Chapter 6
Fuzzy Sets and Fuzzy Logic-Based Methods in Multicriteria Decision Analysis

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Abstract In this chapter, we discuss some fuzzy sets and fuzzy logic-based methods for multicriteria decision aid. Alternatives are identified with score vectors $\mathbf{x} \in [0,1]^n$, and thus they can be seen as fuzzy sets, too. After discussion of integral-based utility functions, we introduce a transformation of score $\mathbf{x}$ into fuzzy quantity $U(\mathbf{x})$. Orderings on fuzzy quantities induce orderings on alternatives. A special attention is paid to defuzzification-based orderings, especially to mean of maxima method. Our approach allows an easy incorporation of importance of criteria. Finally, a fuzzy logic-based construction method to build complete preference structures over set of alternatives is given.

Keywords Fuzzy set · Fuzzy quantity · Fuzzy utility · Dissimilarity · Defuzzification

6.1 Introduction

In this chapter we will deal with alternatives $\mathbf{x}$ from a set of alternatives $\mathcal{A}$. Each alternative $\mathbf{x} \subseteq \mathcal{A}$ is characterized by a score vector $(x_1, \ldots, x_n)$ and we will not distinguish $\mathbf{x}$ and $(x_1, \ldots, x_n)$. Score vector $(x_1, \ldots, x_n) \in [0,1]^n$ summarizes the information about the degrees of fulfilment of criteria $C_1, \ldots, C_n$ by the alternative $\mathbf{x}$. Here $x_i = 1$ means that $\mathbf{x}$ fully satisfies the criterion C_i, while $x_j = 0$ means that $\mathbf{x}$ is completely failed in the criterion C_j. We will not discuss any aspect of commensurability nor of vagueness of the degrees of satisfaction (for this item we recommend Chapter 5 of Bernard De Baets and Janos Fodor in this edited volume). Hence each alternative $\mathbf{x}$ can be seen as a fuzzy subset of the space $\mathcal{C} = \{C_1, \ldots, C_n\}$ of all criteria considered in our decision problems (here n is some fixed integer, the number of all considered criteria). The set $\mathcal{A}$ of discussed alternatives is then a subset of the set of all fuzzy subsets of $\mathcal{C}$. Alternatives from

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$\mathcal{A}$ will be denoted by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. Note that we did not distinguish fuzzy subsets and their membership functions. Thus the same letter $\mathbf{x}$ is used for an alternative from $\mathcal{A}$, for its score vector and for the fuzzy subset of $\mathcal{C}$ with membership function $(x_1, \ldots, x_n) \in [0,1]^n$. We hope that this convention will not create any confusion. In some cases, score values from an interval I (e.g., $I = \mathbb{R}$) will be considered too, and then it will be mentioned explicitly.

We recall first some basic notions and notations from the fuzzy set theory [12, 46]. For a given non-empty set Ω (universe), a fuzzy set V (fuzzy subset V of Ω) is characterized by the membership function $\mu_V : \Omega \to [0,1]$. In this context, classical sets (subsets of Ω) are called crisp sets (crisp subsets of Ω) and they are characterized by the corresponding characteristic function. The height $\mathrm{hgt}(V)$ of a fuzzy set V is given by

$$\mathrm{hgt}(V) = \sup_{\omega \in \Omega} \mu_V(\omega).$$

Evidently, for non-empty crisp set V it holds $\mathrm{hgt}(V) = 1$. Vice versa, on any finite universe Ω, if $\mathrm{hgt}(V) = 1$ then there is a non-empty crisp set V' such that $\mu_{V'} \le \mu_V$. For a given constant $\alpha \in [0,1]$, the corresponding α-cut $V^{(\alpha)}$ of a fuzzy set V is given by

$$V^{(\alpha)} = \{\omega \in \Omega \mid \mu_V(\omega) \ge \alpha\}.$$

Fuzzy set V is called normal if $V^{(1)} \ne \emptyset$, i.e., $\mu_V(\omega) = 1$ for some $\omega \in \Omega$. Fuzzy subsets of the real line $\mathbb{R} =]-\infty, \infty[$ (or any real interval I) are called fuzzy quantities. Moreover, a V is called convex whenever each α-cut $V^{(\alpha)}$ is a convex subset of $\mathbb{R}$, i.e., $V^{(\alpha)}$ is an interval for all $\alpha \in [0,1]$. Equivalently, convexity of a fuzzy quantity V can be characterized by the fulfilment of inequality

$$\mu_V(\lambda r + (1-\lambda)s) \ge \min(\mu_V(r), \mu_V(s)), \tag{6.1}$$

for all $r, s \in \mathbb{R}$ ($r, s \in I$) and $\lambda \in [0,1]$. For more details see [10, 25].

The aim of this chapter is to discuss several methods for building (weak) orderings on $\mathcal{A}$ (in fact, on $[0,1]^n$), based on fuzzy set theory and fuzzy logic. The chapter will bring in the next section an overview of such methods which can be roughly seen as utility function based methods. In Section 6.3, methods based on some orderings of fuzzy quantities are introduced. A special case of MOM (mean of maxima) based defuzzification method of ordering fuzzy quantities is related to utility based decisions and it is investigated in Section 6.4. In Section 6.5, we exploit some fuzzy logic connectives to create a variety of complete preference structures on $[0,1]^n$, which need not be transitive, in general. Finally, some concluding remarks are included.

6.2 Fuzzy Set Based Utility Functions

Global evaluation of a fuzzy event $\mathbf{x}$ (measurable fuzzy subset of some measurable space $(\Omega, \mathcal{X})$) was first introduced by Zadeh [47] as a fuzzy probability measure $\mathcal{P}$, $\mathcal{P}(\mathbf{x}) = \int_\Omega \mathbf{x}\, \mathrm{d}P$, where P is some probability measure on $(\Omega, \mathcal{X})$, i.e., $\mathcal{P}$ is the expected value, $\mathcal{P}(\mathbf{x}) = E(\mathbf{x})$. In our case this means that

$$\mathcal{P}(\mathbf{x}) = \sum_{i=1}^{n} x_i p_i, \tag{6.2}$$

$p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, i.e., $\mathcal{P}$ is a normed additive utility function on $[0,1]^n$. The next step in this direction was done by Klement [19], when discussing normed utility functions $\mathcal{M}$ satisfying the valuation property $\mathcal{M}(\mathbf{x} \vee \mathbf{y}) + \mathcal{M}(\mathbf{x} \wedge \mathbf{y}) = \mathcal{M}(\mathbf{x}) + \mathcal{M}(\mathbf{y})$. Supposing the lower semicontinuity of $\mathcal{M}$, Klement's results for our case yield

$$\mathcal{M}(\mathbf{x}) = \sum_{i=1}^{n} F_i(x_i) p_i, \tag{6.3}$$

where $F_i : [0,1] \to [0,1]$ is a (restriction on $[0,1]$) distribution function of some random variable X_i acting on $]0,1[$, $F_i(u) = P(X_i < u)$.

Evidently, (6.3) generalizes (6.2). In fact, (6.2) corresponds to (6.3) in special case, when all random variables X_i are uniformly distributed over $]0,1[$. Utility function $\mathcal{M}$ given by (6.3) is a general version of additive (lower semicontinuous) utility function over $[0,1]^n$.

