Chapter 3
Preference Function Modelling: The Mathematical Foundations of Decision Theory

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Abstract We establish the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable. The mathematical foundations of decision theory and related theories depend on these conditions which have not been correctly identified in the classical literature. Operations Research and Decision Analysis Societies should act to correct fundamental errors in the mathematical foundations of measurement theory, utility theory, game theory, mathematical economics, decision theory, mathematical psychology, and related disciplines.

Keywords Foundations of science · Measurement theory · Decision theory · Social choice · Group decision making · Utility theory · Game theory · Economic theory · Mathematical psychology

3.1 Introduction

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied to enable the application of the mathematical operations of linear algebra and calculus. We identify these conditions and show that classical measurement and evaluation theories, including utility theory, cannot serve as the mathematical foundation of decision theory, game theory, economics, or other scientific disciplines because they do not satisfy these conditions.

The mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables*, i.e., to variables such as *preference* that describe psychological or subjective properties. Whether psychological properties can be measured (and hence whether

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mathematical operations can be applied to psychological variables) was debated by a Committee that was appointed in 1932 by the British Association for the Advancement of Science but the opposing views in this debate were not reconciled in the Committee's 1940 Final Report.

In 1944, game theory was proposed as the proper instrument with which to develop a theory of economic behavior where utility theory was to be the means for measuring preference. We show that the interpretation of utility theory's lottery operation which is used to construct utility scales leads to an intrinsic contradiction and that the operations of addition and multiplication are not applicable on utility scale values. We present additional shortcomings of utility theory which render it unsuitable to serve as mathematical foundations for economics or other theories and we reconstruct these foundations.

3.2 Measurement of Preference

The applicability of mathematical operations is among the issues implicitly addressed by von Neumann and Morgenstern in [53, §§3.2–3.6] in the context of measurement of individual preferences. Preference, or value, or utility, is not a physical property of the objects being valued, that is, preference is a subjective, i.e. psychological, property. Whether psychological properties can be measured was an open question in 1940 when the Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (Ferguson et al. [30]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the Final Report.

The position that psychological variables cannot be measured was supported by Campbell's view on the role of measurement in physics [24, Part II] which elaborated upon Helmholtz's earlier work on the mathematical modelling of physical measurement [35]. To re-state Campbell's position in current terminology the following is needed.

By an empirical system E we mean a set of empirical *objects* together with *operations* (i.e. functions) and possibly the relation of *order* which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the empirical operations in E as well as the order in E when E is ordered. A scale s is a mapping of the objects in E into the objects in M that reflects the structure of E into M (in technical terms, a scale is a homomorphism from E into M).

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: As Campbell eloquently states [24, pp. 267–268], "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science."

In terms of these concepts, the main elements of Campbell's view are summarized by J. Guild in Ferguson et al. [30, p. 345] in the context of measurement of *sensation* where he states that for psychological variables it is not possible to construct a scale that reflects the empirical operation of addition because such an empirical (or "practical") addition operation has not been defined; if the empirical operation does not exist, its mathematical reflection does not exist as well.

The framework of mathematical modelling is essential. To enable the application of mathematical operations, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects (which typically are the real numbers for the reasons given in Section 3.7.2). Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Therefore, the construction of measurement scales requires that the property-specific empirical operations be identified and reflected in the mathematical model. Moreover, the operations should be chosen so as to achieve the goal of this construction which is the application of mathematical operations in the mathematical model.

3.2.1 Empirical Addition – Circumventing the Issue

Accordingly, von Neumann and Morgenstern had to identify the empirical operations that characterize the property of *preference* and construct a corresponding mathematical model. As we shall see in Section 3.5, the operations of addition and multiplication are not enabled in their mathematical model and their empirical operation requires an interpretation that leads to an intrinsic contradiction.

The task of constructing a model for *preference* measurement is addressed by von Neumann and Morgenstern in [53, §3.4] indirectly in the context of measurement of *individual* preference. While the operation of addition as applies to *length* and *mass* results in scales that are unique up to a positive multiplicative constant, physical variables such as *time* and *potential energy* to which standard mathematical operations do apply are unique up to an additive constant and a positive multiplicative constant. (If s and t are two scales then for *time* or *potential energy* $t = p + q \times s$ for some real numbers p and $q > 0$ while for *length* or *mass* $t = q \times s$ for some $q > 0$.) This observation implies that Guild's argument against the possibility of measurement of psychological variables is not entirely correct. It also seems to indicate the need to identify an empirical – "natural" in von Neumann and Morgenstern's terminology – operation for *preference* measurement for which the resulting scales are unique up to an additive constant and a positive multiplicative constant. Seeking an empirical operation that mimics the “center of gravity” operation, they identified the now-familiar utility theory’s operation of constructing lotteries on “prizes” to serve this purpose.

Von Neumann and Morgenstern’s *uniqueness* argument and *center of gravity* operation are the central elements of their utility theory which is formalized in the axioms of [53, §3.6]. This theory is the basis of game theory which, in turn, was to serve as the mathematical foundation of economic theory. Elaborating upon von Neumann and Morgenstern’s concepts, Stevens [62] proposed a uniqueness-based classification of “scale type” and the focus on the issues of the possibility of measurement of psychological variables and the applicability of mathematical operations to scale values has moved to the construction of “interval” scales, i.e. scales that are unique up to an additive constant and a positive multiplicative constant.

3.2.2 Applicability of Operations on Scale Values Versus Scale Operations

Consider the applicability of the operations of addition and multiplication to scale values for a fixed scale, that is, operations that express facts such as “the weight of an object equals the sum of the weights of two other ones” (which corresponds to addition: $s(a) = s(b) + s(c)$) and “the weight of a given object is two and a half times the weight of another” (which corresponds to multiplication: $s(a) = 2.5 \times s(b)$).

It is important to emphasize the distinction between the application of the operations of addition and multiplication to scale values for a fixed scale (e.g., $s(a) = s(b) + s(c)$) as opposed to what appears to be the same operations when they are applied to an entire scale whereby an equivalent scale is produced (e.g., $t = p + q \times s$ where s and t are two scales and p, q are numbers). In the case of scale values for a fixed scale, the operations of addition and multiplication are applied to elements of the mathematical system M and the result is another element of M. In the case of operations on entire scales, addition or multiplication by a number is applied to an element of the set $S = \{s, t, \ldots\}$ of all possible scales and the result is another element of S rather than M. These are different operations because operations are functions and functions with different domains or ranges are different.

In the case of “interval” scales where the uniqueness of the set of all possible scales is characterized by scale transformations of the form $t = p + q \times s$, it cannot be concluded that the operations of addition and multiplication are applicable to scale values for a fixed scale such as $s(a) = s(b) + s(c)$. It might be claimed that the characterization of scale uniqueness by $t = p + q \times s$ implies the applicability of addition and multiplication to scale values for fixed scales, but this claim requires proof. There is no such proof, nor such claim, in the literature because this claim is false: Consider the automorphisms of the group of integers under addition. The group is a model of itself ($E = M$), and scale transformations are multiplicative: $t = (\pm 1) \times s$. However, by definition, the operation of multiplication which is defined on the set of scales is not defined on the group M.

3.3 The Principle of Reflection

Consider the measurement of *length* and suppose that we can only carry out ordinal measurement on a set of objects, that is, for any pair of objects we can determine which one is longer or whether they are equal in length (in which case we can order the objects by their length). This may be due to a deficiency with the state of technology (appropriate tools are not available) or with the state of science (the state of knowledge and understanding of the empirical or mathematical system is insufficient). We can still construct scales (functions) that map the empirical objects into the real numbers but although the real numbers admit many operations and relations, the only relation on ordinal scale values that is relevant to the property under measurement is the relation of order. Specifically, the operations of addition and multiplication can be carried out on the range of such scales since the range is a subset of the real numbers, but such operations are extraneous because they do not reflect corresponding empirical operations. Extraneous operations may not be carried out on scale values – they are irrelevant and inapplicable; their application to scale values is a modelling error.

