of $F(K)$ belongs to the image of $F(R)$ in $F(K)$. In other words, the following equality holds between subgroups of $F(K)$:

$$\bigcap_{ht\ \mathfrak{p}=1} \mathrm{im}\{F(R_\mathfrak{p}) \to F(K)\} = \mathrm{im}\{F(R) \to F(K)\}.$$

The subgroup of unramified elements (the left-hand side of the equality) will be denoted by $F_{nr}(K)$.

Remark. Purity is a particular case of the following more general problem, called the Gersten conjecture: Given a cohomology theory $F = H^*$ over R, show that the *Gersten complex*

$$0 \to H^*(R) \to H^*(K) \to \bigoplus_{x\in U^{(1)}} H^{*-1}(k(x)) \to \bigoplus_{x\in U^{(2)}} H^{*-2}(k(x)) \to \cdots$$

is exact.

Here H^* is a functor to the category of graded abelian groups which satisfies certain axioms (homotopy invariance, excision, localization, etc.) as in [7, Sect. 5] or [16, Sect. 2]. Observe that exactness at the $H^*(K)$-term gives purity.

For the Witt groups the Gersten conjecture was proven

- for a local regular ring of geometric type over an infinite field of characteristic not 2 – Balmer [3] and Schmid [20]
- for a local regular ring R containing a field of characteristic not 2 – Balmer, Gille, Panin, and Walter [5]

Remark. Purity is closely related to the following problem, called *injectivity*: Given a regular local ring R and a functor F from the category of R-algebras to the category of pointed sets, show that the induced map $F(R) \to F(K)$ has a trivial kernel.

If $F = H^1_{et}(-,G)$, where G is a smooth reductive group scheme over R, it is equivalent to the Grothendieck-Serre conjecture (see [8, 6.5, p. 124]) which is still open. For the Witt groups the injectivity is due to Ojanguren [13] and Pardon.

Our goal is to prove the following general fact (see [22]).

Theorem 3 (Purity Theorem). *Let R be a local regular ring of geometric type obtained by localizing a smooth k-algebra A. Let $F: A$-Alg $\to Ab$ be a functor with transfers described below. Then purity holds for F over R.*

As a corollary of the proof we obtain the following celebrated result.

Theorem 4 (Ojanguren, Panin). *Let R be a regular local ring containing a field k,* $\mathrm{char}(k) \neq 2$. *Then purity holds for the Witt functor $F = W$ over R.*

- The proof of Theorem 3 uses the same techniques as the original proof by Ojanguren–Panin.
- The Witt group formally does not satisfy the condition of being a functor with transfers. Nevertheless, since for all varieties appearing in the proof the canonical sheaves ω turn out to be trivial, the proof works after replacing transfers by Euler traces.

Functors with transfers. Let R be a local regular ring of geometric type obtained by localizing a smooth k-algebra A. Let $F\colon A\text{-Alg} \to Ab$ be a covariant functor. We say the F is a *functor with transfers* if it satisfies the following axioms:

(C) (Continuity) Roughly speaking, it says that F commutes with filtered direct limits of localizations. More precisely, for any A-algebra S essentially smooth over k and for any multiplicative system M in S the canonical map

$$\lim_{g \in M} F(S_g) \to F(M^{-1}S)$$

is an isomorphism, where $M^{-1}S$ denotes the localization of S with respect to M.

(T) (Structure of transfer maps) Let $F_R\colon R\text{-Alg} \to Ab$ denote the restriction of the functor F to the category of R-algebras via the canonical inclusion $A \hookrightarrow R$. For any finite étale R-algebra T, there exist homomorphisms

$$\mathrm{Tr}_R^T\colon F_R(T) \to F_R(R) \quad \text{and} \quad \mathrm{Tr}_K^{T\otimes_R K}\colon F_R(T \otimes_R K) \to F_R(K)$$

called transfer maps which satisfy the following conditions:

