Cohomological Invariants of Central Simple Algebras with Involution

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To Parimala, with gratitude and admiration

Summary This survey reviews the various invariants with values in Galois cohomology groups that have been defined for involutions on central simple algebras following the model of the discriminant, Clifford invariant, and Arason invariant of quadratic forms. From the orthogonal case to the unitary case to the symplectic case, the degree of the invariants increases but their properties are similar.

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University of Hyderabad. The purpose was to survey the cohomological invariants that have been defined for various types of involutions on central simple algebras on the model of quadratic form invariants. I seized the occasion to make explicit some of the classification or structure results that may be expected from future invariants, and to compile a fairly extensive list of references. As this list makes clear, Parimala's contributions to the subject are all-pervasive.

Throughout these notes, F denotes a field of characteristic different from 2 and F_s a separable closure of F. We identify $\mu_2 := \{\pm 1\}$ with $\mathbb{Z}/2\mathbb{Z}$. For any integer $n \geq 0$, we let $H^n(F)$ be the Galois cohomology group:

$$H^n(F) := H^n(\mathrm{Gal}(F_s/F), \mathbb{Z}/2\mathbb{Z}).$$

The Kummer exact sequence

$$1 \to \mu_2 \to F_s^\times \xrightarrow{2} F_s^\times \to 1$$

and Hilbert's Theorem 90 yield identifications $H^1(F) = F^\times/F^{\times 2}$ and $H^2(F) = {}_2\mathrm{Br}(F)$, the 2-torsion subgroup of the Brauer group of F. For $a \in F^\times$, we let $(a) \in H^1(F)$ denote the cohomology class given by the square class $aF^{\times 2}$. We use the notation $\cdot$ for the cup product in the cohomology ring $H^*(F)$, so for $a, b \in F^\times$ the Brauer class of the quaternion algebra $(a,b)_F$ is $(a) \cdot (b)$.

1 Introduction: Classification of Quadratic Forms

Various invariants are classically defined to determine whether quadratic forms over an arbitrary field F are isometric. The first invariant is of course the dimension. To obtain an invariant that vanishes on hyperbolic forms, one considers the dimension modulo 2:

$$e_0(q) = \dim q \pmod 2 \in \mathbb{Z}/2\mathbb{Z}.$$

The next invariant is the discriminant: for q a quadratic form of dimension n, we set

$$e_1(q) = (-1)^{n(n-1)/2} \det q \in F^\times/F^{\times 2}.$$

Thus, e_1 is well defined on the Witt group WF; but it is a group homomorphism only when restricted to the ideal IF of even-dimensional forms, which is the kernel of e_0.

In his foundational paper, Witt defined as a further invariant the Brauer class of the Clifford or even Clifford algebra (depending on which is central simple over F):

$$e_2(q) = \begin{cases} [C(q)] \in {}_2\mathrm{Br}(F) & \text{if } \dim q \text{ is even,} \\ [C_0(q)] \in {}_2\mathrm{Br}(F) & \text{if } \dim q \text{ is odd.} \end{cases}$$

The map e_2 is well defined on WF but it is a group homomorphism only on the square[1] I^2F of IF, which is the kernel of e_1.

Note that the "classical" invariants above take their values in cohomology groups:

$$\mathbb{Z}/2\mathbb{Z} = H^0(F), \quad F^\times/F^{\times 2} = H^1(F), \quad {}_2\mathrm{Br}(F) = H^2(F).$$

They were vastly generalized during the last quarter of the twentieth century: see Pfister's survey [42] for an account of the historical development of the subject. After Voevodsky's proof of the Milnor conjecture and additional work of Orlov–Vishik–Voevodsky, we have surjective homomorphisms

$$e_n\colon I^nF \to H^n(F) \quad \text{for all } n \geq 0$$

defined on n-fold Pfister forms $\langle\langle a_1,\dots,a_n\rangle\rangle := \langle 1,-a_1\rangle \otimes \cdots \otimes \langle 1,-a_n\rangle$ by

$$e_n\big(\langle\langle a_1,\dots,a_n\rangle\rangle\big) = (a_1)\cdot \cdots \cdot (a_n).$$

These homomorphisms satisfy

$$\ker e_n = I^{n+1}F \quad \text{for all } n \geq 0.$$

On the other hand, the Arason–Pfister Hauptsatz [13, Theorem 23.7] states that the dimension of any anisotropic quadratic form q representing a nonzero element in I^nF satisfies $\dim q \geq 2^n$, hence $\bigcap_n I^nF = \{0\}$. Therefore, the problem of deciding whether two quadratic forms q_1 and q_2 over F are isometric can in principle be solved by computing cohomology classes: indeed, q_1 and q_2 are isometric if and only if $\dim q_1 = \dim q_2$ and $q_1 - q_2$ is hyperbolic, and the latter condition can be checked by computing successively[2] $e_0(q_1-q_2), e_1(q_1-q_2), e_2(q_1-q_2), \dots$, which should all vanish. In view of the Arason–Pfister Hauptsatz, it actually suffices to check that $e_d(q_1-q_2) = 0$ for all $d \geq 0$ with $2^d \leq \dim q_1 + \dim q_2$.

This remarkably complete classification result is a model that we want to emulate for other algebraic objects. The problem can be formalized as follows: we are given a base field F and a functor $\mathfrak{A}\colon \mathsf{Fields}_F \to \mathsf{Sets}$ from the category of fields containing F, where the morphisms are F-algebra homomorphisms, to the category of sets. Typically, $\mathfrak{A}(K)$ is the set of K-isomorphism classes of some objects (like quadratic forms) that allow scalar extension. An *invariant* of $\mathfrak{A}$ with values in a functor $\mathfrak{H}\colon \mathsf{Fields}_F \to \mathsf{Sets}$ is a natural transformation of functors:

$$e\colon \mathfrak{A} \to \mathfrak{H}.$$

Typically, $\mathfrak{H}$ is in fact a functor to the category of abelian groups. If it is a Galois cohomology functor $\mathfrak{H}(K) = H^d(K,M)$ for some discrete Galois module M, the invariant is called a *cohomological invariant of degree d*. For examples and background information, we refer to Serre's contribution to the monograph [17]. In particular,

[1] For any integer $n \geq 2$, we let $I^nF = (IF)^n$.

[2] For $d \geq 3$, the element $e_d(q_1-q_2)$ is defined only if $e_{d-1}(q_1-q_2) = 0$.

cohomological invariants of quadratic forms are determined in [17, Sect. 17]: they are generated by the so-called Stiefel–Whitney invariants. (The invariants e_1 and e_2 above can be computed in terms of Stiefel–Whitney classes, but *not* e_i for $i \geq 3$, see [32, pp. 135, 367].)

In the present notes, we consider the case where $\mathfrak{A}(F)$ is the set of isomorphism classes of central simple F-algebras with involution, as explained in the next section. Our aim is not to describe the collection of *all* cohomological invariants of $\mathfrak{A}$, but rather to define a sequence of invariants $e_1, e_2, \dots$, each of which is defined on the subfunctor of $\mathfrak{A}$ on which the preceding invariant vanishes, just as in the case of quadratic forms. Invariants of degree 2 are essentially Brauer classes of the *Tits algebras* that arise in the representation theory of linear algebraic groups [53]. Indeed, central simple algebras with involution define torsors under adjoint classical groups (see Sect. 2), and our approach via central isogenies owes much to Tits's discussion of what he calls the β-invariant in [53, 54].

From the Bayer-Fluckiger–Parimala proof of Serre's conjecture II for classical groups [2], it follows that the invariants of degree 1 and 2 are sufficient to classify involutions over a field of cohomological dimension 1 or 2. We shall not address this topic here, and refer to [8, 35] for this classification. Below, we aim at classification results of a different kind, restricting the dimension of the algebra rather than the cohomological dimension of the center (see Theorem 3.1 for a paradigmatic case).

2 From Quadratic Forms to Involutions

In these notes, an *involution* on a ring is an antiautomorphism of order 2. The involution is said to be *of the first kind* if its restriction to the center is the identity; otherwise, this restriction is an automorphism of order 2 and the involution is said to be *of the second kind*.

Every quadratic form q on an F-vector space V defines an involution ad_q on the endomorphism algebra $\mathrm{End}_F V$, as follows: letting b denote the bilinear polar form of q, the adjoint involution $\mathrm{ad}_q\colon \mathrm{End}_F V \to \mathrm{End}_F V$ is uniquely determined by the property that

$$b\big(x, f(y)\big) = b\big(\mathrm{ad}_q(f)(x), y\big) \quad \text{for all } x, y \in V \text{ and } f \in \mathrm{End}_F V.$$

Note that the involution is really adjoint to the bilinear form, not to the quadratic form, hence we also use the notation ad_b for ad_q. Since every F-automorphism of $\mathrm{End}_F V$ is inner, it is easy to check that every F-linear involution on $\mathrm{End}_F V$ is adjoint to a nonsingular bilinear form that is either symmetric or skew-symmetric. There are therefore two types of involutions of the first kind on $\mathrm{End}_F V$, which can be distinguished by the dimension of their space of symmetric elements: the *orthogonal* involutions are adjoint to quadratic forms (or symmetric bilinear forms) and the *symplectic* involutions are adjoint to skew-symmetric bilinear forms. The involutions actually determine up to a scalar factor the form to which they are adjoint, so mapping each nonsingular symmetric or skew-symmetric form b to the adjoint involution ad_b defines bijections:

nonsingular symmetric bilinear forms on V up to a scalar factor	$\longleftrightarrow$	orthogonal involutions on $\mathrm{End}_F V$

and

nonsingular skew-symmetric bilinear forms on V up to a scalar factor	$\longleftrightarrow$	symplectic involutions on $\mathrm{End}_F V$

Central simple F-algebras are twisted forms (in the sense of Galois cohomology) of endomorphism algebras; therefore, one may also distinguish two types of involutions of the first kind on a central simple F-algebra A: an involution $\sigma\colon A \to A$ is called *orthogonal* (resp., *symplectic*) if for any splitting field K and under any K-isomorphism $A \otimes_F K \simeq \mathrm{End}_K V$, the involution $\sigma \otimes \mathrm{Id}_K$ obtained by scalar extension is adjoint to a nonsingular symmetric (resp., skew-symmetric) bilinear form. Note that symplectic involutions exist only on central simple algebras of even degree, since every skew-symmetric bilinear form on an odd-dimensional vector space is singular.

Involutions on a central simple algebra can also be obtained by adjunction, as in the split case: if we fix a representation $A = \mathrm{End}_D V$, where D is a central division F-algebra Brauer equivalent to A and V is a (right) D-vector space, and choose an involution of the first kind θ on D, then every involution of the first kind on A is adjoint to a nonsingular hermitian or skew-hermitian form on V with respect to θ, and this hermitian form is uniquely determined up to a factor in $F^\times$.

The correspondence between involutions of the first kind and bilinear forms can also be described in terms of the corresponding automorphism groups. Recall that the orthogonal group O_n is the group of isometries of the standard form $\langle 1,\dots,1\rangle$ of dimension n:

$$\mathrm{O}_n(F) = \mathrm{Aut}(\langle 1,\dots,1\rangle).$$

Let $H^1(F,\mathrm{O}_n)$ denote the nonabelian Galois cohomology set:

$$H^1(F,\mathrm{O}_n) = H^1\big(\mathrm{Gal}(F_s/F),\mathrm{O}_n(F_s)\big).$$

The usual technique of nonabelian Galois cohomology (see [49, Chap. III, Sect. 1] or [29, (29.28)]) yields a canonical bijection:

$$\boxed{\text{isometry classes of quadratic forms of dimension } n \text{ over } F} \longleftrightarrow H^1(F,\mathrm{O}_n). \tag{2.1}$$

This bijection maps the isometry class of the standard form to the distinguished element in $H^1(F,\mathrm{O}_n)$.