The Principle of Reflection is an essential element of modelling that states that operations within the mathematical system are applicable *if and only if* they reflect corresponding operations within the empirical system. In technical terms, in order for the mathematical system to be a valid model of the empirical one, the mathematical system must be homomorphic to the empirical system (a homomorphism is a structure-preserving mapping). A mathematical operation is a valid element of the model only if it is the homomorphic image of an empirical operation. Other operations are not applicable on scale values.

By *The Principle of Reflection*, a necessary condition for the applicability of an operation on scale values is the existence of a corresponding empirical operation (the homomorphic pre-image of the mathematical operation). That is, *The Principle of Reflection* applies in both directions and a given operation is applicable in the mathematical image only if the empirical system is equipped with a corresponding operation.

3.4 The Ordinal Utility Claim in Economic Theory

Preference theory, which plays a fundamental role in decision theory, plays the same role under the name utility theory (see Section 3.9.4) in economic theory. We now show that in the context of economic theory, utility theory is founded on errors that have not been detected by decision theorists or other scholars. In his *Manual of Political Economy*, Pareto claims that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility*" [54, p. 393]. More precisely, it is claimed that ordinal utility scales are sufficient to carry out Pareto's development of economic equilibrium. This claim is surprising considering that Pareto's *Manual* is founded on the notions of differentiable utility scales (by different names such as "ophelimity" and "tastes"). This claim is also surprising because a parallel claim stating that *ordinal temperature scales are sufficient to carry out partial differentiation in thermodynamics* is obviously false. It is even more surprising that this false claim has escaped notice for so long and is repeated in current economic literature.

Relying on Pareto's error, Hicks [36, p. 18] states that "The quantitative concept of utility is not necessary in order to explain market phenomena." With the goal of establishing a *Logical foundation of deductive economics* – having identified the *Need for a theory consistently based upon ordinal utility* – (see the titles of Chapter I's sections in *Value and Capital* [36]) he proceeds "to undertake a purge, rejecting all concepts which are tainted by quantitative utility" [36, p. 19]. In essence, Hicks claims that wherever utility appears in economic theory, and in particular in Pareto's theory which employs partial differentiation, it can be replaced by ordinal utility (see also the title *The ordinal character of utility* [36, Chapter I, §4]).

Neither Pareto, who did not act on his claim, nor Hicks, who did proceed to purge "quantitative utility" from economic theory, provide rigorous mathematical justification for this claim and it seems that authors who repeat this claim rely on an incorrect argument in Samuelson's *Foundations of Economic Analysis* [58, pp. 94–95].

3.4.1 Ordinal Utility

An ordinal empirical system E is a set of empirical objects together with the relation of order, which characterize a property under measurement. A mathematical model M of an ordinal empirical system E is an ordered set where the order in M reflects the order in E. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the order of E into M. In general, the purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M and operations that are not defined in E are not applicable in M. In the case of ordinal systems the mathematical image M of the empirical system E is equipped only with order and the operations of addition and multiplication are not applicable in M. In other words, since, by definition, in ordinal systems only order is defined (explicitly – neither addition nor multiplication is defined), addition and multiplication are not applicable on ordinal scale values and it follows that the operation of differentiation is not applicable on ordinal scale values because differentiation requires that the operations of addition and multiplication be applicable.

In summary, if $u(x_1, \ldots, x_n)$ is an ordinal utility function it cannot be differentiated and conversely, a utility function that satisfies a differential condition cannot be an ordinal utility scale.

3.4.2 Optimality Conditions on Indifference Surfaces

In [36, p. 23] Hicks says that "Pure economics has a remarkable way of producing rabbits out of a hat" and that "It is fascinating to try to discover how the rabbits got in; for those of us who do not believe in magic must be convinced that they got in somehow." The following is treated with only that minimal degree of rigor which is necessary to discover how this observation applies to the use of, supposedly ordinal, utility functions in the standard derivation of elementary equilibrium conditions. (A greater degree of rigor is necessary if other errors are to be avoided.)

Consider the problem of maximizing a utility function $u(x_1, \ldots, x_n)$ subject to a constraint of the form $g(x_1, \ldots, x_n) = b$ where the variables $x_1, \ldots, x_n$ represent quantities of goods. Differentiating the Lagrangean $L = u - \lambda(g - b)$ we have

$$\frac{\partial u}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0 \quad \text{for } i = 1, \ldots, n$$

which implies $\frac{\partial u}{\partial x_i} \div \frac{\partial g}{\partial x_i} = \lambda = \frac{\partial u}{\partial x_j} \div \frac{\partial g}{\partial x_j}$ for all $i,\ j$, and therefore

$$\frac{\partial u}{\partial x_j} \div \frac{\partial u}{\partial x_i} = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i}. \tag{3.1}$$

Equation 3.1 is a tangency condition because, in common notation,

$$\frac{\partial x_i}{\partial x_j} = -\left(\frac{\partial f}{\partial x_j} \div \frac{\partial f}{\partial x_i}\right) \tag{3.2}$$

holds on a surface where a function $f(x_1, \ldots, x_n)$ is constant. Since applying this notation to Eq. 3.1 yields

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j},$$

it is preferable to use the explicit notation

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u$$

to indicate that the differentiation is performed on an indifference surface of the function u at the point x. This derivative depends on the function u as well as the point x; the function u is not "eliminated" in this expression. In general, at an arbitrary point x we expect

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u \neq \left.\frac{\partial x_i}{\partial x_j}\right|_x^g$$

but at the solution point x^* Eq. 3.1 implies

$$\left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^u = \left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^g \quad \text{for all } i, j, \tag{3.3}$$

which, together with the constraint $g(x_1, \ldots, x_n) = b$, is a system of equations for the n unknowns $x^* = \left(x_1^*, \ldots, x_n^*\right)$.

In the special case of a budget constraint $p_1 x_1 + \cdots + p_n x_n = b$ where p_i is the price of good i,

$$-\left.\frac{\partial x_i}{\partial x_j}\right|_x^g = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i} = \frac{p_j}{p_i}$$

and the solution satisfies

$$p_1 x_1^* + \cdots + p_n x_n^* = b \qquad \text{and} \qquad -\left.\frac{\partial x_i}{\partial x_j}\right|_{(x_1^*, \cdots, x_n^*)}^u = \frac{p_j}{p_i} \qquad \text{for all } i, j. \tag{3.4}$$

When the number of variables is greater than two, this system of equations cannot be solved by the method of indifference curves, i.e. by using two-dimensional diagrams, because the left hand sides of the equations in (3.4) depend on all the n unknowns. For example, we can construct a family of indifference curves in the (x_1, x_2) plane where the variables $x_3, \ldots, x_n$ are fixed, but $x_3, \ldots, x_n$ must be fixed at the unknown solution values $x_3^*, \ldots, x_n^*$. To emphasize, with each fixed value of the variables $x_3, \ldots, x_n$ is associated a family of (x_1, x_2) indifference curves. To solve for x_1^*, x_2^* by the method of indifference curves, it is necessary to construct the specific family of indifference curves that corresponds to the solution values $x_3^*, \ldots, x_n^*$, but these values are not known. Noting again that the utility function u is not eliminated in Eq. 3.4 and that this equation was derived using the operation of differentiation which is not applicable on ordinal utility functions, we conclude that Hicks's "Generalization to the case of many goods" [36, §9] has no basis.

