Lifting of Coefficients for Chow Motives of Quadrics

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Summary We prove that the natural functor from the category of Chow motives of smooth projective quadrics with integral coefficients to the category with coefficients modulo 2 induces a bijection on the isomorphism classes of objects.

1 Introduction

Alexander Vishik has given a description of the Chow motives of quadrics with integral coefficients in [6]. It uses much subtler methods than the ones used to give a similar description with coefficients in $\mathbb{Z}/2$, found for example in [2], but the description obtained is the same ([2, Theorems 93.1 and 94.1]). The result presented here allows one to recover Vishik's results from the modulo 2 description.

In order to state the main result, we first define the categories involved. Let Λ be a commutative ring. We write $\mathscr{Q}_F$ for the class of smooth projective quadrics over a field F. We consider the additive category $\mathscr{C}(\mathscr{Q}_F,\Lambda)$, where objects are (coproducts of) quadrics in $\mathscr{Q}_F$ and if X,Y are two such quadrics, $\mathrm{Hom}(X,Y)$ is the group of correspondences of degree 0, namely $\mathrm{CH}_{\dim X}(X\times Y,\Lambda)$. We write $\mathscr{C}\mathscr{M}(\mathscr{Q}_F,\Lambda)$ for the idempotent completion of $\mathscr{C}(\mathscr{Q}_F,\Lambda)$. This is the category of graded Chow motives of smooth projective quadrics with coefficients in Λ. If $(X,\rho),(Y,\sigma)$ are two such motives then we have:

$$\mathrm{Hom}((X,\rho),(Y,\sigma)) = \sigma\circ \mathrm{CH}_{\dim X}(X\times Y,\Lambda)\circ\rho.$$

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We will prove the following:

Theorem 1. *The functor* $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}) \to \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2)$ *induces a bijection on the isomorphism classes of objects.*

The proof mostly relies on the low rank of the homogeneous components of the Chow groups of quadrics when passing to a splitting field. These components are almost always indecomposable if we take into account the Galois action. The only exception is the component of rank 2 when the discriminant is trivial but in this case the Galois action on the Chow group is trivial which allows the proof to go through.

It seems that Theorem 1 may be deduced from [6] (see [3, Theorem E.11.2, p. 254]). Here we try to give a more direct and self-contained proof.

2 Chow Groups of Quadrics

We first recall some facts and fix the notations that we will use.

If L/F is a field extension, and S is a scheme over F, we write S_L for the scheme $S \times_{\mathrm{Spec}(F)} \mathrm{Spec}(L)$. Similarly, for an F-vector space U, we write U_L for $U \otimes_F L$, and for a cycle $x \in \mathrm{CH}(S)$, the element $x_L \in \mathrm{CH}(S_L)$ is the pull-back of x along the flat morphism $S_L \to S$.

We say that a cycle in $\mathrm{CH}(S_L)$ is F-*rational* (or simply *rational* when no confusion seems possible) if it can be written as x_L for some cycle $x \in \mathrm{CH}(S)$, i.e., if it belongs to the image of the pull-back homomorphism $\mathrm{CH}(S) \to \mathrm{CH}(S_L)$.

Let F be a field and φ be a nondegenerate quadratic form on an F-vector space V of dimension $D+2$. The associated projective quadric X is smooth of dimension $D = 2d$ or $2d+1$. Let L/F be a splitting extension for X, i.e., a field extension such that V_L has a totally isotropic subspace of dimension $d+1$. We write h^i, l_i for the usual basis of $\mathrm{CH}(X_L)$, where $0 \leq i \leq d$. The class h is the pull-back of the hyperplane class of the projective space of V_L, the class l_i is the class of the projectivisation of a totally isotropic subspace of V_L of dimension $i+1$.

