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Trends in Multiple Criteria Decision Analysis

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Contents

List of Figures

List of Tables

Introduction

Matthias Ehrgott, José Rui Figueira, and Salvatore Greco

1 Introduction

When 5 years ago we edited the book "Multiple Criteria Decision Analysis: State of the Art Surveys" with 24 chapters written by 49 international leading experts, we believed that the book would cover the research field for several years. But over the last 5 years Multiple Criteria Decision Analysis (MCDA) has received an increasing interest and has experienced a development faster than we expected. Thus, what looked like a comprehensive collection of state-of-the-art surveys appears clearly partial and incomplete a few years later. New approaches and new methodologies have been developed which even contribute to change the paradigm of MCDA. A researcher who does not take into account the new contributed risks to be disconnected from the main trends of the discipline and to have a misleading conception of it. These thoughts convinced us to explore the map of the new trends in MCDA in order to recognize the most promising new contributions. This book comprises 13 chapters, once again written by leading international experts, that summarize trends in MCDA that were not covered in our previous book and that describe the development of rapidly evolving sub-fields of MCDA.

Po-Lung Yu and Yen-Chu Chen present the theory of dynamic multiple criteria decision analysis, habitual domains, and competence set analysis. In real life, most decisions are dynamic with multiple criteria. Even though most of the MCDA literature assumes that the parameters involved in decision problems – such as the set of alternatives, the set of criteria, the preference structures of the decision makers – are more or less fixed and steady, in reality – for most nontrivial decision problems – these parameters can change dynamically. In fact, satisfactory solutions are obtained only when those parameters are properly structured. To analyze the decision process in a dynamic context the concepts of habitual domain and competence set are of fundamental importance. A habitual domain is the set of ideas and concepts which we encode and store in our brain, gradually stabilized over a period of time. The competence set is a collection of ideas, knowledge, resources, skills, and effort for the effective solution of a decision problem. Competence set analysis and habitual domain theory suggest how to expand and enrich our competence set and how to maximize the value of our competence set. In this perspective, any decision problem can be dealt with by restructuring its elements and environmental facets in order to gain a broader and richer perception permitting to derive effective solutions.

Andrzej P. Wierzbicki discusses the need for and possible methods of objective ranking after observing that the classical approach in decision analysis and multiple criteria theory concentrates on subjective ranking. However, in many practical situations, the decision maker might not want to use personal preferences, but prefers to have some objective ranking. One reason for objectivity is that decisions of a given class might influence other people, e.g., some decision situations dominating in technology creation, such as constructing a safe bridge or a safe car. Thus, technologists stress objectivity but real managers also know well that there are many managerial situations where stressing objectivity is necessary. Therefore, even if it can be agreed that an absolute objectivity is not attainable, it is reasonable to treat the concept of objectivity as a useful ideal worth striving for, looking for objective ranking interpreted as an approach to ranking that is as objective as possible. Between many possible multiple criteria approaches, the reference point approach (already introduced in the literature to deal with interactive multiple criteria optimization) is mentioned as the best suited methodology for rational objective ranking, because reference levels needed in this approach can be established to some extent objectively – statistically from the given data set.

Jonathan Barzilai in his provocative chapter discusses preference function modelling, i.e., the mathematical foundations of decision theory. He formulates the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable and claims that the mathematical foundations of decision theory and related theories depend on these conditions, which have not been correctly identified in the classical literature. He argues that Operations Research and Decision Analysis Societies should act to correct fundamental errors in the mathematical foundations of measurement theory, utility theory, game theory, mathematical economics, decision theory, mathematical psychology, and related disciplines. Consequences of this approach to some MCDA methodologies such as AHP or value theory are also discussed.

Hassene Aissi and Bernard Roy discuss robustness in MCDA. The term *robust* refers to a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts. Robustness concerns are related to the observation that an action is made, executed, and judged in a real-life context that may not correspond exactly to the model on which the decision analysis is based. The gap between formal representation and real-life context originates frailty points against which the robustness concern attempts to protect. Robustness concerns can be dealt with using approaches involving a single robustness criterion, completing a preference system that has been defined previously, or using several criteria. Robustness can be considered other than by using one or several criteria to compare the solutions in approaches that involve one or several properties designed to characterize the robust solution or to draw robust conclusions. The considerations developed in this chapter show that the use of multiple criteria for apprehending robustness in MCDA is a field of research open to future development, both theoretically and practically.

Bernard De Baets and János Fodor consider preferences expressed in a gradual way. The key concept is that the application of two-valued (yes or-no) preferences, regardless of their sound mathematical theory, is not satisfactory in everyday situations. Therefore, it is desirable to consider a degree of preference. There are two main frameworks in which gradual preferences can be modeled: fuzzy preferences, which are a generalization of Boolean (2-valued) preference structures, and reciprocal preferences, also known as probabilistic relations, which are generalization of the three-valued representation of complete Boolean preference relations. The authors consider both frameworks. Since the whole exposition makes extensive use of (logical) connectives, such as conjunctors, quasi-copulas and copulas, the authors provide an appropriate introduction on the topic.

Radko Mesiar and Lucia Vavríková present fuzzy set and fuzzy logic-based methods for MCDA. Alternatives are evaluated with respect to each criterion on a scale between 0 and 1, which can be seen as membership function of fuzzy sets. Therefore, alternatives can be seen as multidimensional fuzzy evaluations that have to be ordered according to the decision maker's preferences. This chapter considers several methodologies developed within fuzzy set theory to obtain this preference order. After discussion of integral-based utility functions, a transformation of vectors of fuzzy scores x into fuzzy quantity $U(x)$ is presented. Orderings on fuzzy quantities induce orderings on alternatives. Special attention is paid to defuzzification-based orderings, in particular, the mean of maxima method. Moreover, a fuzzy logic-based construction method to build complete preference structures over the set of alternatives is given.

Wassila Ouerdane, Nicolas Maudet, and Alexis Tsoukiàs discuss argumentation theory in MCDA. The main idea is that decision support can be seen as an activity aiming to construct arguments through which a decision maker will convince first herself and then other actors involved in a problem situation that "that action" is the best one. In this context the authors introduce argumentation theory (in an Artificial Intelligence oriented perspective) and review a number of approaches that indeed use argumentative techniques to support decision making, with a specific emphasis on their application to MCDA.

Valerie Belton and Theodor Stewart introduce problem structuring methods (PSM) in MCDA providing an overview of current thinking and practice with regard to PSM for MCDA. Much of the literature on MCDA focuses on methods of analysis that take a well-structured problem as a starting point with a well-defined set of alternatives from which a decision has to be made and a coherent set of criteria against which the alternatives are to be evaluated. It is an erroneous impression that arriving at this point is a relatively trivial task, while in reality this is not so simple even when the decision makers believe to have a clear understanding of the problem. Thus, PSM provides a rich representation of a problematic situation in order to enable effective multicriteria analysis or to conceptualize a decision, which is initially simplistically presented, in order for the multicriteria problem to be appropriately framed. The chapter outlines the key literature, which explores and offers suggestions on how this task might be approached in practice, reviewing several suggested approaches and presenting a selection of case studies.

Salvatore Greco, Roman Słowiński, José Rui Figueira, and Vincent Mousseau present robust ordinal regression. Within the disaggregation–aggregation approach, ordinal regression aims at inducing parameters of a preference model, for example, parameters of a value function, which represent some holistic preference comparisons of alternatives given by the decision maker. Usually, from among many sets of parameters of a preference model representing the preference information given by the DM, only one specific set is selected and used to work out a recommendation. For example, while there exist many value functions representing the holistic preference information given by the DM, only one value function is typically used to recommend the best choice, sorting, or ranking of alternatives. Since the selection of one from among many sets of parameters of the preference model compatible with the preference information given by the DM is rather arbitrary, robust ordinal regression proposes taking into account all the sets of parameters of the preference model compatible with the preference information, in order to give a recommendation in terms of necessary and possible consequences of applying all the compatible preference models on the considered set of alternatives. For example, the necessary weak preference relation holds for any two alternatives a and b if and only if all compatible value functions give to a a value greater than or equal to the value provided to b, and the possible weak preference relation holds for this pair if and only if at least one compatible value function gives to a a value greater than or equal to the value given to b. This approach can be applied to many multiple criteria decision models such as multiple attribute utility theory, fuzzy integral modeling interaction between criteria, and outranking models. Moreover, it can be applied to interactive multiple objective optimization and can be used within an evolutionary multiple objective optimization methodology to take into account preferences of the decision maker. Finally, robust ordinal regression is very useful in group decisions where it permits to detect zones of consensus for decision makers.

Risto Lahdelma and Pekka Salminen present Stochastic Multicriteria Acceptability Analysis (SMAA). SMAA is a family of methods for aiding multicriteria group decision making in problems with uncertain, imprecise, or partially missing information. SMAA is based on simulating different value combinations for uncertain parameters, and computing statistics about how the alternatives are evaluated. Depending on the problem setting, this can mean computing how often each alternative becomes most preferred, how often it receives a particular rank, or obtains a particular classification. Moreover, SMAA proposes inverse weight space analysis, using simulation with randomized weights in order to reveal what kind of weights make each alternative solution most preferred. After discussing several variants of SMAA the authors describe several real-life applications.

D. Marc Kilgour, Ye Chen, and Keith W. Hipel discuss multiple criteria approaches to Group Decision and Negotiation (GDN). After explaining group decision and negotiation, and the differences between them, the applicability of MCDA techniques to problems of group decision and negotiation is discussed. Application of MCDA to GDN is problematic because – as shown by the well-known Condorcet paradox and by Arrow's theorem on collective choices – collective preferences may not exist. While ideas and techniques from MCDA are directly applicable to GDN only rarely, it is clear that many successful systems for the support of negotiators, or the support of group decisions, have borrowed and adapted ideas and techniques from MCDA. The paper presents a review of systems for Group Decision Support and Negotiation Support, then highlights the contributions of MCDA techniques and some suggestions for worthwhile future contributions from MCDA are put forward.

Kalyanmoy Deb presents recent developments in Evolutionary Multi-objective Optimization (EMO). EMO deals with multiobjective optimization using algorithms inspired by natural evolution mechanisms using a population-based approach in which more than one solution participates in an iteration and evolves a new population of solutions at each iteration. This approach is a growing field of research with many applications in several fields. The author discusses the principles of EMO through an illustration of one specific algorithm (NSGA-II) and an application to an interesting real-world bi-objective optimization problem. Thereafter, he provides a list of recent research and application developments of EMO to paint a picture of some salient advancements in EMO research such as hybrids of EMO algorithms and mathematical optimization or multiple criterion decision-making procedures, handling of a large number of objectives, handling of uncertainties in decision variables and parameters, solution of different problem-solving tasks by converting them into multi-objective problems, runtime analysis of EMO algorithms, and others.

Jacek Malczewski introduces MCDA and Geographic Information Systems (GIS). Spatial decision problems typically involve sets of decision alternatives, of multiple, conflicting, and incommensurate evaluation criteria, and, very often, of individuals (decision makers, managers, stakeholders, interest groups). The critical aspect of spatial decision analysis is that it involves evaluation of the spatially defined decision alternative and the decision maker's preferences. This implies that the results of the analysis depend not only on the geographic pattern of decision alternatives, but also on the value judgments involved in the decision-making process. Accordingly, many spatial decision problems give rise to GIS-MCDA, being a process that combines and transforms geographic data (input maps) and the decision maker's preferences into a resultant decision (output map). The major advantage of incorporating MCDA into GIS is that a decision maker can introduce value judgments (i.e., preferences with respect to decision criteria and/or alternatives) into GIS-based decision making enhancing a decision maker's confidence in the likely outcomes of adopting a specific strategy relative to his/her values. Thus, GIS-MCDA helps decision makers to understand the results of GIS-based decision-making procedures, permitting the use of the results in a systematic and defensible way to develop policy recommendations.

The spectrum of arguments, topics, methodologies, and approaches presented in the chapters of this book is surely very large and quite heterogeneous. Indeed MCDA is developing in several directions that probably in the near future would need to be reorganized in a more systematic theoretical scheme. We know that not all new proposals currently discussed in the field are represented in the book and we are sure that new methodologies will appear in the next years. However, we believe that the book represents the main recent ideas in the field and that, together with the above quoted book "Multiple Criteria Decision Analysis – State of the Art Surveys," it gives sufficient resources for an outline of the field of MCDA permitting to understand the most important and characterizing debates in the area being wholly aware of their origins and of their implications.

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Chapter 1
Dynamic MCDM, Habitual Domains and Competence Set Analysis for Effective Decision Making in Changeable Spaces

Po-Lung Yu and Yen-Chu Chen

Abstract This chapter introduces the behavior mechanism that integrates the discoveries of neural science, psychology, system science, optimization theory and multiple criteria decision making. It shows how our brain and mind works and describes our behaviors and decision making as dynamic processes of multicriteria decision making in changeable spaces. Unless extraordinary events occur or special effort exerted, the dynamic processes will be stabilized in certain domains, known as habitual domains. Habitual domains and their expansion and enrichment, which play a vital role in upgrading the quality of our decision making and lives, will be explored. In addition, as important consequential derivatives, concepts of competence set analysis, innovation dynamics and effective decision making in changeable spaces will also be introduced.

Keywords Dynamic MCDM · Dynamics of human behavior · Habitual domains · Competence set analysis · Innovation dynamics · Decision making in changeable spaces

1.1 Introduction

Humans are making decisions all the time. In real life, most decisions are dynamic with multiple criteria. Take “dining” as an example. There are many things we, consciously or subconsciously, consider when we want to dine. Where shall we go?

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Will we eat at home or dining out? What kind of meal shall we have? Location, price, service, etc. might be the factors that affect our decision of choosing the place to eat. Nutrition, flavor and the preference to food might influence our choices, too. Eating, an ordinary human behavior, is a typical multiple criteria decision problem that we all have to face in our daily life. Its decision changes dynamically as time and situation change. Dynamic multiple criteria decision making (MCDM) is, therefore, not unusual.

Indeed, human history is full of literature recording dynamic MCDM events. However, putting MCDM into mathematical analysis started in the nineteenth century by economists and applied mathematicians including Pareto, Edgeworth, Von Neumann, Morgenstern and many more.

Typically, the studies of MCDM are based on the following three patterns of logic. The first is "simple ordering" which states that a good decision should be such that there is no other alternative that can be better in some aspects and not worse in every aspect of consideration. This concept leads to the famous Pareto optimality and nondominated solutions [42] and quotes therein. The second one is based on human goal-setting and goal-seeking behavior, which leads to satisficing and compromise solution [42] and quotes therein. The third pattern is based on value maximization, which leads to the study of value function. The three types of logic lead to an abundant literature of MCDM [12, 37] and quotes therein. Most literature of MCDM assume that the parameters involved in decision problems such as the set of alternatives, the set of criteria, the outcome of each choice, the preference structures of the decision makers, and the players are, more or less, fixed and steady. In reality, for most nontrivial decision problems, these parameters could change dynamically. In fact, great solutions are located only when those parameters are properly restructured. This observation prompts us to study decision making in changeable spaces [38, 43, 48].

Note that the term "dynamic" could have diverse meanings. From the viewpoint of social and management science sense, it carries the implication of "changeable, unpredictable"; however, from the hard science and technological sense, it may also mean "changing according to inner laws of a dynamic process," which might, but not necessarily, imply unpredictability. Much works in MCDM were motivated by applying multiple criteria analysis to dynamic processes (in the second type of meaning), for example, see the concept of ideal point, nondominated decision, cone convexity and compromise solutions in dynamic problems of Yu and Leitmann [50, 51] and in technical control science of Salukvadze [31, 32]. In this article, we use "dynamic" to imply "changes with time and situation." The dimensions and structures of MCDM could dynamically change with time and situations, consistent with the changes of psychological states of the decision makers and new information.

As a living system, each human being has a set of goals or equilibrium points to seek and maintain. Multiple criteria decision problems are part of the problems that the living system must solve. To broaden our understanding of human decision making, it is very important for us to have a good grasp of human behavior. In order to facilitate our presentation, we first briefly describe three nontrivial decision problems which involve changeable parameters in Section 1.2. The examples will be used to illustrate the concepts introduced in the subsequent sections. In Section 1.3 we shall present a dynamic behavioral mechanism to capture how our brain and mind work. The mechanism is essentially a dynamic MCDM in changeable spaces. In Section 1.4, the concepts and expansion of habitual domains (HDs) and their great impact on decision making in changeable spaces will be explored. As important applications of habitual domains, concepts of competence set Analysis and innovation dynamics will be discussed in Section 1.5. Decision parameters for effective decision making in changeable spaces and decision traps will be described in Section 1.6. Finally in Section 1.7 conclusion and further researches will be provided.

1.2 Three Decision Makings in Changeable Spaces

In this section, three nontrivial decision problems in changeable spaces are briefly described in three examples. The examples illustrate how the challenge problems are solved by looking into the possible changes of the relevant parameters. The examples will lubricate our presentation of the concepts to be introduced in the subsequent sections.

Example 1.1. Alinsky's Strategy (Adapted from [1]) During the days of the Johnson-Goldwater campaign (in 1960s), commitments that were made by city authorities to the Woodlawn ghetto organization of Chicago were not being met. The organization was powerless. As the organization was already committed to support the Democratic administration, the president's campaign did not bring them any help. Alinsky, a great social movement leader, came up with a unique solvable situation. He would mobilize a large number of supporters to legally line up and occupy all the restroom facilities of the busy O'Hare Airport. Imagine the chaotic situation of disruption and frustration that occurred when thousands of passengers who were hydraulically loaded (very high level of charge or stress) rushed for restrooms but could not find the facility to relieve the charge or stress.

How embarrassing when the newspapers and media around the world (France, England, Germany, Japan, Soviet Union, Taiwan, China, etc.) headlined and dramatized the situation. The supporters were extremely enthusiastic about the project, sensing the sweetness of revenge against the City. The threat of this tactic was leaked to the administration, and within 48 hours the Woodlawn Organization was meeting with the city authorities, and the problem was, of course, solved graciously with each player releasing a charge and claiming a victory.

Example 1.2. The 1984 Olympics in LA

The 1984 Summer Olympics, officially known as the Games of the XXIII Olympiad, were held in 1984 in Los Angeles, CA, United States of America. Following the news of the massive financial losses of the 1976 Summer Olympics in Montreal, Canada, and that of 1980s Games in Moscow, USSR, few cities wished to host the Olympics. Los Angeles was selected as the host city without voting because it was the only city to bid to host the 1984 Summer Olympics.

Due to the huge financial losses of the Montreal and that of the Moscow Olympics, the Los Angeles government refused to offer any financial support to the 1984 Games. It was then the first Olympic Games that was fully financed by the private sector in the history. The organizers of the Los Angeles Olympics, Chief Executive Officer Peter Ueberroth and Chief Operating Officer Harry Usher, decided to operate the Games like a commercial product. They raised fund from corporations and a great diversity of activities (such as the torch relay) and products (for example, "Sam the Eagle," the symbol and mascot of the Games), and cut operating cost by utilizing volunteers. In the end, the 1984 Olympic Games produced a profit of over $ 220 million.

Peter Ueberroth, who was originally from the area of business, created the chances to let ordinary people (not just the athletes) and corporations to take part in the Olympic Games, and alter people's impression of hosting Olympic Games.

Example 1.3. Chairman Ingenuity (adapted from [43])

A retiring corporate chairman invited to his ranch two finalists (A and B) from whom he would select his replacement using a horse race. A and B, equally skillful in horseback riding, were given a black and white horse, respectively. The chairman laid out the course for the horse race and said, "Starting at the same time now, whoever's horse is slower in completing the course will be selected as the next Chairman!" After a puzzling period, A jumped on B's horse and rode as fast as he could to the finish line while leaving his horse behind. When B realized what was going on, it was too late! Naturally, A was the new Chairman.

In the first two examples, new players, such as the passengers and the media in Example 1.1 and all the potential customers to the Olympic Games besides the athletes in Example 1.2, were introduced into the decision problem. In the third example, new rule/criteria were introduced, too. These examples show us that in reality, the players, criteria and alternatives (part of decision parameters) are not fixed; instead, they are dynamically changed. The dynamic changes of the relevant parameters play an important role in nontrivial decision problems. To help us understand the dynamic changes, let us introduce first the dynamics of human behavior, which basically is a dynamic MCDM in changeable spaces.

1.3 Dynamics of Human Behavior

Multicriteria decision making is only a part of human behaviors. It is a dynamic process because human behaviors are undoubtedly dynamic, evolving, interactive and adaptive processes. The complex processes of human behaviors have a common denominator resulting from a common behavior mechanism. The mechanism depicts the dynamics of human behavior.

In this section, we shall try to capture the behavior mechanism through eight basic hypotheses based on the findings and observations of psychology and neuron science. Each hypothesis is a summary statement of an integral part of a dynamic system describing human behavior. Together they form a fundamental basis for understanding human behavior. This section is a summary sketch of Yu [40–43, 48].

1.3.1 A Sketch of the Behavior Mechanism

Based on the literature of psychology, neural physiology, dynamic optimization theory, and system science, Yu described a dynamic mechanism of human behavior as presented in Fig. 1.1.

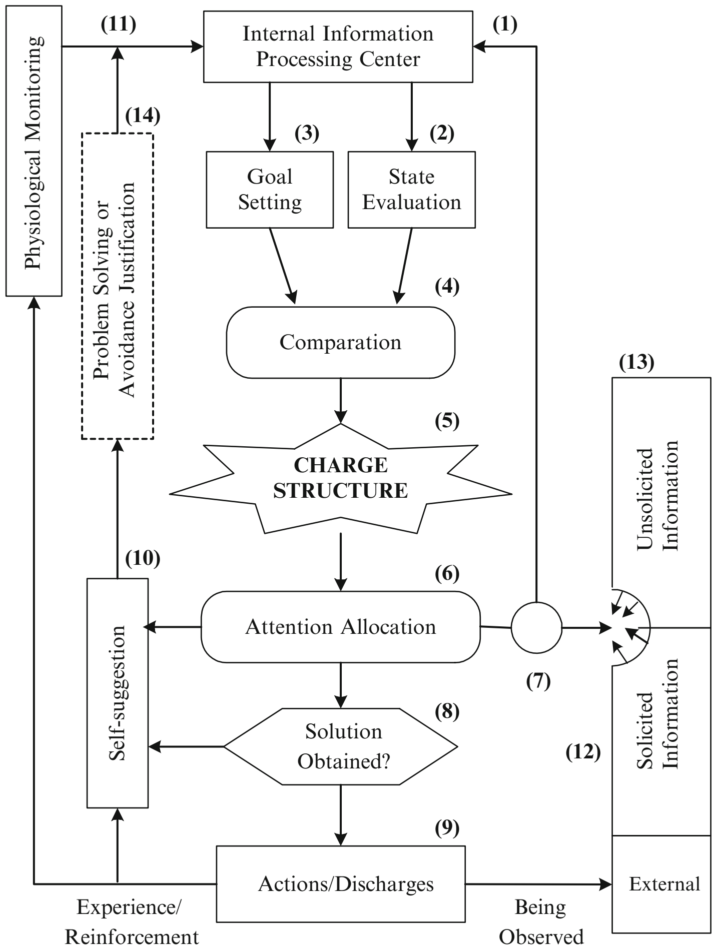

Fig. 1.1 The behavior mechanism

Although Fig. 1.1 is self-explanatory, we briefly explain it as follows:

1. Box (1) is our brain and its extended nerve systems. Its functions may be described by the first four hypotheses (H1–H4) shortly.
2. Boxes (2)–(3) describe a basic function of our mind. We use H5 to explain it.
3. Boxes (4)–(6) describe how we allocate our attention to various events. It will be described by H6.
4. Boxes (8)–(9), (10) and (14) describe a *least resistance principle* that humans use to release their charges. We use H7 to describe it.
5. Boxes (7), (12)–(13) and (11) describe the *information input* to our information processing center (Box (1)). Box (11) is internal information inputs. Boxes (7) and (12)–(13) are for external information inputs, which we use H8 to explain.

The functions described in Fig. 1.1 are interconnected, meaning that through time they can be rapidly interrelated. The outcome of one function can quickly become an input for other functions, from which the outcomes can quickly become an input for the original function.

Observe that the four hypotheses related to Box (1) which describe the information processing functions of the brain are four basic abstractions obtained from the findings of neuron science and psychology. The other Boxes (2)–(14) and hypotheses describe the input, output and dynamics of charges, attention allocation and discharge. They form a complex, dynamic multicriteria optimization system which describes a general framework of our mind. These eight hypotheses will be described in the following subsection.

1.3.2 Eight Hypotheses of Brain and Mind Operation

While the exact mechanism of how the brain works to encode, store and process information is still largely unknown, many neural scientists are still working on the problem with great dedication. We shall summarize what is known into four hypotheses to capture the basic workings of the brain. They are *Circuit Pattern Hypothesis (H1), Unlimited Capacity Hypothesis (H2), Efficient Restructuring Hypothesis (H3) and Analogy/Association Hypothesis (H4).*

The existence of life goals and their mechanism of ideal setting and evaluation lead to dynamic charge structures which not only dictate our attention allocation of time, but also command the action to be taken. This part of the behavior mechanism is related to how our mind works. We shall use another four hypotheses to summarize the main idea: *Goal Setting and State Evaluation Hypothesis (H5), Charge Structure and Attention Allocation Hypothesis (H6), Discharge Hypothesis (H7)* and *Information Inputs Hypothesis (H8).*

1. *Circuit Pattern Hypothesis (H1): Thoughts, concepts or ideas are represented by circuit patterns of the brain. The circuit patterns will be reinforced when the corresponding thoughts or ideas are repeated. Furthermore, the stronger the circuit*

patterns, the more easily the corresponding thoughts or ideas are retrieved in our thinking and decision making processes.

Each thought, concept or message is represented as a circuit pattern or a sequence of circuit patterns. Encoding is accomplished when attention is paid. When thoughts, concepts or messages are repeated, the corresponding circuit patterns will be reinforced and strengthened. The stronger the circuit patterns and the greater the pattern redundancy (or the greater the number of the circuit patterns), the easier the corresponding thoughts, concepts or messages may be retrieved and applied in the thinking and interpretation process.

2. *Unlimited Capacity Hypothesis (H2): Practically, every normal brain has the capacity to encode and store all thoughts, concepts and messages that one intends to.*

   In normal human brains, there are about 100 billion neurons that are interconnected by trillions of synapses. Each neuron has the potential capacity to activate other neurons to form a pattern. To simplify the situation for the moment and to ease computations, let us neglect the number of possible synapses between neurons and simply concentrate on only activated neurons. Since each neuron can be selected or not selected for a particular subset, mathematically the number of possible patterns that can be formed by 100 billion neurons is 2^{10^9}. To appreciate the size of that number, consider the fact that 2^{100} is equal to 1,267,650,600,228,329,401,496,703,205,376 (or 100 neurons). It suggests that the brain has almost infinite capacity, or for practical purposes, all the capacity that will ever be needed to store all that we will ever intend to store. According to neural scientists (see [2, 3, 27, 30]), certain special messages or information may be registered or stored in special sections of the brain, and only a small part of human brain (about percent) is activated and working for us at any moment in time. Therefore, the analogy described above is not a totally accurate representation of how the brain works. However, it does show that even a small section of the brain, which may contain a few hundred to a few million neurons, can create an astronomical number of circuit patterns which can represent an astronomical number of thoughts and ideas. In this sense, our brain still has a practically unlimited capacity for recording and storing information.

3. *Efficient Restructuring Hypothesis (H3): The encoded thoughts, concepts and messages (H1) are organized and stored systematically as data bases for efficient retrieving. Furthermore, according to the dictation of attention they are continuously restructured so that relevant ones can be efficiently retrieved to release charges.*

   Our brain puts all concepts, thoughts and messages into an organizational structure represented by the circuit patterns discussed earlier as H1. Because of charge structure, a concept to be discussed later, the organizational structure within our brain can be reorganized rapidly to accommodate changes in activities and events which can arise rapidly. This hypothesis implies that such restructuring is accomplished almost instantaneously so that all relevant information can be retrieved efficiently to effectively relieve the charge.

4. *Analogy/Association Hypothesis (H4): The perception of new events, subjects or ideas can be learned primarily by analogy and/or association with what is already known. When faced with a new event, subject or idea, the brain first investigates its features and attributes in order to establish a relationship with what is already known by analogy and/or association. Once the right relationship has been established, the whole of the past knowledge (preexisting memory structure) is automatically brought to bear on the interpretation and understanding of the new event, subject or idea.*
   Analogy/Association is a very powerful cognitive ability which enables the brain to process complex information. Note that there is a preexisting code or memory structure which can potentially alter or aid in the interpretation of an arriving symbol. For example, in language use, if we do not have a preexisting code for a word, we have no understanding. A relationship between the arriving symbol and the preexisting code must be established before the preexisting code can play its role in interpreting the arriving symbol.
5. *Goal Setting and State Evaluation (H5): Each one of us has a set of goal functions and for each goal function we have an ideal state or equilibrium point to reach and maintain (goal setting). We continuously monitor, consciously or subconsciously, where we are relative to the ideal state or equilibrium point (state evaluation). Goal setting and state evaluation are dynamic, interactive, and are subject to physiological forces, self-suggestion, external information forces, current data bank (memory) and information processing capacity.*
   There exist a set of goal functions in the internal information processing center which are used to measure the many dimensional aspects of life. Basically our mind works with dynamic multicriteria. A probable set is given in Table 1.1. Goal functions can be mutually associated, interdependent and interrelated.

Table 1.1 A structure of goal functions

1	*Survival and Security*: physiological health (correct blood pressure, body temperature and balance of biochemical states); right level and quality of air, water, food, heat, clothes, shelter and mobility; safety; acquisition of money and other economic goods
2	*Perpetuation of the Species*: sexual activities; giving birth to the next generation; family love; health and welfare
3	*Feelings of Self-Importance*: self-respect and self-esteem; esteem and respect from others; power and dominance; recognition and prestige; achievement; creativity; superiority; accumulation of money and wealth; giving and accepting sympathy and protectiveness
4	*Social Approval*: esteem and respect from others; friendship; affiliation with (desired) groups; conformity with group ideology, beliefs, attitudes and behaviors; giving and accepting sympathy and protectiveness
5	*Sensuous Gratification*: sexual; visual; auditory; smell; taste; tactile
6	*Cognitive Consistency and Curiosity*: consistency in thinking and opinions; exploring and acquiring knowledge, truth, beauty and religion
7	*Self-Actualization*: ability to accept and depend on the self, to cease from identifying with others, to rely on one's own standard, to aspire to the ego-ideal and to detach oneself from social demands and customs when desirable

6. *Charge Structures and Attention Allocation Hypothesis (H6): Each event is related to a set of goal functions. When there is an unfavorable deviation of the perceived value from the ideal, each goal function will produce various levels of charge. The totality of the charges by all goal functions is called the charge structure and it can change dynamically. At any point in time, our attention will be paid to the event which has the most influence on our charge structure.*
   The collection of the charges on all goal functions created by all current events at one point in time is the charge structure at that moment in time. The charge structure is dynamic and changes (perhaps rapidly) over time. Each event can involve many goal functions. Its significance on the charge structure is measured in terms of the extent of which its removal will reduce the levels of charges. Given a fixed set of events, the priority of attention to events at a moment in time depends on the relative significance of the events on the charge structure at that moment in time. The more intense the remaining charge after an event has been removed, the less its relative significance and the lower its relative priority. Thus attention allocation is a dynamic multicriteria optimization problem.
7. *Discharge Hypothesis (H7): To release charges, we tend to select the action which yields the lowest remaining charge (the remaining charge is the resistance to the total discharge) and this is called the least resistance principle.*
   Given the charge structure and the set of alternatives at time t, the selected alternative for discharge will be the one which can reduce the residual charge to the lowest level. This is the least resistance principle which basically is a concept of dynamic multicriteria optimization. When the decision problem involves high stakes and/or uncertainty, active problem solving or avoidance justification can be activated depending on whether or not the decision maker has adequate confidence in finding a satisfactory solution in due time. Either activity can restructure the charge structure and may delay the decision temporarily.
8. *Information Input Hypothesis (H8): Humans have innate needs to gather external information. Unless attention is paid, external information inputs may not be processed.*
   In order to achieve life goals, humans need to continually gather information. Information inputs, either actively sought or arriving without our initiation, will not enter the internal information processing center unless our attention is allotted to them. Allocation of attention to a message depends on the relevancy of the message to the charge structures. Messages which are closely related to long lasting events which have high significance in the charge structures can command a long duration of attention, and can, in turn, impact our charge structures and decision/behavior. Thus information inputs play an important role in dynamic MCDM.

1.3.3 Paradoxical Behavior

The following are some observations of human paradoxical behavior described in [43, 48]. They also appear in the decision making process regularly. We can verify them in terms of H1–H8 and specify under what conditions these statements may or may not hold.

- *Each one of us owns a number of wonderful and fine machines – our brain and body organs. Because they work so well, most of the time we may be unaware of their existence. When we are aware of their existence, it is very likely we are already ill. Similarly, when we are aware of the importance of clean air and water, they most likely have already been polluted.*
  This is mainly because when they work well, the charge structures are low and they will not cause our attention (H6). Once we are "aware" of the problems, the charge structure must be high enough so that we will pay attention to it. The reader may try to explore the charge structures and attention allocation of those people involved in Examples 1.1–1.3. The high levels of charges and dissolution make the examples interesting to us because they go beyond our habitual ways of thinking.
- *Dr. H. Simon, a Nobel Prize laureate, states that people have a bounded rationality. They do not like information overload. They seek satisfying solutions, and not the solution which maximizes the expected utility (see [35, 36]).*
  People gather information from different sources and channels, these messages may have significance in the charge structures and impact our decision making behavior (H8). They do not like information overload because it will create charges (H6). To release charges, people tend to follow the least resistance principle (H7) and seek for satisfying solutions instead of the solution which maximizes the expected utility (because the latter may create high charge structure!) Solutions that make people satisfied are those ones that meet people's goal setting and state evaluations (H5), they might not be the best answers but they are fair enough to solve the problems and make people happy. Again, here we clearly see the impact of the charge structure and attention allocation hypothesis (H6). Note that the challenging problems of Examples 1.1–1.3 were solved by jumping out of our habitual ways of thinking, no utility or expected utility were used.
- *Uncertainty and unknown are no fun until we know how to manage them. If people know how to manage uncertainty and unknown, they do not need probability theory and decision theory.*
  Uncertainty or unknown comes from messages that we are not able to judge or respond by our previous experiences or knowledge (H8). These experiences and knowledge form old circuit patterns (H1) in our brain/mind. When facing decision problems, we do not like the uncertainty and unknown which are new to us and we are unable to find matching circuit patterns to deal with them. This might create charges and makes us feel uncomfortable (H6). However, our brain has unlimited capacity (H2), by restructuring the circuit patterns (H1, H3) and the ability of analogy/association (H4), we can always learn new things and expand our knowledge and competence sets (the concept will be discussed in Section 1.5) to manage uncertainty/unknown. Examples 1.1–1.3 illustrate that much uncertainty and unknown are solved by expanding our competence for generating effective concepts and ideas, rather by using probability or utility theory.

- *Illusions and common beliefs perpetuate themselves through repetition and word of mouth. Once deeply rooted they are difficult to change (see [15]).*
  Illusions and common beliefs are part of the information inputs people receive everyday (H8) and they will form the circuit patterns in our brain. Through repetition and word of mouth, these circuit patterns will be reinforced and strengthened because the corresponding ideas are repeated (H1). Also, the stronger the circuit patterns, the more easily the corresponding thoughts are retrieved in our thinking and decision making processes, this explains why illusions, common beliefs or rumors can usually be accepted easier than the truth. In history, many famous wars were won by creating effective illusions and beliefs. Such creation in fact is an important part of war games.
- *When facing major challenges, people are charged, cautious, exploring, and avoiding making quick conclusions. After major events, people are less charged and tend to take what has happened for granted without careful study and exploration.*
  This is a common flaw when we are making decisions. Major challenges or serious problems are information that have high significance in the charge structures and can command our attention, so we will be cautious and avoiding making rough diagnostic (H6, H8). After major events, the decision maker's stake is low so that his/her attention will be paid to other problems that cause higher charge structures. As we read Examples 1.1–1.3, we are relaxed and enjoying. Those people involved in the examples might, most likely, be fully charged, nervous and exploring all possible alternatives for solving their problems.