The transpose involution t on $M_n(F) = \mathrm{End}_F F^n$ is adjoint to the standard form; its automorphism group consists of the inner automorphisms $\mathrm{Int}(g)$ such that

$$\mathrm{Int}(g) \circ t = t \circ \mathrm{Int}(g).$$

This condition amounts to $\mathrm{Int}(g^{\mathrm{t}} g) = \mathrm{Id}_V$, hence to $g^{\mathrm{t}} g \in F^{\times}$; it defines the group $\mathrm{GO}_n(F)$ of *similitudes* of the standard form:

$$\mathrm{GO}_n(F) = \{g \in \mathrm{GL}_n(F) \mid g^{\mathrm{t}} g \in F^{\times}\}.$$

The automorphisms of the transpose involution are the $\mathrm{Int}(g)$ with $g \in \mathrm{GO}_n(F)$, hence

$$\mathrm{Aut}(t) = \mathrm{PGO}_n(F) := \mathrm{GO}_n(F)/F^{\times}.$$

Again, Galois cohomology yields a bijection (see [29, Sect. 29.F]):

isomorphism classes of orthogonal involutions on central simple algebras of degree n	$\longleftrightarrow$	$H^1(F, \mathrm{PGO}_n)$.	(2.2)

Under this bijection, the isomorphism class of the transpose involution corresponds to the distinguished element of $H^1(F, \mathrm{PGO}_n)$. The canonical map $\mathrm{O}_n(F_{\mathrm{s}}) \to \mathrm{PGO}_n(F_{\mathrm{s}})$ yields a map $H^1(F, \mathrm{O}_n) \to H^1(F, \mathrm{PGO}_n)$. Under the bijections (2.1) and (2.2), this map carries the isometry class of any quadratic form q of dimension n to the isomorphism class of the adjoint involution ad_q on the split algebra of degree n.

In view of the close relationship between quadratic forms and (orthogonal) involutions, it may be expected that invariants for quadratic forms have counterparts for involutions. There is, however, a significant difference to keep in mind: in contrast with quadratic forms, there is no[3] direct sum of involutions. Therefore, the problem of deciding whether two involutions are isomorphic cannot be readily reduced to the problem of deciding whether an involution is hyperbolic (i.e., adjoint to a hyperbolic hermitian or skew-hermitian form).

In the next sections, we successively discuss orthogonal involutions, involutions of the second kind (also called *unitary* involutions), and symplectic involutions. In each case, some invariants with striking common features are defined.

3 Orthogonal Involutions

Any involution of the first kind on a central simple algebra A defines an isomorphism between A and its opposite algebra, hence $2[A] = 0$ in the Brauer group. Therefore, A is split if its degree is odd, and in this case every involution is orthogonal and adjoint to a quadratic form q. Upon scaling, q can be assumed to have trivial discriminant; it is then uniquely determined. (These observations also follow from the equality

[3] More exactly, direct sums of involutions are defined only with some additional data (see [11]).

$\mathrm{PGO}_n = \mathrm{O}_n$ for n odd.) Thus, the case of involutions on central simple algebras of odd degree immediately reduces to the case of quadratic forms; we shall not discuss it further.

In this section, we consider orthogonal involutions on central simple algebras of even degree. We first review in Sects. 3.1 and 3.2 the cases where the algebra has Schur index 1 or 2. A collection of invariants e_i can then be defined for all i. In the following sections, we successively discuss invariants e_1, e_2, and e_3 without restriction on the index of the algebra.

3.1 The Split Case

Just as for involutions on central simple algebras of odd degree, the case of orthogonal involutions on *split* central simple algebras of even degree reduces to the case of quadratic forms. Note that if $q \in I^d F$ for some $d \geq 1$, then for any $\lambda \in F^\times$ we have

$$q \equiv \lambda q \bmod I^{d+1}F,$$

hence $e_d(q) = e_d(\lambda q)$. Since e_d only depends on the similarity class of q, we may consider it as attached to the isomorphism class of the adjoint involution and set

$$e_d(\mathrm{ad}_q) := e_d(q) \quad \text{for } q \in I^d F.$$

We thus have a collection of invariants $(e_d)_{d\geq 1}$ of orthogonal involutions on split central simple algebras of even degree. For a given involution σ, the invariant $e_d(\sigma)$ is defined if and only if all the previous invariants $e_1(\sigma), \dots, e_{d-1}(\sigma)$ are defined and vanish.

The following theorem easily follows from corresponding results for quadratic forms. It provides us with a benchmark against which to compare the invariants we aim to define in the nonsplit case.

Theorem 3.1. *Let σ be an orthogonal involution on a* split *central simple algebra A of even degree. Let also d be an arbitrary integer, $d \geq 1$.*

Hyperbolicity: *If $\deg A < 2^{d+1}$, we have $e_i(\sigma) = 0$ for all $i = 1, \dots, d$ if and only if σ is hyperbolic.*

Decomposition: *If $\deg A = 2^{d+1}$, we have $e_i(\sigma) = 0$ for all $i = 1, \dots, d$ if and only if (A, σ) decomposes into a tensor product of quaternion algebras with involution:*

$$(A,\sigma) = (Q_1,\sigma_1) \otimes_F \cdots \otimes_F (Q_{d+1},\sigma_{d+1}).$$

Classification: *If $\deg A = 2^d$, orthogonal involutions on A with $e_i = 0$ for $i = 1, \dots, d-1$ are classified by their e_d-invariant.*

Proof. Let $\sigma = \mathrm{ad}_q$, hence $\dim q = \deg A$. We have $e_i(\sigma) = 0$ for $i = 1, \dots, d$ if and only if $q \in I^{d+1}F$, hence the hyperbolicity criterion readily follows from the Arason–Pfister Hauptsatz. If $\deg A = 2^{d+1}$ and $e_i(\sigma) = 0$ for $i = 1,\dots,d$, then q is a multiple of a $(d+1)$-fold Pfister form (see [32, Theorem X.5.6]). If $(V_1,q_1),\dots,(V_{d+1},q_{d+1})$ are two-dimensional quadratic spaces such that $q \simeq q_1 \otimes\cdots\otimes q_{d+1}$, then

$$(A,\sigma) \simeq (\operatorname{End}_F V_1, \operatorname{ad}_{q_1}) \otimes_F \cdots \otimes_F (\operatorname{End}_F V_{d+1}, \operatorname{ad}_{q_{d+1}}).$$

The right-hand side is a tensor product of (split) quaternion algebras with (orthogonal) involution. Conversely, if (A,σ) is a tensor product of quaternion algebras with involution, the solution of the Pfister factor conjecture by Becher [5] shows that q is a multiple of some $(d+1)$-fold Pfister form, hence $e_i(q) = 0$ for $i = 1,\dots,d$.

Now, suppose $\deg A = 2^d$ and $\sigma = \operatorname{ad}_q$, $\sigma' = \operatorname{ad}_{q'}$ are orthogonal involutions on A with $e_i(\sigma) = e_i(\sigma')$ for $i = 1,\dots,d-1$, that is, q, $q' \in I^dF$. Let $\lambda \in F^\times$ (resp., $\lambda' \in F^\times$) be a represented value of q (resp., q'). As observed at the beginning of this section, we have

$$e_d(\sigma) = e_d(q) = e_d(\lambda q) \quad \text{and} \quad e_d(\sigma') = e_d(q') = e_d(\lambda' q').$$

If $e_d(\sigma) = e_d(\sigma')$, then $\lambda q - \lambda' q' \in I^{d+1}F$. But since λq and $\lambda' q'$ both represent 1, the dimension of the anisotropic kernel of $\lambda q - \lambda' q'$ is bounded as follows:

$$\dim(\lambda q - \lambda' q')_{\mathrm{an}} \le 2\deg A - 2 < 2^{d+1}.$$

By the Arason–Pfister Hauptsatz it follows that $\lambda q \simeq \lambda' q'$, hence σ and σ' are isomorphic. □

Remark 3.2. A result of Jacobson [24, Theorem 3.12] (see also [37]) on quadratic forms of dimension 6 with trivial discriminant also shows that orthogonal involutions with $e_1 = 0$ on a split central simple algebra of degree 6 are classified by their e_2-invariant (see also Theorem 3.10). As pointed out by Becher, the classification result in Theorem 3.1 may hold under much less stringent conditions on $\deg A$; it might be interesting to determine exactly in which degrees orthogonal involutions with $e_i = 0$ for $i = 1,\dots,d-1$ are classified by their e_d-invariant.

3.2 *The Case of Index 2*

In this section, we assume that the Schur index $\operatorname{ind} A$ is 2, that is, A is Brauer equivalent to some quaternion division F-algebra Q. Then A can be represented as

$$A = \operatorname{End}_Q V$$

for some right Q-vector space V of dimension $\dim_Q V = (1/2)\deg A$, and every orthogonal involution σ on A is adjoint to some skew-hermitian form h on V with respect to the conjugation involution on Q. The form h is uniquely determined up to a factor in $F^\times$. Berhuy [6] defined a complete system of invariants for skew-hermitian forms over Q, which readily yield invariants of orthogonal involutions on A since these invariants are constant on similarity classes of skew-hermitian forms. The idea is to extend scalars to the function field $F(X)$ of the Severi-Brauer variety X of Q (this variety is a projective conic) and to use the fact that the unramified cohomology of $F(X)$ comes from F.

More precisely, if σ is an orthogonal involution on A, then after scalar extension to $F(X)$ the involution $\sigma_{F(X)}$ is an orthogonal involution on the split algebra $A_{F(X)}$. If $e_i(\sigma_{F(X)})$ is defined for some $i \geq 1$, then it actually lies in the unramified subgroup $H^i_{nr}\big(F(X)\big)$, and Berhuy [6, Proposition 9] shows that there is a unique element

$$e_i(\sigma) \in \begin{cases} H^1(F) & \text{if } i = 1, \\ H^i(F, \mu_4^{\otimes i-1})/[A] \cdot H^{i-2}(F) & \text{if } i \geq 2 \end{cases}$$

such that

$$e_i(\sigma)_{F(X)} = e_i(\sigma_{F(X)}).$$

Note that for $i \geq 2$, the group $[A] \cdot H^{i-2}(F)$ can be identified with the kernel of the scalar extension map $H^i(F, \mu_4^{\otimes i-1}) \to H^i(F(X), \mu_4^{\otimes i-1})$, as shown in the proof of Berhuy [6, Proposition 9].

Theorem 3.3. *Let σ be an orthogonal involution on a central simple algebra A of index* 2 *and let d be an arbitrary integer, $d \geq 1$:*

(1) If $\deg A < 2^{d+1}$, we have $e_i(\sigma) = 0$ for all $i = 1, \dots, d$ if and only if σ is hyperbolic.
(2) If $\deg A = 2^{d+1}$, we have $e_i(\sigma) = 0$ for all $i = 1, \dots, d$ if and only if (A, σ) decomposes into a tensor product of quaternion algebras with involution:

$$(A, \sigma) = (Q_1, \sigma_1) \otimes_F \cdots \otimes_F (Q_{d+1}, \sigma_{d+1}).$$

Proof. By definition of $e_i(\sigma)$, we have $e_i(\sigma) = 0$ if and only if $e_i(\sigma_{F(X)}) = 0$. The hyperbolicity criterion (1) follows from the split case (Theorem 3.1) since Parimala–Sridharan–Suresh [41] and Dejaiffe [12] have proved that $\sigma_{F(X)}$ is hyperbolic if and only if σ is hyperbolic. Likewise, in case (2), Theorem 3.1 shows that $(A, \sigma)_{F(X)}$ decomposes into a tensor product of quaternion algebras with involution. By a theorem of Becher [5, Theorem 2], it follows that (A, σ) has a similar decomposition. (Becher's theorem also shows that the decomposition can be chosen in such a way that $Q_1 = Q$ is the quaternion algebra Brauer equivalent to A and Q_i is split for $i \geq 2$.) □

I do not know whether the analogue of the classification result from Theorem 3.1 holds for algebras of index 2 (except for the cases of low degree discussed in Theorems 3.6 and 3.10). The issue is whether skew-hermitian forms over Q are similar when they are similar after scalar extension to $F(X)$.