Returning to Eq. 3.2, we note that $f(x_1, \ldots, x_n)$ and $F(f(x_1, \ldots, x_n))$ have the same indifference surfaces (but with different derivatives) and, by the chain rule, if F and $f(x_1, \cdots, x_n)$ are both differentiable then

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^{F(f)} = \left.\frac{\partial x_i}{\partial x_j}\right|_x^f \tag{3.5}$$

so that this partial derivative is independent of F. However, since both F and f are assumed to be differentiable, Eq. 3.5 does not imply that f is ordinal.

3.4.3 Pareto's Claim

In the Appendix to his *Manual of Political Economy* [54, pp. 392–394] Pareto considers the indifference surfaces of the utility $I = \Psi(x, y, z, \ldots)$ of the goods $x, y, z, \ldots$. Taking for granted the applicability of the operation of differentiation, if $I = F(\Psi)$ "is differentiated with I taken as a constant," Pareto obtains the equation (numbered (8) in his Appendix) $0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \cdots$ independently of F. This equation is followed by the statement that "An equation equivalent to the last mentioned could be obtained directly from observation." Pareto then says that the latter equation (numbered (9) in his Appendix), $0 = q_x dx + q_y dy + q_z dz + \cdots$, "contains nothing which corresponds to ophelimity, or to the indices of ophelimity" (where he uses the term ophelimity for utility) and concludes that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility*" [54, p. 393].

This conclusion has no basis: "direct observation" does not constitute mathematical proof; Pareto does not define the variables $q_x, q_y, q_z, \ldots$; and it is not clear what it is which he directly observes. On the contrary, if Pareto's equation

$$0 = q_x dx + q_y dy + q_z dx + \cdots$$

contains nothing which corresponds to utility, it cannot be equivalent to his equation

$$0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \cdots ,$$

which characterizes utility indifference surfaces. As pointed out in Section 3.4.1, since Ψ satisfies a differential condition it cannot be an ordinal utility scale.

3.4.4 Samuelson's Explanation

Samuelson defines an ordinal utility scale $\varphi(x_1, \ldots, x_n)$ in Eqs. 6–8 of [58, p. 94] and states, correctly, that any function $U = F(\varphi)$ where $F'(\varphi) > 0$ reflects the same order. However, this observation does not imply that φ is ordinal. On the contrary, since this observation is based on differentiating both F and φ, it is only valid if φ is differentiable in which case it cannot be ordinal.

The paragraph that follows this observation in [58, p. 94] consists of one sentence: "To summarize, our ordinal preference field may be written [here Samuelson repeats his Eq. 9 as Eq. 10] where φ is any one cardinal index of utility." Recalling Hicks's comment that "It is fascinating to try to discover how the rabbits got in," this sentence is remarkable, for "those of us who do not believe in magic" will note that the *ordinal* utility at the beginning of the sentence has metamorphosed into *cardinal* utility at the sentence's end. Note that Samuelson does not define the concept of "cardinal" utility, nor does it appear to be defined elsewhere in the literature.

The concepts of tangents, partial derivatives, and differentials that follow on the next page (Samuelson [58, p. 95]) are applicable only if the utility scales in question are differentiable in which case they cannot be ordinal. Additional analysis of the rest of Samuelson's explanation is not necessary, except that it should be noted that the *marginal utilities* that appear in Eq. 25 that follows on [58, p. 98] are partial derivatives of a utility function. If the derivatives of this utility function, i.e. the marginal utilities, exist it cannot be ordinal. Finally, we note that Samuelson's use of preference and utility as synonyms is consistent with the observations in Section 3.9.4 of this chapter.

3.4.5 Counter-Examples

Define an *ordinal utility* function of two variables by $u(x, y) = xy$ if x or y is a rational number, and $u(x, y) = x^3 y$ otherwise. Under the budget constraint $p_1 x + p_2 y = b$ the tangency condition

$$-\frac{\partial y}{\partial x}\bigg|^{u} = \frac{p_1}{p_2}$$

does not hold because (regardless of how the "or" in the definition of $u(x, y)$ is interpreted) the left hand side of this equation is undefined – the derivative does not exist.

More generally, given any finite ordinal system, there exist smooth ordinal utility scales u_1 and u_2 such that

$$\frac{\partial x_i}{\partial x_j}\bigg|_x^{u_1} = -\left(\frac{\partial u_1}{\partial x_j} \div \frac{\partial u_1}{\partial x_i}\right) \neq -\left(\frac{\partial u_2}{\partial x_j} \div \frac{\partial u_2}{\partial x_i}\right) = \frac{\partial x_i}{\partial x_j}\bigg|_x^{u_2}, \tag{3.6}$$

which means that the marginal substitution rate $\frac{\partial x_i}{\partial x_j}$ is undefined at x. These counter-examples show that ordinal utility scales are not sufficient for the derivation of the standard equilibrium conditions of consumer demand theory. In current economic theory (see, e.g., Chapter 3 in Mas-Colell et al. [46]), the claim that ordinal utility theory is sufficient to establish the existence of the partial derivatives that define marginal substitution rates is based on errors. Ordinal systems do not constitute vector spaces; vector differences and norms are undefined in such systems; and there is no basis for the concepts of convexity, continuity, and differentiation in ordinal systems (see, e.g., Definition 3.B.3 in Mas-Colell et al. [46, p. 42]).

3.5 Shortcomings of Utility Theory

Campbell's argument against the possibility of measurement of psychological variables can be rejected on the basis of von Neumann and Morgenstern's uniqueness argument but constructing utility scales that are immune from Campbell's argument is not equivalent to establishing that psychological variables can be measured. In fact, as we show in Section 3.5.2, the operations of addition and multiplication do not apply to utility scale values. This and additional shortcomings of utility theory render it unsuitable to serve as the foundation for the application of mathematical methods in decision theory or in economic theory.

3.5.1 Von Neumann and Morgenstern's Utility Theory

The fundamental role of preference modelling in game theory was recognized by von Neumann and Morgenstern (see [53, §§3.5–3.6]) but their treatment of this difficult problem which is the basis for latter developments in "modern utility theory" (cf. Fishburn [31, §1.3] and Coombs et al. [25, p. 122]) suffers from multiple flaws and this theory cannot serve as a foundation for any scientific theory.

In essence, von Neumann and Morgenstern study a set of objects A equipped with an operation (i.e. a function) and the relation of order (which is not an operation) that satisfy certain assumptions. The operation is of the form $f(\alpha, a, b)$, where a and b are objects in A, α is a real number, and $c = f(\alpha, a, b)$ is an object in A. Their main result is an *existence* and *uniqueness* theorem for scales (homomorphisms) that reflect the structure of the set A into a set B equipped with order and a corresponding operation $g(\alpha,\ s(a),\ s(b))$ where $a \to s(a),\ b \to s(b)$, and $f(\alpha, a, b) \to g(\alpha, s(a), s(b))$.

3.5.2 Addition and Multiplication Are Not Applicable to Utility Scales

The Principle of Reflection implies that the operations of addition and multiplication are not applicable to utility scales despite their "interval" type. These operations are not applicable to von Neumann and Morgenstern's utility model because their axioms include *one* compound empirical *ternary* operation (i.e. the "center of gravity" operation which is a function of *three* variables) instead of the *two binary* operations of addition and multiplication (each of which is a function of *two* variables). Addition and multiplication are not enabled on utility scale values in latter formulations as well because none of these formulations is based on two empirical operations that correspond to addition and multiplication. It should be noted that the goal of constructing the utility framework was to enable the application of mathematical operations rather than to build a system with a certain type of uniqueness.

