Chapter 5
Preference Modelling, a Matter of Degree

Bernard De Baets and János Fodor

Abstract We consider various frameworks in which preferences can be expressed in a gradual way. The first framework is that of fuzzy preference structures as a generalization of Boolean (two-valued) preference structures. A fuzzy preference structure is a triplet of fuzzy relations expressing strict preference, indifference and incomparability in terms of truth degrees. An important issue is the decomposition of a fuzzy preference relation into such a structure. The main tool for doing so is an indifference generator. The second framework is that of reciprocal relations as a generalization of the three-valued representation of complete Boolean preference relations. Reciprocal relations, also known as probabilistic relations, leave no room for incomparability, express indifference in a Boolean way and express strict preference in terms of intensities. We describe properties of fuzzy preference relations in both frameworks, focusing on transitivity-related properties. For reciprocal relations, we explain the cycle-transitivity framework. As the whole exposition makes extensive use of (logical) connectives, such as conjunctors, quasi-copulas and copulas, we provide an appropriate introduction on the topic.

Keywords Fuzzy relation · Preference structure · Transitivity · Reciprocal relation · Cycle-transitivity

5.1 Introduction

Most of the real-world decision problems take place in a complex environment where different forms of incompleteness (such as uncertainty, imprecision, vagueness, partial truth and the like) pervade our knowledge. To face such complexity, an inevitable step is the use of appropriate models of preferences [45, 54].

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The key concept in preference modelling is the notion of a *preference structure*. It represents pairwise comparison in a set of alternatives, and consists of three binary relations expressing strict preference, indifference and incomparability. Nevertheless, the application of two-valued (yes-or-no) preferences, regardless of their sound mathematical theory, is not satisfactory in everyday situations. Therefore, it is desirable to consider a *degree* of preference, which can be represented by fuzzy relations in a natural way.

Thus we face the problem of extending classical preference structures to the fuzzy case. Any proper extension must meet some minimal expectations. For sure, it must allow preference degrees lying anywhere in the unit interval. Roughly speaking, the extension relies mainly on two facts: first, on the right choice of the underlying logical operations (t-norms and t-conorms); second, on the use of an appropriate form of the completeness condition. These are really new features of the extended models since in the Boolean case both the logical operations and the completeness condition are unique.

It has been proved (see [51–53]) that even the above minimal condition is violated unless we use a particular class of t-norms. Within the group of *continuous* t-norms, the only possibility is to use a transform of the Łukasiewicz t-norm [15,53]. This case leads to *additive* fuzzy preference structures, with a rather well-developed theory [11, 26, 27, 52] on functional equations identifying suitable strict preference, indifference and incomparability generators.

We reconsidered the construction of additive fuzzy preference structures [1], by starting from the minimal definition of an additive fuzzy preference structure. We have shown that a given additive fuzzy preference structure is not necessarily the result of applying monotone generators to a large preference relation. In order to cover all additive fuzzy preference structures, we therefore start all over again, looking for the most general strict preference, indifference and incomparability generators. We pinpoint the central role of the indifference generator and clarify that the monotonicity of a generator triplet is totally determined by using a commutative quasi-copula as indifference generator.

Reciprocal relations, satisfying $Q(a,b) + Q(b,a) = 1$, provide another popular tool for expressing the result of the pairwise comparison of a set of alternatives [5] and appear in various fields such as game theory [22] and mathematical psychology [24]. Reciprocal relations are particularly popular in fuzzy set theory where they are used for representing intensities of preference [4, 34]. Compared to additive fuzzy preference structures, however, they leave no room for incomparability.

In the context of preference modelling, transitivity is always an interesting, often desirable property. In fuzzy relational calculus, the notion of T-transitivity is indispensable. Some types of transitivity have been devised specifically for reciprocal relations, such as various types of stochastic transitivity. Although formally reciprocal relations can be seen as a special kind of fuzzy relations, they are not equipped with the same semantics. One should therefore be careful in considering T-transitivity for reciprocal relations, as well as in studying stochastic transitivity of fuzzy relations.

Recently, two general frameworks for studying the transitivity of reciprocal relations have been established, both encompassing various types of T-transitivity and stochastic transitivity. The first framework is that of FG-transitivity, developed by Switalski [48], and is oriented towards reciprocal relations, although formally (but maybe not in a meaningful way) applicable to fuzzy relations. The second framework was developed in [10] and is restricted to reciprocal relations only. For various reasons, this framework has been coined the cycle-transitivity framework.

All the above issues are touched upon in the present chapter. First we summarize some necessary notions and results on fuzzy and probabilistic connectives such as t-norms, t-conorms and (quasi-)copulas. In Section 5.3 we present the basics of fuzzy relations, including their fundamental properties and particular classes such as fuzzy equivalence relations and weak orders. Then we summarize an axiomatic approach to fuzzy preference structures. Closing the section, we show how to build up fuzzy preference structures by the help of an indifference generator and (quasi-)copulas. In Section 5.4 reciprocal relations are introduced. The cycle-transitivity framework is established and studied in considerable detail. Random variables are compared on the basis of winning probabilities, which are shown to be characterizable in the cycle-transitivity framework. They key role played by copulas for (artificially) coupling random variables is emphasized. We conclude by explaining how also mutual ranking probabilities in partially ordered sets fit into this view.

5.2 Fuzzy and Probabilistic Connectives

It is essential to have access to suitable operators for combining the degrees of preference. In this paper, we are mainly interested in two classes of operators: the class of t-norms [37] and the class of (quasi-) copulas [31, 41].

Definition 5.1. A binary operation $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if it satisfies:

(i) Neutral element 1: $(\forall x \in [0,1]) \quad (T(x,1) = T(1,x) = x)$.
(ii) Monotonicity: T is increasing in each variable.
(iii) Commutativity: $(\forall (x,y) \in [0,1]^2) \quad (T(x,y) = T(y,x))$.
(iv) Associativity: $(\forall (x,y,z) \in [0,1]^3) \quad (T(x,T(y,z)) = T(T(x,y),z))$.

The three prototypes of t-norms are the minimum $T_{\mathbf{M}}(x,y) = \min(x,y)$, the product $T_{\mathbf{P}}(x,y) = xy$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. The first one is idempotent, $T_{\mathbf{P}}$ is strict, while $T_{\mathbf{L}}$ is nilpotent.

The following parametric family of t-norms play a key role in fuzzy preference structures. Consider a number $s \in]0,1[\cup]1,\infty[$, and define a binary operation $T_s^{\mathbf{F}}$ on $[0,1]$ by

$$T_s^{\mathbf{F}}(x,y) = \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right).$$

Thus defined $T_s^{\mathbf{F}}$ is a t-norm for the considered parameter values s. Taking the limits in the remaining cases, we get

$$\begin{aligned}\lim_{s\to 0} T_s^{\mathbf{F}}(x,y) &= \min(x,y),\\ \lim_{s\to 1} T_s^{\mathbf{F}}(x,y) &= xy,\\ \lim_{s\to \infty} T_s^{\mathbf{F}}(x,y) &= \max(x+y-1,0).\end{aligned}$$

Thus, we employ also the following notations: $T_{\mathbf{0}}^{\mathbf{F}} = T_{\mathbf{M}}$, $T_{\mathbf{1}}^{\mathbf{F}} = T_{\mathbf{P}}$, $T_{\infty}^{\mathbf{F}} = T_{\mathbf{L}}$.

The parametric family $\left(T_s^{\mathbf{F}}\right)_{s\in[0,\infty]}$ is called the Frank t-norm family, after the author of [29]. Notice that members are positive (i.e., $T_s^{\mathbf{F}}(x,y) > 0$ when $x,y > 0$) for $0 \le s < \infty$, while $T_{\infty}^{\mathbf{F}} = T_{\mathbf{L}}$ has zero divisors (i.e., there are positive x, y such that $T_{\infty}^{\mathbf{F}}(x,y) = 0$).

T-conorms are the dual operations of t-norms, in the sense that for a given t-norm T, the operation $S : [0,1]^2 \to [0,1]$ defined by

$$S(x,y) = 1 - T(1-x, 1-y),$$

is a t-conorm. Formally, the only difference between t-conorms and t-norms is that the former have neutral element 0, while the latter have neutral element 1.

Concerning the duals of the prototypes, we have $S_{\mathbf{M}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x + y - xy$, and $S_{\mathbf{L}}(x,y) = \min(x+y,1)$.

The t-conorm family consisting of duals of members of the Frank t-norm family is called the Frank t-conorm family: $\left(S_s^{\mathbf{F}}\right)_{s\in[0,\infty]}$. Corresponding pairs $\left(T_s^{\mathbf{F}}, S_s^{\mathbf{F}}\right)$ are ordinally irreducible solutions of the Frank equation:

$$T(x,y) + S(x,y) = x + y.$$

For more details see [29].

Now we turn to the probabilistic connectives quasi-copulas and copulas.

Definition 5.2. A binary operation $C : [0,1]^2 \to [0,1]$ is called a *quasi-copula* if it satisfies:

(i) Neutral element 1: $(\forall x \in [0,1])(C(x,1) = C(1,x) = x)$.
(i′) Absorbing element 0: $(\forall x \in [0,1])(C(x,0) = C(0,x) = 0)$.
(ii) Monotonicity: C is increasing in each variable.
(iii) 1-Lipschitz property: $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$(|C(x_1,y_1) - C(x_2,y_2)| \le |x_1 - x_2| + |y_1 - y_2|).$$

If instead of (iii), C satisfies

(iv) Moderate growth: $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$(x_1 \leq x_2 \ \wedge \ y_1 \leq y_2) \quad \text{imply}$$
$$C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1),$$

then it is called a *copula*.

