Chapter 4
Robustness in Multi-criteria Decision Aiding

Hassene Aissi and Bernard Roy

Abstract After bringing precisions to the meaning we give to several of the terms used in this chapter (e.g., robustness, result, procedure, method, etc.), we highlight the principal characteristics of most of the publications about robustness. Subsequently, we present several partial responses to the question, "Why is robustness a matter of interest in Multi-Criteria Decision Aiding (MCDA)?" (see Section 4.2). Only then do we provide an outline for this chapter. At this point, we introduce the concept of *variable setting*, which serves to connect what we define as the formal representation of the decision-aiding problem and the real-life decisional context. We then introduce five typical problems that will serve as reference problems in the rest of the chapter. Section 4.3 deals with recent approaches that involve a single robustness criterion completing (but not replacing) a preference system that has been defined previously, independently of the robustness concern. The following section deals with approaches in which the robustness concern is modelled using several criteria. Section 4.5 deals with the approaches in which robustness is considered other than by using one or several criteria to compare the solutions. These approaches generally involve using one or several properties destined to characterize the robust solution or to draw robust conclusions. In the last three sections, in addition to describing the appropriate literature, we suggest some avenues for new development and in some cases, we present some new approaches.

Keywords Robustness · Multi-criteria methods · Decision aiding · MAUT · ELECTRE methods · Mathematical programming

H. Aissi (✉) and B. Roy
LAMSADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75116, Paris, France
e-mail: aissi@lamsade.dauphine.fr; roy@lamsade.dauphine.fr

M. Ehrgott et al. (eds.), *Trends in Multiple Criteria Decision Analysis*, International Series in Operations Research and Management Science 142,
DOI 10.1007/978-1-4419-5904-1_4,

4.1 Introduction

In the field of Multi-Criteria Decision Aiding (MCDA), the subject of robustness is increasingly present in scientific journals. This subject is also present more and more in much of the less formal works done by companies applying operational research tools about concrete decision-aiding problems. In MCDA, the multiple meanings accorded to the term "robust" are open to debate. This subject is discussed in detail in the Newsletter of the European Working Group "Multiple Criteria Decision Aiding" [25] in the contributions of Aloulou et al. (n° 12, 2005), Dias (n° 13, 2006), Fernandez Barberis (n° 13, 2006), Pictet (n° 15, 2007), Rios Insua (n° 9, 2004), Rosenhead (n° 6, 2003), Roy (n° 6, 2002), Roy (n° 8, 2003), Sayin (n° 11, 2005), Sevaux, Sörensen (n° 10, 2004) and Vincke (n° 8, 2003). This series of perspectives highlights the polysemic character of the notion of robustness. This polysemic character is primarily due to the fact that, depending on the situation, this notion can be similar to, and sometimes compared to, the notion of flexibility, stability, sensitivity and even equity.

In this chapter, we use the term **robust** as a qualifier meaning **a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts, notably the degradation of the properties that must be maintained** (see Roy Roy [29]). The research on robustness seeks to insure this capacity to the greatest degree possible. Consequently, robustness stems from a process that responds to a concern: a need for resistance or self-protection.

For this reason, we prefer to use the expression **robustness concern**, rather than robustness analysis because the latter can give the impression of work done a posteriori, as is the case with sensitivity analysis, for example. Robustness more often involves a concern that must be taken into account a priori, when formulating the problem. (Of course, this does not exclude the use of sensitivity analysis to respond to such a concern, if necessary.)

In the following section, we will endeavour to explain in detail the multiple reasons for the existence of the robustness concern. Our perspective, like that of this book, is primarily multi-criteria. We will show that robustness and multiple criteria can be expressed in a variety of forms. At this point, we will present the outline of the rest of the chapter. But, before doing so, it is necessary to provide some important explanations and call back to memory some basic notions.

First, let us explain briefly the meaning that we assign to certain terms (see Roy Roy [29] for more details). We designate as **procedure** P a set of instructions used for handling a problem. **Result** is used to refer to the outcome of applying P to a rigorously formulated problem. This result can have diverse forms: solutions, bundles of solutions possessing the required properties, or simple statements (e.g., "there is no solution with this property" or "this solution is non-dominated"). Like Vincke [42, 43], we use **method** M to designate a family of $\hat{P}$ procedures that have enough similar features (i.e., structure, process, concept, action or hypothesis) that they can only be differentiated by the value attributed to certain parameters or by diverse options dealing with the way certain rules are formulated (e.g., the role of the different criteria).

Most publications dealing with robustness use the term "robust" to characterize solutions. This term is also used to qualify a statement (or a conclusion), a method (see Billaut et al. [8], Roy [30, 32], Vincke [43], for example).

Among works targeting the search for robust solutions, many of them have the following characteristics (see Roy [32]):

(i) The problem studied is one of the standard OR models: job shop, flow shop, knapsack, spanning tree, shortest path, travelling salesman, maximum flow, maximum stable, p-median and p-centre in location and/or the standard mathematical programming models, notably linear programming. These problems are studied in a mono-criterion context.
(ii) A **scenario** set is defined by considering the value of some parameters as uncertain. These parameters are either those present in the definition of the optimization criterion, or those that intervene in certain constraints. Such parameters are assumed capable of taking a few or all of the values in one interval. A scenario is defined by attributing one of the possible values to each of these uncertain parameters.
(iii) Feasible solutions that optimize a criterion $r(x)$, used to indicate the relative robustness of solution x, are qualified as robust. Frequently, $r(x)$ is one of the three measures introduced by Kouvelis and Yu [23]. Since we will refer to them in the rest of this chapter, the definitions of these indicators are given below.

These measures are based on the unique optimization criterion v of the standard model considered. This criterion attributes a value $v_s(x)$ to x in scenario s. Here, optimum is assumed to mean maximum.

- Absolute robustness: The robustness measure that must be maximized is defined by the value of the solution in the worst scenario: $r(x) = \min_s\{v_s(x)\}$.
- Absolute deviation: The robustness measure that must be minimized is defined by the value of the absolute regret in the worst scenario, due to the fact that the solution differs from that which would be optimal in this scenario: $r(x) = \max_s \{v_s^* - v_s(x)\}$, where v_s^* is the value of the optimal solution in scenario s.
- Relative deviation: The robustness measure that must be minimized is defined by the value of the relative regret in the worst scenario, due to the fact that the solution is not optimal in this scenario: $r(x) = \max_s \left\{\frac{v_s^* - v_s(x)}{v_s^*}\right\}$.

Let us underline that these measures correspond to the classical and criteria in decision under uncertainty. The complexity and the approximation of the underlying problems are studied in Aissi et al. [1].

4.2 Why Is Robustness of Interest in MCDA?

In our opinion, in decision aiding, the desire to take our own ignorance into account as much as possible explains why the robustness concern exists. From this perspective, it is important to remember that the decisions for which decision aiding is performed will be:

1. executed in a real-life context that may not correspond exactly to the model on which the decision aiding is based; and
2. judged in terms of a system of values that will appear to be pertinent (and not necessarily stable) for a future that may not be well defined; as a result, this system of values may not correspond exactly to the one used to create and exploit the model.

These are two of the possible reasons for a non-perfect conformity, and thus a gap between:

- on the one hand, the **formal representation** (FR), including the model and the processing procedures that are applied to it; and
- on the other hand, **the real-life context** (RLC) in which decisions will be made, executed and judged.

"State of nature" could be used instead of real-life context, but because the latter expression refers to real life, it seems more appropriate in the context of decision aiding than the expression referring to nature.

In decision aiding, it is important to try to take into account the vague approximations and zones of ignorance responsible for the formal representation's non-perfect conformity to the real-life context: FR $\neq$ RLC. In this section, we illustrate these vague approximations and zones of ignorance, though without any pretence of exhaustivity.

In the formal representation, the vague approximations and zones of ignorance against which the robustness concern attempts to protect appear in the form of **frailty points** (Roy [29]). To highlight these frailty points, the formal representation, adopted as the problem formalization, can be examined from four different perspectives:

1. *The way that imperfect knowledge is treated*: imperfect knowledge may be ignored, for example by treating uncertain data as certain, or it may be modelled using elements of arbitrariness, for example using probability distribution, fuzzy numbers or thresholds. In a third possibility, imperfect knowledge may be incorporated in the procedure when the latter has been conceived to take into account imprecise and/or ambiguous data even non-necessarily coherent and complete.
2. *The preferential attribution of questionable, even inappropriate, meaning to certain data*: preferential attributions of meaning can be made by moving from qualitative or numerical analysis to quantitative analysis without justification, or by attributing inappropriate meanings to so-called objective measurements, using data generated through a questioning procedure.
3. *The modelling of complex aspects of reality (notably introduction of parameters), which are difficult to grasp because imperfectly defined*: the choice of model parameters (e.g., sets of weights, capacity indicators, utility functions, reference levels or aspiration levels) has a great influence.
4. *The way that essentially technical parameters and/or selection rules with little or no concrete meaning are introduced*: these parameters are notably those imposed by the processing procedure, for example, the minimum deviation guaranteeing

the strict nature of the inequality, the bounds limiting the domain of investigation or the parameters required by a metaheuristic. These rules can be for instance related to the way the selection of a solution among several ones is conceived (solution in the neighbourhood of an optimum).

Taking a robustness concern into account implies first identifying the frailty points in FR. These points obviously depend on the way that the decision-aiding problem was formulated and modelled. They can also depend on the processing procedures that will be used. In general, these frailty points appear to be connected to sources of contingency, uncertainty or arbitrariness (see Roy [29, 32], Section 2.2). We believe that, used in conjunction with these sources (which are on a higher hierarchical level), the four perspectives described above can help OR researchers confronted with real-world problems to inventory these frailty points.

In many cases, establishing an inventory by concentrating only on the elements in the FR that reflect uncertainty can lead to the exclusion of a certain number of frailty points. In fact, the term "uncertainty" does not cover all the forms of vague approximations and zones of ignorance that must be resisted or protected against. For example, approximations due to simplifications, imperfect determinations, or arbitrary options are not uncertain, nor are zones of ignorance due to imperfect knowledge about the complexity of the phenomena or the systems of values.

Limiting robustness concern to considerations of uncertainty generally accompanies the use of scenario-based concepts for understanding the relationship between the formal representation and the real-life context (see end of Section 4.1, ii). From this somewhat limited viewpoint, the search for robustness is based on defining a finite or infinite set of scenarios. **This set must allow the different real-life contexts that should be considered to be incorporated into the formal representation**: it is the uncertainty with which real values are assigned to certain data or parameters that makes it necessary to consider these different realities. Each scenario is thus defined by attributing a precise value to each of the data elements and parameters.