Comonotone additive models are related to [8, 16]. In such a case, instead of probability measure P on $\mathcal{C}$ (p_i is the weight of criterion C_i) one needs to know a fuzzy measure M on $\mathcal{C}$ [18, 36], $M : 2^{\mathcal{C}} \to [0,1]$, M is non-decreasing and $M(\emptyset) = 0$, $M(\mathcal{C}) = 1$ (here $M(E)$ is the weight of group E of some criteria from $\mathcal{C}$), and then the corresponding utility function $Ch : [0,1]^n \to [0,1]$ is given by

$$Ch(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)}(M(A_i) - M(A_{i+1})), \tag{6.4}$$

where σ is a permutation of $(1,\ldots,n)$ such that $(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ is a non-decreasing permutation of $(x_1,\ldots,x_n)$, and $A_i = \{j \in \{1,\ldots,n\} \,|\, x_j \geq x_{\sigma(i)}\}$, with a convention $A_{n+1} = \emptyset$. A trivial generalization of (6.4) can follow the same line as the way relating (6.2) and (6.3), namely

$$Chg(\mathbf{x}) = \sum_{i=1}^{n} F_{\sigma(i)}(x_{\sigma(i)})(M(A_i) - M(A_{i+1})), \tag{6.5}$$

where σ is a permutation of $1,\ldots,n$ such that $(F_{\sigma(1)}(x_{\sigma(1)}),\ldots,F_{\sigma(n)}(x_{\sigma(n)}))$ is non-decreasing. Note that if fuzzy measure M is additive (i.e., it is a probability measure), then (6.2) coincides with (6.4), while (6.3) coincides with (6.5). For discussion of comonotone maxitive models we recommend [29].

Example 6.1. Let $n = 2$, $M(C_1) = M(C_2) = \frac{1}{3}$, $F_1(u) = u$, $F_2(u) = u^2$. Then (6.5) yields the utility function $Chg : [0,1]^2 \to [0,1]$ given by $Chg(x_1,x_2) = 2\left(x_1 \wedge x_2^2\right)/3 + \left(x_1 \vee x_2^2\right)/3 = \left(x_1 + x_2^2 + x_1 \wedge x_2^2\right)/3$.

The most general form of a normed utility function U, i.e., $U : [0,1]^n \to [0,1]$ is non-decreasing and $U(0) = 0$, $U(1) = 1$, represents aggregation operators discussed, e.g., in [5, 6, 22]. In fact, normed utility functions are exactly n-ary aggregation operators, and only expected properties of these functions restrict our possible choice to some well-known classes (this is, e.g., the case of formulas (6.2), (6.3) and (6.4), while the characterization of normed utility functions Chg introduced in (6.5) is related to the valuation property restricted to comonotone alternatives only). Among well-known classes of aggregation operators, we recall (weighted) quasi-arithmetic means characterized by the unanimity and the bisymmetry [1, 13], OWA operators [41] characterized by the anonymity and comonotone additivity, triangular norms, triangular conorms and uninorms characterized by the anonymity, associativity and neutral element [21, 43], quasi-copulas and related operators characterized by 1-Lipschitz property and neutral element [32], etc. A deep discussion and state-of-art overview of aggregation operators is the topic of a forthcoming monograph [17], and a lot of useful material on aggregation operators can be found in recent handbook [3] and monograph [37].

In any case when a normed utility function (an aggregation operator) $U:[0,1]^n \to [0,1]$ is exploited to build a preference structure on $\mathcal{A}$, this preference structure is evidently transitive and does not possess any incomparable pairs of alternatives, i.e., it is a weak ordering on $\mathcal{A}$ given by $\mathbf{x} \leq_U \mathbf{y}$ if and only if $U(\mathbf{x}) \leq U(\mathbf{y})$. However, we cannot avoid many possible ties in such a case. Among several ways of refining such kinds of preference structures (recall, e.g., Lorentz [24] approach to the problem how to refine the standard arithmetic mean), we focus now on a modification of a recent method based on limit approach we have introduced in [23]. The next result is based on an infinite sequence of $\mathcal{U} = (U_k)_{k\in N}$ of n-ary aggregation operators. Partial ordering $\precsim^{\mathcal{U}}$ induced by $\mathcal{U}$ is given as follows: $\mathbf{x} \precsim^{\mathcal{U}} \mathbf{y}$ if and only if there is a $k_0 \in N$ such that for all $k \in N, k \geq k_0$, it holds $U_k(\mathbf{x}) \leq U_k(\mathbf{y})$. Note that denoting by R_k the relation on $[0,1]^n \times [0,1]^n$ given by $R_k(\mathbf{x},\mathbf{y})$ if and only if $\mathbf{x} \leq_{U_k} \mathbf{y}, k \in N$ and R the relation given by $R(\mathbf{x},\mathbf{y})$ if and only if $\mathbf{x} \precsim^{\mathcal{U}} \mathbf{y}$ then $R = \liminf R_k$.

Proposition 6.1. *Let $\mathcal{U} = (U_k)_{k\in N}$ be a system of n-ary aggregation operators with pointwise limit U. Then $U : [0,1]^n \to [0,1]$ is an aggregation operator and the partial order $\precsim^{\mathcal{U}}$ on $[0,1]^n$ x $[0,1]^n$ is related to the weak order $\leq_U$ as follows:*

$$\boldsymbol{x} <_U \boldsymbol{y} \Rightarrow \boldsymbol{x} \prec^{\mathcal{U}} \boldsymbol{y},$$
$$\boldsymbol{x} \approx^{\mathcal{U}} \boldsymbol{y} \Rightarrow \boldsymbol{x} \approx_U \boldsymbol{y}.$$

Moreover, if $\precsim^{\mathcal{U}}$ does not admit incomparable pairs then it is a refinement of $\leq_U$. Note that the original roots of Proposition 6.1 can be found in [9] where the refinements of $\leq_{Min}$ and $\leq_{Max}$ weak orderings were discussed.

Example 6.2.
(i) Let $\mathcal{B} = (B_k)_{k\in N}$, where

$$B_k(x_1,\ldots,x_n) = k \log \left(\frac{\sum_{i=1}^n \exp\left(\frac{x_i}{k}\right)}{n} \right).$$

Then $\lim_{k\to\infty} B_k = M$ is the arithmetic mean. Note that for $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $B_k(\mathbf{x}) < B_k(\mathbf{y})$ if and only if

$$\frac{\sum_{i=1}^n \exp\left(\frac{x_i}{k}\right)}{n} < \frac{\sum_{i=1}^n \exp\left(\frac{y_i}{k}\right)}{n}.$$

By means of Taylor's series, we see that $B_k(\mathbf{x}) \le B_k(\mathbf{y})$ if and only if

$$\frac{1}{k}\left(\sum_{i=1}^n x_i\right) + \frac{1}{2!k^2}\left(\sum_{i=1}^n x_i^2\right) + \frac{1}{3!k^3}\left(\sum_{i=1}^n x_i^3\right) + \cdots$$
$$\le \frac{1}{k}\left(\sum_{i=1}^n y_i\right) + \frac{1}{2!k^2}\left(\sum_{i=1}^n y_i^2\right) + \frac{1}{3!k^3}\left(\sum_{i=1}^n y_i^3\right) + \cdots.$$

Then, for $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \precsim^{\mathcal{B}} \mathbf{y}$ if and only if $MOF_{\mathbf{x}} \le MOF_{\mathbf{y}}$, where $MOF_{\mathbf{x}}$: N$\to$ $[0,1]$ is the moment function given by $MOF_{\mathbf{x}}(m) = \left(\frac{1}{n}\right)\sum_{i=1}^n x_i^m$, i.e., $MOF_{\mathbf{x}}(m)$ is the mth initial moment of a random variable described by the uniform sample $x = (x_1, \ldots, x_n)$. Observe that $\mathbf{x} \approx^{\mathcal{B}} y$ if and only if $\mathbf{x}$ is a permutation of $\mathbf{y}$.