Note that in case of a quasi-copula condition (i′) is superfluous. For a copula, condition (ii) can be omitted (as it follows from (iv) and (i′)). As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true. It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. Finally, note that for any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$, where $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ is the Łukasiewicz t-norm and $T_{\mathbf{M}}(x, y) = \min(x, y)$ is the minimum operator.

We consider a *continuous De Morgan triplet* (T, S, N) on [0, 1], consisting of a continuous t-norm T, a strong negation N (i.e., a decreasing involutive permutation of [0, 1]) and the N-dual t-conorm S defined by

$$S(x, y) = N(T(N(x), N(y))).$$

Note that a strong negation is uniquely determined by the corresponding automorphism ϕ of the unit interval, $N_\phi(x) := \phi^{-1}(1 - \phi(x))$.

For a t-norm T that is at least left-continuous,

$$I_T(x, y) = \sup\{u \in [0, 1] \mid T(x, u) \leq y\}$$

denotes the unique residual implication (R-implication) of T. This operation plays an important role in Section 5.3.1.

5.3 Fuzzy Preference Structures

Binary relations, especially different kinds of orderings and equivalence relations, play a central role in various fields of science such as decision making, measurement theory and social sciences. Fuzzy logics provide a natural framework for extending the concept of crisp binary relations by assigning to each ordered pair of elements a number from the unit interval – the strength of the link between the two elements. This idea was already used in the first paper on fuzzy sets by Zadeh [55].

In the whole section we assume that A is a given set and (T, S, N) is a continuous De Morgan triplet interpreting logical operations **AND**, **OR** and **NOT**, respectively.

5.3.1 Fuzzy Relations

Fuzzy relations are introduced naturally in the following way.

Definition 5.3. A *binary fuzzy relation* R on the set A is a function $R : A \times A \to [0, 1]$.

That is, R is a fuzzy subset of $A \times A$. For any $a, b \in A$ the value $R(a, b)$ is understood as the degree to which the elements a and b are in relation.

For a given $\lambda \in [0, 1]$ the crisp relation R_λ is defined as the set of ordered pairs with values not less than λ:

$$R_\lambda = \{(a, b) \in A^2 \mid R(a, b) \geq \lambda\}.$$

These λ*-cuts* R_λ form a chain (a nested family) of relations.

The *complement* $\mathrm{co}_N\ R$, the *converse* R^t and the *dual* R^d of a given fuzzy relation R are defined as follows ($a, b \in A$):

$$\mathrm{co}_N\ R(a, b) := N(R(a, b)),\quad R^t(a, b) := R(b, a),\quad R^d(a, b) := N(R(b, a)).$$

Notice that $R^d = \mathrm{co}_N\ R^t = (\mathrm{co}_N\ R)^t$.

For two binary fuzzy relations R and Q on A, we can define their T-intersection $R \cap_T Q$ and S-union $R \cup_S Q$ as follows:

$$(R \cap_T Q)(a, b) := T(R(a, b), Q(a, b)),$$
$$(R \cup_S Q)(a, b) := S(R(a, b), Q(a, b)).$$

Since we deal only with binary fuzzy relations, we often omit the adjective and simply write fuzzy relation.

5.3.1.1 Properties of Fuzzy Relations

In this section we consider and explain the most basic properties of fuzzy relations.

Definition 5.4. A binary fuzzy relation R on A is called

- *reflexive* if $R(a, a) = 1$ for all $a \in A$;
- *irreflexive* if $R(a, a) = 0$ for all $a \in A$;
- *symmetric* if $R(a, b) = R(b, a)$ for all $a, b \in A$.

If a fuzzy relation R on A is reflexive (irreflexive) then all crisp relations R_λ are reflexive (irreflexive for $\lambda \in\ ]0, 1]$). It is obvious that R is irreflexive if and only if R^d is reflexive, which holds if and only if $\mathrm{co}_N\ R$ is reflexive.

Definition 5.5. A fuzzy relation R on A is called

- T*-asymmetric* if $T(R(a, b), R(b, a)) = 0$ holds for all $a, b \in A$;
- T*-antisymmetric* if $T(R(a, b), R(b, a)) = 0$ holds for all $a, b \in A$ such that $a \neq b$.

One can prove easily that if a fuzzy relation R on A is $T_{\mathbf{M}}$-antisymmetric ($T_{\mathbf{M}}$-asymmetric) then its cut relations R_λ are antisymmetric (asymmetric) crisp relations for $\lambda \in]0, 1]$.

Obviously, if a fuzzy relation R on A is T-antisymmetric (T-asymmetric) for a certain t-norm T, then R is also T'-antisymmetric (T'-asymmetric) for any t-norm T' such that $T' \leq T$. Therefore, a $T_{\mathbf{M}}$-antisymmetric relation is T-antisymmetric for any t-norm T.

If T is a positive t-norm then a fuzzy relation R is T-antisymmetric (T-asymmetric) if and only if R is $T_{\mathbf{M}}$-antisymmetric ($T_{\mathbf{M}}$-asymmetric). For positive T, T-asymmetry implies irreflexivity.

Remarkable new definitions (that is, different from the case $T_{\mathbf{M}}$) can only be obtained by using t-norms with zero divisors. In such cases both values $R(a, b)$ and $R(b, a)$ can be positive, but cannot be too high simultaneously. More exactly, if one considers the Łukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max\{x + y - 1, 0\}$, $T_{\mathbf{L}}$-antisymmetry ($T_{\mathbf{L}}$-asymmetry) of R is equivalent to the following inequality: $R(a, b) + R(b, a) \leq 1$.

Definition 5.6. A fuzzy relation R on A is called

- *strongly S-complete* if $S(R(a, b), R(b, a)) = 1$ for all $a, b \in A$;
- *S-complete* if $S(R(a, b), R(b, a)) = 1$ for all $a, b \in A$ such that $a \neq b$.

Obviously, if R is S-complete (strongly S-complete) on A then it is S'-complete (strongly S'-complete) on A for any t-conorm S' such that $S' \geq S$.

Since (T, S, N) is a De Morgan triplet, S-completeness and T-antisymmetry (strong S-completeness and T-asymmetry) are dual properties. That is, a fuzzy relation R on A is S-complete (strongly S-complete) if and only if its dual R^d is T-antisymmetric (T-asymmetric) on A. Using duality, it is easy to prove that when T is a positive t-norm in the De Morgan triplet (T, S, N) then a fuzzy relation R on A is S-complete (strongly S-complete) if and only if R is $S_{\mathbf{M}}$-complete (strongly $S_{\mathbf{M}}$-complete) on A. Strong S-completeness implies reflexivity if and only if T is a positive t-norm.

If a fuzzy relation R on A is $S_{\mathbf{M}}$-complete (strongly $S_{\mathbf{M}}$-complete) then its cut relations R_λ are complete (strongly complete) crisp binary relations.

Now we turn to transitivity, which is certainly one of the most important properties concerning either equivalences or different types of orders. Since classical transitivity can be introduced by using the composition of relations, first we define the corresponding notion of T-composition for binary fuzzy relations.

Definition 5.7. Let R_1, R_2 be fuzzy relations on A. The *T-composition* of R_1 and R_2 is a fuzzy relation denoted as $R_1 \circ_T R_2$, and defined by

$$(R_1 \circ_T R_2)(a, b) = \sup_{c \in A} T(R_1(a, c), R_2(c, b)). \tag{5.1}$$

This definition is natural. Indeed, if Q_1 and Q_2 are crisp binary relations on A then $a(Q_1 \circ Q_2)b$ if and only if there exists an element $c \in A$ such that aQ_1c and cQ_2b. This corresponds to (5.1) in the fuzzy case.

Consider two fuzzy relations R_1, R_2 on A. We say that R_1 is *contained in* R_2 and denote by $R_1 \subseteq R_2$ if and only if for all $a, b \in A$ we have $R_1(a,b) \leq R_2(a,b)$. Fuzzy relations R_1 and R_2 are said to be *equal* if and only if $R_1(a,b) = R_2(a,b)$ for all $a, b \in A$.

It is easy to prove that

(a) $R_1 \circ_T (R_2 \circ_T R_3) = (R_1 \circ_T R_2) \circ_T R_3$
(b) $R_1 \subseteq R_2$ implies $R_1 \circ_T R_3 \subseteq R_2 \circ_T R_3$ and $R_3 \circ_T R_1 \subseteq R_3 \circ_T R_2$

for all fuzzy relations R_1, R_2 and R_3 on A. In other words, composition of fuzzy relations is an associative and increasing operation. For proof see [27].

Turning back to transitivity, the idea behind it is that "the strength of the link between two elements must be greater than or equal to the strength of any indirect chain (i.e., involving other elements)", see [21]. This is expressed in the following definition (see also [56]).

Definition 5.8. A fuzzy relation R on A is called *T-transitive* if

$$T(R(a,c), R(c,b)) \leq R(a,b) \tag{5.2}$$

holds for all $a, b, c \in A$.

General representation theorems of T-transitive fuzzy relations have been established in [28]. One of those is recalled now.

Theorem 5.1. *[28] Let R be a fuzzy relation on A. Then R is T-transitive if and only if there exist two families $\{f_\gamma\}_{\gamma \in \Gamma}$, $\{g_\gamma\}_{\gamma \in \Gamma}$ of functions from A to* $[0,1]$ *such that $f_\gamma(a) \geq g_\gamma(a)$ for all $a \in A$, $\gamma \in \Gamma$ and*

$$R(a,b) = \inf_{\gamma \in \Gamma} I_T(f_\gamma(a), g_\gamma(b)). \tag{5.3}$$

It is easy to see that if R is a $T_{\mathbf{M}}$-transitive fuzzy relation on A then each λ-cut of R is a transitive relation for $\lambda \in]0,1]$.