Roy [28, 29] showed that, in order to respond to the reasons for the existence of robustness concern, it is preferable to go beyond the limited viewpoint described above. To avoid limiting the search for robustness to a simple consideration of uncertainty, the scenario concept must be left behind, especially since this concept has the additional disadvantage of causing confusion in certain professional milieus. Roy proposed replacing this scenario view of robustness with a view centred on a **version** that is strongly connected to the decision-aiding problem formulation. Each version represents a reality that should be considered and is defined using a combination of the options related to the model's frailty points. In some cases, the version set thus defined is not enough to clarify the relationship between FR and RLC, primarily because the robustness concern can make it necessary to take into account all the processing procedures in a certain family, and not just a single one. The frailty points that make it necessary to take such a family into account can be due both to the technical parameters that are part of the procedure definition and to the personality of the experts who are in charge of processing the model (see Roy [29]). It is even possible that the robustness concern relates only to a single version of the problem formulation, to which the entire procedure family must be applied.

This wider view of the robustness concern can make it appropriate to replace the scenario set by a set comprised of all the pertinent pairs (**procedure**, **version**). Such a pair (p, v) is defined by a set of values and options that characterize the procedure p and the version v that are under consideration. In the following, we denote any pertinent pair as $s = (p, v)$ and refer to this pair as a **variable setting**, an expression primarily employed in reliability engineering (see Salazar and Rocco [37] for example).

We denote the set of pertinent variable settings S. When the robustness concern is based on a single procedure, S is simply a set of versions, and in many cases, a set of scenarios. However, when the focus is on the robustness of a method as opposed to a single version v of a problem, S refers to the family $\hat{P}$ of procedures that characterize this method. In any case, S is the intermediary through which the formal representation (FR) incorporates the different real-life contexts (RLC) that the robustness concern requires be taken into account.

$\forall s = (p, v) \in S$, the procedure p applied to the version v produces a result, $R(s)$. This result can take extremely varied, non-exclusive forms, as suggested in the introduction.

Once S is finite, it is possible, in some cases, to associate a subjective probability to each element $s \in S$. This probability must reflect the chances that this variable setting s will be able to correctly describe what the RLC will be. In this case, S is said to be probabilized.

In order to illustrate the robustness concern in **multi-criteria** decision aiding (MCDA) more concretely, a few problem types, chosen more or less arbitrarily from those that exist, are briefly described below.

Problem 1. Choosing a supplier following a Call to Bid for the acquisition and installation of new equipment

Suppose that around 15 bids were received and that each one was evaluated according to the following criteria: cost; deadline; two satisfaction levels, each one related to a specific property and a possible veto effect; and the confidence that the supplier will respect the deadlines and the specifications. Here, the vague approximations and the zones of ignorance affect the way that the bids received are evaluated in terms of these five criteria, especially the last three. They also affect the role that each criteria plays in the final decision (i.e., the relative weights and the possibility of a veto). Thus, for some of the responses, an analyst might retain not just a single evaluation of given criteria, but two or three. By combining these evaluations, he/she can define a set V of the versions of the problem. If, for example, the analyst chooses a decision-aiding method like ELECTRE, he/she might decide to take several sets of weights into account, and once a veto threshold criterion is justified, to retain two or three values, thus defining a set P of procedures. S would thus be defined by the Cartesian product $P \times V$. It would also be possible to consider that the different sets of weights allow differentiating versions instead of procedures. The definition of S would be unchanged.

The decision maker may expect the analyst to recommend as few bid proposals as the vague approximations and zones of ignorance permit, along with the arguments that justify why each of the bids was selected. These arguments must, for example, allow the decision maker to understand under what conditions (i.e., the hypotheses related to the vague approximations and zones of ignorance) the bid in question is at least as acceptable as the others, while also explaining the risks taken if these conditions are not satisfied.

Problem 2. Setting the structural characteristics of a water treatment system for a municipality that currently has no system

Determining the optimal value for these structural characteristics requires sufficiently precise knowledge of the needs that will have to be satisfied throughout the expected life of the system. These needs are, in fact, not very well known because they depend on multiple factors, including but not limited to the evolution of the population, of the population's use of the system, and of the laws regulating system discharges, as well as the arrival of new activities in the sector. If the analyst tries to formulate the problem in terms of the optimization of a single criterion, this criterion must not take into account only the provisional costs of constructing and maintaining the system. It is also necessary to take into account the cost of adapting the system if the municipality's needs were underestimated and cannot be satisfied without modifying the initial structural characteristics. In addition, the analyst must take into account the negative consequences of budget overruns for the initial construction and maintenance operations if the municipality's needs were overestimated. This example shows that the formulation of a single optimization criterion can run up against serious difficulties. Even if the OR researcher manages to overcome these difficulties and develops a suitable scenario set, this formulation of the decision-aiding problem may not respond to the decision maker's expectations. Here, the robustness concern stems from a desire to be able to justify the decision in the future, if necessary, as well to avoid any cases in which needs were left critically unsatisfied, except in unforeseeable circumstances. This example shows that, in certain cases, the robustness concern may play a crucial role in the formulation of the decision-aiding problem, taking into account multiple criteria.

Problem 3. Scheduling airline flight crews for all the flights of an airline company

The robustness concern in this example is the need to take into account unanticipated crew absences (e.g., illnesses, injuries during a mission) and/or flight plan modifications (e.g., a plane type other than the one expected). The question that must be answered is how can these risks be handled given the potential conflicts between the following two points of view:

- The point of view of the airline company, which seeks an economic optimum in the context of highly complex legislation that leaves very little room for interpretation;
- The point of view of the crew, which includes the desires that the crew would like to see satisfied while avoiding scheduling perturbations that would make the crew's life difficult.

Problem 4. Controlling the execution of a vast project

Diverse software using a variety of methods for establishing a provisional schedule of execution are available. To use these software, it is necessary to take multiple data elements into consideration (e.g., task duration, supply delivery deadlines, skilled worker availability, weather, etc.). However, the real-life context of the project's execution may not correspond to the initial values predicted for each of these data elements. If this is the case, both the execution cost and the completion time can undergo tremendous modifications. Thus, execution cost and the completion time are two criteria which should be taken into account when choosing a provisional schedule liable to best withstand the vague approximations and zones of ignorance that affect these data, where "best withstand" means "has the potential to allow acceptable local adaptations".

Problem 5. Reliability of a complex system

Let us consider the case of a complex system whose reliability depends on the values that will be attributed to certain variables during the design phase. In this system, the relationship between the values retained and the reliability of each of the large system components is highly complex and thus imperfectly known; furthermore, the relationship between these values and the reliability of overall system is even less well known. In these conditions, in order to enlighten the choice of these values during the design phase, it may be appropriate to take into account as many reliability criteria as there are large system components.

The above examples (to which we will refer later in the chapter) underline the often multi-criteria character of the preference models that can be used to guide the choice of a solution. However, in these examples, no criterion for characterizing the relative robustness of a solution has been considered. This is generally how preferences of a decision maker are modelled, especially when there is a single criterion. To take the robustness concern into account, one of the three following families of approaches can be considered:

(a) Define a mono-dimensional robustness measure that will make sense of such statements as "solution x is at least as robust as solution y". This measure is then used to introduce a new criterion linked to a preference model that has been defined previously without taking the robustness concern into account.
(b) Apprehend robustness multi-dimensionally, in such a way that it is expressed through several criteria, not just one. These criteria can then constitute the preference model itself, or as in (a) above, they can complete an initial preference model that has no criterion to express the robustness concern.
(c) Apprehend robustness other than by describing one or more criteria designed to allow the solutions to be compared. This last family of approaches leads, more or less explicitly, to make intervene one or more properties intended to characterize the solutions that are qualified as robust. These properties can also serve to establish robust conclusions. Defining these properties can, in some cases, bring one or more robustness measures into play. Thus, this family of approaches serves as a constraint and not as a criterion of comparison.

The three sections that follow deal with each of the above families of approaches. In these last three sections, in addition to describing the appropriate literature, we suggest some avenues for new development and in some cases, we present some new approaches.

4.3 Robustness in MCDA: Mono-dimensional Approaches

4.3.1 Characterizing Mono-dimensional Approaches

In this section, we examine the approaches that lead to apprehending the robustness by completing a preference model that was previously defined with no direct link to the robustness concern. The robustness measure $r(x)$ is introduced to give meaning to the statement "solution x is at least as robust as solution y".

At the end of the introduction, we called back to memory three measures defined by Kouvelis and Yu [23]. These measures are appropriate when the previously defined preference model is mono-criterion. Most of the works that have used one of these three measures have done so by substituting the robustness criterion induced by the chosen robustness measure for the initial criterion. This kind of approach remains mono-criterion and consequently is not within the scope of this chapter.

In the following sections, we explore the works or new avenues of research that use this robustness measure to define a new criterion, which is added to the initially defined preference model. We first consider two cases, one in which the initial model is mono-criterion (see Section 4.3.2) and one in which it is multi-criteria (see Section 4.3.3). Then, in Section 4.3.4, we present a new approach that, under the specified conditions, can be applied to both the mono-criterion and multi-criteria cases.

4.3.2 With an Initial Mono-criterion Preference Model

In this kind of approach, two criteria are considered to guide the choice of a solution. In addition to the single preference criterion (e.g., gain, cost, duration), a robustness criterion is added to take the different frailty points inherent to the formal representation (FR) into account. Since these two criteria are in conflict, in all but certain particularly auspicious cases, this naturally leads to a consideration of the efficient frontier or an approximation of it.

By hypothesis, the preference criterion is intended to attribute a value $v(x)$ to each solution x by ignoring the vague approximations and the zones of ignorance against which robustness is supposed to withstand. To define $v(x)$ in such conditions, it is possible to use the values $v_s(x)$ that this criterion attributes to solution x with the variable setting $s \in S$. For example, $v(x)$ can be the median or the arithmetic mean of the values $v_s(x)$, or even their expected value if S is probabilized. It is also possible to set $v(x) = v_{s_0}(x)$, where s_0 is a variable setting characterizing a description of the reality chosen as reference for its high credibility. In these conditions, the robustness measure can be one of the robustness measures proposed by Kouvelis and Yu (see end of Section 4.1) or any other criterion appropriate for dealing with the impact of imperfect knowledge.

In the classic mono-criterion approaches that take into account one of the criteria proposed by Kouvelis and Yu, robustness focuses on the worst case and assigns no importance to the solution performances in the other variable settings. The approaches presented in this section try to remedy these drawbacks by simultaneously considering the performance in the worst case and in the median or average case. Thus, these approaches make it possible for decision makers to choose from several compromise solutions.

Below, we present several papers from the literature that use this kind of approach.