(ii) Let $\mathcal{MAX} = (M_k)_{k\in N}$ be the system of root-power operators [11]

$$M_k(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^n x_i^k\right)^{\frac{1}{k}}.$$

Then $\lim_{k\to+\infty} M_k = Max$.
For $\mathbf{x} \in [0,1]^n$, define the occurrence function $\sigma_{\mathbf{x}} : [0,1] \to N \cup \{0\}$, $\sigma_{\mathbf{x}}(u)$ =card$\{i \in \{1,\ldots,n\}\,|x_i = u\}$. Then $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if $\sigma_{\mathbf{x}} \circ \eta \le_{Lex} \sigma_{\mathbf{y}} \circ \eta$, where $\eta : [0,1] \to [0,1]$ is given by $\eta(u) = 1 - u$.
Note that though $[0,1]$ is uncountable, the supports of both $\sigma_{\mathbf{x}} \circ \eta$ and $\sigma_{\mathbf{y}} \circ \eta$ are finite. The lexicographic relation $\sigma_{\mathbf{x}} \circ \eta <_{Lex} \sigma_{\mathbf{y}} \circ \eta$ means that there is $u \in [0,1]$ such that $\sigma_{\mathbf{x}}(1-u) <_{Lex} \sigma_{\mathbf{y}}(1-u)$ and for all $v \in [0,u[$ it holds $\sigma_{\mathbf{x}}(1-v) = \sigma_{\mathbf{y}}(1-v)$. Observe that on $[0,1]^n$, $\precsim^{\mathcal{MAX}} \equiv \le_{LexiMax}$ is just the LexiMax preorder [11].

(iii) Starting from an arbitrary continuous Archimedean t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, see [21], also $t^k : [0,1] \to [0,\infty]$, $k \in N$, is an additive generator and it generates a continuous Archimedean t-norm T_k. Then $\lim_{k\to\infty} T_k = Min$ and for $\mathcal{MIN} = (T_k)_{k\in N}$, $\mathbf{x} \precsim^{\mathcal{MIN}} \mathbf{y}$ if and only if $Min(\mathbf{x}) = Min(\mathbf{y}) = 0$, or $\sigma_{\mathbf{x}}|[0,1[\le_{Lex} \sigma_{\mathbf{y}}|[0,1[$.

Note that though we focus in this chapter on the comparison of alternatives described by score vectors with a fixed dimension (n is fixed), aggregation operator based approach allows to compare also the alternatives having score vectors with different dimension. Recall, e.g., classical comparison by means of the arithmetic mean. Interestingly, when applying Proposition 2.1 in such situation, one can obtain different partial orderings $\precsim^{\mathcal{U}_1}$ and $\precsim^{\mathcal{U}_2}$ of score vectors with non-fixed dimension, though for the fixed dimension n they coincide.

Example 6.3. Taking the system $\mathcal{MAX} = (M_k)_{k\in N}$ of general root-power operators defined for any arity n, we have seen in Example 2.2(ii) that $\precsim^{\mathcal{MAX}}$ on $[0,1]^n$ is the LexiMax preorder, whenever the arity n is fixed. However, when we have to compare alternatives $\mathbf{x} \in [0,1]^n$ and $\mathbf{y} \in [0,1]^m$ with $m \neq n$, then $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if

$$\frac{\sigma_{\mathbf{x}} \circ \eta}{n} \leq_{Lex} \frac{\sigma_{\mathbf{y}} \circ \eta}{m},$$

see [23]. Hence $\precsim^{\mathcal{MAX}}$ extends the standard LexiMax preorder in that sense that the score vector of $\mathbf{x}$ repeated m-times $\widetilde{\mathbf{x}} = \underbrace{(\mathbf{x},\ldots,\mathbf{x})}_{m-times}$ and the score vector of $\mathbf{y}$ repeated n-times $\widetilde{\mathbf{y}} = \underbrace{(\mathbf{y},\ldots,\mathbf{y})}_{n-times}$ are of the same $n.m$-arity and then we apply the standard LexiMax to compare $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$, $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if $\widetilde{\mathbf{x}} \leq_{LexiMax} \widetilde{\mathbf{y}}$.

On the other hand, when modifying the Example 6.2(iii) by duality, we can start from an arbitrary continuous Archimedean t-conorm S with an additive generator $s : [0,1] \to [0,\infty]$, and work with S_k generated by s^k, $k \in N$. Take, for example $s(t) = t$, $t \in [0,1]$, i.e., $S_k(u_1,\ldots,u_n) = \min\left(1, \left(\sum_{i=1}^n u_i^k\right)^{\frac{1}{k}}\right)$. Let $\mathcal{S} = (S_k)_{k\in N}$. Then for $\mathbf{x}, \mathbf{y} \in [0,1[^n$ it holds $\mathbf{x} \precsim^{\mathcal{S}} \mathbf{y}$ if and only if $\mathbf{x} \leq_{LexiMax} \mathbf{y}$, i.e., $\precsim^{\mathcal{S}}$ and $\precsim^{\mathcal{MAX}}$ coincide on $[0,1[^n$ for each $n \in N$. However, if $\mathbf{x} \in [0,1[^n$ and $\mathbf{y} \in [0,1[^m$ with $n \neq m$, then $\mathbf{x} \precsim^{\mathcal{S}} \mathbf{y}$ if and only if $\mathbf{x}^* \leq_{LexiMax} \mathbf{y}^*$, where both $\mathbf{x}^* = (\mathbf{x}, \underbrace{0,\ldots,0}_{m-times})$ and $\mathbf{y}^* = (\mathbf{y}, \underbrace{0,\ldots,0}_{n-times})$ have the same dimension $n+m$.

Example 6.4. Put $\mathbf{x} = (0.8, 0.2)$ and $\mathbf{y} = (0.8, 0.3, 0.1)$. Then $\widetilde{\mathbf{x}} = (0.8, 0.2, 0.8, 0.2, 0.8, 0.2) >_{LexiMax} \widetilde{\mathbf{y}} = (0.8, 0.3, 0.1, 0.8, 0.3, 0.1)$ and $\mathbf{x}^* = (0.8, 0.2, 0, 0, 0) <_{LexiMax} \mathbf{y}^* = (0.8, 0.3, 0.1, 0, 0)$, i.e., $\mathbf{x} \succ^{\mathcal{MAX}} \mathbf{y}$ but $\mathbf{x} \prec^{\mathcal{S}} \mathbf{y}$.

6.3 Fuzzy Quantities Based Preference Structures Constructions

In several decision-making models, the choice of an appropriate alternative is transformed to the problem of comparison of some available quantitative information. Recall, e.g., the standard optimization problems arising from the minimal costs or the maximal profit, several ordering approaches such as leximin or discrimin, see [9], etc. The simplest situation occurs when each alternative is described by a single real value (say the costs), in which case from two alternatives we choose the cheaper one. From a mathematical point of view, we exploit here the standard ordering on the real line. Much more complex is the situation when alternatives are described by fuzzy reals [10, 25]. In that case, there are several orderings known so far. For an exhaustive overview we recommend [39, 40]. Because of greater flexibility and modelling power, we focus our attention to the last case, i.e., to the decision problems for alternatives characterized by fuzzy quantities.