(a) $\mathrm{Tr}_R^R = id_R$ and for any finite étale R-algebras T_1 and T_2 the following relation holds

$$\mathrm{Tr}_R^{T_1 \times T_2}(x) = \mathrm{Tr}_R^{T_1}(x_1) + \mathrm{Tr}_R^{T_2}(x_2).$$

(b1) Let $R[t]$ denote a polynomial ring over R. For a finite $R[t]$-algebra S such that $S/(t)$ and $S/(t-1)$ are finite étale over R the following diagram commutes:

$$\begin{array}{ccc} F_R(S) & \longrightarrow & F_R(S/(t)) \\ \downarrow & & \downarrow \mathrm{Tr} \\ F_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}]{} & F_R(R) \end{array}$$

(b2) The transfer map $\mathrm{Tr}_K^{T\otimes_R K}$ satisfies conditions (a) and (b1) above and the following diagram induced by extension of scalars via the canonical inclusion $R \hookrightarrow K$ commutes:

$$\begin{array}{ccc} F_R(T) & \longrightarrow & F_R(T\otimes_R K) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ F_R(R) & \longrightarrow & F_R(K) \end{array}$$

Observe that if F is *homotopy invariant*, that is, if the canonical inclusion induces an isomorphism $F_R(R') \simeq F_R(R'[t])$ for any R-algebra R', and transfer maps satisfy the general base change, that is, (b2) holds for any extension R'/R, then the condition (b1) follows automatically.

(E) (Finite monodromy) A certain technical condition which, roughly speaking, means that restrictions of F via canonical maps $A \to A\otimes_k A$, $a \mapsto a\otimes 1$ and $a \mapsto 1\otimes a$, become locally isomorphic in the étale topology. This condition holds automatically for the Witt group and for any functor defined over a field.

Examples. Let S be a k-algebra.

- $F = \mathbb{G}_m$, that is, $F\colon S \mapsto S^*$. Purity is equivalent to the exact sequence

$$R^* \to K^* \overset{\oplus v_{\mathfrak{p}}}{\to} \bigoplus_{ht\,\mathfrak{p}=1} \mathbb{Z}$$

- For the functors K_* of K-theory purity means that the sequence

$$K_*(R) \to K_*(K) \overset{\partial_{\mathfrak{p}}}{\to} \bigoplus_{ht\,\mathfrak{p}=1} K_{*-1}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$$

is exact.
- $F = \mathrm{coker}(\mu)$, where $\mu\colon G \to \mathbb{G}_m$ is a surjective morphism of group schemes and G is a linear algebraic group over k, $\mathrm{char}(k) = 0$, which is rational as a k-variety.
- $F = H^*_{et}(-,\mathscr{C})$, where $\mathscr{C}$ is a locally constant sheaf with finite stalks of $\mathbb{Z}/n$-modules over R, $(n, \mathrm{char}(k)) = 1$.
- $F = \mathrm{coker}(\mathrm{Nrd})$, where $\mathrm{Nrd}\colon \mathrm{GL}_{1,\mathscr{A}} \to \mathbb{G}_m$ is the reduced norm of an Azumaya algebra $\mathscr{A}$ over R.
- $F = \mathrm{coker}(\mathrm{Sn})$, where $\mathrm{Sn}\colon \mathrm{SO}_q \to H^1_{et}(-,\mu_2)$ is the spinor norm of a nonsingular quadratic form q defined over R.
- $F = \mathrm{coker}(\mathrm{Nrd})$, where $\mathrm{Nrd}\colon U(\mathscr{A},\sigma) \to U(Z,\sigma|_Z)$ is the reduced norm from the unitary group of an Azumaya algebra $\mathscr{A}$ with involution of the second kind σ over R to the unitary group of its center Z.

We also have the torsion versions of the previous cases. Here $d > 1$ and S is an R-algebra.

- $F\colon S \mapsto S^*/\mathrm{Nrd}(\mathscr{A}_S^*)\cdot(S^*)^d$
- $F\colon S \mapsto U(Z,\sigma)/\mathrm{Nrd}(U(\mathscr{A},\sigma))\cdot U(Z,\sigma)^d$

4 The Proof of the Purity Theorem

First, we discuss two main geometric ingredients of the proof: the Geometric Presentation Lemma and the Quillen trick.