Chen et al. [9] studied the problem of industrial system design. Designers have always tried to take into account variations in the properties of the object to be designed, even when these variations are due to uncontrollable factors, such as temperature or humidity. These factors can cause the overall system performance to deteriorate sharply during operation. It is thus important to integrate the possible variations as early as possible in the design process, allowing the possible impact of these variations to be anticipated so as to minimize their effect on system performance. A solution is qualified as robust if its performance varies little under the influence of these variation-provoking factors. The possible variations of a material property are modelled using a set S of probabilized variable settings. A reference value and a neighbourhood defined around this value are associated to this property. The preference criterion of initial model is defined by the expected value of the performance in this neighbourhood. The added robustness criterion corresponds to the variance of the performance in this same neighbourhood. Decision makers are attracted to the solutions that offer a compromise between global performance and robustness.

Ehrgott and Ryan [10] studied the robustness of crew schedules at Air New Zealand. In the current systems for airline planning and management, optimizing crew schedules involves only a single criterion, the cost criterion. This criterion $v(x)$ takes into account the costs engendered supposing that a plan x is perfectly respected. However, in reality, the sources of the risks likely to perturb traffic are numerous. If aircraft downtimes are not sufficient to withstand these perturbations, plan x will not be respected, which will provoke penalties for the airlines. For this reason, the airlines are also interested in robust solutions that are able to withstand these perturbations. Optimizing the criterion $v(x)$ yields solutions that cannot be considered robust because they also make it necessary to minimize aircraft downtimes. Ehrgott and Ryan considered that the robustness of a solution increased as the total penalties caused by the probable delays decreased. For this reason, in addition to the criterion $v(x)$, they introduced a robustness criterion $r(x)$ based on the sum of the penalties that the "predictable delays" were likely to provoke. These predictable delays were introduced for each flight based on statistical observations that allowed an average delay and a standard deviation to be defined. The predictable delay is defined as the average delay increased by three standard deviations. The set of delays thus constructed constitutes a single variable setting s that is taken into account when defining $r(x)$ as the sum of the penalties assigned to each flight according to this single variable setting. The efficient frontier is then generated using the ε-constraints method.

Salazar and Rocco [37] studied reliable system design (see also Problem 5). The design of a product is often initially limited to finding the characteristics that meet the required specifications. Nevertheless, product reliability can vary due to uncontrollable external perturbations (e.g., aging, environmental changes) or due to *design variables*, which could have negative consequences. In this context, the stability of the reliability plays an important role, as does the design cost. In order to illustrate the problem, the authors considered the case of a complex system with several components. In their study, the reliability of the system and the reliability of the different components are related and are expressed with a complex mathematical formulation. By setting an admissibility interval for overall reliability, it is possible to determine, exactly or approximately, the feasible domain of the different components' reliabilities. Clearly, the points that are close to the borders of this domain are less interesting than those that are near the centre since a small variation in the frailty point values can push the system out of the acceptable reliability interval. Given a system reliability value, robustness can be apprehended, on the one hand, through the volume of the biggest parallelepiped included in the feasibility domain containing this value and, on the other hand, through the cost corresponding to the maximum cost in this volume. Decision makers are naturally interested in solutions that offer a compromise between design costs and the stability of the reliability.

Kennington et al. [22] studied Dense Wavelength Division Multiplexing (DWDM) routing and provisioning. DWDM is an optical transmission technology that allows data from different sources to be circulated over an optical fibre by assigning a wavelength to each source. Thus, in theory, several dozen different data flows can be transmitted at the same time. The transmission speeds are those of fibre optics: several billion bits per second. Given a network and an estimated demand, the DWDM routing and provisioning problem seeks to design a low-cost fibre optics network that will allow data to be sent to different demand centres. However, the process of estimating demand includes vague approximation and zone of ignorance. Under-estimating the demand, or over-estimating it, can have troublesome consequences. In this study, the imperfect knowledge of the demand is taken into account through a probabilized scenario set. The robustness concern is taken into account through a penalty measure that avoids the solutions proposing a capacity that is significantly under or over the demand in all the scenarios. This measure is based on the subjective costs corresponding to the positive or negative deviations from the demand. The installation costs and the robustness criterion help to enlighten the decision maker's choices. Since robustness plays an important role, Kennington et al. [22] transformed the bi-criteria problem into a lexicographical problem.

These three examples show the degree to which the additional robustness criterion can depend on the nature of the problem studied. For this reason, it hardly seems possible to formulate rules to facilitate the criterion's design. Thus, modellers must use their imagination to make the criterion correspond correctly to the problem at hand.

To bring this section to a close, we suggest an approach that is different from the ones described above. We consider the case in which S is finite and $v(x)$ is defined either based on a reference variable setting or on an average or median of S. In these conditions, it is possible to adopt one of the robustness criteria proposed by Roy [30]: bw-absolute robustness, bw-absolute deviation, or bw-relative deviation. We present what we think is an especially interesting case of project scheduling (see Problem 4). The robustness criterion can be defined as the proportion or the probability of the variable setting $s \in S$ for which $v_s(x) \leq v(x) + \Delta$ where Δ is a given constant. When controlling the execution of a vast project, the single preference criterion may be a cost criterion that includes the penalties engendered if the project is not completed on time. In this approach, the efficient frontier or an approximation of this frontier appears to be quite interesting for the decision maker. For this reason, it could be useful to study the sensitivity of this efficient frontier to variations of Δ.

4.3.3 With an Initial Multi-criteria Preference Model

In this section, we consider the case in which the initial preference model is multi-criterion, and not mono-criterion as in Section 4.3.2. Let F be a family of $n \geq 2$ criteria defined with no reference to a robustness concern. For the ith criterion, the performance can be defined as in Section 4.3.2. Again, we are interested in approaches that use a single additional criterion to give meaning to the statement "solution x is at least as robust as solution y".

This criterion must synthesize, in a single dimension expressing robustness, the impact of the variable settings in S on the performances of each of the n criteria in F. Unfortunately, we were unable to find a single publication in the literature proposing such a criterion. We describe below one possible approach.

This approach consists of:

1. First to consider each of the n criteria separately, and to assign a specific robustness measure to each one (see Section 4.3.2)
2. Second to aggregate these measures in order to define the additional robustness criterion

If the n criteria have a common scale, the aggregation can, with a few precautions, use the operators Max, OWA (Yager [44]), and even the Choquet integral (Grabisch [16], see also Roy [30]) to differentiate between the roles of each of the different robustness measures. In these conditions, the efficient frontier has $n + 1$ dimensions. For $n \geq 3$, it might be better to try to aggregate the criteria of the initial preference model in order to reduce the complexity of the calculation and make the results easier to interpret. It must be pointed out that aggregating all n criteria of the initial preference model into a single criterion before defining the single robustness criterion falls within the framework defined in Section 4.3.2 and thus constitutes a different approach from the one presented here.

4.3.4 With an Initial Preference Model That Is Either Mono-criterion or Multi-criteria

The approach presented below is applicable only to a finite set A of actions (the term "action" is here substituted for "solution"), which are evaluated using one or more criteria in each of the variable settings in set S, which is also assumed to be finite. In the case of a single criterion v, the values $v_s(x)$ define for A a complete preorder P_s, $\forall s \in S$. In the case of multiple criteria, the same result can be obtained (except that the preorders may be only partial) by applying an aggregation procedure (e.g., an ELECTRE-type method). Let P be a set of complete or partial preorders thus defined. We propose defining the robustness measure $r(x)$, associated to an action x, by the proportion (or the probability if S is probabilized) of the preorders P_s in P in which x occupies a rank at least equal to α, where α defines an imposed rank. It is also possible to imagine another imposed rank β, penalizing solutions that are not very well ranked in some variable settings. The robustness measure $r(x)$ can be defined by substituting from the previously defined measure the proportion (or probability) of the P_s in P in which x occupies a rank at least equal to β. The greater this measure, the more robust the action x is judged to be. This approach can be very useful for helping a decision maker to choose a supplier following a Call to Bid (see Problem 1 in Section 4.2). It might also be useful for studying the sensitivity of the results to the values of α and β. The results obtained must be able to be easily summarized as robust conclusions (see Section 4.5) that decision makers can easily understand.

4.4 Robustness in MCDA: Multi-dimensional Approaches

4.4.1 Characterizing Multi-dimensional Approaches

In this section, we survey the approaches that involve not a single measure of the robustness concern, but several. Each of these measures is designed to look at robustness from a specific point of view. These measures are used to define a set R ($|R| > 1$) of criteria intended to judge the more or less robustness of the solution.

In order to present the research pertaining to this kind of approaches, as well as to propose some new paths to explore, we distinguish the following three cases:

- The family R constitutes a preference model intended to enlighten the decision in the absence of any other previously defined preference model.
- The family R is substituted for or completes a previously defined mono-criterion preference model that has no links to the robustness concern.
- The family R is substituted for or completes a previously defined multi-criteria preference model whose criteria do not represent the robustness concern.

4.4.2 Without Any Initial Preference Model

Surprisingly, our bibliographic search did not reveal any studies about the kind of approach discussed in this section. As mentioned above, this kind of approach takes preferences into account directly by defining a priori a family of criteria in which each member expresses a different point of view of robustness; these criteria are not, however, based on a multi-criteria preference model initially conceived with no link to robustness. Nevertheless, one of the authors (see Pomerol et al. [26]) helped to develop and implement such an approach for dealing with concrete problems. The following paragraph provides a brief summary of the study by Pomerol et al. [26].

The concrete context was a large Parisian railway station that had to cope with intense rail traffic. Minor perturbations (e.g., a delayed gate closing due to an obstruction) frequently caused delays. Despite the actions of the dispatchers, who intervened to re-establish normal traffic patterns as quickly as possible, more serious accidents (e.g., damaged equipment) provoked a snowballing effect, leading to cancelled trains. To resolve the problem, new timetables, as well as local improvements to the rail network and the rolling stock, were envisioned. Combining these suggestions led to defining a set X of solutions to be studied. The goal of the study was to compare the robustness of these solutions when faced with different kinds of perturbations, while also taking into account the way that the dispatchers intervened to lessen the negative effects as much as possible. A set S, called the "incidence benchmark", was built; this set contained a set of representative incidences, each of them described precisely with a weight assigned according to its frequency. The family R was composed of the following six criteria:

$g_0(x)$: maximum delay allowed to any train without any perturbation being provoked;
$g_1(x)$: total number of trains including timetable concern by a delay from the original incident to the return to the theoretical schedule;
$g_2(x)$: the total duration of the perturbation;
$g_3(x)$: the total number of travellers concern by the perturbation;
$g_4(x)$: average delay of the travelling time;
$g_5(x)$: the total number of the trains concerned.

The first three criteria essentially reflect the viewpoints of the operator in charge of train traffic, while the others are directly concerned with traveller satisfaction. The performance $g_0(x)$ is completely determined by the *timetable* component that is part of the definition of x. In no way does $g_0(x)$ depend on the different variable settings $s \in S$, which is not true of the other five criteria. In addition, $\forall x \in X$, the calculation of $g_j(x)$, where $j \neq 0$, requires that the behaviour of the dispatchers facing each of the incidents in S be taken into account. To calculate these performances, it was necessary to call upon an expert system to reproduce this type of behaviour.