On the set of alternatives $\mathcal{A}$, let each alternative $\mathbf{x}$ be described by the score vector $(x_1, \ldots, x_n)$, where n is the number of applied criteria and $x_1, \ldots, x_n \in I$ are the single score from some prescribed real interval I (usually, $I = [0, 1]$ or $I = \mathbb{R}$). In the criterion i, the dissimilarity $D_i(x, y)$ of a score x and another score y, with $x, y \in I$, is described by the $D_i : I^2 \to \mathbb{R}$, such that

$$D_i(x, y) = K_i(f_i(x) - f_i(y)),$$

where $K_i : \mathbb{R} \to \mathbb{R}$ is a convex function with the unique minimum $K_i(0) = 0$ (shape function), and $f_i : I \to \mathbb{R}$ is a strictly monotone continuous function (scale transformation). Evidently, each D_i is then continuous. Observe that this approach to dissimilarity is based on the ideas of verbal fuzzy quantities as proposed and discussed in [26–28]. Note also that the concept of dissimilarity functions is closely related to the penalty functions proposed by Yager and Rybalov [44], compare also [4]. Finally, remark that the dissimilarity function D is related to some standard metric on the interval I whenever it is symmetric, i.e., if K is an even function. To be more precise, for any even shape function K, let L be the inverse function to $K|_{[0,\infty[}$. Then for the dissimilarity function $D(x, y) = K(f(x) - f(y))$ we have $L \circ D(x, y) = |f(x) - f(y)|$, i.e., $L \circ D$ is a metric. Typical examples of such dissimilarity functions are (on any real interval I):

- $D(x, y) = (f(x) - f(y))^2$
- $D(x, y) = |f(x) - f(y)|$
- $D(x, y) = 1 - \cos\left(\frac{\arctan x - \arctan y}{2}\right)$

As an example of a dissimilarity D which is not a transformed metric (only the symmetry is violated) we introduce functions D_c, with $c \in]0, \infty[$, $c \neq 1$, given by

$$D_c(x, y) = \begin{cases} c(y - x) & \text{if } x \leq y, \\ x - y & \text{else.} \end{cases} \tag{6.6}$$

The dissimilarity of score $(x_1, \ldots, x_n)$ and the unanimous score $(r, \ldots, r)$ is described by the real vector $(D_1(x_1, r), \ldots, D_n(x_n, r))$. The fuzzy utility function U, compare [14, 45], assigns to each alternative $\mathbf{x}$ (with score $(x_1, \ldots, x_n)$) the fuzzy quantity $U(\mathbf{x})$ with membership function $\mu_{\mathbf{x}} : I \to [0, 1]$,

$$\mu_{\mathbf{x}}(r) = \frac{1}{1 + \sum_{i=1}^{n} D_i(x_i, r)}. \tag{6.7}$$

Proposition 6.2. *For each alternative $\boldsymbol{x} \in \mathcal{A}$, the fuzzy utility function value $U(\boldsymbol{x})$ with membership function given by (6.7) is a convex fuzzy quantity with continuous membership function.*

Proof. The continuity of $\mu_{\mathbf{x}}$ follows from the continuity of each dissimilarity function D_i. For arbitrary $r, s \in I$ and $\lambda \in [0,1]$, the convexity of $K_i, i = 1, \dots, n$, and the strict monotonicity and the continuity of $f_i, i = 1, \dots, n$, ensure the following facts: the function $g : I \to \mathbb{R}$, $g(r) = 1 + \sum_{i=1}^{n} D_i(x_i, r) \geq 1$, is continuous and convex, and thus $g(\lambda r + (1-\lambda)s) \leq \lambda g(r) + (1-\lambda) g(s) \leq \max(g(r), g(s))$, compare also [7]. As far as $\mu_\alpha = \frac{1}{g}$, the inequality (6.1) follows. □

Note that the convexity of all shapes K_i was crucial in the above proof, justifying the restriction of possible shapes to convex ones. Observe that applying the above-described procedures, introduced fuzzy utility function $U : I^n \to \mathcal{F}(\mathbb{R})$ assigns to n-tuple $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ continuous convex quantity $U(\mathbf{x})$ with membership function $\mu_{\mathbf{x}}$ described by (6.7).

Example 6.5.

(i) Let $D_1 = \cdots = D_n = D$, $D(x,y) = (x-y)^2$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then $U(\mathbf{x})$ has membership function

$$\mu_{\mathbf{x}}(r) = \frac{1}{n(\overline{x} - r)^2 + 1 - n\overline{x}^2 + \sum_{i=1}^{n} x_i^2} = \frac{1}{1 + n((\overline{x} - r)^2 + \sigma^2)},$$

which is symmetric w.r.t. point $r = \overline{x}$ (the arithmetic mean of $(x_1, \dots, x_n)$, $\overline{x} = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$), where the dispersion $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \overline{x}^2$ (i.e., $\mu_{\mathbf{x}}(\overline{x} - \epsilon) = \mu(\overline{x} + \epsilon)$ for all $\epsilon \in \mathbb{R}$). Moreover, $U(\mathbf{x})$ is unimodal fuzzy number with height $\frac{1}{1+n\sigma^2}$ attained in the point $r = \overline{x}$.

(ii) Let $D_1 = \cdots = D_n = D$, $D(x,y) = |x-y|$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\mu_{\mathbf{x}}(r) = \frac{1}{1 + \sum_{i=1}^{n} |x_i - r|}$$

is symmetric w.r.t. point $r = \overline{x}$ if and only if for the order statistics $x'_1, \dots, x'_n$ of $(x_1, \dots, x_n)$ it holds $x'_1 + x'_n = x'_2 + x'_{n-1} = \cdots = 2\overline{x}$. Moreover, $U(\mathbf{x})$ always attains its height in the median $\text{med}(x_1, \dots, x_n)$ and it is unimodal if and only if n is odd, or if n is even, $n = 2k$, and $x'_k = x'_{k+1}$.

(iii) Let $D_1(x,y) = |x-y|$, $D_2(x,y) = (x-y)^2$ and $(x_1, x_2) \in \mathbb{R}^2$. Then

$$\mu_{(x_1,x_2)}(r) = \frac{1}{1 + |x_1 - r| + (x_2 - r)^2}$$

is symmetric w.r.t. point $r = \overline{x}$ if and only if $x_1 = x_2$. Moreover, it is unimodal and $U(x_1, x_2)$ attains its height in the point $r = \text{med}\left(x_1, x_2 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$.

Remark 6.1. Formula (6.7) arises from the global dissimilarity $\sum_{i=1}^{n} D_i(x_i, r) \in [0, \infty[$ and the decreasing bijection $\varphi : [0, \infty] \to [0, 1]$ given by $\varphi(t) = \frac{1}{1+t}$. Concerning Proposition 6.2, the sum operator can be replaced by any other operator $H : [0, \infty[^n \to [0, \infty[$ such that $h(r) = H(D_1(x_1, r), \dots, D_n(x_n, r))$ is a convex function. For example, we can take $H(t_1, \dots, t_n) = \sum_{i=1}^{n} t_i^2$. Similarly, φ can be replaced by any other decreasing bijection $\eta : [0, \infty] \to [0, 1]$, not violating the validity of Proposition 6.2. For example, we can take $\eta(t) = e^{-t}$.

On the set of all fuzzy quantities $\mathcal{F}$, respectively of all continuous convex fuzzy quantities $\mathcal{Q}$, there were introduced many types of orderings. For an exhaustive overview we recommend [10, 39, 40]. Based on any such ordering $\leq$, we can derive a preference relation $\precsim$ on the set $\mathcal{A}$ of all alternatives, $\mathbf{x} \precsim \mathbf{y}$ if and only if $U(\mathbf{x}) \leq U(\mathbf{y})$. Obviously, if $\leq$ is a fuzzy ordering, then $\precsim$ is a fuzzy ordering relation (for more details on fuzzy orderings and fuzzy preference structures see Chapter 5 in this edited volume). However, in this paper, we will deal with crisp preference relations (crisp orderings) only, i.e., only crisp orderings of fuzzy quantities will be taken into account. In such a case, we can even refine the derived weak ordering relation.