3.3 Discriminant

Let n be an even integer, $n = 2m$. A first invariant of orthogonal involutions on central simple algebras of degree n arises from the fact that PGO_n is not connected: if $g \in \mathrm{GO}_n(F)$, say $g^{\mathrm{t}}g = \lambda \in F^\times$, then $(\det g)^2 = \lambda^n$, hence $\det g = \pm\lambda^m$. The map $g \mapsto \lambda^m(\det g)^{-1} \in \mu_2$ yields a homomorphism $\delta\colon \mathrm{PGO}_n(F) \to \mu_2$, hence an exact sequence of algebraic groups

$$1 \to \mathrm{PGO}_n^+ \to \mathrm{PGO}_n \xrightarrow{\delta} \mu_2 \to 1. \tag{3.4}$$

We thus get a map

$$\delta^1 : H^1(F, \mathrm{PGO}_n) \to H^1(F) = F^\times/F^{\times 2}, \tag{3.5}$$

which defines the *determinant* of orthogonal involutions on central simple algebras of degree n. The *discriminant* is the determinant with a change of sign if m is odd: for σ an orthogonal involution on a central simple algebra of degree n, we set

$$e_1(\sigma) = \operatorname{disc}\sigma = (-1)^{n/2}\det\sigma \in F^\times/F^{\times 2}.$$

Alternatively, the discriminant can be directly obtained by substituting in the discussion earlier the group $\mathrm{PGO}(m\mathbb{H})$ of the hyperbolic space of dimension n for PGO_n. *Since the discriminant is more useful than the determinant for our purposes, we change notation and let henceforth*

$$\mathrm{PGO}_{2m} = \mathrm{PGO}(m\mathbb{H}).$$

The determinant and discriminant of quadratic forms can be defined similarly from the exact sequence

$$1 \to \mathrm{O}_n^+ \to \mathrm{O}_n \xrightarrow{\delta} \mu_2 \to 1.$$

Therefore, for any quadratic form q of dimension n,

$$\det \mathrm{ad}_q = \det q \quad \text{and} \quad \operatorname{disc}\mathrm{ad}_q = \operatorname{disc} q.$$

It follows that the e_1-invariant defined earlier coincides with the e_1-invariant defined in Sects. 3.1 and 3.2 when $\operatorname{ind} A \le 2$.

Knus–Parimala–Sridharan [30] give a nice direct definition of the determinant: if σ is an orthogonal involution on a central simple algebra A of even degree, they show that all the skew-symmetric units have the same reduced norm up to squares and that

$$\det\sigma = \mathrm{Nrd}_A(a)F^{\times 2} \quad \text{for any } a \in A^\times \text{ such that } \sigma(a) = -a.$$

In a slightly different form, this formula already appears in Tits's seminal work [52, Sect. 2.6].

This first invariant is of course rather weak. Yet, we have the following result.

Theorem 3.6. *Let σ be an orthogonal involution on a central simple algebra A:*

(a) If $\deg A < 4$ *(i.e., A is a quaternion algebra), σ is hyperbolic if and only if* $e_1(\sigma) = 0$.

(b) If $\deg A = 4$, *then* $e_1(\sigma) = 0$ *if and only if A decomposes as a tensor product of quaternion algebras with involution:*

$$(A,\sigma) = (Q_1,\sigma_1) \otimes (Q_2,\sigma_2). \tag{3.7}$$

(c) If $\deg A = 2$, *orthogonal involutions on A are classified by their e_1-invariant.*

Part (b) was first observed by Knus–Parimala–Sridharan [30]. Note that in any decomposition of the form (3.7), the involutions σ_1 and σ_2 are of the same type (orthogonal or symplectic) since σ is orthogonal. However, when there is any decomposition of the type (3.7), there is one where σ_1 and σ_2 are the quaternion conjugations (which are the unique symplectic involutions on Q_1 and Q_2). The subalgebras Q_1 and Q_2 are then uniquely determined by σ (see [29, p. 215]) (or (3.11)).

3.4 Clifford Algebras

An analogue of the even Clifford algebra of quadratic forms has been defined for orthogonal involutions by Jacobson [23] (by Galois descent) and by Tits [52] (rationally) (see also [29, Sect. 8]). For σ an orthogonal involution on a central simple F-algebra A of degree $n = 2m$, the Clifford algebra $C(A,\sigma)$ has dimension 2^{n-1} and its center $Z(A,\sigma)$ is isomorphic to $F(\sqrt{\operatorname{disc}\sigma})$ if $\operatorname{disc}\sigma \neq 1$, to $F \times F$ if $\operatorname{disc}\sigma = 1$. It is simple if $Z(A,\sigma)$ is a field and is a direct product of two central simple F-algebras $C_+(A,\sigma)$ and $C_-(A,\sigma)$ of degree 2^{m-1} if $Z(A,\sigma) \simeq F \times F$. For any quadratic form q on an F-vector space V of even dimension, we have

$$C(\operatorname{End}_F V, \operatorname{ad}_q) = C_0(q).$$

If $\operatorname{disc} q = 1$, then $C_+(\operatorname{End}_F V, \operatorname{ad}_q)$ and $C_-(\operatorname{End}_F V, \operatorname{ad}_q)$ are isomorphic and Brauer equivalent to the full Clifford algebra $C(q)$.

The construction of the Clifford algebra $C(A,\sigma)$ is functorial and may be used to obtain a second invariant of orthogonal involutions of even degree and trivial discriminant in the same spirit as the Witt invariant of quadratic forms, as we proceed to show.

The set of isomorphism classes of orthogonal involutions of trivial discriminant on central simple algebras of even degree $n = 2m$ corresponds under the bijection (2.2) to the kernel of the map δ^1 of (3.5), which is the image of $H^1(F, \mathrm{PGO}_n^+)$ in $H^1(F, \mathrm{PGO}_n)$. These involutions can therefore be classified by $H^1(F, \mathrm{PGO}_n^+)$, but the map $H^1(F,\mathrm{PGO}_n^+) \to H^1(F,\mathrm{PGO}_n)$ is generally *not* injective: a given central simple F-algebra A of degree n with an orthogonal involution σ of trivial discriminant can be the image of two different elements in $H^1(F,\mathrm{PGO}_n^+)$, depending on the choice of an F-algebra isomorphism $\varphi\colon Z(A,\sigma) \xrightarrow{\sim} F \times F$. We thus have a bijection:

$$\boxed{\begin{array}{l}\text{isomorphism classes of triples}\\ (A,\sigma,\varphi) \text{ as above}\end{array}} \quad \longleftrightarrow \quad H^1(F,\mathrm{PGO}_n^+).$$

If φ, $\varphi'\colon Z(A,\sigma) \xrightarrow{\sim} F \times F$ are the two different isomorphisms, the triples (A,σ,φ) and (A,σ,φ') are isomorphic if and only if (A,σ) has an automorphism whose induced action on $C(A,\sigma)$ swaps the two components or, equivalently, if A contains an improper similitude, that is, an element g such that $\sigma(g)g \in F^\times$ and $\mathrm{Nrd}_A(g) = -(\sigma(g)g)^{n/2}$. This readily follows from the exact sequence below, which is a part of the cohomology sequence derived from a twisted version of (3.4):

$$\mathrm{PGO}(A,\sigma) \xrightarrow{\delta} \mu_2 \to H^1(F,\mathrm{PGO}^+(A,\sigma)) \to H^1(F,\mathrm{PGO}(A,\sigma)) \xrightarrow{\delta^1} H^1(F).$$

In particular, if A is split, then the triples (A,σ,φ) and (A,σ,φ') are isomorphic.

The group PGO_n^+ has a simply connected cover Spin_n and we have an exact sequence of algebraic groups:

$$1 \to \mu \to \mathrm{Spin}_n \to \mathrm{PGO}_n^+ \to 1, \tag{3.8}$$

where μ is the center of Spin_n:

$$\mu = \begin{cases} \mu_4 & \text{if } n \equiv 2 \bmod 4,\\ \mu_2 \times \mu_2 & \text{if } n \equiv 0 \bmod 4.\end{cases}$$

The cohomology sequence associated to (3.8) yields a map

$$\partial\colon H^1(F,\mathrm{PGO}_n^+) \to H^2(F,\mu),$$

which can be viewed as a cohomological invariant of degree 2 of algebras with orthogonal involution of trivial discriminant – except that one has to factor out the effect of changing (A,σ,φ) into (A,σ,φ'). The images under ∂ of the elements in $H^1(F,\mathrm{PGO}_n^+)$ corresponding to (A,σ,φ) and (A,σ,φ') are

$$[C_+(A,\sigma)] \text{ and } [C_-(A,\sigma)] \in H^2(F,\mu_4) \subset \mathrm{Br}(F) \quad \text{if } n \equiv 2 \bmod 4,$$

and, if $n \equiv 0 \bmod 4$,

$$\big([C_+(A,\sigma)],[C_-(A,\sigma)]\big) \text{ and } \big([C_-(A,\sigma)],[C_+(A,\sigma)]\big) \in H^2(F) \times H^2(F).$$

When $n \equiv 2 \bmod 4$, we have (see [29, (9.15) and (9.16)])

$$2[C_+(A,\sigma)] = 2[C_-(A,\sigma)] = [A] \quad \text{and} \quad [C_+(A,\sigma)] + [C_-(A,\sigma)] = 0. \tag{3.9}$$

When $n \equiv 0 \bmod 4$, we have (see [29, (9.13) and (9.14)])

$$2[C_+(A,\sigma)] = 2[C_-(A,\sigma)] = 0 \quad \text{and} \quad [C_+(A,\sigma)] + [C_-(A,\sigma)] = [A].$$

Thus, in both cases, we have

$$[C_+(A,\sigma)] - [C_-(A,\sigma)] = [A].$$

Therefore, letting B_A^2 denote the subgroup of $\operatorname{Br}(F)$ generated by $[A]$, that is, $B_A^2 = \{0, [A]\} = [A] \cdot H^0(F)$, we may set

$$e_2(\sigma) = [C_+(A,\sigma)] + B_A^2 = [C_-(A,\sigma)] + B_A^2 \in \operatorname{Br}(F)/B_A^2.$$

If q is a quadratic form of trivial discriminant on an even-dimensional vector space V, we have $B^2_{\operatorname{End} V} = \{0\}$ and

$$e_2(\mathrm{ad}_q) = e_2(q) \in H^2(F).$$

Therefore, the e_2-invariant defined earlier coincides with the e_2-invariant of Sects. 3.1 and 3.2 when $\operatorname{ind} A \leq 2$.

Theorem 3.10. *Let σ be an orthogonal involution on a central simple F-algebra A of even degree:*

(a) If $\deg A < 8$, *then $e_1(\sigma) = e_2(\sigma) = 0$ if and only if σ is hyperbolic.*
(b) If $\deg A = 8$, *then $e_1(\sigma) = e_2(\sigma) = 0$ if and only if (A,σ) has a decomposition of the form*

$$(A,\sigma) = (Q_1,\sigma_1) \otimes_F (Q_2,\sigma_2) \otimes_F (Q_3,\sigma_3)$$

for some quaternion F-algebras Q_1, Q_2, Q_3.
(c) If $\deg A = 4$ *or* 6, *orthogonal involutions on A with $e_1 = 0$ are classified by their e_2-invariant.*
(d) If $\deg A \equiv 2 \bmod 4$, *then* $(A,\sigma) \simeq (\operatorname{End}_F V, \mathrm{ad}_q)$ *for some quadratic form $q \in I^3F$ if and only if $e_2(\sigma) = 0$.*

Sketch of proof. If $\deg A = 4$, the two components $C_\pm(A,\sigma)$ of the Clifford algebra are quaternion algebras. Letting $\sigma_\pm$ denote the quaternion conjugation on $C_\pm(A,\sigma)$, we have a decomposition

$$(A,\sigma) \simeq (C_+(A,\sigma),\sigma_+) \otimes_F (C_-(A,\sigma),\sigma_-), \tag{3.11}$$

which arises from the coincidence of Dynkin diagrams $D_2 = A_1 \times A_1$ (see [29, (15.12)]). Since $e_2(\sigma)$ determines the pair $\{C_+(A,\sigma), C_-(A,\sigma)\}$, it follows that orthogonal involutions of trivial discriminant on A are classified by their e_2-invariant. If $C_+(A,\sigma)$ or $C_-(A,\sigma)$ is split, then the corresponding involution σ_+ or σ_- is hyperbolic, hence σ also is hyperbolic.