To end this section, let us underline that, in the kind of approach considered here, testing the coherence of family R is essential (i.e., verifying the following properties: exhaustivity, cohesion and non-redundancy; see Roy [27], Roy and Bouyssou [33]) because the preference model here is characterized by the family R.

4.4.3 With an Initial Mono-criterion Preference Model

This kind of approach is characterized by a family R containing several criteria to take robustness concern into account, rather than a single criterion as in Section 4.3.2. The criteria in R must reflect different non-correlated points of view. Consequently, if one of them can be chosen from the three proposed by Kouvelis and Yu (see Section 4.1), given the dependencies that exist between these criteria, we do not feel that the intervention of a second one would be pertinent.

Below, we first present an approach that substitutes several robustness criteria for the initial single preference criterion, and then we describe several approaches in which the robustness criteria complete the initial preference criteria.

Hites et al. [20] studied the connections between the robustness concern and multi-criteria analysis. Defining the elements in S as scenarios, these authors proposed substituting the set $R = \{v_s(x)/s \in S\}$ for the single criterion $v(x)$. Each of the criteria thus defined provides pertinent information for determining the relative robustness of solution x. By considering this set, these authors showed that an approach that applies a classic multi-criteria method is not appropriate for identifying robust solutions. One of the reasons comes from the cardinality of S: classic multi-criteria methods are only appropriate for criteria families containing 20 or at most 30 criteria. When the number of scenarios is small (a few units), considering the efficient frontier or an approximation of this frontier can in some cases help to respond to the robustness concern. It is useful to note that in all other approaches to the problem, a solution presented as robust must necessarily be a non-dominated solution in the multi-criteria problem defined by the set R considered here.

Let us move on to approaches in which the initial preference criteria are completed by a family R of criteria. The following paragraph briefly presents the only study that we were able to find involving this kind of approach.

Jia and Ierapetritou [21] studied the problem of *batch scheduling* in the chemical industry. The discrete process represents an ideal operational mode for synthesizing chemical products in small or intermediate quantities. This process is able to produce several composites by batch, using standard equipment and is also able to adapt to the variations in the nature and quality of the primary materials, which is a major advantage in terms of flexibility. In order to insure that any resource used in the production process is exploited efficiently, it is important to take *a detailed plant schedule* into account in the design phase. The design objective is to determine the number and types of equipment to be used and to build a feasible schedule of the operations that will maximize a performance criterion, given the following elements:

- The production guidelines (e.g., production time for each task, quantities of materials involved in the manufacturing of the different products)
- The availability and the capacity of the production equipment and storage facilities
- The production requirements;
- The time horizon of the study

In this study, the performance criterion corresponds to the total production time. However, during the design phase, it is almost impossible to obtain precise information about the production conditions. Thus, the information needed to calculate the expected performance exactly is not available. To remedy this lack, the various possible production conditions are modelled by a set S of variable settings. The production time associated to each of these variable settings can be calculated. In order to quantify the effect of the variations in the production conditions, two additional criteria are considered. The first tends to support feasible solutions in most of the variable settings by seeking to minimize the expected value of the unmet demand. The second attempts to measure the stability of the solution performance by seeking to minimize the expected value of the positive deviation between the real duration and the expected value of that duration. The main advantage of this measure with respect to the variance is its simplicity, since unlike variance, this measure can be written linearly. The efficient frontier provides the decision maker with a set of interesting compromise solutions.

The work presented above shows that combining a performance criterion and a set of robustness criteria can have numerous practical applications. To conclude this section, we suggest a new approach of the same type.

This new approach is concerned with the case in which S is finite. We assume that the initial preference model criterion $v(x)$ expresses a gain. We assume besides that $v(x)$ is defined by a variable setting s_1 judged particularly convincing: $v(x) = v_{s_1}(x)$. The value $v(x)$ could also be defined by the median or the arithmetic average of the values $v_s(x)$. The approach proposed here consists of modelling the robustness concern using two criteria. The first is the minimum gain of all the variable settings, and the second is defined by the number of variable settings, such as $v_s(x) \geq b$, where b corresponds to an objective that the decision maker hopes to reach, and even to exceed with a maximum of chance. Depending on the decision-making context, the second criterion can be replaced by the number of variable settings in which the absolute or relative regret is limited to b. The presence of these two robustness criteria, in addition to the expected gain with the reference variable setting s_1, can help the decision makers to be aware of how subjective the notion of robustness is. By discussing the value to be attributed to the bound b with the analyst, the decision maker can clarify the meaning that he/she assigns to the term "robust solution". For these reasons, studying the efficient frontier, possibly parameterized by b, seems to be a useful support tool for the final choice. This new approach is different than the one described in Section 4.3.2. In fact, in this new approach, we have retained a gain criterion as such (i.e., a gain in the variable setting or an average or median gain), and we have also added a second robustness criterion: the minimum gain. Cancelling the gain criterion would create a bi-criteria approach similar to the one proposed in Section 4.4.2. The advantages of this new approach are illustrated in Appendix A.

4.4.4 With an Initial Multi-criteria Preference Model

In this section, we consider the kind of approach in which the initial preference model is a family F containing $n \geq 2$ criterion. A criteria family R is introduced; the robustness concern is taken into account either by substituting R for F, or by using R to complete F. Each criterion in R may refer either to a specific aspect of robustness or to a criterion of the initial preference model. Unlike the approach described in Section 4.3.3, this kind of approach does not seek to aggregate the R criteria into a single summary criterion, but rather attempts to consider the criteria jointly. The most interesting case is the one in which the set R is substituted for F, since the case in which the set F is completed by R is difficult to interpret and can involve implementing algorithms that require a lot of computer resources and high calculation times.

In practice, researchers generally use only a single measure to model the robustness concern, undoubtedly for reasons of simplicity. Nonetheless, we found two papers in the literature that deal with the kind of approach considered in this section. We present them below.

Fernández et al. [13] examined *multi-criteria Weber location problem*. The problem dealt with in the paper consists of choosing the location of a super-server in a municipality where n servers are already in operation. This super-server includes k servers, each with its own individual characteristics. To server i is associated a vector with n components. Each of these components p_{ij}, called weights, is used to take into account the relative importance that the decision maker assigns to the distance that separates the already established server j from server i. In reality, these weights are ill-determined, and for this reason, a set S_i of vectors with plausible weights p^s_{ij} is defined for $i = 1, \ldots, k$. The decision maker's preference in terms of the choice of location for server i at place h is taken into account for each $s \in S$, by the criterion of the weighted sum d^s_{ij}, defined as $d^s_{ij} = \sum_{j=1}^{n} p^s_{ij} ||x_h - x_j||^2$, where x_h and x_j, respectively, represent the vectors of the coordinates for place h and those for server j. The servers $i = 1, \ldots, k$ should be located in the same place. This location is chosen by finding a compromise between the preference components referring to the different weighted sums d^s_{ij}. The authors begin by selecting, as the only possible locations, the places h for which the quantities d_{ij}^s have an acceptable value in all the variable settings, where $i = 1, \ldots, k$. Then, to facilitate the choice among the selected places, the authors bring into play k robustness criteria r_{ih}, $i = 1, \ldots, k$. Each of these criteria is a maximum regret criterion, defined as $r_{ih} = \max_{s \in S_i} \left\{ d_{ih}^s - \min_q d_{iq}^s \right\}$. The efficient frontier or the approximation of this frontier can help the decision maker to choose the best location possible.

Besharati and Azarm [7] studied the problem of *engineering design optimization*. This problem is similar to the one described in Section 4.3.2 (Chen et al. [9]), but the approach used is different. The initial preference model has n criteria (and not a single criterion) f_i for which the value $i = 1, \ldots, n$ must be minimized. In order to prevent undesirable consequences due to uncontrollable factors, the authors propose a method based on a generalization of the robustness criteria proposed by Kouvelis and Yu for the case in which the initial preference model is formed by a family F and for the case in which the values of certain constraint coefficients are imperfectly known. More precisely, the imperfect knowledge of the value of the frailty points is modelled with a set of variable settings S, where each element characterizes a possible version of criterion f_i, and the robustness of a solution is evaluated using two criteria.

The first criterion measures, for the worst variable setting, a p-distance between a given solution x and a point of reference x^*:

$$\max_{s \in S} \left[\sum_{i=1}^{n} \left| \frac{f_i^s(x) - f_i(x^*)}{f_i(x^w) - f_i(x^*)} \right|^p \right]^{\frac{1}{p}},$$

where x^w corresponds to a solution deemed particularly bad for all criteria and all variable settings. Let us notice that this p-distance insures that all the initial criteria play the same role.

The second criterion measures the performance variability of a solution x by calculating the p-distance between the points corresponding to the best and the worst variable setting for solution x, denoted s^b and s^w, respectively:

$$\left[\sum_{i=1}^{n} \left| \frac{f_i^{s^w}(x) - f_i^{s^b}(x)}{f_i(x^w) - f_i(x^*)} \right|^p \right]^{\frac{1}{p}}.$$

Let us note that in this paper, the frailty points do not affect only the coefficients of the objective function, but also the coefficients of the constraint matrix. For this reason, the authors are interested in efficient solutions that remain feasible in all the variable settings.

These two robustness criteria seem to be interesting since they can be applied in many contexts. In fact, the first is a generalization of the multi-criteria case of the absolute regret, and the second can be seen as a dispersion measure.

4.5 Robustness in MCDA: Other Approaches

4.5.1 Preliminaries

The approaches discussed in this section differ from the ones presented above in the sense that they are not intended to identify the most robust solutions in terms of one or more previously defined criteria. These approaches have their place in this chapter because they apply to formal representations of decision-aiding problems involving an initial multi-criteria preference model without any link to robustness.

Most of these approaches assign a determinant role to the fact that a solution, a set of solutions, or a method possesses (or does not possess) certain properties characterizing robustness, properties that are formulated in terms other than to maximize a criterion or to be on the efficient frontier (as was the case in the two sections above). In some cases, these properties make one or more robustness measures and their associated thresholds intervene so as to define the conditions under which the property(ies) will be judged satisfied. In most cases, these approaches yield results that allow conclusions about the robustness concern to be drawn.

Before presenting some of these approaches, it is necessary to call back to memory what Roy [31, 32] has called *robust conclusions*. By definition, each variable setting $s \in S$ is associated to an exactly defined formal representation of the problem and an exactly defined processing procedure. Applying this procedure to the problem's formal representation provides what has been defined under the general term **result** (see Section 4.1). Let us denote this result $\mathcal{R}(s)$.