Definition 6.1. Let $\mathcal{A}$ be a set of alternatives and let $U : \mathcal{A} \to \mathcal{F}(\mathbb{R})$ be a fuzzy utility function given by (6.7). Let $\leq$ be a crisp ordering on the set of all continuous convex fuzzy quantities. Then we define a weak ordering relation $\precsim$ on $\mathcal{A}$ as follows: $\mathbf{x} \precsim \mathbf{y}$ whenever $U(\mathbf{x}) < U(\mathbf{y})$ or $U(\mathbf{x}) = U(\mathbf{y})$ and $\mathrm{hgt}(U(\mathbf{x})) \geq \mathrm{hgt}(U(\mathbf{x}))$.

It is evident that $\precsim$ given in the above definition is really a weak ordering relation on $\mathcal{A}$. However, it need not fit the Pareto principle, in general, i.e., for two alternatives $\mathbf{x}$ and $\mathbf{y}$ characterized by the respective scores $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_i \leq y_i$, $i = 1, \ldots, n$, we do not have necessarily $\mathbf{x} \precsim \mathbf{y}$.

A huge class of crisp orderings on fuzzy quantities (continuous, convex) is linked to defuzzification methods, see, e.g., [42], i.e., to mappings $\mathrm{DF} : \mathcal{F} \to \mathbb{R}$ or $\mathrm{DF} : \mathcal{Q} \to \mathbb{R}$. For a defuzzification DF, we simply have then $U(\mathbf{x}) \leq U(\mathbf{y})$ if and only if $\mathrm{DF}(U_{\mathbf{x}}) \leq \mathrm{DF}(U_{\mathbf{y}})$ (where the last inequality is the standard inequality among real numbers). Each such defuzzification method induces an operator $A_{\mathrm{DF}} : I^n \to \mathbb{R}$, $A_{\mathrm{DF}}(x_1, \ldots, x_n) = \mathrm{DF}(U_{\mathbf{x}})$. As already mentioned, the operator A_{DF} need not be monotone, in general, and thus the Pareto principle may fail.

Example 6.6. We continue Example 6.5. As a defuzzification method DF for a fuzzy quantity F with $(\mathrm{hgt}(F))$-cut $= [u, v]$ we take $\mathrm{DF}(F) = u + \frac{(v-u)^3}{1+(v-u)^2}$. Then:

(i) if $D_1 = \cdots = D_n = D$, $D(x, y) = (x - y)^2$, we obtain $A_{\mathrm{DF}}(x_1, \ldots, x_n) = \overline{x} = M(x_1, \ldots, x_n)$, where M is the standard arithmetic mean operator (acting on $\mathbb{R}$). Evidently, M is a monotone idempotent operator, i.e., the Pareto principle is satisfied.

(ii) if $D_1 = \cdots = D_n = D$, $D(x, y) = |x - y|$, then A_{DF} is an idempotent operator on $\mathbb{R}$ which is not monotone, and thus it violates the Pareto principle. For example $A_{\mathrm{DF}}(1, 3, 5, 6) = \frac{23}{5}$ but $A_{\mathrm{DF}}(2, 4, 5, 10) = \frac{9}{2}$.

Observe, however, that any defuzzification method DF compatible with the fuzzy maximum $\widetilde{\max}$ (i.e., $\mathrm{DF}(U_{\mathbf{x}}) \leq \mathrm{DF}(U_{\mathbf{y}})$ whenever $\widetilde{\max}\,(U_{\mathbf{x}}, U_{\mathbf{y}}) = U_{\mathbf{y}}$) yields a non-decreasing idempotent operator A_{DF}. Recall that

$$\mu_{\widetilde{\max}(U_{\mathbf{x}}, U_{\mathbf{y}})}(r) = \sup\left(\min(U_{\mathbf{x}}(t), U_{\mathbf{y}}(s)) \mid \max(t, s) = r\right),$$

see [12].

6.4 Mean of Maxima Defuzzification Approach

One of the simplest defuzzification methods is the MOM method [42], i.e., the centre of the (hgt(F))-cut.

Definition 6.2. For the MOM method, the operator $A_{\mathrm{MOM}} = A : I^n \to I$ is given by

$$A(x_1, \ldots, x_n) = \frac{\inf\{r \mid \mu_{\mathbf{x}}(r) = \mathrm{hgt}(U_{\mathbf{x}})\} + \sup\{r \mid \mu_{\mathbf{x}}(r) = \mathrm{hgt}(U_{\mathbf{x}})\}}{2}. \quad (6.8)$$

For unanimous score $\mathbf{x} = (x, \ldots, x)$ it is obvious that for arbitrary dissimilarity functions $D_1, \ldots, D_n$, the membership function $\mu_{\mathbf{x}}(r) = \frac{1}{1+\sum_{i=1}^{n} D_i(x,r)}$ of $U(\mathbf{x})$ is normal and unimodal with the unique maximum $\mu_{\mathbf{x}}(x) = 1$, and hence the operator A is idempotent, $A(x, \ldots, x) = x$ for all $x \in I$. To show the monotonicity of A (and thus the fitting to the Pareto principle), we need first some lemmas.

Lemma 6.1. *Let $K : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for any $u, v, w \in \mathbb{R}$, $v \geq 0, w \geq 0$ it holds*

$$K(u+v) + K(u+w) \leq K(u) + K(u+v+w). \quad (6.9)$$

Proof. If $v = 0$ or $w = 0$, (6.9) trivially holds (even with equality). Suppose that $v > 0, w > 0$. Then $u + v = \frac{w}{v+w}u + \frac{v}{v+w}(u+v+w)$ and from the convexity of K it holds

$$K(u+v) \leq \frac{w}{v+w}K(u) + \frac{v}{v+w}K(u+v+w).$$

Similarly,

$$K(u+w) \leq \frac{v}{v+w}K(u) + \frac{w}{v+w}K(u+v+w).$$

Summation of the two last inequalities gives just the desired inequality (6.9). □

Lemma 6.2. *Let $K : \mathbb{R} \to \mathbb{R}$ be a convex function and let $f : I \to \mathbb{R}$ be a strictly monotone continuous function. Then for any $x, y, s, p \in I$, $x \leq y$ and $s \leq p$ it holds $K(f(y) - f(p)) - K(f(x) - f(p)) \leq K(f(y) - f(s)) - K(f(x) - f(s))$, i.e.,*

$$D(y, p) - D(x, p) \leq D(y, s) - D(x, s). \quad (6.10)$$

Proof. Applying Lemma 6.1, it is enough to put $f(x) - f(p) = u$, $f(p) - f(s) = v$ and $f(y) - f(x) = w$ if f is increasing; in the case when f is decreasing, it is enough to put $f(y) - f(s) = u$, $f(s) - f(p) = v$ and $f(x) - f(y) = w$. □

Theorem 6.1. *Let $D_i(x, y) = K_i(f_i(x) - f_i(y))$, $i = 1, \ldots, n$, $x, y \in I$, be given dissimilarity functions. Then the idempotent operator $A : I^n \to I$ given by (6.8) is monotone and thus an aggregation operator fitting the Pareto principle.*