Lemma 5 (Geometric Presentation Lemma). *Let R be a local essentially smooth k-algebra with maximal ideal $\mathfrak{m}$ over an infinite field k, and S an essentially smooth integral k-algebra, finite over the polynomial algebra $R[t]$. Suppose that $\varepsilon\colon S \to R$ is an R-augmentation and let $I = \ker \varepsilon$. Assume that $S/\mathfrak{m}S$ is smooth over the residue field $R/\mathfrak{m}$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$. Then, given a regular function $f \in S$ such that $S/(f)$ is finite over R, we can find a $t' \in I$ such that:*

- *S is finite over $R[t']$.*
- *There is an ideal J comaximal with I such that $I \cap J = (t')$.*
- *The ideals J and $(t'-1)$ are comaximal with the ideal (f).*
- *$S/(t')$ and $S/(t'-1)$ are étale over R.*

Proof. We will only show how to construct this t' and establish the first property and the part of the last.

Replacing t by $t - \varepsilon(t)$ we may assume that $t \in I$. We denote by "bar" the reduction modulo $\mathfrak{m}$. By the assumptions made on S the quotient $\bar{S} = S/\mathfrak{m}S$ is smooth over the residue field $\bar{R} = R/\mathfrak{m}$ at its maximal ideal $\bar{I} = \varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$.

Choose an $\alpha \in S$ such that $\bar{\alpha}$ is a local parameter of the localization of $\bar{S}$ at $\bar{I}$. By the Chinese Remainder theorem we may assume that $\bar{\alpha}$ does not vanish at the zeros of $\bar{f}$ different from $\bar{I}$.

Without changing $\bar{\alpha}$ we may replace α by $\alpha - \varepsilon(\alpha)$ and assume that $\alpha \in I$. Since S is integral over $R[t]$, there exists a relation of integral dependence

$$\alpha^n + p_1(t)\alpha^{n-1} + \cdots + p_n(t) = 0. \qquad (*)$$

For any $r \in k^\times$ and any N larger than the degree of each $p_i(t)$ we put

$$t' = \alpha - rt^N.$$

By the equation $(*)$, t is integral over $R[t']$. Hence S, which is integral over $R[t]$, is integral over $R[t']$ and the first property is proven.

By Bertini's theorem we may choose α such that the algebra $\bar{S}/(\bar{\alpha})$ is étale over $\bar{R}$. Consider the fiber product diagram

$$\begin{array}{ccc}
\bar{S}[u]/(\bar{\alpha} - u\bar{t}^N) & \longrightarrow & \bar{S}/(t') \\
\uparrow \pi & & \uparrow \\
k[u] & \xrightarrow{u \mapsto r} & k
\end{array}$$

Since the fiber of π at $u = 0$ is étale, the fibers of π at almost all rational points $u \in k^\times$ are étale. In other words, $\bar{S}/(\bar{t}')$ is étale over k and, hence, over $\bar{R}$ for almost all $r \in k^\times$.

By assumption S and $R[t']$ are smooth. Since S is finite over $R[t']$, S is finitely generated projective as an $R[t']$-module and, hence, $S/(t')$ is free as an R-module. In particular, $S/(t')$ is flat over R. Finally, the fact that $\bar{S}/(\bar{t}')$ is étale over $\bar{R}$ implies that $S/(t')$ is étale over R. □

Observe that in the original statement of the geometric presentation lemma by Panin–Ojanguren the last condition (that the fibers are étale) was missing. Hence, our lemma (see [22, Theorem 6.1]) provides a slightly stronger version. This difference becomes important when one looks for transfer maps related with certain group schemes (e.g., the spinor norm case discussed in [23]).

The following fact, which is due to Quillen, can be viewed as a generalization of Noether's Normalization Lemma.

Lemma 6 (Quillen's trick). *Let A be a smooth finite-type algebra of dimension d over a field k. Let $f \in A$ be a regular function. Let $\mathscr{I}$ be a finite subset of* $\operatorname{Spec}(A)$.