Although modern utility models (e.g., Luce and Raiffa [45, §2.5], Fishburn [31, pp. 7–9], Coombs et al. [25, pp. 122–129], French [32, Ch. 5]) are not equivalent to von Neumann and Morgenstern's model, *The Principle of Reflection* implies that all utility models are weak: despite the fact that they produce "interval" scales, none of these models enables the operations of addition and multiplication.

3.5.3 Barzilai's Paradox: Utility's Intrinsic Contradiction

As an abstract mathematical system, von Neumann and Morgenstern's utility axioms are consistent. However, while von Neumann and Morgenstern establish the *existence* and *uniqueness* of scales that satisfy these axioms, they do not address utility scale *construction*. This construction requires a specific interpretation of the empirical operation in the context of preference measurement (in terms of lotteries) and although the axioms are consistent in the abstract, *the interpretation of the empirical utility operation creates an intrinsic contradiction*. Utility theory constrains the values of utility scales for lotteries while the values of utility scales for prizes are unconstrained. The theory permits lotteries that are prizes (cf. Luce and Raiffa's "neat example" [45, pp. 26–27]) and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained. In other words, utility theory has one rule for assigning values to prizes and a different, conflicting, rule for assigning values to lotteries. For a prize which is a lottery ticket, the conflicting rules are contradictory. For a numerical example see Barzilai [11] or [14].

3.5.4 Utility Theory Is Neither Prescriptive Nor Normative

Coombs et al. [25, p. 123]) state that "utility theory was developed as a prescriptive theory." This claim has no basis since von Neumann and Morgenstern's utility theory as well as its later variants (e.g., Luce and Raiffa [45, §2.5], Fishburn [31, pp. 7–9], Coombs et al. [25, pp. 122–129], French [32, Ch. 5], Luce [43, p. 195]) are mathematical theories. These theories are of the form $P \rightarrow Q$, that is, if the assumptions P hold then the conclusions Q follow. In other words, these theories are not of the form "*Thou Shalt Assume P*" but rather "*if* you assume P." Since mathematical theories do not dictate to decision makers what sets of assumptions they *should* satisfy, the claim that utility theory is prescriptive has no basis in mathematical logic nor in modern utility theory.

Howard says that a normative theory establishes norms for how things should be (*In Praise of the Old Time Religion* [38, p. 29]) and appears to suggest that decision theory says how you should act in compliance with von Neumann and Morgenstern's assumptions [53, p. 31]. His comments on "second-rate thinking" and education [38, p. 30] seem to indicate that he believes that those who do not share his praise for the old time utility religion need to be re-educated. In the context of logic and science this position is untenable – mathematical theories do not dictate assumptions to decision makers. Furthermore, educating decision makers to follow flawed theories is not a remedy for "second-rate thinking." Flawed theories should be corrected rather than be taught as the norm.

Unfortunately, according to Edwards [28, pp. 254–255], Howard is not alone. Edwards reports as editor of the proceedings of a conference on utility theories that the attendees of the conference unanimously agreed that the experimental and observational evidence has established as a fact the assertion that people do not maximize "subjective expected utility" and the attendees also unanimously stated that they consider "subjective expected utility" to be the appropriate normative rule for decision making under risk or uncertainty. These utility theorists are saying that although decision makers reject the assumptions of the *mathematical theory* of utility, they should accept the conclusions which these assumptions imply. This position is logically untenable.

3.5.5 Von Neumann and Morgenstern's Structure Is Not Operational

The construction of utility functions requires the interpretation of the operation $f(\alpha, a_1, a_0)$ as the construction of a lottery on the prizes a_1, a_0 with probabilities $\alpha, 1-\alpha$, respectively. The utility of a prize a is then assigned the value α where $u(a_1) = 1, u(a_0) = 0$ and $a = f(\alpha, a_1, a_0)$.

In order for $f(\alpha, a_1, a_0)$ to be an operation, it must be a single-valued function. Presumably with this in mind, von Neumann and Morgenstern interpret the relation of equality on elements of the set A as *true identity*: in [53, A.1.1–A.1.2, p. 617] they remark in the hope of "dispelling possible misunderstanding" that "[w]e do not axiomatize the relation =, but interpret it as *true identity*." If equality is interpreted as true identity, equality of the form $a = f(\alpha, a_1, a_0)$ cannot hold when a is a prize since a lottery and a prize are not identical objects. Consequently, von Neumann and Morgenstern's interpretation of their axioms does not enable the practical construction of utility functions.

Possibly for this reason, later variants of utility theory (e.g., Luce and Raiffa [45]) interpret equality as indifference rather than true identity. This interpretation requires the extension of the set A to contain the lotteries in addition to the prizes. In this model, lotteries are elements of the set A rather than an operation on A so that this extended set is no longer equipped with any operations but rather with the relations of order and indifference (see, e.g., Coombs et al. [25, p. 122]). This utility structure is not homomorphic (and therefore is not equivalent) to von Neumann and Morgenstern's structure and the utility functions it generates are weak (i.e. do not enable the operations of addition and multiplication) and only enable the relation of order despite their "interval" type of uniqueness.

3.6 Shortcomings of Game Theory

As a branch of decision theory, game theory is an operations research discipline that was founded by von Neumann and Morgenstern [53] with the aim of serving as the proper instrument with which to develop a theory of economic behavior. Unfortunately, game theory is founded on multiple errors and while its utility foundations can be replaced with proper ones, other fundamental game theory errors must be corrected if it is to serve as the mathematical foundation of economic theory (see Barzilai [12–14]). In particular, preference measurement plays a fundamental role, and is necessary in order to introduce the real numbers and operations on them, in game theory and economics and it is not possible to escape the need to construct preference functions by assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money. The mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, preference scales are not unique, and preference spaces are not vector spaces. See Barzilai [15–18].

3.6.1 Undefined Sums

The expression $v(S) + v(T)$ which represents the sum of coalition values in von Neumann and Morgenstern's definition of the characteristic function of a game has no basis since, by *The Principle of Reflection*, addition is undefined for utility or value scales. The sum of points on a straight line in an affine geometry, which is the correct model for preference measurement (see Section 3.7.1), is undefined as well. For the same reasons, the sum of imputations, which are utilities, is undefined. In consequence, throughout the literature of game theory, the treatment of the topic of the division of the "payoff" among the players in a coalition has no foundation.

3.6.2 The Utility of a Coalition

The definition of the characteristic function of a game depends on a reduction to "the value" of a two-person (a coalition vs. its complement) game. In turn, the construction of a two-person-game value depends on the concept of expected utility of a player. The reduction treats a coalition, i.e. a group of players, as a single player but there is no foundation in the theory for *the utility of a group of players*.

3.6.3 "The" Value of a Two-Person Zero-Sum Game Is Ill-Defined

To construct von Neumann and Morgenstern's characteristic function, a coalition and its complement are treated as players in a two-person zero-sum game, and the coalition is assigned its "single player" value in this reduced game. However, the concept of "the" value of two-person zero-sum game theory is not unique and consequently is ill-defined.

The minimax theorem which states that every two-person zero-sum game with finitely many pure strategies has optimal mixed strategies is a cornerstone of game theory. Given a two-person zero-sum game, denote by x^* and y^* the minimax optimal strategies and by u the utility function of player 1. Utility functions are not unique and for any p and positive q, u is equivalent to $p + q \times u$ but since the minimax optimal strategies do not depend on the choice of p and q, x^* and y^* are well defined. However, the value of the game varies when p and q vary so that it depends on the choice of the utility function u and given an arbitrary real number v, the numbers p and q can be chosen so that the value of the game equals v. As a result, the concept of "the" value of a game is ill-defined and any game theoretic concept that depends on "the" value of a game is ill-defined as well.