Proof. For a given score $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\varepsilon > 0$, $i \in \{1, \ldots, n\}$, suppose that $\mathbf{y} = (x_1, \ldots, x_i + \varepsilon, \ldots, x_n) \in I^n$. Denote $P_{\mathbf{x}}(r) = \sum_{j=1}^{n} D_j(x_j, r)$, $r \in I$. Then $P_{\mathbf{y}}(r) = P_{\mathbf{x}}(r) + D_i(x_i + \varepsilon, r) - D_i(x_i, r)$. Take an arbitrary element $p \in I$ such that for all $r \in I$, $P_{\mathbf{x}}(r) \geq P_{\mathbf{x}}(p)$, i.e., p is an element minimizing $P_{\mathbf{x}}$ (existence of p follows from the continuity of all dissimilarities D_i and the convexity of all shapes K_i). Then for arbitrary $s \in I$, $s \leq p$ it holds $P_{\mathbf{x}}(s) \geq P_{\mathbf{x}}(p)$ and because of Lemma 6.2, inequality (6.10), also $D_i(x_i + \varepsilon, s) - D_i(x_i, s) \geq D_i(x_i + \varepsilon, p) - D_i(x_i, p)$. Summing the two last inequalities we obtain $P_{\mathbf{y}}(s) \geq P_{\mathbf{y}}(p)$. However, this means that there is a minimal element p' of $P_{\mathbf{y}}$ such that $p' \geq p$, and thus the centre of minimal elements of $P_{\mathbf{x}}$ is less or equal to the centre of minimal elements of $P_{\mathbf{y}}$. Observe that the set of minimal elements of $P_{\mathbf{x}}$ is the same as the set of maximal elements of $\mu_{\mathbf{x}} = \frac{1}{1+P_{\mathbf{x}}}$, and thus because of (6.8), $A(\mathbf{x}) \leq A(\mathbf{y})$. □

Note that if all dissimilarity functions D_i are equal, $D_1 = \cdots = D_n = D$, the concept of MOM-based aggregation operators coincides with the penalty based approach proposed by Yager and Rybalov [44] and further developed in [4], in which case the aggregation operator A is anonymous, i.e., for all $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and any permutation σ of $(1, \ldots, n)$ it holds $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Observe also that the convexity of function K_i involved in dissimilarities D_i is crucial to ensure that $\mu_{\mathbf{x}}$ is a quasi-convex membership function and that A is monotone. For example, for $K_1 = \cdots = K_n = K$,

$$K(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{else,} \end{cases} \qquad \text{and for any } f_i : I \to \mathbb{R},$$

the operator A derived by (6.8) is the modus operator which is not monotone. Indeed, $A(0, 0.2, 0.5, 1, 1) = 1$ while $A(0.5, 0.5, 0.5, 1, 1) = 0.5$. Note also that following the ideas of quantitative weights in aggregation discussed in [4], for any non-zero weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ we can derive the corresponding weighted aggregation operator $A_{\mathbf{w}}$ applying Definition 6.2 to the weighted dissimilarity functions $w_1 D_1, \ldots, w_n D_n$. For further generalization see [30, 31].

Remark 6.2. Definition 6.1 brings a refinement of the weak ordering $\leq_A$ on the set of alternatives $\mathcal{A}$ given by $\mathbf{x} \leq_A \mathbf{y}$ whenever $A(\mathbf{x}) \leq A(\mathbf{y})$, where A is an aggregation operator given by (6.8), see also Theorem 6.1. Indeed, for the weak ordering $\precsim$ introduced in Definition 6.1 and based on MOM fuzzification method and dissimilarity functions $D_1, \ldots, D_n$:

$$\mathbf{x} <_A \mathbf{y} \Rightarrow \mathbf{x} \prec \mathbf{y},$$
$$\mathbf{x} \approx \mathbf{y} \Rightarrow \mathbf{x} \approx_A \mathbf{y}.$$

Moreover, if $\mathbf{x} <_A \mathbf{y}$, then:

- if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) | \quad r \in I\right\} < \min\left\{\sum_{i=1}^{n} D_i(y_i, r) | \quad r \in I\right\}$ then $\mathbf{x} \prec \mathbf{y}$;

- if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) \mid r \in I\right\} > \min\left\{\sum_{i=1}^{n} D_i(y_i, r) \mid r \in I\right\}$ then $\mathbf{x} \succ \mathbf{y}$;
- and if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) \mid r \in I\right\} = \min\left\{\sum_{i=1}^{n} D_i(y_i, r) \mid r \in I\right\}$ then $\mathbf{x} \approx \mathbf{y}$.

Example 6.7. We continue the previous examples.

(i) For $D_1 = \ldots = D_n = D$, $D(x, y) = (x-y)^2$, the corresponding aggregation operator $A : \mathbb{R} \to \mathbb{R}$ is the standard arithmetic mean, $A = M$. The corresponding weighted aggregation operator $A_{\mathbf{w}}$ is the weighted arithmetic mean linked to the weighting vector $\mathbf{w}$,

$$A_{\mathbf{w}}(\mathbf{x}) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \quad .$$

Evidently $p = (p_1, \ldots, p_n)$ where $p_j = \frac{w_j}{\sum_{i=1}^{n} w_i}$, $j = 1, \ldots, n$, is a probability distribution on the set of criteria $\mathcal{C} = \{C_1, \ldots, C_n\}$ and then the weighted mean $A_{\mathbf{w}}$ is the expected value operator on the set of alternatives $\mathcal{A}$ (compare with formula (6.2)). Moreover, denoting the corresponding variance operator by $\sigma_{\mathbf{w}}^2$, we have the next probabilistic interpretation for the weak order $\precsim$ on $\mathcal{A}$ as given in Definition 6.1 for this case, namely $\mathbf{x} \precsim \mathbf{y}$ if and only if either $A_{\mathbf{w}}(\mathbf{x}) < A_{\mathbf{w}}(\mathbf{y})$ or $A_{\mathbf{w}}(\mathbf{x}) = A_{\mathbf{w}}(\mathbf{y})$ and $\sigma_{\mathbf{w}}^2(\mathbf{x}) \leq \sigma_{\mathbf{w}}^2(\mathbf{y})$, i.e., $\precsim$ is just the $\left(A_{\mathbf{w}}, \sigma_{\mathbf{w}}^2\right)$ – lexicographical ordering as introduced in [23], $\mathbf{x} \precsim \mathbf{y}$ if and only if $\left(A_{\mathbf{w}}(\mathbf{x}), \sigma_{\mathbf{w}}^2(\mathbf{x})\right) \leq_{Lex} \left(A_{\mathbf{w}}(\mathbf{y}), \sigma_{\mathbf{w}}^2(\mathbf{y})\right)$.
Modifying D into $D_f(x, y) = (f(x) - f(y))^2$, the quasi-arithmetic mean M_f is recovered, $A_f(\mathbf{x}) = M_f(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$. Finally, the corresponding weighted aggregation operator is the weighted quasi-arithmetic mean,

$$A_{f,\mathbf{w}}(\mathbf{x}) = f^{-1}\left(\frac{\sum_{i=1}^{n} w_i f(x_i)}{\sum_{i=1}^{n} w_i}\right).$$

In this case, the weak order $\precsim$ introduced in Definition 6.1 can be represented as follows: $\mathbf{x} \precsim \mathbf{y}$ if and only if $\left(A_{f,\mathbf{w}}(\mathbf{x}), \sigma_{\mathbf{w}}^2(f(\mathbf{x}))\right) \leq_{Lex} \left(A_{f,\mathbf{w}}(\mathbf{y}), \sigma_{\mathbf{w}}^2(f(\mathbf{y}))\right)$.