For more paradoxical behaviors, please refer to [43, 48].

1.4 Habitual Domains

Our behavior and thinking are dynamic as described in the previous section. This dynamic change of charge makes it difficult to predict human behavior. Fortunately, these dynamic changes will gradually stabilize within certain domains. Formerly, the set of ideas and concepts which we encode and store in our brain can over a period of time gradually stabilize in certain domain, known as *habitual domains*, and unless there is an occurrence of extraordinary events, our thinking processes will reach some steady state or may even become fixed. This phenomenon can be proved mathematically [4, 42] as a natural consequence of the basic behavior mechanism (H1–H8) described in Section 1.3. As a consequence of this stabilization, we can observe that every individual has his or her own set of habitual ways of thinking, judging and responding to different problems, events and issues. Understanding the habitual ways of making decisions by ourselves and others is certainly important for us to make better decisions or avoid expensive mistakes. Habitual domains was first suggested in 1977 [38] and further developed [4, 40–44, 48] and quotes therein by Yu and his associates.

In this section, we shall discuss the stability and concepts of habitual domains and introduce the tool boxes for their expansion and enrichment so that we can make good use of our habitual domains to improve the quality of decision making and upgrade our lives. In fact, the concept of habitual domains is the underlying concept of competence set analysis to be introduced subsequently.

1.4.1 Definition and Stability of Habitual Domains

By the *habitual domain at time t*, denoted by HD_t, we mean the collection of ideas and actions that can be activated at time t. In view of Fig. 1.1, we see that habitual domains involve self-suggestion, external information, physiological monitoring, goal setting, state evaluation, charge structures, attention allocation and discharges. They also concern encoding, storing, retrieving and interpretation mechanisms (H1–H4). When a particular aspect or function is emphasized, it will be designated as "habitual domain on that function." Thus, habitual domain on self-suggestion, habitual domains on charge structures, habitual domain on attention, habitual domain on making a particular decision, etc. all make sense. When the responses to a particular event are of interest, we can designate it as "habitual domains on the responses to that event," etc. Note that conceptually habitual domains are dynamic sets which evolve with time.

Recall from H1 that each idea (thought, concept, and perception) is represented by a circuit pattern or a sequence of circuit patterns; otherwise, it is not encoded and not available for retrieving. From H2, we see that the brain has an infinite capacity for storing encoded ideas. Thus, $|HD_t|$, the number of elements in the habitual domain at time t, is a monotonic nondecreasing function of time t.

From H4 (analogy and association), new ideas are perceived and generated from existing ideas. The larger the number of existing ideas, the larger the probability that a new arriving idea is one of them; therefore, the smaller the probability that a new idea can be acquired. Thus, $|HD_t|$, although increasing, is increasing at a decreasing rate. If we eliminate the rare case that $|HD_t|$ can forever increase at a rate above a positive constant, we see that $|HD_t|$ will eventually level off and reach its steady state. Once $|HD_t|$ reaches its steady state, unless extraordinary events occur, habitual ways of thinking and responses to stimuli can be expected.

Theoretically our mind is capable of almost unlimited expansion (H2) and with sufficient effort one can learn almost anything new over a period of time. However, the amount of knowledge or ideas that exist in one's mind may increase with time, but the rate of increment tends to decrease as time goes by. This may be due to the fact that the probability of learning new ideas or concepts becomes lower as a number of ideas or actions in the habitual domain are larger. These observations enable us to show that the number of ideas in one's HD_t converges when suitable conditions are met. The followings are mathematically precise models which describe conditions for stability on the number of elements in habitual domains.

Let us introduce the following notation:

1. Let a_t be the number of additional new ideas or concepts acquired during the period $(t-1,\ t]$. Note that the timescale can be in seconds, minutes, hours, or days, etc. Assume that $a_t \geq 0$, and that once an idea is registered or learned, it will not be erased from the memory, no matter whether it can be retrieved easily or not. When a particular event is emphasized, a_t designates the additional ideas or concepts acquired during $(t-1,\ t]$ concerning that event.
2. For convenience, denote the sequence of a_t throughout a period of time by a_t. Note that due to the biophysical and environmental conditions of the individuals, a_t is not necessarily monotonic. It can be up or down and subject to certain fluctuation. For instance, people may function better and more effectively in the morning than at night. Consequently, the a_t in the morning will be larger than that at night. Also observe that a_t may display a pattern of periodicity (day/night for instance) which is unique for each individual. The periodicity can be a result of biophysical rhythms or rhythms of the environment.

The following can readily be proved by applying the ratio test of power series. The interested readers please refer to [4, 42] for further proof.

Theorem 1.1. *Suppose there exists T such that whenever* $t > T$, $\frac{a_t+1}{a_t} \leq r < 1$. *Then as* $t \to \infty$, $\sum_{t=0}^{\infty} a_t$ *converges.*

Theorem 1.2. *Assume that (i) there exists a time index s, periodicity constant* $m > 0$, *and constants D and M, such that* $\sum_{n=0}^{\infty} a_{s+nm} \leq D$, *where* a_{s+nm} *is a subsequence of* a_t *with periodicity m, and (ii) for any period n,* $\left(\sum_{i=1}^{m} a_{s+nm+i}\right)/ma_{s+nm} \leq M$. *Then* $\sum_{t=0}^{\infty} a_t$ *converges.*

Note that for habitual domain to converge, Theorem 1.2 does not require a_t to be monotonically decreasing as required in Theorem 1.1. As long as there exists a convergent subsequence, and the sum of a_t within a time period of length m is bounded, then a_t can fluctuate up and down without affecting the convergence of HD_t. Thus the assumptions in Theorem 1.2 are a step closer to reality than those in Theorem 1.1.

Another aspect of the stability of habitual domains is the "strength" of the elements in HD_t to be activated, which is called *activation probability*.

Define $x_i(t),\ i \in HD_t$, to be the *activation probability* of element i at time t. For simplicity let $HD_t = 1, 2, \ldots, n$ and $x = (x_1, \ldots, x_n)$. Note that n, the number of elements in HD_t, can be very large. As $x_i(t)$ is a measurement of the force for idea i to be activated, we can assume that $x_i(t) \geq 0$. Also $x_i(t) = 0$ means that idea i cannot be activated at time t, by assigning $x_i(t) = 0$ we may assume that HD_t contains all possible ideas of interest that may be acquired now and in the future.

Similar to charge structure, $x_i(t)$ may be a measurement of charge or force for idea i to occupy the "attention" at time t. Note that $x_i(t)/\sum_i x_i(t)$ will be a measurement of relative strength for idea i to be activated. If all $x_i(t)$ become stable after some time, the relative strength of each i to be activated will also be stable. For stability of $x_i(t)$, the interested reader may refer to [4, 42] for mathematical derivation and further discussion.

1.4.2 Elements of Habitual Domains

There are two kinds of thoughts or memory stored in our brain or mind: (1) the ideas that can be activated in thinking processes; and (2) the operators which transform the activated ideas into other ideas. The operators are related to thinking processes or judging methods. In a broad sense, operators are also ideas. But because of their ability to transform or generate (new) ideas, we call them operators. For instance, let us consider the numerical system. The integers $0, 1, 2, \ldots$ are ideas, but the operation concepts of $+, -, \times, \div$, are operators, because they transform numbers into other numbers.

Habitual domains at time t, HD_t, have the following four subconcepts:

1. *Potential domain*, designated by PD_t, which is the collection of all ideas and operators that can be potentially activated with respect to specific events or problems by one person or by one organization at time t. In general, the larger the PD_t, the more likely that a larger set of ideas and operators will be activated, holding all other things equal.
2. *Actual domain*, designated by AD_t, which is the collection of ideas and operators which actually occur at time t. Note that not all the ideas and operators in the potential domain can actually occur. Also note that the actual domain is a subset of the potential domain. That is $AD_t \subset PD_t$.
3. *Activation probability*, designated by AP_t, which is defined for each subset of PD_t and is the probability that a subset of PD_t is actually activated or is in AD_t. For example, people who emphasize profit may have a greater frequency to activate the idea of money. Similarly, people who study mathematics may have a greater frequency to generate equations.
4. *Reachable domain*, designated by $R(I_t, O_t)$, which is the collection of ideas and operators that can be generated from the initial idea set (I_t) and the initial operator set (O_t). In general, the larger the idea set and/or operator set, the larger the reachable domain.

At any point in time, without specification, habitual domains (HD_t) will mean the collection of the above four subsets. That is $HD_t = \{PD_t, AD_t, AP_t, R(I_t, O_t)\}$. In general, the actual domain is only a small portion of the reachable domain, while the reachable domain is only a small portion of the potential domain, and only a small portion of the actual domain is observable. This makes it very difficult for us to observe other people's habitual domains and/or even our own habitual domains. A lot of work and attention is therefore needed in order to accomplish that. For further discussion, see [42, 43, 48].

As a mental exercise, it might be of interest for the reader to answer: "With respect to the players or rules of games, how the PD_t, AD_t and RD_t evolve over time in Examples 1.1–1.3?" Note that it is the expansion of the relevant HD_t that get the challenge problems solved. We will further discuss this later.

1.4.3 Expansion and Enrichment of Habitual Domains

As our habitual domains expand and enrich, we tend to have more ideas and operators to deal with problems. As a consequence, we could be more effective in solving problems, routine or challenges, and more capable to release the pains and frustrations of our own and others. In Example 1.1, Alinsky could solve the challenging problem graciously because he could see through people's charge structure in their potential domains. To be able to do so, his habitual domains must be very broad, rich and flexible as to find the solution that could release everyone's charge. In Example 1.2, Peter Ueberroth also owns a large habitual domain, so he could incorporate the potential domains of potential players as to create the winning strategy to solve the challenge problems of the Olympic. Thus, it is important to expand and enrich our habitual domains. By doing so, we can understand the problems better, and make decisions more effectively and efficiently. Without doing so, we might unwillingly get stuck and trapped in the process, feel powerless and frustrated.

There are three Toolboxes coined by Yu [43, 45, 48] to help us enrich and expand our potential domain and actual domain:

- Toolbox 1: "Seven Self-Perpetuating Operators" to change our minds in positive ways
- Toolbox 2: Eight Methods for expanding the habitual domains
- Toolbox 3: Nine Principles of Deep Knowledge

We will briefly introduce them in the following three subsections.

1.4.3.1 Seven Self-Perpetuating Operators

Just as the plus and minus operators in mathematics help us to arrive at another set of numbers, the following seven operators, the circuit patterns, are not right or wrong, but they can help us reach another set of ideas or concepts. These operators are self-perpetuating because once they are firmly implanted in our habitual domain and used repetitively, they will continuously grow and help us continuously expand and enrich our habitual domains.

1. *Everyone is a priceless living entity. We all are unique creations who carry the spark of the divine*. (Goal setting for self-image)
   Once this idea of the pricelessness of ourselves and every other living person becomes such a strong circuit pattern as to be a core element of our belief system, it will be the source of tremendous power. If we are all sparks of the divine, we will have high level of self-respect and respect others. We can try to be polite and humble to others, to listen to their ideas and problems. Our habitual domain will become absorbing, continuously being expanded and enriched.
2. *Clear, specific and challenging goals produce energy for our lives. I am totally committed to doing and learning with confidence. This is the only way I can reach the goals*. (Self-empowering)

Goals energize us. They fill us with vitality. But to be effective they must be clear, specific, measurable, reachable and challenging. Without clear, specific and challenging goals, in the rush of daily events, we may find ourselves being attracted to randomly arriving goals and desires. We may even be controlled by them. With a clear, specific and challenging goal, we create a high level of charge that focuses our efforts. Through positive problem solving, the charge becomes a strong drive to reach our goal. The accomplishment will enhance our confidence, competence and courage to undertake further challenges. In the process, our habitual domain will continuously be expanded and enriched.

3. *There are reasons for everything that occurs. One major reason is to help us grow and develop*. (Meaning of events (state evaluation))
   Because we carry the divine spark (since we are transformed from God or Buddha), everything that happens to us has a reason; i.e., to help us grow and mature. Therefore we must pay close attention to the events in our lives. We must be concerned and look for understanding. As a consequence, our habitual domain will be expanded and enriched.
4. *Every task is part of my life's mission. I have the enthusiasm and confidence to accomplish this mission*. (Meaning of works)
   All the work we do is important. In fact, everything we do matters. Basically, there is no such thing as an unimportant task. We must regard every task as important to our lives. Whatever we are doing at a given moment is occupying 100% of our lives at that time. We must learn to say, "All my life is being given right now to what I am doing. I must do this as though it were worth taking up my life. The work I do has value when I bring to it my best efforts." Once this operator has taken root in your habitual domain, you will find yourself approach each job with a total dedication of mind and effort, and you will experience feelings of accomplishment and satisfaction. As a result your habitual domain will be expanded and enriched.
5. *I am the owner of my living domain. I take responsibility for everything that happens in it*. (Attitude toward living domain)
   We are the masters of the domains wherein we live, act and connect with the outside world. Therefore, we must take responsibility – that is, agree to be a positive force – in everything within our world. We cannot simply let things happen around us and expect to succeed at reaching our own potential. When this circuit pattern or belief becomes strong, it will push us on to a keener understanding of the habitual domain of ourselves and others with whom we interact. We will be willing to take the initiative to be our best self. Our habitual domain, as a result, will be expanded and enriched.
6. *Be appreciative and grateful, and do not forget to give back to society*. (Attitude toward the world)
   Appreciation, gratification and making contributions are all circuit patterns, modes of behavior which can be cultivated until they become second nature. They benefit us first, because such circuit patterns make us feel good. But they also benefit others. Through reciprocation, by making a contribution and giving back some of what we have gained we will assure that our circuit patterns create

an upward spiral of joy and satisfaction that affects not only ourselves but those around us. In the process, we expand and enrich our habitual domains.

7. *Our remaining lifetime is our most valuable asset. I want to enjoy it 100% and make a 100% contribution to society in each moment of my remaining life*. (Attitude toward life)
   When we say to ourselves that we want to enjoy our lifetime here and now and make a 100% contribution to society in each moment of our remaining life, our brain will restructure the circuit patterns, allowing us more likelihood to do so. Our life and habitual domain will consequently be expanded and enriched. So enjoy our life and make our contributions here and now.

1.4.3.2 Eight Methods for Expanding the Habitual Domains

The followings are eight basic methods for expanding the habitual domains, shown in Table 1.2.

Each can be extremely useful when used alone. Their power is only multiplied when we combine two or more. Let us briefly discuss three of them, the interested readers please refer to [43, 45, 48] for more.

1. *Learning Actively*
   By active learning we mean obtaining those concepts, ideas and thoughts from various channels including consultations with experts, reading relevant books and following the radio, television, journals, etc. Libraries are filled with literatures which contain human wisdom, experiences and thoughts. As long as we are willing to study, we certainly can learn from this abundant literature contributed by our ancestors and current scholars (refer to Unlimited Capacity Hypothesis (H2) in Section 1.3). According to the Circuit Pattern Hypothesis (H1), to effectively acquire new knowledge, we must first pay attention to new ideas and information and then repeatedly rehearse them, so that the new corresponding circuit patterns will become strong and numerous. This will make the ideas and information easily retrievable so that they can be integrated systematically into our existing memory and become part of our knowledge.
2. *Projecting from a Higher Position*
   Projecting from a higher position in the hierarchy of living systems is an effective way to expand our habitual domains because it permits us to see those things

Table 1.2 Eight methods for expanding and enriching habitual domains

1.	Learning Actively
2.	Take the Higher Position
3.	Active Association
4.	Changing the Relative Parameters
5.	Changing the Environment
6.	Brainstorming
7.	Retreat in Order to Advance
8.	Praying or Meditation

which we could not see otherwise. It is very useful especially when we are facing multiple criteria decision making problems. By standing and projecting from a higher position, we could make better observations and comparisons and see into the future more clearly. One of the consequences is that we may be less satisfied with our current abilities and achievements. This will prompt us to be humble and willing to work hard and to learn more. This process usually, through self-suggestion, will force us to generate new ideas, new concepts and, consequently, to expand our habitual domains. In observing, thinking or projecting from the higher hierarchical position, we must again first try to dissolve our prejudices and wishful thinking. Otherwise, our view or projection could be strongly influenced by them. This would prevent us from seeing things objectively and from generating good ideas.

3. *Changing the Relevant Parameters*
   There are many parameters which are involved in a particular problem or event. If we are willing to change the parameters, we may be able to obtain new ideas. Recall Example 1.1, the passengers, the toilets in the airport, the medias, etc. were the parameters. Alinsky made good use of them and therefore could solve the problem. Many of us have a habitual way of looking at a particular problem with a fixed parameter value (for instance, our assumption that the recovery time of an investment is 7 years), which can fix our mind, perhaps unconsciously, in dealing with our problems. Being willing to search for the parameters and change their values can usually help us expand our view of the problem. In many important decision problems, our criteria, alternatives, possible outcomes and preferences can all be changed over time. Allowing the parameters to change in our thinking process will allow us to reconsider what a good decision should be and to construct better strategies for the problems.

1.4.3.3 Nine Principles of Deep Knowledge

The following nine principles, shown in Table 1.3, can help us sharpen our mind and, in the process, expand our habitual domains.

We shall briefly discuss two of them here, the interested readers please refer to [43, 45, 48] for more.

Table 1.3 Nine principles of deep knowledge

1.	Deep and Down Principle
2.	Alternating Principle
3.	Contrasting and Complementing Principle
4.	Revolving and Cycling Principle
5.	Inner Connection Principle
6.	Changing and Transforming Principle
7.	Contradiction Principle
8.	Cracking and Ripping Principle
9.	Void Principle

1. *Deep and Down Principle*
   This principle has two meanings. First, every so often one needs to reduce his/her charges to as low as possible. When we are very busy and deeply concentrating, only ideas carrying high activation probability will occupy our mind. We will be too preoccupied with those situations to consider anything else. By reducing our charges, we can attend to those ideas with low activation probability. Thus, our actual domain will be expanded. The second is to take "the humble position" when dealing with others. By being sincerely humble, we could make other people to offer their ideas and operators to us more willingly and absorb new ideas more easily and effectively. Consequently our habitual domain will be expanded and enriched.
2. *The Alternating Principle*
   Sometimes we have to omit or change our combined assumptions so that we can create new ideas from different sets of assumptions. It is easy to think of examples of how varying the combination of elements can create beneficial results. By combining 0 and 1 in different orders, mathematicians can create numerical systems and digital systems upon which electronic devices and computers are based. Different combinations of the three primary colors (red, blue and yellow) can create an unlimited variety of colors and patterns. By alternating seven basic tones, one can compose an infinite number of songs. Recall in Example 1.3, speed of riding horses has been alternated. If the candidate can utilize this principle, he would have known how to solve the problem. In multiple criteria decision making, the decision maker can alternate the parameters (as combining with the "*Changing the Relevant Parameters*" we mentioned above) or assumptions to see if new ideas or solutions are produced. By doing so, the habitual domains will be expanded and enriched, and the decision makers can make better decisions.

Stable habitual domains exist for every living system including individuals, organizations, societies and nations. Wherever we go, our habitual domains go with us and have a great impact on our behaviors and decisions. The three tool boxes we have explored in this section will help us expand and enrich our habitual domains as to find and/or create great ideas to solve problems, routines or challenges, more effectively and efficiently. For more detailed applications, see [43, 45–48].

1.5 Competence Set Analysis

We have explored the essence and relationship of the dynamics of decision making and habitual domains. Generally, decision making is a manifestation of habitual domains, and expanding and enriching the habitual domains will help people make decisions more effectively. Now, if we are facing a particular problem, how do we handle it? Recall Example 1.2, why was there no city other than Los Angeles willing to hold the 1984 Summer Olympics? Because they were threatened by the previous financial disastrous experience of Olympic Games at Montreal and Moscow (being affected by the corresponding circuit patterns), and they were afraid of not being able to bear the possible financial loss.

Why could the Los Angeles Olympics be so successful and produced a profit of over $220 million? Because Peter Ueberroth, the Chief Executive Officer, effectively made use of all potential resources and integrated all competences in the potential domains.

For each decision problem there is a competence set of ideas, knowledge, skills and resources for its effective solution. Such a set, denoted by CS^*, like habitual domain, implicitly contains potential domain, actual domain, activation probability, and reachable domain as discussed in Section 1.4. When the decision maker thinks he/she has already acquired and mastered the CS^* as perceived, he/she would feel comfortable making the decision and/or undertaking the challenge.

In this section, based on [43, 48], we shall discuss the concept and elements of competence sets, and the relationship among them. Several important research issues will be described. In the last subsection we shall introduce the concept of innovation dynamics as an integration of habitual domains theory and competence set analysis.

1.5.1 Concept of Competence Set Analysis

The concept of competence set has been prevailing in our daily life. In order to increase our competence, we go to schools to study and get diplomas or degrees when we graduate from schools with satisfactory performance. Diplomas or degrees symbolize that we have certain set of competence. In order to help people reduce uncertainty and unknown, or verify that certain professionals or organizations have certain competence sets, various certificates through examinations are issued to certify the qualifications. Hundreds of billions of dollars annually have been spent in this acquiring and in verifying competence sets.

Given a problem, different people might see the needed competence set differently. Indeed, competence set for a problem is a habitual domain projecting to the problem. Note that competence sets are dynamic and can change with time t. For ease of presentation, without confusion, in the following discussion we shall not use t as a subscript to signify the time dynamics. In order to more precisely understand CS, we shall distinguish "perceived" and "real" CS, and "perceived" and "real" skill set Sk that we have acquired. Thus, there are four basic elements of competence set for a given problem E, described as follows:

1. *The true competence set* $(Tr(E))$: consists of ideas, knowledge, skills, information and resources that are truly needed for solving problem E successfully.
2. *The perceived competence set* $(Tr^*(E))$: the true competence set as perceived by the decision maker (DM).
3. *The DM's acquired skill set* $(Sk(E))$: consists of ideas, knowledge, skills, information and resources that have actually been acquired by the DM.
4. *The perceived acquired skill set* $(Sk^*(E))$: the acquired skill set as perceived by the DM.

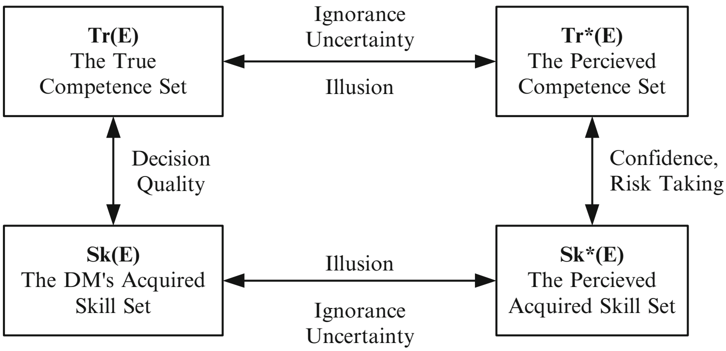

Fig. 1.2 The interrelationships among four elements of competence set

Note that each of the above sets inherently involves with actual, reachable and potential domains, and activation probabilities. This fact makes competence set analysis rich, interesting and complex. Without special mention, in the following discussion we shall focus on actual domains of the competence sets. The interrelationships of the above four elements are shown in Fig. 1.2.

Note that the above four elements are some special subsets of the habitual domain of a decision problem E. Without confusion, we shall drop E in the following general discussion. According to the different relations among the four elements, we have the following observations:

1. The gaps between the true competence set (Tr or Sk) and *perceived* competence set (Tr^* or Sk^*) are due to ignorance, uncertainty and illusion.
2. If Tr^* is much larger than Sk^* (i.e. $Tr^* \supset\supset Sk^*$), the DM would feel uncomfortable and lack of confidence to make good decisions; conversely, if Sk^* is much larger than Tr^* (i.e. $Sk^* \supset\supset Tr^*$), the DM would be fully confident in making decisions.
3. If Sk is much larger than Sk^* (i.e. $Sk \supset\supset Sk^*$), the DM underestimates his own competence; conversely, if Sk^* is much larger than Sk (i.e. $Sk^* \supset\supset Sk$), the DM overestimates his own competence.
4. If Tr is much larger than Tr^* (i.e. $Tr \supset\supset Tr^*$), the DM underestimates the difficulty of the problem; conversely, if Tr^* is much larger than Tr (i.e. $Tr^* \supset\supset Tr$), the DM overestimates the difficulty of the problem.
5. If Tr is much larger than Sk (i.e. $Tr \supset\supset Sk$), and decision is based on Sk, then the decision can be expected to be of low quality; conversely, if Sk is much larger than Tr (i.e. $Sk \supset\supset Tr$), then the decision can be expected to be of high quality.

The following observations are worth mentioning (for further discussion, please see Chapter 8 of [48]):

First, *core competence* is the collection of ideas or skills that would almost surely be activated when problem E is presented. To be powerful, the core competence should be flexible and adaptable.

Next, the *ideal competence* set (similar to ideal habitual domain, see [43,45,48]) is the one that can instantly retrieve a subset of it to solve each arriving problem successfully and instantaneously.

Finally, a competence is *competitive* if it is adequately flexible, adaptable, and can be easily integrated or disintegrated as needed to solve the arriving problems faster and more effectively than that of the competitors.

1.5.2 Research Issues of Competence Set Analysis

Competence set analysis has two inherent domains: competence domain and problem domain. Like habitual domain, each domain has its actual domain and potential domain, as depicted in Fig. 1.3.

From these two domains, there are two main research directions:

1. *Given a problem or set of problems, what is the needed competence set? and how to acquire it?*
   For example, how to produce and deliver a quality product or service to satisfy customers' needs is a main problem of supply chain management. To successfully solve this problem, each participant in a supply chain including suppliers, manufacturers, distributors, and retailers must provide the chain with its own set of competence, so that the collected set of competence can effectively achieve the goal of satisfying customers' needs.
   To analyze the competence sets of individuals or corporations, we can decompose the CS as follows:

$$CS_t = \left(CS_t^t, CS_t^2, CS_t^3, \ldots, CS_t^4\right), \quad (1.1)$$

   where CS_t^k denotes the kth item of the CS at time t. Note that CS will be dynamically changed as time goes by. When confronting a decision problem, one can evaluate its current CS to see if the problem can be solved by the CS. If the CS is not adequate to solve the problem, one should try to expand or transform the competence set:

$$CS_{t+1} = T_t\left(CS_t, E_t\right), \quad (1.2)$$

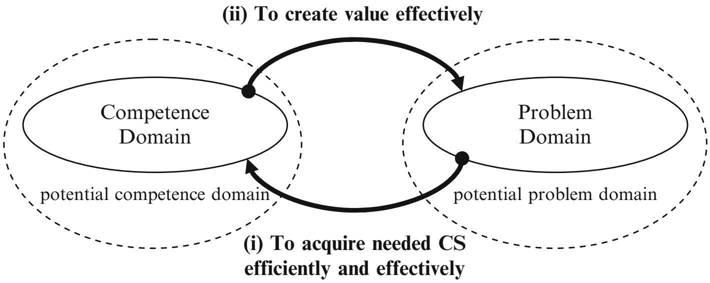

Fig. 1.3 Two domains of competence set analysis

where E_t denotes the event or decision problem that the corporation or individual is confronted. After transforming by the function T_t, the original competence set CS_t is expanded into a new one, CS_{t+1}.

The expansion of competence set can be achieved internally or externally. Through internal adjustment or development on the resources, time and processes, corporations can improve or transform the existent competence set so as to solve the problems or achieve the goal [8, 21]. On the other hand, by integrating with external competence, corporations can expand their competence sets. Outsourcing, strategic alliance, merging, acquisition, and utilization of information technology and consulting services are means to expand competence set externally [9, 10, 33].

How to expand the existent competence set to the needed competence set in most effective and efficient way? A mathematical foundation for such competence analysis is provided by Yu and Zhang [53]. Under some suitable assumptions, the problem can be formulated and solved by decision tree, graph theory, spanning trees, spanning tables and mathematical programming [13,19,23–25,34,52]. Most earlier researches have focused only on the deterministic situation. However, one could remove this assumption to include uncertainty, fuzziness and unknowns. In the recent studies, some heuristic methods, such as genetic algorithm (GA), hybrid genetic algorithm (hGA), multicriteria genetic algorithm, multiobjective evolutionary algorithm (MOEA), data mining technology and forest learning technique have also been incorporated into the analysis of competence set expansion [7, 16, 20, 26, 28, 29].

2. *Given a competence set, how to locate a set of problems to solve as to maximize the value of the competence?*

Given a competence set, what is the best set of problems that can be solved by the competence set as to maximize its value? If someone has already acquired a particular competence set, what are the problems he/she should focus to solve as to maximize its value? For instance, if we get a doctoral degree from certain university, which symbolize we have a certain set of competence, how do we maximize the value of this degree? Think of the opportunities in actual domains and potential domains. There are lots of studies of competence set analysis working in this direction. For example, Chen [6] established several indices that have impact on consumers decision making and provided a framework for helping firms in expanding the benefits of their products to fully address the consumer's needs. Hu et al. [17, 18] generate learning sequences for decision makers through competence set expansion to help them make better decisions. Chang and Chen [5] develop an analytical model of competence sets to assist drivers in routing decisions. Chiang-Lin et al. [8] studied the change of value when competence sets are changed in linear patterns so that the corporations can create value by taking loss at the ordering time and making profit at the delivery time.

The products and/or services that corporations provide, in abstract, could be regarded as main results of competence set transformation. Their *value* depends on two major factors:

(a) *Targets*: who are the customers? Whom will the products or services be served to, currently and potentially?
(b) *Functions*: what kind of pains and frustrations, both in actual and potential domains, can the products and/or services help release?

For simplicity, we can use a value function to represent the above idea:

$$V\left(CS^k\right) = \tilde{f}\,(x, u)\,, \tag{1.3}$$

where k denotes the kth products and/or services provided by the corporations, and CS^k represents the competence set of the kth products and/or services. The value created by the products or services could be a fuzzy set function of the targets (x) and the functions (utilities) (u). Note that both x and u have actual domains, reachable domains and potential domains. To create maximum value, one should not only look for the target customers and understand their problems, pains or frustrations from actual domains, but also discover the customers and their problems, pains, and frustrations hidden in the potential domains and reachable domains (for illustration, see Example 1.4: YouTube – Broadcast Yourself to be described in Section 1.5.3).

1.5.3 Innovation Dynamics

Without innovation, corporations will stand still and eventually get out of business. Without creating value, corporations cannot realize their meaning of existence and cannot sustain their continued growth. Innovation indeed is the key factor for the corporate sustained growth and is the key activity for value creation. *Corporate innovation* itself is a dynamic process involving corporate goal setting, state evaluation, understanding customers' need, output of products and services and creating values for the targeted customers and the stakeholders. *Innovation dynamics* (see Fig. 1.4), based on habitual domains theory and competence set analysis, is a comprehensive and integrated framework to help people understand corporate innovation and creation on maximal values for the targeted customers and themselves.

The dynamics can be interpreted clockwise, according to the index of Fig. 1.4, as follows:

(i) According to habitual domain Theory, when there exist unfavorable discrepancies (for instance, the corporations are losing money instead of making money, or they are technologically behind, instead of ahead of the competitors) between the current states and the ideal goals of individuals or organizations, it will create charges which can prompt the individuals or corporations to work harder to reach their ideal goals (see Section 1.3). During this stage, the corporations will evaluate their current competence sets to see if the sets are sufficient to solve the problems. If the problems cannot be solved or the goals cannot be reached by the current competence sets, the corporations will tend to transform

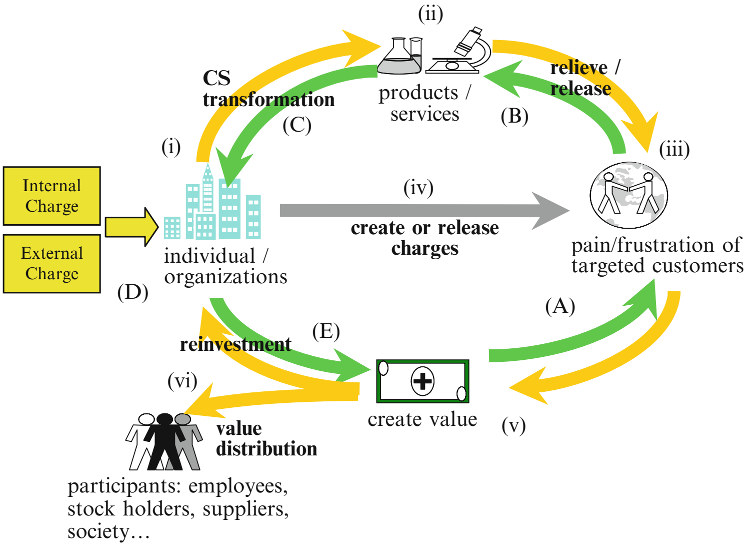

Fig. 1.4 Innovation dynamics

their competence sets by internal adjustment or development (such as reengineering) or integrate with external competence (e.g., outsourcing or strategic alliance) to expand the competence sets (see Eq. 1.2 of Section 1.5.2)

(ii) The transformation of competence sets will be presented in visible or invisible ways, which results in a new set of the products or services produced by the corporations.

(iii) The products or services produced by corporations must carry the capability to relieve / release the pain and frustration of targeted customers. Note that there are actual domains, reachable domains and potential domains for the targeted customers, their pains, frustrations and problems.

(iv) Besides discharge, corporations or organizations can create charges to the targeted customers by means of marketing, advertisement or promotion.

(v) The targeted customers will experience the change of charges. When their pains and frustrations are relieved, by buying our products or services, the customers become happy, and the products and services create their value.

(vi) The value will be distributed to the participants such as employees, stock holders, suppliers, society, etc. In addition, to gain the competitive edge, products and services have to be continuously upgraded and improved. The reinvestment therefore is needed in order to develop or produce new products and services.

In contrast to the clockwise cycle, the innovation dynamics can be interpreted counter-clockwise, according to the indexing of Fig. 1.4, as follows:

(A) To create values, the corporations must consider who will be the targeted customers, and what kind of pain and frustration they have, both in actual and potential domains.

(B) In order to ease the pains and frustrations for the targeted customers, what products or services, in actual and potential domains, are needed? Competitiveness becomes an important issue in the final selection of the products and services to produce.
(C) How do the corporations transform their internal and external competence and resource to develop or provide the selected products and services effectively and efficiently?
(D) When the transformation of competence sets succeed, the corporation's internal and external charge will be released.
(E) New goals as to create new values can be reestablished. The innovation cycle: (A) → (B) → (C) → (D) → (E) → (A) will go round and round.

The concept of innovation dynamics describes the dynamics of how to solve a set of problems with our existent or acquired competence (to relieve the pains or frustrations of "targeted customers or decision makers" at certain situations) as to create value, and how to distribute this created value so that we can continuously expand and enrich the competence set to solve more challenging problems and create more value.

Note, while we describe innovation dynamics in terms of corporation, it can also be applied to individual person as to continuously expand and enrich his/her habitual domains and maximize the value of his/her life.

Let us use the following example to illustrate innovation dynamics further.

Example 1.4. YouTube – Broadcast Yourself

YouTube is a video sharing web site where users can upload, view and share video clips. It was created in mid-February 2005 by three former PayPal employees, Steve Chen, Chad Hurley and Jawed Karim. Its service uses Adobe Flash technology to display a wide variety of user-generated video content, including movie clips, TV clips and music videos, as well as amateur content such as videoblogging and short original videos. In November 2006, YouTube was acquired by Google Inc. for US$1.65 billion. (Adopted from *http://en.wikipedia.org/wiki/YouTube.)*

At the very beginning, all these young men wanted to do was to share some videos from a dinner party with a half-dozen friends in San Francisco, but they could not figure out a good solution. Sending the clips by e-mail kept getting rejected because they were too big. Posting the videos online was a headache, too. So the buddies decided to work in Hurley's garage, determined to design something simpler. It was YouTube.