If $\deg A = 6$, then $C_+(A,\sigma)$ and $C_-(A,\sigma)$ are central simple F-algebras of degree 4. The exterior power construction λ^2 yields a central simple F-algebra

$\lambda^2 C_+(A,\sigma)$ (resp., $\lambda^2 C_-(A,\sigma)$) of degree 6 endowed with a canonical involution γ_+ (resp., γ_-). As a consequence of the coincidence of Dynkin diagrams $D_3 = A_3$ (see [29, (15.32)]), we have isomorphisms

$$(A,\sigma) \simeq (\lambda^2 C_+(A,\sigma),\gamma_+) \simeq (\lambda^2 C_-(A,\sigma),\gamma_-).$$

Since $e_2(\sigma)$ determines the pair $\{[C_+(A,\sigma)],[C_-(A,\sigma)]\}$, it follows that orthogonal involutions of trivial discriminant on A are classified by their e_2-invariant.

Now, suppose $\deg A \equiv 2 \bmod 4$. If $C_+(A,\sigma)$ is split, it follows from (3.9) that $[A] = 0$. Similarly, if $[C_+(A,\sigma)] = [A]$, then since (3.9) shows that $2[C_+(A,\sigma)] = [A]$ it follows that $[C_+(A,\sigma)] = [A] = 0$. Therefore, in each case we have $(A,\sigma) \simeq (\operatorname{End}_F V, \operatorname{ad}_q)$ for some quadratic form q, and

$$e_2(q) = e_2(\sigma) = 0,$$

hence $q \in I^3F$. In particular, if $\deg A = 6$ we have $\dim q = 6$, hence q is hyperbolic.

Finally, (b) is a consequence of triality (see [29, Sect. 42.B]), which shows that for the canonical involutions σ_+, σ_- on $C_+(A,\sigma)$, $C_-(A,\sigma)$, we have

$$C\big(C_+(A,\sigma),\sigma_+\big) \simeq (A,\sigma) \times (C_-(A,\sigma),\sigma_-).$$

If $C_+(A,\sigma)$ is split, then (A,σ) is isomorphic to one of the components of the even Clifford algebra of a quadratic form of dimension 8, from which the existence of a decomposition easily follows. □

In view of part (d) of Theorem 3.10, the case of orthogonal involutions with $e_1 = e_2 = 0$ on central simple algebras of degree $2 \pmod 4$ is reduced to the split case. We thus have invariants e_i as in Sect. 3.1, and Theorem 3.1 applies. Therefore, in the rest of this section, we only consider central simple algebras of degree $0 \pmod 4$.

3.5 Higher Invariants

The hope to define further invariants of orthogonal involutions on the model of the Arason e_3-invariant of quadratic forms was dashed by an example of Quéguiner-Mathieu [45] or [4, Sect. 3.4]. The following variation on Quéguiner-Mathieu's example was suggested by Becher: consider quaternion algebras with orthogonal involutions (Q_1,σ_1), (Q_2,σ_2), and (Q_3,σ_3) over an arbitrary field F, and let

$$(A,\sigma) = (Q_1,\sigma_1) \otimes_F (Q_2,\sigma_2) \otimes_F (Q_3,\sigma_3).$$

Suppose A is division. We have $\operatorname{disc}\sigma = 1$ and, as observed in Theorem 3.10, $e_2(\sigma) = 0$. Suppose we could find a functorial invariant $e_3(\sigma)$ in some quotient $H^3(F)/E(A)$ of $H^3(F)$, and let

$$e_3(\sigma) = \sum_i (a_i)\cdot(b_i)\cdot(c_i) + E_A \quad \text{for some } a_i, b_i, c_i \in F^\times.$$

Let K be the function field of the product of the quadrics:

$$x_0^2 - a_i x_1^2 - b_i x_2^2 + a_i b_i x_3^2 - c_i x_4^2 = 0.$$

We have $e_3(\sigma)_K = 0$, hence $e_3(\sigma_K) = 0$ by functoriality. Now, a theorem of Merkurjev on the index reduction of central simple algebras over the function field of quadrics [13, Corollary 30.10] shows that A_K is a division algebra. After scalar extension to its generic splitting field L, which is the field of functions on the Severi-Brauer variety of A_K, we have

$$(A_K, \sigma_K)_L \simeq (\operatorname{End}_L V, \operatorname{ad}_{q_\sigma})$$

for some eight-dimensional quadratic form q_σ over L, which must be anisotropic by a theorem of Karpenko [27]. But since $e_1(\sigma_K) = e_2(\sigma_K) = e_3(\sigma_K) = 0$, functoriality yields $e_i(q_\sigma) = 0$ for $i = 1, 2, 3$, hence q_σ is hyperbolic: this is a contradiction.

The same idea can be used more generally to prove that there is no functorial invariant $e_3(\sigma)$ in a quotient of a cohomology group $H^3(F, M)$ of any torsion module M, because F has an extension K of cohomological dimension 2 such that A_K is a division algebra (see [39, Proof of Theorem 4]).

On the other hand, Rost defined for all simply connected algebraic groups a cohomological invariant of degree 3 that generalizes the Arason invariant in the following sense: the Rost invariant of any element $\xi \in H^1(F, \mathrm{Spin}_n)$ is the Arason invariant of the quadratic form of dimension n corresponding to the image of ξ in $H^1(F, \mathrm{O}_n)$. The Rost invariant of Spin groups can be used to define a *relative* invariant of orthogonal involutions, as was shown by Garibaldi [15]. (Bayer-Fluckiger and Parimala [3] used a similar technique to obtain a relative invariant of hermitian forms.) We next describe his procedure.

Suppose σ_0 is a given orthogonal involution with $e_1(\sigma_0) = e_2(\sigma_0) = 0$ on a central simple algebra A of degree $0 \pmod 4$. We consider the twisted version of (3.8)

$$1 \to \mu_2 \times \mu_2 \to \mathrm{Spin}(A, \sigma_0) \to \mathrm{PGO}^+(A, \sigma_0) \to 1$$

and the associated cohomology sequence

$$H^1(F) \times H^1(F) \to H^1\big(F, \mathrm{Spin}(A, \sigma_0)\big) \to H^1\big(F, \mathrm{PGO}^+(A, \sigma_0)\big) \xrightarrow{\partial} H^2(F) \times H^2(F). \tag{3.12}$$

Let σ be another involution on A with $e_1(\sigma) = e_2(\sigma) = 0$. As observed in Sect. 3.4 (in the nontwisted case), the cohomology class in $H^1\big(F, \mathrm{PGO}(A, \sigma_0)\big)$ associated with (A, σ) lifts in two ways to $H^1\big(F, \mathrm{PGO}^+(A, \sigma_0)\big)$, to cohomology classes η, η' corresponding to triples (A, σ, φ), (A, σ, φ'), where φ and φ' are the two F-algebra isomorphisms $Z(A, \sigma) \xrightarrow{\sim} Z(A, \sigma_0)$ ($\simeq F \times F$). The images of η, η' under ∂ are the two components of $C(A, \sigma) \otimes_{Z(A, \sigma_0)} C(A, \sigma_0)$, where the tensor product is taken with respect to the isomorphism φ or φ'; we thus get for $\partial(\eta)$ and $\partial(\eta')$ the pairs

$$\big([C_+(A, \sigma) \otimes_F C_+(A, \sigma_0)], [C_-(A, \sigma) \otimes_F C_-(A, \sigma_0)]\big)$$

and

$$\big([C_-(A,\sigma)\otimes_F C_+(A,\sigma_0)],[C_+(A,\sigma)\otimes_F C_-(A,\sigma_0)]\big).$$

Since $e_2(\sigma)=0$, one of the components $C_\pm(A,\sigma)$ is split and the other is Brauer equivalent to A. The same holds for σ_0 since $e_2(\sigma_0)=0$; therefore, we have $\partial(\eta)=0$ or $\partial(\eta')=0$, and at least one of η,η' lifts to some $\xi\in H^1\big(F,\mathrm{Spin}(A,\sigma_0)\big)$. The analysis of the exact sequence (3.12) given in [15] shows how the Rost invariant of ξ varies when a different lift ξ' is chosen: the difference of Rost invariants $R(\xi)-R(\xi')$ lies in the subgroup $B_A^3\subset H^3(F,\mu_4^{\otimes 2})$, which is defined to be the image under the injection $H^3(F)\hookrightarrow H^3(F,\mu_4^{\otimes 2})$ of the subgroup

$$[A]\cdot H^1(F)=\{[A]\cdot(\lambda)\mid \lambda\in F^\times\}\subset H^3(F).$$

Therefore, a relative e_3-invariant of involutions with $e_1=e_2=0$ on A is defined by

$$e_3(\sigma/\sigma_0)=R(\xi)+B_A^3\in H^3(F,\mu_4^{\otimes 2})/B_A^3.$$

In the particular case where the Schur index $\operatorname{ind}A$ divides $(1/2)\deg A$, that is, when $A\simeq M_2(A')$ for some central simple algebra A', we may choose for σ_0 a hyperbolic involution and thus consider $e_3(\sigma/\sigma_0)$ as an absolute invariant $e_3(\sigma)$. When $(A,\sigma)\simeq(\operatorname{End}_F V,\mathrm{ad}_q)$, we have $B_A^3=\{0\}$ and

$$e_3(\mathrm{ad}_q)=e_3(q)$$

in view of the relation between the Rost and the Arason invariants. Therefore, the e_3-invariant defined earlier coincides with the e_3-invariant of Sects. 3.1 and 3.2 when $\operatorname{ind}A\le 2$.

In the case where $\deg A=8$ (and $\operatorname{ind}A$ divides 4), an explicit computation of the e_3-invariant was recently obtained by Quéguiner-Mathieu and Tignol: if $e_1(\sigma)=e_2(\sigma)=0$, we may find a decomposition

$$(A,\sigma)\simeq(M_2(F),\mathrm{ad}_{\langle\langle\lambda\rangle\rangle})\otimes_F(D,\theta)$$

for some $\lambda\in F^\times$, some central simple F-algebra D of degree 4, and some orthogonal involution θ on D such that the quaternion F-algebra $(\operatorname{disc}\theta,\lambda)_F$ is split: see [5] if $\operatorname{ind}A=1$ or 2 and [38] if $\operatorname{ind}A=4$. The Clifford algebra $C(D,\theta)$ is a quaternion algebra over a quadratic étale F-algebra Z, which is isomorphic to $F(\sqrt{\operatorname{disc}\theta})$ if $\operatorname{disc}\theta\neq 1$ and to $F\times F$ if $\operatorname{disc}\theta=1$. Therefore, we may find an element $\ell\in Z$ such that $N_{Z/F}(\ell)=\lambda$. The e_3-invariant is

$$e_3(\sigma)=\mathrm{cor}_{Z/F}\big((\ell)\cdot[C(D,\theta)]\big)+B_A^3\in H^3(F)/B_A^3,$$

as can be seen by extending scalars to the function field F_A of the Severi-Brauer variety of A, since the map $H^3(F)/B_A^3\to H^3(F_A)$ induced by scalar extension from F to F_A is injective. (Note that in this special case where $\deg A=8$ and one of the components of the Clifford algebra is split, the Rost invariant has exponent 2, see [17, p. 146].)

Theorem 3.13. *Let σ be an orthogonal involution on a central simple algebra A of even degree with* $\operatorname{ind} A \mid (1/2)\deg A$. *If* $\deg A < 16$*, the involution σ is hyperbolic if and only if* $e_1(\sigma) = e_2(\sigma) = e_3(\sigma) = 0$.