Negative S-transitivity is the dual concept of T-transitivity and vice versa. Therefore, only some main points are explained in detail. The others can be obtained by corresponding results on T-transitivity.

Definition 5.9. A fuzzy relation R on A is called *negatively S-transitive* if $R(a,b) \leq S(R(a,c), R(c,b))$ for all $a, b, c \in A$.

It is easily seen that a fuzzy relation R on A is negatively S-transitive if and only if its dual R^d is T-transitive, where (T, S, N) is still a De Morgan triplet.

Suppose that R is negatively S-transitive for a given S. Then R is negatively S'-transitive for any t-conorm S' such that $S' \geq S$. In particular, a negatively $S_{\mathbf{M}}$-transitive relation is negatively S'-transitive for any t-conorm S'.

If R is strongly S-complete and T-transitive on A then R is negatively S-transitive on A (for the proof see [27]).

5.3.1.2 Special Types of Fuzzy Relations

Fuzzy relations that are reflexive and T-transitive are called *fuzzy preorders* with respect to T, short *T-preorders*. Symmetric T-preorders are called *fuzzy equivalence relations* with respect to T, short *T-equivalences*. Note that the term *T-similarity relation* is also used for a T-equivalence relation. Similarity relations have been introduced and investigated by Zadeh [56] (see also [43, 44]).

The following result is a characterization and representation of T-equivalence relations (published under the term "indistinguishability" instead of equivalence, see [50]). Compare also with Theorem 5.1.

Theorem 5.2. [50] *Let R be a binary fuzzy relation on A. Then R is a T-equivalence relation on A if and only if there exists a family $\{h_\gamma\}_{\gamma\in\Gamma}$ of functions from A to $[0,1]$ so that for all $a, b \in A$*

$$R(a,b) = \inf_{\gamma\in\Gamma} I_T(\max\{h_\gamma(a), h_\gamma(b)\}, \min\{h_\gamma(a), h_\gamma(b)\}), \tag{5.4}$$

where I_T is the R-implication defined by T.

Equivalence classes of a T-equivalence relation consist of elements being close to each other, and formally are defined as follows. Let R be a T-equivalence relation on A. For any given $a \in A$, an *equivalence class* of a is a fuzzy set $R[a] : A \to [0,1]$ defined by $R[a](c) = R(a,c)$ for all $c \in A$. It may happen that $R[a] = R[b]$ for different elements $a, b \in A$. It is easy to verify that $R[a] = R[b]$ holds if and only if $R(a,b) = 1$. Each λ-cut of a fuzzy $T_{\mathbf{M}}$-equivalence relation is a crisp equivalence relation, as one can check it easily.

Strongly complete T-preorders are called *fuzzy weak orders* with respect to T, short *weak T-orders*.

Given a T-equivalence $E : X^2 \to [0,1]$, a binary fuzzy relation $L : X^2 \to [0,1]$ is called a *fuzzy order* with respect to T and E, short *T-E-order*, if it is T-transitive and additionally has the following two properties:

- *E-reflexivity:* $E(x,y) \le L(x,y)$ for all $x, y \in X$
- *T-E-antisymmetry:* $T(L(x,y), L(y,x)) \le E(x,y)$ for all $x, y \in X$

We are ready to state the first – score function-based – representation theorem of weak T-orders. For more details and proofs see [3].

Theorem 5.3. [3] *A binary fuzzy relation $R : X^2 \to [0,1]$ is a weak T-order if and only if there exists a non-empty domain Y, a T-equivalence $E : Y^2 \to [0,1]$, a strongly $S_{\mathbf{M}}$-complete T-E-order $L : Y^2 \to [0,1]$ and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:*

$$R(x,y) = L(f(x), f(y)). \tag{5.5}$$

This result is an extension of the following well-known classical theorem.

Theorem 5.4. *A binary relation $\lesssim$ on a non-empty domain X is a weak order if and only if there exists a linearly ordered non-empty set $(Y, \preceq)$ and a mapping $f : X \to Y$ such that $\lesssim$ can be represented in the following way for all $x, y \in X$:*

$$x \lesssim y \quad \text{if and only if} \quad f(x) \preceq f(y). \tag{5.6}$$

The standard crisp case consists of the unit interval [0, 1] equipped with its natural linear order. Given a left-continuous t-norm T, the canonical fuzzification of the natural linear order on [0, 1] is the residual implication I_T [2,32,33]. The following proposition, therefore, provides us with a construction that can be considered as a straightforward counterpart of (5.6).

Proposition 5.1. *Given a function $f : X \to [0, 1]$, the binary fuzzy relation $R : X^2 \to [0, 1]$ defined by*

$$R(x, y) = I_T\big(f(x), f(y)\big) \tag{5.7}$$

is a weak T-order.

The function f is called a *score function*. Note that there are weak T-orders that cannot be represented by means of a single score function [3]. Therefore, a weak T-order $R : X^2 \to [0, 1]$ is called *representable* if there exists a function $f : X \to [0, 1]$, called *generating score function*, such that Eq. 5.7 holds. A representable weak $T_{\mathbf{M}}$-order is called *Gödel-representable* [12]. The following result is a unique characterization of representable fuzzy weak orders for continuous t-norms.

Theorem 5.5. [3] *Assume that T is continuous. Then a weak T-order R is representable if and only if the following function is a generating score function of R:*

$$\bar{f}(x) = \inf_{z \in X} R(z, x).$$

The following well-known theorem shows that fuzzy weak orders can be represented by more than one score function. Compare also with Theorems 5.1 and 5.2.

Theorem 5.6. [50] *Consider a binary fuzzy relation $R : X^2 \to [0, 1]$. Then the following two statements are equivalent:*

(i) *R is a T-preorder.*
(ii) *There exists a non-empty family of $X \to [0, 1]$ score functions $(f_i)_{i \in I}$ such that the following representation holds:*

$$R(x, y) = \inf_{i \in I} I_T(f_i(x), f_i(y)). \tag{5.8}$$

Now the following theorem provides us with a unique characterization of weak T-orders.

Theorem 5.7. [3] *Consider a binary fuzzy relation* $R : X^2 \to [0,1]$. *Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp weak order $\lesssim$ *and a non-empty family of* $X \to [0,1]$ *score functions* $(f_i)_{i \in I}$ *that are non-decreasing with respect to* $\lesssim$ *such that representation* (5.8) *holds.*

The following theorem finally characterizes weak T-orders as intersections of representable weak T-orders that are generated by score functions that are monotonic at the same time with respect to the same crisp linear order.

Theorem 5.8. [3] *Consider a binary fuzzy relation* $R : X^2 \to [0,1]$. *Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp linear order $\preceq$ *and a non-empty family of* $X \to [0,1]$ *score functions* $(f_i)_{i \in I}$ *that are non-decreasing with respect to* $\preceq$ *such that representation* (5.8) *holds.*

The interested reader can find further representation and construction results in [3]. This includes inclusion-based representations, and representations by decomposition into crisp linear orders and fuzzy equivalence relations, which also facilitates a pseudo-metric-based construction.

5.3.2 Additive Fuzzy Preference Structures: Bottom-Up Approach

5.3.2.1 Classical Preference Structures

Consider a set of alternatives A and suppose that a decision maker wants to judge these alternatives by pairwise comparison. Given two alternatives, the decision maker can act in one of the following three ways:

(i) He/she clearly prefers one to the other.
(ii) The two alternatives are indifferent to him/her.
(iii) He/she is unable to compare the two alternatives.

According to these cases, three binary relations can be defined in A: the *strict preference relation* P, the *indifference relation* I and the *incomparability relation* J. For any $(a,b) \in A^2$, we classify:

$$(a,b) \in P \quad \Leftrightarrow \quad \text{he/she prefers } a \text{ to } b;$$
$$(a,b) \in I \quad \Leftrightarrow \quad a \text{ and } b \text{ are indifferent to him/her;}$$
$$(a,b) \in J \quad \Leftrightarrow \quad \text{he/she is unable to compare } a \text{ and } b.$$

One easily verifies that the triplet (P, I, J) defined above satisfies the conditions formulated in the following definition of a preference structure. For a binary relation R in A, we denote its *converse* by R^t and its *complement* by $\text{co}\ R$.

Definition 5.10. [45] A preference structure on A is a triplet (P, I, J) of binary relations in A that satisfy:

(B1) P is irreflexive, I is reflexive and J is irreflexive.
(B2) P is asymmetrical, I is symmetrical and J is symmetrical.
(B3) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$.
(B4) $P \cup P^t \cup I \cup J = A^2$.

This definition is exhaustive: it lists all properties of the components P, I and J of a preference structure. The asymmetry of P can also be written as $P \cap P^t = \emptyset$. Condition (B4) is called the *completeness condition* and can be expressed equivalently (up to symmetry) in the following alternative ways: $\text{co}\,(P \cup I) = P^t \cup J$, $\text{co}\,(P \cup P^t) = I \cup J$, $\text{co}\,(P \cup P^t \cup I) = J$, $\text{co}\,(P \cup P^t \cup J) = I$ and $\text{co}\,(P^t \cup I \cup J) = P$.