When the platform was built up, the founders intended to offer their services to the sellers in eBay (a popular auction web site) so that those sellers could introduce and demonstrate their goods through video clips instead of photos and text. In this stage, the value YouTube created could be represented as

$$V(CS_t) = \sum_{x_j \in X_e Bay} c_j x_j,$$

where x_j denotes the seller j in eBay, c_j is the expected payoff that YouTube could obtain from seller j and X_{eBay} is the set of the customers of eBay.

Unfortunately the eBay sellers did not like the idea. The founders of YouTube searched for potential customers in their potential domains. They found that many internet users have strong desire to create video clips and share with others of interest. This discovery prompts the founders to expand their target customers to all internet users and provide their service free to them. Since then, YouTube allows people to upload their video clips to the internet and share them with friends worldwide. By sharing and viewing these video clips, more and more people know this web site. The shift of YouTube's target customers from the eBay sellers (*actual domain*) to the internet users worldwide (*potential domain*) indeed is a key factor of its major success.

As described in the last subsection, the target customers and the functions (utilities) have their actual domains, potential domains and reachable domains. The original idea of YouTube is to solve the problem of video sharing, which is the *actual domain* of YouTube's function. To enhance its competence, each video is accompanied by a full HTML markup for linking to it and/or embedding it within another page, unless the submitter of a video chooses to disable the embedding feature. By adding an embeddable markup, the video provider can have his/her video play automatically when his/her webpage is loaded. These simple cut-and-paste options are especially popular with users of social-networking sites. By offering this embedding function, YouTube not only solves the problem of video sharing by providing a web site, but also releases the charges and frustrations of those users who want to play the video directly on their webpage. The former is the need existing in the actual domain of the internet users, while the latter is the desire and anxiety hidden in the potential domain. The value created by this enhancement can be represented as follows:

$$V\left(CS_{t+1}\right) = V\left(CS_t \oplus \Delta\right),$$

where $\Delta = \{u^1, u^2, u^3, \ldots, u_n\}$ represents the enhanced functions or the competences in terms of utilities newly added. Similar to Eq. 1.3, the value created by YouTube after the transformation of its competence set could be described by a fuzzy set function:

$$V\left(CS_{t+1}\right) = \sum_{x_j \in X} \tilde{f}_j\left(x_j, u_j\right), \tag{1.4}$$

where X is the set of all the potential users, x_j is the jth user/target customer, and u_j is a vector that represents the functions and/or utilities provided to user (or target group) x_j.

It is worth noting that a successful value creation is a virtuous cycle. The more people's pain and frustration is released, the more value the product or service can create; the more value it creates, the more people can be served and the more pain and frustration can be relieved or released. As time and situation dynamically change, both sides of Eq. 1.4 can be a function of time t as expressed in (1.5).

$$V_{t+1}\left(CS_{t+1}\right)=\sum_{x_{jt}\in X_t}\tilde{f}_t\left(x_{jt},u_{jt}\right), \tag{1.5}$$

where X_t represents the set of all the potential users at time t, x_{jt} is the jth user/target customer at time t, and u_{jt} is the utilities and/or functions provided to the user (or target group) x_{jt} at time t. Note that usually V_{t+1} is increasing as u_{jt} expands, which leads to the resource reinvestment to CS_{t+1}, so u_{jt+1} and x_{jt+1} will also be expanded, which, in turn, will increase V_{t+2}. In the YouTube example, x_{jt} and V_{t+1} are expanded exponentially.

Innovation can be defined as the work and process to transform the creative ideas into reality as to create the value expected. It includes planning, executing (building structures, organization, processes, etc.) and adjustment. It could demand hard working, perseverance, persistence and competences. The innovation dynamics provides us a comprehensive and integrated framework, which can help the corporations create value as to realize their meaning of existence. The framework can also be applied to individuals because everyone owns competence sets. By active applying the framework, one can continuously expand and enrich his/her competence set and habitual domains, and make best use (create maximum value) of his/her competence and life.

1.6 Decision Making in Changeable Spaces

Recall that for most nontrivial decision problems, there are a number of parameters that could be changeable in the process of decision making. As the parameters change, the features of the decision problems change. Treating these parameters as control variable, looking into their reachable domains and potential domains (habitual domains), the challenging problems may be solved more effectively. Otherwise, we may get trapped in the process without satisfying solution.

Consider Example 1.1. Before Alinsky got involved, the relevant parameters of the problem had no much change and the Woodlawn Organization got trapped and was powerless. With Alinsky's involvement, the decision situation changed dramatically. Alinsky's habitual domain penetrated into the potential domains of the problem, and changed the rule, the strategies and the minds of the potential players (these are part of the relevant parameters). The challenging problem was solved gracefully with all players claiming victory after the parameters changed. The readers are invited to explore those changes of the relevant parameters in Examples 1.2–1.4.

A superior strategist restructures the decision situations by changing the value of relevant parameters to find a solution that all players can claim victory, not just to find an optimal solution with the parameters fixed at certain values. Making decision for the problems with changeable parameters will be called "Decision Making in Changeable Spaces." As pointed out before, knowing the potential domains, reachable domains of the decision problems play a key role for solving challenging problem. In Section 1.6.1, we will briefly list those parameters that have great impact in decision making. By understanding their potential domains, we will be more capable of knowing and solving the problems. In Section 1.6.2, we will discuss decision blinds and decision traps and how to overcome them using competence set analysis discussed in Section 1.5.

1.6.1 Parameters in Decision Processes

We will list those parameters that are changeable and can shape up the decision situations in terms of Decision Elements and Decision Environments. For each decision maker there are five basic elements involved in decision processes. These are decision alternatives, decision criteria, decision outcomes, decision preference and decision information inputs. For further details, see [38, 43, 48].

1. Decision Alternatives
   Alternatives are those choices which we can select or control in order to achieve our decision goals. It can be regarded as a habitual domain which may vary with time and situation. Therefore, the set of alternatives may not be a fixed set, contrary to the implicit assumption that the set is fixed in decision science. In many high-stake decision problems, such as those discussed in Examples 1.1–1.4, the set is usually not fixed, especially when the decision problems are in their transition state. New alternatives can and should continuously be generated and suggested. As a habitual domain, the set tends to be stable over time. Being able to create new and innovative alternatives outside of the existing habitual domain is very important in solving nontrivial decision problems. See Examples 1.1–1.4.
2. Decision Criteria
   Each decision problem involves a set of criteria for measuring the effectiveness or efficiency of the decision. To be effective, the chosen decision criteria should be able to create a high level of charge in our charge structure. Toward this end, the criteria, ideally, should be clear, measurable and specific.
   Note that the set of criteria used to measure the performance or effectiveness of a decision maker can be a function of time and situation and will depend on the habitual domains of the individual decision makers.
3. Decision Outcomes
   The measurement of decision outcomes in terms of the criteria can be deterministic, probabilistic, fuzzy or unknown. When our decision criteria change, the perceptions of the possible outcomes of our decisions also change. Such changes can have important effects on the final decision. Note that the outcomes may be invisibly hidden in the potential domains. This includes trusting or grudging, which has great impact on the execution of the decision.
4. Preferences
   Implicitly or explicitly, we have preferences over the possible decision outcomes of our decisions. Preferences may be represented by numerical orderings. One

may also define a value function for each possible outcome such that the outcome with a higher value is better than that with a lower value. There are a larger number of methods for constructing numerical ordering for preferences. The interested reader is referred to [11, 14, 42]. Note that the preference may vary over time. The shift from the sellers of eBay to the internet users as target customers of YouTube (Example 1.4) is an obvious example.

5. Information Inputs
   Each information input can potentially affect our perception of the decision elements described above. Observe that information inputs can be regarded as habitual domains. Although they can vary with time and situations, they can be stabilized. In order to maintain effective decision making, we need to purposely be aware of whether or not our information inputs are adequate and alive (not stagnant) to prevent ourselves from being trapped in a stagnant habitual domain. Observe that information input can shape up the decision situations and be a powerful control variable in decision making as implicitly shown in Examples 1.1–1.2 and 1.4.

Decision environments may be described by four facets:

1. Decisions as a Part of the Behavior Mechanism
   Decision itself is a part of our behavior mechanism, as we described in Section 1.3. Decision problems may be regarded as events which can catch our attention. Because we have to attend to a number of events over time, the same decision problem cannot occupy our attention all of the time. The complex and important problems which do not have satisfactory solutions may create a relatively high level of charge on us. Our attention is caught more often and for longer periods of time by these kinds of problems until their final solutions are found. Recall in Example 1.1, Alinsky understood that decision as a part of human behavior, so he could artfully utilize "human hydraulic load and restrooms" to design its grand winning strategy. In this respect, we encourage the readers to review dynamics of human behavior as to become more capable of using the concepts to solve challenging problems.
2. Stages of Decision Processes
   Decision processes may take time to complete. In order to facilitate analysis, the entire process may be decomposed into subprocesses; or instead of one large process, we may solve the problem in a number of substages. Once we start to decompose the processes into substages, we begin to get a more concrete feeling and gain more control over the decision processes. In Examples 1.1–1.2, 1.4, we could see the stages of decision processes when key players or key events occur.
3. Players in the Decision Processes
   Players play an important role in the decision process. People make decisions and interact with others continuously through the interaction of different habitual domains. This can make the decision process very interesting and complex. In Examples 1.1–1.2 and 1.4, we see that more potential players (participants or customers) were introduced into the problems, which eventually changed the decision situations and solved the problems. Note, in Example 1.1, Alinsky planed

to get the whole world into the game through media, which made city authority comply with the agreement graciously.

4. Unknowns in Decision Processes
   Unknowns can create charge and excitement in our decision processes. If we know the unknowns and how to manage them, they add satisfaction to our decision processes; otherwise, they can create fear, frustration and bitterness. The unknowns could exist in any decision element. When our habitual domain expanded, outside of our habitual domain (unknown) will be reduced. As long as we are aware these decision elements are habitual domain and can be stabilized and expanded, our perception of the decision problems will be changed.

These five decision elements and four decision environmental facets not only vary with time, but also interact with each other through time. This can make the processes of decision making quite dynamic, complex and interesting. The dynamics of human behaviors and three toolboxes of Habitual Domains described in Sections 1.3 and 1.4 can help us understand how these elements and facets can be changed and interact over time, and how to use them to ensure high quality decisions.

1.6.2 Decision Blinds and Decision Traps

Recall that given a decision problem (E), there is a competence set for its satisfactory solution. Competence set is a projection of our habitual domain on that particular problem. It also has actual, reachable and potential domain (see Section 1.5). Recall that the truly needed competence set for a problem E is denoted by $Tr(E)$, while our perceived competence set for E is by $Tr^*(E)$. For simplicity of presentation, let us assume that the perceived true competence sets $Tr^*(E) = Sk^*(E) = Sk(E)$. Then $Tr(E)\backslash Tr^*(E)$ would be the decision blind, the set of competence needed for solving problem E and we do not see it. See Fig. 1.5.

Note that the larger the decision blind, the more likely to make dangerous mistakes.

As $Tr^*(E)$ and $Tr(E)$ are dynamic, we may denote them as $Tr_t^*(E)$ and $Tr_t(E)$. Suppose that $Tr_t^*(E)$ is fixed or trapped in a certain domain, while $Tr_t(E)$ changes with time and $Tr_t(E)\backslash Tr_t^*(E)$ gets large. Obviously the decision blinds get larger

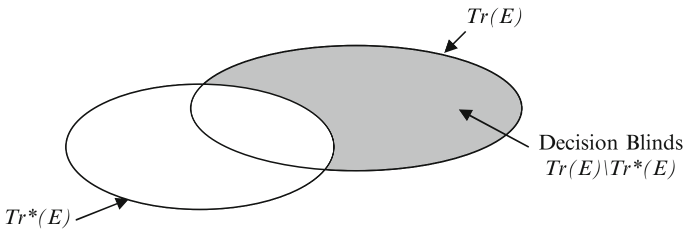

Fig. 1.5 Decision blinds

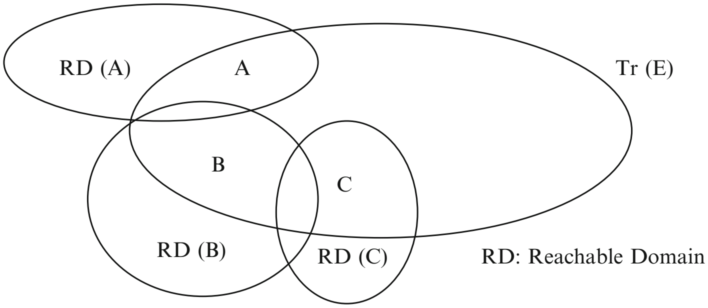

Fig. 1.6 Decision blind is reducing as we move from A to B then to C

and the probability to make dangerous mistake would get larger, too. When we are in this kind of situations we say we are in decision traps.

Note that $Tr_t^*(E)$ is fixed or trapped in a certain domain is equivalent to that the corresponding actual domain and reachable domain are fixed or trapped in a certain domain. This situation can occur when we are in a very highly charged state of mind or we are over confident, respond quickly, and unwittingly and habitually commit the behavior of decision traps. In Example 1.1, before Alinsky got involved, the Woodlawn organization might get in a decision trap. So were the Olympic organizers at Montreal (1976) and Moscow (1980) as to have disastrous financial loss.

By changing our actual domain, we can change and expand our reachable domain. As actual domains and their corresponding reachable domains can be changed, we can avoid decision traps and reduce the decision blinds by systematically changing the actual domains. As an illustration, in Fig. 1.6, if $Tr(E)$ and RDs are given, as depicted, then as we move actual domain from A to B, then to C . . ., our decision blind reduces step by step. If we could systematically move our consideration over the five decision elements and four facets of decision environment discussed in Section 1.6.1, we could systematically reduce our decision blind and avoid dangerous decision traps.

Finally, the three tool boxes for expanding and enriching our habitual domain can let us tap those ideas in the potential domains of our own and others. They are effective tools for us to expand our competence set as to reduce decision blinds and avoid decision traps. For further discussion, see [43, 46–48].

1.7 Conclusion

In this chapter, we introduce the dynamics of human behavior, the concepts of habitual domains, competence set analysis and decision making in changeable spaces. We first explore the dynamics of human behavior through eight basic hypotheses, which is a dynamic MCDM in changeable spaces. The stability of this behavior dynamics leads to the concept of habitual domains. Habitual domains follow us whenever we go and have a great impact on our decision making and lives. In order to improve the quality and effectiveness of decisions and upgrade our lives, we need to expand and enrich our habitual domains. Three tool boxes for expanding and enriching our habitual domains are introduced so that we can have better ideas to deal with problems, routine or challenge, and enjoy the process of solving the problems.

Decision problems implicitly have their habitual domains. For each decision problem, there is a collection of ideas, knowledge, resource, skills and effort for its effective solution. This collection is called "Competence Sets." We described the concept of competence set analysis. Based on competence set analysis and habitual domains theory, a framework of innovation dynamics is introduced. It describes the dynamics of how we can expand and enrich our competence set on one hand and maximize the value of our competence set on the other hand.

Finally, we describe decision making in changeable spaces. The parameters of decision problems, including the five elements and four environmental facets of decision problems, are explored. These parameters, like habitual domains, have their own actual, reachable and potential domains, and can be changing with time and situation. By restructuring the parameters, we may gain a broader and richer perception to the decision problems as to be able to derive effective solutions for challenging decision problem in changeable spaces. In the process we also introduce decision blinds and traps and how to deal with them using competence set analysis and habitual domain concept.

Many research problems remain open. For instances, in competence set analysis, the interaction among Tr, Tr^*, Sk and Sk^* and their impact on decision making in changeable spaces need to be further explored. Effective and concrete means to maximize the value of our competence sets need further studies. For innovation dynamics, mathematical analysis for specific cases would be of great interest to explore. How to early detect decision traps and decision blinds, and to locate effective methods to deal with them certainly, would bring value to practical decision making in changeable spaces and to academic research in decision science. In second order games [39] in which relevant parameters and players' state of mind can change with time and situation, how to restructure the games so that each player can declare a victory? Some significant results based on habitual domains theory can be found in [22, 49].

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Chapter 2
The Need for and Possible Methods of Objective Ranking

Andrzej P. Wierzbicki

Abstract The classical approach in decision analysis and multiple criteria theory concentrates on subjective ranking, at most including some aspects of intersubjective ranking (ranking understood here in a wide sense, including the selection or a classification of decision options). Intuitive subjective ranking should be distinguished here from rational subjective ranking, based on the data relevant for the decision situation and on an approximation of personal preferences. However, in many practical situations, the decision maker might not want to use personal preferences, but prefers to have some objective ranking. This need of rational objective ranking might have many reasons, some of which are discussed in this chapter. Decision theory avoided the problem of objective ranking partly because of the general doubt in objectivity characteristic for the twentieth century; the related issues are also discussed. While an absolute objectivity is not attainable, the concept of objectivity can be treated as a useful ideal worth striving for; in this sense, we characterize objective ranking as an approach to ranking that is as objective as possible. Between possible multiple criteria approaches, the reference point approach seems to be most suited for rational objective ranking. Some of the basic assumptions and philosophy of reference point approaches are recalled in this chapter. Several approaches to define reference points based on statistical data are outlined. Examples show that such objective ranking can be very useful in many management situations.

Keywords Rational subjective ranking · Rational objective ranking · Objectivity · Reference point approaches

2.1 Introduction

While there exists a need for *objective ranking* in some management situations, the classical approach in decision analysis and multiple criteria theory concentrates solely on *subjective ranking*, at most including some aspects of *intersubjective*

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ranking. This is because, in a popular belief of management science, decision making is usually based on personal experience, memory, thoughts, thinking paradigms and the psychological states (sometimes called *habitual domains*, see [35]) of the decision maker. Management science maintains that all individual decisions are subjective; it might be only admitted that there are situations where the decision may have impact on many other people, in which case, showing a kind of objectivity is needed. Objectivity might be considered desirable but, since the *true state of nature* and the *perceived state of nature* usually are not the same, and people use their perceived state of nature to make decisions, it is not possible to achieve full objectivity and thus not essential to seek objectivity.

While correct in basic arguments and dominating in management science, the above described perception is far from completeness. There are classes of individual decision situations where objectivity is needed, because practically all decisions of a given class might influence other people. Such kind of decision situations is dominating in technology creation, because all creation of technological tools assumes impacts on many other people; consider, for example, the issue of constructing a safe bridge or a safe car. Thus, technologists stress objectivity much more than management scientists – while real managers also know well that there are many managerial situations where stressing objectivity is necessary. Technologists also know, since the works of Heisenberg [9] discussed in more detail later, that a full precision of measurement is impossible, thus the concept of a *true state of nature* can be an approximation only and full objectivity is not attainable. However, they interpret this fact quite differently than social scientists, seeing in this fact not a reason to dismiss objectivity, but a constraint to objectivity. We see that different disciplines perceive the issue of objectivity versus subjectivity quite differently and that an interdisciplinary, even philosophical discussion of these concepts is needed; we shall return to such a discussion in the next section.

We must also stress to use here the concept of *ranking* in a wide sense, including the *selection* of one or several best, or worst decision options, or a *classification* of all decision options. All classical approaches of multi-attribute decision analysis – whether presented in [12], or in [24], or in [11] – concentrate on subjective ranking. By this we do not mean *intuitive subjective ranking*, which can be done by any experienced decision maker based on her/his intuition, but *rational subjective ranking*, based on the data relevant for the decision situation – however, using an approximation of personal preferences in aggregating multiple criteria.

And therein is the catch: in many practical situations, if the decision maker wants to have a computerized decision support and rational ranking, she/he does not want to use personal preferences, prefers to have some objective ranking. This is, as suggested above both from social science and technological perspectives, usually because the decision is not only a personal one, but affects many people – and it is often very difficult to achieve an intersubjective rational ranking, accounting for personal preferences of all people involved. We shall discuss in more detail the reasons for the need of objective ranking in the next section.

Decision theory avoided – to some extent, we comment on this issue later – the problem of objective ranking partly because of the general doubt in objectivity characteristic for the twentieth century. Thus, we recall also some of philosophical foundations and contemporary approaches to the issue of objectivity. While it can be agreed that an absolute objectivity is not attainable, the concept of objectivity can be treated as a goal, a higher-level value, a useful ideal worth striving for; in this sense, we characterize *objective ranking as an approach to ranking that is as objective as possible*.

Several multiple criteria decision analysis approaches are recalled in relation to the problem of objective ranking. Between such possible multiple criteria approaches, the reference point approach seems to be most suited for rational objective ranking, because reference levels needed in this approach can be established – to some extent objectively – statistically from the given data set. Some of the basic assumptions and philosophy of reference point approaches are recalled, stressing their unique concentration on the sovereignty of the subjective decision maker. However, precisely this sovereignty makes it possible also to postulate a proxy, virtual objective decision maker that is motivated only by statistical data. Several approaches to define reference points based on statistical data are outlined. Examples show that such objective ranking can be very useful in many management situations.

2.2 The Need for Objective Ranking and the Issue of Objectivity

Objectivity as a goal and objective ranking are needed not only in technology creation, but also – as we show here – in management. For an individual decision maker, this might mean that she/he needs some independent reasons for ranking, such as a dean cannot rank the laboratories in her/his school fully subjectively, must have some reasonable, objective grounds that can be explained to entire faculty, see one of further examples. For a ranking that expresses the preferences of a group, diverse methods of aggregating group preferences might be considered; but they must be accepted as fair – thus objective in the sense of intersubjective fairness – by the group, and the task of achieving a consensus about the fairness might be difficult. One of acceptable methods of such aggregation might be the specification of a *proxy, virtual decision maker that is as objective as possible, e.g., motivated only by statistical data*.

The need for objective ranking is expressed also in business community by the prevalent practice of hiring external consulting companies to give independent advice, including ranking, to the chief executive officer (CEO) of a company. The CEO obviously could use her/his detailed, tacit knowledge about the company and intuition to select a solution or ranking (either intuitive or rational); but she/he apparently prefers, if the situation is serious enough, not to use personal preferences and to ask for an independent evaluation instead.

There are many other situations where we need ranking, broadly understood thus including also classification and selection of either best or worst options (decisions, alternatives, etc.), performed as objectively as possible. This particularly concerns the task of selecting the worst options, often encountered in management (some opinions suggest that best management is concentrated on patching the worst observed symptoms); if we have to restructure the worst parts of an organization, we prefer to select them possibly objectively. These obvious needs have been neglected by decision theory that assumed subjectivity of a decision maker because of many reasons, partly paradigmatic, partly related to the anti-positivist and antiscientism turn in the philosophy of twentieth century.

Here we must add some philosophical comments on subjectivity and objectivity. The industrial era episteme – sometimes called not quite precisely positivism or scientism – valued objectivity; today we know that absolute objectivity does not exist. The destruction of this episteme started early, e.g., since Heisenberg [9] has shown that not only a measurement depends on a theory and on instruments, but also the very fact of measurement distorts the measured variable. This was followed by diverse philosophical debates, summarized, e.g., by Van Orman Quine [21] who has shown that the logical empiricism (neo-positivism) is logically inconsistent itself, that all human knowledge "is a man-made fabric that impinges on existence only along the edges". This means that there is no absolute objectivity; however, this was quite differently interpreted by hard sciences and by technology, which nevertheless tried to remain as objective as possible, and by social sciences which, in some cases, went much further to maintain that all knowledge is subjective – results from a discourse, is constructed, negotiated, relativist, depends on power and money, that the very concept of *"Nature"* is only a construction of our minds, see, e.g., [14]. This has led to a general divergence of the episteme – understood after Michel Foucault as the way of constructing and justifying knowledge, characteristic for a historical era or a cultural sphere, see [41] – of the three different cultural spheres of hard and natural sciences, of technology, and of social sciences and humanities, see [27].

Full objectivity is obviously – after Heisenberg and Quine – not attainable; but in many situations we must try to be as much objective as possible. This concerns not only technology that cannot advance without trying to be objective and, in fact, pursues Popperian falsificationism [20] in everyday practice when submitting technological artifacts to destructive tests in order to increase their reliability – while postmodern social sciences ridicule falsificationism as an utopian description how science develops. However, objectivity is needed also – as indicated above – in management.

In order to show that the postmodern episteme is not the only possible one, we present here another description of the relation of human knowledge to nature [32]. First, from a technological perspective we do not accept the assumption of postmodern philosophy that "*Nature*" is only a construction of our minds and has only local character. Of course, the word *nature* refers both to the construction of our minds and to something more – to some persisting, universal (to some degree) aspects of the world surrounding us. People are not alone in the world; in addition to other people, there exists another part of reality, that of nature, although part of this reality has been converted by people to form human-made, mostly technological systems. There are aspects of reality that are local and multiple, there are aspects that are more or less universal. To some of our colleagues who believe that there is no universe, only a *multiverse*, we propose the following *hard wall test*: we position ourselves against a hard wall, close our eyes and try to convince ourselves that there is no wall before us or that it is not hard. If we do not succeed in convincing ourselves, it means that there is no multi-verse, because nature apparently has some universal aspects. If we succeed in convincing ourselves, we can try to verify or falsify this conviction by running ahead with closed eyes.

Second, the general relation of human knowledge to reality might be described as follows. People, motivated by curiosity and aided by intuition and emotions, observe reality and formulate hypotheses about properties of nature, of other people, of human relations; they also construct tools that help them to deal with nature (such as cars) or with other people (such as telephones); together, we call all this knowledge. People test and evaluate the knowledge constructed by them by applying it to reality: perform destructive tests of tools, devise critical empirical tests of theories concerning nature, apply and evaluate theories concerning social and economic relations; in general, we can consider this as a generalized principle of falsification, broader than defined by Popper even in his later works [20].

Such a process can be represented as a general spiral of evolutionary knowledge creation, see Fig. 2.1. We observe reality (either in nature or in society) and its changes, compare our observations with human heritage in knowledge (the transition *Observation*). Then our intuitive and emotive knowledge helps us to generate new hypotheses (*Enlightenment*) or to create new tools; we apply them to existing reality (*Application*), usually with the goal of achieving some changes, modifications of reality (*Modification*); we observe them again.

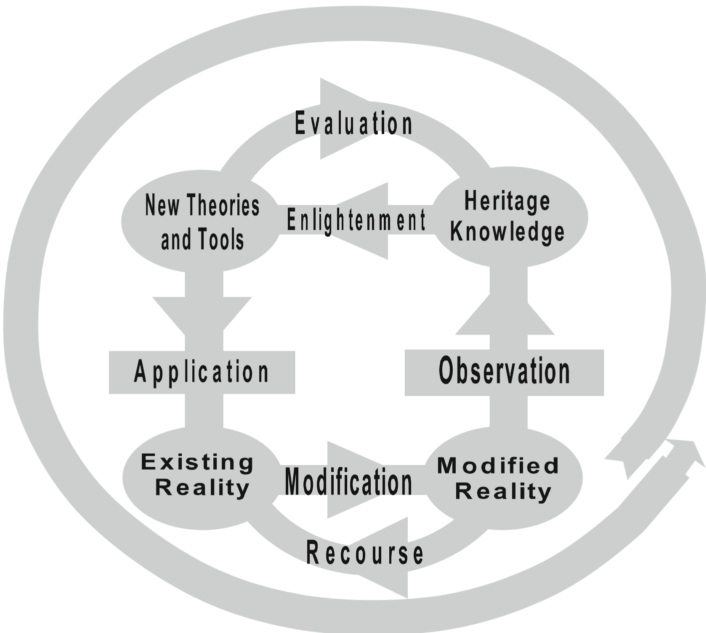

Fig. 2.1 The general OEAM spiral of evolutionary knowledge creation

It is important, however, to note that many other transitions enhance this spiral. First is the natural evolution in time: modified reality becomes existing reality through *Recourse*. Second is the evolutionary selection of tested knowledge: most new knowledge might be somehow recorded, but only the positively tested knowledge, resilient to falsification attempts, remains an important part of human heritage (*Evaluation*); this can be interpreted as an objectifying, stabilizing feedback. Naturally, there might be also other transitions between the nodes indicated in the spiral model, but the transitions indicated in Fig. 2.1 are the most essential ones.

Thus, nature is not only the effect of construction of knowledge by people, nor is it only the cause of knowledge: it is both cause and effect in a positive feedback loop, where more knowledge results in more modifications of nature and more modifications result in more knowledge. As in most positive feedback loops, the overall result is an avalanche-like growth; and this avalanche-like growth, if unchecked by stabilizing feedbacks, beside tremendous opportunities creates also diverse dangers, usually not immediately perceived but lurking in the future. Thus, the importance of selecting knowledge that is as objective as possible relates also to the fact that avalanche-like growth creates diverse threats: we must leave to our children best possible knowledge in order to prepare them for dealing with unknown future.

This description of a spiral-like, evolutionary character of knowledge creation presented in Fig. 2.1 was proposed first in [31] as consistent with our technological cognitive horizon, and different than presented in [10] from a position of an economic cognitive horizon; we are aware that there are many theories and schools of thought concerning philosophy of life and development of science, but we present this description as an extension of one of them. It is an extension of the concept of *objective knowledge* as presented in [20] which, however, admits relativistic interpretations. It only postulates objectivity as a higher level value, similar to justice: both absolute justice and absolute objectivity might be not attainable, but are important, worth striving for, particularly if we take into account uncertainty about future (see also [22]). This description is, however, concentrating not on individual knowledge creation, but on the evolutionary value of well-tested, as objectively as possible, knowledge for human societies and for humanity as a whole, including future generations.

2.3 Basic Formulations and Assumptions

We turn now to the main subject of this paper. We assume that we have a decision problem with n criteria, indexed by $i = 1, \ldots, n$ (also denoted by $i \in I$), and m alternative decisions called also alternatives, indexed by $j = 1, \ldots, m$ or $j = A, B, \ldots, H$ (also denoted by $j \in J$). The corresponding criteria values are denoted by q_{ij}; we assume that all are maximized or converted to maximized variables. The maximal values $\max_{j \in J} q_{ij} = q_i^{\text{up}}$ are called upper bounds for criteria and are often equivalent to the components of so called ideal or utopia point $\mathbf{q}^{\text{uto}} = \mathbf{q}^{\text{up}} = \left(q_1^{\text{up}}, \ldots, q_i^{\text{up}}, \ldots, q_n^{\text{up}}\right)$ – except for cases when they were established a priori as a measurement scale, see further comments. The minimal values $\min_{j \in J} q_{ij} = q_i^{\mathrm{lo}}$ are called lower bounds and, generally, are not equivalent to the components of so called nadir point $\mathbf{q}^{\mathrm{nad}} \approx \mathbf{q}^{\mathrm{lo}} = \left(q_1^{\mathrm{lo}}, \ldots, q_i^{\mathrm{lo}}, \ldots, q_n^{\mathrm{lo}}\right)$; the nadir point $\mathbf{q}^{\mathrm{nad}}$ is defined similarly as the lower bound point q^{lo}, but with minimization restricted to Pareto optimal or efficient or nondominated alternatives, see, e.g., [3]. An alternative $j^* \in J$ is Pareto optimal (Pareto-nondominated or shortly nondominated, also called efficient), if there is no other alternative $j \in J$ that dominates j^*, that is, if we denote $\mathbf{q}_j = (q_{1j}, \ldots, q_{ij}, \ldots, q_{nj})$, there is no $j \in J$ such that $\mathbf{q}_j \geq \mathbf{q}_{j^*}$, $\mathbf{q}_j \neq \mathbf{q}_{j^*}$.

While there is an extensive literature how to select the best alternative (usually between nondominated ones) or to rank or classify all alternatives in response to the preferences of a decision maker, this literature usually makes several tacit assumptions:

1. A standard and usually undisputed assumption is that there is a decision maker (DM) that does not mind to reveal her/his preferences – either a priori, before the computer system proposes her/his supposedly best decision (in this case, we should actually not speak about decision support, only about decision automation), or interactively, exchanging information with a computerized decision support system (in this case, truly supporting decisions). In group decision making, it is often assumed that the group members do not mind discussing their preferences. However, highly political decision makers might intuitively (using their experience in political negotiations) refuse to discuss their preferences, and do not have time for a long interaction with the decision support system. Moreover, as discussed above, there are also many rational reasons why a decision maker might want to obtain an advice on the best decision or ranking of decisions that is as objective as possible, thus independent from her/his preferences, particularly if the final decision will be highly political, or there is actually a large group of decision makers or stakeholders in the decision situation.
2. Another standard and usually undisputed assumption is that there is an analyst (AN) that knows well decision theory and practice, interacts with decision makers on the correct definition and modeling of the decision situation, thus influences, e.g., the choice of criteria, further programs or fine-tunes the decision support system, etc. (even if the role of the analyst might be hidden just by an assumed approach used for constructing the decision support system). However, the role of an analyst is essential even if it should not be dominant; for example, the choice of criteria might be a result of a political process, and even if the analyst would know the extensive literature how to select criteria reasonably from decision theoretical point of view, she/he has just to accept even unreasonable criteria.

In further discussions, we assume that there are decision makers and analysts, but their roles should be interpreted more broadly than usually.

2.4 Why Classical Approaches Are Not Applicable in This Case

We discuss here two classes of methods taught usually – for historical reasons – as "the basic approach" to multiple criteria decision making. The first of them is the weighted sum aggregation of criteria: determining by diverse approaches, between which the AHP [24] is one of the most widely known, weighting coefficients w_i for all $i \in I$, with the additional requirement on the scaling of weighting coefficients that $\sum_{i \in I} w_i = 1$, and then using them to aggregate all criteria by a weighted sum:

$$\sigma_{jsum} = \sum_{i \in I} w_i q_{ij}. \tag{2.1}$$

We use the aggregated values σ_{jsum} to select the best alternative (maximizing σ_{jsum} between $j \in J$) or to rank alternatives (ordering them from the largest to the lowest value of σ_{jsum}). Such an aggregation might be sometimes necessary, but it has several limitations, particularly for the problem of objective ranking. The most serious between them are the following:

1. The weighted sum is based on a tacit (unstated) assumption that a compensatory trade-off analysis is applicable to all criteria: a worsening of the value of one criterion might be compensated by the improvement of the value of another one. While often encountered in economic applications, this compensatory character of criteria is usually not encountered in interdisciplinary applications.
2. Changes of weighting coefficients in interactive decision processes with more than two criteria often lead to counter-intuitive changes of criteria values [69] explained below.
3. The linear aggregation of preferences expressed by the weighted sum tends to promote decisions with unbalanced criteria, as illustrated by the Korhonen paradox quoted below; in order to accommodate the natural human preference for balanced solutions, a nonlinear aggregation is necessary.
4. In the weighted sum approach, it is not easy to propose a way of defining weighting coefficients that are as objective as possible (except if all criteria have the same importance and we assume simply equal weighting coefficients).

The Korhonen paradox can be illustrated by the following example. Suppose we select a product and consider two criteria: quality and cost, while using an assessment scale 0–10 points for both criteria (0 points for cost means very expensive, 10 points means very cheap products). Suppose we have three alternative decisions. Alternative A has 10 points for quality, 0 points for cost. Alternative B has 0 points for quality, 10 points for cost. Alternative C has 4.5 points for quality and 4.5 points for cost. It is easy to prove that when using a weighted sum for ranking the alternatives, alternative C will be never ranked first – no matter what weighting coefficients we use. Thus, weighted sum indeed tends to promote decisions with unbalanced criteria; in order to obtain a balanced solution (the first rank for alternative product C), we have either to use additional constraints or a nonlinear aggregation scheme.

Educated that weighting coefficients methods are basic, the legislators in Poland introduced a public tender law. This law requires that any institution preparing a tender using public money should publish beforehand all criteria of ranking the offers and all weighting coefficients used to aggregate the criteria. This legal innovation backfired: while the law was intended to make public tenders more transparent and accountable, the practical outcome was opposite because of effects similar to the Korhonen paradox. Organizers of the tenders soon discovered that they are forced either to select the offer that is cheapest and worst in quality or the best in quality but most expensive one. In order to counteract, they either limited the solution space drastically by diverse side constraints (which is difficult but consistent with the spirit of the law) or added additional poorly defined criteria such as the degree of satisfaction (which is simple and legal but fully inconsistent with the spirit of the law, since it makes the tender less transparent and opens hidden door for graft).