If D is even, then $\mathrm{CH}_d(X_L)$ is freely generated by h^d and l_d. In this case, there are exactly two classes of maximal totally isotropic spaces, l_d and $l_d{}'$. They correspond to spaces exchanged by a reflection and verify the relation $l_d + l_d{}' = h^d$.

The group $\mathrm{Aut}(L/F)$ acts on $\mathrm{CH}(X_L)$. It acts trivially on the ith homogeneous component of $\mathrm{CH}(X_L)$, as long as $2i \neq D$.

See [2] for proofs of all these facts.

In the next proposition, X is a smooth projective quadric of dimension $D = 2d$ associated with a quadratic space (V,φ) over a field F, L/F is a splitting extension for X, and $\mathrm{disc}\,X$ is the discriminant algebra of φ.

Proposition 2. *Under the natural* $\mathrm{Aut}(L/F)$*-actions, the pair* $\{l_d, l_d{}'\}$ *can be identified with the connected components of* $\mathrm{Spec}(\mathrm{disc}\,X \otimes L)$.

Proof. We consider the scheme $G(\varphi)$ of maximal totally isotropic subspaces of V, i.e., the grassmannian variety of isotropic $(d+1)$-dimensional subspaces of V. The scheme $G(\varphi)_L$ has two connected components exchanged by any reflection of the quadratic space (V,φ). There is a faithfully flat morphism $G(\varphi) \to \mathrm{Spec}(\mathrm{disc}\, X)$ (see [2, §85, p. 357]), hence the connected components of $G(\varphi)_L$ are in correspondence with those of $\mathrm{Spec}(\mathrm{disc}\, X \otimes L)$, in a way respecting the natural $\mathrm{Aut}(L/F)$-actions.

Now two maximal totally isotropic subspaces lie in the same connected component of $G(\varphi)_L$ if and only if the corresponding d-dimensional closed subvarieties of the quadric X_L are rationally equivalent (see [2, §86, p. 358]). Therefore, the pair $\{l_d, l_d'\}$ is $\mathrm{Aut}(L/F)$-isomorphic to the pair of connected components of $G(\varphi)_L$. The statement follows. □

3 Lifting of Coefficients

We now give a useful characterization of rational cycles (Proposition 7). The proof will rely on the following theorem from [5, Proposition 9]:

Theorem 3 (Rost's nilpotence for quadrics). *Let X be a smooth projective quadric over a field F, and let $\alpha \in \mathrm{End}_{\mathscr{CM}(\mathscr{Q}_F,\Lambda)}(X)$. If $\alpha_L \in \mathrm{CH}(X_L^2)$ vanishes for some field extension L/F then α is nilpotent.*

We will use the following classical corollaries:

Corollary 4 ([2, Corollary 92.5]). *Let X be a smooth projective quadric over a field F and L/F a field extension. Let π a projector in $\mathrm{End}_{\mathscr{CM}(\mathscr{Q}_L,\Lambda)}(X_L)$ that is the restriction of some element in $\mathrm{End}_{\mathscr{CM}(\mathscr{Q}_F,\Lambda)}(X)$. Then there exist a projector φ in $\mathrm{End}_{\mathscr{CM}(\mathscr{Q}_F,\Lambda)}(X)$ such that $\varphi_L = \pi$.*

Corollary 5 ([2, Corollary 92.7]). *Let $f\colon (X,\rho) \to (Y,\sigma)$ be a morphism in the category $\mathscr{CM}(\mathscr{Q}_F,\Lambda)$. If f_L is an isomorphism for some field extension L/F then f is an isomorphism.*

Proposition 6. *For any $n \geq 1$, the functor $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2^n) \to \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2)$ is bijective on the isomorphism classes of objects.*

Proof. We are clearly in the situation $(\star)$ of [7, Sect. 2, p. 587] for the obvious functor $\mathscr{C}(\mathscr{Q}_F,\mathbb{Z}/2^n) \to \mathscr{C}(\mathscr{Q}_F,\mathbb{Z}/2)$. The statement then follows from [7, Propositions 2.5 and 2.2] (see also [4, Corollary 2.7]). □

Any smooth projective quadric admits a (noncanonical) finite Galois splitting extension, of degree a power of 2. This will be used together with the following proposition when we will need to prove that a given cycle with integral coefficients (and defined over some extension of the base field) is rational.