It is possible to associate a single reflexive relation to any preference structure so that it completely characterizes this structure. A preference structure (P, I, J) on A is characterized by the reflexive binary relation $R = P \cup I$, its large preference relation, in the following way:

$$(P, I, J) = (R \cap \text{co}\ R^t, R \cap R^t, \text{co}\ R \cap \text{co}\ R^t).$$

Conversely, a triplet (P, I, J) constructed in this way from a reflexive binary relation R in A is a preference structure on A. The interpretation of the large preference relation is

$$(a, b) \in R \quad \Leftrightarrow \quad b \text{ is considered at most as good as } a.$$

The above definition of a preference structure can be written in the following minimal way, identifying a relation with its characteristic mapping [13]: I is reflexive and symmetrical, and for any $(a, b) \in A^2$:

$$P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1.$$

Thus, classical preference structures can also be considered as *Boolean preference structures*, employing 1 and 0 for describing presence or absence of strict preferences, indifferences and incomparabilities. Complement, intersection and union then correspond to Boolean negation, conjunction (i.e. minimum) and disjunction (i.e. maximum) on characteristic mappings.

5.3.2.2 The Quest for Fuzzy Preference Structures: The Axiomatic Approach

As preference structures are based on classical set theory and are therefore restricted to two-valued relations, they do not allow to express degrees of strict preference, indifference or incomparability. This is seen as an important drawback to their practical use, leading researchers already at an early stage to the theory of fuzzy sets, and in particular to the calculus of fuzzy relations. In that case, preference degrees are expressed on the continuous scale [0, 1] and operations from fuzzy logic are used for manipulating these degrees.

A *fuzzy preference structure (FPS)* on A is a triplet (P, I, J) of binary fuzzy relations in A satisfying:

(F1) P is irreflexive, I is reflexive and J is irreflexive.
(F2) P is T-asymmetrical, I and J are symmetrical.
(F3) $P \cap_T I = \emptyset$, $P \cap_T J = \emptyset$ and $I \cap_T J = \emptyset$.
(F4) a completeness condition, such as $\text{co}_N\,(P \cup_S I) = P^t \cup_S J$, $\text{co}_N\,(P \cup_S P^t \cup_S I) = J$ or $P \cup_S P^t \cup_S I \cup_S J = A^2$.

Invoking the **assignment principle:** *for any pair of alternatives* (a, b) *the decision maker is allowed to assign at least one of the degrees* $P(a, b)$*,* $P(b, a)$*,* $I(a, b)$ *and* $J(a, b)$ *freely in the unit interval*, shows that only a *nilpotent t-norm* T is acceptable, i.e. a ϕ'-transform of the Łukasiewicz t-norm: $T(x, y) := \phi'^{-1}(\max(\phi'(x) + \phi'(y) - 1, 0))$ [52]. For the sake of simplicity, we consider $\phi = \phi'$. Consequently, we will be working with a Łukasiewicz triplet $\left(T_\phi^\infty, S_\phi^\infty, N_\phi\right)$. The latter notation is used to indicate that the Łukasiewicz t-norm belongs to the Frank t-norm family $(T^s)_{s\in[0,\infty]}$ (which is also a family of copulas) and corresponds to the parameter value $s = \infty$ (note that the minimum operator and the algebraic product correspond to the parameter values $s = 0$ and $s = 1$, respectively). Moreover, in that case, the completeness conditions $\text{co}_\phi \left(P \cup_\phi^\infty I\right) = P^t \cup_\phi^\infty J$ and $\text{co}_\phi \left(P \cup_\phi^\infty P^t\right) = I \cup_\phi^\infty J$ become equivalent and turn out to be stronger than the other completeness conditions, with $P \cup_\phi^\infty P^t \cup_\phi^\infty I \cup_\phi^\infty J = A^2$ as weakest condition [52]. Restricting to the **strongest completeness condition(s)**, we then obtain the following definition.

Definition 5.11. Given a [0, 1]-automorphism ϕ, a ϕ-FPS (a ϕ-fuzzy preference structure) on A is a triplet of binary fuzzy relations (P, I, J) in A satisfying:

(F1) P is irreflexive, I is reflexive and J is irreflexive.
(F2) P is T_ϕ^∞-asymmetrical, I and J are symmetrical.
(F3) $P \cap_\phi^\infty I = \emptyset$, $P \cap_\phi^\infty J = \emptyset$ and $I \cap_\phi^\infty J = \emptyset$.
(F4) $\text{co}_\phi \left(P \cup_\phi^\infty I\right) = P^t \cup_\phi^\infty J$.

Moreover, a minimal formulation of this definition, similar to the classical one, exists: a triplet (P, I, J) of binary fuzzy relations in A is a ϕ-FPS on A if and only if I is reflexive and symmetrical, and for any $(a, b) \in A^2$:

$$\phi(P(a, b)) + \phi(P^t(a, b)) + \phi(I(a, b)) + \phi(J(a, b)) = 1.$$

In view of the above equality, fuzzy preference structures are also called *additive fuzzy preference structures*.

Axiomatic Constructions

Again choosing a continuous de Morgan triplet (T, S, N), we could transport the classical construction formalism to the fuzzy case and define, given a reflexive binary fuzzy relation R in A:

$$(P, I, J) = (R \cap_T \text{co}_N\ R^t, R \cap_T R^t, \text{co}_N\ R \cap_T \text{co}_N\ R^t).$$

At the same time, we want to keep R as the fuzzy large preference relation of the triplet (P, I, J), i.e. $R = P \cup_S I$ and $\text{co}_N R = P^t \cup_S J$. Fodor and Roubens observed that the latter is not possible in general, and proposed four axioms for defining fuzzy strict preference, indifference and incomparability relations [26, 27]. According to the first axiom, the *independence of irrelevant alternatives*, there exist three $[0,1]^2 \to [0,1]$ mappings p, i, j such that $P(a,b) = p(R(a,b), R(b,a))$, $I(a,b) = i(R(a,b), R(b,a))$ and $J(a,b) = j(R(a,b), R(b,a))$. The second and third axioms state that the mappings $p(x, N(y))$, $i(x, y)$ and $j(N(x), N(y))$ are increasing in both x and y, and that i and j are symmetrical. The fourth and main axiom requires that $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$, or explicitly, for any $(x, y) \in [0,1]^2$:

$$S(p(x,y), i(x,y)) = x,$$
$$S(p(x,y), j(x,y)) = N(y).$$

The latter axiom implies that $\text{co}_N (P \cup_S I) = P^t \cup_S J$, i.e. the first completeness condition.

Theorem 5.9. [26,27] *If (T, S, N) and (p, i, j) satisfy the above axioms, then there exists a $[0,1]$-automorphism ϕ such that*

$$(T, S, N) = \left(T_\phi^\infty, S_\phi^\infty, N_\phi\right)$$

and, for any $(x, y) \in [0,1]^2$:

$$\begin{aligned} T_\phi^\infty(x, N_\phi(y)) \le \quad & p(x,y) && \le T^0(x, N_\phi(y)),\\ T_\phi^\infty(x,y) \le \quad & i(x,y) && \le T^0(x,y),\\ T_\phi^\infty(N_\phi(x), N_\phi(y)) \le \quad & j(x,y) && \le T^0(N_\phi(x), N_\phi(y)). \end{aligned}$$

Moreover, for any reflexive binary fuzzy relation R in A, the triplet (P, I, J) of binary fuzzy relations in A defined by

$$P(a,b) = p(R(a,b), R(b,a)),$$
$$I(a,b) = i(R(a,b), R(b,a)),$$
$$J(a,b) = j(R(a,b), R(b,a))$$

is a ϕ-FPS on A such that $R = P \cup_\phi^\infty I$ and $\text{co}_\phi\ R = P^t \cup_\phi^\infty J$.

Although in general the function i is of two variables, and there is no need to extend it for more than two arguments, it might be a t-norm. The following theorem states that the only construction methods of the above type based on continuous t-norms are the ones using two Frank t-norms with reciprocal parameters.

Theorem 5.10. [26, 27] *Consider a* $[0,1]$*-automorphism* ϕ *and two continuous t-norms* T_1 *and* T_2. *Define* p *and* i *by* $p(x,y) = T_1(x, N_\phi(y))$ *and* $i(x,y) = T_2(x,y)$. *Then* (p,i,j) *satisfies the above axioms if and only if there exists a parameter* $s \in [0,\infty]$ *such that* $T_1 = T_\phi^{1/s}$ *and* $T_2 = T_\phi^s$. *In this case, we have that* $j(x,y) = i(N_\phi(x), N_\phi(y))$.

Summarizing, we have that for any reflexive binary fuzzy relation R in A the triplets

$$(P_s, I_s, J_s) = \left(R \cap_\phi^{\frac{1}{s}} \mathrm{co}_\phi\, R^t, R \cap_\phi^s R^t, \mathrm{co}_\phi\, R \cap_\phi^s \mathrm{co}_\phi\, R^t\right),$$

with $s \in [0,\infty]$, are the only t-norm-based constructions of fuzzy preference structures that satisfy $R = P \cup_\phi^\infty I$ and $\mathrm{co}_\phi\, R = P^t \cup_\phi^\infty J$. Consequently, R is again called the *large preference relation*. Note that

$$\phi(R(a,b)) = \phi(P(a,b)) + \phi(I(a,b)).$$

In fact, in [27] it was only shown that ordinal sums of Frank t-norms should be used. For the sake of simplicity, only the ordinally irreducible ones were considered. However, we can prove that this is the only option.

Finally, we deal with the *reconstruction* of a ϕ-FPS from its large preference relation. As expected, an additional condition is required. A ϕ-FPS (P,I,J) on A is called:

(i) an (s,ϕ)-FPS, with $s \in \{0,1,\infty\}$, if $P \cap_\phi^s P^t = I \cap_\phi^{\frac{1}{s}} J$;
(ii) an (s,ϕ)-FPS, with $s \in]0,1[\cup]1,\infty[$, if

$$s^{\phi\left(P \cap_\phi^s P^t\right)} + s^{-\phi\left(I \cap_\phi^{1/s} J\right)} = 2.$$

One can verify that the triplet (P_s, I_s, J_s) constructed above is an (s,ϕ)-FPS. Moreover, any (s,ϕ)-FPS can be reconstructed from its large preference relation by means of the corresponding construction. The characterizing condition of a $(0,\phi)$-FPS, respectively (∞,ϕ)-FPS, can also be written as $P \cap^0 P^t = \emptyset$, i.e. $\min(P(a,b), P(b,a)) = 0$ for any (a,b), respectively. $I \cap^0 J = \emptyset$, i.e. $\min(I(a,b), J(a,b)) = 0$ for any (a,b).