The example of counter-intuitive effects of changing weighting coefficients given by Nakayama [16] is simple: suppose $n = 3$ and the criteria values for many alternatives are densely (or continuously) spread over the positive part of the surface of a sphere, $q_1^2 + q_2^2 + q_3^2 = 1$. Suppose we select first $w_1 = w_2 = w_3 = 0.3333$, which results in the best alternative with criteria values $q_1 = q_2 = q_3 = 0.577$. Suppose we want next to increase the values of q_1 strongly and of q_2 slightly, while agreeing to decrease q_3; what modifications of weighting coefficients would do the job? If we choose $w_1 = 0.55,\ w_2 = 0.35$ and $w_3 = 0.1$, the result will be a strong increase of $q_1 = 0.8338$ accompanied by a decrease of both $q_2 = 0.5306$ and $q_3 = 0.1516$; in order to increase q_1 strongly and q_2 slightly we must increase w_2 almost as strongly as w_1. If we have more criteria, it might be sometimes very difficult to choose a change of weighting coefficients resulting in a desired change of criteria values.

Both such theoretical examples and recent practical experience presented above show that we should be very careful when using weighted sum aggregation. In short summary, a linear weighted sum aggregation is simple mathematically but too simplistic in representing typical human preferences that are usually nonlinear; using this simplistic approach resulted in practice in adverse and unforeseen side-effects. For objective ranking, weighted sum aggregation is not applicable, except in the most simplest case of equal weighting coefficients.

Thus, we should rather look for nonlinear approximations of the preferences of decision makers. There are many highly developed methods of the elicitation of nonlinear utility or value functions, see, e.g., [11, 12]. However, these classical methods are not directly applicable for objective ranking, because they are developed precisely in order to express the subjectivity of the decision maker. As noted above, in decisions involving political processes such elicitations of utility or value functions might be not applicable because of several reasons:

1. Politically minded decision makers might be adverse to a disclosure and detailed specifications of their preferences.
2. Such elicitations of utility or value functions require a large number of pairwise comparisons of alternatives, done in the form of questions addressed to the decision maker and her/his answers; this number is nonlinearly growing with the number of criteria.

For these and other reasons, we should further look for more ad hoc and rough nonlinear approximations of preferences of decision makers, which do not require much time nor a detailed specification or identification of preferences. However, it is not obvious how to define the grounds of an objective selection or ranking. In multiple criteria optimization, one of similar issues was to propose compromise solutions, see, e.g., [5, 34, 36]; however, such solutions might depend too strongly on the assumed metric of the distance from the utopia or ideal point. In [28] it is proposed to define objective selection and ranking as dependent only on a given set of data, agreed upon to be relevant for the decision situation (generally, for any selected *data information system*, see [19]), and independent of any more detailed specification of personal preferences than that given by defining criteria and the partial order in criteria space. The specification of criteria and their partial order (whether to minimize, or maximize them) can be also easily be agreed upon, be objective in the sense of intersubjective fairness.

It is also not obvious how an objective selection and ranking might be achieved, because almost all the tradition of aggregation of multiple criteria concentrated on rational subjective aggregation of preferences and thus subjective selection and ranking. While we could try, in the sense of intersubjective fairness, identify group utility functions or group weighting coefficients, both these concepts are too abstract to be reasonably debated by an average group (imagine a stockholder meeting trying to define their aggregate utility function under uncertainty). Thus, neither of these approaches is easily adaptable for rational objective selection or ranking. The approach that can be easily adapted for rational objective selection and ranking, also classification, is reference point approach as described below, because reference levels needed in this approach can be either defined subjectively by the decision maker, or established objectively statistically from the given data set.

2.5 Reference Point Approaches for Objective Ranking

A rough approximation of decision maker preferences is provided by reference point approaches. In these approaches, we note that:

1. The preferences of decision maker can be approximated using several degrees of specificity, and the reference point approaches assume that this specification should be as general as possible, since a more detailed specification violates the sovereign right of a decision maker to change her/his mind.
2. The most general specification of preferences contains a selection of outcomes of a model of decision situation that are chosen by the decision maker (or analyst) to measure the quality of decisions, called criteria (quality measures, quality indicators) or sometimes objectives (values of objective functions) and denoted here by $q_i,\ i \in I$. This specification is accompanied by defining a partial order in the space of criteria – simply asking the decision maker which criteria should be maximized and which minimized, while another option, stabilizing some criteria around given reference levels, is also possible in reference point approaches, see

[30]. Here we consider – in order to simplify presentation – the simplest case when all criteria are maximized.

3. The second level of specificity in reference point approaches is assumed to consist of specification of reference points – generally, desired levels of criteria. These reference points might be interval-type, double, including aspiration levels, denoted here by a_i (levels of criteria values that the decision maker would like to achieve) and reservation levels r_i (levels of criteria values that should be achieved according to the decision maker). Specification of reference levels is treated as an alternative to trade off or weighting coefficient information that leads usually to linear representation of preferences and unbalanced decisions as discussed above, although some reference point approaches – see, e.g., [16, 23] – combine reference levels with trade-off information.
4. While the detailed specification of preferences might include full or gradual identification of utility or value functions, as shortly indicated above, this is avoided in reference point approaches that stress learning instead of value identification – according to the reference point philosophy, the decision maker should learn during the interaction with a decision support system, hence her/his preferences might change in the decision making process and she/he has full, sovereign right or even necessity to be inconsistent.
5. Thus, instead of a nonlinear value function, reference point approaches approximate the preferences of the decision maker by a nonlinear *achievement function* which is an ad hoc, easily adaptable nonlinear approximation of the value function of decision maker consistent with the information contained in criteria specification, their partial order and the position of reference point (or points) between the lower and upper bounds for criteria. As opposed to goal programming, similar in approach to reference point methods but using distance concepts instead of achievement functions, the latter functions preserve strict monotonicity with respect to the partial order in criteria space – because they are not equivalent to distances, see later comments.
6. The particular form of this nonlinear approximation of value function is determined essentially by max–min terms that favor solutions with balanced deviations from reference points and express the Rawlsian principle of justice (concentrating the attention on worst off members of society or on issues worst provided for, see [22]; these terms are slightly corrected by regularizing terms, resulting in nondomination (Pareto optimality) of alternatives that maximize achievement functions. It can be shown [26] that such achievement functions have the property of *full controllability*, independently of convexity assumptions. This means that, also for discrete decision problems, any nondominated (Pareto optimal) alternative can be selected by the decision maker when modifying reference points and maximizing the achievement function; this provides for the full sovereignty of the decision maker.

While there are many variants of reference point approaches, see [15, 23], we concentrate here on a reference point approach that requires the specification of interval-type reference, that is, two reference levels (aspiration and reservation) for each criterion. After this specification, the approach uses a nonlinear aggregation of criteria by an achievement function that is performed in two steps:

1. We first count achievements for each individual criterion or satisfaction with its values by transforming it (strictly monotonically and piece-wise linearly), e.g., in the case of maximized criteria as shown in Eq. 2.2. For problems with a continuous (nonempty interior) set of options, for an easy transformation to a linear programming problem, such a function needs additional specific parameters selected to assure the concavity of this function, see [10]. In a discrete decision problem, however, we do not necessarily need concavity and can choose these coefficients to have a reasonable interpretation of the values of the *partial (or individual) achievement function*:

$$\sigma_i\left(q_i, a_i, r_i\right)=\begin{cases}\alpha\left(q_i-q_i^{\mathrm{lc}}\right) / \left(r_i-q_i^{lo}\right) & \text{if } q_i^{lo} \leq q_i<r_i, \\ \alpha+(\beta-\alpha)\left(q_i-r_i\right) /\left(a_i-r_i\right) & \text{if } r_i \leq q_i<a_i, \\ \beta+(10-\beta)\left(q_i-a_i\right) / \left(q_i^{\mathrm{up}}-a_i\right) & \text{if } a_i \leq q_i \leq q_i^{\mathrm{up}}.\end{cases} \tag{2.2}$$

Since the range of [0; 10] points is often used for eliciting expert opinions about subjectively evaluated criteria or achievements, we adopted this range in Eq. 2.2 for the values of a partial achievement function $\sigma_i(q_i, a_i, r_i)$. The parameters α and β, $0<\alpha<\beta<10$, in this case denote correspondingly the values of the partial achievement function for $q_i=r_i$ and for $q_i=a_i$. The value $\sigma_{ij}=\sigma_i(q_{ij},\ a_i,\ r_i)$ of this achievement function for a given alternative $j \in J$ signifies the satisfaction level with the criterion value for this alternative. Thus, the above transformation assigns satisfaction levels from 0 to α (say, $\alpha=3$) for criterion values between q_i^{lo} and r_i, from α to β (say, $\beta=7$) for criterion values between r_i and a_i, from β to 10 for criterion values between a_i and q_i^{up}.

2. After this transformation of all criteria values, we might use then the following form of the overall achievement function:

$$\sigma(\mathbf{q}, \mathbf{a}, \mathbf{r})=\min_{i \in I} \sigma_i\left(q_i, a_i, r_i\right)+\varepsilon / n \sum_{i \in I} \sigma_i\left(q_i, a_i, r_i\right), \tag{2.3}$$

where $\mathbf{q}=(q_1, \ldots, q_i, \ldots, q_n)$ is the vector of criteria values, $\mathbf{a}=(a_1, \ldots, a_i, \ldots, a_n)$ and $\mathbf{r}=(r_1, \ldots, r_i, \ldots, r_n)$ are the vectors of aspiration and reservation levels, while $\varepsilon>0$ is a small regularizing coefficient. The achievement values $\sigma_j=\sigma(\mathbf{q}_j, \mathbf{a}, \mathbf{r})$ for all $j \in J$ can be used either to select the best alternative, or to order the options in an overall ranking list or classification list, starting with the highest achievement value.

The formulae (2.2), (2.3) do not express the only form of an achievement function; there are many possible forms of such functions as shown in [30]. All of them, however, are not equivalent to a distance: a distance, say, from the aspiration point $\mathbf{a}$ has the value 0 when $\mathbf{q}=\mathbf{a}$ and looses its monotonicity when crossing this point, while the overall achievement function maintains its strict monotonicity as a strictly monotone function of strictly monotone partial achievement functions. Moreover, all of them have an important property of partial order approximation: their level sets approximate closely the positive cone defining the partial order in criteria space (see [26]). As indicated above, the achievement function has also a very important theoretical property of *controllability*, not possessed by utility functions nor by weighted sums: for sufficiently small values of ε, given any point $\mathbf{q}^*$ in criteria space that is (ε-properly) Pareto-nondominated and corresponds to some alternative decision (such as the alternative C in the Korhonen paradox), we can always choose such reference levels – in fact, it suffices to set aspiration levels equal to the components of $\mathbf{q}^*$ – that the maximum of the achievement function (3) is attained precisely at this point. Conversely, if $\varepsilon > 0$, all maxima of achievement function (2.3) correspond to Pareto-nondominated alternatives – because of the monotonicity of this function with respect to the partial order in the criteria space, mentioned above, similarly as in the case of utility functions and weighted sums, but not in the case of a distance norm used in goal programming, since the norm is not monotone when passing zero. As noted above, precisely the controllability property results in a fully sovereign control of the decision support system by the user.

We turn now to the question how to use reference point approaches for objective ranking. Since an achievement function models a proxy decision maker, it is sufficient to define – as objectively as possible – the corresponding aspiration and reservation levels. Several ways of such definition were listed in [6]: *neutral, statistical, voting*; we shall concentrate here on statistical determination. A statistical determination of reference levels concerns values q_i^{av} that would be used as basic reference levels, a modification of these values to obtain aspiration levels a_i, and another modification of these values to obtain reservation levels r_i; these might be defined (for the case of maximization of criteria) as follows:

$$q_i^{\mathrm{av}} = \sum_{j \in J} q_{ij}/m; \qquad r_i = 0.5\left(q_i^{\mathrm{lo}} + q_i^{\mathrm{av}}\right); \qquad a_i = 0.5\left(q_i^{\mathrm{up}} + q_i^{\mathrm{av}}\right). \quad (2.4)$$

Recall that m is just the number of alternative decision options, hence q_i^{av} is just an average criterion value between all alternatives, and aspiration and reservation levels – just averages of these averages and the lower and upper bounds, respectively. However, as shown by examples presented later, there are no essential reasons why we should limit the averaging to the set of alternative options ranked; we could use as well a larger set of data in order to define more adequate (say, historically meaningful) averages, or a smaller set, e.g., only the Pareto-nondominated alternatives.

Thus, we are ready to propose one basic version of an objectified reference point approach for discrete decision alternatives. Here are our advices for the analyst:

1. Accept the criteria and their character (which to maximize, which to minimize) proposed by decision maker(s), but insist on a reasonable definition of their upper and lower bounds.

2. Gather (the evaluation of) all criteria values for all alternative decisions. In the case that some criteria have to be assessed by expert opinions, organize an objectifying process for these assessments (e.g., voting on these assessments as if judging ski-jumping, with deleting extreme assessments or even with using median score, allowing for a dispute and a repeated vote in cases of divergent assessments).
3. Compute the averages of criteria values, the statistically objective reservation and aspiration points as in Eq. 2.4. Assuming $\alpha = 3$ and $\beta = 7$ for all criteria and using the achievement functions as defined by (2.2), (2.3), compute achievement factors σ_j for all alternatives and order alternatives in a decreasing fashion of these factors (say, randomly if $\sigma_j = \sigma_{j'}$ for some j and j'; we shall suggest in the next section a way of improving such ordering). Use this ordering either for a suggested (objective and neutral) selection of the best alternative, or a classification of alternatives (say, into projects accepted and rejected), or an objective and neutral ranking.
4. Discuss with decision maker(s) the suggested objective and neutral outcome. If she/he wants to modify it, several ways of interaction are possible, starting with subjective modifications of reference levels, or an intersubjective definition of importance factors for every criterion (see [29]).

2.6 Examples

The first example concerns international business management. Suppose an international corporation consists of six divisions A, ..., F. Suppose these units are characterized by diverse data items, such as name, location, number of employees, etc. However, suppose that the CEO of this corporation is really interested in ranking or classification of these divisions taking into account the following attributes used as criteria:

1. Profit (p., in percent of revenue)
2. Market share (m.s., in percent of supplying a specific market sector, e.g., global market for a type of products specific for this division)
3. Internal collaboration (i.t., in percent of revenue coming from supplying other divisions of the corporation)
4. Local social image (l.s.i., meaning public relations and the perception of this division – e.g., of its friendliness to local environment – in the society where it is located, evaluated on a scale 0–100 points)

All these criteria are maximized, improve when increased. An example of decision table of this type is shown in Table 2.1 (with data distorted for privacy reasons), while Pareto-nondominated divisions are distinguished by mark *.

The CEO obviously could propose an intuitive, subjective ranking of these divisions – and this ranking might be even better than a rational one resulting from Table 2.1, if the CEO knows all these divisions in minute detail. However,

Table 2.1 Data for an example on international business management (Empl. = employees)

Division	Name	Location	Empl.	q_1: p.	q_2: m.s.	q_3: i.t.	q_4: l.s.i.
A	Alpha	USA	250	11%	8%	10%	40
B*	Beta	Brasilia	750	23%	40%	34%	60
C*	Gamma	China	450	16%	50%	45%	70
D*	Delta	Dubai	150	35%	20%	20%	44
E*	Epsilon	C. Europe	350	18%	30%	20%	80
F	Fi	France	220	12%	8%	9%	30

when preparing a discussion with her/his stockholders, (s)he might prefer to ask a consulting firm for an objective ranking.

Thus, we first illustrate the issue of objective ranking and statistical determination of reservation and aspiration levels. The principle that all criteria improve when increasing is easy to agree upon; similarly, the stockholders would easily accept the principle that the details of ranking should be determined mostly by the data contained in Table 2.1 and not by any personal preferences. The question how to statistically define reservations and aspirations is actually technical, but interesting for illustration. There are no essential reasons why we should limit the averaging to the set of alternatives ranked; we could use as well a larger set of data in order to define more adequate (say, historically meaningful) averages, or a smaller set – for example, only the Pareto-nondominated alternatives denoted by * in Table 2.1 – in order to define, say, more demanding averages and aspirations. For the data from Table 2.1, we can thus present two variants of objective ranking: A – based on averages of data from this table; B – based on averages from Pareto optimal options – see Table 2.2. We use here the achievement function from Eq. 2.3 with $\varepsilon = 0.4 (n = 4)$.

We do not observe changes of ranking and classification when shifting from average A to more demanding B aspirations and reservations; this is confirmed by other applications and shows that objective ranking gives – at least, on the examples considered – rather robust results. Generally, we might expect rank reversals, although usually not very significant, when shifting to more demanding aspirations. This is, however, a natural phenomenon: average aspirations favor standard though good solutions, truly interesting solutions result from demanding aspirations. Note that we did not change the estimates of the lower and upper bounds and thus measurement ranges when averaging over Pareto-nondominated solutions; although the lower bounds for Pareto-nondominated alternatives (so called nadir point) are in this case different than the lower bounds for all alternatives, a change of ranges would mean a change of measurement units and should be avoided, see also [11].

The second example concerns knowledge management at a university. It illustrates a management application where the worst ranked options are the most interesting, because they indicate the need of a corrective action. Objective ranking was actually motivated originally by this specific application when evaluating scientific creativity conditions in a Japanese research university, JAIST, see [25]. The evaluation was based on survey results. The survey included 48 questions with diverse answers and over 140 respondents with diverse characteristics: school

Table 2.2 An example of objective ranking and classification for the data from Table 2.1

Criterion	q1	q2	q3	q4			
Upper bound	35%	50%	45%	80			
Lower bound	11%	8%	9%	30			
Reference A							
(average)	19.2%	26%	23%	54			
Aspiration A	27.1%	38%	34%	67			
Reservation A	15.1%	17%	16%	42			
Reference B							
(Pareto average)	23%	35.0%	29.7%	63.5			
Aspiration B	29%	42.5%	37.4%	71.7			
Reservation B	17%	17%	19.4%	46.7			
Ranking A: Division	σ_1	σ_2	σ_3	σ_4	σ	Rank	Class
A	0.00	0.00	0.37	2.50	0.29	5	III
B	5.63	7.50	7.00	5.88	8.23	1	I
C	3.30	10.0	10.0	7.62	6.39	2	II
D	10.0	3.57	3.89	3.32	5.40	4	II
E	3.97	5.48	3.89	10.0	6.30	3	II
F	0.73	0.00	0.00	0.00	0.07	6	III
Ranking B: Division							
A	0.00	0.00	0.29	1.80	0.21	5	III
B	5.00	6.61	6.24	5.13	7.30	1	I
C	2.50	10.0	10.0	6.73	5.42	2	II
D	10.0	3.47	3.13	2.51	4.42	4	II
E	3.33	5.04	3.13	10.0	5.28	3	II
F	0.50	0.00	0.00	0.00	0.05	6	III

attachment (JAIST consists of three schools), nationality (Japanese or foreign – the latter constitute over 10% of young researchers at JAIST), research position (master students, doctoral students, research associates, etc.). In total, the data base was not very large, but large enough to create computational problems.

The questions were of three types. The first type was assessment questions, assessing the situation between students and at the university; the most critical questions of this type might be selected as those that correspond to worst responses. The second type was important questions, assessing importance of a given subject; the most important questions might be considered as those that correspond to best responses. For those two types of questions, responders were required to tick appropriate responses in the scale *vg (very good), g (good), a (average), b (bad), vb (very bad)* – sometimes in an inverted scale if the questions were negatively formulated. The third type was controlling questions, testing the answers to the first two types by indirect questioning revealing responder attitudes or asking for a detailed explanation.

Answers to all questions of first two types were evaluated on a common scale, as a percentage distribution (histogram) of answers *vg – g – a – b – vb*. It is good if there are many answers specifying positive evaluations *very good* and *good*, and if there are only few answers specifying negative evaluations *bad* and *very bad*. The interpretation of the evaluation *average* was *almost bad*; if we want most answers to be *very good* and *good*, we admit only a few answers to be *average*. Therefore, in this case $I = G \cup B,\ G = \{vg, g\},\ B = \{a, b, vb\}$; the statistical distributions (percentage histograms) of answers were interpreted in the sense of multiple criteria optimization, with $i \in G = \{vg, g\}$ counted as positive outcomes (quality indicators that should be maximized) and $i \in B = \{a, b, vb\}$ counted as negative outcomes (quality indicators to be minimized).

A reference point approach (similar as described here, only using single reference point r) was proposed for this particular case of ranking probability distributions; other approaches are usually more complicated (see, e.g., [18]). However, when the dean of the School of Knowledge Science in JAIST, himself a well-known specialist in multiple criteria decision support, was asked to define his preferences or preferred aspiration levels, the reality of the managerial situation overcome his theoretical background: he responded "in this case, I want the ranking to be as objective as possible – I must discuss the results with the deans of other schools and with all professors". This was the origin of reflection on objective versus subjective rational ranking.

Thus, a statistical average of the percentages of answers in the entire data set was taken as the reference distribution or profile. Since it was realized that such a reference profile might result in good but standard answers, some artificial reference distributions were also constructed as more demanding than the average one; averages over Pareto optimal options were not computed because of the complexity of the data set.

The reference distribution called *Average* above ($\mathbf{r}_D$) represents the actual average of percentages of answers for all questions (of the first and second type) and all responders. This distribution might be taken as the basic one, because it results from the experimental data and might be considered as independent from the preferences of the decision maker, thus resulting in a ranking of questions that is as objective as possible – although, theoretically, average aspirations result only in average, not necessarily interesting answers (actually, this theoretical conclusion was later confirmed in practice). Truly interesting results might correspond to more demanding aspirations, hence beside the average distribution we postulated synthetic users and considered three more demanding ones, which were characterized by the types of neutral reference distributions. The one called *Regular* ($\mathbf{r}_A$) was almost linearly decreasing; the one called *Stepwise* ($\mathbf{r}_C$) was almost uniform for positive and for negative outcomes; while the one called *Demanding* ($\mathbf{r}_B$) was almost hyperbolically decreasing and actually the most demanding (Table 2.3).

The detailed results of the survey were not only very interesting theoretically, but also very useful for university management, see [25]. It was found that seven questions of the first (assessment) type ranked as worst practically did not depend on the variants of reference distributions and ranking, on the schools or on the characteristics of respondents; thus, the objective ranking gave robust results as to the problems that required most urgent intervention by the university management. The

Table 2.3 Four different types of reference profile distributions

Name	Symbol	*vg* (%)	*g* (%)	*a* (%)	*b* (%)	*vb* (%)
Regular	$\mathbf{r}_A$	36	28	20	12	4
Demanding	$\mathbf{r}_B$	48	26	14	8	4
Stepwise	$\mathbf{r}_C$	42	42	7	5	4
Average	$\mathbf{r}_D$	21	38	22	14	5

best ranked questions of the second (importance) type were more changeable, only three of them consistently were ranked among the best ones in diverse ranking profiles. Moreover, a rank reversal phenomenon was observed: if the average reference distribution was used, best ranked were questions of rather obvious type, more interesting results were obtained when using more demanding reference profile. This rank reversal, however, influenced more the best ranked questions than worst ranked questions, more significant for university management.

In [25], the more demanding reference distributions or profiles were constructed by an arbitrary modification of the statistical average reference profile. However, can we construct them more objectively? The answer is positive, as shown in the preceding example, provided we have a good algorithm for finding all Pareto optimal (nondominated) options in a complex data set. In classical approaches, Pareto optimal points in complex data sets are found by envelope analysis using appropriate linear and mixed integer programming formulations. However, envelope analysis results only in the envelope – a convex hull of Pareto optimal points, while discrete alternative problems are known to possess many Pareto optimal points in the interior of the convex hull. Thus, EMO algorithms are a natural candidate to resolve the problem of a sufficiently fine approximation of the Pareto set (that can have many elements for complex data sets) to estimate well the averages of criteria values over Pareto set. Naturally, because of the discrete character of the problem, the genetic variant of the evolutionary algorithms should be also considered, see, e.g., [2].

2.7 Conclusions and Further Research

While absolute objectivity is known not to be attainable, postmodern sociology of science is wrong in reducing scientific objectivity to power and money: *we must transfer knowledge that is as objective as possible to future generations, because only this way we can help them in facing uncertain future*. In the same sense, we use *the concept of objective ranking as* such a ranking (including classification and selection of decision options) that is *not absolutely objective, but as objective as possible*.

We define here *objective ranking as dependent only on a given set of data, relevant for the decision situation, and independent of any more detailed specification of personal preferences than that given by defining criteria and the partial order in criterion space*.

Rational objective ranking can be based on reference point approach, because reference levels needed in this approach can be established statistically from the given data set.

Examples show that such objective ranking can be very useful in many management situations. A technical problem of finding objective but demanding reference levels can be solved by averaging over Pareto set and using, e.g., EMO algorithms for this purpose.

The concept of objective ranking opens many avenues of possible future research, such as: the use of equitable aggregation, see [13, 17] and the use of ordered weighted averaging (OWA, see [36]), both in objective ranking; possible extensions of rough set theory [8, 19] for objective ranking; multiobjective comparison of empirical statistical profiles [7], and many other possibilities.

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Chapter 3
Preference Function Modelling: The Mathematical Foundations of Decision Theory

Jonathan Barzilai

Abstract We establish the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable. The mathematical foundations of decision theory and related theories depend on these conditions which have not been correctly identified in the classical literature. Operations Research and Decision Analysis Societies should act to correct fundamental errors in the mathematical foundations of measurement theory, utility theory, game theory, mathematical economics, decision theory, mathematical psychology, and related disciplines.

Keywords Foundations of science · Measurement theory · Decision theory · Social choice · Group decision making · Utility theory · Game theory · Economic theory · Mathematical psychology

3.1 Introduction

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied to enable the application of the mathematical operations of linear algebra and calculus. We identify these conditions and show that classical measurement and evaluation theories, including utility theory, cannot serve as the mathematical foundation of decision theory, game theory, economics, or other scientific disciplines because they do not satisfy these conditions.

The mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables*, i.e., to variables such as *preference* that describe psychological or subjective properties. Whether psychological properties can be measured (and hence whether

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mathematical operations can be applied to psychological variables) was debated by a Committee that was appointed in 1932 by the British Association for the Advancement of Science but the opposing views in this debate were not reconciled in the Committee's 1940 Final Report.

In 1944, game theory was proposed as the proper instrument with which to develop a theory of economic behavior where utility theory was to be the means for measuring preference. We show that the interpretation of utility theory's lottery operation which is used to construct utility scales leads to an intrinsic contradiction and that the operations of addition and multiplication are not applicable on utility scale values. We present additional shortcomings of utility theory which render it unsuitable to serve as mathematical foundations for economics or other theories and we reconstruct these foundations.

3.2 Measurement of Preference

The applicability of mathematical operations is among the issues implicitly addressed by von Neumann and Morgenstern in [53, §§3.2–3.6] in the context of measurement of individual preferences. Preference, or value, or utility, is not a physical property of the objects being valued, that is, preference is a subjective, i.e. psychological, property. Whether psychological properties can be measured was an open question in 1940 when the Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (Ferguson et al. [30]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the Final Report.

The position that psychological variables cannot be measured was supported by Campbell's view on the role of measurement in physics [24, Part II] which elaborated upon Helmholtz's earlier work on the mathematical modelling of physical measurement [35]. To re-state Campbell's position in current terminology the following is needed.

By an empirical system E we mean a set of empirical *objects* together with *operations* (i.e. functions) and possibly the relation of *order* which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the empirical operations in E as well as the order in E when E is ordered. A scale s is a mapping of the objects in E into the objects in M that reflects the structure of E into M (in technical terms, a scale is a homomorphism from E into M).

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: As Campbell eloquently states [24, pp. 267–268], "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science."

In terms of these concepts, the main elements of Campbell's view are summarized by J. Guild in Ferguson et al. [30, p. 345] in the context of measurement of *sensation* where he states that for psychological variables it is not possible to construct a scale that reflects the empirical operation of addition because such an empirical (or "practical") addition operation has not been defined; if the empirical operation does not exist, its mathematical reflection does not exist as well.

The framework of mathematical modelling is essential. To enable the application of mathematical operations, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects (which typically are the real numbers for the reasons given in Section 3.7.2). Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Therefore, the construction of measurement scales requires that the property-specific empirical operations be identified and reflected in the mathematical model. Moreover, the operations should be chosen so as to achieve the goal of this construction which is the application of mathematical operations in the mathematical model.

3.2.1 Empirical Addition – Circumventing the Issue

Accordingly, von Neumann and Morgenstern had to identify the empirical operations that characterize the property of *preference* and construct a corresponding mathematical model. As we shall see in Section 3.5, the operations of addition and multiplication are not enabled in their mathematical model and their empirical operation requires an interpretation that leads to an intrinsic contradiction.

The task of constructing a model for *preference* measurement is addressed by von Neumann and Morgenstern in [53, §3.4] indirectly in the context of measurement of *individual* preference. While the operation of addition as applies to *length* and *mass* results in scales that are unique up to a positive multiplicative constant, physical variables such as *time* and *potential energy* to which standard mathematical operations do apply are unique up to an additive constant and a positive multiplicative constant. (If s and t are two scales then for *time* or *potential energy* $t = p + q \times s$ for some real numbers p and $q > 0$ while for *length* or *mass* $t = q \times s$ for some $q > 0$.) This observation implies that Guild's argument against the possibility of measurement of psychological variables is not entirely correct. It also seems to indicate the need to identify an empirical – "natural" in von Neumann and Morgenstern's terminology – operation for *preference* measurement for which the resulting scales are unique up to an additive constant and a positive multiplicative constant. Seeking an empirical operation that mimics the "center of gravity" operation, they identified the now-familiar utility theory's operation of constructing lotteries on "prizes" to serve this purpose.

Von Neumann and Morgenstern's *uniqueness* argument and *center of gravity* operation are the central elements of their utility theory which is formalized in the axioms of [53, §3.6]. This theory is the basis of game theory which, in turn, was to serve as the mathematical foundation of economic theory. Elaborating upon von Neumann and Morgenstern's concepts, Stevens [62] proposed a uniqueness-based classification of "scale type" and the focus on the issues of the possibility of measurement of psychological variables and the applicability of mathematical operations to scale values has moved to the construction of "interval" scales, i.e. scales that are unique up to an additive constant and a positive multiplicative constant.

3.2.2 Applicability of Operations on Scale Values Versus Scale Operations

Consider the applicability of the operations of addition and multiplication to scale values for a fixed scale, that is, operations that express facts such as "the weight of an object equals the sum of the weights of two other ones" (which corresponds to addition: $s(a) = s(b) + s(c)$) and "the weight of a given object is two and a half times the weight of another" (which corresponds to multiplication: $s(a) = 2.5 \times s(b)$).

It is important to emphasize the distinction between the application of the operations of addition and multiplication to scale values for a fixed scale (e.g., $s(a) = s(b) + s(c)$) as opposed to what appears to be the same operations when they are applied to an entire scale whereby an equivalent scale is produced (e.g., $t = p + q \times s$ where s and t are two scales and p, q are numbers). In the case of scale values for a fixed scale, the operations of addition and multiplication are applied to elements of the mathematical system M and the result is another element of M. In the case of operations on entire scales, addition or multiplication by a number is applied to an element of the set $S = \{s, t, \ldots\}$ of all possible scales and the result is another element of S rather than M. These are different operations because operations are functions and functions with different domains or ranges are different.

In the case of "interval" scales where the uniqueness of the set of all possible scales is characterized by scale transformations of the form $t = p + q \times s$, it cannot be concluded that the operations of addition and multiplication are applicable to scale values for a fixed scale such as $s(a) = s(b) + s(c)$. It might be claimed that the characterization of scale uniqueness by $t = p + q \times s$ implies the applicability of addition and multiplication to scale values for fixed scales, but this claim requires proof. There is no such proof, nor such claim, in the literature because this claim is false: Consider the automorphisms of the group of integers under addition. The group is a model of itself ($E = M$), and scale transformations are multiplicative: $t = (\pm 1) \times s$. However, by definition, the operation of multiplication which is defined on the set of scales is not defined on the group M.

3.3 The Principle of Reflection

Consider the measurement of *length* and suppose that we can only carry out ordinal measurement on a set of objects, that is, for any pair of objects we can determine which one is longer or whether they are equal in length (in which case we can order the objects by their length). This may be due to a deficiency with the state of technology (appropriate tools are not available) or with the state of science (the state of knowledge and understanding of the empirical or mathematical system is insufficient). We can still construct scales (functions) that map the empirical objects into the real numbers but although the real numbers admit many operations and relations, the only relation on ordinal scale values that is relevant to the property under measurement is the relation of order. Specifically, the operations of addition and multiplication can be carried out on the range of such scales since the range is a subset of the real numbers, but such operations are extraneous because they do not reflect corresponding empirical operations. Extraneous operations may not be carried out on scale values – they are irrelevant and inapplicable; their application to scale values is a modelling error.

The Principle of Reflection is an essential element of modelling that states that operations within the mathematical system are applicable *if and only if* they reflect corresponding operations within the empirical system. In technical terms, in order for the mathematical system to be a valid model of the empirical one, the mathematical system must be homomorphic to the empirical system (a homomorphism is a structure-preserving mapping). A mathematical operation is a valid element of the model only if it is the homomorphic image of an empirical operation. Other operations are not applicable on scale values.

By *The Principle of Reflection*, a necessary condition for the applicability of an operation on scale values is the existence of a corresponding empirical operation (the homomorphic pre-image of the mathematical operation). That is, *The Principle of Reflection* applies in both directions and a given operation is applicable in the mathematical image only if the empirical system is equipped with a corresponding operation.

3.4 The Ordinal Utility Claim in Economic Theory

Preference theory, which plays a fundamental role in decision theory, plays the same role under the name utility theory (see Section 3.9.4) in economic theory. We now show that in the context of economic theory, utility theory is founded on errors that have not been detected by decision theorists or other scholars. In his *Manual of Political Economy*, Pareto claims that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility*" [54, p. 393]. More precisely, it is claimed that ordinal utility scales are sufficient to carry out Pareto's development of economic equilibrium. This claim is surprising considering that Pareto's *Manual* is founded on the notions of differentiable utility scales (by different names such as "ophelimity" and "tastes"). This claim is also surprising because a parallel claim stating that *ordinal temperature scales are sufficient to carry out partial differentiation in thermodynamics* is obviously false. It is even more surprising that this false claim has escaped notice for so long and is repeated in current economic literature.

Relying on Pareto's error, Hicks [36, p. 18] states that "The quantitative concept of utility is not necessary in order to explain market phenomena." With the goal of establishing a *Logical foundation of deductive economics* – having identified the *Need for a theory consistently based upon ordinal utility* – (see the titles of Chapter I's sections in *Value and Capital* [36]) he proceeds "to undertake a purge, rejecting all concepts which are tainted by quantitative utility" [36, p. 19]. In essence, Hicks claims that wherever utility appears in economic theory, and in particular in Pareto's theory which employs partial differentiation, it can be replaced by ordinal utility (see also the title *The ordinal character of utility* [36, Chapter I, §4]).

Neither Pareto, who did not act on his claim, nor Hicks, who did proceed to purge "quantitative utility" from economic theory, provide rigorous mathematical justification for this claim and it seems that authors who repeat this claim rely on an incorrect argument in Samuelson's *Foundations of Economic Analysis* [58, pp. 94–95].