Proposition 7. *Let $X,Y \in \mathscr{Q}_F$ and L/F be a splitting Galois extension of degree m for X and Y. A correspondence in $\mathrm{CH}((X\times Y)_L,\mathbb{Z})$ is rational if and only if it is invariant under the group $\mathrm{Gal}(L/F)$ and its image in $\mathrm{CH}((X\times Y)_L,\mathbb{Z}/m)$ is rational.*

Proof. We write Z for $X \times Y$. We first prove that if x is a $\mathrm{Gal}(L/F)$-invariant cycle in $\mathrm{CH}(Z_L)$ then $[L:F]\cdot x$ is rational.

Let $\tau\colon L \to \bar{F}$ be a separable closure so that we have an $\bar{F}$-isomorphism $L \otimes \bar{F} \to \bar{F} \times \cdots \times \bar{F}$ given by $u \otimes 1 \mapsto (\tau \circ \gamma(u))_{\gamma \in \mathrm{Gal}(L/F)}$.

We have a cartesian square:

$$\begin{array}{ccc} Z_L & \longleftarrow & Z_{L\otimes \bar{F}} \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Z_{\bar{F}} \end{array}$$

It follows that we have a commutative diagram of pull-backs and push-forward:

$$\begin{array}{ccc} \mathrm{CH}(Z_L) & \longrightarrow & \mathrm{CH}(Z_{L\otimes \bar{F}}) \\ \downarrow & & \downarrow \\ \mathrm{CH}(Z) & \longrightarrow & \mathrm{CH}(Z_{\bar{F}}) \end{array}$$

The top map followed by the map on the right is:

$$x \mapsto \sum_{\gamma \in \mathrm{Gal}(L/F)} t^*({}^{\gamma}x)$$

where $t: Z_{\bar{F}} \to Z_L$ is the map induced by τ. Using the commutativity of the diagram and the injectivity of t^*, we see that the composite $\mathrm{CH}(Z_L) \to \mathrm{CH}(Z) \to \mathrm{CH}(Z_L)$ maps x to $\sum {}^{\gamma}x$, where γ runs in $\mathrm{Gal}(L/F)$. The claim follows.

Now suppose that u is a cycle in $\mathrm{CH}(Z_L,\mathbb{Z})$ invariant under $\mathrm{Gal}(L/F)$, and that its image in $\mathrm{CH}(Z_L,\mathbb{Z}/m)$ is rational. We can find a rational cycle v in $\mathrm{CH}(Z_L,\mathbb{Z})$ and a cycle δ in $\mathrm{CH}(Z_L,\mathbb{Z})$ such that $m\delta = v-u$. As $\mathrm{CH}(Z_L,\mathbb{Z})$ is torsion-free, δ is invariant under $\mathrm{Gal}(L/F)$. The first claim ensures that $v-u$ is rational, hence u is rational, and we have proven the proposition. □

Let us remark that if $X \in \mathscr{Q}_F$, L/F is a splitting extension, and $2i < \dim X$ then $2l_i = h^{\dim X - i} \in \mathrm{CH}(X_L)$ is always rational. It follows that $2\,\mathrm{CH}_i(X_L)$ consists of rational cycles when $2i \neq \dim X$.

4 Surjectivity in the Main Theorem

Proposition 8. *The functor $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}) \to \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2)$ is surjective on the isomorphism classes of objects.*

Proof. Let $(X,\pi) \in \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2)$ and L/F a finite splitting Galois extension for X of degree 2^n. By Proposition 6, we can lift the isomorphism class of (X,π) to the isomorphism class of some $(X,\tau) \in \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2^n)$.