5.3.2.3 Additive Fuzzy Preference Structures and Indifference Generators

Now we reconsider the construction of additive fuzzy preference structures, not by rephrasing the conclusions resulting from an axiomatic study, but by starting from the minimal definition of an additive fuzzy preference structure. For the sake of brevity, we consider the case $\phi(x) = x$. For motivation and more details we refer to [1].

Definition 5.12. A triplet (p, i, j) of $[0, 1]^2 \to [0, 1]$ mappings is called a *generator triplet* compatible with a continuous t-conorm S and a strong negator N if and only if for any reflexive binary fuzzy relation R on a set of alternatives A it holds that the triplet (P, I, J) of binary fuzzy relations on A defined by:

$$P(a, b) = p(R(a, b), R(b, a)),$$

$$I(a, b) = i(R(a, b), R(b, a)),$$

$$J(a, b) = j(R(a, b), R(b, a))$$

is a FPS on A such that $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$.

The above conditions $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$ require the reconstructability of the fuzzy large preference relation R from the fuzzy preference structure it generates. The following theorem expresses that for that purpose only nilpotent t-conorms can be used.

Theorem 5.11. *If (p, i, j) is a generator triplet compatible with a continuous t-conorm S and a strong negator $N = N_\phi$, then $S = S_\psi^\infty$, i.e. S is nilpotent.*

Let us again consider the case $\psi(x) = x$. The above theorem implies that we can omit the specification "compatible with a continuous t-conorm S and strong negation N" and simply talk about generator triplets. The minimal definition of a fuzzy preference structure then immediately leads to the following proposition.

Proposition 5.2. *A triplet (p, i, j) is a generator triplet if and only if, for any $(x, y) \in [0, 1]^2$:*

(i) $i(1, 1) = 1$
(ii) $i(x, y) = i(y, x)$
(iii) $p(x, y) + p(y, x) + i(x, y) + j(x, y) = 1$
(iv) $p(x, y) + i(x, y) = x$

From this proposition it follows that a generator triplet is uniquely determined by, for instance, the generator i. Indeed, for any generator triplet (p, i, j) it holds that

$$p(x, y) = x - i(x, y),$$

$$j(x, y) = i(x, y) - (x + y - 1).$$

The fact that p and j take values in $[0, 1]$ implies that $T^\infty \leq i \leq T^0$. Moreover, from any symmetrical i such that $T^\infty \leq i \leq T^0$ a generator triplet can be built. It is therefore not surprising that additional properties of generator triplets (p, i, j) are completely determined by additional properties of i. In fact, in practice, it would be sufficient to talk about a single generator i. We could simply talk about *the generator* of the FPS. Note that the symmetry of i implies the symmetry of j.

Firstly, we try to characterize generator triplets fitting into the axiomatic framework of Fodor and Roubens.

Definition 5.13. A generator triplet (p, i, j) is called *monotone* if:

(i) p is increasing in the first and decreasing in the second argument.
(ii) i is increasing in both arguments.
(iii) j is decreasing in both arguments.

Inspired by the paper [37], we can show that monotone generator triplets are characterized by a 1-Lipschitz indifference generator, i.e. by a commutative quasi-copula.

Theorem 5.12. *A generator triplet (p, i, j) is monotone if and only if i is a commutative quasi-copula.*

The following theorem shows that when i is a symmetrical ordinal sum of Frank t-norms, $j(1-x, 1-y)$ is also a t-norm and $p(x, 1-y)$ is symmetrical. Note that by symmetrical ordinal sum we mean the following: if (a, b, T) is a summand, then also $(1-b, 1-a, T)$ is a summand.

The associativity of $p(x, 1-y)$, however, can only be guaranteed in case of an ordinally irreducible i, i.e. a Frank t-norm.

Theorem 5.13. *Consider a generator triplet (p, i, j) such that i is a t-norm, then the following statements are equivalent:*

(i) *The mapping $j(1-x, 1-y)$ is a t-norm.*
(ii) *The mapping $p(x, 1-y)$ is symmetrical.*
(iii) *i is a symmetrical ordinal sum of Frank t-norms.*

Theorem 5.14. *Consider a generator triplet (p, i, j) such that i is a t-norm, then the following statements are equivalent:*

(i) *The mapping $p(x, 1-y)$ is a t-norm.*
(ii) *i is a Frank t-norm.*

In the latter case, i.e. when i is a Frank t-norm, say $i = T^s$, $s \in [0, \infty]$, it holds that

$$p(x, y) = T^{1/s}(x, 1-y),$$

$$j(x, y) = T^s(1-x, 1-y).$$

This result closes the loop, and brings us back to the conclusions drawn from the axiomatic study of Fodor and Roubens expressed in Theorem 5.10.

5.4 Reciprocal Preference Relations

5.4.1 Reciprocal Relations

5.4.1.1 Definition

Another interesting class of $A^2 \to [0, 1]$ mappings is the class of *reciprocal relations Q* (also called *ipsodual relations* or *probabilistic relations* [20]) satisfying the condition $Q(a, b) + Q(b, a) = 1$, for any $(a, b) \in A^2$. For such relations, it holds in particular that $Q(a, a) = 1/2$. Many authors like viewing them as particular kinds of fuzzy relations, but we do not adhere to that view, as reciprocal relations are of an inherent bipolar nature. The usual operations (such as intersection, union and composition) on fuzzy relations simply make no sense on reciprocal relations. Not surprisingly then, as will be shown further on, also other notions of transitivity apply to them.

Reciprocity is intimately linked with completeness. Let R be a complete ($\{0, 1\}$-valued) relation on A, i.e. for any $(a, b) \in A^2$ it holds that $\max(R(a, b), R(b, a)) = 1$, then R has an equivalent $\{0, 1/2, 1\}$-valued reciprocal representation Q given by

$$Q(a, b) = \begin{cases} 1, & \text{if } R(a, b) = 1 \text{ and } R(b, a) = 0, \\ 1/2, & \text{if } R(a, b) = R(b, a) = 1, \\ 0, & \text{if } R(a, b) = 0 \text{ and } R(b, a) = 1, \end{cases}$$

or in a more compact arithmetic form:

$$Q(a, b) = \frac{1 + R(a, b) - R(b, a)}{2}. \tag{5.9}$$

One easily verifies that R is transitive if and only if its reciprocal representation Q satisfies, for any $(a, b, c) \in A^3$:

$$(Q(a, b) \geq 1/2 \ \wedge \ Q(b, c) \geq 1/2) \ \Rightarrow \ Q(a, c) = \max(Q(a, b), Q(b, c)). \tag{5.10}$$

Reciprocal relations generalize the above representation by taking values also in the intervals $]0, 1/2[$ and $]1/2, 1[$.

5.4.1.2 A Fuzzy Set Viewpoint

In the fuzzy set community (see e.g., [23, 30, 35]), the semantics attributed to a reciprocal relation Q is as follows:

$$Q(a,b) \in \begin{cases}]1/2,1], & \text{if } a \text{ is rather preferred to } b, \\ \{1/2\}, & \text{if } a \text{ and } b \text{ are indifferent,} \\ [0,1/2[, & \text{if } b \text{ is rather preferred to } a. \end{cases}$$

Hence, a reciprocal relation can be seen as a compact representation of an additive fuzzy preference structure in which the indifference relation I is a crisp relation, the incomparability relation J is empty and the strict preference relation P and its converse P^t are fuzzy relations complementing each other. Reciprocal relations are therefore closely related to fuzzy weak orders.

Note that, similarly as for a complete relation, a weakly complete fuzzy relation R on A can be transformed into a reciprocal representation $Q = P + 1/2I$, with P and I the strict preference and indifference components of the additive fuzzy preference structure (P, I, J) generated from R by means of the generator $i = T_{\mathbf{L}}$ [14, 52]:

$$\begin{aligned} P(a,b) &= T_{\mathbf{M}}(R(a,b), 1 - R(b,a)) = 1 - R(b,a), \\ I(a,b) &= T_{\mathbf{L}}(R(a,b), R(b,a)) = R(a,b) + R(b,a) - 1, \\ J(a,b) &= T_{\mathbf{L}}(1 - R(a,b), 1 - R(b,a)) = 0. \end{aligned}$$

Note that the corresponding expression for Q is formally the same as (5.9). This representation is not equivalent to the fuzzy relation R, as many weakly complete fuzzy relations R may have the same representation. Recall that, if the fuzzy relation R is also strongly complete, then the generator i used is immaterial.

5.4.1.3 A Frequentist View

However, the origin of reciprocal relations is not to be found in the fuzzy set community. For several decades, reciprocal relations are used as a convenient tool for expressing the results of the pairwise comparison of a set of alternatives in fields such as game theory [22], voting theory [42] and psychology [20]. A typical use is that where an individual is asked, in a controlled experimental set-up, to compare the same set of alternatives multiple times, where each time he can either prefer alternative a to b or b to a. The fraction of times a is preferred to b then yields $Q(a,b)$. In what follows, we will stay close to that frequentist view. However, we prefer to use the more neutral term reciprocal relation, rather than the term probabilistic relation.