3.4.1 Ordinal Utility

An ordinal empirical system E is a set of empirical objects together with the relation of order, which characterize a property under measurement. A mathematical model M of an ordinal empirical system E is an ordered set where the order in M reflects the order in E. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the order of E into M. In general, the purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M and operations that are not defined in E are not applicable in M. In the case of ordinal systems the mathematical image M of the empirical system E is equipped only with order and the operations of addition and multiplication are not applicable in M. In other words, since, by definition, in ordinal systems only order is defined (explicitly – neither addition nor multiplication is defined), addition and multiplication are not applicable on ordinal scale values and it follows that the operation of differentiation is not applicable on ordinal scale values because differentiation requires that the operations of addition and multiplication be applicable.

In summary, if $u(x_1, \ldots, x_n)$ is an ordinal utility function it cannot be differentiated and conversely, a utility function that satisfies a differential condition cannot be an ordinal utility scale.

3.4.2 Optimality Conditions on Indifference Surfaces

In [36, p. 23] Hicks says that "Pure economics has a remarkable way of producing rabbits out of a hat" and that "It is fascinating to try to discover how the rabbits got in; for those of us who do not believe in magic must be convinced that they got in somehow." The following is treated with only that minimal degree of rigor which is necessary to discover how this observation applies to the use of, supposedly ordinal, utility functions in the standard derivation of elementary equilibrium conditions. (A greater degree of rigor is necessary if other errors are to be avoided.)

Consider the problem of maximizing a utility function $u(x_1, \ldots, x_n)$ subject to a constraint of the form $g(x_1, \ldots, x_n) = b$ where the variables $x_1, \ldots, x_n$ represent quantities of goods. Differentiating the Lagrangean $L = u - \lambda(g - b)$ we have

$$\frac{\partial u}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0 \quad \text{for } i = 1, \ldots, n$$

which implies $\frac{\partial u}{\partial x_i} \div \frac{\partial g}{\partial x_i} = \lambda = \frac{\partial u}{\partial x_j} \div \frac{\partial g}{\partial x_j}$ for all $i,\ j$, and therefore

$$\frac{\partial u}{\partial x_j} \div \frac{\partial u}{\partial x_i} = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i}. \tag{3.1}$$

Equation 3.1 is a tangency condition because, in common notation,

$$\frac{\partial x_i}{\partial x_j} = -\left(\frac{\partial f}{\partial x_j} \div \frac{\partial f}{\partial x_i}\right) \tag{3.2}$$

holds on a surface where a function $f(x_1, \ldots, x_n)$ is constant. Since applying this notation to Eq. 3.1 yields

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j},$$

it is preferable to use the explicit notation

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u$$

to indicate that the differentiation is performed on an indifference surface of the function u at the point x. This derivative depends on the function u as well as the point x; the function u is not "eliminated" in this expression. In general, at an arbitrary point x we expect

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u \neq \left.\frac{\partial x_i}{\partial x_j}\right|_x^g$$

but at the solution point x^* Eq. 3.1 implies

$$\left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^u = \left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^g \quad \text{for all } i, j, \tag{3.3}$$

which, together with the constraint $g(x_1, \ldots, x_n) = b$, is a system of equations for the n unknowns $x^* = \left(x_1^*, \ldots, x_n^*\right)$.

In the special case of a budget constraint $p_1 x_1 + \cdots + p_n x_n = b$ where p_i is the price of good i,

$$-\left.\frac{\partial x_i}{\partial x_j}\right|_x^g = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i} = \frac{p_j}{p_i}$$

and the solution satisfies

$$p_1 x_1^* + \cdots + p_n x_n^* = b \qquad \text{and} \qquad -\left.\frac{\partial x_i}{\partial x_j}\right|_{(x_1^*, \cdots, x_n^*)}^u = \frac{p_j}{p_i} \qquad \text{for all } i, j. \tag{3.4}$$

When the number of variables is greater than two, this system of equations cannot be solved by the method of indifference curves, i.e. by using two-dimensional diagrams, because the left hand sides of the equations in (3.4) depend on all the n unknowns. For example, we can construct a family of indifference curves in the (x_1, x_2) plane where the variables $x_3, \ldots, x_n$ are fixed, but $x_3, \ldots, x_n$ must be fixed at the unknown solution values $x_3^*, \ldots, x_n^*$. To emphasize, with each fixed value of the variables $x_3, \ldots, x_n$ is associated a family of (x_1, x_2) indifference curves. To solve for x_1^*, x_2^* by the method of indifference curves, it is necessary to construct the specific family of indifference curves that corresponds to the solution values $x_3^*, \ldots, x_n^*$, but these values are not known. Noting again that the utility function u is not eliminated in Eq. 3.4 and that this equation was derived using the operation of differentiation which is not applicable on ordinal utility functions, we conclude that Hicks's "Generalization to the case of many goods" [36, §9] has no basis.

Returning to Eq. 3.2, we note that $f(x_1, \ldots, x_n)$ and $F(f(x_1, \ldots, x_n))$ have the same indifference surfaces (but with different derivatives) and, by the chain rule, if F and $f(x_1, \cdots, x_n)$ are both differentiable then

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^{F(f)} = \left.\frac{\partial x_i}{\partial x_j}\right|_x^f \tag{3.5}$$

so that this partial derivative is independent of F. However, since both F and f are assumed to be differentiable, Eq. 3.5 does not imply that f is ordinal.

3.4.3 Pareto's Claim

In the Appendix to his *Manual of Political Economy* [54, pp. 392–394] Pareto considers the indifference surfaces of the utility $I = \Psi(x, y, z, \ldots)$ of the goods $x, y, z, \ldots$. Taking for granted the applicability of the operation of differentiation, if $I = F(\Psi)$ "is differentiated with I taken as a constant," Pareto obtains the equation (numbered (8) in his Appendix) $0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \cdots$ independently of F. This equation is followed by the statement that "An equation equivalent to the last mentioned could be obtained directly from observation." Pareto then says that the latter equation (numbered (9) in his Appendix), $0 = q_x dx + q_y dy + q_z dz + \cdots$, "contains nothing which corresponds to ophelimity, or to the indices of ophelimity" (where he uses the term ophelimity for utility) and concludes that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility*" [54, p. 393].

This conclusion has no basis: "direct observation" does not constitute mathematical proof; Pareto does not define the variables $q_x, q_y, q_z, \ldots$; and it is not clear what it is which he directly observes. On the contrary, if Pareto's equation

$$0 = q_x dx + q_y dy + q_z dx + \cdots$$

contains nothing which corresponds to utility, it cannot be equivalent to his equation

$$0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \cdots ,$$

which characterizes utility indifference surfaces. As pointed out in Section 3.4.1, since Ψ satisfies a differential condition it cannot be an ordinal utility scale.

3.4.4 Samuelson's Explanation

Samuelson defines an ordinal utility scale $\varphi(x_1, \ldots, x_n)$ in Eqs. 6–8 of [58, p. 94] and states, correctly, that any function $U = F(\varphi)$ where $F'(\varphi) > 0$ reflects the same order. However, this observation does not imply that φ is ordinal. On the contrary, since this observation is based on differentiating both F and φ, it is only valid if φ is differentiable in which case it cannot be ordinal.

The paragraph that follows this observation in [58, p. 94] consists of one sentence: "To summarize, our ordinal preference field may be written [here Samuelson repeats his Eq. 9 as Eq. 10] where φ is any one cardinal index of utility." Recalling Hicks's comment that "It is fascinating to try to discover how the rabbits got in," this sentence is remarkable, for "those of us who do not believe in magic" will note that the *ordinal* utility at the beginning of the sentence has metamorphosed into *cardinal* utility at the sentence's end. Note that Samuelson does not define the concept of "cardinal" utility, nor does it appear to be defined elsewhere in the literature.

The concepts of tangents, partial derivatives, and differentials that follow on the next page (Samuelson [58, p. 95]) are applicable only if the utility scales in question are differentiable in which case they cannot be ordinal. Additional analysis of the rest of Samuelson's explanation is not necessary, except that it should be noted that the *marginal utilities* that appear in Eq. 25 that follows on [58, p. 98] are partial derivatives of a utility function. If the derivatives of this utility function, i.e. the marginal utilities, exist it cannot be ordinal. Finally, we note that Samuelson's use of preference and utility as synonyms is consistent with the observations in Section 3.9.4 of this chapter.

3.4.5 Counter-Examples

Define an *ordinal utility* function of two variables by $u(x, y) = xy$ if x or y is a rational number, and $u(x, y) = x^3 y$ otherwise. Under the budget constraint $p_1 x + p_2 y = b$ the tangency condition

$$-\frac{\partial y}{\partial x}\bigg|^{u} = \frac{p_1}{p_2}$$

does not hold because (regardless of how the "or" in the definition of $u(x, y)$ is interpreted) the left hand side of this equation is undefined – the derivative does not exist.

More generally, given any finite ordinal system, there exist smooth ordinal utility scales u_1 and u_2 such that

$$\frac{\partial x_i}{\partial x_j}\bigg|_{x}^{u_1} = -\left(\frac{\partial u_1}{\partial x_j} \div \frac{\partial u_1}{\partial x_i}\right) \neq -\left(\frac{\partial u_2}{\partial x_j} \div \frac{\partial u_2}{\partial x_i}\right) = \frac{\partial x_i}{\partial x_j}\bigg|_{x}^{u_2}, \tag{3.6}$$

which means that the marginal substitution rate $\frac{\partial x_i}{\partial x_j}$ is undefined at x. These counter-examples show that ordinal utility scales are not sufficient for the derivation of the standard equilibrium conditions of consumer demand theory. In current economic theory (see, e.g., Chapter 3 in Mas-Colell et al. [46]), the claim that ordinal utility theory is sufficient to establish the existence of the partial derivatives that define marginal substitution rates is based on errors. Ordinal systems do not constitute vector spaces; vector differences and norms are undefined in such systems; and there is no basis for the concepts of convexity, continuity, and differentiation in ordinal systems (see, e.g., Definition 3.B.3 in Mas-Colell et al. [46, p. 42]).

3.5 Shortcomings of Utility Theory

Campbell's argument against the possibility of measurement of psychological variables can be rejected on the basis of von Neumann and Morgenstern's uniqueness argument but constructing utility scales that are immune from Campbell's argument is not equivalent to establishing that psychological variables can be measured. In fact, as we show in Section 3.5.2, the operations of addition and multiplication do not apply to utility scale values. This and additional shortcomings of utility theory render it unsuitable to serve as the foundation for the application of mathematical methods in decision theory or in economic theory.

3.5.1 Von Neumann and Morgenstern's Utility Theory

The fundamental role of preference modelling in game theory was recognized by von Neumann and Morgenstern (see [53, §§3.5–3.6]) but their treatment of this difficult problem which is the basis for latter developments in "modern utility theory" (cf. Fishburn [31, §1.3] and Coombs et al. [25, p. 122]) suffers from multiple flaws and this theory cannot serve as a foundation for any scientific theory.

In essence, von Neumann and Morgenstern study a set of objects A equipped with an operation (i.e. a function) and the relation of order (which is not an operation) that satisfy certain assumptions. The operation is of the form $f(\alpha, a, b)$, where a and b are objects in A, α is a real number, and $c = f(\alpha, a, b)$ is an object in A. Their main result is an *existence* and *uniqueness* theorem for scales (homomorphisms) that reflect the structure of the set A into a set B equipped with order and a corresponding operation $g(\alpha,\ s(a),\ s(b))$ where $a \to s(a),\ b \to s(b)$, and $f(\alpha, a, b) \to g(\alpha, s(a), s(b))$.

3.5.2 Addition and Multiplication Are Not Applicable to Utility Scales

The Principle of Reflection implies that the operations of addition and multiplication are not applicable to utility scales despite their "interval" type. These operations are not applicable to von Neumann and Morgenstern's utility model because their axioms include *one* compound empirical *ternary* operation (i.e. the "center of gravity" operation which is a function of *three* variables) instead of the *two binary* operations of addition and multiplication (each of which is a function of *two* variables). Addition and multiplication are not enabled on utility scale values in latter formulations as well because none of these formulations is based on two empirical operations that correspond to addition and multiplication. It should be noted that the goal of constructing the utility framework was to enable the application of mathematical operations rather than to build a system with a certain type of uniqueness.

Although modern utility models (e.g., Luce and Raiffa [45, §2.5], Fishburn [31, pp. 7–9], Coombs et al. [25, pp. 122–129], French [32, Ch. 5]) are not equivalent to von Neumann and Morgenstern's model, *The Principle of Reflection* implies that all utility models are weak: despite the fact that they produce "interval" scales, none of these models enables the operations of addition and multiplication.

3.5.3 Barzilai's Paradox: Utility's Intrinsic Contradiction

As an abstract mathematical system, von Neumann and Morgenstern's utility axioms are consistent. However, while von Neumann and Morgenstern establish the *existence* and *uniqueness* of scales that satisfy these axioms, they do not address utility scale *construction*. This construction requires a specific interpretation of the empirical operation in the context of preference measurement (in terms of lotteries) and although the axioms are consistent in the abstract, *the interpretation of the empirical utility operation creates an intrinsic contradiction*. Utility theory constrains the values of utility scales for lotteries while the values of utility scales for prizes are unconstrained. The theory permits lotteries that are prizes (cf. Luce and Raiffa's "neat example" [45, pp. 26–27]) and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained. In other words, utility theory has one rule for assigning values to prizes and a different, conflicting, rule for assigning values to lotteries. For a prize which is a lottery ticket, the conflicting rules are contradictory. For a numerical example see Barzilai [11] or [14].

3.5.4 Utility Theory Is Neither Prescriptive Nor Normative

Coombs et al. [25, p. 123]) state that "utility theory was developed as a prescriptive theory." This claim has no basis since von Neumann and Morgenstern's utility theory as well as its later variants (e.g., Luce and Raiffa [45, §2.5], Fishburn [31, pp. 7–9], Coombs et al. [25, pp. 122–129], French [32, Ch. 5], Luce [43, p. 195]) are mathematical theories. These theories are of the form $P \to Q$, that is, if the assumptions P hold then the conclusions Q follow. In other words, these theories are not of the form "*Thou Shalt Assume P*" but rather "*if* you assume P." Since mathematical theories do not dictate to decision makers what sets of assumptions they *should* satisfy, the claim that utility theory is prescriptive has no basis in mathematical logic nor in modern utility theory.

Howard says that a normative theory establishes norms for how things should be (*In Praise of the Old Time Religion* [38, p. 29]) and appears to suggest that decision theory says how you should act in compliance with von Neumann and Morgenstern's assumptions [53, p. 31]. His comments on "second-rate thinking" and education [38, p. 30] seem to indicate that he believes that those who do not share his praise for the old time utility religion need to be re-educated. In the context of logic and science this position is untenable – mathematical theories do not dictate assumptions to decision makers. Furthermore, educating decision makers to follow flawed theories is not a remedy for "second-rate thinking." Flawed theories should be corrected rather than be taught as the norm.

Unfortunately, according to Edwards [28, pp. 254–255], Howard is not alone. Edwards reports as editor of the proceedings of a conference on utility theories that the attendees of the conference unanimously agreed that the experimental and observational evidence has established as a fact the assertion that people do not maximize "subjective expected utility" and the attendees also unanimously stated that they consider "subjective expected utility" to be the appropriate normative rule for decision making under risk or uncertainty. These utility theorists are saying that although decision makers reject the assumptions of the *mathematical theory* of utility, they should accept the conclusions which these assumptions imply. This position is logically untenable.

3.5.5 Von Neumann and Morgenstern's Structure Is Not Operational

The construction of utility functions requires the interpretation of the operation $f(\alpha, a_1, a_0)$ as the construction of a lottery on the prizes a_1, a_0 with probabilities $\alpha, 1-\alpha$, respectively. The utility of a prize a is then assigned the value α where $u(a_1) = 1$, $u(a_0) = 0$ and $a = f(\alpha, a_1, a_0)$.

In order for $f(\alpha, a_1, a_0)$ to be an operation, it must be a single-valued function. Presumably with this in mind, von Neumann and Morgenstern interpret the relation of equality on elements of the set A as *true identity*: in [53, A.1.1–A.1.2, p. 617] they remark in the hope of "dispelling possible misunderstanding" that "[w]e do not axiomatize the relation =, but interpret it as *true identity*." If equality is interpreted as true identity, equality of the form $a = f(\alpha, a_1, a_0)$ cannot hold when a is a prize since a lottery and a prize are not identical objects. Consequently, von Neumann and Morgenstern's interpretation of their axioms does not enable the practical construction of utility functions.

Possibly for this reason, later variants of utility theory (e.g., Luce and Raiffa [45]) interpret equality as indifference rather than true identity. This interpretation requires the extension of the set A to contain the lotteries in addition to the prizes. In this model, lotteries are elements of the set A rather than an operation on A so that this extended set is no longer equipped with any operations but rather with the relations of order and indifference (see, e.g., Coombs et al. [25, p. 122]). This utility structure is not homomorphic (and therefore is not equivalent) to von Neumann and Morgenstern's structure and the utility functions it generates are weak (i.e. do not enable the operations of addition and multiplication) and only enable the relation of order despite their "interval" type of uniqueness.

3.6 Shortcomings of Game Theory

As a branch of decision theory, game theory is an operations research discipline that was founded by von Neumann and Morgenstern [53] with the aim of serving as the proper instrument with which to develop a theory of economic behavior. Unfortunately, game theory is founded on multiple errors and while its utility foundations can be replaced with proper ones, other fundamental game theory errors must be corrected if it is to serve as the mathematical foundation of economic theory (see Barzilai [12–14]). In particular, preference measurement plays a fundamental role, and is necessary in order to introduce the real numbers and operations on them, in game theory and economics and it is not possible to escape the need to construct preference functions by assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money. The mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, preference scales are not unique, and preference spaces are not vector spaces. See Barzilai [15–18].

3.6.1 Undefined Sums

The expression $v(S) + v(T)$ which represents the sum of coalition values in von Neumann and Morgenstern's definition of the characteristic function of a game has no basis since, by *The Principle of Reflection*, addition is undefined for utility or value scales. The sum of points on a straight line in an affine geometry, which is the correct model for preference measurement (see Section 3.7.1), is undefined as well. For the same reasons, the sum of imputations, which are utilities, is undefined. In consequence, throughout the literature of game theory, the treatment of the topic of the division of the "payoff" among the players in a coalition has no foundation.

3.6.2 The Utility of a Coalition

The definition of the characteristic function of a game depends on a reduction to "the value" of a two-person (a coalition vs. its complement) game. In turn, the construction of a two-person-game value depends on the concept of expected utility of a player. The reduction treats a coalition, i.e. a group of players, as a single player but there is no foundation in the theory for *the utility of a group of players*.

3.6.3 "The" Value of a Two-Person Zero-Sum Game Is Ill-Defined

To construct von Neumann and Morgenstern's characteristic function, a coalition and its complement are treated as players in a two-person zero-sum game, and the coalition is assigned its "single player" value in this reduced game. However, the concept of "the" value of two-person zero-sum game theory is not unique and consequently is ill-defined.

The minimax theorem which states that every two-person zero-sum game with finitely many pure strategies has optimal mixed strategies is a cornerstone of game theory. Given a two-person zero-sum game, denote by x^* and y^* the minimax optimal strategies and by u the utility function of player 1. Utility functions are not unique and for any p and positive q, u is equivalent to $p + q \times u$ but since the minimax optimal strategies do not depend on the choice of p and q, x^* and y^* are well defined. However, the value of the game varies when p and q vary so that it depends on the choice of the utility function u and given an arbitrary real number v, the numbers p and q can be chosen so that the value of the game equals v. As a result, the concept of "the" value of a game is ill-defined and any game theoretic concept that depends on "the" value of a game is ill-defined as well.

3.6.4 The Characteristic Function of Game Theory is Ill-Defined

The *construction* of the characteristic function of a game is ignored in the literature where it is assumed that a characteristic function is "given" and conclusions are drawn from its numerical values. This is not surprising since without specifying whose values are being measured the characteristic function of a game cannot be constructed.

The assignment of values to objects such as outcomes and coalitions, i.e. the construction of value functions, is a fundamental concept of game theory. *Value* (or utility, or preference) is not a physical property of the objects being valued, that is, *value* is a subjective (or psychological, or personal) property. Therefore, the definition of *value* requires specifying both *what* is being valued and *whose* values are being measured.

Game theory's characteristic function assigns values to coalitions but von Neumann and Morgenstern do not specify *whose* values are being measured in the construction of this function. Since it is not possible to construct a value (or utility) scale of an unspecified person or a group of persons, game theory's characteristic function is not well defined. All game theory concepts that depend on values where it is not specified whose values are being measured are ill-defined (see also Barzilai [21]). This includes the concept of imputations, von Neumann and Morgenstern's solution of a game, and Shapley's value [33, 60] and [5, Chapter 3] in all its variants and generalizations (e.g., McLean [47], Monderer and Samet [52], and Winter [63]). Moreover, since the current definition of an n-person game employs the ill-defined concept of the characteristic function (see, e.g., Monderer and Samet [52, p. 2058]), the very definition of a game has no foundation.

3.6.5 The Essential Role of Preference

Under the heading "The Mathematical Method in Economics" von Neumann and Morgenstern state in *Theory of Games and Economic Behavior* [53, §1.1.1] that the purpose of the book was "to present a discussion of some fundamental questions of economic theory." *The role of preference measurement in game theory is essential* because the outcomes of economic activity are empirical objects rather than real numbers such as $\sqrt{\pi}$ and the application of mathematical operations such as addition and multiplication requires the mathematical modelling of economic systems by corresponding mathematical systems. In other words, *the purpose of preference measurement is to introduce the real numbers and operations on them in order to enable the application of The Mathematical Method*.

Consider Guild's statement in support of the position that mathematical operations are not applicable to non-physical variables (his position as well as the opposing position were based on incorrect arguments concerning the applicability of mathematical operations to non-physical variables – see Section 3.7.1) as summarized in [30, p. 345] in the context of measurement of *sensation*:

> I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of addition as applied to sensation. No such meaning has ever been defined. When we say that one length is twice another or one mass is twice another we know what is meant: we know that certain practical operations have been defined for the addition of lengths or masses, and it is in terms of these operations, and in no other terms whatever, that we are able to interpret a numerical relation between lengths and masses. But if we say that one sensation intensity is twice another nobody knows what the statement, if true, would imply.

Note that the *property* (length, mass, etc.) of the objects must be specified in order for the mathematical operations to be applicable and that addition and multiplication are applied on lengths and masses of objects. It is not possible to "add objects" without knowing whether what is being added is their mass, length, temperature, etc. Observing that *preference* is the only property of relevance in the context of the mathematical foundations of game theory, we conclude that preference measurement is not a cosmetic issue but a fundamental one in this context.

3.6.6 Implications

The fact that *preference modelling is of the essence* in game theory implies that much of the theory is in error. Under the title "What is game theory trying to accomplish?" Aumann [3] says that game theory is not a branch of abstract mathematics but is rather motivated by and related to the world around us. As pointed out above, economic transactions are not performed in order to attain as an outcome the number $\sqrt{\pi}$. Stated differently, the outcome of a real-world economic transaction is seldom a real number. One therefore cannot simply "assume" (see, e.g., Definition 2.3 in Aumann [5]) that the outcome of an economic transaction which is modelled as a play of a game is a numerical payoff function. The only way to introduce the real numbers, and thereby *The Mathematical Method*, into game theory is through the construction of preference functions which represent preference for empirical objects including outcomes of games. As we shall see in Section 3.6.7, it is not possible to escape the need to construct preference functions by "assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [5, p. 106]). Note that this statement implies that utility is a property of money so that in the fundamental structure of preference modelling (see Section 3.2), money, in the form of a $20 bill, or 20 coconuts, cocoa beans, dried fish, salt bars, or a beaver pelt (cf. Shubik [61, p. 361]), is an object rather than a property of empirical objects. In the context of mathematical modelling the distinction between objects and properties of objects is fundamental. (In addition to these considerations, the mathematical operations of game theory must be performed on the preferences of the players because what matters to them is their preferences for the outcomes rather than the physical outcomes.)

Having concluded that the mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, recall that (1) preference functions are not unique (they are unique up to affine transformations) and (2) the sum of values of a preference function is undefined (see Section 3.7.1).

3.6.7 On "Utility Functions That Are Linear in Money"

Consider again the assumption that "payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [5, p. 106]). In addition to the obvious reasons for rejecting this assumption (e.g., the St. Petersburg Paradox which implies that this is an unrealistic assumption; it is also necessary to make the even more unrealistic assumption that the additive and multiplicative constants in the players' utility scales are all identical) we re-emphasize that money is not a property of objects and preference functions are unique up to affine rather than linear transformations. This implies that in the case of monetary outcomes it is still necessary to construct the decision maker's preference function for money.

It is correct to say that a given decision maker (who must be identified since preference is a subjective property) is indifferent between the objects A and B where B is a sum of money, which means that $f(A) = f(B)$ where f is the decision maker's preference function. However, the statement that the outcome of a play is the object A and $f(A) = f(B)$ requires the determination of the preference value $f(A)$ and, *in addition*, $f(B)$ as well as the identification of the object B for which $f(A) = f(B)$. It follows that this indirect and more laborious procedure does not eliminate the need to construct the decision maker's preference function and *game theory cannot be divorced from preference modelling*. It follows that there is no escape from the fact that utility sums are undefined.

3.6.8 The Minimax Solution of Two-Person Zero-Sum Games

In [22, 23] we give examples that show that even for repeated games, the minimax solution of two-person zero-sum game theory prescribes to the players "optimal" strategies that cannot be described as conservative or rational. In addition, since the minimax probabilities do not depend on the outcomes of the game (they only depend on the numerical payoffs which are associated with the outcomes), they are invariant with respect to a change of payoff unit. For example, denote the outcomes of a two-person zero-sum game by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.7}$$

where player 1 can choose between $R1$ and $R2$ (rows) and player 2 between $C1$ and $C2$ (columns) and consider the case where player 1's utility values for these outcomes are given by

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}. \tag{3.8}$$

According to the minimax rule, player 1 is to play $R1$ or $R2$ with probabilities (0.75, 0.25) regardless of whether the numbers in this game represent the payoffs in cents, dollars, euros, millions of dollars, billions of dollars, or any other unit. Probabilities cannot be assigned to scale values on an indefinite scale; when the scale is changed, the probabilities assigned to scale values must change. Denote by x the probability that the temperature in a given process will reach 20°. Then x must depend on the choice of the temperature scale – it cannot be the case that the temperature will reach 20° on the Celsius scale, the Fahrenheit scale, and any arbitrary other scale with the same probability. In the minimax solution of two-person zero-sum games choice probabilities are divorced from choice consequences because probabilities are assigned to indefinite scale values. This is a fundamental error which indicates that this problem is formulated incorrectly.

The following should be noted: Aumann tells us in the General Summary and Conclusions of his 1985 paper entitled "What Is Game Theory Trying to Accomplish?" [3, p. 65] that "Game-theoretic solution concepts should be understood in terms of their applications, and should be judged by the quantity and quality of their applications." More recently, in their paper entitled "When All Is Said and Done, How Should You Play and What Should You Expect?" Aumann and Dreze [6, p. 2] tell us that 77 years after it was born in 1928, strategic game theory has not gotten beyond the optimal strategies which rational players should play according to von Neumann's minimax theorem of two-person zero-sum games; that when the game is not two-person zero-sum none of the equilibrium theories tell the players how to play; and that the "Harsanyi-Selten selection theory does choose a unique equilibrium, composed of a well-defined strategy for each player and having a well-defined expected outcome. But nobody – least of all Harsanyi and Selten themselves – would actually recommend using these strategies."

This implies that while the meaning of n-person "solutions" is in question, game theorists universally accept the minimax strategy as a reasonable (in fact, *the only*) solution for rational players in two-person zero-sum games. Consistent with this view is Aumann's characterization of the minimax theorem as a vital cornerstone of game theory in his survey of game theory [4, p. 6], yet this solution, too, is a flawed game theory concept.

3.6.9 Errors Not Corrected

It has been suggested that the errors uncovered here have been corrected in recent times, but this is not the case. In the Preface to his 1989 *Lectures on Game Theory* [5], Aumann states that its material has not been superseded. This material includes a discussion of game theory without its preference underpinning, the use of undefined sums, ill-defined concepts, and additional errors.

For example, the "payoff functions" h^i in part (3) of Definition 2.3 in Aumann [5] are not unique and there is no basis for assuming that the outcomes of games are real numbers. Moreover, these functions are unique up to additive and multiplicative constants which are not independent of the index i. As a result, the very definition of a game has no basis even in the simplest two-person case. In the absence of the property of preference, no operations are applicable in game theory but when preference is modelled the sum of values of a preference function is undefined. Such sums appear in Aumann [5] (Definition 3.9 p. 28, Definitions 4.3 and 4.6, p. 38) and throughout game theory's literature.

While Aumann's discussion of Shapley's value ignores utility theory altogether, Hart introduces his 1989 *Shapley Value* [33] as an evaluation, "in the [sic] participant's utility scale," of the prospective outcomes. He then refers explicitly to utility theory and to measuring the value of each player in the game. Note that in addition to the use of undefined sums and ill-defined concepts in the context of Shapley's value, it is not clear whether Shapley's value is intended to represent the evaluation of prospective outcomes of a game by a player or the evaluation of the players themselves (not surprisingly, the question who evaluates the players is not addressed in the literature).

More recently Hart (2004, [34, pp. 36–37]), denoting by x^i the utility of an outcome to player i, refers to the set of utilities as the set of *feasible payoff vectors* and uses the undefined sum of these utilities $\Sigma_{i \in S} x^i$ in the definition of a "transferable utility" game. As pointed out earlier, utility spaces are not vector spaces and utility sums are undefined.

3.7 Reconstructing the Foundations

3.7.1 Proper Scales – Straight Lines

In order to enable the "powerful weapon of mathematical analysis" to be applied to any scientific discipline it is necessary, at a minimum, to construct models that enable the operations of addition and multiplication, for without these operations the tools of linear algebra and elementary statistics cannot be applied. This construction, which leads to the well-known geometrical model of points on a straight line, is based on two observations:

- If the operations of addition and multiplication are to be enabled in the mathematical system M, these operations must be defined in M. The empirical system E must then be equipped with corresponding operations in order for M to be a model of E.
- Mathematical systems with an absolute *zero* or *one* are not homogeneous: these special, distinguishable, elements are unlike others. On the other hand, since the existence of an absolute *zero* or *one* for empirical systems that characterize subjective properties has not been established, they must be modelled by homogeneous mathematical systems.

Sets that are equipped with the operations of addition and multiplication, including the inverse operations of subtraction and division, are studied in abstract algebra and are called *fields*. The axioms that define fields are listed in Section 3.7.3. A field is a not a homogeneous system since it contains two special elements, namely an absolute *zero* and an absolute *one* which are the additive and multiplicative identities of the field (in technical terms, they are invariant under field automorphisms). It follows that to model a subjective property by a mathematical system where the operations of addition and multiplication are defined we need to modify a field in order to homogenize its special elements, i.e. we need to construct a *homogeneous field*. To homogenize the multiplicative identity, we construct a one-dimensional vector space which is a *partially homogeneous field* (it is homogeneous with respect to the multiplicative identity but not with respect to the additive identity) where the elements of the field serve as the set of scalars in the vector space. To homogenize the additive identity as well, we combine points with the vectors and scalars and construct a one-dimensional affine space, which is a homogeneous field, over the previously constructed vector space. The axioms characterizing vector and affine spaces are listed in Section 3.7.3. The end result of this construction, the one-dimensional affine space, is the algebraic formulation of the familiar straight line of elementary (affine) geometry so that for the operations of addition and multiplication to be enabled on models that characterize subjective properties, the empirical objects must correspond to points on a straight line of an affine geometry. For details see Section 3.7.3, or the equivalent formulations in [2, p. 79], and [55, pp. 46–47].

In an affine space, the difference of two points is a vector and no other operations are defined on points. In particular, it is important to note that the ratio of two points as well as the sum of two points are undefined. The operation of addition is defined on *point differences*, which are vectors, and this operation satisfies the *group* axioms listed in Section 3.7.3. Multiplication of a vector by a scalar is defined and the result is a vector. In the one-dimensional case, and only in this case, the ratio of a vector divided by another non-zero vector is a scalar.

It follows that Campbell's argument is correct with respect to the application of *The Principle of Reflection* and the identification of addition as a fundamental operation, but that argument does not take into account the role of the multiplication operation and the modified forms of addition and multiplication when the models correctly account for the degree of homogeneity of the relevant systems. Note also that it is not sufficient to model the operation of addition since, except for the natural numbers, multiplication is not repeated addition: In general, and in particular for the real numbers, multiplication is not defined as repeated addition but through field axioms.

Since the purpose of modelling is to enable the application of mathematical operations, we classify scales by the type of mathematical operations that they enable. We use the terms *proper scales* to denote scales where the operations of addition and multiplication are enabled on scale values, and *weak scales* to denote scales where these operations are not enabled. This partition is of fundamental importance and we note that it follows from *The Principle of Reflection* that all the models in the literature are weak because they are based on operations that do not correspond to addition and multiplication.

3.7.2 Strong Scales – the Real Numbers

Proper scales enable the application of the operations of linear algebra but are not necessarily equipped with the relation of order which is needed to indicate a direction on the straight line (e.g., to indicate that an object is more preferable, or heavier, or more beautiful than another). To construct proper ordered scales the underlying field must be ordered (e.g., the field of complex numbers is unordered while the field of the rational numbers is ordered). For a formal definition of an ordered field see Section 3.7.3.1.

Physics, as well as other sciences, cannot be developed without the mathematical "weapons" of calculus. For example, the basic concept of acceleration in Newton's Second Law is defined as a (second) derivative; in statistics, the standard deviation requires the use of the square root function whose definition requires the limit operation; and marginal rates of change, defined by partial derivatives, are used in economics. If calculus is to be enabled on ordered proper scales, the underlying field must be an ordered field where any limit of elements of the field is itself an element of the field. In technical terms, the underlying field must be *complete* (see McShane and Botts [48, Ch. 1, §5] for a formal definition). Since the only ordered complete field is the field of real numbers, in order to enable the operations of addition and multiplication, the relation of order, and the application of calculus on subjective scales, the objects must be mapped into the real, ordered, homogeneous field, i.e. a one-dimensional, real, ordered, affine space, and the set of objects must be a subset of points on an empirical ordered real straight line. We use the term *strong models* to denote such models and *strong scales* to denote scales produced by strong models.

The application of the powerful weapon of mathematical analysis requires a system in which addition and multiplication, order, and limits are enabled. The reason for the central role played by the real numbers in science is that the field of real numbers is the only ordered complete field.

3.7.3 *The Axioms of an Affine Straight Line*

3.7.3.1 Groups and Fields

A group is a set G with an operation that satisfies the following requirements (i.e. axioms or assumptions):

- The operation is *closed*: the result of applying the operation to any two elements a and b in G is another element c in G. We use the notation $c = a \circ b$ and since the operation is applicable to pairs of elements of G, it is said to be a binary operation.
- The operation is *associative*: $(a \circ b) \circ c = a \circ (b \circ c)$ for any elements in G.
- The group has an *identity*: there is an element e of G such that $a \circ e = a$ for any element a in G.
- *Inverse elements*: for any element a in G, the equation $a \circ x = e$ has a unique solution x in G. If $a \circ x = e,\ x$ is called the inverse of a.

If $a \circ b = b \circ a$ for all elements of a group, the group is called *commutative*. We re-emphasize that a group is an algebraic structure with *one* operation and we also note that a group is not a homogeneous structure because it contains an element, namely its identity, which is unlike any other element of the group since the identity of a group G is the only element of the group that satisfies $a \circ e = a$ for all a in G.

A *field* is a set F with two operations that satisfy the following assumptions:

- The set F with one of the operations is a commutative group. This operation is called *addition* and the identity of the additive group is called zero (denoted '0').
- The set of all non-zero elements of F is a commutative group under the other operation on F. That operation is called *multiplication* and the multiplicative identity is called one (denoted '1').
- For any element a of the field, $a \times 0 = 0$.
- For any elements of the field the *distributive* law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

Two operations are called addition and multiplication only if they are related to one another by satisfying the requirements of a field; a single operation on a set is not termed addition nor multiplication. The additive inverse of the element a is denoted $-a$, and the multiplicative inverse of a non-zero element is denoted a^{-1} or $1/a$. Subtraction and division are defined by $a - b = a + (-b)$ and $a/b = a \times b^{-1}$.