Assume that we have found a $\mathrm{Gal}(L/F)$-invariant projector ρ in $\mathrm{CH}_{\dim X}(X \times X)_L$ which gives modulo 2^n the projector τ_L. By Proposition 7, ρ is a rational cycle, and Corollary 4 provides a projector p such that $\rho = p_L$. Write $\tilde{p} \in \mathrm{CH}(X \times X, \mathbb{Z}/2^n)$ for the image of p. Consider the morphism $(X,\tau) \to (X,\tilde{p})$ given by $\tilde{p} \circ \tau$. Since $\tilde{p}_L = \rho$ mod $2^n = \tau_L$, this morphism becomes an isomorphism (the identity) after extending scalars to L hence is an isomorphism by Corollary 5. It follows that the isomorphism class of $(X,p) \in \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z})$ is a lifting of the class of $(X,\tau) \in \mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2^n)$.

We now build the projector ρ. For any commutative ring Λ, projectors in $\mathrm{CH}((X\times X)_L,\Lambda)$ are in bijective correspondence with ordered pairs of subgroups of $\mathrm{CH}(X_L,\Lambda)$ which form a direct sum decomposition. This bijection is compatible with the natural $\mathrm{Gal}(L/F)$-actions. A projector is of degree 0 if and only if the two summands in the associated decomposition are graded subgroups of $\mathrm{CH}(X_L,\Lambda)$.

When $\dim X$ is odd or when $\mathrm{disc}X$ is a field, each homogeneous component of $\mathrm{CH}(X_L,\Lambda)$ is $\mathrm{Gal}(L/F)$-indecomposable, hence $\mathrm{Gal}(L/F)$-invariant projectors of degree 0 of $\mathrm{CH}(X_L,\Lambda)$ are in one-to-one correspondence with the subsets of $\{0,\dots,\dim X\}$. It follows that we can lift any $\mathrm{Gal}(L/F)$-invariant projector of degree 0 with coefficients in $\mathbb{Z}/2^n$ to an integral $\mathrm{Gal}(L/F)$-invariant projector of degree 0.

When $\dim X$ is even and $\mathrm{disc}X$ is trivial, $\mathrm{CH}_i(X_L,\Lambda)$ is indecomposable if $2i \neq 2d_X = \dim X$. The group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{d_X}(X_L,\Lambda)$. If the rank of the restriction of $(\tau_L)_*$ to $\mathrm{CH}_{d_X}(X_L,\mathbb{Z}/2^n)$ is 0 or 2, the projector τ_L clearly lifts to a $\mathrm{Gal}(L/F)$-invariant projector in $\mathrm{CH}_{\dim X}(X\times X)_L$.

The last case is when the rank is 1. We fix a decomposition of the group $\mathrm{CH}_{d_X}(X_L,\mathbb{Z})$ into the direct sum of rank 1 summands. Any such decomposition of $\mathrm{CH}_{d_X}(X_L,\Lambda)$ is then given by some element of $SL_2(\Lambda)$. The next lemma ensures that we can lift any element of $SL_2(\mathbb{Z}/2^n)$ to $SL_2(\mathbb{Z})$, thus that τ_L lifts to a projector with integral coefficients. It remains to notice that $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{\dim X}(X\times X)_L$ since $\mathrm{disc}X$ is trivial, to conclude the proof. □

Lemma 9 ([4, Lemma 2.14]). *For any positive integers k and p, the reduction homomorphism $SL_k(\mathbb{Z}) \to SL_k(\mathbb{Z}/p)$ is surjective.*

Proof. As $\mathbb{Z}/p$ is a semilocal commutative ring, it follows from [1, Corollary 9.3, Chap. V, p. 267], applied with $A = \mathbb{Z}$ and $\underline{q} = p\mathbb{Z}$, that any matrix in $SL_k(\mathbb{Z}/p)$ is the image modulo p of a product of elementary matrices with integral coefficients. Such a product in particular belongs to $SL_k(\mathbb{Z})$, as required. □

5 Injectivity in the Main Theorem

In order to prove injectivity in Theorem 1, we may assume that we are given two motives $(X,\rho),(Y,\sigma)$ in $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z})$ and an isomorphism between their images in $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2)$. We will build an isomorphism with integral coefficients between the two motives (which will not, in general, be an integral lifting of the original isomorphism with finite coefficients).