5.4.2 The Cycle-Transitivity Framework

5.4.2.1 Stochastic Transitivity

Transitivity properties for reciprocal relations rather have the conditional flavor of (5.10). There exist various kinds of stochastic transitivity for reciprocal relations [5, 40]. For instance, a reciprocal relation Q on A is called *weakly stochastic transitive* if for any $(a, b, c) \in A^3$ it holds that $Q(a, b) \geq 1/2 \ \wedge$ $Q(b, c) \geq 1/2$ implies $Q(a, c) \geq 1/2$. In [10], the following generalization of stochastic transitivity was proposed.

Definition 5.14. Let g be an increasing $[1/2, 1]^2 \to [0, 1]$ mapping such that $g(1/2, 1/2) \leq 1/2$. A reciprocal relation Q on A is called g-stochastic transitive if for any $(a, b, c) \in A^3$ it holds that

$$(Q(a, b) \geq 1/2 \ \wedge \ Q(b, c) \geq 1/2) \ \Rightarrow \ Q(a, c) \geq g(Q(a, b), Q(b, c)).$$

Note that the condition $g(1/2, 1/2) \leq 1/2$ ensures that the reciprocal representation Q of any transitive complete relation R is always g-stochastic transitive. In other words, g-stochastic transitivity generalizes transitivity of complete relations. This definition includes the standard types of stochastic transitivity [40]:

(i) *Strong* stochastic transitivity when $g = \max$
(ii) *Moderate* stochastic transitivity when $g = \min$
(iii) *Weak* stochastic transitivity when $g = 1/2$

In [10], also a special type of stochastic transitivity was introduced.

Definition 5.15. Let g be an increasing $[1/2, 1]^2 \to [0, 1]$ mapping such that $g(1/2, 1/2) = 1/2$ and $g(1/2, 1) = g(1, 1/2) = 1$. A reciprocal relation Q on A is called g-isostochastic transitive if for any $(a, b, c) \in A^3$ it holds that

$$(Q(a, b) \geq 1/2 \ \wedge \ Q(b, c) \geq 1/2) \ \Rightarrow \ Q(a, c) = g(Q(a, b), Q(b, c)).$$

The conditions imposed upon g again ensure that g-isostochastic transitivity generalizes transitivity of complete relations. Note that for a given mapping g, the property of g-isostochastic transitivity obviously is much more restrictive than the property of g-stochastic transitivity.

5.4.2.2 *FG*-Transitivity

The framework of FG-transitivity, developed by Switalski [47, 51], formally generalizes g-stochastic transitivity in the sense that $Q(a, c)$ is bounded both from below and above by $[1/2, 1]^2 \to [0, 1]$ mappings.

Definition 5.16. Let F and G be two $[1/2, 1]^2 \to [0, 1]$ mappings such that $F(1/2, 1/2) \leq 1/2 \leq G(1/2, 1/2)$ and $G(1/2, 1) = G(1, 1/2) = G(1, 1) = 1$ and $F \leq G$. A reciprocal relation Q on A is called FG-transitive if for any $(a, b, c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2)$$
$$\Downarrow$$
$$F(Q(a,b), Q(b,c)) \leq Q(a,c) \leq G(Q(a,b), Q(b,c)).$$

5.4.2.3 Cycle-Transitivity

Similarly as the FG-transitivity framework, the cycle-transitivity framework involves two bounds. However, these bounds are not independent, and moreover, the arguments are subjected to a reordering before they are applied. More specifically, for a reciprocal relation Q, we define for all $(a,b,c) \in A^3$ the following quantities [10]:

$$\alpha_{abc} = \min(Q(a,b), Q(b,c), Q(c,a)),$$
$$\beta_{abc} = \text{median}(Q(a,b), Q(b,c), Q(c,a)),$$
$$\gamma_{abc} = \max(Q(a,b), Q(b,c), Q(c,a)).$$

Let us also denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\}$. A function $U : \Delta \to \mathbb{R}$ is called an if it satisfies:

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;
(ii) for any $(\alpha,\beta,\gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geq 1. \quad (5.11)$$

The function $L : \Delta \to \mathbb{R}$ defined by $L(\alpha,\beta,\gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha)$ is called the *dual* of a given upper bound function U. Inequality (5.11) then simply expresses that $L \leq U$. Condition (i) again guarantees that cycle-transitivity generalizes transitivity of complete relations.

Definition 5.17. A reciprocal relation Q on A is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}), \quad (5.12)$$

where L is the dual lower bound function of U.

Due to the built-in duality, it holds that if (5.12) is true for some (a,b,c), then this is also the case for any permutation of (a,b,c). In practice, it is therefore sufficient to check (5.12) for a single permutation of any $(a,b,c) \in A^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in A^3$ (not being cyclic permutations of one another), e.g., (a,b,c) and (c,b,a). Hence, (5.12) can be replaced by

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}). \tag{5.13}$$

Note that a value of $U(\alpha, \beta, \gamma)$ equal to 2 is used to express that for the given values there is no restriction at all (as $\alpha + \beta + \gamma - 1$ is always bounded by 2).

Two upper bound functions U_1 and U_2 are called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that $\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$ is equivalent to $\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma)$.

If it happens that in (5.11) the equality holds for all $(\alpha, \beta, \gamma) \in \Delta$, then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, also (5.12) and (5.13) can only hold with equality then. Furthermore, it then holds that $U(0, 0, 1) = 0$ and $U(0, 1, 1) = 1$.

The simplest example of a self-dual upper bound function is the median, i.e. $U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta$. Another example of a self-dual upper bound function is the function U_E defined by

$$U_E(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma.$$

Cycle-transitivity w.r.t. U_E of a reciprocal relation Q on A can also be expressed as

$$\alpha_{ijk}\beta_{ijk}\gamma_{ijk} = (1 - \alpha_{ijk})(1 - \beta_{ijk})(1 - \gamma_{ijk}).$$

It is then easy to see that cycle-transitivity w.r.t. U_E is equivalent to the notion of multiplicative transitivity [49]. Recall that a reciprocal relation Q on A is called *multiplicatively transitive* if for any $(a, b, c) \in A^3$ it holds that

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)}.$$

The cycle-transitive formulation is more appropriate as it avoids division by zero.

5.4.2.4 Cycle-Transitivity Is a General Framework

Although C-transitivity is not intended to be applied to reciprocal relations, it can be formally cast quite nicely into the cycle-transitivity framework.

Proposition 5.3. [10] *Let C be a commutative conjunctor such that $C \leq T_{\mathbf{M}}$. A reciprocal relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha, \beta, \gamma) = \min(\alpha + \beta - C(\alpha, \beta), \beta + \gamma - C(\beta, \gamma), \gamma + \alpha - C(\gamma, \alpha)).$$

Moreover, if C is 1-Lipschitz, then U_C is given by

$$U_C(\alpha, \beta, \gamma) = \alpha + \beta - C(\alpha, \beta). \tag{5.14}$$

This proposition applies in particular to commutative quasi-copulas and copulas. In case of a copula, the expression in (5.14) is known as the dual of the copula. Consider the three basic t-norms/copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$:

(i) For $C = T_{\mathbf{M}}$, we immediately obtain as upper bound function the median

$$U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta.$$

(ii) For $C = T_{\mathbf{P}}$, we find

$$U_{\mathbf{P}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta.$$

(iii) For $C = T_{\mathbf{L}}$, we obtain

$$U_{\mathbf{L}}(\alpha, \beta, \gamma) = \begin{cases} \alpha + \beta, & \text{if } \alpha + \beta < 1, \\ 1, & \text{if } \alpha + \beta \geq 1. \end{cases}$$

An equivalent upper bound function is given by $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$.

Cycle-transitivity also incorporates stochastic transitivity, although the latter fits more naturally in the FG-transitivity framework. We list just one interesting proposition under mild conditions on the function g.

Proposition 5.4. *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping such that $g(1/2, x) \leq x$ for any $x \in [1/2, 1]$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma), & \text{if } \beta \geq 1/2 \wedge \alpha < 1/2, \\ 1/2, & \text{if } \alpha \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases} \tag{5.15}$$

A final simplification, eliminating the special case $\alpha = 1/2$ in (5.15), is obtained by requiring g to have as neutral element $1/2$, i.e. $g(1/2, x) = g(x, 1/2) = x$ for any $x \in [1/2, 1]$.

Proposition 5.5. *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma), & \text{if } \beta \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases} \tag{5.16}$$

This proposition implies in particular that strong stochastic transitivity ($g = \max$) is equivalent to cycle-transitivity w.r.t. the simplified upper bound function U'_{ss} defined by

$$U'_{ss}(\alpha,\beta,\gamma) = \begin{cases} \beta, & \text{if } \beta \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases}$$

Note that g-stochastic transitivity w.r.t. a function $g \geq \max$ always implies strong stochastic transitivity. This means that any reciprocal relation that is cycle-transitive w.r.t. an upper bound function U_g of the form (5.16) is at least strongly stochastic transitive. It is obvious that $T_{\mathbf{M}}$-transitivity implies strong stochastic transitivity and that moderate stochastic transitivity implies $T_{\mathbf{L}}$-transitivity.

One particular form of stochastic transitivity deserves our attention. A probabilistic relation Q on A is called *partially stochastic transitive* [24] if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) > 1/2 \wedge Q(b,c) > 1/2) \Rightarrow Q(a,c) \geq \min(Q(a,b), Q(b,c)).$$

Clearly, it is a slight weakening of moderate stochastic transitivity. Interestingly, also this type of transitivity can be expressed elegantly in the cycle-transitivity framework [17] by means of a simple upper bound function.

Proposition 5.6. *Cycle-transitivity w.r.t. the upper bound function U_{ps} defined by*

$$U_{ps}(\alpha,\beta,\gamma) = \gamma$$

is equivalent to partial stochastic transitivity.