A field F is ordered if it contains a subset P such that if $a, b \in P$, then $a+b \in P$ and $a \times b \in P$, and for any $a \in F$ exactly one of $a = 0$, or $a \in P$, or $-a \in P$ holds.

3.7.3.2 Vector and Affine Spaces

A vector space is a pair of sets (V, F) together with associated operations as follows. The elements of F are termed *scalars* and F is a field. The elements of V are termed *vectors* and V is a commutative group under an operation termed vector addition.

These sets and operations are connected by the additional requirement that for any scalars $j, k \in F$ and vectors $u, v \in V$ the scalar product $k \cdot v \in V$ is defined and satisfies, in the usual notation, $(j+k)v = jv + kv$, $k(u+v) = ku + kv$, $(jk)v = j(kv)$ and $1 \cdot v = v$.

An *affine space* (or a *point space*) is a triplet of sets (P, V, F) together with associated operations as follows (see also Artzy [2] or Postnikov [55]). The pair (V, F) is a vector space. The elements of P are termed *points* and two functions are defined on points: a one-to-one and onto function $h : P \to V$ and a "difference" function $\Delta : P^2 \to V$ that is defined by $\Delta(a, b) = h(a) - h(b)$. Note that this difference mapping is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \to V$, the sets of points and vectors are equipped with different operations and the operations of addition and multiplication are not defined on points. If $\Delta(a, b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that a point space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences*. Note that in an affine space no point is distinguishable from any other.

The dimension of the affine space (P, V, F) is the dimension of the vector space V. By a homogeneous field we mean a *one-dimensional* affine space. A homogeneous field is therefore an affine space (P, V, F) such that for any pair of vectors $u, v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$. In a homogeneous field (P, V, F) the set P is termed a *straight line* and the vectors and points are said to be collinear. Division in a homogeneous field is defined as follows. For $u, v \in V$, $u/v = \alpha$ means that $u = \alpha v$ provided that $v \neq 0$. Therefore, in an affine space, the expression $\Delta(a, b)/\Delta(c, d)$ for the points $a, b, c, d \in P$ where $\Delta(c, d) \neq 0$ is defined and is a scalar:

$$\frac{\Delta(a, b)}{\Delta(c, d)} \in F \tag{3.9}$$

if and only if the space is one-dimensional, i.e. it is a straight line or a homogeneous field. When the space is a straight line, $\Delta(a, b)/\Delta(c, d) = \alpha$ (where $a, b, c, d \in P$) means by definition that $\Delta(a, b) = \alpha\Delta(c, d)$.

3.8 Measurement Theory

Beginning with Stevens [62] in 1946, measurement theory (which only deals with the *mathematical modelling* of measurement) has centered on issues of scale uniqueness rather than applicability of operations. As a result of the shift of focus from applicability of operations to uniqueness, the operations of addition and multiplication are not applicable on scale values for any scale constructed on the basis of this theory regardless of their "scale type" including "ratio" scales and "interval" scales (see Section 3.2.2 and Barzilai [13]).

The focus of this theory was further narrowed when Scott and Suppes [59] in 1958 adopted a system with a single set of objects as the foundation of the theory. Vector and affine spaces cannot be modelled by such systems because the construction of vector and affine spaces requires two or three sets, respectively (the sets of scalars, vectors, and points). The operations on points, vectors, and scalars are not closed operations: the difference of two points in an affine space is a vector rather than a point and, in a one-dimensional space, the ratio of two vectors is a scalar rather than a vector. Because proper scales for psychological variables are affine scales that are based on three sets, the operations of addition and multiplication are not enabled on scales constructed on the basis of classical measurement theory for any psychological variable for in this theory no model is based on three sets. In particular, this is the case for *preference* which is the fundamental variable of decision theory. In consequence, the mathematical foundations of decision theory must be replaced in order to enable the application of mathematical operations including addition and multiplication.

The mathematical models in *Foundations of Measurement* (Krantz et al. [41] and Luce et al. [44]) and Roberts [56] are incorrect even for the most elementary variable of physics – *position* of points on an affine straight line. Derived from the model of *position*, the correct model for *length* of segments (position differences) on this line is a one-dimensional vector space. Likewise, "extensive measurement" (see, e.g., Roberts [56, §3.2]) is not the correct model for the measurement of *mass*, another elementary physical variable. In essence, "extensive measurement" is the "vector half" of a one-dimensional vector space where multiplication and the scalars are lost. Not surprisingly, the second half of a one-dimensional affine space is then lost in the classical theory's "difference measurement" where the scalars and vectors are both lost together with vector addition and scalar multiplication (see Roberts [56, §3.2–3.3]). In his 1992 paper [42, p. 80], Luce acknowledges the inadequacy of the models of the classical theory: "Everybody involved in this research has been aware all along that the class of homogeneous structures fails to include a number of scientifically important examples of classical physical measurement and, quite possibly, some that are relevant to the behavioral sciences." But despite the evidence of inadequacy, these models have not been corrected in the classical theory.

In summary, the fundamental problem of applicability of mathematical operations to scale values has received no attention in the classical theory of measurement after 1944; the theory does not provide the tools and insight necessary for identifying shortcomings and errors of evaluation and decision methodologies including utility theory and the Analytic Hierarchy Process; the basic model of Scott and Suppes [59] is flawed; and the operations of addition and multiplication are not applicable to scale values produced by any measurement theory model.

3.9 Classical Decision Theory

3.9.1 Utility Theory

Barzilai's paradox (see Section 3.5.3, [14, §6.4.2] and [11, §4.2]) and the inapplicability of addition and multiplication on utility scale values imply that utility theory cannot serve as a foundation for any scientific discipline. In addition, von Neumann and Morgenstern's utility theory was not developed as, and is not, a prescriptive theory neither is it a normative theory (see [14, §6.4.3]). Moreover, the interpretation by von Neumann and Morgenstern of utility equality as a true identity precludes the possibility of indifference between a prize and a lottery which is utilized in the construction of utility scales while under the interpretation of utility equality as indifference the construction of lotteries is not single-valued and is therefore not an operation (see [14, §6.4.4]).

In the context of decision theory, despite the evidence to the contrary (e.g., Barzilai [14, §6.4.3] and [11]), utility theory is still treated by some as the foundation of decision theory and is considered a normative theory. Howard in particular refers to utility theory in the non-scientific term "The Old Time Religion" [38] while elsewhere he refers to "Heathens, Heretics, and Cults: The Religious Spectrum of Decision Aiding" [37]. A recent publication entitled "Advances in Decision Analysis" [29] does not contribute to correcting these errors.

3.9.2 Undefined Ratios and Pairwise Comparisons

In order for the operations of addition and multiplication to be applicable, the mathematical system M must be (1) a field if it is a model of a system with an absolute *zero* and *one*, (2) a one-dimensional vector space when the empirical system has an absolute *zero* but not an absolute *one*, or (3) a one-dimensional affine space which is the case for all non-physical properties with neither an absolute *zero* nor absolute *one*. This implies that for proper scales, scale ratios are undefined for subjective variables including *preference*. In particular, this invalidates all decision methodologies that apply the operations of addition and multiplication to scale values and are based on preference ratios. For example, in the absence of an absolute zero for *time*, it must be modelled as a homogeneous variable and the ratio of two times (the expression t_1/t_2) is undefined. For the same reason, the ratio of two potential energies e_1/e_2 is undefined while the *ratios of the differences* $\Delta t_1/\Delta t_2$ and $\Delta e_1/\Delta e_2$ are properly defined. We saw that the sum of von Neumann and Morgenstern's utility scale values is undefined. Since the sum of two points in an affine space is undefined, the sum of proper preference scale values is undefined as well.

The expression $(a - b)/(c - d) = k$ where a, b, c, d are points on an affine straight line and k is a scalar is used in the construction of proper scales. The number of points in the left hand side of this expression can be reduced from four to three (e.g., if $b = d$) but it cannot be reduced to two and this implies that pairwise comparisons cannot be used to construct preference scales where the operations of addition and multiplication are enabled.

3.9.3 The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP, see Saaty [57]) is not a valid methodology. More than 30 years after the publication of Miller's work in the 1960s [49–51], there is still no acknowledgement in the AHP literature (or elsewhere) of his contribution to decision theory in general and the AHP in particular. Miller was not a mathematician and his methodology is based on mathematical errors although some of its non-mathematical elements are valuable. The AHP is based on these mathematical errors and additional ones (see Barzilai [7–10, 19] and the references there).

Not surprisingly, these errors have been mis-identified in the literature and some of these errors appear in decision theory. For example, Kirkwood [40, p. 53] relies on Dyer and Sarin which repeats the common error that the coefficients of a linear value function correspond to relative importance [27, p. 820]. Furthermore, "difference measurement" which is the topic of Dyer and Sarin is not the correct model of preference measurement. More specifically, in his *Remarks on the Analytic Hierarchy Process* [26, p. 250] Dyer's major focus is in Section 3 where he argues that the AHP "generates rank orderings that are not meaningful" and states that "[a] symptom of this deficiency is the phenomenon of rank reversal" but his argument is circular since the only AHP deficiency presented in Section 3 of his paper is rank reversal. Moreover, the AHP suffers from multiple methodological flaws that cannot be corrected by "its synthesis with the concepts of multiattribute utility theory" (which suffers from its own flaws) as stated by Dyer [26, p. 249].

The AHP is a method for constructing preference scales and, as is the case for other methodologies, the operations of addition and multiplication are not applicable on AHP scale values. The applicability of addition and multiplication must be established before these operations are used to compute AHP eigenvectors and, as we saw in Section 3.2.2, the fact that eigenvectors are unique up to a multiplicative constant does not imply the applicability of addition and multiplication.

In order for addition and multiplication to be applicable on preference scale values, the alternatives must correspond to points on a straight line in an affine geometry (see Section 3.7.1 or Barzilai [11, 12]). Since the ratio of points on an affine straight line is undefined, preference ratios, which are the building blocks of AHP scales, are undefined. In addition, pairwise comparisons cannot be used to construct affine straight lines.

The fundamental mathematical error of using inapplicable operations to construct AHP scales renders the numbers generated by the AHP meaningless. Other AHP errors include the fact that the coefficients of a linear preference function cannot correspond to weights representing relative importance and therefore cannot be decomposed using Miller's criteria tree; the eigenvector method is not the correct method for constructing preference scales; the assignment of the numbers 1–9 to AHP's "verbal scales" is arbitrary, and there is no foundation for these "verbal scales" (see Barzilai [7–10, 19, 20]).

3.9.4 Value Theory

Scale construction for physical variables requires the specification of the empirical objects and the property under measurement. For example, if the property under measurement is *temperature*, the construction results in a *temperature* scale and, clearly, the measurement of *length* does not produce a *mass* scale. In the case of subjective measurement too, the property under measurement must be explicitly specified. If the property under measurement is *preference*, the resulting scales are *preference* scales. Noting that von Neumann and Morgenstern's measurement of preference [53, §3.1] results in utility scales, we conclude that *preference* and *utility* (and, for the same reason, *value, worth, ophelimity*, etc.) are synonyms for the same underlying subjective property. It follows that the distinction between utility theory and value theory has no foundation in logic and science. For example, Keeney and Raiffa's notion of "the utility of value" of an object ($u[v(x)]$, in [39, p. 221]) is as meaningless as "the temperature of the temperature of water" or "the length of the length of a pencil" are.

Likewise, although the notions of "strength of preference" (Dyer and Sarin [27]) and "difference measurement" (e.g., Krantz et al. [41], Roberts [56]) are intuitively appealing, these measurement models of *value, utility, priorities*, etc., are based on measurement theory errors as shown above. Similarly, the utility theories in Edwards [28] are founded on errors as well and, although the issues have been known for a few years, the more recent "Advances in Decision Analysis" (Edwards et al. [29]) does not contribute to correcting these methodological errors.

3.9.5 Group Decision Making

The common view in the classical literature concerning group decision making is based on a misinterpretation of the implications of Arrow's Impossibility Theorem [1] which is a negative result. Constructive theories cannot be founded on negative results and, in addition, this theorem deals with ordinal scales which enable the relation of order but do not enable the operations of addition and multiplication. The concepts of trade-off and cancellation are not applicable to ordinal scales – see Barzilai [14, §6.5] for details.

3.10 Summary

Classical decision and measurement theories are founded on errors which have been propagated throughout the literature and have led to a large number of methodologies that are based on flawed mathematical foundations and produce meaningless numbers. The fundamental issue of applicability of the operations of addition and multiplication to scale values was not resolved by von Neumann and Morgenstern's utility theory and the literature of classical decision and measurement theory offers no insight into this and other fundamental problems. Decision theory is not a prescriptive theory and decision analysis will not be a sound methodology until these errors are corrected.

We identified the conditions that must be satisfied in order to enable the application of linear algebra and calculus, and established that there is only one model for strong measurement of subjective variables. The mathematical foundations of the social sciences need to be corrected to account for these conditions. In particular, foundational errors in utility theory, game theory, mathematical economics, decision theory, measurement theory, and mathematical psychology need to be corrected. It is hoped that the leaders of INFORMS and its Decision Analysis Society, who have been aware of these errors for the last few years, will act to bring these errors to the attention of their followers and correct the educational literature.

This chapter includes the results of very recent research. The development of the theory, methodology, and software tools continues and updates will be posted at www.scientificmetrics.com.

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Chapter 4
Robustness in Multi-criteria Decision Aiding

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Abstract After bringing precisions to the meaning we give to several of the terms used in this chapter (e.g., robustness, result, procedure, method, etc.), we highlight the principal characteristics of most of the publications about robustness. Subsequently, we present several partial responses to the question, "Why is robustness a matter of interest in Multi-Criteria Decision Aiding (MCDA)?" (see Section 4.2). Only then do we provide an outline for this chapter. At this point, we introduce the concept of *variable setting*, which serves to connect what we define as the formal representation of the decision-aiding problem and the real-life decisional context. We then introduce five typical problems that will serve as reference problems in the rest of the chapter. Section 4.3 deals with recent approaches that involve a single robustness criterion completing (but not replacing) a preference system that has been defined previously, independently of the robustness concern. The following section deals with approaches in which the robustness concern is modelled using several criteria. Section 4.5 deals with the approaches in which robustness is considered other than by using one or several criteria to compare the solutions. These approaches generally involve using one or several properties destined to characterize the robust solution or to draw robust conclusions. In the last three sections, in addition to describing the appropriate literature, we suggest some avenues for new development and in some cases, we present some new approaches.

Keywords Robustness · Multi-criteria methods · Decision aiding · MAUT · ELECTRE methods · Mathematical programming

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4.1 Introduction

In the field of Multi-Criteria Decision Aiding (MCDA), the subject of robustness is increasingly present in scientific journals. This subject is also present more and more in much of the less formal works done by companies applying operational research tools about concrete decision-aiding problems. In MCDA, the multiple meanings accorded to the term "robust" are open to debate. This subject is discussed in detail in the Newsletter of the European Working Group "Multiple Criteria Decision Aiding" [25] in the contributions of Aloulou et al. (n° 12, 2005), Dias (n° 13, 2006), Fernandez Barberis (n° 13, 2006), Pictet (n° 15, 2007), Rios Insua (n° 9, 2004), Rosenhead (n° 6, 2003), Roy (n° 6, 2002), Roy (n° 8, 2003), Sayin (n° 11, 2005), Sevaux, Sörensen (n° 10, 2004) and Vincke (n° 8, 2003). This series of perspectives highlights the polysemic character of the notion of robustness. This polysemic character is primarily due to the fact that, depending on the situation, this notion can be similar to, and sometimes compared to, the notion of flexibility, stability, sensitivity and even equity.

In this chapter, we use the term **robust** as a qualifier meaning **a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts, notably the degradation of the properties that must be maintained** (see Roy Roy [29]). The research on robustness seeks to insure this capacity to the greatest degree possible. Consequently, robustness stems from a process that responds to a concern: a need for resistance or self-protection.

For this reason, we prefer to use the expression **robustness concern**, rather than robustness analysis because the latter can give the impression of work done a posteriori, as is the case with sensitivity analysis, for example. Robustness more often involves a concern that must be taken into account a priori, when formulating the problem. (Of course, this does not exclude the use of sensitivity analysis to respond to such a concern, if necessary.)

In the following section, we will endeavour to explain in detail the multiple reasons for the existence of the robustness concern. Our perspective, like that of this book, is primarily multi-criteria. We will show that robustness and multiple criteria can be expressed in a variety of forms. At this point, we will present the outline of the rest of the chapter. But, before doing so, it is necessary to provide some important explanations and call back to memory some basic notions.

First, let us explain briefly the meaning that we assign to certain terms (see Roy Roy [29] for more details). We designate as **procedure** P a set of instructions used for handling a problem. **Result** is used to refer to the outcome of applying P to a rigorously formulated problem. This result can have diverse forms: solutions, bundles of solutions possessing the required properties, or simple statements (e.g., "there is no solution with this property" or "this solution is non-dominated"). Like Vincke [42, 43], we use **method** M to designate a family of $\hat{P}$ procedures that have enough similar features (i.e., structure, process, concept, action or hypothesis) that they can only be differentiated by the value attributed to certain parameters or by diverse options dealing with the way certain rules are formulated (e.g., the role of the different criteria).

Most publications dealing with robustness use the term "robust" to characterize solutions. This term is also used to qualify a statement (or a conclusion), a method (see Billaut et al. [8], Roy [30, 32], Vincke [43], for example).

Among works targeting the search for robust solutions, many of them have the following characteristics (see Roy [32]):

(i) The problem studied is one of the standard OR models: job shop, flow shop, knapsack, spanning tree, shortest path, travelling salesman, maximum flow, maximum stable, p-median and p-centre in location and/or the standard mathematical programming models, notably linear programming. These problems are studied in a mono-criterion context.
(ii) A **scenario** set is defined by considering the value of some parameters as uncertain. These parameters are either those present in the definition of the optimization criterion, or those that intervene in certain constraints. Such parameters are assumed capable of taking a few or all of the values in one interval. A scenario is defined by attributing one of the possible values to each of these uncertain parameters.
(iii) Feasible solutions that optimize a criterion $r(x)$, used to indicate the relative robustness of solution x, are qualified as robust. Frequently, $r(x)$ is one of the three measures introduced by Kouvelis and Yu [23]. Since we will refer to them in the rest of this chapter, the definitions of these indicators are given below.

These measures are based on the unique optimization criterion v of the standard model considered. This criterion attributes a value $v_s(x)$ to x in scenario s. Here, optimum is assumed to mean maximum.

- Absolute robustness: The robustness measure that must be maximized is defined by the value of the solution in the worst scenario: $r(x) = \min_s\{v_s(x)\}$.
- Absolute deviation: The robustness measure that must be minimized is defined by the value of the absolute regret in the worst scenario, due to the fact that the solution differs from that which would be optimal in this scenario: $r(x) = \max_s \{v_s^* - v_s(x)\}$, where v_s^* is the value of the optimal solution in scenario s.
- Relative deviation: The robustness measure that must be minimized is defined by the value of the relative regret in the worst scenario, due to the fact that the solution is not optimal in this scenario: $r(x) = \max_s \left\{\frac{v_s^* - v_s(x)}{v_s^*}\right\}$.

Let us underline that these measures correspond to the classical and criteria in decision under uncertainty. The complexity and the approximation of the underlying problems are studied in Aissi et al. [1].

4.2 Why Is Robustness of Interest in MCDA?

In our opinion, in decision aiding, the desire to take our own ignorance into account as much as possible explains why the robustness concern exists. From this perspective, it is important to remember that the decisions for which decision aiding is performed will be:

1. executed in a real-life context that may not correspond exactly to the model on which the decision aiding is based; and
2. judged in terms of a system of values that will appear to be pertinent (and not necessarily stable) for a future that may not be well defined; as a result, this system of values may not correspond exactly to the one used to create and exploit the model.

These are two of the possible reasons for a non-perfect conformity, and thus a gap between:

- on the one hand, the **formal representation** (FR), including the model and the processing procedures that are applied to it; and
- on the other hand, **the real-life context** (RLC) in which decisions will be made, executed and judged.

"State of nature" could be used instead of real-life context, but because the latter expression refers to real life, it seems more appropriate in the context of decision aiding than the expression referring to nature.

In decision aiding, it is important to try to take into account the vague approximations and zones of ignorance responsible for the formal representation's non-perfect conformity to the real-life context: FR $\neq$ RLC. In this section, we illustrate these vague approximations and zones of ignorance, though without any pretence of exhaustivity.

In the formal representation, the vague approximations and zones of ignorance against which the robustness concern attempts to protect appear in the form of **frailty points** (Roy [29]). To highlight these frailty points, the formal representation, adopted as the problem formalization, can be examined from four different perspectives:

1. *The way that imperfect knowledge is treated*: imperfect knowledge may be ignored, for example by treating uncertain data as certain, or it may be modelled using elements of arbitrariness, for example using probability distribution, fuzzy numbers or thresholds. In a third possibility, imperfect knowledge may be incorporated in the procedure when the latter has been conceived to take into account imprecise and/or ambiguous data even non-necessarily coherent and complete.
2. *The preferential attribution of questionable, even inappropriate, meaning to certain data*: preferential attributions of meaning can be made by moving from qualitative or numerical analysis to quantitative analysis without justification, or by attributing inappropriate meanings to so-called objective measurements, using data generated through a questioning procedure.
3. *The modelling of complex aspects of reality (notably introduction of parameters), which are difficult to grasp because imperfectly defined*: the choice of model parameters (e.g., sets of weights, capacity indicators, utility functions, reference levels or aspiration levels) has a great influence.
4. *The way that essentially technical parameters and/or selection rules with little or no concrete meaning are introduced*: these parameters are notably those imposed by the processing procedure, for example, the minimum deviation guaranteeing

the strict nature of the inequality, the bounds limiting the domain of investigation or the parameters required by a metaheuristic. These rules can be for instance related to the way the selection of a solution among several ones is conceived (solution in the neighbourhood of an optimum).

Taking a robustness concern into account implies first identifying the frailty points in FR. These points obviously depend on the way that the decision-aiding problem was formulated and modelled. They can also depend on the processing procedures that will be used. In general, these frailty points appear to be connected to sources of contingency, uncertainty or arbitrariness (see Roy [29, 32], Section 2.2). We believe that, used in conjunction with these sources (which are on a higher hierarchical level), the four perspectives described above can help OR researchers confronted with real-world problems to inventory these frailty points.

In many cases, establishing an inventory by concentrating only on the elements in the FR that reflect uncertainty can lead to the exclusion of a certain number of frailty points. In fact, the term "uncertainty" does not cover all the forms of vague approximations and zones of ignorance that must be resisted or protected against. For example, approximations due to simplifications, imperfect determinations, or arbitrary options are not uncertain, nor are zones of ignorance due to imperfect knowledge about the complexity of the phenomena or the systems of values.

Limiting robustness concern to considerations of uncertainty generally accompanies the use of scenario-based concepts for understanding the relationship between the formal representation and the real-life context (see end of Section 4.1, ii). From this somewhat limited viewpoint, the search for robustness is based on defining a finite or infinite set of scenarios. **This set must allow the different real-life contexts that should be considered to be incorporated into the formal representation**: it is the uncertainty with which real values are assigned to certain data or parameters that makes it necessary to consider these different realities. Each scenario is thus defined by attributing a precise value to each of the data elements and parameters.

Roy [28, 29] showed that, in order to respond to the reasons for the existence of robustness concern, it is preferable to go beyond the limited viewpoint described above. To avoid limiting the search for robustness to a simple consideration of uncertainty, the scenario concept must be left behind, especially since this concept has the additional disadvantage of causing confusion in certain professional milieus. Roy proposed replacing this scenario view of robustness with a view centred on a **version** that is strongly connected to the decision-aiding problem formulation. Each version represents a reality that should be considered and is defined using a combination of the options related to the model's frailty points. In some cases, the version set thus defined is not enough to clarify the relationship between FR and RLC, primarily because the robustness concern can make it necessary to take into account all the processing procedures in a certain family, and not just a single one. The frailty points that make it necessary to take such a family into account can be due both to the technical parameters that are part of the procedure definition and to the personality of the experts who are in charge of processing the model (see Roy [29]). It is even possible that the robustness concern relates only to a single version of the problem formulation, to which the entire procedure family must be applied.

This wider view of the robustness concern can make it appropriate to replace the scenario set by a set comprised of all the pertinent pairs (**procedure**, **version**). Such a pair (p, v) is defined by a set of values and options that characterize the procedure p and the version v that are under consideration. In the following, we denote any pertinent pair as $s = (p, v)$ and refer to this pair as a **variable setting**, an expression primarily employed in reliability engineering (see Salazar and Rocco [37] for example).

We denote the set of pertinent variable settings S. When the robustness concern is based on a single procedure, S is simply a set of versions, and in many cases, a set of scenarios. However, when the focus is on the robustness of a method as opposed to a single version v of a problem, S refers to the family $\hat{P}$ of procedures that characterize this method. In any case, S is the intermediary through which the formal representation (FR) incorporates the different real-life contexts (RLC) that the robustness concern requires be taken into account.

$\forall s = (p, v) \in S$, the procedure p applied to the version v produces a result, $R(s)$. This result can take extremely varied, non-exclusive forms, as suggested in the introduction.

Once S is finite, it is possible, in some cases, to associate a subjective probability to each element $s \in S$. This probability must reflect the chances that this variable setting s will be able to correctly describe what the RLC will be. In this case, S is said to be probabilized.

In order to illustrate the robustness concern in **multi-criteria** decision aiding (MCDA) more concretely, a few problem types, chosen more or less arbitrarily from those that exist, are briefly described below.

Problem 1. Choosing a supplier following a Call to Bid for the acquisition and installation of new equipment

Suppose that around 15 bids were received and that each one was evaluated according to the following criteria: cost; deadline; two satisfaction levels, each one related to a specific property and a possible veto effect; and the confidence that the supplier will respect the deadlines and the specifications. Here, the vague approximations and the zones of ignorance affect the way that the bids received are evaluated in terms of these five criteria, especially the last three. They also affect the role that each criteria plays in the final decision (i.e., the relative weights and the possibility of a veto). Thus, for some of the responses, an analyst might retain not just a single evaluation of given criteria, but two or three. By combining these evaluations, he/she can define a set V of the versions of the problem. If, for example, the analyst chooses a decision-aiding method like ELECTRE, he/she might decide to take several sets of weights into account, and once a veto threshold criterion is justified, to retain two or three values, thus defining a set P of procedures. S would thus be defined by the Cartesian product $P \times V$. It would also be possible to consider that the different sets of weights allow differentiating versions instead of procedures. The definition of S would be unchanged.

The decision maker may expect the analyst to recommend as few bid proposals as the vague approximations and zones of ignorance permit, along with the arguments that justify why each of the bids was selected. These arguments must, for example, allow the decision maker to understand under what conditions (i.e., the hypotheses related to the vague approximations and zones of ignorance) the bid in question is at least as acceptable as the others, while also explaining the risks taken if these conditions are not satisfied.

Problem 2. Setting the structural characteristics of a water treatment system for a municipality that currently has no system

Determining the optimal value for these structural characteristics requires sufficiently precise knowledge of the needs that will have to be satisfied throughout the expected life of the system. These needs are, in fact, not very well known because they depend on multiple factors, including but not limited to the evolution of the population, of the population's use of the system, and of the laws regulating system discharges, as well as the arrival of new activities in the sector. If the analyst tries to formulate the problem in terms of the optimization of a single criterion, this criterion must not take into account only the provisional costs of constructing and maintaining the system. It is also necessary to take into account the cost of adapting the system if the municipality's needs were underestimated and cannot be satisfied without modifying the initial structural characteristics. In addition, the analyst must take into account the negative consequences of budget overruns for the initial construction and maintenance operations if the municipality's needs were overestimated. This example shows that the formulation of a single optimization criterion can run up against serious difficulties. Even if the OR researcher manages to overcome these difficulties and develops a suitable scenario set, this formulation of the decision-aiding problem may not respond to the decision maker's expectations. Here, the robustness concern stems from a desire to be able to justify the decision in the future, if necessary, as well to avoid any cases in which needs were left critically unsatisfied, except in unforeseeable circumstances. This example shows that, in certain cases, the robustness concern may play a crucial role in the formulation of the decision-aiding problem, taking into account multiple criteria.

Problem 3. Scheduling airline flight crews for all the flights of an airline company

The robustness concern in this example is the need to take into account unanticipated crew absences (e.g., illnesses, injuries during a mission) and/or flight plan modifications (e.g., a plane type other than the one expected). The question that must be answered is how can these risks be handled given the potential conflicts between the following two points of view:

- The point of view of the airline company, which seeks an economic optimum in the context of highly complex legislation that leaves very little room for interpretation;
- The point of view of the crew, which includes the desires that the crew would like to see satisfied while avoiding scheduling perturbations that would make the crew's life difficult.

Problem 4. Controlling the execution of a vast project

Diverse software using a variety of methods for establishing a provisional schedule of execution are available. To use these software, it is necessary to take multiple data elements into consideration (e.g., task duration, supply delivery deadlines, skilled worker availability, weather, etc.). However, the real-life context of the project's execution may not correspond to the initial values predicted for each of these data elements. If this is the case, both the execution cost and the completion time can undergo tremendous modifications. Thus, execution cost and the completion time are two criteria which should be taken into account when choosing a provisional schedule liable to best withstand the vague approximations and zones of ignorance that affect these data, where "best withstand" means "has the potential to allow acceptable local adaptations".

Problem 5. Reliability of a complex system

Let us consider the case of a complex system whose reliability depends on the values that will be attributed to certain variables during the design phase. In this system, the relationship between the values retained and the reliability of each of the large system components is highly complex and thus imperfectly known; furthermore, the relationship between these values and the reliability of overall system is even less well known. In these conditions, in order to enlighten the choice of these values during the design phase, it may be appropriate to take into account as many reliability criteria as there are large system components.

The above examples (to which we will refer later in the chapter) underline the often multi-criteria character of the preference models that can be used to guide the choice of a solution. However, in these examples, no criterion for characterizing the relative robustness of a solution has been considered. This is generally how preferences of a decision maker are modelled, especially when there is a single criterion. To take the robustness concern into account, one of the three following families of approaches can be considered:

(a) Define a mono-dimensional robustness measure that will make sense of such statements as "solution x is at least as robust as solution y". This measure is then used to introduce a new criterion linked to a preference model that has been defined previously without taking the robustness concern into account.
(b) Apprehend robustness multi-dimensionally, in such a way that it is expressed through several criteria, not just one. These criteria can then constitute the preference model itself, or as in (a) above, they can complete an initial preference model that has no criterion to express the robustness concern.
(c) Apprehend robustness other than by describing one or more criteria designed to allow the solutions to be compared. This last family of approaches leads, more or less explicitly, to make intervene one or more properties intended to characterize the solutions that are qualified as robust. These properties can also serve to establish robust conclusions. Defining these properties can, in some cases, bring one or more robustness measures into play. Thus, this family of approaches serves as a constraint and not as a criterion of comparison.

The three sections that follow deal with each of the above families of approaches. In these last three sections, in addition to describing the appropriate literature, we suggest some avenues for new development and in some cases, we present some new approaches.

4.3 Robustness in MCDA: Mono-dimensional Approaches

4.3.1 Characterizing Mono-dimensional Approaches

In this section, we examine the approaches that lead to apprehending the robustness by completing a preference model that was previously defined with no direct link to the robustness concern. The robustness measure $r(x)$ is introduced to give meaning to the statement "solution x is at least as robust as solution y".

At the end of the introduction, we called back to memory three measures defined by Kouvelis and Yu [23]. These measures are appropriate when the previously defined preference model is mono-criterion. Most of the works that have used one of these three measures have done so by substituting the robustness criterion induced by the chosen robustness measure for the initial criterion. This kind of approach remains mono-criterion and consequently is not within the scope of this chapter.

In the following sections, we explore the works or new avenues of research that use this robustness measure to define a new criterion, which is added to the initially defined preference model. We first consider two cases, one in which the initial model is mono-criterion (see Section 4.3.2) and one in which it is multi-criteria (see Section 4.3.3). Then, in Section 4.3.4, we present a new approach that, under the specified conditions, can be applied to both the mono-criterion and multi-criteria cases.

4.3.2 With an Initial Mono-criterion Preference Model

In this kind of approach, two criteria are considered to guide the choice of a solution. In addition to the single preference criterion (e.g., gain, cost, duration), a robustness criterion is added to take the different frailty points inherent to the formal representation (FR) into account. Since these two criteria are in conflict, in all but certain particularly auspicious cases, this naturally leads to a consideration of the efficient frontier or an approximation of it.

By hypothesis, the preference criterion is intended to attribute a value $v(x)$ to each solution x by ignoring the vague approximations and the zones of ignorance against which robustness is supposed to withstand. To define $v(x)$ in such conditions, it is possible to use the values $v_s(x)$ that this criterion attributes to solution x with the variable setting $s \in S$. For example, $v(x)$ can be the median or the arithmetic mean of the values $v_s(x)$, or even their expected value if S is probabilized. It is also possible to set $v(x) = v_{s_0}(x)$, where s_0 is a variable setting characterizing a description of the reality chosen as reference for its high credibility. In these conditions, the robustness measure can be one of the robustness measures proposed by Kouvelis and Yu (see end of Section 4.1) or any other criterion appropriate for dealing with the impact of imperfect knowledge.

In the classic mono-criterion approaches that take into account one of the criteria proposed by Kouvelis and Yu, robustness focuses on the worst case and assigns no importance to the solution performances in the other variable settings. The approaches presented in this section try to remedy these drawbacks by simultaneously considering the performance in the worst case and in the median or average case. Thus, these approaches make it possible for decision makers to choose from several compromise solutions.

Below, we present several papers from the literature that use this kind of approach.

Chen et al. [9] studied the problem of industrial system design. Designers have always tried to take into account variations in the properties of the object to be designed, even when these variations are due to uncontrollable factors, such as temperature or humidity. These factors can cause the overall system performance to deteriorate sharply during operation. It is thus important to integrate the possible variations as early as possible in the design process, allowing the possible impact of these variations to be anticipated so as to minimize their effect on system performance. A solution is qualified as robust if its performance varies little under the influence of these variation-provoking factors. The possible variations of a material property are modelled using a set S of probabilized variable settings. A reference value and a neighbourhood defined around this value are associated to this property. The preference criterion of initial model is defined by the expected value of the performance in this neighbourhood. The added robustness criterion corresponds to the variance of the performance in this same neighbourhood. Decision makers are attracted to the solutions that offer a compromise between global performance and robustness.

Ehrgott and Ryan [10] studied the robustness of crew schedules at Air New Zealand. In the current systems for airline planning and management, optimizing crew schedules involves only a single criterion, the cost criterion. This criterion $v(x)$ takes into account the costs engendered supposing that a plan x is perfectly respected. However, in reality, the sources of the risks likely to perturb traffic are numerous. If aircraft downtimes are not sufficient to withstand these perturbations, plan x will not be respected, which will provoke penalties for the airlines. For this reason, the airlines are also interested in robust solutions that are able to withstand these perturbations. Optimizing the criterion $v(x)$ yields solutions that cannot be considered robust because they also make it necessary to minimize aircraft downtimes. Ehrgott and Ryan considered that the robustness of a solution increased as the total penalties caused by the probable delays decreased. For this reason, in addition to the criterion $v(x)$, they introduced a robustness criterion $r(x)$ based on the sum of the penalties that the “predictable delays” were likely to provoke. These predictable delays were introduced for each flight based on statistical observations that allowed an average delay and a standard deviation to be defined. The predictable delay is defined as the average delay increased by three standard deviations. The set of delays thus constructed constitutes a single variable setting s that is taken into account when defining $r(x)$ as the sum of the penalties assigned to each flight according to this single variable setting. The efficient frontier is then generated using the ε-constraints method.