Definition 4.1. A robust conclusion related to a sub-set $\hat{S}(S)$ is a statement that summarizes the result set $\{\mathcal{R}(s)/s \in \hat{S}\}$.

To illustrate this definition, we give several typical forms of robust conclusions that are interesting in the context of decision aiding (in cases when the preference model may or may not be multi-criteria).

(i) $\forall s \in \hat{S}$, x is a solution for which the deviation from the optimum (or from an efficient frontier) never exceeds a given threshold.
(ii) If the variable settings $s \in \hat{S}$ are taken into account, the results that follow (e.g., guaranteed cost, guaranteed completion time) are incompatible.
(iii) The results that follow ... are validated by the results $\mathcal{R}(s)$ obtained with a sample $\hat{S}$ of variable settings; since the sample is considered to be representative of S, it can be inferred that these statements are valid for all S.
(iv) For "almost" all $s \in \hat{S}$, x is a solution for which the deviation from the optimum (or from an efficient frontier) never exceeds a given threshold. Here, "almost" means that exceptions apply to the variable settings that, without necessarily being completely and perfectly identified, are considered to be negligible in the sense that they bring into play combinations of unlikely frailty points options.

(v) The results $\mathcal{R}(s)$ obtained $\forall s \in \hat{S}$ highlight a solution set $\{x_1, \ldots, x_q\}$ that responds to the robustness concern as it was formulated (this formulation may be relatively imprecise).

These examples show that:

- Stating robust conclusions does not necessarily lead to recommending the implementation of one solution over another (or even the choice of one method over another, see Section 4.5.4), but simply provides a framework for the decision maker's choices, and even sometimes restricts those choices.
- A robust conclusion may be more or less rigorous depending on whether it is validated over a relatively well-defined set and whether its formulation more or less permits interpretation (see Roy [31, 32] for an explanation of the distinction between perfectly-, approximately-, and pseudo-robust conclusions).

In the next section, we present a certain number of approaches that are included in this chapter either because they are recent, or because, despite being proposed in the past, they merit further consideration with respect to the above considerations, allowing them to be broadened, thus removing them from the restricted context in which they were proposed.

4.5.2 Robustness in Mathematical Programming

In mathematical programming, the search for a solution able to resist to vague approximations and zones of ignorance in order to withstand negative impacts is both a practical concern and a source of interesting theoretical problems. Different concepts and methods have been proposed in the literature for organizing and integrating this imperfect knowledge into the decision-making process (Ben-Tal and Nemirovski [3,4], Bertsimas and Sim [5], Bertsimas et al. [6], El Ghaoui and Lebret [11], El Ghaoui et al. [12], Soyster [39, 40]). When the objective function coefficients are not known exactly, the classic criteria from decision-making theory (e.g., worst case, absolute and relative regret) have often been used to define the robustness concern in linear programming as well as in integer programming.

When the imperfect knowledge concerns constraint matrix coefficients, the models studied in the literature primarily deal with the imperfect knowledge about either the columns or the lines of the matrix. These models assume that the constraint matrix columns (or lines) have coefficients able to vary in well-defined sets.

Imperfect knowledge in the constraint matrix columns was initially studied by Soyster [39, 40]. In this model, each column $A_j = (a_{ij})$ of the constraint matrix $m \times n$ can have values from a set $K_j \subset R^m$. The objective function coefficients, as well as the right-hand members, are assumed to be known exactly. The author deems robust any solution that is feasible for all the possible values of the vectors A_j chosen from K_j. The search for a robust solution is thus equivalent to solving a new mathematical program of the same nature for a constraint matrix $A' = \left(a'_{ij}\right)$ defined as follows:

- $a'_{ij} = \max_{a_j \in K_j} a_{ij}$ if the constraint is of the type $\leq$;
- $a'_{ij} = \min_{a_j \in K_j} a_{ij}$ if the constraint is of the type $\geq$.

The Soyster model is very conservative. The new mathematical program does not always allow feasible solutions although certain robust solutions of the type defined above may exist. In fact, although for certain j, the vector a'_{ij}, where $i = 1, ..., n$, described above does not belong to K_j, the set of feasible solutions of the new linear program does not necessarily contain all the robust solutions. In addition, even when feasible solutions exist, the one that optimizes the objective function may have an incorrect value and thus not be optimal in the set of robust solutions.

According to Ben-Tal and Nemirovski [3, 4], when the constraint matrix coefficients (and possibly the objective function coefficients) are not exactly known, the robust solution must remain feasible for all possible values of the unknown inputs. In the general case, it is possible that this intersection is empty. In the case this intersection is not empty, the resulting solution is very conservative.

Bertsimas and Sim [5] presented an approach that allows the degree of conservatism of the model's recommendation to be controlled when the imperfect knowledge is related to the lines of the constraint matrix. More specifically, each coefficient in the constraint matrix can have any value in the interval $\left[a_{ij} - \alpha_{ij}, a_{ij} + \alpha_{ij}\right]$ and for each line i, a number Γ_i is considered, where Γ_i cannot exceed the number n of variables. The model is based on the hypothesis that it is not very likely that all the model parameters will reach the worst values simultaneously. A solution is deemed Γ-robust if it respects the constraint i, for all $i = 1, \ldots, m$, when at most Γ_i coefficients are likely to reach the interval's higher bound $a_{ij} + \alpha_{ij}$ in cases with a $\leq$-type constraint (or likely to reach the interval's lower bound $a_{ij} - \alpha_{ij}$ in cases with a $\geq$-type constraint), and the other coefficient values are set to the average value of the interval. Bertsimas and Sim showed that, unlike the min–max versions of the absolute or relative regret, this approach generates a robust version with the same complexity as the starting problem. Specifically, the robust version of the shortest path, spanning tree and assignment problems are solvable in polynomial time. In addition, the robust version of the NP-hard problem that is β-approximable is also β-approximable. Nevertheless, this approach does have limitations. In fact, it generates a program that is parameterized by quantities, and it is not easy to specify the appropriate values for this program in advance. In the absence of information facilitating the choice of these values, setting $\Gamma_i = n$, for all $i = 1, \ldots, m$, produces a model similar to Soyster's conservative model.

In the studies cited above, the robustness concern does not bring into play criteria that permit the degree of robustness of the solution to be apprehended. This concern leads to considering any solution that is feasible in the defined conditions as robust. To conclude this section, we suggest a different approach. Let us consider the following linear program:

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = b_j, \quad i = 1, \ldots, m$$

$$x_j \geq 0, \quad j = 1, \ldots, n.$$

Let us also suppose that only the objective function coefficients are known exactly and that, by hypothesis, all the constraint matrix lines correspond to quantities that are expressed in the same type of unit (e.g., physical, monetary). The values of the constraint matrix coefficients are uncertain. A finite set S of variable settings allows this imperfect knowledge to be modelled. In the general case, it is possible for the intersection of the feasible domains of all the variable settings to be empty. In addition, even if this intersection is not empty, the price of robustness, as Soyster refers to it, can be high. The decision maker might accept an unfeasible robust solution in a small subset of S, but only if the cost is relatively low. In fact, it is often acceptable in practice to not respect equality; however, in this case, it is important for the non-zero deviations between the right and left members to be "small" and few in number. Thus, an unfeasible mathematical solution may be preferable to a much more costly solution that perfectly satisfies all the equalities. For a solution x, the deviations that must be taken into account are defined as follows:

$$e_i^s = b_i^s - \sum_{j=1}^{n} a_{ij}^s x_j.$$

A solution may be judged even more robust if these deviations remain small in the greatest possible number of variable settings. From this perspective, we propose adding one of the following three robustness criteria to the cost criteria:

- $\frac{1}{|S|} \sum_{s \in S} \max_{i=1,\ldots,m} |e_i^s|$;
- $\frac{1}{m|S|} \sum_{s \in S} \sum_{i=1}^{m} |e_i^s|$; and
- $\frac{1}{m|S|} \sum_{s \in S} |e_i^s|^2$.

In some cases, it may be appropriate to incorporate weights into these criteria to indicate the importance of the different deviations according to whether they are positive or negative or whether they are related to one line i or another.

Ultimately, this approach seeks solutions that provide a compromise between, on the one hand, the value of the objective function and, on the other hand, the deviations representing imperfect satisfaction of the constraints. Let us notice that the frontier can be built using the simplex method by optimizing a linear combination of the two criteria chosen, unless if the chosen robustness criterion is the third one above. This combination is possible because there are no unsupported solutions.

4.5.3 Obtaining Robust Conclusions from a Representative Subset S

At the beginning of the 1980s, two studies were completed in order to obtain robust conclusions. The first examined the execution priorities for extending Metro lines in the Paris region (see Roy and Hugonnard [34]). The second dealt with choosing a company to automate the sorting centres in the French Postal Service (see Roy and Bouyssou [33], Chapter 8). For these studies, Roy designed and implemented an approach that, although relatively informal, allowed a series of robust conclusions to be obtained. These two concrete studies were multi-criteria. Frailty points (not referred to by this term) appeared for two reasons: on the one hand, the presence of diverse data with ill-determined values in a certain interval and, on the other hand, the choice of a method that, due to technical reasons, justified the use of several processing procedures. In both cases, the robust conclusions obtained turned out to be quite interesting. In our opinion, this approach deserves to be broadened and extended within the formal framework described below.

Before describing this framework, we should specify that this type of approach is appropriate only for cases with a finite set A of possibilities, which we call *actions* rather than solutions. The results $\mathcal{R}(s)$ which have to be exploited can be those obtained by applying (Roy [27], Chapter 6):

- A selection procedure (a choice problematic)
- An assignment procedure (a sorting problematic) or
- An ordering procedure (a ranking problematic)

We propose to structure this approach into three steps.

Step 1: Moving from S to $\hat{S}$ (see Section 4.5.1)

In step 1, S always designates the set of variable settings derived, on the one hand, from the possible versions retained when formulating the problem and, on the other hand, from the various possible processing procedures that are envisioned. $\hat{S}$ designates a finite subset of S fulfilling the following two requirements:

- *Calculability requirement*: $\mathcal{R}(s)$ must be able to be determined, $\forall s \in \hat{S}$.
- *Representativity requirement*: Studying $\{\mathcal{R}(s)/s \in \hat{S}\}$ permits conclusions to be drawn, conclusions that can, with a negligible risk of error, be considered as valid for all of S.

Since these two requirements are generally in conflict, elaborating $\hat{S}$ means finding a compromise. To fulfil the representativity requirement (which in many cases will be highly subjective), a combinatorial approach or a probabilistic approach, or possibly a combination of the two, may be used.

The combinatorial approach involves retaining a very limited set f of possible options (e.g., two or three) for each frailty point e_f. $\hat{S}$ is then defined as the

Cartesian product of these sets or as a part of this Cartesian product, by eliminating the least likely combinations in order to respect the calculability requirement.