Proof. The "only if" part is clear. We may thus assume $e_i(\sigma) = 0$ for $i = 1,2,3$, and we have to show that σ is hyperbolic. If $\deg A < 8$, the theorem follows from Theorem 3.10(a). If A is split (which, by Theorem 3.10(d), occurs in particular when $\deg A \equiv 2 \pmod 4$), the theorem follows from Theorem 3.1. Likewise, the theorem follows from Theorem 3.3 if $\operatorname{ind} A = 2$. Therefore, it suffices to consider the case where $\deg A = 8$ and $\operatorname{ind} A = 4$. Let F_A be the function field of the Severi-Brauer variety of A. Since A splits over F_A, and since the theorem holds in the split case, $(A,\sigma)_{F_A}$ is hyperbolic. It follows that σ is hyperbolic by a theorem of Sivatski [50, Proposition 3] (based on a result of Laghribi [31, Théorème 4]). □

Theorem 3.13 yields the expected hyperbolicity criterion for $\deg A < 16$, under the hypothesis that $\operatorname{ind} A \mid (1/2)\deg A$ (which is necessary for σ to be hyperbolic). Whether the decomposition criterion holds when $\deg A = 16$ is an open question, which is addressed by Garibaldi [15]. By contrast, Quéguiner-Mathieu and Tignol have used an example of Hoffmann [22] and a result of Sivatski [50, Proposition 4] to construct nonisomorphic orthogonal involutions σ, σ' on a central simple algebra A of degree 8 and index 4 such that

$$e_1(\sigma) = e_1(\sigma') = e_2(\sigma) = e_2(\sigma') = 0 \quad \text{and} \quad e_3(\sigma) = e_3(\sigma'),$$

hence the classification criterion does *not* hold in degree 8. If σ_0 is a fixed orthogonal involution with $e_1(\sigma_0) = e_2(\sigma_0) = 0$ on a central simple algebra A of degree 8 and index 4, orthogonal involutions σ on A with $e_1(\sigma) = e_2(\sigma) = 0$ and $e_3(\sigma) = e_3(\sigma_0)$ are classified by a relative invariant:

$$e_4(\sigma/\sigma_0) \in H^4(F)/B^4_{\sigma_0},$$

where

$$B^4_{\sigma_0} = \{(a)\cdot e_3(\sigma_0) \mid a \in F^\times,\ (a)\cdot[A] = 0\}.$$

Garibaldi [15] shows how to use the Rost invariant of groups of type E_8 to obtain an *absolute* e_3-invariant for orthogonal involutions on central simple algebras of degree 16.

4 Unitary Involutions

Let $K = F(\sqrt{a})$ be a quadratic field extension of F with nontrivial automorphism ι. For each integer $n \geq 1$, let h_n be the n-dimensional hermitian form over K with maximal Witt index, defined for $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n) \in K^n$ by

$$h_n(x,y) = \iota(x_1)y_1 - \iota(x_2)y_2 + \cdots - (-1)^n\iota(x_n)y_n = \iota(x)\cdot d_n\cdot y^{\mathrm{t}},$$

where d_n is the diagonal matrix of order n:

$$d_n := \mathrm{diag}(1, -1, \ldots, -(-1)^n).$$

Let also $\tau\colon M_n(K) \to M_n(K)$ be the unitary involution defined by

$$\tau(m) = d_n^{-1} \cdot \iota(m)^{\mathrm{t}} \cdot d_n \quad \text{for } m \in M_n(K).$$

The unitary group $\mathrm{U}_{n,K}$ is the group of K-automorphisms of h_n:

$$\mathrm{U}_{n,K}(F) := \mathrm{Aut}_K(h_n) = \{u \in \mathrm{GL}_n(K) \mid \tau(u) = u^{-1}\}.$$

As in the orthogonal case, we may consider the group of similitudes

$$\mathrm{GU}_{n,K}(F) := \{g \in \mathrm{GL}_n(K) \mid \tau(g)g \in F^\times\}$$

and the corresponding projective group

$$\mathrm{PGU}_{n,K}(F) := \mathrm{Aut}_K(M_n(K), \tau) = \mathrm{GU}_{n,K}/K^\times.$$

Galois cohomology (see [29, Sect. 29.D]) yields a bijection:

K-isomorphism classes of central simple K-algebras of degree n with unitary involution	$\longleftrightarrow$	$H^1(F, \mathrm{PGU}_{n,K})$.

More precisely, the set on the left consists of the isomorphism classes of triples (B, τ, φ), where B is a central simple algebra of degree n over a quadratic extension Z of F, τ is an involution on B that restricts to the nontrivial F-automorphism of Z, and $\varphi\colon Z \xrightarrow{\sim} K$ is an F-algebra isomorphism.

4.1 The (Quasi)split Case

When B is split, the study of unitary involutions reduces to the quadratic form case by an observation due to Jacobson: every hermitian form $h\colon V \times V \to K$ on a K-vector space V yields a quadratic form $q_h\colon V \to F$ on V (viewed as an F-vector space) by

$$q_h(x) = h(x,x) \quad \text{for } x \in V.$$

The quadratic form uniquely determines h because it is easily verified that if $\alpha = \sqrt{a} \in K$ satisfies $\iota(\alpha) = -\alpha$, then

$$h(x,y) = \frac{1}{2}\big(q_h(x+y) - q_h(x) - q_h(y) + q_h(x+y\alpha)\alpha^{-1} - q_h(x)\alpha^{-1} - q_h(y\alpha)\alpha^{-1}\big).$$

Therefore, the invariants of h and q_h are the same and the invariants of ad_h are the invariants of the similarity class of q_h (which is always even dimensional since $\dim_F V = 2\dim_K V$). If $e_i(q_h)$ is defined for some i, we set

$$e_i(\mathrm{ad}_h) = e_i(q_h) \in H^i(F).$$

The correspondence between the involutions adjoint to h and to q_h is made explicit in the following lemma (which is a very special case of a result of Garibaldi and Quéguiner-Mathieu [19, Proposition 1.9]).

Lemma 4.1. *Let B be a split central simple algebra over $K = F(\sqrt{a})$ and let τ be a unitary involution on B. There is a split central simple F-algebra A of degree* $\deg A = 2\deg B$ *and an orthogonal involution σ on A such that (B,τ) embeds in (A,σ):*

$$(B,\tau) \hookrightarrow (A,\sigma).$$

For a given embedding $B \subset A$, the orthogonal involution σ is uniquely determined by the condition that $\sigma|_B = \tau$. For any integer i, the invariant $e_i(\tau)$ is defined if and only if $e_i(\sigma)$ is defined, and

$$e_i(\tau) = e_i(\sigma).$$

Proof. Let $B = \mathrm{End}_K V$ for some K-vector space V and $\tau = \mathrm{ad}_h$ for some hermitian form h on V. Define $A = \mathrm{End}_F V$, so there is a canonical embedding $B \hookrightarrow A$, and let $\sigma = \mathrm{ad}_{q_h}$, the orthogonal involution on A adjoint to the quadratic form q_h. It is readily verified that $\sigma|_B = \tau$. If σ' is another orthogonal involution on A such that $\sigma'|_B = \tau$, then $\sigma' = \mathrm{Int}(s)\circ\sigma$ for some $s \in A^\times$ such that $\sigma(s) = s$ and $sx = xs$ for all $x \in B$. Since s centralizes B, we must have $s \in K^\times$; the condition $\sigma(s) = s$ then amounts to $\iota(s) = s$, hence $s \in F^\times$ and therefore $\sigma' = \sigma$. By definition of $e_i(\tau)$, we have

$$e_i(\tau) = e_i(q_h) = e_i(\sigma).$$

□

If $(v_1,\dots,v_n)$ is a K-base of V where the hermitian form h is diagonal,

$$h = \langle a_1,\dots,a_n\rangle_K,$$

then $a_1,\dots,a_n \in F^\times$ and $(v_1, v_1\alpha,\dots,v_n,v_n\alpha)$ is an orthogonal F-base of V where the quadratic form q_h is

$$q_h = \langle 1,-a\rangle \otimes \langle a_1,\dots,a_n\rangle.$$

Therefore,

$$e_1(\mathrm{ad}_h) = \begin{cases} (a) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

and, when n is even,

$$e_2(\mathrm{ad}_h) = (a, (-1)^{n/2}a_1,\dots,a_n)_F \in H^2(F).$$

The *discriminant* $\operatorname{disc} h$ is defined as the image of $(-1)^{n/2}a_1,\dots,a_n$ in the factor group $F^\times/N_{K/F}(K^\times)$; the preceding equation may thus be rewritten as

$$e_2(\mathrm{ad}_h) = (a, \operatorname{disc} h)_F.$$

To discuss decompositions into tensor products of quaternion algebras, it is useful to keep in mind the following theorem of Albert: if τ is a unitary involution on a quaternion algebra Q over $K = F(\sqrt{a})$, then there is a unique quaternion F-algebra Q' with conjugation involution γ such that

$$(Q,\tau) = (Q',\gamma) \otimes_F (K,\iota),$$

where ι is the nontrivial F-automorphism of K. (The quaternion algebra Q' is in fact the discriminant algebra $D(Q,\tau)$: see the proof of Theorem 4.5.) Therefore, every tensor product of quaternion K-algebras with unitary involution has an alternative decomposition:

$$(Q_1,\tau_1)\otimes_K \cdots \otimes_K (Q_n,\tau_n) = (Q'_1,\gamma_1)\otimes_F \cdots \otimes_F (Q'_n,\gamma_n)\otimes_F (K,\iota).$$

Theorem 4.2. *Let τ be a unitary involution on a* split *central simple algebra B of even degree over $K = F(\sqrt{a})$, and let d be an arbitrary integer, $d \geq 1$:*

(a) If $\deg B < 2^d$, we have $e_i(\tau) = 0$ for $i = 2,\dots,d$ if and only if τ is hyperbolic.
(b) If $\deg B = 2^d$, we have $e_i(\tau) = 0$ for $i = 2,\dots,d$ if and only if (B,τ) decomposes into a tensor product of quaternion algebras with unitary involutions:

$$(B,\tau) = (Q_1,\tau_1)\otimes_K \cdots \otimes_K (Q_d,\tau_d).$$

(c) If $\deg B = 2^{d-1}$, unitary involutions on B with $e_i = 0$ for $i = 2,\dots,d-1$ are classified by their e_d-invariant.

Comparing with Theorem 3.1, note the shift in the degree of the algebra.

Proof. Let $\tau = \mathrm{ad}_h$, so $e_i(\tau) = e_i(q_h)$ when defined. Note that $\dim q_h = 2\deg B$, so in case (a) q_h is hyperbolic if and only if $q_h \in I^{d+1}F$, and this condition is equivalent to $e_i(q_h) = 0$ for $i = 2,\dots,d$. If $\deg B = 2^d$ and $e_i(\tau) = 0$ for $i = 2,\dots,d$, then q_h is a $(d+1)$-Pfister form of the type

$$q_h = \langle\langle a, a_1,\dots,a_d\rangle\rangle \quad \text{for some } a_1,\dots,a_d \in F^\times,$$

hence

$$h = \langle\langle a_1,\dots,a_d\rangle\rangle_K.$$

Therefore,

$$\begin{aligned}(B,\tau) &= (M_2(F),\mathrm{ad}_{\langle\langle a_1\rangle\rangle})\otimes_F \cdots \otimes_F (M_2(F),\mathrm{ad}_{\langle\langle a_d\rangle\rangle})\otimes_F (K,\iota)\\ &= (M_2(K),\mathrm{ad}_{\langle\langle a_1\rangle\rangle_K})\otimes_K \cdots \otimes_K (M_2(K),\mathrm{ad}_{\langle\langle a_d\rangle\rangle_K}).\end{aligned}$$

Conversely, suppose there are quaternion F-algebras $Q'_1,\dots,Q'_d$ with conjugation involutions $\gamma_1,\dots,\gamma_d$ such that

$$(B,\tau) = (Q'_1,\gamma_1)\otimes_F \cdots \otimes_F (Q'_d,\gamma_d)\otimes_F (K,\iota).$$

Since B is split, the product $Q'_1 \otimes_F \cdots \otimes_F Q'_d$ is split by K, so we may find $b \in F^\times$ such that

$$Q'_1 \otimes_F \cdots \otimes_F Q'_d = (a,b)_F \quad \text{in } \mathrm{Br}(F).$$

Embed $K \hookrightarrow (a,b)_F$ and pick an involution θ on $(a,b)_F$ that restricts to ι on K and that is orthogonal if d is even and symplectic if d is odd. (So, in the latter case θ is the conjugation involution.) We then have an embedding

$$(B,\tau) \hookrightarrow (A,\sigma) := (Q'_1,\gamma_1) \otimes_F \cdots \otimes_F (Q'_d,\gamma_d) \otimes_F \big((a,b)_F,\theta\big).$$

By Theorem 3.1 we have $e_i(\sigma) = 0$ for $i = 1,\dots,d$, hence $e_i(\tau) = 0$ for $i = 2,\dots,d$ by Lemma 4.1.