Salazar and Rocco [37] studied reliable system design (see also Problem 5). The design of a product is often initially limited to finding the characteristics that meet the required specifications. Nevertheless, product reliability can vary due to uncontrollable external perturbations (e.g., aging, environmental changes) or due to *design variables*, which could have negative consequences. In this context, the stability of the reliability plays an important role, as does the design cost. In order to illustrate the problem, the authors considered the case of a complex system with several components. In their study, the reliability of the system and the reliability of the different components are related and are expressed with a complex mathematical formulation. By setting an admissibility interval for overall reliability, it is possible to determine, exactly or approximately, the feasible domain of the different components' reliabilities. Clearly, the points that are close to the borders of this domain are less interesting than those that are near the centre since a small variation in the frailty point values can push the system out of the acceptable reliability interval. Given a system reliability value, robustness can be apprehended, on the one hand, through the volume of the biggest parallelepiped included in the feasibility domain containing this value and, on the other hand, through the cost corresponding to the maximum cost in this volume. Decision makers are naturally interested in solutions that offer a compromise between design costs and the stability of the reliability.

Kennington et al. [22] studied Dense Wavelength Division Multiplexing (DWDM) routing and provisioning. DWDM is an optical transmission technology that allows data from different sources to be circulated over an optical fibre by assigning a wavelength to each source. Thus, in theory, several dozen different data flows can be transmitted at the same time. The transmission speeds are those of fibre optics: several billion bits per second. Given a network and an estimated demand, the DWDM routing and provisioning problem seeks to design a low-cost fibre optics network that will allow data to be sent to different demand centres. However, the process of estimating demand includes vague approximation and zone of ignorance. Under-estimating the demand, or over-estimating it, can have troublesome consequences. In this study, the imperfect knowledge of the demand is taken into account through a probabilized scenario set. The robustness concern is taken into account through a penalty measure that avoids the solutions proposing a capacity that is significantly under or over the demand in all the scenarios. This measure is based on the subjective costs corresponding to the positive or negative deviations from the demand. The installation costs and the robustness criterion help to enlighten the decision maker's choices. Since robustness plays an important role, Kennington et al. [22] transformed the bi-criteria problem into a lexicographical problem.

These three examples show the degree to which the additional robustness criterion can depend on the nature of the problem studied. For this reason, it hardly seems possible to formulate rules to facilitate the criterion's design. Thus, modellers must use their imagination to make the criterion correspond correctly to the problem at hand.

To bring this section to a close, we suggest an approach that is different from the ones described above. We consider the case in which S is finite and $v(x)$ is defined either based on a reference variable setting or on an average or median of S. In these conditions, it is possible to adopt one of the robustness criteria proposed by Roy [30]: bw-absolute robustness, bw-absolute deviation, or bw-relative deviation. We present what we think is an especially interesting case of project scheduling (see Problem 4). The robustness criterion can be defined as the proportion or the probability of the variable setting $s \in S$ for which $v_s(x) \leq v(x) + \Delta$ where Δ is a given constant. When controlling the execution of a vast project, the single preference criterion may be a cost criterion that includes the penalties engendered if the project is not completed on time. In this approach, the efficient frontier or an approximation of this frontier appears to be quite interesting for the decision maker. For this reason, it could be useful to study the sensitivity of this efficient frontier to variations of Δ.

4.3.3 With an Initial Multi-criteria Preference Model

In this section, we consider the case in which the initial preference model is multi-criterion, and not mono-criterion as in Section 4.3.2. Let F be a family of $n \geq 2$ criteria defined with no reference to a robustness concern. For the ith criterion, the performance can be defined as in Section 4.3.2. Again, we are interested in approaches that use a single additional criterion to give meaning to the statement "solution x is at least as robust as solution y".

This criterion must synthesize, in a single dimension expressing robustness, the impact of the variable settings in S on the performances of each of the n criteria in F. Unfortunately, we were unable to find a single publication in the literature proposing such a criterion. We describe below one possible approach.

This approach consists of:

1. First to consider each of the n criteria separately, and to assign a specific robustness measure to each one (see Section 4.3.2)
2. Second to aggregate these measures in order to define the additional robustness criterion

If the n criteria have a common scale, the aggregation can, with a few precautions, use the operators Max, OWA (Yager [44]), and even the Choquet integral (Grabisch [16], see also Roy [30]) to differentiate between the roles of each of the different robustness measures. In these conditions, the efficient frontier has $n + 1$ dimensions. For $n \geq 3$, it might be better to try to aggregate the criteria of the initial preference model in order to reduce the complexity of the calculation and make the results easier to interpret. It must be pointed out that aggregating all n criteria of the initial preference model into a single criterion before defining the single robustness criterion falls within the framework defined in Section 4.3.2 and thus constitutes a different approach from the one presented here.

4.3.4 With an Initial Preference Model That Is Either Mono-criterion or Multi-criteria

The approach presented below is applicable only to a finite set A of actions (the term "action" is here substituted for "solution"), which are evaluated using one or more criteria in each of the variable settings in set S, which is also assumed to be finite. In the case of a single criterion v, the values $v_s(x)$ define for A a complete preorder P_s, $\forall s \in S$. In the case of multiple criteria, the same result can be obtained (except that the preorders may be only partial) by applying an aggregation procedure (e.g., an ELECTRE-type method). Let P be a set of complete or partial preorders thus defined. We propose defining the robustness measure $r(x)$, associated to an action x, by the proportion (or the probability if S is probabilized) of the preorders P_s in P in which x occupies a rank at least equal to α, where α defines an imposed rank. It is also possible to imagine another imposed rank β, penalizing solutions that are not very well ranked in some variable settings. The robustness measure $r(x)$ can be defined by substituting from the previously defined measure the proportion (or probability) of the P_s in P in which x occupies a rank at least equal to β. The greater this measure, the more robust the action x is judged to be. This approach can be very useful for helping a decision maker to choose a supplier following a Call to Bid (see Problem 1 in Section 4.2). It might also be useful for studying the sensitivity of the results to the values of α and β. The results obtained must be able to be easily summarized as robust conclusions (see Section 4.5) that decision makers can easily understand.

4.4 Robustness in MCDA: Multi-dimensional Approaches

4.4.1 Characterizing Multi-dimensional Approaches

In this section, we survey the approaches that involve not a single measure of the robustness concern, but several. Each of these measures is designed to look at robustness from a specific point of view. These measures are used to define a set R ($|R| > 1$) of criteria intended to judge the more or less robustness of the solution.

In order to present the research pertaining to this kind of approaches, as well as to propose some new paths to explore, we distinguish the following three cases:

- The family R constitutes a preference model intended to enlighten the decision in the absence of any other previously defined preference model.
- The family R is substituted for or completes a previously defined mono-criterion preference model that has no links to the robustness concern.
- The family R is substituted for or completes a previously defined multi-criteria preference model whose criteria do not represent the robustness concern.

4.4.2 Without Any Initial Preference Model

Surprisingly, our bibliographic search did not reveal any studies about the kind of approach discussed in this section. As mentioned above, this kind of approach takes preferences into account directly by defining a priori a family of criteria in which each member expresses a different point of view of robustness; these criteria are not, however, based on a multi-criteria preference model initially conceived with no link to robustness. Nevertheless, one of the authors (see Pomerol et al. [26]) helped to develop and implement such an approach for dealing with concrete problems. The following paragraph provides a brief summary of the study by Pomerol et al. [26].

The concrete context was a large Parisian railway station that had to cope with intense rail traffic. Minor perturbations (e.g., a delayed gate closing due to an obstruction) frequently caused delays. Despite the actions of the dispatchers, who intervened to re-establish normal traffic patterns as quickly as possible, more serious accidents (e.g., damaged equipment) provoked a snowballing effect, leading to cancelled trains. To resolve the problem, new timetables, as well as local improvements to the rail network and the rolling stock, were envisioned. Combining these suggestions led to defining a set X of solutions to be studied. The goal of the study was to compare the robustness of these solutions when faced with different kinds of perturbations, while also taking into account the way that the dispatchers intervened to lessen the negative effects as much as possible. A set S, called the "incidence benchmark", was built; this set contained a set of representative incidences, each of them described precisely with a weight assigned according to its frequency. The family R was composed of the following six criteria:

$g_0(x)$: maximum delay allowed to any train without any perturbation being provoked;
$g_1(x)$: total number of trains including timetable concern by a delay from the original incident to the return to the theoretical schedule;
$g_2(x)$: the total duration of the perturbation;
$g_3(x)$: the total number of travellers concern by the perturbation;
$g_4(x)$: average delay of the travelling time;
$g_5(x)$: the total number of the trains concerned.

The first three criteria essentially reflect the viewpoints of the operator in charge of train traffic, while the others are directly concerned with traveller satisfaction. The performance $g_0(x)$ is completely determined by the *timetable* component that is part of the definition of x. In no way does $g_0(x)$ depend on the different variable settings $s \in S$, which is not true of the other five criteria. In addition, $\forall x \in X$, the calculation of $g_j(x)$, where $j \neq 0$, requires that the behaviour of the dispatchers facing each of the incidents in S be taken into account. To calculate these performances, it was necessary to call upon an expert system to reproduce this type of behaviour.

To end this section, let us underline that, in the kind of approach considered here, testing the coherence of family R is essential (i.e., verifying the following properties: exhaustivity, cohesion and non-redundancy; see Roy [27], Roy and Bouyssou [33]) because the preference model here is characterized by the family R.

4.4.3 With an Initial Mono-criterion Preference Model

This kind of approach is characterized by a family R containing several criteria to take robustness concern into account, rather than a single criterion as in Section 4.3.2. The criteria in R must reflect different non-correlated points of view. Consequently, if one of them can be chosen from the three proposed by Kouvelis and Yu (see Section 4.1), given the dependencies that exist between these criteria, we do not feel that the intervention of a second one would be pertinent.

Below, we first present an approach that substitutes several robustness criteria for the initial single preference criterion, and then we describe several approaches in which the robustness criteria complete the initial preference criteria.

Hites et al. [20] studied the connections between the robustness concern and multi-criteria analysis. Defining the elements in S as scenarios, these authors proposed substituting the set $R = \{v_s(x)/s \in S\}$ for the single criterion $v(x)$. Each of the criteria thus defined provides pertinent information for determining the relative robustness of solution x. By considering this set, these authors showed that an approach that applies a classic multi-criteria method is not appropriate for identifying robust solutions. One of the reasons comes from the cardinality of S: classic multi-criteria methods are only appropriate for criteria families containing 20 or at most 30 criteria. When the number of scenarios is small (a few units), considering the efficient frontier or an approximation of this frontier can in some cases help to respond to the robustness concern. It is useful to note that in all other approaches to the problem, a solution presented as robust must necessarily be a non-dominated solution in the multi-criteria problem defined by the set R considered here.

Let us move on to approaches in which the initial preference criteria are completed by a family R of criteria. The following paragraph briefly presents the only study that we were able to find involving this kind of approach.

Jia and Ierapetritou [21] studied the problem of *batch scheduling* in the chemical industry. The discrete process represents an ideal operational mode for synthesizing chemical products in small or intermediate quantities. This process is able to produce several composites by batch, using standard equipment and is also able to adapt to the variations in the nature and quality of the primary materials, which is a major advantage in terms of flexibility. In order to insure that any resource used in the production process is exploited efficiently, it is important to take *a detailed plant schedule* into account in the design phase. The design objective is to determine the number and types of equipment to be used and to build a feasible schedule of the operations that will maximize a performance criterion, given the following elements:

- The production guidelines (e.g., production time for each task, quantities of materials involved in the manufacturing of the different products)
- The availability and the capacity of the production equipment and storage facilities
- The production requirements;
- The time horizon of the study

In this study, the performance criterion corresponds to the total production time. However, during the design phase, it is almost impossible to obtain precise information about the production conditions. Thus, the information needed to calculate the expected performance exactly is not available. To remedy this lack, the various possible production conditions are modelled by a set S of variable settings. The production time associated to each of these variable settings can be calculated. In order to quantify the effect of the variations in the production conditions, two additional criteria are considered. The first tends to support feasible solutions in most of the variable settings by seeking to minimize the expected value of the unmet demand. The second attempts to measure the stability of the solution performance by seeking to minimize the expected value of the positive deviation between the real duration and the expected value of that duration. The main advantage of this measure with respect to the variance is its simplicity, since unlike variance, this measure can be written linearly. The efficient frontier provides the decision maker with a set of interesting compromise solutions.

The work presented above shows that combining a performance criterion and a set of robustness criteria can have numerous practical applications. To conclude this section, we suggest a new approach of the same type.

This new approach is concerned with the case in which S is finite. We assume that the initial preference model criterion $v(x)$ expresses a gain. We assume besides that $v(x)$ is defined by a variable setting s_1 judged particularly convincing: $v(x) = v_{s_1}(x)$. The value $v(x)$ could also be defined by the median or the arithmetic average of the values $v_s(x)$. The approach proposed here consists of modelling the robustness concern using two criteria. The first is the minimum gain of all the variable settings, and the second is defined by the number of variable settings, such as $v_s(x) \geq b$, where b corresponds to an objective that the decision maker hopes to reach, and even to exceed with a maximum of chance. Depending on the decision-making context, the second criterion can be replaced by the number of variable settings in which the absolute or relative regret is limited to b. The presence of these two robustness criteria, in addition to the expected gain with the reference variable setting s_1, can help the decision makers to be aware of how subjective the notion of robustness is. By discussing the value to be attributed to the bound b with the analyst, the decision maker can clarify the meaning that he/she assigns to the term "robust solution". For these reasons, studying the efficient frontier, possibly parameterized by b, seems to be a useful support tool for the final choice. This new approach is different than the one described in Section 4.3.2. In fact, in this new approach, we have retained a gain criterion as such (i.e., a gain in the variable setting or an average or median gain), and we have also added a second robustness criterion: the minimum gain. Cancelling the gain criterion would create a bi-criteria approach similar to the one proposed in Section 4.4.2. The advantages of this new approach are illustrated in Appendix A.

4.4.4 With an Initial Multi-criteria Preference Model

In this section, we consider the kind of approach in which the initial preference model is a family F containing $n \geq 2$ criterion. A criteria family R is introduced; the robustness concern is taken into account either by substituting R for F, or by using R to complete F. Each criterion in R may refer either to a specific aspect of robustness or to a criterion of the initial preference model. Unlike the approach described in Section 4.3.3, this kind of approach does not seek to aggregate the R criteria into a single summary criterion, but rather attempts to consider the criteria jointly. The most interesting case is the one in which the set R is substituted for F, since the case in which the set F is completed by R is difficult to interpret and can involve implementing algorithms that require a lot of computer resources and high calculation times.

In practice, researchers generally use only a single measure to model the robustness concern, undoubtedly for reasons of simplicity. Nonetheless, we found two papers in the literature that deal with the kind of approach considered in this section. We present them below.

Fernández et al. [13] examined *multi-criteria Weber location problem*. The problem dealt with in the paper consists of choosing the location of a super-server in a municipality where n servers are already in operation. This super-server includes k servers, each with its own individual characteristics. To server i is associated a vector with n components. Each of these components p_{ij}, called weights, is used to take into account the relative importance that the decision maker assigns to the distance that separates the already established server j from server i. In reality, these weights are ill-determined, and for this reason, a set S_i of vectors with plausible weights p^s_{ij} is defined for $i = 1, \ldots, k$. The decision maker's preference in terms of the choice of location for server i at place h is taken into account for each $s \in S$, by the criterion of the weighted sum d^s_{ij}, defined as $d^s_{ij} = \sum_{j=1}^{n} p^s_{ij} ||x_h - x_j||^2$, where x_h and x_j, respectively, represent the vectors of the coordinates for place h and those for server j. The servers $i = 1, \ldots, k$ should be located in the same place. This location is chosen by finding a compromise between the preference components referring to the different weighted sums d^s_{ij}. The authors begin by selecting, as the only possible locations, the places h for which the quantities d_{ij}^s have an acceptable value in all the variable settings, where $i = 1, \ldots, k$. Then, to facilitate the choice among the selected places, the authors bring into play k robustness criteria r_{ih}, $i = 1, \ldots, k$. Each of these criteria is a maximum regret criterion, defined as $r_{ih} = \max_{s \in S_i} \left\{ d_{ih}^s - \min_q d_{iq}^s \right\}$. The efficient frontier or the approximation of this frontier can help the decision maker to choose the best location possible.

Besharati and Azarm [7] studied the problem of *engineering design optimization*. This problem is similar to the one described in Section 4.3.2 (Chen et al. [9]), but the approach used is different. The initial preference model has n criteria (and not a single criterion) f_i for which the value $i = 1, \ldots, n$ must be minimized. In order to prevent undesirable consequences due to uncontrollable factors, the authors propose a method based on a generalization of the robustness criteria proposed by Kouvelis and Yu for the case in which the initial preference model is formed by a family F and for the case in which the values of certain constraint coefficients are imperfectly known. More precisely, the imperfect knowledge of the value of the frailty points is modelled with a set of variable settings S, where each element characterizes a possible version of criterion f_i, and the robustness of a solution is evaluated using two criteria.

The first criterion measures, for the worst variable setting, a p-distance between a given solution x and a point of reference x^*:

$$\max_{s \in S} \left[\sum_{i=1}^{n} \left| \frac{f_i^s(x) - f_i(x^*)}{f_i(x^w) - f_i(x^*)} \right|^p \right]^{\frac{1}{p}},$$

where x^w corresponds to a solution deemed particularly bad for all criteria and all variable settings. Let us notice that this p-distance insures that all the initial criteria play the same role.

The second criterion measures the performance variability of a solution x by calculating the p-distance between the points corresponding to the best and the worst variable setting for solution x, denoted s^b and s^w, respectively:

$$\left[\sum_{i=1}^{n} \left| \frac{f_i^{s^w}(x) - f_i^{s^b}(x)}{f_i(x^w) - f_i(x^*)} \right|^p \right]^{\frac{1}{p}}.$$

Let us note that in this paper, the frailty points do not affect only the coefficients of the objective function, but also the coefficients of the constraint matrix. For this reason, the authors are interested in efficient solutions that remain feasible in all the variable settings.

These two robustness criteria seem to be interesting since they can be applied in many contexts. In fact, the first is a generalization of the multi-criteria case of the absolute regret, and the second can be seen as a dispersion measure.

4.5 Robustness in MCDA: Other Approaches

4.5.1 Preliminaries

The approaches discussed in this section differ from the ones presented above in the sense that they are not intended to identify the most robust solutions in terms of one or more previously defined criteria. These approaches have their place in this chapter because they apply to formal representations of decision-aiding problems involving an initial multi-criteria preference model without any link to robustness.

Most of these approaches assign a determinant role to the fact that a solution, a set of solutions, or a method possesses (or does not possess) certain properties characterizing robustness, properties that are formulated in terms other than to maximize a criterion or to be on the efficient frontier (as was the case in the two sections above). In some cases, these properties make one or more robustness measures and their associated thresholds intervene so as to define the conditions under which the property(ies) will be judged satisfied. In most cases, these approaches yield results that allow conclusions about the robustness concern to be drawn.

Before presenting some of these approaches, it is necessary to call back to memory what Roy [31, 32] has called *robust conclusions*. By definition, each variable setting $s \in S$ is associated to an exactly defined formal representation of the problem and an exactly defined processing procedure. Applying this procedure to the problem's formal representation provides what has been defined under the general term **result** (see Section 4.1). Let us denote this result $\mathcal{R}(s)$.

Definition 4.1. A robust conclusion related to a sub-set $\hat{S}(S)$ is a statement that summarizes the result set $\{\mathcal{R}(s)/s \in \hat{S}\}$.

To illustrate this definition, we give several typical forms of robust conclusions that are interesting in the context of decision aiding (in cases when the preference model may or may not be multi-criteria).

(i) $\forall s \in \hat{S}$, x is a solution for which the deviation from the optimum (or from an efficient frontier) never exceeds a given threshold.
(ii) If the variable settings $s \in \hat{S}$ are taken into account, the results that follow (e.g., guaranteed cost, guaranteed completion time) are incompatible.
(iii) The results that follow ... are validated by the results $\mathcal{R}(s)$ obtained with a sample $\hat{S}$ of variable settings; since the sample is considered to be representative of S, it can be inferred that these statements are valid for all S.
(iv) For "almost" all $s \in \hat{S}$, x is a solution for which the deviation from the optimum (or from an efficient frontier) never exceeds a given threshold. Here, "almost" means that exceptions apply to the variable settings that, without necessarily being completely and perfectly identified, are considered to be negligible in the sense that they bring into play combinations of unlikely frailty points options.

(v) The results $\mathcal{R}(s)$ obtained $\forall s \in \hat{S}$ highlight a solution set $\{x_1, \ldots, x_q\}$ that responds to the robustness concern as it was formulated (this formulation may be relatively imprecise).

These examples show that:

- Stating robust conclusions does not necessarily lead to recommending the implementation of one solution over another (or even the choice of one method over another, see Section 4.5.4), but simply provides a framework for the decision maker's choices, and even sometimes restricts those choices.
- A robust conclusion may be more or less rigorous depending on whether it is validated over a relatively well-defined set and whether its formulation more or less permits interpretation (see Roy [31, 32] for an explanation of the distinction between perfectly-, approximately-, and pseudo-robust conclusions).

In the next section, we present a certain number of approaches that are included in this chapter either because they are recent, or because, despite being proposed in the past, they merit further consideration with respect to the above considerations, allowing them to be broadened, thus removing them from the restricted context in which they were proposed.

4.5.2 Robustness in Mathematical Programming

In mathematical programming, the search for a solution able to resist to vague approximations and zones of ignorance in order to withstand negative impacts is both a practical concern and a source of interesting theoretical problems. Different concepts and methods have been proposed in the literature for organizing and integrating this imperfect knowledge into the decision-making process (Ben-Tal and Nemirovski [3,4], Bertsimas and Sim [5], Bertsimas et al. [6], El Ghaoui and Lebret [11], El Ghaoui et al. [12], Soyster [39, 40]). When the objective function coefficients are not known exactly, the classic criteria from decision-making theory (e.g., worst case, absolute and relative regret) have often been used to define the robustness concern in linear programming as well as in integer programming.

When the imperfect knowledge concerns constraint matrix coefficients, the models studied in the literature primarily deal with the imperfect knowledge about either the columns or the lines of the matrix. These models assume that the constraint matrix columns (or lines) have coefficients able to vary in well-defined sets.

Imperfect knowledge in the constraint matrix columns was initially studied by Soyster [39, 40]. In this model, each column $A_j = (a_{ij})$ of the constraint matrix $m \times n$ can have values from a set $K_j \subset R^m$. The objective function coefficients, as well as the right-hand members, are assumed to be known exactly. The author deems robust any solution that is feasible for all the possible values of the vectors A_j chosen from K_j. The search for a robust solution is thus equivalent to solving a new mathematical program of the same nature for a constraint matrix $A' = \left(a'_{ij}\right)$ defined as follows:

- $a'_{ij} = \max_{a_j \in K_j} a_{ij}$ if the constraint is of the type $\leq$;
- $a'_{ij} = \min_{a_j \in K_j} a_{ij}$ if the constraint is of the type $\geq$.

The Soyster model is very conservative. The new mathematical program does not always allow feasible solutions although certain robust solutions of the type defined above may exist. In fact, although for certain j, the vector a'_{ij}, where $i = 1, ..., n$, described above does not belong to K_j, the set of feasible solutions of the new linear program does not necessarily contain all the robust solutions. In addition, even when feasible solutions exist, the one that optimizes the objective function may have an incorrect value and thus not be optimal in the set of robust solutions.

According to Ben-Tal and Nemirovski [3, 4], when the constraint matrix coefficients (and possibly the objective function coefficients) are not exactly known, the robust solution must remain feasible for all possible values of the unknown inputs. In the general case, it is possible that this intersection is empty. In the case this intersection is not empty, the resulting solution is very conservative.

Bertsimas and Sim [5] presented an approach that allows the degree of conservatism of the model's recommendation to be controlled when the imperfect knowledge is related to the lines of the constraint matrix. More specifically, each coefficient in the constraint matrix can have any value in the interval $\left[a_{ij} - \alpha_{ij}, a_{ij} + \alpha_{ij}\right]$ and for each line i, a number Γ_i is considered, where Γ_i cannot exceed the number n of variables. The model is based on the hypothesis that it is not very likely that all the model parameters will reach the worst values simultaneously. A solution is deemed Γ-robust if it respects the constraint i, for all $i = 1, \ldots, m$, when at most Γ_i coefficients are likely to reach the interval's higher bound $a_{ij} + \alpha_{ij}$ in cases with a $\leq$-type constraint (or likely to reach the interval's lower bound $a_{ij} - \alpha_{ij}$ in cases with a $\geq$-type constraint), and the other coefficient values are set to the average value of the interval. Bertsimas and Sim showed that, unlike the min–max versions of the absolute or relative regret, this approach generates a robust version with the same complexity as the starting problem. Specifically, the robust version of the shortest path, spanning tree and assignment problems are solvable in polynomial time. In addition, the robust version of the NP-hard problem that is β-approximable is also β-approximable. Nevertheless, this approach does have limitations. In fact, it generates a program that is parameterized by quantities, and it is not easy to specify the appropriate values for this program in advance. In the absence of information facilitating the choice of these values, setting $\Gamma_i = n$, for all $i = 1, \ldots, m$, produces a model similar to Soyster's conservative model.

In the studies cited above, the robustness concern does not bring into play criteria that permit the degree of robustness of the solution to be apprehended. This concern leads to considering any solution that is feasible in the defined conditions as robust. To conclude this section, we suggest a different approach. Let us consider the following linear program:

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = b_j, \quad i = 1, \ldots, m$$

$$x_j \geq 0, \quad j = 1, \ldots, n.$$

Let us also suppose that only the objective function coefficients are known exactly and that, by hypothesis, all the constraint matrix lines correspond to quantities that are expressed in the same type of unit (e.g., physical, monetary). The values of the constraint matrix coefficients are uncertain. A finite set S of variable settings allows this imperfect knowledge to be modelled. In the general case, it is possible for the intersection of the feasible domains of all the variable settings to be empty. In addition, even if this intersection is not empty, the price of robustness, as Soyster refers to it, can be high. The decision maker might accept an unfeasible robust solution in a small subset of S, but only if the cost is relatively low. In fact, it is often acceptable in practice to not respect equality; however, in this case, it is important for the non-zero deviations between the right and left members to be "small" and few in number. Thus, an unfeasible mathematical solution may be preferable to a much more costly solution that perfectly satisfies all the equalities. For a solution x, the deviations that must be taken into account are defined as follows:

$$e_i^s = b_i^s - \sum_{j=1}^{n} a_{ij}^s x_j.$$

A solution may be judged even more robust if these deviations remain small in the greatest possible number of variable settings. From this perspective, we propose adding one of the following three robustness criteria to the cost criteria:

- $\frac{1}{|S|} \sum_{s \in S} \max_{i=1,\ldots,m} |e_i^s|$;
- $\frac{1}{m|S|} \sum_{s \in S} \sum_{i=1}^{m} |e_i^s|$; and
- $\frac{1}{m|S|} \sum_{s \in S} |e_i^s|^2$.

In some cases, it may be appropriate to incorporate weights into these criteria to indicate the importance of the different deviations according to whether they are positive or negative or whether they are related to one line i or another.

Ultimately, this approach seeks solutions that provide a compromise between, on the one hand, the value of the objective function and, on the other hand, the deviations representing imperfect satisfaction of the constraints. Let us notice that the frontier can be built using the simplex method by optimizing a linear combination of the two criteria chosen, unless if the chosen robustness criterion is the third one above. This combination is possible because there are no unsupported solutions.

4.5.3 Obtaining Robust Conclusions from a Representative Subset S

At the beginning of the 1980s, two studies were completed in order to obtain robust conclusions. The first examined the execution priorities for extending Metro lines in the Paris region (see Roy and Hugonnard [34]). The second dealt with choosing a company to automate the sorting centres in the French Postal Service (see Roy and Bouyssou [33], Chapter 8). For these studies, Roy designed and implemented an approach that, although relatively informal, allowed a series of robust conclusions to be obtained. These two concrete studies were multi-criteria. Frailty points (not referred to by this term) appeared for two reasons: on the one hand, the presence of diverse data with ill-determined values in a certain interval and, on the other hand, the choice of a method that, due to technical reasons, justified the use of several processing procedures. In both cases, the robust conclusions obtained turned out to be quite interesting. In our opinion, this approach deserves to be broadened and extended within the formal framework described below.

Before describing this framework, we should specify that this type of approach is appropriate only for cases with a finite set A of possibilities, which we call *actions* rather than solutions. The results $\mathcal{R}(s)$ which have to be exploited can be those obtained by applying (Roy [27], Chapter 6):

- A selection procedure (a choice problematic)
- An assignment procedure (a sorting problematic) or
- An ordering procedure (a ranking problematic)

We propose to structure this approach into three steps.

Step 1: Moving from S to $\hat{S}$ (see Section 4.5.1)

In step 1, S always designates the set of variable settings derived, on the one hand, from the possible versions retained when formulating the problem and, on the other hand, from the various possible processing procedures that are envisioned. $\hat{S}$ designates a finite subset of S fulfilling the following two requirements:

- *Calculability requirement*: $\mathcal{R}(s)$ must be able to be determined, $\forall s \in \hat{S}$.
- *Representativity requirement*: Studying $\{\mathcal{R}(s)/s \in \hat{S}\}$ permits conclusions to be drawn, conclusions that can, with a negligible risk of error, be considered as valid for all of S.

Since these two requirements are generally in conflict, elaborating $\hat{S}$ means finding a compromise. To fulfil the representativity requirement (which in many cases will be highly subjective), a combinatorial approach or a probabilistic approach, or possibly a combination of the two, may be used.

The combinatorial approach involves retaining a very limited set f of possible options (e.g., two or three) for each frailty point e_f. $\hat{S}$ is then defined as the

Cartesian product of these sets or as a part of this Cartesian product, by eliminating the least likely combinations in order to respect the calculability requirement.

The probabilistic approach involves choosing a random sorting procedure defined on S and applying this procedure repeatedly to gather a number of variable settings compatible with the calculability requirement. If S is the Cartesian product of a certain number of intervals, the sorting can be done independently for each of these intervals according to a uniform law. (For more information about this type of procedure, notably its representativity, see for example, Steuer [41].)

Step 2: Moving from $\hat{S}$ to $\hat{S}'$

After calculating $\mathcal{R}(s)$, $\forall s \in \hat{S}$, a preliminary examination of these results is conducted in order to highlight two categories of frailty points.

- Category 1 contains the points that can have a significant influence on the results. These are the points that produce a result $\mathcal{R}(s)$ that is greatly influenced by the option (relative to the points being examined) present in s when a subset of $\hat{S}$ is examined. This subset is such that each component is, for every frailty point other than the one being examined, either identical or very similar.
- Category 2 contains the points with a negligible influence on the results. These are the points that produce a result $\mathcal{R}(s)$ that is very little influenced by the option (relative to the points being examined) present in s when a subset of $\hat{S}$ is examined. This subset is such that each component is, for every frailty point other than the one being examined, either identical or very similar.

To conduct such an examination, it is possible, in some cases, to use classic data analysis tools. The presence of reference variable settings s^* in $\hat{S}$ (that have particular importance to the decision maker) can also be quite useful. This is especially true if all or a part of the variable settings that differ from s^* only in terms of a single component are introduced into $\hat{S}$.

The examination described above is done in order to replace $\hat{S}$ with a set $\hat{S}'$ at least as representative and, if possible, smaller. In fact, only one option (possibly two) can be retained for category 2 frailty points. This case leads to the withdrawal of a certain number of variable settings from $\hat{S}$. Category 1 frailty points can nonetheless justify adding certain variable settings to better highlight the influence of these category 1 points.

Step 3: Obtaining robust conclusions

A careful analysis of $\{\mathcal{R}(s)/s \in \hat{S}'\}$, possibly facilitated by a systematic procedure, must allow pertinent robust conclusions to be drawn for the problem being studied. Below, we provide several typical examples of conclusions that could be validated depending on the nature of the procedure that is used to determine $\mathcal{R}(s)$. These studies are inspired from the conclusions obtained for the two concrete examples given at the beginning of this section.

With a selection procedure

- action $a_1 \in \mathcal{R}(s), \forall s \in \hat{S}'$;
- action $a_2 \notin \mathcal{R}(s), \forall s \in \hat{S}'$;
- depending on whether the frailty point option f is ... or ..., the action a_3 belongs or does not belong to $\mathcal{R}(s)$; and
- actions a_4 and a_5 are always associated since $\mathcal{R}(s)$ either contains both of the actions, or neither.

With an assignment procedure

- $\forall s \in S$, c_k is the worst category to which can be assigned, and as soon as, the frailty point f option is at least equal to ..., then the worst category is not c_k but c_h;
- action a_2 is always assigned to a higher category than the one to which action a_3, $\forall s \in \hat{S}'$, is assigned, and as soon as the frailty point f option is at least equal to ..., two categories, at least, separate their assignments.

With an ordering procedure

- none of the actions in $B \subset A$ is among the first 10 in $\mathcal{R}(s), \forall s \in \hat{S}'$;
- the actions in $C \subset A$ are the only ones that are always among the first 12 in $\mathcal{R}(s), \forall s \in \hat{S}'$.

In many cases, the conclusions that can be validated cannot be formulated as rigorously as the ones above (perfectly robust conclusions). Exceptions could be tolerated. The latter may not be clearly defined. If these exceptions are due to variable settings combining the extreme options of several frailty points, they may be judged negligible since they are not very likely (i.e., approximately-robust or pseudo-robust conclusions). Taking as a starting point a statement similar to the ones proposed above, it should be possible to design a procedure capable of identifying under what conditions and for which actions this type of statement can be validated.

4.5.4 Approaches for Judging the Robustness of a Method

As mentioned in the introduction, "method" here refers to a family $\hat{P}$ of procedures that can be differentiated by the options chosen with respect to some of the method's frailty points. This could be, for example,

- the concordance levels or the cut thresholds in the ELECTRE methods;
- the thresholds making certain inequalities strict in the MACBETH or UTA methods; and
- the multiple parameters involved in the tabu, simulated annealing or genetic methods.

In addition to these frailty points, which can be described as techniques, many multi-criteria methods involve parameters that are supposed to take into account an aspect of reality without referring to the existence of a true value that they should have in this reality. This is notably the case, for example, with substitution rates, intrinsic weights, indifference, preference and veto thresholds, and the analytical form of a probability distribution or those defining a fuzzy number. This second kind of frailty point can be viewed either as part of a method (in this case connected to the procedure), or as part of a model (in this case connected to the version of the problem).

Once the frailty points of a method have been defined, a procedure P_s is characterized by a variable setting s that describes the option retained for each of these frailty points. Using a method to either implement a repetitive application or simply to enlighten a single decision can lead to consider all the variable settings in a certain set S (which can leave out certain irrelevant procedures of $\hat{P}$) as equally legitimate. Considering the robustness of the method implies a desire to protect oneself from the arbitrariness involved in choosing one element in S rather than another. Vincke [42,43] proposed basing the evaluation of a method's robustness on the relative similarity of the results obtained with the different procedures $P_s, s \in S$.

This approach to the robustness of a multi-criteria decision-aiding method requires an exact definition of "similarity". Vincke proposed defining this similarity using a distance measure applied to result pairs, with the distance obviously depending on the nature of the results produced by the multi-criteria method (e.g., utility functions, action selections, complete or partial preorders, category assignments). The criterion used to evaluate the robustness of a method can thus be defined by the maximum value of this distance for the set of variable setting pairs belonging to S when the method is applied to a specific version of the problem. Accordingly, a method can be qualified as robust on such basis if this maximum remains under a fixed threshold for the set of versions retained for the problem studied. As Vincke underlined, this definition of a method's robustness should not be used to judge whether or not a method is "good" or "bad" because, in fact, a method that systematically produces the same "bad" results could nonetheless be robust.