The probabilistic approach involves choosing a random sorting procedure defined on S and applying this procedure repeatedly to gather a number of variable settings compatible with the calculability requirement. If S is the Cartesian product of a certain number of intervals, the sorting can be done independently for each of these intervals according to a uniform law. (For more information about this type of procedure, notably its representativity, see for example, Steuer [41].)

Step 2: Moving from $\hat{S}$ to $\hat{S}'$

After calculating $\mathcal{R}(s)$, $\forall s \in \hat{S}$, a preliminary examination of these results is conducted in order to highlight two categories of frailty points.

- Category 1 contains the points that can have a significant influence on the results. These are the points that produce a result $\mathcal{R}(s)$ that is greatly influenced by the option (relative to the points being examined) present in s when a subset of $\hat{S}$ is examined. This subset is such that each component is, for every frailty point other than the one being examined, either identical or very similar.
- Category 2 contains the points with a negligible influence on the results. These are the points that produce a result $\mathcal{R}(s)$ that is very little influenced by the option (relative to the points being examined) present in s when a subset of $\hat{S}$ is examined. This subset is such that each component is, for every frailty point other than the one being examined, either identical or very similar.

To conduct such an examination, it is possible, in some cases, to use classic data analysis tools. The presence of reference variable settings s^* in $\hat{S}$ (that have particular importance to the decision maker) can also be quite useful. This is especially true if all or a part of the variable settings that differ from s^* only in terms of a single component are introduced into $\hat{S}$.

The examination described above is done in order to replace $\hat{S}$ with a set $\hat{S}'$ at least as representative and, if possible, smaller. In fact, only one option (possibly two) can be retained for category 2 frailty points. This case leads to the withdrawal of a certain number of variable settings from $\hat{S}$. Category 1 frailty points can nonetheless justify adding certain variable settings to better highlight the influence of these category 1 points.

Step 3: Obtaining robust conclusions

A careful analysis of $\{\mathcal{R}(s)/s \in \hat{S}'\}$, possibly facilitated by a systematic procedure, must allow pertinent robust conclusions to be drawn for the problem being studied. Below, we provide several typical examples of conclusions that could be validated depending on the nature of the procedure that is used to determine $\mathcal{R}(s)$. These studies are inspired from the conclusions obtained for the two concrete examples given at the beginning of this section.

With a selection procedure

- action $a_1 \in \mathcal{R}(s), \forall s \in \hat{S}'$;
- action $a_2 \notin \mathcal{R}(s), \forall s \in \hat{S}'$;
- depending on whether the frailty point option f is ... or ..., the action a_3 belongs or does not belong to $\mathcal{R}(s)$; and
- actions a_4 and a_5 are always associated since $\mathcal{R}(s)$ either contains both of the actions, or neither.

With an assignment procedure

- $\forall s \in S$, c_k is the worst category to which can be assigned, and as soon as, the frailty point f option is at least equal to ..., then the worst category is not c_k but c_h;
- action a_2 is always assigned to a higher category than the one to which action a_3, $\forall s \in \hat{S}'$, is assigned, and as soon as the frailty point f option is at least equal to ..., two categories, at least, separate their assignments.

With an ordering procedure

- none of the actions in $B \subset A$ is among the first 10 in $\mathcal{R}(s), \forall s \in \hat{S}'$;
- the actions in $C \subset A$ are the only ones that are always among the first 12 in $\mathcal{R}(s), \forall s \in \hat{S}'$.

In many cases, the conclusions that can be validated cannot be formulated as rigorously as the ones above (perfectly robust conclusions). Exceptions could be tolerated. The latter may not be clearly defined. If these exceptions are due to variable settings combining the extreme options of several frailty points, they may be judged negligible since they are not very likely (i.e., approximately-robust or pseudo-robust conclusions). Taking as a starting point a statement similar to the ones proposed above, it should be possible to design a procedure capable of identifying under what conditions and for which actions this type of statement can be validated.

4.5.4 Approaches for Judging the Robustness of a Method

As mentioned in the introduction, "method" here refers to a family $\hat{P}$ of procedures that can be differentiated by the options chosen with respect to some of the method's frailty points. This could be, for example,

- the concordance levels or the cut thresholds in the ELECTRE methods;
- the thresholds making certain inequalities strict in the MACBETH or UTA methods; and
- the multiple parameters involved in the tabu, simulated annealing or genetic methods.

In addition to these frailty points, which can be described as techniques, many multi-criteria methods involve parameters that are supposed to take into account an aspect of reality without referring to the existence of a true value that they should have in this reality. This is notably the case, for example, with substitution rates, intrinsic weights, indifference, preference and veto thresholds, and the analytical form of a probability distribution or those defining a fuzzy number. This second kind of frailty point can be viewed either as part of a method (in this case connected to the procedure), or as part of a model (in this case connected to the version of the problem).

Once the frailty points of a method have been defined, a procedure P_s is characterized by a variable setting s that describes the option retained for each of these frailty points. Using a method to either implement a repetitive application or simply to enlighten a single decision can lead to consider all the variable settings in a certain set S (which can leave out certain irrelevant procedures of $\hat{P}$) as equally legitimate. Considering the robustness of the method implies a desire to protect oneself from the arbitrariness involved in choosing one element in S rather than another. Vincke [42,43] proposed basing the evaluation of a method's robustness on the relative similarity of the results obtained with the different procedures $P_s, s \in S$.

This approach to the robustness of a multi-criteria decision-aiding method requires an exact definition of "similarity". Vincke proposed defining this similarity using a distance measure applied to result pairs, with the distance obviously depending on the nature of the results produced by the multi-criteria method (e.g., utility functions, action selections, complete or partial preorders, category assignments). The criterion used to evaluate the robustness of a method can thus be defined by the maximum value of this distance for the set of variable setting pairs belonging to S when the method is applied to a specific version of the problem. Accordingly, a method can be qualified as robust on such basis if this maximum remains under a fixed threshold for the set of versions retained for the problem studied. As Vincke underlined, this definition of a method's robustness should not be used to judge whether or not a method is "good" or "bad" because, in fact, a method that systematically produces the same "bad" results could nonetheless be robust.

These considerations show that it is not easy to assign a meaning to the notion of robustness of a multi-criteria method. This notion cannot have an absolute character. The definition depends on both the version set of the problem studied and the way that the set S is defined. In the approach proposed by Vincke, it also depends on the distance measure chosen.

The subject of the robustness of a method could lead to interesting theoretical research. It would not be necessary to expect such research to help researchers confronted with real-life problems to choose the most robust method for dealing with these problems. In fact, for a given problem, the way that the version set is defined is frequently influenced by the method. In addition, the set S is strongly conditioned by the method. For these reasons, we cannot see how and on what basis one method can be declared more robust than another. The practitioner can nonetheless expect research about the robustness of different methods to provide guidance in order:

- to better take into account the frailty point set for the chosen method, notably when formulating robust conclusions, when it is necessary to enlighten a decision;

- to decide which option to retain for each of the frailty points chosen, when it is necessary to implement a method for repeated application (in time or space).

In Roy et al. [34, 35], the interested reader will find a description of a case in which the way of assigning numerical values to some parameters for a repetitive application of the ELECTRE III method was based on a comparison of the relative similarities of the top ranking elements. The objective of the method was to aid decision makers to periodically select, from among several hundred Metro stations, a small group of stations (a maximum of eight) whose renovation should be given priority. The k stations ranked highest by the method are referred to as the "top ranking" stations. The comparison of these top ranking stations (setting $k = 20$) highlighted, in a first step, the fact that, for most of the frailty points, the choice of the option retained had little or no influence (i.e., top ranking highly similar in terms of the symmetric difference). This, in turn, in a second step allowed (setting $k = 10$) the impact of the options retained for the remaining frailty points to be studied more precisely and the additional information to be taken into account when making the final choices.

4.5.5 Approach Allowing to Formulate Robust Conclusions in the Framework of Additive Utility Functions

Figueira et al. [14] have proposed diverse multi-criteria aggregation procedures, based on the principle of ordinal regression, allowing certain statements of robust conclusions to be validated in terms of the concepts of what is "possible" and what is "necessary" (see also Chapter 9). The subject of these conclusions varies with the aggregation procedure proposed:

- UTADIS (Greco et al. [18]) deals with the category among a set of totally ranked categories to which an action a can be assigned.
- UTAGMS (Greco et al. [19]) deals with assertions such as "action a outranks action b".
- GRIP (Figueira et al. [15]) deals with the intensity with which an action a outranks an action b.
- GRIP-MOO (Figueira et al. [14]) deals with the best actions that are not outranked by any other feasible action in a particular step of an interactive multi-objective optimization procedure.

These four aggregation procedures were designed to help one or several decision maker(s) (DM) (see the extensions presented in Greco et al. [18]) in the following circumstances.

The DM is interested in a set A of actions evaluated with n criteria. The DM can provide preference information for some reference actions, but this information differs depending on the procedure in question. Essentially, this information explains how the DM ranks these reference actions from best to worst, how certain actions are compared to others, or, in GRIP, the intensity with which action a is preferred to action b for certain criteria considered separately and/or comprehensively. Let I denote the set of information provided. The envisaged aggregation is additive as in the UTA procedures (Siskos et al. [38]), yet with various improvements, notably in the form of the marginal utility functions, which are no longer piecewise linear but simply non-decreasing. This means that the authors seek to take the information I into consideration with synthetic utility functions, each interpretable as a weighted sum of n marginal utility functions associated with different criteria. An adjusting algorithm makes it possible to identify a set $U(I)$ of synthetic utility functions said to be "compatible with I". The set U is defined by a set of linear constraints: a compatible utility function is associated to each interior point of the polyhedron S delimited by these constraints. Here, any point $s \in S$ constitutes one of the variable settings taken into account. It is not impossible for S to be empty, which means that the additive utility model considered is inappropriate for taking the DM's preferences into account as they were expressed in the set I.

In all cases, each of the aggregation procedures described above leads to present to the DM conclusions in terms of what is "necessary" and what is "possible" (see Greco et al. [17]). A conclusion is said to be necessary if it is validated by all the functions of $U(I)$; it is said to be possible if it is validated by at least one of these $U(I)$ functions. Any conclusion that is necessary is thus possible. Ruling out any situation of incomparability, this additive utility model identifies as possible any conclusion of the above types that is not necessary. After showing the results obtained to the DM, it can be interesting to ask if I can be enriched either by adding complementary information about the same reference set, or by adding other reference actions. The enrichment of I leads to new conclusions that in turn lead to new responses to the DM's robustness concern. This enrichment also reduces S, which may possibly become empty.