Then there exist functions $x_1, \dots, x_d$ in A algebraically independent over k such that if $\mathfrak{i}\colon W = k[x_1, \dots, x_{d-1}] \to A$ denotes the inclusion, then

- *$A/(f)$ is finite over W.*
- *A is smooth over W at the points of $\mathscr{I}$.*
- *The inclusion $\mathfrak{i}$ factors as $\mathfrak{i}\colon W \hookrightarrow W[x_d] \to A$, where the last map is finite.*

A generalization of this lemma involving some support condition was proven recently by Panin and the author in [17].

Proof of the Purity Theorem. Let $R = A_{\mathfrak{q}}$ be a localization of a smooth k-algebra A at a prime ideal $\mathfrak{q}$ and let K be its quotient field. We have to prove that any unramified element $\alpha \in F_{nr}(K)$ belongs to the image of the canonical map $i_K^*\colon F(R) \to F(K)$. We give only a brief sketch of the proof.

The proof consists of three steps.

Step 1. We may choose $f \in A$ in such a way that the element α is the image of some element α_f by means of the canonical map $F(A_f) \to F_{nr}(K)$. By applying Quillen's trick to the algebra A, the function f and the subset $\mathscr{I} = \{\mathfrak{q}\}$ we obtain the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto a \otimes 1} & A \otimes_W R \\ \uparrow & & \uparrow {\scriptstyle r \mapsto 1 \otimes r} \\ W & \xrightarrow{\mathfrak{i}} & R \end{array}$$

where $\mathfrak{i}\colon W \to A \to A_{\mathfrak{q}}$ is the respective inclusion of the ring of polynomials W. Let

$$\varphi\colon F(A_f) \to F((A \otimes_W R)_f)$$

be the map induced by the upper horizontal arrow.

Step 2. Let $S = A \otimes_W R$. It is easy to check that S/R and $f = f \otimes 1 \in S$ satisfy the hypothesis of the Geometric Presentation Lemma. Applying that lemma (with $t = x_d$) we then construct $t' \in R[t]$ and the ideal J. Let

$$\psi\colon F(S_f) \to F(R)$$

be the map defined by $\psi = \mathrm{Tr}_1 \circ p_1^* - \mathrm{Tr}_J \circ p_J^*$, where $p_1\colon S_f \to S/(t'-1)$, $p_J\colon S_f \to S/J$ are the quotient maps and Tr_1, Tr_J are the respective transfers.

This is a key point of the proof: There is no map $F(S_f) \to F(R)$ which comes from a regular one, however, by the Geometric Presentation Lemma one can define its substitute ψ using transfers.

Step 3. Consider the commutative diagram

$$\begin{array}{ccccc} F(A_f) & \xrightarrow{\varphi} & F(S_f) & \xrightarrow{i_K^*} & F((S \otimes_R K)_f) \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_K} \\ & & F(R) & \xrightarrow{i_K^*} & F(K) \end{array}$$

where the square is obtained by the base change and the maps φ and ψ were defined before. By commutativity, the image of $\alpha' = \psi(\varphi(\alpha_f))$ in $F(K)$ coincides with $\psi_K(\beta)$, where $\beta = i_K^*(\varphi(\alpha_f))$. By the homotopy invariance *(b2)* we obtain that

$$\psi_K(\beta) = \mathrm{Tr}_1 \circ p_1^*(\beta) - \mathrm{Tr}_J \circ p_J^*(\beta) = \mathrm{Tr}_0 \circ p_0^*(\beta) - \mathrm{Tr}_J \circ p_J^*(\beta),$$

and by the additivity *(a)* the latter coincides with the image of the augmentation map $\varepsilon_K^*(\beta)$. Here $\varepsilon_K\colon (S \otimes_R K)_f \to K$ is given by the usual multiplication.

On the other hand, one can show that $\varepsilon_K(\beta) = \varepsilon_K(i_K^*(\varphi(\alpha_f))) = \alpha$. Therefore, α is the image of α' by means of the canonical map i_K^*, and the proof is finished. □

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