We fix a finite Galois splitting extension L/F for X and Y of degree 2^n. Using Proposition 6 we may assume that there exists an isomorphism α between (X,ρ) and (Y,σ) in $\mathscr{CM}(\mathscr{Q}_F,\mathbb{Z}/2^n)$. By Proposition 7 and Corollary 5, it is enough to build an isomorphism $(X_L,\rho_L) \to (Y_L,\sigma_L)$ which reduces to a rational correspondence modulo 2^n and which is equivariant under $\mathrm{Gal}(L/F)$.

Let d_X be such that $\dim X = 2d_X$ or $2d_X+1$ and d_Y defined similarly for Y. Let $r(X,\rho)$ be the rank of $\mathrm{CH}_{d_X}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ if $\dim X$ is even and $r(X,\rho)=0$ if $\dim X$ is odd. We define $r(Y,\sigma)$ in a similar fashion. We will distinguish cases using these integers.

A basis of $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ gives an isomorphism of (X_L,ρ_L) with twists of Tate motives, thus choosing bases for the groups $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ and $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$, we can see morphisms between the two motives as matrices.

We fix a basis (e_i) of $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ as follows: we choose $e_i \in \mathrm{CH}_i(X_L)$ among the cycles $h^{\dim X - i}, l_i$ for $2i \neq \dim X$. We are done when $r(X,\rho)=0$.

If $r(X,\rho)=2$ we complete the basis with $e_{d_X} = l_{d_X}, e'_{d_X} = l'_{d_X} \in \mathrm{CH}_{d_X}(X_L)$.

If $r(X,\rho)=1$ we choose a generator e_{d_X} of $\mathrm{CH}_{d_X}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ to complete the basis.

We choose a basis (f_i) for $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$ in a similar way.

If we write $\tilde{\rho}$ and $\tilde{\sigma}$ for the reduction modulo 2^n of ρ and σ, these bases reduce to bases $(\tilde{e}_i)$ of $\mathrm{CH}(X_L,\mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\rho}_L)_*$ and $(\tilde{f}_i)$ of $\mathrm{CH}(Y_L,\mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\sigma}_L)_*$. In these homogeneous bases the matrix of a correspondence of degree 0 is diagonal by blocks. The sizes of the blocks are the ranks of the homogeneous components of $\mathrm{im}\,(\rho_L)_*$.

Lemma 10. *If* $r(X,\rho)=1$ *then* $\mathrm{disc}\,X$ *is trivial.*

Proof. Assume $\mathrm{disc}\,X$ is not trivial. The correspondence ρ induces a projection of $\mathrm{CH}_{d_X}(X_L)$ which is equivariant under the action of $\mathrm{Gal}(L/F)$. But $\mathrm{CH}_{d_X}(X_L)$ is indecomposable as a $\mathrm{Gal}(L/F)$-module. It follows that $(\rho_L)_*$ is either the identity or 0 when restricted to $\mathrm{CH}_{d_X}(X_L)$, hence $r(X,\rho) \neq 1$. □

Corollary 11. *If* $r(X,\rho) \neq 2$ *then* $\mathrm{Gal}(L/F)$ *acts trivially on* $\mathrm{im}\,(\rho_L)_*$.

Lemma 12. *If* $r(X,\rho)=2$ *then* $r(Y,\rho)=2$*,* $\dim Y = \dim X$ *and* $\mathrm{disc}\,Y = \mathrm{disc}\,X$.