Based on the possible and the necessary, this kind of approaches can be exploited in other contexts. Let us consider, for example, a set X (not necessarily finite) of solutions evaluated with n criteria $v_1, \ldots, v_n$ or a set S (finite) of variable settings that helps to define a performance v_{is} for each solution $x, i = 1, \ldots, n$. All efficient solutions, $\forall s \in S$, can be qualified as *necessarily efficient*, and all solutions that are efficient for at least one $s \in S$ can be qualified as *possibly efficient*. In many cases, it could be predicted that the sets of *necessarily efficient* solutions will be empty, and the set of possibly efficient solutions will be excessively large. This could lead to considering the greatest value of λ for which solutions are efficient for at least λ variable settings $s \in S$. Such solutions can, in a certain sense, be qualified as robust.

Other approaches for exploiting this approach based on the possible and the necessary have been presented (Greco et al. [17]). These methods primarily concern certain classic mono-criterion models in OR-DA (e.g., minimal spanning trees) and the multi-criteria outranking models.

4.5.6 Approaches to Robustness Based on the Concept of Prudent Order

The concept of prudent order was introduced by Arrow and Raynaud [2]. In this section, we first provide a brief reminder of what constitutes a prudent order. Then, we highlight the elements of this concept that are appropriate to bring an answer to robustness concern. Our explanations are based on Lamboray's research [24].

Let $A = \{a_1, \ldots, a_n\}$ denote a set of actions and F a family of q individuals (these individuals may be criteria). The individual i $(i \in \{1, \ldots, q\})$ ranks the actions in A with respect to his/her preferences according to a complete ranking O_i.

The concept of prudent order is designed to highlight rankings defined on A that minimize the oppositions. The meaning of "minimize the oppositions" will be explained later.

Let S denote a relation that counts the number of rankings that prefer a_i over a_j : $S_{ij} = |\{k \in \{1, \ldots, q\} : (a_i, a_j) \in O_k\}|$. Let $R_{\geq\lambda}$ $(R_{>\lambda})$ be a cut-relation of S defined as follows: $R_{\geq\lambda} = \{(a_i, a_j) : S_{ij} \geq \lambda\}$ $(R_{>\lambda} = \{(a_i, a_j) : S_{ij} > \lambda\})$. Clearly, increasing the value of λ decreases the cardinality of $R_{\geq\lambda}$. When $\lambda = 1$, this relation contains q complete orders (linear order) O_i. So there is a maximum value of λ, denoted α, such that $R_{\geq\alpha}$ contains a complete order ($R_{\geq(\alpha+1)}$ does not contain any complete order).

The relation $R_{>q}$ is empty. Consequently, it contains no cycle. So there is also a minimum value of λ, denoted β, such that $R_{>\beta}$ contains no cycle, and as a result, $R_{>(\beta-1)}$ contains at least one cycle. In the case of unanimity $(O_1 = O_2 = \cdots = O_q)$, $\beta = z0$, and $\alpha = q$.

By definition, a prudent order $\mathbf{O}$ is a complete order verifying $R_{>\beta} \subseteq \mathbf{O} \subseteq R_{\geq\alpha}$. In the case of unanimity, the common order is a prudent order.

Before interpreting the concept of prudent order, let us provide some results.

Arrow and Raynaud showed that $\alpha + \beta = q$. In the particular case where $\alpha \geq \beta$, it is easy to verify that only one prudent order exists. In the opposite case, $\alpha < \frac{q}{2} < \beta$ is necessarily verified. Thus, several prudent orders can exist. This number could be very high when n is large. In the general case, Arrow and Raynaud justify the fact that these orders are said to be **prudent** as follows (an analogous justification is valid for the single prudent order in the case $\alpha \geq \beta$).

First of all, an ordered pair $(a_i, a_j) \in R_{>\beta}$ belongs by definition to all prudent orders. These ordered pairs create no cycle between them, and consequently no contradiction. For Arrow and Raynaud, a prudent order must highlight elements of consensus. From this perspective, not retaining a pair $(a_i, a_j) \in R_{>\beta}$ would lead to retaining the opposite pair (a_j, a_i), this solution would create a great opposition when the pair (a_i, a_j) is supported by a majority at least equal to $\beta > \frac{q}{2}$. Let us now consider a ranking that contains a pair $(a_i, a_j) \notin R_{>\beta}$. The number of individuals that support such a pair is $<\alpha$. These pairs in the prudent order are all supported by at least α individuals. Eliminating the pairs not found in $R_{\geq\alpha}$ leads to qualifying as prudent only the orders that minimize the greatest opposition, which is equal to $q - \alpha$ in all prudent orders.

In the exceptional case when there is only one prudent order, this order can be viewed as a robust ranking. In the opposite case, Lamboray [24] proposes elaborating robust conclusions based on the multiplicity of the prudent orders.

An initial form of robust conclusions can be obtained by building assertions that are valid for all the prudent orders. From this perspective, it is possible to examine the pairs (a_i, a_j) that are contained in all the prudent orders. It is also possible to look at the best and worst ranks of action a_i in the entire prudent order set. Lamboray have shown how these extreme ranks can be computed.

Another form of robust conclusions can be obtained by looking only at the prudent orders that possess a given property, for example, those that contain one or more pairs (a_i, a_j) or those that assign to action a_i a rank at least equal or at most equal to an imposed rank. Looking at only this type of prudent orders leads to drawing conditional robust conclusions. Such conclusions can facilitate a dialogue whose objective is to find a consensus ranking.

Let us observe that the multiplicity of the prudent orders can be seen as a consequence of the difficulties and the ambiguities (i.e., the arbitrariness) that are encountered when attempting to aggregate purely ordinal information. Let us underline in conclusion that the concept of prudent orders is defined based on a set of complete orders. It would be interesting to try to generalize this concept for the case of complete pre-orders or semi-orders. Complete orders guarantee $s_{ij} + s_{ji} = 1$. Unless the definition of s_{ij} is modified, this equality is no longer verified if there are ties. The verification of this equality, unfortunately, plays an important role in the definition and interpretation of prudent orders.

4.6 Conclusion

In MCDA, robustness is a practical and theoretical concern of great importance. The term robust refers to a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts, notably the degradation of the properties that must be maintained.

The objective of the first two sections of this chapter was to call back to memory a certain number of basic ideas and introduce a few definitions in order to establish the framework for examining the new trends discussed herein. Section 4.3 was devoted to an approach in which robustness is considered through a single criterion that completes a preference model that has been defined previously, independently of the robustness concern. In the first part of Section 4.3, we characterized this type of approach, and then in the next two sections (4.3.2 and 4.3.3), we described two sub-types of this approach, presenting several articles dealing with these two sub-types. In the last section, we propose a new approach (Section 4.3.4). In Section 4.4, we examined how robustness can be taken into account using several criteria. After characterizing this second type of approach, we broke it down into three sub-types (see Sections 4.4.2, 4.4.3 and 4.4.4). Very few publications dealing with this type of approach were found in the literature, but those that were available were mentioned in each section. After presenting these three sub-types, we introduced new approaches, one of which is illustrated in the Appendix. In a last section before the conclusion, we presented approaches that allow robustness to be considered other than with a single criterion or multiple criteria serving to compare the solutions. Following several preliminary explanations (Section 4.5.1), we described (Section 4.5.2) a new robustness approach in mathematical programming. Section 4.5.3 presented a procedure for obtaining robust conclusions from a representative subset of the set S of variable settings. The manner in which the robustness of a method should be apprehended is the focus of Section 4.5.4. Section 4.5.5 aims to formulate the robust conclusions related to the additive utility functions. Section 4.5.6 examines robustness approaches based on prudent orders.

The considerations developed in this chapter show that the use of multiple criteria for apprehending robustness in MCDA is a field of research open to future development, both theoretically and practically. These future developments should contribute to increasing the use of operational research tools.

Appendix: A Numerical Example

In this example, we consider 20 actions evaluated in 20 scenarios (see Table 4.3) according to a criterion v used to express a gain.

We suppose that the scenario s_1 involves, for each frailty point, a value that the decision maker (DM) judges particularly plausible. The other scenarios were built by taking diverse possible combinations of values that deviate significantly from those retained in s_1. The case that is of interest here is the one in which, to enlighten his/her choice, the DM would like, in addition to considering the gain solution x corresponding to scenario s_1 (gain denoted $v_1(x)$), to consider two criteria deemed pertinent to determine the robustness of solution x:

- The criterion $r_1(x)$ expressing the worst gain that solution x could yield for the 20 scenarios considered
- The criterion $r_2(x)$ indicating the number of scenarios that will guarantee a gain at least equal to the value $b = 190$, a value that reflects an objective that the DM would like to have a maximum chance of attaining and, if possible, exceeding

For the three criteria retained, only 6 of the 20 actions are efficient (see Table 4.1). Table 4.1 gives the DM the information necessary to make a responsible choice, while also making the DM aware of the subjectivity that is inherent to any of the choices made based on the three retained criteria. Depending on the DM's attitude towards risk and the way that the likelihood of the different scenarios is evaluated, the DM could:

- Choose x_4, which would suppose that he/she accepts the risk of only earning 120, a value that is quite far from the objective of 190. The fact that this objective is not only attained, but considerably exceeded in all the scenarios except s_2 could convince the DM to take this risk. However, the DM could also refuse this choice

Table 4.1 Performance matrix of potential actions

Action	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}
1	150	180	170	175	165	160	170	175	160	175	170	165	155	160	170	165	170	165	180	180
2	110	145	120	155	165	195	145	150	165	170	155	160	165	170	155	155	170	165	195	90
3	180	130	175	170	175	175	175	175	170	175	190	190	195	190	195	190	195	190	195	190
4	200	120	190	195	190	195	190	195	190	195	190	195	190	195	190	200	190	195	200	210
5	90	90	210	205	200	205	195	200	190	195	195	200	205	210	215	210	210	205	220	210
6	130	125	135	150	145	150	155	160	165	195	155	145	170	155	145	140	145	135	160	185
7	190	125	175	175	175	190	175	170	175	175	190	195	190	195	190	195	190	185	200	210
8	170	140	190	195	190	195	190	195	190	195	180	185	180	185	180	185	180	185	125	100
9	95	125	190	165	165	155	175	160	165	150	150	165	170	165	160	155	165	170	185	120
10	100	130	175	195	160	155	155	150	170	155	145	140	155	160	155	160	155	160	185	110
11	105	135	110	120	195	150	160	155	150	130	145	155	145	160	155	150	135	145	90	190
12	170	140	190	190	170	175	170	160	165	160	190	165	170	190	190	190	190	180	170	190
13	115	150	150	165	160	150	195	145	140	150	165	170	145	160	170	155	170	165	185	120
14	120	125	155	160	155	145	165	195	155	160	155	160	165	170	155	145	165	170	200	110
15	125	130	145	150	165	170	165	160	195	170	155	150	165	160	155	170	165	160	170	185
16	160	140	190	195	200	195	190	195	190	190	195	160	175	160	175	160	150	170	170	175
17	135	130	155	135	130	120	135	150	145	125	195	135	125	155	135	120	110	175	180	190
18	140	135	145	130	125	145	140	135	125	130	135	195	160	155	150	145	140	135	150	185
19	145	130	135	155	150	145	125	155	150	145	135	140	195	150	145	155	145	150	190	160
20	145	125	125	135	155	150	140	145	155	150	145	135	155	195	140	145	160	165	180	170