3.6.4 The Characteristic Function of Game Theory is Ill-Defined

The *construction* of the characteristic function of a game is ignored in the literature where it is assumed that a characteristic function is "given" and conclusions are drawn from its numerical values. This is not surprising since without specifying whose values are being measured the characteristic function of a game cannot be constructed.

The assignment of values to objects such as outcomes and coalitions, i.e. the construction of value functions, is a fundamental concept of game theory. *Value* (or utility, or preference) is not a physical property of the objects being valued, that is, *value* is a subjective (or psychological, or personal) property. Therefore, the definition of *value* requires specifying both *what* is being valued and *whose* values are being measured.

Game theory's characteristic function assigns values to coalitions but von Neumann and Morgenstern do not specify *whose* values are being measured in the construction of this function. Since it is not possible to construct a value (or utility) scale of an unspecified person or a group of persons, game theory's characteristic function is not well defined. All game theory concepts that depend on values where it is not specified whose values are being measured are ill-defined (see also Barzilai [21]). This includes the concept of imputations, von Neumann and Morgenstern's solution of a game, and Shapley's value [33, 60] and [5, Chapter 3] in all its variants and generalizations (e.g., McLean [47], Monderer and Samet [52], and Winter [63]). Moreover, since the current definition of an n-person game employs the ill-defined concept of the characteristic function (see, e.g., Monderer and Samet [52, p. 2058]), the very definition of a game has no foundation.

3.6.5 The Essential Role of Preference

Under the heading "The Mathematical Method in Economics" von Neumann and Morgenstern state in *Theory of Games and Economic Behavior* [53, §1.1.1] that the purpose of the book was "to present a discussion of some fundamental questions of economic theory." *The role of preference measurement in game theory is essential* because the outcomes of economic activity are empirical objects rather than real numbers such as $\sqrt{\pi}$ and the application of mathematical operations such as addition and multiplication requires the mathematical modelling of economic systems by corresponding mathematical systems. In other words, *the purpose of preference measurement is to introduce the real numbers and operations on them in order to enable the application of The Mathematical Method*.

Consider Guild's statement in support of the position that mathematical operations are not applicable to non-physical variables (his position as well as the opposing position were based on incorrect arguments concerning the applicability of mathematical operations to non-physical variables – see Section 3.7.1) as summarized in [30, p. 345] in the context of measurement of *sensation*:

> I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of addition as applied to sensation. No such meaning has ever been defined. When we say that one length is twice another or one mass is twice another we know what is meant: we know that certain practical operations have been defined for the addition of lengths or masses, and it is in terms of these operations, and in no other terms whatever, that we are able to interpret a numerical relation between lengths and masses. But if we say that one sensation intensity is twice another nobody knows what the statement, if true, would imply.

Note that the *property* (length, mass, etc.) of the objects must be specified in order for the mathematical operations to be applicable and that addition and multiplication are applied on lengths and masses of objects. It is not possible to "add objects" without knowing whether what is being added is their mass, length, temperature, etc. Observing that *preference* is the only property of relevance in the context of the mathematical foundations of game theory, we conclude that preference measurement is not a cosmetic issue but a fundamental one in this context.

3.6.6 Implications

The fact that *preference modelling is of the essence* in game theory implies that much of the theory is in error. Under the title "What is game theory trying to accomplish?" Aumann [3] says that game theory is not a branch of abstract mathematics but is rather motivated by and related to the world around us. As pointed out above, economic transactions are not performed in order to attain as an outcome the number $\sqrt{\pi}$. Stated differently, the outcome of a real-world economic transaction is seldom a real number. One therefore cannot simply "assume" (see, e.g., Definition 2.3 in Aumann [5]) that the outcome of an economic transaction which is modelled as a play of a game is a numerical payoff function. The only way to introduce the real numbers, and thereby *The Mathematical Method*, into game theory is through the construction of preference functions which represent preference for empirical objects including outcomes of games. As we shall see in Section 3.6.7, it is not possible to escape the need to construct preference functions by "assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [5, p. 106]). Note that this statement implies that utility is a property of money so that in the fundamental structure of preference modelling (see Section 3.2), money, in the form of a $20 bill, or 20 coconuts, cocoa beans, dried fish, salt bars, or a beaver pelt (cf. Shubik [61, p. 361]), is an object rather than a property of empirical objects. In the context of mathematical modelling the distinction between objects and properties of objects is fundamental. (In addition to these considerations, the mathematical operations of game theory must be performed on the preferences of the players because what matters to them is their preferences for the outcomes rather than the physical outcomes.)

Having concluded that the mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, recall that (1) preference functions are not unique (they are unique up to affine transformations) and (2) the sum of values of a preference function is undefined (see Section 3.7.1).

3.6.7 On "Utility Functions That Are Linear in Money"

Consider again the assumption that "payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [5, p. 106]). In addition to the obvious reasons for rejecting this assumption (e.g., the St. Petersburg Paradox which implies that this is an unrealistic assumption; it is also necessary to make the even more unrealistic assumption that the additive and multiplicative constants in the players' utility scales are all identical) we re-emphasize that money is not a property of objects and preference functions are unique up to affine rather than linear transformations. This implies that in the case of monetary outcomes it is still necessary to construct the decision maker's preference function for money.

It is correct to say that a given decision maker (who must be identified since preference is a subjective property) is indifferent between the objects A and B where B is a sum of money, which means that $f(A) = f(B)$ where f is the decision maker's preference function. However, the statement that the outcome of a play is the object A and $f(A) = f(B)$ requires the determination of the preference value $f(A)$ and, *in addition*, $f(B)$ as well as the identification of the object B for which $f(A) = f(B)$. It follows that this indirect and more laborious procedure does not eliminate the need to construct the decision maker's preference function and *game theory cannot be divorced from preference modelling*. It follows that there is no escape from the fact that utility sums are undefined.

3.6.8 The Minimax Solution of Two-Person Zero-Sum Games

In [22, 23] we give examples that show that even for repeated games, the minimax solution of two-person zero-sum game theory prescribes to the players "optimal" strategies that cannot be described as conservative or rational. In addition, since the minimax probabilities do not depend on the outcomes of the game (they only depend on the numerical payoffs which are associated with the outcomes), they are invariant with respect to a change of payoff unit. For example, denote the outcomes of a two-person zero-sum game by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.7}$$

where player 1 can choose between $R1$ and $R2$ (rows) and player 2 between $C1$ and $C2$ (columns) and consider the case where player 1's utility values for these outcomes are given by

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}. \tag{3.8}$$

According to the minimax rule, player 1 is to play $R1$ or $R2$ with probabilities (0.75, 0.25) regardless of whether the numbers in this game represent the payoffs in cents, dollars, euros, millions of dollars, billions of dollars, or any other unit. Probabilities cannot be assigned to scale values on an indefinite scale; when the scale is changed, the probabilities assigned to scale values must change. Denote by x the probability that the temperature in a given process will reach 20°. Then x must depend on the choice of the temperature scale – it cannot be the case that the temperature will reach 20° on the Celsius scale, the Fahrenheit scale, and any arbitrary other scale with the same probability. In the minimax solution of two-person zero-sum games choice probabilities are divorced from choice consequences because probabilities are assigned to indefinite scale values. This is a fundamental error which indicates that this problem is formulated incorrectly.

The following should be noted: Aumann tells us in the General Summary and Conclusions of his 1985 paper entitled "What Is Game Theory Trying to Accomplish?" [3, p. 65] that "Game-theoretic solution concepts should be understood in terms of their applications, and should be judged by the quantity and quality of their applications." More recently, in their paper entitled "When All Is Said and Done, How Should You Play and What Should You Expect?" Aumann and Dreze [6, p. 2] tell us that 77 years after it was born in 1928, strategic game theory has not gotten beyond the optimal strategies which rational players should play according to von Neumann's minimax theorem of two-person zero-sum games; that when the game is not two-person zero-sum none of the equilibrium theories tell the players how to play; and that the "Harsanyi-Selten selection theory does choose a unique equilibrium, composed of a well-defined strategy for each player and having a well-defined expected outcome. But nobody – least of all Harsanyi and Selten themselves – would actually recommend using these strategies."