Proof. As the isomorphism $(\alpha_L)_*$ is graded, the d_X-th homogeneous component of $\mathrm{im}(\alpha_L)_*$ has rank 2. This image is a subgroup of the Chow group with coefficients in $\mathbb{Z}/2^n$ of a split quadric, thus the only possibility is that $\dim Y$ is even, $d_X = d_Y$ and $r(Y,\sigma)=2$.

The isomorphism $(\alpha_L)_*$ is equivariant under the action of $\mathrm{Gal}(L/F)$. It follows that an element of the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}(X_L,\mathbb{Z}/2^n)$ if and only if it acts trivially on $\mathrm{CH}(Y_L,\mathbb{Z}/2^n)$. But such an element acts trivially on $\mathrm{CH}(X_L,\mathbb{Z}/2^n)$ (respectively, $\mathrm{CH}(Y_L,\mathbb{Z}/2^n)$) if and only if it acts trivially on the pair of integral cycles $\{l_{d_X}, {l_{d_X}}'\} \subset \mathrm{CH}(X_L)$ (respectively, $\{l_{d_Y}, {l_{d_Y}}'\} \subset \mathrm{CH}(Y_L)$). Therefore, the pair of integral cycles $\{l_{d_X}, {l_{d_X}}'\}$ is $\mathrm{Gal}(L/F)$-isomorphic to the pair $\{l_{d_Y}, {l_{d_Y}}'\}$. By proposition 2, we have a $\mathrm{Gal}(L/F)$-isomorphism between the split étale algebras $\mathrm{disc}\,X \otimes L$ and $\mathrm{disc}\,Y \otimes L$. Hence, $\mathrm{disc}\,X$ and $\mathrm{disc}\,Y$ correspond to the same cocycle in $H^1(\mathrm{Gal}(L/F),\mathbb{Z}/2)$, thus are isomorphic. □

We now proceed with the proof of the injectivity.

Let us first assume that $r(X,\rho) \neq 2$. Then $r(Y,\sigma) \neq 2$ by the preceding lemma. By Corollary 11 the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{im}\,(\rho_L)_*$ and on $\mathrm{im}\,(\sigma_L)_*$, therefore any morphism $(X_L,\rho_L) \to (Y_L,\sigma_L)$ is defined by a cycle invariant under $\mathrm{Gal}(L/F)$.

Because the isomorphism α_L is of degree 0, its matrix in our graded bases of the modulo 2^n Chow groups is diagonal. Let $\lambda_i \in (\mathbb{Z}/2^n)^\times$ be the coefficients in the diagonal so that we have $(\alpha_L)_*(\tilde{e}_i) = \lambda_i \tilde{f}_i$ for all i such that $\mathrm{CH}_i(X_L) \cap \mathrm{im}\,(\rho_L)_* \neq \emptyset$.

If $r(X,\rho) = 1$ then λ_{d_X} is defined and we consider the cycle $\beta = (\lambda_{d_X})^{-1}\alpha_L$. If $r(X,\rho) = 0$, we just put $\beta = \alpha_L$.

Now we take $k_i \in \mathbb{Z}/2^n$ such that $\lambda_i^{-1} = 2k_i + 1$. Let $\Delta \in \mathrm{End}(X_L,\tilde{\rho}_L)$ be the identity morphism. We consider the rational cycle

$$\gamma = \Delta + 2\sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i},$$

where the sum is taken over all i such that $\mathrm{CH}_i(X_L) \subset \mathrm{im}\,(\rho_L)_*$ (which implies in case $r(X,\rho) = 1$ that we do not take $i = d_X$). The composite $\beta \circ \gamma$ is rational, and its matrix in our bases is the identity matrix. This correspondence lifts to an isomorphism with integral coefficients $(X_L,\rho_L) \to (Y_L,\sigma_L)$.