To prove (c), let $B = \mathrm{End}_K V$ with $\dim_K V = 2^{d-1}$ and consider unitary involutions $\tau = \mathrm{ad}_h$ and $\tau' = \mathrm{ad}_{h'}$ on B with $e_i = 0$ for $i = 2,\dots,d-1$. The corresponding quadratic forms q_h and $q_{h'}$ satisfy $e_i(q_h) = e_i(q_{h'}) = 0$ for $i = 1,\dots,d-1$; hence, Theorem 3.1 shows that q_h and $q_{h'}$ are similar if and only if $e_d(q_h) = e_d(q_{h'})$. Therefore, τ and τ' are isomorphic if and only if $e_d(\tau) = e_d(\tau')$. □

4.2 The Discriminant Algebra

Since the group $\mathrm{PGU}_{n,K}$ is connected, the procedure that yields the discriminant of orthogonal involutions in Sect. 3.3 does not apply here. The group $\mathrm{PGU}_{n,K}$ is not simply connected however, so we may obtain cohomological invariants of degree 2 as in the orthogonal case. The simply connected cover of $\mathrm{PGU}_{n,K}$ is the special unitary group:

$$\mathrm{SU}_{n,K} := \{u \in \mathrm{U}_{n,K} \mid \det(u) = 1\}.$$

Its center is a twisted version of the group of nth roots of unity:

$$\mu_{n,K} := \{z \in K^\times \mid N_{K/F}(z) = 1 = z^n\}.$$

Henceforth, we assume that the characteristic of F does not divide n, so that $\mu_{n,K}$ is a smooth algebraic group. Its group of rational points over F_s is isomorphic to the group of nth roots of unity in F_s, with a twisted Galois action that disappears after scalar extension to K. From the exact sequence

$$1 \to \mu_{n,K} \to \mathrm{SU}_{n,K} \to \mathrm{PGU}_{n,K} \to 1, \tag{4.3}$$

we obtain a connecting map in cohomology:

$$\partial\colon H^1(F,\mathrm{PGU}_{n,K}) \to H^2(F,\mu_{n,K}).$$

The features of this map are significantly different according to whether n is odd or even.

4.2.1 The Odd Degree Case

When n is odd, the scalar extension map

$$\mathrm{res}_{K/F}\colon H^2(F,\mu_{n,K}) \to H^2(K,\mu_n)$$

is injective, and identifies $H^2(F,\mu_{n,K})$ with the kernel of the corestriction map

$$\mathrm{cor}_{K/F}\colon H^2(K,\mu_n) \to H^2(F,\mu_n),$$

see [29, (30.12)]. Comparing the exact sequence (4.3) with the corresponding exact sequence after scalar extension to K, which is

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1,$$

it is easy to see that the cohomology class in $H^1(F,\mathrm{PGU}_{n,K})$ represented by an algebra with involution (B,τ) is mapped by ∂ to the Brauer class $[B] \in H^2(K,\mu_n)$.

4.2.2 The Even Degree Case

Let $n = 2m$. Besides the restriction map $\mathrm{res}_{K/F}$, we may also consider the map defined by raising to the mth power:

$$m\colon H^2(F,\mu_{n,K}) \to H^2(F,\mu_2).$$

The following description of $H^2(F,\mu_{n,K})$ is due to Colliot-Thélène–Gille–Parimala [10, Proposition 2.10].

Proposition 4.4. *The map* $(\mathrm{res}_{K/F},m)\colon H^2(F,\mu_{n,K}) \to H^2(K,\mu_n) \times H^2(F,\mu_2)$ *is injective. Its image is the group of pairs* (ξ,η) *such that* $\xi^m = \mathrm{res}_{K/F}(\eta)$ *and* $\mathrm{cor}_{K/F}(\xi) = 0$.

Let (B,τ) be a central simple K-algebra of degree n with unitary involution, and let $\Delta(B,\tau) \in H^2(F,\mu_{n,K})$ be the image under ∂ of the corresponding cohomology class in $H^1(F,\mathrm{PGU}_{n,K})$. As in the case where n is odd, we have

$$\mathrm{res}_{K/F}\,\Delta(B,\tau) = [B],$$

which is an obvious invariant that gives no information on the involution τ. The other component of $\Delta(B,\tau)$ under the map of Proposition 4.4 is more interesting; let

$$e_2(\tau) = \Delta(B,\tau)^m \in H^2(F).$$

This cohomology class turns out to be the Brauer class of a central simple F-algebra $D(B,\tau)$ of degree $\binom{n}{m}$ first considered by Tits [53, Sect. 6.3] and by Tamagawa [51]. When $B = \mathrm{End}_K V$ and $\tau = \mathrm{ad}_h$ for some hermitian form h, the algebra $D(B,\tau)$ is Brauer equivalent to the quaternion algebra $(a,\operatorname{disc} h)_F$. Therefore, $D(B,\tau)$ is called the *discriminant algebra* of (B,τ) in [29, Sect. 10], and the e_2-invariant defined earlier coincides with the e_2-invariant of unitary involutions on split algebras defined in Sect. 4.1.

Theorem 4.5. *Let τ be a unitary involution on a central simple K-algebra B of even degree:*

(a) If $\deg B = 2$, then τ is hyperbolic if and only if $e_2(\tau) = 0$.
(b) If $\deg B = 4$, then $e_2(\tau) = 0$ if and only if there is a decomposition

$$(B,\tau) = (B_1,\tau_1) \otimes_K (B_2,\tau_2)$$

for some subalgebras B_1, $B_2 \subset B$ of degree 2.
(c) If $\deg B = 2$, unitary involutions on B are classified by their e_2-invariant.

Proof. If $\deg B = 2$, then $D(B,\tau)$ is the quaternion F-algebra such that

$$(B,\tau) = (D(B,\tau),\gamma) \otimes_F (K,\iota),$$

where γ is the conjugation involution (see [29, p. 129]). Therefore, τ is hyperbolic if $D(B,\tau)$ is split, and $D(B,\tau)$ determines τ uniquely. Part (b) is due to Karpenko–Quéguiner [26]. It is based on the coincidence of Dynkin diagrams $A_3 = D_3$, which can be used to show that every central simple algebra of degree 4 with unitary involution is isomorphic to the Clifford algebra of a canonical orthogonal involution on its discriminant algebra (see [29, Sect. 15.D]). □

4.3 Higher Invariants

The same method as in Sect. 3.5 allows one to derive from Rost's invariant for $\mathrm{SU}_{n,K}$ a relative invariant of unitary involutions on a given central simple algebra B over $K = F(\sqrt{a})$. Suppose $\deg B = n$ is even and not divisible by the characteristic, and let τ_0 be a unitary involution on B with $e_2(\tau_0) = 0$. By Proposition 4.4, the cohomology class in $H^1\big(F, \mathrm{PGU}(B,\tau_0)\big)$ corresponding to a unitary involution τ on B with $e_2(\tau) = 0$ lifts to some $\xi \in H^1\big(F, \mathrm{SU}(B,\tau_0)\big)$. The Rost invariant $R(\xi)$ lies in $H^3(F,\mu_d^{\otimes 2})$, where d is the Dynkin index of $\mathrm{SU}(B,\tau_0)$, determined in [17, (12.6), p. 142]:

$$d = \begin{cases} \exp B & \text{if } n \text{ is a 2-power and } \exp B = n \text{ or } n/2,\\ 2\exp B & \text{otherwise.} \end{cases}$$

In view of the description of the Rost invariant on the center of $\mathrm{SU}(B,\tau_0)$ given by Merkurjev–Parimala–Tignol [40], one should be able to define

$$e_3(\tau/\tau_0) = R(\xi) + B_B^3 \in H^3(F,\mu_d^{\otimes 2})/B_B^3,$$

where

$$B_B^3 = \operatorname{cor}_{K/F}\big([B]\cdot H^1(K,\mu_d)\big).$$

As far as I know, this invariant has not been investigated yet.

Although the case where the degree n is odd does not pertain to the line of investigation developed so far in this text, it is worth mentioning that a similar construction based on the Rost invariant is better documented in this case: consider the F-vector space $\operatorname{Sym}(\tau)$ of τ-symmetric elements in B and the quadratic form

$$Q_\tau\colon \operatorname{Sym}(\tau) \to F, \quad x \mapsto \operatorname{Trd}_B(x^2).$$

Assuming $\deg B = n$ is odd (and not divisible by the characteristic), it is shown in [29, (31.45)] that the Rost invariant of $\mathrm{SU}(B,\tau_0)$ yields a relative invariant of unitary involutions on B defined by

$$f_3(\tau/\tau_0) = e_3(Q_\tau - Q_{\tau_0}) \in H^3(F).$$

By a result of Garibaldi–Gille [16, Proposition 7.2], we may in fact define an *absolute* invariant by

$$f_3(\tau) = e_3\left(Q_\tau - \langle 1\rangle - \frac{n-1}{2}\langle 2,2a\rangle\right) \in H^3(F),$$

so that $f_3(\tau/\tau_0) = f_3(\tau) - f_3(\tau_0)$. (The form $\langle 1\rangle + \frac{n-1}{2}\langle 2,2a\rangle$ is Witt equivalent to the form Q_τ for τ the adjoint involution of the n-dimensional hermitian form of maximal Witt index.) The absolute invariant f_3 was first investigated in the particular case where $\deg B = 3$ by Haile–Knus–Rost–Tignol [20]. It classifies the unitary involutions on a given central simple algebra of degree 3 up to isomorphism (see [29, Sect. 19.B and (30.21)]).

There is also an absolute invariant of degree 4 defined just for $\deg B = 4$ and $K = F(\sqrt{-1})$ by Rost–Serre–Tignol [46]: letting n_D denote the norm form of the quaternion F-algebra Brauer equivalent to the discriminant algebra $D(B,\tau)$, the invariant is

$$f_4(\tau) = e_4(Q_\tau - n_D) \in H^4(F).$$

It vanishes if and only if B is generated by two elements $x, y \in \operatorname{Sym}(\tau)$ such that x^4, $y^4 \in F^\times$ and $yx = ixy$, where $i = \sqrt{-1} \in K$. If B is split and $\tau = \operatorname{ad}_h$ with

$$h = \langle a_1,a_2,a_3,a_4\rangle_K,$$

we have

$$f_4(\tau) = (-1)\cdot(-a_1a_2)\cdot(-a_1a_3)\cdot(-a_1a_4).$$

Note that the fourfold Pfister form $\langle\langle -1,-a_1a_2,-a_1a_3,-a_1a_4\rangle\rangle$ is indeed an invariant of the hermitian form h: in the notation of Garibaldi et al. [17, p. 67], we have

$$\langle\langle -1,-a_1a_2,-a_1a_3,-a_1a_4\rangle\rangle = 2 + \lambda_2^2(h) + \lambda_2^4(h).$$

5 Symplectic Involutions

Let n be an even integer, $n = 2m$, and let s_n be the following skew-symmetric matrix of order n:

$$s_n = \mathrm{diag}\big(\underbrace{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)}_{m}\big).$$

Define a symplectic involution $\sigma\colon M_n(F) \to M_n(F)$ by

$$\sigma(m) = s_n^{-1} \cdot m \cdot s_n \quad \text{for } m \in M_n(F).$$

The symplectic group is the automorphism group of the bilinear form with Gram matrix s_n:

$$\mathrm{Sp}_n(F) := \{u \in \mathrm{GL}_n(F) \mid u^{\mathrm{t}} s_n u = s_n\} = \{u \in \mathrm{GL}_n(F) \mid \sigma(u) = u^{-1}\}.$$

As in the orthogonal and symplectic cases, the group of similitudes is defined by

$$\mathrm{GSp}_n(F) := \{g \in \mathrm{GL}_n(F) \mid \sigma(g)g \in F^\times\};$$

its projective version is

$$\mathrm{PGSp}_n(F) := \mathrm{Aut}(M_n(F), \sigma) = \mathrm{GSp}_n(F)/F^\times.$$

Galois cohomology (see [29, (29.22)]) yields a bijection:

isomorphism classes of central simple F-algebras of degree n with symplectic involution	$\longleftrightarrow$	$H^1(F, \mathrm{PGSp}_n)$.