This implies that while the meaning of n-person "solutions" is in question, game theorists universally accept the minimax strategy as a reasonable (in fact, *the only*) solution for rational players in two-person zero-sum games. Consistent with this view is Aumann's characterization of the minimax theorem as a vital cornerstone of game theory in his survey of game theory [4, p. 6], yet this solution, too, is a flawed game theory concept.

3.6.9 Errors Not Corrected

It has been suggested that the errors uncovered here have been corrected in recent times, but this is not the case. In the Preface to his 1989 *Lectures on Game Theory* [5], Aumann states that its material has not been superseded. This material includes a discussion of game theory without its preference underpinning, the use of undefined sums, ill-defined concepts, and additional errors.

For example, the "payoff functions" h^i in part (3) of Definition 2.3 in Aumann [5] are not unique and there is no basis for assuming that the outcomes of games are real numbers. Moreover, these functions are unique up to additive and multiplicative constants which are not independent of the index i. As a result, the very definition of a game has no basis even in the simplest two-person case. In the absence of the property of preference, no operations are applicable in game theory but when preference is modelled the sum of values of a preference function is undefined. Such sums appear in Aumann [5] (Definition 3.9 p. 28, Definitions 4.3 and 4.6, p. 38) and throughout game theory's literature.

While Aumann's discussion of Shapley's value ignores utility theory altogether, Hart introduces his 1989 *Shapley Value* [33] as an evaluation, "in the [sic] participant's utility scale," of the prospective outcomes. He then refers explicitly to utility theory and to measuring the value of each player in the game. Note that in addition to the use of undefined sums and ill-defined concepts in the context of Shapley's value, it is not clear whether Shapley's value is intended to represent the evaluation of prospective outcomes of a game by a player or the evaluation of the players themselves (not surprisingly, the question who evaluates the players is not addressed in the literature).

More recently Hart (2004, [34, pp. 36–37]), denoting by x^i the utility of an outcome to player i, refers to the set of utilities as the set of *feasible payoff vectors* and uses the undefined sum of these utilities $\Sigma_{i \in S} x^i$ in the definition of a "transferable utility" game. As pointed out earlier, utility spaces are not vector spaces and utility sums are undefined.

3.7 Reconstructing the Foundations

3.7.1 Proper Scales – Straight Lines

In order to enable the "powerful weapon of mathematical analysis" to be applied to any scientific discipline it is necessary, at a minimum, to construct models that enable the operations of addition and multiplication, for without these operations the tools of linear algebra and elementary statistics cannot be applied. This construction, which leads to the well-known geometrical model of points on a straight line, is based on two observations:

- If the operations of addition and multiplication are to be enabled in the mathematical system M, these operations must be defined in M. The empirical system E must then be equipped with corresponding operations in order for M to be a model of E.
- Mathematical systems with an absolute *zero* or *one* are not homogeneous: these special, distinguishable, elements are unlike others. On the other hand, since the existence of an absolute *zero* or *one* for empirical systems that characterize subjective properties has not been established, they must be modelled by homogeneous mathematical systems.

Sets that are equipped with the operations of addition and multiplication, including the inverse operations of subtraction and division, are studied in abstract algebra and are called *fields*. The axioms that define fields are listed in Section 3.7.3. A field is a not a homogeneous system since it contains two special elements, namely an absolute *zero* and an absolute *one* which are the additive and multiplicative identities of the field (in technical terms, they are invariant under field automorphisms). It follows that to model a subjective property by a mathematical system where the operations of addition and multiplication are defined we need to modify a field in order to homogenize its special elements, i.e. we need to construct a *homogeneous field*. To homogenize the multiplicative identity, we construct a one-dimensional vector space which is a *partially homogeneous field* (it is homogeneous with respect to the multiplicative identity but not with respect to the additive identity) where the elements of the field serve as the set of scalars in the vector space. To homogenize the additive identity as well, we combine points with the vectors and scalars and construct a one-dimensional affine space, which is a homogeneous field, over the previously constructed vector space. The axioms characterizing vector and affine spaces are listed in Section 3.7.3. The end result of this construction, the one-dimensional affine space, is the algebraic formulation of the familiar straight line of elementary (affine) geometry so that for the operations of addition and multiplication to be enabled on models that characterize subjective properties, the empirical objects must correspond to points on a straight line of an affine geometry. For details see Section 3.7.3, or the equivalent formulations in [2, p. 79], and [55, pp. 46–47].

In an affine space, the difference of two points is a vector and no other operations are defined on points. In particular, it is important to note that the ratio of two points as well as the sum of two points are undefined. The operation of addition is defined on *point differences*, which are vectors, and this operation satisfies the *group* axioms listed in Section 3.7.3. Multiplication of a vector by a scalar is defined and the result is a vector. In the one-dimensional case, and only in this case, the ratio of a vector divided by another non-zero vector is a scalar.

It follows that Campbell's argument is correct with respect to the application of *The Principle of Reflection* and the identification of addition as a fundamental operation, but that argument does not take into account the role of the multiplication operation and the modified forms of addition and multiplication when the models correctly account for the degree of homogeneity of the relevant systems. Note also that it is not sufficient to model the operation of addition since, except for the natural numbers, multiplication is not repeated addition: In general, and in particular for the real numbers, multiplication is not defined as repeated addition but through field axioms.

Since the purpose of modelling is to enable the application of mathematical operations, we classify scales by the type of mathematical operations that they enable. We use the terms *proper scales* to denote scales where the operations of addition and multiplication are enabled on scale values, and *weak scales* to denote scales where these operations are not enabled. This partition is of fundamental importance and we note that it follows from *The Principle of Reflection* that all the models in the literature are weak because they are based on operations that do not correspond to addition and multiplication.

3.7.2 Strong Scales – the Real Numbers

Proper scales enable the application of the operations of linear algebra but are not necessarily equipped with the relation of order which is needed to indicate a direction on the straight line (e.g., to indicate that an object is more preferable, or heavier, or more beautiful than another). To construct proper ordered scales the underlying field must be ordered (e.g., the field of complex numbers is unordered while the field of the rational numbers is ordered). For a formal definition of an ordered field see Section 3.7.3.1.

Physics, as well as other sciences, cannot be developed without the mathematical "weapons" of calculus. For example, the basic concept of acceleration in Newton's Second Law is defined as a (second) derivative; in statistics, the standard deviation requires the use of the square root function whose definition requires the limit operation; and marginal rates of change, defined by partial derivatives, are used in economics. If calculus is to be enabled on ordered proper scales, the underlying field must be an ordered field where any limit of elements of the field is itself an element of the field. In technical terms, the underlying field must be *complete* (see McShane and Botts [48, Ch. 1, §5] for a formal definition). Since the only ordered complete field is the field of real numbers, in order to enable the operations of addition and multiplication, the relation of order, and the application of calculus on subjective scales, the objects must be mapped into the real, ordered, homogeneous field, i.e. a one-dimensional, real, ordered, affine space, and the set of objects must be a subset of points on an empirical ordered real straight line. We use the term *strong models* to denote such models and *strong scales* to denote scales produced by strong models.

The application of the powerful weapon of mathematical analysis requires a system in which addition and multiplication, order, and limits are enabled. The reason for the central role played by the real numbers in science is that the field of real numbers is the only ordered complete field.