(ii) For $D_1 = \cdots = D_n = D$, $D(x, y) = |x - y|$, Definition 6.2 leads to the standard median operator $A = \text{med}$,

$$A(\mathbf{x}) = \text{med}(x_1, \ldots, x_n) = \begin{cases} x'_{\frac{n+1}{2}}, & \text{if } n \text{ is odd}, \\ \frac{x'_{\frac{n}{2}} + x'_{\frac{n}{2}+1}}{2}, & \text{if } n \text{ is even}, \end{cases}$$

where $\left(x'_1, \ldots, x'_n\right)$ is a non-decreasing permutation of $(x_1, \ldots, x_n)$. Note that a modification of D into D_f, $D_f(x, y) = |f(x) - f(y)|$, has no influence on the resulting aggregation operator A in this case whenever n is odd. The corresponding weighted median $A_{\mathbf{w}} = \text{med}_{\mathbf{w}}$ corresponds to the weighted median proposed in [4, 44], and in the case of integer weights it is given by

$$\mathrm{med}_{\mathbf{w}}(\mathbf{x}) = \mathrm{med}(\underbrace{x_1,\ldots,x_1}_{w_1\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{w_n\text{-times}}).$$

Denote by w_i' the weight corresponding to x_i', and by W_i' the cumulative weight, i.e., $W_i' = \sum_{j=1}^{i} w_j'$. Define two functions: $C_{\mathbf{x},\mathbf{w}}, D_{\mathbf{x},\mathbf{w}} : \left[0, W_n'\right] \to I$ by $C_{\mathbf{x},\mathbf{w}}(0) = 0$ and $C_{\mathbf{x},\mathbf{w}}(u) = x_i'$ whenever $W_{i-1}' < u \leq W_i'$ with convention $W_0' = 0$, and $D_{\mathbf{x},\mathbf{w}}(u) = C_{\mathbf{x},\mathbf{w}}\left(W_n' - u\right)$. Denote $R_{\mathbf{x},\mathbf{w}} = \int_0^{W_n'} |C_{\mathbf{x},\mathbf{w}}(u) - D_{\mathbf{x},\mathbf{w}}(u)|\, du$.

Then the weak order $\precsim$ given by Definition 6.1 and related to this framework can be represented in the following form: $\mathbf{x} \precsim \mathbf{y}$ if and only if $(\mathrm{med}_{\mathbf{w}}(\mathbf{x}), R(\mathbf{x},\mathbf{w})) \leq_{Lex} (\mathrm{med}_{\mathbf{w}}(\mathbf{y}), R(\mathbf{y},\mathbf{w}))$.

Note that in the case when $w_1 = \cdots = w_n = 1$ (i.e., when the standard median is considered), then $R(\mathbf{x}) = R(\mathbf{x},\mathbf{w}) = \sum_{i=1}^{n} |x_i - x_{n+1-i}|$ and we have a new refinement of the median weak order.

(iii) For $D_1(x,y) = |x-y|$ and $D_2(x,y) = (x-y)^2$ the aggregation operator A defined by (6.8) is given by $A(x_1,x_2) = \mathrm{med}\left(x_1, x_2 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$. The corresponding weighted aggregation operator $A_{\mathbf{w}}$ is given by

$$A_{\mathbf{w}}(x_1,x_2) = \mathrm{med}\left(x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2}\right).$$

Finally, observe that if $D_1 = \cdots = D_n = D_c$, $c \in]0,\infty[$, $c \neq 1$ see formula (6.6), then the corresponding aggregation operator A_c is the order statistic corresponding to the $q = \frac{1}{1+c}.100\%$-quantile, and $A_1 = A$ from Example 6.7 (ii).

Remark 6.3. The operator A_{MOM} introduced in Definition 6.2, see (6.8), can be seen also as a solution of the minimization problem $A_{\mathrm{MOM}}(\mathbf{x}) = r_{\mathbf{x}}$, where

$$\sum_{i=1}^{n} D_i(x_i, r_{\mathbf{x}}) = \min\left\{\sum_{i=1}^{n} D_i(x_i, r) | r \in I\right\}. \tag{6.11}$$

This approach leads to minimization problem of $\sum_{i=1}^{n} w_i D_i(x_i, r)$ when the weights $w_1,\ldots,w_n$ have to be incorporated. Moreover, if weights are input-dependent, i.e., when $w_i = w(x_i)$ for a given weighting function $w : I \to [0,\infty[$, then we have to minimize the expression $\sum_{i=1}^{n} w_i(x_i) D_i(x_i, r)$. However, the resulting operator $A_{\mathbf{w}} : I^n \to I$ need not fulfil Pareto principle, in general. For example, if $D_1 = \ldots = D_n = D$, $D(x,y) = (x-y)^2$ then

$$A_{\mathbf{w}}(\mathbf{x}) = \frac{\sum_{i=1}^{n} w_i(x_i) x_i}{\sum_{i=1}^{n} w_i(x_i)}$$

is a mixture operator [33–35]. For $I = [0,1]$ and $w = id_{[0,1]}$, $A_{\mathbf{w}}(0,1) = \frac{0+1}{0+1} = 1$ but $A_{\mathbf{w}}(0.5,1) = \frac{0.25+1}{0.5+1} = \frac{5}{6}$. For deeper discussion of mixture operators and related concepts we recommend [30, 31].

6.5 Fuzzy Logic-Based Construction of Preference Relations

Let $T : [0,1]^2 \to [0,1]$ be a left-continuous [21], i.e., T is associative, commutative, non-decreasing and $T(x,1) = x$ for all $x \in [0,1]$, and let $I_T : [0,1]^2 \to [0,1]$ be the adjoint residual implication,

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}.$$

Then $x \le y$ if and only if $I_T(x,y) \ge I_T(y,x)$. This fact allows to introduce preference relations on $\mathcal{A}$ as follows:

Definition 6.3. Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm and let $H : [0,1]^n \to [0,1]$ be an aggregation operator. Then the preference relation $R_{T,H} \subseteq \mathcal{A}^2$ is given by $(\mathbf{x},\mathbf{y}) \in R_{T,H}$, i.e., $\mathbf{x} \ge_{T,H} \mathbf{y}$ if and only if

$$H(I_T(x_1,y_1),\ldots,I_T(x_n,y_n)) \le H(I_T(y_1,x_1),\ldots,I_T(y_n,x_n)). \qquad (6.12)$$

$R_{T,H}$ is a complete preference relation (i.e., there are no incomparable alternatives $\mathbf{x}$ and $\mathbf{y}$), but not a weak ordering, in general.

Note that if H has neutral element 1 or if H is strictly monotone then the decision about relation of $\mathbf{x}$ and $\mathbf{y}$ depends only on those score for which $x_i \neq y_i$, i.e., the discriminative approach to decision making as discussed in [11] is applied. Observe that though in some cases $\ge_{T,H}$ can be represented in the form of a transitive complete weak preference relation $\ge_Q$ (Q is an aggregation operator) and thus the preference relation $R_{T,H}$ is also transitive, in general this is not true. Note also that H need not be anonymous (symmetric).

Recall that the strongest t-norm $T_M : [0,1]^2 \to [0,1]$ is given by $T_M(x,y) = \min\{x,y\}$ and that the related implication $I_{T_M} : [0,1]^2 \to [0,1]$ which is called the Gödel implication is given by

$$I_{T_M}(x,y) = \begin{cases} 1 & \text{if} \quad x \le y, \\ y & \text{else.} \end{cases}$$

An important class of continuous t-norm is the class of continuous Archimedean t-norms $T : [0,1]^2 \to [0,1]$ characterized by additive generators $t : [0,1] \to [0,\infty]$, which are continuous, strictly decreasing and $t(1) = 0$, in such a way that for all $(x,y) \in [0,1]^2$ it holds

$$T(x,y) = t^{-1}(\min\{t(0), t(x) + t(y)\}). \qquad (6.13)$$

Then the corresponding residual implication $I_T : [0,1]^2 \to [0,1]$ is given by

$$I_T(x,y) = t^{-1}(\max\{0, t(y) - t(x)\}). \tag{6.14}$$

If $t(0)$ is finite then T is called a nilpotent t-norm. Prototypical example of nilpotent t-norm is the Łukasiewicz t-norm $T_L : [0,1]^2 \to [0,1]$ given by $T_L(x,y) = \max\{0, x + y - 1\}$ with an additive generator $t_L : [0,1] \to [0,\infty]$, $t_{T_L}(x) = 1-x$. The corresponding Łukasiewicz implication $I_{T_L} : [0,1]^2 \to [0,1]$ is given by $I_{T_L}(x,y) = \min\{1, 1 - x + y\}$.