As pointed out in Sect. 2, symplectic involutions on split central simple algebras are adjoint to nonsingular skew-symmetric bilinear forms. Since every such form is hyperbolic, all the symplectic involutions on a split central simple algebra are hyperbolic (and conjugate). The classification problem thus arises only when the index is at least 2.

5.1 The Case of Index 2

Let A be a central simple F-algebra of degree n and index 2, that is, A is Brauer equivalent to a quaternion F-algebra Q. As in Sect. 3.2, we may represent A as

$$A = \mathrm{End}_Q V$$

for some right Q-vector space V with $\dim_Q V = n/2$. Every symplectic involution on A is adjoint to some hermitian form h on V with respect to the conjugation involution

γ on Q. As in Sect. 4.1, a result of Jacobson yields a reduction to quadratic forms since the map

$$q_h \colon V \to F, \quad x \mapsto h(x,x)$$

is a quadratic form on V viewed as an F-vector space, which determines h uniquely (see [48, Theorem 10.1.7, p. 352]). The invariants of ad_h are the invariants of the similarity class of h, which are also the invariants of the similarity class of q_h. If $e_i(q_h)$ is defined for some i, let

$$e_i(\operatorname{ad}_h) = e_i(q_h) \in H^i(F).$$

The correspondence between the symplectic involution ad_h and the orthogonal involution adjoint to q_h was used by MacDonald [36, Theorem 4.7] to determine all the invariants of symplectic involutions on central simple algebras A with $\operatorname{ind} A = 2$ and $\deg A \equiv 2 \bmod 4$. This correspondence can also be described directly, as the next lemma shows.

Lemma 5.1. *Let A be a central simple F-algebra Brauer equivalent to a quaternion F-algebra Q. Let σ be a symplectic involution on A and let γ be the conjugation involution on Q. Then $\sigma \otimes \gamma$ is an orthogonal involution on the split algebra $A \otimes_F Q$. If $(A,\sigma) = (\operatorname{End}_Q V, \operatorname{ad}_h)$ for some hermitian form h, then there is a canonical isomorphism*

$$(A \otimes_F Q, \sigma \otimes \gamma) \simeq (\operatorname{End}_F V, \operatorname{ad}_{q_h})$$

which carries $x \otimes y \in A \otimes_F Q$ to the map $v \mapsto x(v)\gamma(y)$. Therefore, for any integer $i \geq 1$, the invariant $e_i(\sigma)$ is defined if and only if $e_i(\sigma \otimes \gamma)$ is defined, and then

$$e_i(\sigma) = e_i(\sigma \otimes \gamma).$$

Proof. The lemma follows by a straightforward verification. □

Let $m = n/2$ and let $(v_\alpha)_{\alpha=1}^m$ be an orthogonal Q-base of V with respect to h, in which h has the diagonalization

$$h = \langle \lambda_1, \dots, \lambda_m \rangle_Q.$$

We have $\lambda_\alpha \in F^\times$ for all α. If $(1,i,j,k)$ is a quaternion F-base of Q with $i^2 = a$ and $j^2 = b$, then $(v_\alpha, v_\alpha i, v_\alpha j, v_\alpha k)_{\alpha=1}^m$ is an F-base of V in which q_h has the diagonal form

$$q_h = \langle\langle a,b \rangle\rangle \otimes \langle \lambda_1, \dots, \lambda_m \rangle.$$

Therefore,

$$e_1(\operatorname{ad}_h) = 0, \quad e_2(\operatorname{ad}_h) = \begin{cases} [Q] & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases} \tag{5.2}$$

and, if m is even,

$$e_3(\operatorname{ad}_h) = [Q] \cdot \big((-1)^{m/2}\lambda_1, \dots, \lambda_m\big) \in H^3(F).$$

Note that the involution ad_h always decomposes: if $V_0 \subset V$ denotes the F-span of $(v_\alpha)_{\alpha=1}^m$, then $V = V_0 \otimes_F Q$ and

$$(\mathrm{End}_Q V, \mathrm{ad}_h) = (\mathrm{End}_F V_0, \mathrm{ad}_{\langle\lambda_1,\ldots,\lambda_m\rangle}) \otimes_F (Q,\gamma).$$

Theorem 5.3. *Let σ be a symplectic involution on a central simple algebra of index 2, and let d be an arbitrary integer, $d \geq 2$:*

(a) If $\deg A < 2^d$, we have $e_i(\sigma) = 0$ for all $i = 2,\ldots,d$ if and only if σ is hyperbolic.
(b) If $\deg A = 2^d$, we have $e_i(\sigma) = 0$ for all $i = 2,\ldots,d$ if and only if (A,σ) decomposes into a tensor product of quaternion algebras with involution:

$$(A,\sigma) = (Q_1,\sigma_1) \otimes_F \cdots \otimes_F (Q_d,\sigma_d). \tag{5.4}$$

(c) If $\deg A = 2^{d-1}$, symplectic involutions on A with $e_i = 0$ for $i = 2,\ldots,d-1$ are classified by their e_d-invariant.

Proof. Using the same notation as in Lemma 5.1, part (a) readily follows from Theorem 3.1 since $e_i(\sigma) = e_i(\sigma \otimes \gamma)$ and σ is hyperbolic if and only if $\sigma \otimes \gamma$ is hyperbolic.

If $\deg A = 2^d$ and $e_i(\sigma) = 0$ for $i = 2,\ldots,d$, then q_h is a $(d+1)$-Pfister form

$$q_h = \langle\langle a,b,\lambda_1,\ldots,\lambda_{d-1}\rangle\rangle \quad \text{for some } \lambda_1,\ldots,\lambda_{d-1} \in F^\times,$$

and we have a decomposition

$$(A,\sigma) \simeq (M_2(F), \mathrm{ad}_{\langle\langle\lambda_1\rangle\rangle}) \otimes_F \cdots \otimes_F (M_2(F), \mathrm{ad}_{\langle\langle\lambda_{d-1}\rangle\rangle}) \otimes_F (Q,\gamma).$$

Conversely, if we have a decomposition (5.4), then

$$(A \otimes_F Q, \sigma\otimes\gamma) \simeq (Q_1,\sigma_1) \otimes_F \cdots \otimes_F (Q_{d-1},\sigma_{d-1}) \otimes_F (Q,\gamma).$$

By Theorem 3.1, we have $e_i(\sigma\otimes\gamma) = 0$ for $i = 2,\ldots,d$, hence $e_i(\sigma) = 0$ for $i = 2,\ldots,d$ by Lemma 5.1.

Similarly, part (c) reduces to the split orthogonal case by Lemma 5.1. □

5.2 Invariant of Degree 2

The group PGSp_n is connected, so there is no analogue of the discriminant of orthogonal involutions as defined in Sect. 3.3. The simply connected cover of PGSp_n is the symplectic group, whose center is μ_2. The exact sequence

$$1 \to \mu_2 \to \mathrm{Sp}_n \to \mathrm{PGSp}_n \to 1 \tag{5.5}$$

yields a connecting map in cohomology:

$$\partial : H^1(F, \mathrm{PGSp}_n) \to H^2(F).$$

The cohomology class corresponding to an algebra with involution (A, σ) is mapped by ∂ to the Brauer class $[A]$, which does not yield any information on σ. Yet, we have a result analogous to Theorems 3.6 and 4.5.

Theorem 5.6. *Let σ be a symplectic involution on a central simple F-algebra A of even degree:*

(a) If $\deg A = 2$*, then σ is unique. It is hyperbolic if and only if A is split.*
(b) If $\deg A = 4$*, there is a decomposition*

$$(A, \sigma) = (Q_1, \sigma_1) \otimes_F (Q_2, \sigma_2)$$

for some quaternion subalgebras Q_1, $Q_2 \subset A$.

Part (a) is clear. Part (b) was first observed by Rowen [47, Theorem B]. For a proof based on the coincidence of Dynkin diagrams $B_2 = C_2$, see [54, p. 131] or [29, (16.16)].

5.3 The Discriminant

Quéguiner-Mathieu's construction of Sect. 3.5 has an analogue for symplectic involutions, showing that there is no absolute invariant of degree 3 for symplectic involutions on central division algebras of degree 4. Start with an arbitrary division algebra A of degree 4 with symplectic involution σ over a field F, and suppose there is a functorial invariant $e_3(\sigma) \in H^3(F)$. Let

$$e_3(\sigma) = \sum_i (a_i) \cdot (b_i) \cdot (c_i) \quad \text{for some } a_i, b_i, c_i \in F^\times.$$

Let K be the function field of the product of the quadrics:

$$x_0^2 - a_i x_1^2 - b_i x_2^2 + a_i b_i x_3^2 - c_i x_4^2 = 0.$$

We have $e_3(\sigma)_K = 0$, hence $e_3(\sigma_K) = 0$, but A_K is a division algebra by a theorem of Merkurjev [13, Corollary 30.10]. It is a tensor product of quaternion algebras by Theorem 5.6; let Q_1, Q_2 be quaternion K-algebras such that $A_K \simeq Q_1 \otimes_K Q_2$. The difference of the norm forms $n_{Q_1} - n_{Q_2}$ is Witt equivalent to a six-dimensional quadratic form φ known as an *Albert form* of A_K. Let L be the function field of the projective quadric $\varphi = 0$. Over L, the quaternion algebras Q_1, Q_2 have isomorphic maximal subfields, hence A_L is not division. It is not split either, since the function field of a quadric does not split any algebra of index 4, hence $\operatorname{ind} A_L = 2$. Since $e_3(\sigma_L) = 0$, Theorem 5.3(a) shows that σ_L is hyperbolic. In view of the coincidence of Dynkin diagrams $B_2 = C_2$, this property translates into a condition on a five-dimensional quadratic F-form q such that[4] $A \simeq C_0(q)$ under an isomorphism that carries σ to the canonical involution of $C_0(q)$: the form q_L is isotropic (see [29, (15.21)]). However, q_K is anisotropic and is not a Pfister neighbor since A_K is a division algebra (see [29, p. 270, Exercise 8]). We obtain a contradiction because Hoffmann [21] has shown that such a form cannot become isotropic by scalar extension to the function field of a six-dimensional quadratic form.

On the other hand, as in the orthogonal case (see Sect. 3.5), the Rost invariant of Sp_n can be used to define a relative invariant of symplectic involutions on central simple algebras of degree $n \equiv 0 \bmod 4$. In the symplectic case, this relative invariant has an easy description.