Next assume that $r(X,\rho) = 2$. Then we have $\dim X = \dim Y$, $r(Y,\sigma) = 2$, and $\mathrm{disc}\,X = \mathrm{disc}\,Y$ by Lemma 12. The matrix of $(\alpha_L)_*$ is diagonal by blocks:

$$\begin{pmatrix} \nu_{i_1} & & & & & & \\ & \ddots & & & & & \\ & & \nu_{i_r} & & & & \\ & & & B & & & \\ & & & & \nu_{i_{r+1}} & & \\ & & & & & \ddots & \\ & & & & & & \nu_{i_p} \end{pmatrix},$$

where $\nu_i \in (\mathbb{Z}/2^n)^\times$ and $B \in GL_2(\mathbb{Z}/2^n)$.

Now if $\mathrm{disc}\,X = \mathrm{disc}\,Y$ is a field, there is an element in $\mathrm{Gal}(L/F)$ that simultaneously exchanges the cycles in the bases $\{l_{d_X}, {l_{d_X}}'\}$ of $\mathrm{CH}_{d_X}(X_L,\mathbb{Z}/2^n)$ and $\{l_{d_Y}, {l_{d_Y}}'\}$ of $\mathrm{CH}_{d_Y}(Y_L,\mathbb{Z}/2^n)$. It follows that we may write B as:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

for some a and b in $\mathbb{Z}/2^n$. The determinant of B is $(a+b)(a-b) \in (\mathbb{Z}/2^n)^\times$, hence $(a-b) \in (\mathbb{Z}/2^n)^\times$. Thus, we may replace α_L by $(a-b)^{-1}\alpha_L$ and assume that $a = b+1$.

As before we may write $\nu_i^{-1} = 2k_i + 1$ and replace α_L by the rational cycle:

$$\alpha_L \circ \left(\Delta + 2\sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i}\right)$$

the sum being taken over all i such that $\mathrm{CH}_i(X_L) \cap \mathrm{im}(\rho_L)_* \neq \emptyset$ and $i \neq d_X$. Therefore, we may assume that $v_i = 1$ for all i and that we have a matrix of the shape (I_r being the identity block of size r):

$$\begin{pmatrix} I_s & & & 0 \\ & a+1 & a & \\ & a & a+1 & \\ 0 & & & I_t \end{pmatrix}.$$

The matrix of the rational cycle $h^{d_X} \times h^{d_Y} \in \mathrm{CH}((X \times Y)_L, \mathbb{Z}/2^n)$ is:

$$\begin{pmatrix} 0 & & & 0 \\ & 1 & 1 & \\ & 1 & 1 & \\ 0 & & & 0 \end{pmatrix}$$

Now $\alpha_L - a(h^{d_X} \times h^{d_Y})$ is rational and its matrix is the identity. This cycle is invariant under $\mathrm{Gal}(L/F)$ and lifts to an isomorphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$.

It remains to treat the case when $\mathrm{disc}\, X$ is trivial. In this case the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}(X_L)$ (and on $\mathrm{CH}(Y_L)$ as $\mathrm{disc}\, Y$ is also trivial). As before, composing with a rational cycle, we may assume that $v_i = 1$ for all i. We write $\det B^{-1} = 2k+1$.

The cycle $\Delta + k(h^{d_X} \times h^{d_X}) \in \mathrm{End}(X_L, \tilde{\rho}_L)$ is rational and its matrix in our basis is:

$$\begin{pmatrix} I_p & & & 0 \\ & 1+k & k & \\ & k & 1+k & \\ 0 & & & I_r \end{pmatrix}.$$

We see that the determinant of this matrix is $1+2k$. Therefore, the composite $\alpha_L \circ (\Delta + k(h^{d_X} \times h^{d_X}))$ has determinant 1. We use Lemma 9 to conclude, which completes the proof of Theorem 1.

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