If $t(0) = \infty$ then the corresponding t-norm T is called a strict t-norm. A prototypical example of a strict t-norm is the product t-norm $T_P : [0,1]^2 \to [0,1]$, $T_P(x,y) = xy$, where adjoint residual implication $I_{T_P} : [0,1]^2 \to [0,1]$ is called the Goguen implication and it is given by

$$I_{T_P}(x,y) = \begin{cases} 1 & \text{if} \quad x \le y, \\ \frac{y}{x} & \text{else.} \end{cases}$$

For more details about t-norms we recommend monographs [2, 20, 21].

Example 6.8.

(i) For any nilpotent t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, see [21], and the quasi-arithmetic mean M_t generated by t, see [5], $\ge_{T,M_t}$ is exactly $\ge_{M_t}$, and $\mathbf{x} = (x_1,\ldots,x_n) \ge_{T,M_t}, \mathbf{y} = (y_1,\ldots,y_n)$ if and only if $\sum_{i=1}^n t(x_i) \le \sum_{i=1}^n t(y_i)$. Thus the transitivity of $\ge_{T,M_t}$ is obvious.
Note that considering $T = T_L$ we have $M_t = M$ the standard arithmetic mean and $t : [0,1] \to [0,\infty]$ is given by $t(x) = 1 - x$. Then $\sum_{i=1}^n t(x_i) = n - \sum_{i=1}^n x_i \le n - \sum_{i=1}^n y_i = \sum_{i=1}^n t(y_i)$ if and only if $\frac{1}{n}\sum_{i=1}^n x_i = M(\mathbf{x}) \ge M(\mathbf{y}) = \frac{1}{n}\sum_{i=1}^n y_i$.

(ii) Similarly for any strict t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, $\ge_{T,M_t}$ is transitive, but there is no aggregation operator Q such that $\ge_{T,M_t} \equiv \ge_Q$. Observe that now $\mathbf{x} \ge_{T,M_t} \mathbf{y}$ if and only if $\sum_{x_i \ne y_i} t(x_i) \le \sum_{x_i \ne y_i} t(y_i)$.
In the particular case $T = T_P$, $M_t = G$ is the geometric mean and $t : [0,1] \to [0,\infty]$ is given by $t(x) = -\log x$. Then $G(\mathbf{x}) > G(\mathbf{y})$ implies $\mathbf{x} >_{T,M_t} \mathbf{y}$ and $G(\mathbf{x}) = G(\mathbf{y}) > 0$ implies $\mathbf{x} \approx_{T,M_t} \mathbf{y}$. In the case when $G(\mathbf{x}) = G(\mathbf{y}) = 0$, we should apply Discri-G comparison, i.e., we omit coordinates i where $x_i = y_i$ $(= 0)$ and then we apply the geometric mean to the remaining scores for both alternatives. Summarizing, we see that $\le_{T,M_t}$ is now the Discri-M_t weak order. Due to the isomorphism of strict t-norms with the product t-norm T_P, the same conclusion is true also in general, i.e., for any strict t-norm T with an additive generator t, the weak ordering $\le_{T,M_t}$ is just the Dicsri-M_t ordering as introduced in [12].

(iii) For any nilpotent t-norm T with an additive generator t, $\geq_{T,T_M}$ is not transitive. Observe that $\mathbf{x} \geq_{T,T_M} \mathbf{y}$ if and only if

$$\min\left\{t^{-1}(\max(t(y_1)-t(x_1),0)),\ldots,t^{-1}(\max(t(y_n)-t(x_n),0))\right\}$$
$$\leq \min\left\{t^{-1}(\max(t(x_1)-t(y_1),0)),\ldots,t^{-1}(\max(t(x_n)-t(y_n),0))\right\}.$$

For $T = T_L$ (Łukasiewicz t-norm), we have $\mathbf{x} \geq_{T_L,T_M} \mathbf{y}$ if and only if max $\{x_i - y_i | i = 1,\ldots,n\} \geq \max\{y_i - x_i | i = 1,\ldots,n\}$. Let $\mathbf{x} = (1, 0, 0.5)$, $\mathbf{y} = (0, 0.2, 0.6)$ and $\mathbf{z} = (0.5, 0.6, 0)$. Then $\mathbf{x} >_{T_L,\,T_M} \mathbf{y} >_{T_L,T_M} \mathbf{z} >_{T_L,T_M} \mathbf{x}$, visualizing the non-transitivity of $\geq_{T_L,T_M}$.

In some cases, the approach introduced in Definition 6.3 can be related to the bipolar aggregation B of the vector $\mathbf{x} - \mathbf{y}$, and then $\mathbf{x} \geq_{T,H} \mathbf{y}$ if and only if $B(\mathbf{x} - \mathbf{y}) \geq 0$.

Example 6.9. Consider $H = Ch$ the Choquet integral based on a fuzzy measure M, see (6.4), and let $T = T_L$ be the Łukasiewicz t-norm. Then for any $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $H(I_T(x_1, y_1), \ldots, I_T(x_n, y_n)) = Ch(\min(1, 1-x_i+y_i))$. Considering the [3,6,18, 37,38], Ši with respect to M, it holds $Ch(\min(1, 1-\mathbf{x}+\mathbf{y})) = 1-\text{Ši}(\max(0, \mathbf{x}-\mathbf{y}))$, and thus $\mathbf{x} \geq_{Ch,T_L} \mathbf{y}$ if and only if $1 - \text{Ši}(\max(0, \mathbf{x} - \mathbf{y})) \leq 1 - \text{Ši}(\max(0, \mathbf{y} - \mathbf{x}))$.

However, due to the properties of the Šipoš integral, the latest inequality is equivalent to the inequality $\text{Ši}(\mathbf{x}, \mathbf{y}) \geq 0$, i.e., $\mathbf{x} \geq_{Ch,T_L} \mathbf{y}$ if and only if $\text{Ši}(\mathbf{x}, \mathbf{y}) \geq 0$.

6.6 Concluding Remarks

We have introduced and discussed several methods of multicriteria decision making based on the fuzzy set theory and on the fuzzy logic. In some cases, well-established methods were rediscovered, such as the comparison of alternatives by means of the aggregation of the corresponding score vectors, or by means of the discriminative aggregation (i.e., aggregating only those score values where both compared alternatives differ). Definition 6.1 has brought as a corollary a refinement of aggregation based comparison for those cases when Theorem 6.1 applies (note that there our approach covers a big part of aggregation functions known from multicriteria decision-making problems). Based on fuzzy logic ideas, also non-transitive preference structure on the set of alternatives $\mathcal{A}$ was proposed. Approaches discussed in Sections 6.3 and 6.4 enable an easy implementation of weights/importance of single criteria into the corresponding processing. Evidently, our overview of fuzzy set/logic based methods is only a small part of numerous decision-making methods proposed in the framework of multicriteria characterization of alternatives. We have chosen only some of recently proposed methods, and in some particular cases this is the first place where the discussed methods appear. In some cases, presented approaches allow to compare also alternatives described by score vectors with different dimension (i.e., not each criterion was evaluated for all compared alternatives). As a by-product, two different extensions of well-known LexiMax (LexiMin) methods were introduced. Among several new approaches with promising potential for multicriteria decision making, but still to be developed, recall the level-dependent fuzzy measures and the related fuzzy integrals, such as the Choquet integral with respect to the level-dependent fuzzy measures proposed and discussed in [15], or the extended Sugeno integral representing the comonotone maxitive utility functions proposed in [29].

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