Finally, not surprisingly, isostochastic transitivity corresponds to cycle-transitivity w.r.t. particular self-dual upper bound functions [10]. An interesting way of constructing a self-dual upper bound function goes as follows.

Proposition 5.7. *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. It then holds that any $\Delta \to \mathbb{R}$ function U of the form*

$$U_g^s(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma), & \text{if } \beta \geq 1/2, \\ \alpha+\beta-1+g(1-\beta,1-\alpha), & \text{if } \beta < 1/2, \end{cases}$$

is a self-dual upper bound function.

Note that the function g in Proposition 5.7 has the same properties as the function g in Proposition 5.5.

Proposition 5.8. *A reciprocal relation Q on A is cycle-transitive w.r.t. a self-dual upper bound function of type U_g^s if and only if it is g-isostochastic transitive.*

In particular, a reciprocal relation Q is $T_{\mathbf{M}}$-transitive if and only if

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2) \Rightarrow Q(a,c) = \max(Q(a,b), Q(b,c)),$$

for any $(a,b,c) \in A^3$. Note that this is formally the same as (5.10) with the difference that in the latter case Q was only $\{0,1/2,1\}$-valued.

If the function g is a commutative, associative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$, then the $[0, 1]^2 \to [0, 1]$ mapping S_g defined by

$$S_g(x, y) = 2g\left(\frac{1+x}{2}, \frac{1+y}{2}\right) - 1$$

is a t-conorm. For the self-dual upper bound function U_E, the associated t-conorm S_E is given by

$$S_E(x, y) = \frac{x + y}{1 + xy},$$

which belongs to the parametric Hamacher t-conorm family, and is the co-copula of the Hamacher t-norm with parameter value 2 [37].

We have shown that the cycle-transitivity and FG-transitivity frameworks cannot easily be translated into one another, which underlines that these are two essentially different frameworks [6].

5.4.3 Comparison of Random Variables

5.4.3.1 Dice-Transitivity of Winning Probabilities

Consider three dice A, B and C which, instead of the usual numbers, carry the following integers on their faces:

$$A = \{1, 3, 4, 15, 16, 17\},\ B = \{2, 10, 11, 12, 13, 14\},\ C = \{5, 6, 7, 8, 9, 18\}.$$

Denoting by $\mathcal{P}(X, Y)$ the probability that dice X wins from dice Y, we have $\mathcal{P}(A, B) = 20/36$, $\mathcal{P}(B, C) = 25/36$ and $\mathcal{P}(C, A) = 21/36$. It is natural to say that dice X is strictly preferred to dice Y if $\mathcal{P}(X, Y) > 1/2$, which reflects that dice X wins from dice Y in the long run (or that X statistically wins from Y, denoted $X >_s Y$). Note that $\mathcal{P}(Y, X) = 1 - \mathcal{P}(X, Y)$ which implies that the relation $>_s$ is asymmetric. In the above example, it holds that $A >_s B$, $B >_s C$ and $C >_s A$: the relation $>_s$ is not transitive and forms a cycle. In other words, if we interpret the probabilities $\mathcal{P}(X, Y)$ as constituents of a reciprocal relation on the set of alternatives $\{A, B, C\}$, then this reciprocal relation is even not weakly stochastic transitive.

This example can be generalized as follows: we allow the dice to possess any number of faces (whether or not this can be materialized) and allow identical numbers on the faces of a single or multiple dice. In other words, a generalized dice can be identified with a multiset of integers. Given a collection of m such generalized dice, we can still build a reciprocal relation Q containing the *winning probabilities* for each pair of dice [19]. For any two such dice A and B, we define

$$Q(A, B) = \mathcal{P}\{A \text{ wins from } B\} + \frac{1}{2}\mathcal{P}\{A \text{ and } B \text{ end in a tie}\}.$$

The dice or integer multisets may be identified with independent discrete random variables that are uniformly distributed on these multisets (i.e. the probability of an integer is proportional to its number of occurrences); the reciprocal relation Q may be regarded as a quantitative description of the pairwise comparison of these random variables.

In the characterization of the transitivity of this reciprocal relation, a type of cycle-transitivity, which can neither be seen as a type of C-transitivity, nor as a type of FG-transitivity, has proven to play a predominant role. For obvious reasons, this new type of transitivity has been called dice-transitivity.

Definition 5.18. Cycle-transitivity w.r.t. the upper bound function U_D defined by

$$U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma,$$

is called *dice-transitivity*.

Dice-transitivity is closely related to $T_{\mathbf{P}}$-transitivity. However, it uses the quantities β and γ instead of the quantities α and β, and is therefore less restrictive. Dice-transitivity can be situated between $T_{\mathbf{L}}$-transitivity and $T_{\mathbf{P}}$-transitivity, and also between $T_{\mathbf{L}}$-transitivity and moderate stochastic transitivity.

Proposition 5.9. [19] *The reciprocal relation generated by a collection of generalized dice is dice-transitive.*

5.4.3.2 A Method for Comparing Random Variables

Many methods can be established for the comparison of the components (random variables, r.v.) of a random vector $(X_1, \ldots, X_n)$, as there exist many ways to extract useful information from the joint cumulative distribution function (c.d.f.) $F_{X_1,\ldots,X_n}$ that characterizes the random vector. A first simplification consists in comparing the r.v. two by two. It means that a method for comparing r.v. should only use the information contained in the bivariate c.d.f. F_{X_i,X_j}. Therefore, one can very well ignore the existence of a multivariate c.d.f. and just describe mutual dependencies between the r.v. by means of the bivariate c.d.f. Of course one should be aware that not all choices of bivariate c.d.f. are compatible with a multivariate c.d.f. The problem of characterizing those ensembles of bivariate c.d.f. that can be identified with the marginal bivariate c.d.f. of a single multivariate c.d.f. is known as the *compatibility problem* [41].

A second simplifying step often made is to bypass the information contained in the bivariate c.d.f. to devise a comparison method that entirely relies on the one-dimensional marginal c.d.f. In this case there is even not a compatibility problem, as for any set of univariate c.d.f. F_{X_i}, the product $F_{X_1} F_{X_2} \cdots F_{X_n}$ is a valid joint c.d.f., namely the one expressing the independence of the r.v. There are many ways to compare one-dimensional c.d.f., and by far the simplest one is the method that builds a partial order on the set of r.v. using the principle of first order stochastic dominance [49]. It states that a r.v. X is weakly preferred to a r.v. Y if for all $u \in \mathbb{R}$ it holds that $F_X(u) \leq F_Y(u)$. At the extreme end of the chain of simplifications are the methods that compare r.v. by means of a characteristic or a function of some characteristics derived from the one-dimensional marginal c.d.f. The simplest example is the weak order induced by the expected values of the r.v.

Proceeding along the line of thought of the previous section, a random vector $(X_1, X_2, \ldots, X_m)$ generates a reciprocal relation by means of the following recipe.

Definition 5.19. Given a random vector $(X_1, X_2, \ldots, X_m)$, the binary relation Q defined by

$$Q(X_i, X_j) = \mathcal{P}\{X_i > X_j\} + \frac{1}{2}\mathcal{P}\{X_i = X_j\}$$

is a reciprocal relation.

For two discrete r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k} p_{X_i,X_j}(k,k),$$

with p_{X_i,X_j} the joint probability mass function (p.m.f.) of (X_i, X_j). For two continuous r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{x} f_{X_i,X_j}(x,y)\,\mathrm{d}y,$$

with f_{X_i,X_j} the joint probability density function (p.d.f.) of (X_i, X_j).

For this pairwise comparison, one needs the two-dimensional marginal distributions. Sklar's theorem [41,46] tells us that if a joint cumulative distribution function F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C_{ij} such that for all x, y:

$$F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y)).$$

If X_i and X_j are continuous, then C_{ij} is unique; otherwise, C_{ij} is uniquely determined on $\mathrm{Ran}(F_{X_i}) \times \mathrm{Ran}(F_{X_j})$.

As the above comparison method takes into account the bivariate marginal c.d.f. it takes into account the dependence of the components of the random vector. The information contained in the reciprocal relation is therefore much richer than if, for instance, we would have based the comparison of X_i and X_j solely on their expected values. Despite the fact that the dependence structure is entirely captured by the multivariate c.d.f., the pairwise comparison is only apt to take into account pairwise dependence, as only bivariate c.d.f. are involved. Indeed, the bivariate c.d.f. do not fully disclose the dependence structure; the r.v. may even be pairwise independent while not mutually independent.

Since the copulas C_{ij} that couple the univariate marginal c.d.f. into the bivariate marginal c.d.f. can be different from another, the analysis of the reciprocal relation and in particular the identification of its transitivity properties appear rather cumbersome. It is nonetheless possible to state in general, without making any assumptions on the bivariate c.d.f., that the probabilistic relation Q generated by an arbitrary random vector always shows some minimal form of transitivity.

Proposition 5.10. [8] *The reciprocal relation Q generated by a random vector is $T_{\mathbf{L}}$-transitive.*

5.4.3.3 Artificial Coupling of Random Variables

Our further interest is to study the situation where abstraction is made that the r.v. are components of a random vector, and all bivariate c.d.f. are enforced to depend in the same way upon the univariate c.d.f., in other words, we consider the situation of all copulas being the same, realizing that this might not be possible at all. In fact, this simplification is equivalent to considering instead of a random vector, a collection of r.v. and to artificially compare them, all in the same manner and based upon a same copula. The pairwise comparison then relies upon the knowledge of the one-dimensional marginal c.d.f. solely, as is the case in stochastic dominance methods. Our comparison method, however, is not equivalent to any known kind of stochastic dominance, but should rather be regarded as a graded variant of it (see also [7]).