3.7.3 The Axioms of an Affine Straight Line

3.7.3.1 Groups and Fields

A group is a set G with an operation that satisfies the following requirements (i.e. axioms or assumptions):

- The operation is *closed*: the result of applying the operation to any two elements a and b in G is another element c in G. We use the notation $c = a \circ b$ and since the operation is applicable to pairs of elements of G, it is said to be a binary operation.
- The operation is *associative*: $(a \circ b) \circ c = a \circ (b \circ c)$ for any elements in G.
- The group has an *identity*: there is an element e of G such that $a \circ e = a$ for any element a in G.
- *Inverse elements*: for any element a in G, the equation $a \circ x = e$ has a unique solution x in G. If $a \circ x = e,\ x$ is called the inverse of a.

If $a \circ b = b \circ a$ for all elements of a group, the group is called *commutative*. We re-emphasize that a group is an algebraic structure with *one* operation and we also note that a group is not a homogeneous structure because it contains an element, namely its identity, which is unlike any other element of the group since the identity of a group G is the only element of the group that satisfies $a \circ e = a$ for all a in G.

A *field* is a set F with two operations that satisfy the following assumptions:

- The set F with one of the operations is a commutative group. This operation is called *addition* and the identity of the additive group is called zero (denoted '0').
- The set of all non-zero elements of F is a commutative group under the other operation on F. That operation is called *multiplication* and the multiplicative identity is called one (denoted '1').
- For any element a of the field, $a \times 0 = 0$.
- For any elements of the field the *distributive* law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

Two operations are called addition and multiplication only if they are related to one another by satisfying the requirements of a field; a single operation on a set is not termed addition nor multiplication. The additive inverse of the element a is denoted $-a$, and the multiplicative inverse of a non-zero element is denoted a^{-1} or $1/a$. Subtraction and division are defined by $a - b = a + (-b)$ and $a/b = a \times b^{-1}$.

A field F is ordered if it contains a subset P such that if $a, b \in P$, then $a + b \in P$ and $a \times b \in P$, and for any $a \in F$ exactly one of $a = 0$, or $a \in P$, or $-a \in P$ holds.

3.7.3.2 Vector and Affine Spaces

A vector space is a pair of sets (V, F) together with associated operations as follows. The elements of F are termed *scalars* and F is a field. The elements of V are termed *vectors* and V is a commutative group under an operation termed vector addition.

These sets and operations are connected by the additional requirement that for any scalars $j, k \in F$ and vectors $u, v \in V$ the scalar product $k \cdot v \in V$ is defined and satisfies, in the usual notation, $(j + k)v = jv + kv$, $k(u + v) = ku + kv$, $(jk)v = j(kv)$ and $1 \cdot v = v$.

An *affine space* (or a *point space*) is a triplet of sets (P, V, F) together with associated operations as follows (see also Artzy [2] or Postnikov [55]). The pair (V, F) is a vector space. The elements of P are termed *points* and two functions are defined on points: a one-to-one and onto function $h : P \to V$ and a "difference" function $\Delta : P^2 \to V$ that is defined by $\Delta(a, b) = h(a) - h(b)$. Note that this difference mapping is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \to V$, the sets of points and vectors are equipped with different operations and the operations of addition and multiplication are not defined on points. If $\Delta(a, b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that a point space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences*. Note that in an affine space no point is distinguishable from any other.

The dimension of the affine space (P, V, F) is the dimension of the vector space V. By a homogeneous field we mean a *one-dimensional* affine space. A homogeneous field is therefore an affine space (P, V, F) such that for any pair of vectors $u, v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$. In a homogeneous field (P, V, F) the set P is termed a *straight line* and the vectors and points are said to be collinear. Division in a homogeneous field is defined as follows. For $u, v \in V$, $u/v = \alpha$ means that $u = \alpha v$ provided that $v \neq 0$. Therefore, in an affine space, the expression $\Delta(a, b)/\Delta(c, d)$ for the points $a, b, c, d \in P$ where $\Delta(c, d) \neq 0$ is defined and is a scalar:

$$\frac{\Delta(a, b)}{\Delta(c, d)} \in F \tag{3.9}$$

if and only if the space is one-dimensional, i.e. it is a straight line or a homogeneous field. When the space is a straight line, $\Delta(a, b)/\Delta(c, d) = \alpha$ (where $a, b, c, d \in P$) means by definition that $\Delta(a, b) = \alpha\Delta(c, d)$.

3.8 Measurement Theory

Beginning with Stevens [62] in 1946, measurement theory (which only deals with the *mathematical modelling* of measurement) has centered on issues of scale uniqueness rather than applicability of operations. As a result of the shift of focus from applicability of operations to uniqueness, the operations of addition and multiplication are not applicable on scale values for any scale constructed on the basis of this theory regardless of their "scale type" including "ratio" scales and "interval" scales (see Section 3.2.2 and Barzilai [13]).

The focus of this theory was further narrowed when Scott and Suppes [59] in 1958 adopted a system with a single set of objects as the foundation of the theory. Vector and affine spaces cannot be modelled by such systems because the construction of vector and affine spaces requires two or three sets, respectively (the sets of scalars, vectors, and points). The operations on points, vectors, and scalars are not closed operations: the difference of two points in an affine space is a vector rather than a point and, in a one-dimensional space, the ratio of two vectors is a scalar rather than a vector. Because proper scales for psychological variables are affine scales that are based on three sets, the operations of addition and multiplication are not enabled on scales constructed on the basis of classical measurement theory for any psychological variable for in this theory no model is based on three sets. In particular, this is the case for *preference* which is the fundamental variable of decision theory. In consequence, the mathematical foundations of decision theory must be replaced in order to enable the application of mathematical operations including addition and multiplication.

The mathematical models in *Foundations of Measurement* (Krantz et al. [41] and Luce et al. [44]) and Roberts [56] are incorrect even for the most elementary variable of physics – *position* of points on an affine straight line. Derived from the model of *position*, the correct model for *length* of segments (position differences) on this line is a one-dimensional vector space. Likewise, "extensive measurement" (see, e.g., Roberts [56, §3.2]) is not the correct model for the measurement of *mass*, another elementary physical variable. In essence, "extensive measurement" is the "vector half" of a one-dimensional vector space where multiplication and the scalars are lost. Not surprisingly, the second half of a one-dimensional affine space is then lost in the classical theory's "difference measurement" where the scalars and vectors are both lost together with vector addition and scalar multiplication (see Roberts [56, §3.2–3.3]). In his 1992 paper [42, p. 80], Luce acknowledges the inadequacy of the models of the classical theory: "Everybody involved in this research has been aware all along that the class of homogeneous structures fails to include a number of scientifically important examples of classical physical measurement and, quite possibly, some that are relevant to the behavioral sciences." But despite the evidence of inadequacy, these models have not been corrected in the classical theory.

In summary, the fundamental problem of applicability of mathematical operations to scale values has received no attention in the classical theory of measurement after 1944; the theory does not provide the tools and insight necessary for identifying shortcomings and errors of evaluation and decision methodologies including utility theory and the Analytic Hierarchy Process; the basic model of Scott and Suppes [59] is flawed; and the operations of addition and multiplication are not applicable to scale values produced by any measurement theory model.

3.9 Classical Decision Theory

3.9.1 Utility Theory

Barzilai's paradox (see Section 3.5.3, [14, §6.4.2] and [11, §4.2]) and the inapplicability of addition and multiplication on utility scale values imply that utility theory cannot serve as a foundation for any scientific discipline. In addition, von Neumann and Morgenstern's utility theory was not developed as, and is not, a prescriptive theory neither is it a normative theory (see [14, §6.4.3]). Moreover, the interpretation by von Neumann and Morgenstern of utility equality as a true identity precludes the possibility of indifference between a prize and a lottery which is utilized in the construction of utility scales while under the interpretation of utility equality as indifference the construction of lotteries is not single-valued and is therefore not an operation (see [14, §6.4.4]).