Let σ_0 be a fixed symplectic involution on a central simple F-algebra A of even degree $n = 2m$. If m is odd, the index of A is 1 or 2. As pointed out at the beginning of this section, the split case is uninteresting since every symplectic involution is hyperbolic. If $\operatorname{ind} A = 2$, then (5.2) yields $e_2(\sigma_0) = [A] \neq 0$, hence there is no e_i-invariant for $i \geq 3$. Therefore, *throughout this section we assume that m is even.* On the vector space $\mathrm{Sym}(\sigma_0)$ of symmetric elements, the reduced norm has a square root given by a polynomial map Nrp analogous to the Pfaffian of skew-symmetric matrices under the transpose involution. The map

$$\mathrm{Nrp}\colon\ \mathrm{Sym}(A,\sigma_0) \to F$$

is a form of degree m such that $\mathrm{Nrp}(1) = 1$, $\mathrm{Nrd}(s) = \mathrm{Nrp}(s)^2$ and

$$\mathrm{Nrp}\big(as\sigma_0(a)\big) = \mathrm{Nrp}(s)\,\mathrm{Nrd}(a) \quad \text{for } s \in \mathrm{Sym}(\sigma_0) \text{ and } a \in A.$$

Every symplectic involution σ on A has the form $\sigma = \mathrm{Int}(s) \circ \sigma_0$ for some unit $s \in \mathrm{Sym}(\sigma_0)$ uniquely determined up to a scalar factor. Since $\mathrm{Nrp}(\lambda s) = \lambda^m \,\mathrm{Nrp}(s)$ for $\lambda \in F$, and since m is even, the square class $\big(\mathrm{Nrp}(s)\big) \in H^1(F)$ is uniquely determined by σ. We may thus set

$$e_3(\sigma/\sigma_0) = \big(\mathrm{Nrp}(s)\big) \cdot [A] \in H^3(F).$$

If σ' is a symplectic involution on A such that (A,σ) and (A,σ') are isomorphic, then σ and σ' are conjugate so there exists $a \in A^\times$ such that

$$\sigma' = \mathrm{Int}(a) \circ \sigma \circ \mathrm{Int}(a)^{-1} = \mathrm{Int}\big(as\sigma_0(a)\big) \circ \sigma_0.$$

Since $\mathrm{Nrp}\big(as\sigma_0(a)\big) = \mathrm{Nrp}(s)\,\mathrm{Nrd}(a)$ and $\big(\mathrm{Nrd}(a)\big) \cdot [A] = 0$, we have

$$\big(\mathrm{Nrp}(as\sigma_0(a))\big) \cdot [A] = \big(\mathrm{Nrp}(s)\big) \cdot [A],$$

[4] This condition determines q uniquely up to a scalar factor (see [29, Sect. 15.C]).

hence

$$e_3(\sigma'/\sigma_0) = e_3(\sigma/\sigma_0).$$

Therefore, $e_3(\sigma/\sigma_0)$ depends only on the isomorphism class of σ: it is an invariant of symplectic involutions on A.

Alternatively, as pointed out by Garibaldi, the invariant $e_3(\sigma/\sigma_0)$ may be obtained by mimicking the argument in Sect. 3.5, using the following twisted version of (5.5):

$$1 \to \mu_2 \to \mathrm{Sp}(A,\sigma_0) \to \mathrm{PGSp}(A,\sigma_0) \to 1.$$

The involution σ defines a cohomology class $\xi \in H^1\big(F, \mathrm{PGSp}(A,\sigma_0)\big)$ that can be lifted to $H^1\big(F, \mathrm{Sp}(A,\sigma_0)\big)$. The Rost invariant of the lift is the invariant $e_3(\sigma/\sigma_0)$. Note that in this case the Rost invariant does not depend on the choice of the lift of ξ since it vanishes on cocycles that come from the center of $\mathrm{Sp}(A,\sigma_0)$ (see [40]). Therefore, we do not have to factor out $H^3(F)$ by a subgroup depending on A to obtain a well-defined relative invariant.

Theorem 5.7. *If* $\deg A = 4$*, then the symplectic involutions* σ *and* σ_0 *are conjugate if and only if* $e_3(\sigma/\sigma_0) = 0$*.*

This theorem was proved by Knus–Lam–Shapiro–Tignol [28] with a slightly different definition of the invariant (see also [29, Sect. 16.B]). It also follows from the exceptional isomorphism $B_2 = C_2$ together with Garibaldi [14, Lemma 1.2].

If the Schur index $\operatorname{ind} A$ divides $(1/2)\deg A$, then A carries hyperbolic involutions. Taking for σ_0 a hyperbolic involution, we may consider $e_3(\sigma/\sigma_0)$ as an absolute invariant. It coincides with the e_3-invariant of Sect. 5.1 when $\operatorname{ind} A = 2$, as was shown by Berhuy–Monsurrò–Tignol [7, Example 2].

If $\deg A \equiv 0 \bmod 8$, the invariant e_3 can be turned into an absolute invariant by reduction to the case where $\operatorname{ind} A$ divides $(1/2)\deg A$, as was shown by Garibaldi–Parimala–Tignol [18].

Theorem 5.8. *If* $n \equiv 0 \bmod 8$*, there is a unique invariant* e_3 *of central simple algebras of degree* n *with symplectic involution with values in* H^3 *such that for any extension* K *of* F *and any symplectic involutions* σ*,* σ_0 *on a central simple* K*-algebra* A *of degree* n*:*

(i) $e_3(\sigma) = 0$ *if* σ *is hyperbolic.*
(ii) $e_3(\sigma/\sigma_0) = e_3(\sigma) - e_3(\sigma_0)$*.*

Proof. Let A be a central simple F-algebra of degree 8. If A is not division, then it carries a hyperbolic involution σ_0. On the set of isomorphism classes of symplectic involutions on A, the unique map e_3 satisfying the conditions of the theorem is given by

$$e_3(\sigma) = e_3(\sigma/\sigma_0).$$

Now, suppose A is division and decomposes into a tensor product of quaternion subalgebras:

$$A = Q_1 \otimes_F Q_2 \otimes_F Q_3.$$

Then $Q_1 \otimes Q_2$ is division, and remains division over the generic splitting field F_{Q_3} of Q_3. Let X be the projective quadric defined by the vanishing of an Albert form of $Q_1 \otimes Q_2$. Over the function field $F(X)$, the product $Q_1 \otimes Q_2$ is not division, hence we may find $e_3(\sigma_{F(X)}) \in H^3(F(X))$. Similarly, since Q_3 splits over F_{Q_3}, we may find $e_3(\sigma_{F_{Q_3}}) \in H^3(F_{Q_3})$. One may check that $e_3(\sigma_{F(X)})$ is unramified over F, that is, it is in the kernel of all the residue maps corresponding to points of codimension 1 on X. Arason [1, (5.6)] proved that the scalar extension map $H^3(F) \to H^3_{\mathrm{nr}}(F(X))$ is injective, and Kahn [25] showed that its cokernel is $\mathbb{Z}/2\mathbb{Z}$, the nontrivial element being represented by the relative invariant $e_3(\rho_{F(X)}/\rho_0)$, where ρ is any symplectic involution on $Q_1 \otimes Q_2$ and ρ_0 is a hyperbolic symplectic involution on $(Q_1 \otimes Q_2)_{F(X)}$. Since $Q_1 \otimes Q_2$ remains division after scalar extension to F_{Q_3}, the form φ is anisotropic over F_{Q_3}, and we have a commutative diagram where the vertical maps are given by scalar extension:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^3(F) & \longrightarrow & H^3_{\mathrm{nr}}(F(X)) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ & & H^3(F_{Q_3}) & \longrightarrow & H^3_{\mathrm{nr}}(F_{Q_3}(X)) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0. \end{array}$$

By uniqueness of e_3 over $F_{Q_3}(X)$, we must have

$$e_3\left(\sigma_{F_{Q_3}}\right)_{F_{Q_3}(X)} = e_3(\sigma_{F(X)})_{F_{Q_3}(X)} = e_3\left(\sigma_{F_{Q_3}(X)}\right),$$

hence a diagram chase shows that $e_3(\sigma_{F(X)})$ is the image of a unique element $e_3(\sigma) \in H^3(F)$. Thus, e_3 is well defined and unique on the set of isomorphism classes of symplectic involutions on tensor products of three quaternion algebras. The proof in the general case relies on the same arguments, using induction on the minimal number of terms in a decomposition of $[A]$ into a sum of Brauer classes of quaternion algebras. See [18, Sect. 2]. □

The e_3-invariant of symplectic involutions has the same property regarding tensor decompositions as the invariants e_1 and e_2 of orthogonal involutions (see Theorems 3.6(b) and 3.10(b)) and the invariant e_2 of unitary involutions (Theorem 4.5(b)), as was shown in Garibaldi et al. [18].

Theorem 5.9. *Let σ be a symplectic involution on a central simple F-algebra of degree 8. There is a decomposition*

$$(A,\sigma) = (Q_1,\sigma_1) \otimes_F (Q_2,\sigma_2) \otimes_F (Q_3,\sigma_3)$$

for some quaternion subalgebras Q_1, Q_2, $Q_3 \subset A$ if and only if $e_3(A,\sigma) = 0$.

Appendix: Trace Form Invariants

Besides those that are explicitly defined in terms of the trace form (at the end of Sect. 4.3), several invariants defined earlier[5] have an alternative description in terms of the trace form. Throughout this appendix, we use the following notation: for σ an involution of arbitrary type on a central simple algebra A, we let Q_σ denote the quadratic form

$$Q_\sigma\colon \operatorname{Sym}(\sigma) \to F, \quad x \mapsto \operatorname{Trd}_A(x^2),$$

where F is the subfield fixed under σ in the center of A.

Suppose first σ is orthogonal and $\deg A = n = 2m$. Lewis [33, Theorem 1] and Quéguiner [44, Sect. 2.2] computed

$$\det Q_\sigma = 2^m \det\sigma.$$

(See also [29, (11.5)].) The Witt–Clifford invariant of Q_σ was computed by Quéguiner [44, Sect. 2.3]; it turns out to depend only on the discriminant of σ, the Brauer class of A, and the residue of m modulo 8. Therefore, if σ and σ_0 are two orthogonal involutions with the same discriminant on A, then $Q_\sigma - Q_{\sigma_0} \in I^3F$. The e_3-invariant of that form was computed in a particular case in [9, Lemma 4]. Note that $e_3(Q_\sigma - Q_{\sigma_0})$ lies in $H^3(F)$ whereas $e_3(\sigma/\sigma_0)$ lies in $H^3(F, \mu_4^{\otimes 2})/B_A^3$.

Suppose next τ is a unitary involution on a central simple algebra B of degree n over a field $K = F(\sqrt{a})$. Then by Quéguiner [44, Lemma 13] (see also [29, (11.16)])

$$\det Q_\tau = (-a)^{n(n-1)/2} \cdot F^{\times 2} \in F^\times / F^{\times 2}.$$

If $n = 2m$, the Witt–Clifford invariant $e_2(Q_\tau)$ is related to the Brauer class $e_2(\tau)$ of the discriminant algebra $D(B,\tau)$ as follows:

$$e_2(Q_\tau) = e_2(\tau) + (-a) \cdot (2^m (-1)^{m(m-1)/2}),$$

see [44, Sect. 3.4] or [29, (11.17)].

Finally, assume σ is a symplectic involution on a central simple F-algebra A of degree $n = 2m$. Then by Lewis [33, Theorem 1] or Quéguiner [44, Sect. 2.2] $\det Q_\sigma = 1$, and by Quéguiner [44, Sect. 2.3]

$$e_2(Q_\sigma) = \begin{cases} 0 & \text{if } m \equiv 0,\ 1 \bmod 8, \\ [A] & \text{if } m \equiv 2,\ 7 \bmod 8, \\ [A] + (-1)\cdot(-1) & \text{if } m \equiv 3,\ 6 \bmod 8, \\ (-1)\cdot(-1) & \text{if } m \equiv 4,\ 5 \bmod 8. \end{cases}$$

[5] Essentially the first nontrivial invariant in each of the orthogonal, unitary, and symplectic case.

If σ, σ_0 are symplectic involutions on A, then by Berhuy et al. [7, Theorem 4] we have

$$e_3(Q_\sigma - Q_{\sigma_0}) = \begin{cases} e_3(\sigma/\sigma_0) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

In a different direction, note that the trace form is also used to define the *signature* of an involution (see [34], [43], or [29, Sect. 11]).

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