The case $C = T_{\mathbf{P}}$ generalizes Proposition 5.9, and applies in particular to a collection of independent r.v. where all copulas effectively equal $T_{\mathbf{P}}$.

Proposition 5.11. [18, 19] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{P}}$ is dice-transitive,* i.e. *it is cycle-transitive w.r.t. the upper bound function given by $U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma$.*

We discuss next the case when using one of the extreme copulas to artificially couple the r.v. In case $C = T_{\mathbf{M}}$, the r.v. are coupled comonotonically. Note that this case is possible in reality.

Proposition 5.12. [16, 17] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{M}}$ is cycle-transitive w.r.t. to the upper bound function U given by $U(\alpha, \beta, \gamma) = \min(\beta + \gamma, 1)$. Cycle-transitivity w.r.t. the upper bound function U is equivalent to $T_{\mathbf{L}}$-transitivity.*

In case $C = T_{\mathbf{L}}$, the r.v. are coupled countermonotonically. This assumption can never represent a true dependence structure for more than two r.v., due to the compatibility problem.

Proposition 5.13. [16, 17] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{L}}$ is partially stochastic transitive,* i.e. *it is cycle-transitive w.r.t. to the upper bound function defined by $U_{ps}(\alpha, \beta, \gamma) = \max(\beta, \gamma) = \gamma$.*

The proofs of these propositions were first given for discrete uniformly distributed r.v. [16, 19]. It allowed for an interpretation of the values $Q(X_i, X_j)$ as winning probabilities in a hypothetical dice game, or equivalently, as a method for the pairwise comparison of ordered lists of numbers. Subsequently, we have shown that as far as transitivity is concerned, this situation is generic and therefore characterizes the type of transitivity observed in general [17, 18].

The above results can be seen as particular cases of a more general result.

Proposition 5.14. [8] *Let C be a commutative copula such that for any $n > 1$ and for any $0 \leq x_1 \leq \cdots \leq x_n \leq 1$ and $0 \leq y_1 \leq \cdots \leq y_n \leq 1$, it holds that*

$$\sum_i C(x_i, y_i) - \sum_i C(x_{n-2i}, y_{n-2i-1}) - \sum_i C(x_{n-2i-1}, y_{n-2i})$$
$$\leq C\Bigg(x_n + \sum_i C(x_{n-2i-2}, y_{n-2i-1}) - \sum_i C(x_{n-2i}, y_{n-2i-1}),$$
$$y_n + \sum_i C(x_{n-2i-1}, y_{n-2i-2}) - \sum_i C(x_{n-2i-1}, y_{n-2i})\Bigg), \quad (5.17)$$

where the sums extend over all integer values that lead to meaningful indices of x and y. Then the reciprocal relation Q generated by a collection of random variables pairwise coupled by C is cycle-transitive w.r.t. to the upper bound function U^C defined by:

$$U^C(\alpha, \beta, \gamma) = \max(\beta + C(1-\beta, \gamma), \gamma + C(\beta, 1-\gamma)).$$

Inequality (5.17) is called the *twisted staircase condition* and appears to be quite complicated. Although its origin is well understood [8], it is not yet clear for which commutative copulas it holds. We strongly conjecture that it holds for all Frank copulas.

5.4.3.4 Comparison of Special Independent Random Variables

Dice-transitivity is the generic type of transitivity shared by the reciprocal relations generated by a collection of independent r.v. If one considers independent r.v. with densities all belonging to one of the one-parameter families in Table 5.1, then the corresponding reciprocal relation shows the corresponding type of cycle-transitivity listed in Table 5.2 [18].

Note that all upper bound functions in Table 5.2 are self-dual. More striking is that the two families of power-law distributions (one-parameter subfamilies of the two-parameter Beta and Pareto families) and the family of Gumbel distributions all yield the same type of transitivity as exponential distributions, namely cycle-transitivity w.r.t. the self-dual upper bound function U_E, or, in other words, multiplicative transitivity.

Table 5.1 Parametric families of continuous distributions

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, \infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, \infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, \infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R},\ a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$(e^{-\lvert x-\lambda\rvert/\mu})/(2\mu)$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, \infty[$
Normal	$(e^{-(x-\lambda)^2/2\sigma^2})/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}, \sigma > 0$	$x \in]-\infty, \infty[$

Table 5.2 Cycle-transitivity for the continuous distributions in Table 5.1

Name	Upper bound function $U(\alpha, \beta, \gamma)$
Exponential Beta Pareto Gumbel	$\alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$
Uniform	$\begin{cases} \beta + \gamma - 1 + \frac{1}{2}\left[\max(\sqrt{2(1-\beta)} + \sqrt{2(1-\gamma)} - 1, 0)\right]^2 & \beta \geq 1/2 \\ \alpha + \beta - \frac{1}{2}\left[\max(\sqrt{2\alpha} + \sqrt{2\beta} - 1, 0)\right]^2, & \beta < 1/2 \end{cases}$
Laplace	$\begin{cases} \beta + \gamma - 1 + f^{-1}(f(1-\beta) + f(1-\gamma)), & \beta \geq 1/2 \\ \alpha + \beta - f^{-1}(f(\alpha) + f(\beta)), & \beta < 1/2 \end{cases}$ with $f^{-1}(x) = \frac{1}{2}\left(1 + \frac{x}{2}\right)e^{-x}$
Normal	$\begin{cases} \beta + \gamma - 1 + \Phi(\Phi^{-1}(1-\beta) + \Phi(1-\gamma)), & \beta \geq 1/2 \\ \alpha + \beta - \Phi(\Phi^{-1}(\alpha) + \Phi^{-1}(\beta)), & \beta < 1/2 \end{cases}$ with $\Phi(x) = (\sqrt{2\pi})^{-1}\int_{-\infty}^{x} e^{-t^2/2}dt$

In the cases of the unimodal uniform, Gumbel, Laplace and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions. In [18], we have shown that the corresponding reciprocal relation is in that case moderately stochastic transitive.

5.4.4 Mutual Ranking Probabilities in Posets

Consider a finite poset $(P, \leq)$ with $P = \{x_1, \ldots, x_n\}$. A linear extension of P is an order-preserving permutation of its elements (hence, also a ranking of the elements compatible with the partial order). We denote by $p(x_i < x_j)$ the fraction of linear extensions of P in which x_i precedes x_j. If the space of all linear extensions of P is equipped with the uniform measure, the position of x in a linear extension can be regarded as a discrete random variable X with values in $\{1, \ldots, n\}$.

Since $p(x_i < x_j) = \mathcal{P}\{X_i < X_j\}$, the latter value is called a mutual rank probability. Note that P uniquely determines a random vector $X = (X_1, \ldots, X_n)$ with multivariate distribution function $F_{X_1,\ldots,X_n}$, whereas the mutual rank probabilities $p(x_i < x_j)$ are then computed from the bivariate marginal distributions F_{X_i,X_j}. Note that for general P, despite the fact that the multivariate distribution function $F_{X_1,\ldots,X_n}$, or equivalently, the n-dimensional copula, can be very complex, certain pairwise couplings are trivial. Indeed, if in P it holds that $x_i < y_j$, then x_i precedes y_j in all linear extensions and X_i and X_j are comonotone, which means that X_i and X_j are coupled by (a discretization of) $T_{\mathbf{M}}$. For pairs of elements in P that are incomparable, the bivariate couplings can vary from pair to pair. The copulas are not all equal to $T_{\mathbf{L}}$, as can be seen already from the example where P is an antichain with three elements.

Definition 5.20. Given a poset $P = \{x_1, \ldots, x_n\}$, consider the reciprocal relation Q_P defined by

$$Q_P(x_i, x_j) = \mathcal{P}\{X_i < X_j\} = p(x_i < x_j). \tag{5.18}$$

The problem of probabilistic transitivity in a finite poset P was raised by Fishburn [25]. It can be rephrased as follows: find the largest function $\delta : [0,1]^2 \to [0,1]$ such that for any finite poset and any x_i, x_j, x_k in it, it holds that

$$\delta(Q_P(x_i, x_j), Q_P(x_j, x_k)) \leq Q_P(x_i, x_k).$$

Fishburn has shown in particular that

$$\big(Q_P(x_i, x_j) \geq u \ \wedge\ Q_P(x_j, x_k) \geq u\big) \ \Rightarrow\ Q_P(x_i, x_k) \geq u$$

for $u \geq \rho \approx 0.78$.

A non-trivial lower bound for δ was obtained by Kahn and Yu [36] via geometric arguments. They have shown that $\delta^* \leq \delta$ with δ^* the conjunctor

$$\delta^*(u,v) = \begin{cases} 0, & \text{if } u + v < 1 \\ \min(u,v), & \text{if } u + v - 1 \geq \min(u^2, v^2) \\ \dfrac{(1-u)(1-v)}{(1-\sqrt{u+v-1})^2}, & \text{elsewhere} \end{cases}$$

Interestingly, the particular form of this function allows to state δ^*-transitivity also as

$$\begin{aligned} &Q_P(x_i, x_j) + Q_P(x_j, x_k) \geq 1 \\ &\quad \Rightarrow Q_P(x_i, x_k) \geq \delta^*(Q_P(x_i, x_j), Q_P(x_j, x_k)), \end{aligned}$$

which can be seen to be closely related to stochastic transitivity. Moreover, δ^*-transitivity can be positioned within the cycle-transitivity framework.

Proposition 5.15. [9] δ^*-Transitivity implies cycle-transitivity w.r.t. the upper bound function U_P defined by

$$U_P(\alpha, \beta, \gamma) = \alpha + \gamma - \alpha\gamma,$$

and hence also dice-transitivity.

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