In the context of decision theory, despite the evidence to the contrary (e.g., Barzilai [14, §6.4.3] and [11]), utility theory is still treated by some as the foundation of decision theory and is considered a normative theory. Howard in particular refers to utility theory in the non-scientific term "The Old Time Religion" [38] while elsewhere he refers to "Heathens, Heretics, and Cults: The Religious Spectrum of Decision Aiding" [37]. A recent publication entitled "Advances in Decision Analysis" [29] does not contribute to correcting these errors.

3.9.2 Undefined Ratios and Pairwise Comparisons

In order for the operations of addition and multiplication to be applicable, the mathematical system M must be (1) a field if it is a model of a system with an absolute *zero* and *one*, (2) a one-dimensional vector space when the empirical system has an absolute *zero* but not an absolute *one*, or (3) a one-dimensional affine space which is the case for all non-physical properties with neither an absolute *zero* nor absolute *one*. This implies that for proper scales, scale ratios are undefined for subjective variables including *preference*. In particular, this invalidates all decision methodologies that apply the operations of addition and multiplication to scale values and are based on preference ratios. For example, in the absence of an absolute zero for *time*, it must be modelled as a homogeneous variable and the ratio of two times (the expression t_1/t_2) is undefined. For the same reason, the ratio of two potential energies e_1/e_2 is undefined while the *ratios of the differences* $\Delta t_1/\Delta t_2$ and $\Delta e_1/\Delta e_2$ are properly defined. We saw that the sum of von Neumann and Morgenstern's utility scale values is undefined. Since the sum of two points in an affine space is undefined, the sum of proper preference scale values is undefined as well.

The expression $(a - b)/(c - d) = k$ where a, b, c, d are points on an affine straight line and k is a scalar is used in the construction of proper scales. The number of points in the left hand side of this expression can be reduced from four to three (e.g., if $b = d$) but it cannot be reduced to two and this implies that pairwise comparisons cannot be used to construct preference scales where the operations of addition and multiplication are enabled.

3.9.3 The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP, see Saaty [57]) is not a valid methodology. More than 30 years after the publication of Miller's work in the 1960s [49–51], there is still no acknowledgement in the AHP literature (or elsewhere) of his contribution to decision theory in general and the AHP in particular. Miller was not a mathematician and his methodology is based on mathematical errors although some of its non-mathematical elements are valuable. The AHP is based on these mathematical errors and additional ones (see Barzilai [7–10, 19] and the references there).

Not surprisingly, these errors have been mis-identified in the literature and some of these errors appear in decision theory. For example, Kirkwood [40, p. 53] relies on Dyer and Sarin which repeats the common error that the coefficients of a linear value function correspond to relative importance [27, p. 820]. Furthermore, "difference measurement" which is the topic of Dyer and Sarin is not the correct model of preference measurement. More specifically, in his *Remarks on the Analytic Hierarchy Process* [26, p. 250] Dyer's major focus is in Section 3 where he argues that the AHP "generates rank orderings that are not meaningful" and states that "[a] symptom of this deficiency is the phenomenon of rank reversal" but his argument is circular since the only AHP deficiency presented in Section 3 of his paper is rank reversal. Moreover, the AHP suffers from multiple methodological flaws that cannot be corrected by "its synthesis with the concepts of multiattribute utility theory" (which suffers from its own flaws) as stated by Dyer [26, p. 249].

The AHP is a method for constructing preference scales and, as is the case for other methodologies, the operations of addition and multiplication are not applicable on AHP scale values. The applicability of addition and multiplication must be established before these operations are used to compute AHP eigenvectors and, as we saw in Section 3.2.2, the fact that eigenvectors are unique up to a multiplicative constant does not imply the applicability of addition and multiplication.

In order for addition and multiplication to be applicable on preference scale values, the alternatives must correspond to points on a straight line in an affine geometry (see Section 3.7.1 or Barzilai [11, 12]). Since the ratio of points on an affine straight line is undefined, preference ratios, which are the building blocks of AHP scales, are undefined. In addition, pairwise comparisons cannot be used to construct affine straight lines.

The fundamental mathematical error of using inapplicable operations to construct AHP scales renders the numbers generated by the AHP meaningless. Other AHP errors include the fact that the coefficients of a linear preference function cannot correspond to weights representing relative importance and therefore cannot be decomposed using Miller's criteria tree; the eigenvector method is not the correct method for constructing preference scales; the assignment of the numbers 1–9 to AHP's "verbal scales" is arbitrary, and there is no foundation for these "verbal scales" (see Barzilai [7–10, 19, 20]).

3.9.4 Value Theory

Scale construction for physical variables requires the specification of the empirical objects and the property under measurement. For example, if the property under measurement is *temperature*, the construction results in a *temperature* scale and, clearly, the measurement of *length* does not produce a *mass* scale. In the case of subjective measurement too, the property under measurement must be explicitly specified. If the property under measurement is *preference*, the resulting scales are *preference* scales. Noting that von Neumann and Morgenstern's measurement of preference [53, §3.1] results in utility scales, we conclude that *preference* and *utility* (and, for the same reason, *value, worth, ophelimity*, etc.) are synonyms for the same underlying subjective property. It follows that the distinction between utility theory and value theory has no foundation in logic and science. For example, Keeney and Raiffa's notion of "the utility of value" of an object ($u[v(x)]$, in [39, p. 221]) is as meaningless as "the temperature of the temperature of water" or "the length of the length of a pencil" are.

Likewise, although the notions of "strength of preference" (Dyer and Sarin [27]) and "difference measurement" (e.g., Krantz et al. [41], Roberts [56]) are intuitively appealing, these measurement models of *value, utility, priorities*, etc., are based on measurement theory errors as shown above. Similarly, the utility theories in Edwards [28] are founded on errors as well and, although the issues have been known for a few years, the more recent "Advances in Decision Analysis" (Edwards et al. [29]) does not contribute to correcting these methodological errors.

3.9.5 Group Decision Making

The common view in the classical literature concerning group decision making is based on a misinterpretation of the implications of Arrow's Impossibility Theorem [1] which is a negative result. Constructive theories cannot be founded on negative results and, in addition, this theorem deals with ordinal scales which enable the relation of order but do not enable the operations of addition and multiplication. The concepts of trade-off and cancellation are not applicable to ordinal scales – see Barzilai [14, §6.5] for details.

3.10 Summary

Classical decision and measurement theories are founded on errors which have been propagated throughout the literature and have led to a large number of methodologies that are based on flawed mathematical foundations and produce meaningless numbers. The fundamental issue of applicability of the operations of addition and multiplication to scale values was not resolved by von Neumann and Morgenstern's utility theory and the literature of classical decision and measurement theory offers no insight into this and other fundamental problems. Decision theory is not a prescriptive theory and decision analysis will not be a sound methodology until these errors are corrected.

We identified the conditions that must be satisfied in order to enable the application of linear algebra and calculus, and established that there is only one model for strong measurement of subjective variables. The mathematical foundations of the social sciences need to be corrected to account for these conditions. In particular, foundational errors in utility theory, game theory, mathematical economics, decision theory, measurement theory, and mathematical psychology need to be corrected. It is hoped that the leaders of INFORMS and its Decision Analysis Society, who have been aware of these errors for the last few years, will act to bring these errors to the attention of their followers and correct the educational literature.

This chapter includes the results of very recent research. The development of the theory, methodology, and software tools continues and updates will be posted at www.scientificmetrics.com.

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