Table 4.2 Efficient set of actions for $b = 190$

Action	v_1	r_1	r_2
1	150	150	3
16	160	140	9
12	170	140	8
3	180	130	10
7	190	125	10
4	200	120	19

Table 4.3 Efficient set of actions for $b = 180$

Action	v_1	r_1	r_2
1	150	150	3
8	170	140	16
3	180	130	11
7	190	125	10
4	200	120	19

if he/she thinks that scenario s_2 is plausible enough and that a gain of only 120 would be highly detrimental

- Not choose x_4, for the reasons outlined above. This would normally lead the DM to eliminate x_7, which, in the scenario s_2, could lead to a slightly higher gain than the one obtained with x_4 while producing gains at best equal to the ones for most of the other scenarios. A desire to maximize the gain would lead to choosing x_1. The DM could judge this choice to be "bad" since, in the scenario s_1 that the DM appears to favour, the objective is far from being attained (150 instead of 190); in addition, the objective is not attained in any of the 19 other scenarios. Observing that, whatever the choice among the 5 other efficient actions, choosing scenario s_2 entails running a risk, the DM could decide that the best compromise between the chance of attaining his/her objective and the risks of a mediocre gain would be either x_3 or x_{16}, with the latter solution appearing preferable to x_{12}, which produces the same result in scenario s_2.

The analyst should point out to the DM that lowering the objective $b = 190$ would allow the set of efficient actions to be modified, thus highlighting other possible compromises. In fact, this is the case if b is set to 180 (see Table 4.2). Action x_8 guarantees a gain of 140, as does x_{16}, but in addition to being better in scenario s_1, this action also guarantees a gain of 180 in 16 scenarios, instead of the 9 obtained with x_{16}.

Thus, depending on the DM's ambitions (i.e., the desired objective level) and his/her attitude towards risk (worst case), the DM may find the choice is between x_4 and x_8.

References

1. H. Aissi, C. Bazgan, and D. Vanderpooten. Min–max and min–max regret versions of combinatorial optimization problems: A survey. *European Journal of Operational Research*, 197(2): 427–438, 2009.
2. K. Arrow and H. Raynaud. *Social Choice and Multicriteria Decision-Making*. MIT Press, 1986.
3. A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23:769–805, 1998.
4. A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.
5. D. Bertsimas and M. Sim. Robust discrete optimization and network flows. *Mathematical Programming Series B*, 98:49–71, 2003.
6. D. Bertsimas, D. Pachamanova, and M. Sim. Robust linear optimization under general norms. *Operations Research Letters*, 32:510–516, 2004.
7. B. Besharati and S. Azarm. Worst case deterministic feasibility and multiobjective robustness measures for engineering design optimisation. *International Journal of Reliability and Safety*, 1:40–58, 2006.
8. J.C. Billaut, A. Moukrim, and E. Sanlaville, editors. *Flexibilité et Robustesse en Ordonnancement*. Paris, Lavoisier, 2005.
9. W. Chen, J. Allen, K.L. Tsui, and F. Mistree. A procedure for robust design: Minimizing variations caused by noise factors and control factors. *Journal of Mechanical Design*, 118: 478–493, 1996.
10. M. Ehrgott and D.M. Ryan. Constructing robust crew schedules with bicriteria optimization. *Journal of Multi-Criteria Decision Analysis*, 11:139–150, 2002.
11. L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertainty data. *SIAM Journal on Matrix Analysis and Applications*, 18:1035–1064, 1997.

12. L. El Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal of Optimization*, 9:33–52, 1998.
13. F.R. Fernández, S. Nickel, J. Puerto, and A.M. Rodríguez-Chía. Robustness in the Pareto-solutions for the multicriteria Weber location problem. *Journal of Multi-Criteria Decision Analysis*, 10:191–203, 2001.
14. J. Figueira, S. Greco, V. Mousseau, and R. Slowinski. Interactive multiobjective optimization using a set of additive value functions. In J. Branke, K. Deb, K. Miettinen, and R. Slowinski, editors, *Multiobjective Optimization: Interactive and Evolutionary Approaches*, volume 5252 of *Lecture Notes in Computer Science*, pages 97–120. Springer-Verlag, Berlin, 2008.
15. J. Figueira, S. Greco, and R. Slowinski. Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. *European Journal of Operational Research*, 195(2):460–486, 2009.
16. M. Grabisch. The application of a fuzzy integral in multicriteria decision making. *European Journal of Operational Research*, 89:445–457, 1996.
17. S. Greco, V. Mousseau, and R. Slowinski. The necessary and the possible in a perspective of robust decision aiding. In *66th Meeting of the EURO Working Group "Multiple Criteria Decision Aiding", Marrakech, Morroco, October 18–20*, 2007.
18. S. Greco, V. Mousseau, and R. Slowinski. Multicriteria sorting with a set of value functions. In *Joint Conference FRANCORO V / ROADEF 2007, Grenoble, France, February 20-23, 2007*, 2007.
19. S. Greco, V. Mousseau, and R. Slowinski. Ordinal regression revisited: Multiple criteria ranking with a set of additive value functions. *European Journal of Operational Research*, 191: 416–435, 2008.
20. R. Hites, Y. De Smet, N. Risse, M. Salazar-Neumann, and P. Vincke. About the applicability of MCDA to some robustness problems. *European Journal of Operational Research*, 74:322–332, 2006.
21. Z. Jia and M.G. Ierapetritou. Generate pareto optimal solutions of scheduling problems using normal boundary intersection technique. *Computers & Chemical Engineering*, 31:268–280, 2007.
22. J. Kennington, K. Lewis, E. Olinick, A. Ortynski, and G. Spiride. Robust solutions for the WDM routing and provisioning problem with uncertain demands and single link failure protection. Technical Report 01-EMIS-07, EMIS Departement, SMU, Dallas, 2001.
23. P. Kouvelis and G. Yu. *Robust Discrete Optimization and its Applications*. Kluwer Academic Publishers, Boston, 1997.
24. C. Lamboray. Supporting the search for a group ranking with robust conclusions on prudent orders. In B. Roy, M.A. Aloulou, and R. Kalaï, editors, *Robustness in OR-DA*, pages 145–172. Université Paris-Dauphine, Annales du LAMSADE n° 7, 2007.
25. Newsletter of the European Working Group "Multiple Criteria Decision Aiding". Newletter of the European Working Group "Multiple Criteria Decision Aiding". Available online at http://www.inescc.pt/~ewgmcda/Articles.html, 2002-2007.
26. J.C. Pomerol, B. Roy, C. Rosenthal-Sabroux, and A. Saad. An "intelligent" DSS for the multicriteria evaluation of railway timetables. *Foundations of Computing and Decision Sciences*, 20:219–238, 1995.
27. B. Roy. *Multicriteria Methodology for Decision Analysis*. Kluwer Academic Publishers, 1996.
28. B. Roy. Robustesse de quoi et vis-à-vis de quoi mais aussi robustesse pourquoi en aide à la décision ? In C. Henggeler Antunes, J. Figueira, and J. Clímaco, editors, *Aide Multicritère á la Décision- Multiple Criteria Decision Aiding, 56e Journées du Groupe de Travail Européen "Aide Multicritère à la Décision" – 56th Meeting of the European Working Group "Multiple Criteria Decison Aiding", Coimbra, Portugal, October 3–5, 2002, Comissão de Coordenação e Desenvolvimento Regional do Centro, INESC Coimbra, Faculdade de Economia da Universidade de Coimbra*, pages 29–39, 2004.
29. B. Roy. A propos de robustesse en recherche opérationnelle et aide à la décision. In J.C. Billaut, A. Moukrim, and E. Sanlaville, editors, *Flexibilité et robustesse en ordonnancement*, pages 35–50. Paris, Lavoisier, 2005.

30. B. Roy. Robustness in operational research and decision aiding: A multi-faceted issue. *European Journal of Operational Research*, 200:629–638, 2010. Also in French: La robustesse en recherche opérationnelle et aide à la décision: une préoccupation multi-facettes. *In Robustness in OR-DA*, Roy, B. and Aloulou, M.A. and Kalaï, R., editors, pages 209–236, Université Paris-Dauphine, Annales du LAMSADE n° 7, 2007.
31. B. Roy. Une lacune en RO-AD: les conclusions robustes. Cahier du LAMSADE 144, Université Paris-Dauphine, 1997.
32. B. Roy. A missing link in OR-DA, robustness analysis. *Foundations of Computing and Decision Sciences*, 23:141–160, 1998.
33. B. Roy and D. Bouyssou. *Aide Multicritère à la Décision : Méthodes et Cas*. Paris, Economica, 1993.
34. B. Roy and J.C. Hugonnard. Ranking of suburban line extension projects on the Paris metro system by a multicriteria method. *Transportation Research*, 16A:301–312, 1982.
35. B. Roy, M. Présent, and D. Silhol. Programmation de la rénovation des stations du métro parisien : Un cas d'application de la méthode ELECTRE III. Document du LAMSADE 24, Université Paris-Dauphine, 1983.
36. B. Roy, M. Présent, and D. Silhol. A programming method for determining which paris metro stations should be renovated. *European Journal of Operational Research*, 24:318–334, 1986.
37. D.E. Salazar and C.M.S. Rocco. Solving advanced multi-objective robust designs by means of multiple objective evolutionary algorithms (MOEA): A reliability application. *Reliability Engineering & System Safety*, 92:697–706, 2007.
38. Y. Siskos, E. Grigoroudis, and N.F. Matsatsinis. UTA methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis – State of the Art Surveys*, pages 297–343. Springer, Berlin, 2005.
39. L.A. Soyster. Convex programming with set-inclusive constraints and application to inexact linear programming. *Operations Research*, 2:1154–1157, 1973.
40. L.A. Soyster. A duality theory for convex programming with set-inclusive constraints. *Operations Research*, 22:892–898, 1974.
41. R.E. Steuer. *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley, New York, 1986.
42. P. Vincke. Robust and neutral methods for aggregating preferences into an outranking relation. *European Journal of Operational Research*, 112:405–412, 1999.
43. P. Vincke. Robust solutions and methods in decision-aid. *Journal of Multi-Criteria Decision Analysis*, 8:181–187, 1999.
44. R.R. Yager. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics*, 18:183–190, 1988.