These considerations show that it is not easy to assign a meaning to the notion of robustness of a multi-criteria method. This notion cannot have an absolute character. The definition depends on both the version set of the problem studied and the way that the set S is defined. In the approach proposed by Vincke, it also depends on the distance measure chosen.

The subject of the robustness of a method could lead to interesting theoretical research. It would not be necessary to expect such research to help researchers confronted with real-life problems to choose the most robust method for dealing with these problems. In fact, for a given problem, the way that the version set is defined is frequently influenced by the method. In addition, the set S is strongly conditioned by the method. For these reasons, we cannot see how and on what basis one method can be declared more robust than another. The practitioner can nonetheless expect research about the robustness of different methods to provide guidance in order:

- to better take into account the frailty point set for the chosen method, notably when formulating robust conclusions, when it is necessary to enlighten a decision;

- to decide which option to retain for each of the frailty points chosen, when it is necessary to implement a method for repeated application (in time or space).

In Roy et al. [34, 35], the interested reader will find a description of a case in which the way of assigning numerical values to some parameters for a repetitive application of the ELECTRE III method was based on a comparison of the relative similarities of the top ranking elements. The objective of the method was to aid decision makers to periodically select, from among several hundred Metro stations, a small group of stations (a maximum of eight) whose renovation should be given priority. The k stations ranked highest by the method are referred to as the "top ranking" stations. The comparison of these top ranking stations (setting $k = 20$) highlighted, in a first step, the fact that, for most of the frailty points, the choice of the option retained had little or no influence (i.e., top ranking highly similar in terms of the symmetric difference). This, in turn, in a second step allowed (setting $k = 10$) the impact of the options retained for the remaining frailty points to be studied more precisely and the additional information to be taken into account when making the final choices.

4.5.5 Approach Allowing to Formulate Robust Conclusions in the Framework of Additive Utility Functions

Figueira et al. [14] have proposed diverse multi-criteria aggregation procedures, based on the principle of ordinal regression, allowing certain statements of robust conclusions to be validated in terms of the concepts of what is "possible" and what is "necessary" (see also Chapter 9). The subject of these conclusions varies with the aggregation procedure proposed:

- UTADIS (Greco et al. [18]) deals with the category among a set of totally ranked categories to which an action a can be assigned.
- UTAGMS (Greco et al. [19]) deals with assertions such as "action a outranks action b".
- GRIP (Figueira et al. [15]) deals with the intensity with which an action a outranks an action b.
- GRIP-MOO (Figueira et al. [14]) deals with the best actions that are not outranked by any other feasible action in a particular step of an interactive multi-objective optimization procedure.

These four aggregation procedures were designed to help one or several decision maker(s) (DM) (see the extensions presented in Greco et al. [18]) in the following circumstances.

The DM is interested in a set A of actions evaluated with n criteria. The DM can provide preference information for some reference actions, but this information differs depending on the procedure in question. Essentially, this information explains how the DM ranks these reference actions from best to worst, how certain actions are compared to others, or, in GRIP, the intensity with which action a is preferred to action b for certain criteria considered separately and/or comprehensively. Let I denote the set of information provided. The envisaged aggregation is additive as in the UTA procedures (Siskos et al. [38]), yet with various improvements, notably in the form of the marginal utility functions, which are no longer piecewise linear but simply non-decreasing. This means that the authors seek to take the information I into consideration with synthetic utility functions, each interpretable as a weighted sum of n marginal utility functions associated with different criteria. An adjusting algorithm makes it possible to identify a set $U(I)$ of synthetic utility functions said to be "compatible with I". The set U is defined by a set of linear constraints: a compatible utility function is associated to each interior point of the polyhedron S delimited by these constraints. Here, any point $s \in S$ constitutes one of the variable settings taken into account. It is not impossible for S to be empty, which means that the additive utility model considered is inappropriate for taking the DM's preferences into account as they were expressed in the set I.

In all cases, each of the aggregation procedures described above leads to present to the DM conclusions in terms of what is "necessary" and what is "possible" (see Greco et al. [17]). A conclusion is said to be necessary if it is validated by all the functions of $U(I)$; it is said to be possible if it is validated by at least one of these $U(I)$ functions. Any conclusion that is necessary is thus possible. Ruling out any situation of incomparability, this additive utility model identifies as possible any conclusion of the above types that is not necessary. After showing the results obtained to the DM, it can be interesting to ask if I can be enriched either by adding complementary information about the same reference set, or by adding other reference actions. The enrichment of I leads to new conclusions that in turn lead to new responses to the DM's robustness concern. This enrichment also reduces S, which may possibly become empty.

Based on the possible and the necessary, this kind of approaches can be exploited in other contexts. Let us consider, for example, a set X (not necessarily finite) of solutions evaluated with n criteria $v_1, \ldots, v_n$ or a set S (finite) of variable settings that helps to define a performance v_{is} for each solution x, $i = 1, \ldots, n$. All efficient solutions, $\forall s \in S$, can be qualified as *necessarily efficient*, and all solutions that are efficient for at least one $s \in S$ can be qualified as *possibly efficient*. In many cases, it could be predicted that the sets of *necessarily efficient* solutions will be empty, and the set of possibly efficient solutions will be excessively large. This could lead to considering the greatest value of λ for which solutions are efficient for at least λ variable settings $s \in S$. Such solutions can, in a certain sense, be qualified as robust.

Other approaches for exploiting this approach based on the possible and the necessary have been presented (Greco et al. [17]). These methods primarily concern certain classic mono-criterion models in OR-DA (e.g., minimal spanning trees) and the multi-criteria outranking models.

4.5.6 Approaches to Robustness Based on the Concept of Prudent Order

The concept of prudent order was introduced by Arrow and Raynaud [2]. In this section, we first provide a brief reminder of what constitutes a prudent order. Then, we highlight the elements of this concept that are appropriate to bring an answer to robustness concern. Our explanations are based on Lamboray's research [24].

Let $A = \{a_1, \ldots, a_n\}$ denote a set of actions and F a family of q individuals (these individuals may be criteria). The individual i ($i \in \{1, \ldots, q\}$) ranks the actions in A with respect to his/her preferences according to a complete ranking O_i.

The concept of prudent order is designed to highlight rankings defined on A that minimize the oppositions. The meaning of "minimize the oppositions" will be explained later.

Let S denote a relation that counts the number of rankings that prefer a_i over a_j : $S_{ij} = |\{k \in \{1, \ldots, q\} : (a_i, a_j) \in O_k\}|$. Let $R_{\geq\lambda}$ ($R_{>\lambda}$) be a cut-relation of S defined as follows: $R_{\geq\lambda} = \{(a_i, a_j) : S_{ij} \geq \lambda\}$ ($R_{>\lambda} = \{(a_i, a_j) : S_{ij} > \lambda\}$). Clearly, increasing the value of λ decreases the cardinality of $R_{\geq\lambda}$. When $\lambda = 1$, this relation contains q complete orders (linear order) O_i. So there is a maximum value of λ, denoted α, such that $R_{\geq\alpha}$ contains a complete order ($R_{\geq(\alpha+1)}$ does not contain any complete order).

The relation $R_{>q}$ is empty. Consequently, it contains no cycle. So there is also a minimum value of λ, denoted β, such that $R_{>\beta}$ contains no cycle, and as a result, $R_{>(\beta-1)}$ contains at least one cycle. In the case of unanimity ($O_1 = O_2 = \cdots = O_q$), $\beta = z0$, and $\alpha = q$.

By definition, a prudent order $\mathbf{O}$ is a complete order verifying $R_{>\beta} \subseteq \mathbf{O} \subseteq R_{\geq\alpha}$. In the case of unanimity, the common order is a prudent order.

Before interpreting the concept of prudent order, let us provide some results.

Arrow and Raynaud showed that $\alpha + \beta = q$. In the particular case where $\alpha \geq \beta$, it is easy to verify that only one prudent order exists. In the opposite case, $\alpha < \frac{q}{2} < \beta$ is necessarily verified. Thus, several prudent orders can exist. This number could be very high when n is large. In the general case, Arrow and Raynaud justify the fact that these orders are said to be **prudent** as follows (an analogous justification is valid for the single prudent order in the case $\alpha \geq \beta$).

First of all, an ordered pair $(a_i, a_j) \in R_{>\beta}$ belongs by definition to all prudent orders. These ordered pairs create no cycle between them, and consequently no contradiction. For Arrow and Raynaud, a prudent order must highlight elements of consensus. From this perspective, not retaining a pair $(a_i, a_j) \in R_{>\beta}$ would lead to retaining the opposite pair (a_j, a_i), this solution would create a great opposition when the pair (a_i, a_j) is supported by a majority at least equal to $\beta > \frac{q}{2}$. Let us now consider a ranking that contains a pair $(a_i, a_j) \notin R_{>\beta}$. The number of individuals that support such a pair is $<\alpha$. These pairs in the prudent order are all supported by at least α individuals. Eliminating the pairs not found in $R_{\geq\alpha}$ leads to qualifying as prudent only the orders that minimize the greatest opposition, which is equal to $q - \alpha$ in all prudent orders.

In the exceptional case when there is only one prudent order, this order can be viewed as a robust ranking. In the opposite case, Lamboray [24] proposes elaborating robust conclusions based on the multiplicity of the prudent orders.

An initial form of robust conclusions can be obtained by building assertions that are valid for all the prudent orders. From this perspective, it is possible to examine the pairs (a_i, a_j) that are contained in all the prudent orders. It is also possible to look at the best and worst ranks of action a_i in the entire prudent order set. Lamboray have shown how these extreme ranks can be computed.

Another form of robust conclusions can be obtained by looking only at the prudent orders that possess a given property, for example, those that contain one or more pairs (a_i, a_j) or those that assign to action a_i a rank at least equal or at most equal to an imposed rank. Looking at only this type of prudent orders leads to drawing conditional robust conclusions. Such conclusions can facilitate a dialogue whose objective is to find a consensus ranking.

Let us observe that the multiplicity of the prudent orders can be seen as a consequence of the difficulties and the ambiguities (i.e., the arbitrariness) that are encountered when attempting to aggregate purely ordinal information. Let us underline in conclusion that the concept of prudent orders is defined based on a set of complete orders. It would be interesting to try to generalize this concept for the case of complete pre-orders or semi-orders. Complete orders guarantee $s_{ij} + s_{ji} = 1$. Unless the definition of s_{ij} is modified, this equality is no longer verified if there are ties. The verification of this equality, unfortunately, plays an important role in the definition and interpretation of prudent orders.

4.6 Conclusion

In MCDA, robustness is a practical and theoretical concern of great importance. The term robust refers to a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts, notably the degradation of the properties that must be maintained.

The objective of the first two sections of this chapter was to call back to memory a certain number of basic ideas and introduce a few definitions in order to establish the framework for examining the new trends discussed herein. Section 4.3 was devoted to an approach in which robustness is considered through a single criterion that completes a preference model that has been defined previously, independently of the robustness concern. In the first part of Section 4.3, we characterized this type of approach, and then in the next two sections (4.3.2 and 4.3.3), we described two sub-types of this approach, presenting several articles dealing with these two sub-types. In the last section, we propose a new approach (Section 4.3.4). In Section 4.4, we examined how robustness can be taken into account using several criteria. After characterizing this second type of approach, we broke it down into three sub-types (see Sections 4.4.2, 4.4.3 and 4.4.4). Very few publications dealing with this type of approach were found in the literature, but those that were available were mentioned in each section. After presenting these three sub-types, we introduced new approaches, one of which is illustrated in the Appendix. In a last section before the conclusion, we presented approaches that allow robustness to be considered other than with a single criterion or multiple criteria serving to compare the solutions. Following several preliminary explanations (Section 4.5.1), we described (Section 4.5.2) a new robustness approach in mathematical programming. Section 4.5.3 presented a procedure for obtaining robust conclusions from a representative subset of the set S of variable settings. The manner in which the robustness of a method should be apprehended is the focus of Section 4.5.4. Section 4.5.5 aims to formulate the robust conclusions related to the additive utility functions. Section 4.5.6 examines robustness approaches based on prudent orders.

The considerations developed in this chapter show that the use of multiple criteria for apprehending robustness in MCDA is a field of research open to future development, both theoretically and practically. These future developments should contribute to increasing the use of operational research tools.

Appendix: A Numerical Example

In this example, we consider 20 actions evaluated in 20 scenarios (see Table 4.3) according to a criterion v used to express a gain.

We suppose that the scenario s_1 involves, for each frailty point, a value that the decision maker (DM) judges particularly plausible. The other scenarios were built by taking diverse possible combinations of values that deviate significantly from those retained in s_1. The case that is of interest here is the one in which, to enlighten his/her choice, the DM would like, in addition to considering the gain solution x corresponding to scenario s_1 (gain denoted $v_1(x)$), to consider two criteria deemed pertinent to determine the robustness of solution x:

- The criterion $r_1(x)$ expressing the worst gain that solution x could yield for the 20 scenarios considered
- The criterion $r_2(x)$ indicating the number of scenarios that will guarantee a gain at least equal to the value $b = 190$, a value that reflects an objective that the DM would like to have a maximum chance of attaining and, if possible, exceeding

For the three criteria retained, only 6 of the 20 actions are efficient (see Table 4.1). Table 4.1 gives the DM the information necessary to make a responsible choice, while also making the DM aware of the subjectivity that is inherent to any of the choices made based on the three retained criteria. Depending on the DM's attitude towards risk and the way that the likelihood of the different scenarios is evaluated, the DM could:

- Choose x_4, which would suppose that he/she accepts the risk of only earning 120, a value that is quite far from the objective of 190. The fact that this objective is not only attained, but considerably exceeded in all the scenarios except s_2 could convince the DM to take this risk. However, the DM could also refuse this choice

Table 4.1 Performance matrix of potential actions

Action	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}
1	150	180	170	175	165	160	170	175	160	175	170	165	155	160	170	165	170	165	180	180
2	110	145	120	155	165	195	145	150	165	170	155	160	165	170	155	155	170	165	195	90
3	180	130	175	170	175	175	175	175	170	175	190	190	195	190	195	190	195	190	195	190
4	200	120	190	195	190	195	190	195	190	195	190	195	190	195	190	200	190	195	200	210
5	90	90	210	205	200	205	195	200	190	195	195	200	205	210	215	210	210	205	220	210
6	130	125	135	150	145	150	155	160	165	195	155	145	170	155	145	140	145	135	160	185
7	190	125	175	175	175	190	175	170	175	175	190	195	190	195	190	195	190	185	200	210
8	170	140	190	195	190	195	190	195	190	195	180	185	180	185	180	185	180	185	125	100
9	95	125	190	165	165	155	175	160	165	150	150	165	170	165	160	155	165	170	185	120
10	100	130	175	195	160	155	155	150	170	155	145	140	155	160	155	160	155	160	185	110
11	105	135	110	120	195	150	160	155	150	130	145	155	145	160	155	150	135	145	90	190
12	170	140	190	190	170	175	170	160	165	160	190	165	170	190	190	190	190	180	170	190
13	115	150	150	165	160	150	195	145	140	150	165	170	145	160	170	155	170	165	185	120
14	120	125	155	160	155	145	165	195	155	160	155	160	165	170	155	145	165	170	200	110
15	125	130	145	150	165	170	165	160	195	170	155	150	165	160	155	170	165	160	170	185
16	160	140	190	195	200	195	190	195	190	190	195	160	175	160	175	160	150	170	170	175
17	135	130	155	135	130	120	135	150	145	125	195	135	125	155	135	120	110	175	180	190
18	140	135	145	130	125	145	140	135	125	130	135	195	160	155	150	145	140	135	150	185
19	145	130	135	155	150	145	125	155	150	145	135	140	195	150	145	155	145	150	190	160
20	145	125	125	135	155	150	140	145	155	150	145	135	155	195	140	145	160	165	180	170

Table 4.2 Efficient set of actions for $b = 190$

Action	v_1	r_1	r_2
1	150	150	3
16	160	140	9
12	170	140	8
3	180	130	10
7	190	125	10
4	200	120	19

Table 4.3 Efficient set of actions for $b = 180$

Action	v_1	r_1	r_2
1	150	150	3
8	170	140	16
3	180	130	11
7	190	125	10
4	200	120	19

if he/she thinks that scenario s_2 is plausible enough and that a gain of only 120 would be highly detrimental

- Not choose x_4, for the reasons outlined above. This would normally lead the DM to eliminate x_7, which, in the scenario s_2, could lead to a slightly higher gain than the one obtained with x_4 while producing gains at best equal to the ones for most of the other scenarios. A desire to maximize the gain would lead to choosing x_1. The DM could judge this choice to be "bad" since, in the scenario s_1 that the DM appears to favour, the objective is far from being attained (150 instead of 190); in addition, the objective is not attained in any of the 19 other scenarios. Observing that, whatever the choice among the 5 other efficient actions, choosing scenario s_2 entails running a risk, the DM could decide that the best compromise between the chance of attaining his/her objective and the risks of a mediocre gain would be either x_3 or x_{16}, with the latter solution appearing preferable to x_{12}, which produces the same result in scenario s_2.

The analyst should point out to the DM that lowering the objective $b = 190$ would allow the set of efficient actions to be modified, thus highlighting other possible compromises. In fact, this is the case if b is set to 180 (see Table 4.2). Action x_8 guarantees a gain of 140, as does x_{16}, but in addition to being better in scenario s_1, this action also guarantees a gain of 180 in 16 scenarios, instead of the 9 obtained with x_{16}.

Thus, depending on the DM's ambitions (i.e., the desired objective level) and his/her attitude towards risk (worst case), the DM may find the choice is between x_4 and x_8.

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Chapter 5
Preference Modelling, a Matter of Degree

Bernard De Baets and János Fodor

Abstract We consider various frameworks in which preferences can be expressed in a gradual way. The first framework is that of fuzzy preference structures as a generalization of Boolean (two-valued) preference structures. A fuzzy preference structure is a triplet of fuzzy relations expressing strict preference, indifference and incomparability in terms of truth degrees. An important issue is the decomposition of a fuzzy preference relation into such a structure. The main tool for doing so is an indifference generator. The second framework is that of reciprocal relations as a generalization of the three-valued representation of complete Boolean preference relations. Reciprocal relations, also known as probabilistic relations, leave no room for incomparability, express indifference in a Boolean way and express strict preference in terms of intensities. We describe properties of fuzzy preference relations in both frameworks, focusing on transitivity-related properties. For reciprocal relations, we explain the cycle-transitivity framework. As the whole exposition makes extensive use of (logical) connectives, such as conjunctors, quasi-copulas and copulas, we provide an appropriate introduction on the topic.

Keywords Fuzzy relation · Preference structure · Transitivity · Reciprocal relation · Cycle-transitivity

5.1 Introduction

Most of the real-world decision problems take place in a complex environment where different forms of incompleteness (such as uncertainty, imprecision, vagueness, partial truth and the like) pervade our knowledge. To face such complexity, an inevitable step is the use of appropriate models of preferences [45, 54].

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The key concept in preference modelling is the notion of a *preference structure*. It represents pairwise comparison in a set of alternatives, and consists of three binary relations expressing strict preference, indifference and incomparability. Nevertheless, the application of two-valued (yes-or-no) preferences, regardless of their sound mathematical theory, is not satisfactory in everyday situations. Therefore, it is desirable to consider a *degree* of preference, which can be represented by fuzzy relations in a natural way.

Thus we face the problem of extending classical preference structures to the fuzzy case. Any proper extension must meet some minimal expectations. For sure, it must allow preference degrees lying anywhere in the unit interval. Roughly speaking, the extension relies mainly on two facts: first, on the right choice of the underlying logical operations (t-norms and t-conorms); second, on the use of an appropriate form of the completeness condition. These are really new features of the extended models since in the Boolean case both the logical operations and the completeness condition are unique.

It has been proved (see [51–53]) that even the above minimal condition is violated unless we use a particular class of t-norms. Within the group of *continuous* t-norms, the only possibility is to use a transform of the Łukasiewicz t-norm [15,53]. This case leads to *additive* fuzzy preference structures, with a rather well-developed theory [11, 26, 27, 52] on functional equations identifying suitable strict preference, indifference and incomparability generators.

We reconsidered the construction of additive fuzzy preference structures [1], by starting from the minimal definition of an additive fuzzy preference structure. We have shown that a given additive fuzzy preference structure is not necessarily the result of applying monotone generators to a large preference relation. In order to cover all additive fuzzy preference structures, we therefore start all over again, looking for the most general strict preference, indifference and incomparability generators. We pinpoint the central role of the indifference generator and clarify that the monotonicity of a generator triplet is totally determined by using a commutative quasi-copula as indifference generator.

Reciprocal relations, satisfying $Q(a,b) + Q(b,a) = 1$, provide another popular tool for expressing the result of the pairwise comparison of a set of alternatives [5] and appear in various fields such as game theory [22] and mathematical psychology [24]. Reciprocal relations are particularly popular in fuzzy set theory where they are used for representing intensities of preference [4, 34]. Compared to additive fuzzy preference structures, however, they leave no room for incomparability.

In the context of preference modelling, transitivity is always an interesting, often desirable property. In fuzzy relational calculus, the notion of T-transitivity is indispensable. Some types of transitivity have been devised specifically for reciprocal relations, such as various types of stochastic transitivity. Although formally reciprocal relations can be seen as a special kind of fuzzy relations, they are not equipped with the same semantics. One should therefore be careful in considering T-transitivity for reciprocal relations, as well as in studying stochastic transitivity of fuzzy relations.

Recently, two general frameworks for studying the transitivity of reciprocal relations have been established, both encompassing various types of T-transitivity and stochastic transitivity. The first framework is that of FG-transitivity, developed by Switalski [48], and is oriented towards reciprocal relations, although formally (but maybe not in a meaningful way) applicable to fuzzy relations. The second framework was developed in [10] and is restricted to reciprocal relations only. For various reasons, this framework has been coined the cycle-transitivity framework.

All the above issues are touched upon in the present chapter. First we summarize some necessary notions and results on fuzzy and probabilistic connectives such as t-norms, t-conorms and (quasi-)copulas. In Section 5.3 we present the basics of fuzzy relations, including their fundamental properties and particular classes such as fuzzy equivalence relations and weak orders. Then we summarize an axiomatic approach to fuzzy preference structures. Closing the section, we show how to build up fuzzy preference structures by the help of an indifference generator and (quasi-)copulas. In Section 5.4 reciprocal relations are introduced. The cycle-transitivity framework is established and studied in considerable detail. Random variables are compared on the basis of winning probabilities, which are shown to be characterizable in the cycle-transitivity framework. They key role played by copulas for (artificially) coupling random variables is emphasized. We conclude by explaining how also mutual ranking probabilities in partially ordered sets fit into this view.

5.2 Fuzzy and Probabilistic Connectives

It is essential to have access to suitable operators for combining the degrees of preference. In this paper, we are mainly interested in two classes of operators: the class of t-norms [37] and the class of (quasi-) copulas [31, 41].

Definition 5.1. A binary operation $T : [0,1]^2 \to [0,1]$ is called a *t-norm* if it satisfies:

(i) Neutral element 1: $(\forall x \in [0,1]) \quad (T(x,1) = T(1,x) = x)$.
(ii) Monotonicity: T is increasing in each variable.
(iii) Commutativity: $(\forall (x,y) \in [0,1]^2) \quad (T(x,y) = T(y,x))$.
(iv) Associativity: $(\forall (x,y,z) \in [0,1]^3) \quad (T(x,T(y,z)) = T(T(x,y),z))$.

The three prototypes of t-norms are the minimum $T_{\mathbf{M}}(x,y) = \min(x,y)$, the product $T_{\mathbf{P}}(x,y) = xy$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. The first one is idempotent, $T_{\mathbf{P}}$ is strict, while $T_{\mathbf{L}}$ is nilpotent.

The following parametric family of t-norms play a key role in fuzzy preference structures. Consider a number $s \in]0,1[\cup]1,\infty[$, and define a binary operation $T_s^{\mathbf{F}}$ on $[0,1]$ by

$$T_s^{\mathbf{F}}(x,y) = \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right).$$

Thus defined $T_s^{\mathbf{F}}$ is a t-norm for the considered parameter values s. Taking the limits in the remaining cases, we get

$$\begin{aligned}\lim_{s\to 0} T_s^{\mathbf{F}}(x,y) &= \min(x,y),\\ \lim_{s\to 1} T_s^{\mathbf{F}}(x,y) &= xy,\\ \lim_{s\to \infty} T_s^{\mathbf{F}}(x,y) &= \max(x+y-1,0).\end{aligned}$$

Thus, we employ also the following notations: $T_{\mathbf{0}}^{\mathbf{F}} = T_{\mathbf{M}}$, $T_{\mathbf{1}}^{\mathbf{F}} = T_{\mathbf{P}}$, $T_{\infty}^{\mathbf{F}} = T_{\mathbf{L}}$.

The parametric family $\left(T_s^{\mathbf{F}}\right)_{s\in[0,\infty]}$ is called the Frank t-norm family, after the author of [29]. Notice that members are positive (i.e., $T_s^{\mathbf{F}}(x,y) > 0$ when $x,y > 0$) for $0 \le s < \infty$, while $T_{\infty}^{\mathbf{F}} = T_{\mathbf{L}}$ has zero divisors (i.e., there are positive x,y such that $T_{\infty}^{\mathbf{F}}(x,y) = 0$).

T-conorms are the dual operations of t-norms, in the sense that for a given t-norm T, the operation $S : [0,1]^2 \to [0,1]$ defined by

$$S(x,y) = 1 - T(1-x, 1-y),$$

is a t-conorm. Formally, the only difference between t-conorms and t-norms is that the former have neutral element 0, while the latter have neutral element 1.

Concerning the duals of the prototypes, we have $S_{\mathbf{M}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x + y - xy$, and $S_{\mathbf{L}}(x,y) = \min(x+y,1)$.

The t-conorm family consisting of duals of members of the Frank t-norm family is called the Frank t-conorm family: $\left(S_s^{\mathbf{F}}\right)_{s\in[0,\infty]}$. Corresponding pairs $\left(T_s^{\mathbf{F}}, S_s^{\mathbf{F}}\right)$ are ordinally irreducible solutions of the Frank equation:

$$T(x,y) + S(x,y) = x + y.$$

For more details see [29].

Now we turn to the probabilistic connectives quasi-copulas and copulas.

Definition 5.2. A binary operation $C : [0,1]^2 \to [0,1]$ is called a *quasi-copula* if it satisfies:

(i) Neutral element 1: $(\forall x \in [0,1])(C(x,1) = C(1,x) = x)$.
(i′) Absorbing element 0: $(\forall x \in [0,1])(C(x,0) = C(0,x) = 0)$.
(ii) Monotonicity: C is increasing in each variable.
(iii) 1-Lipschitz property: $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$(|C(x_1,y_1) - C(x_2,y_2)| \le |x_1 - x_2| + |y_1 - y_2|).$$

If instead of (iii), C satisfies

(iv) Moderate growth: $(\forall (x_1,x_2,y_1,y_2) \in [0,1]^4)$

$$(x_1 \leq x_2 \ \wedge \ y_1 \leq y_2) \quad \text{imply}$$
$$C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1),$$

then it is called a *copula*.

Note that in case of a quasi-copula condition (i′) is superfluous. For a copula, condition (ii) can be omitted (as it follows from (iv) and (i′)). As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true. It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. Finally, note that for any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$, where $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ is the Łukasiewicz t-norm and $T_{\mathbf{M}}(x, y) = \min(x, y)$ is the minimum operator.

We consider a *continuous De Morgan triplet* (T, S, N) on [0, 1], consisting of a continuous t-norm T, a strong negation N (i.e., a decreasing involutive permutation of [0, 1]) and the N-dual t-conorm S defined by

$$S(x, y) = N(T(N(x), N(y))).$$

Note that a strong negation is uniquely determined by the corresponding automorphism ϕ of the unit interval, $N_\phi(x) := \phi^{-1}(1 - \phi(x))$.

For a t-norm T that is at least left-continuous,

$$I_T(x, y) = \sup\{u \in [0, 1] \mid T(x, u) \leq y\}$$

denotes the unique residual implication (R-implication) of T. This operation plays an important role in Section 5.3.1.

5.3 Fuzzy Preference Structures

Binary relations, especially different kinds of orderings and equivalence relations, play a central role in various fields of science such as decision making, measurement theory and social sciences. Fuzzy logics provide a natural framework for extending the concept of crisp binary relations by assigning to each ordered pair of elements a number from the unit interval – the strength of the link between the two elements. This idea was already used in the first paper on fuzzy sets by Zadeh [55].

In the whole section we assume that A is a given set and (T, S, N) is a continuous De Morgan triplet interpreting logical operations **AND**, **OR** and **NOT**, respectively.

5.3.1 Fuzzy Relations

Fuzzy relations are introduced naturally in the following way.

Definition 5.3. A *binary fuzzy relation* R on the set A is a function $R : A \times A \to [0,1]$.

That is, R is a fuzzy subset of $A \times A$. For any $a, b \in A$ the value $R(a,b)$ is understood as the degree to which the elements a and b are in relation.

For a given $\lambda \in [0,1]$ the crisp relation R_λ is defined as the set of ordered pairs with values not less than λ:

$$R_\lambda = \{(a,b) \in A^2 \mid R(a,b) \geq \lambda\}.$$

These λ*-cuts* R_λ form a chain (a nested family) of relations.

The *complement* $\mathrm{co}_N\, R$, the *converse* R^t and the *dual* R^d of a given fuzzy relation R are defined as follows ($a, b \in A$):

$$\mathrm{co}_N\, R(a,b) := N(R(a,b)), \quad R^t(a,b) := R(b,a), \quad R^d(a,b) := N(R(b,a)).$$

Notice that $R^d = \mathrm{co}_N\, R^t = (\mathrm{co}_N\, R)^t$.

For two binary fuzzy relations R and Q on A, we can define their T-intersection $R \cap_T Q$ and S-union $R \cup_S Q$ as follows:

$$(R \cap_T Q)(a,b) := T(R(a,b), Q(a,b)),$$
$$(R \cup_S Q)(a,b) := S(R(a,b), Q(a,b)).$$

Since we deal only with binary fuzzy relations, we often omit the adjective and simply write fuzzy relation.

5.3.1.1 Properties of Fuzzy Relations

In this section we consider and explain the most basic properties of fuzzy relations.

Definition 5.4. A binary fuzzy relation R on A is called

- *reflexive* if $R(a,a) = 1$ for all $a \in A$;
- *irreflexive* if $R(a,a) = 0$ for all $a \in A$;
- *symmetric* if $R(a,b) = R(b,a)$ for all $a, b \in A$.

If a fuzzy relation R on A is reflexive (irreflexive) then all crisp relations R_λ are reflexive (irreflexive for $\lambda \in]0,1]$). It is obvious that R is irreflexive if and only if R^d is reflexive, which holds if and only if $\mathrm{co}_N\, R$ is reflexive.

Definition 5.5. A fuzzy relation R on A is called

- T*-asymmetric* if $T(R(a,b), R(b,a)) = 0$ holds for all $a, b \in A$;
- T*-antisymmetric* if $T(R(a,b), R(b,a)) = 0$ holds for all $a, b \in A$ such that $a \neq b$.

One can prove easily that if a fuzzy relation R on A is $T_\mathbf{M}$-antisymmetric ($T_\mathbf{M}$-asymmetric) then its cut relations R_λ are antisymmetric (asymmetric) crisp relations for $\lambda \in]0,1]$.

Obviously, if a fuzzy relation R on A is T-antisymmetric (T-asymmetric) for a certain t-norm T, then R is also T'-antisymmetric (T'-asymmetric) for any t-norm T' such that $T' \leq T$. Therefore, a $T_\mathbf{M}$-antisymmetric relation is T-antisymmetric for any t-norm T.

If T is a positive t-norm then a fuzzy relation R is T-antisymmetric (T-asymmetric) if and only if R is $T_\mathbf{M}$-antisymmetric ($T_\mathbf{M}$-asymmetric). For positive T, T-asymmetry implies irreflexivity.

Remarkable new definitions (that is, different from the case $T_\mathbf{M}$) can only be obtained by using t-norms with zero divisors. In such cases both values $R(a,b)$ and $R(b,a)$ can be positive, but cannot be too high simultaneously. More exactly, if one considers the Łukasiewicz t-norm $T_\mathbf{L}(x,y) = \max\{x+y-1, 0\}$, $T_\mathbf{L}$-antisymmetry ($T_\mathbf{L}$-asymmetry) of R is equivalent to the following inequality: $R(a,b) + R(b,a) \leq 1$.

Definition 5.6. A fuzzy relation R on A is called

- *strongly S-complete* if $S(R(a,b), R(b,a)) = 1$ for all $a, b \in A$;
- *S-complete* if $S(R(a,b), R(b,a)) = 1$ for all $a, b \in A$ such that $a \neq b$.

Obviously, if R is S-complete (strongly S-complete) on A then it is S'-complete (strongly S'-complete) on A for any t-conorm S' such that $S' \geq S$.

Since (T, S, N) is a De Morgan triplet, S-completeness and T-antisymmetry (strong S-completeness and T-asymmetry) are dual properties. That is, a fuzzy relation R on A is S-complete (strongly S-complete) if and only if its dual R^d is T-antisymmetric (T-asymmetric) on A. Using duality, it is easy to prove that when T is a positive t-norm in the De Morgan triplet (T, S, N) then a fuzzy relation R on A is S-complete (strongly S-complete) if and only if R is $S_\mathbf{M}$-complete (strongly $S_\mathbf{M}$-complete) on A. Strong S-completeness implies reflexivity if and only if T is a positive t-norm.

If a fuzzy relation R on A is $S_\mathbf{M}$-complete (strongly $S_\mathbf{M}$-complete) then its cut relations R_λ are complete (strongly complete) crisp binary relations.

Now we turn to transitivity, which is certainly one of the most important properties concerning either equivalences or different types of orders. Since classical transitivity can be introduced by using the composition of relations, first we define the corresponding notion of T-composition for binary fuzzy relations.

Definition 5.7. Let R_1, R_2 be fuzzy relations on A. The *T-composition* of R_1 and R_2 is a fuzzy relation denoted as $R_1 \circ_T R_2$, and defined by

$$(R_1 \circ_T R_2)(a,b) = \sup_{c \in A} T(R_1(a,c), R_2(c,b)). \tag{5.1}$$

This definition is natural. Indeed, if Q_1 and Q_2 are crisp binary relations on A then $a(Q_1 \circ Q_2)b$ if and only if there exists an element $c \in A$ such that aQ_1c and cQ_2b. This corresponds to (5.1) in the fuzzy case.

Consider two fuzzy relations R_1, R_2 on A. We say that R_1 is *contained in* R_2 and denote by $R_1 \subseteq R_2$ if and only if for all $a, b \in A$ we have $R_1(a,b) \leq R_2(a,b)$. Fuzzy relations R_1 and R_2 are said to be *equal* if and only if $R_1(a,b) = R_2(a,b)$ for all $a, b \in A$.

It is easy to prove that

(a) $R_1 \circ_T (R_2 \circ_T R_3) = (R_1 \circ_T R_2) \circ_T R_3$
(b) $R_1 \subseteq R_2$ implies $R_1 \circ_T R_3 \subseteq R_2 \circ_T R_3$ and $R_3 \circ_T R_1 \subseteq R_3 \circ_T R_2$

for all fuzzy relations R_1, R_2 and R_3 on A. In other words, composition of fuzzy relations is an associative and increasing operation. For proof see [27].

Turning back to transitivity, the idea behind it is that "the strength of the link between two elements must be greater than or equal to the strength of any indirect chain (i.e., involving other elements)", see [21]. This is expressed in the following definition (see also [56]).

Definition 5.8. A fuzzy relation R on A is called T*-transitive* if

$$T(R(a,c), R(c,b)) \leq R(a,b) \tag{5.2}$$

holds for all $a, b, c \in A$.

General representation theorems of T-transitive fuzzy relations have been established in [28]. One of those is recalled now.

Theorem 5.1. *[28] Let R be a fuzzy relation on A. Then R is T-transitive if and only if there exist two families $\{f_\gamma\}_{\gamma \in \Gamma}$, $\{g_\gamma\}_{\gamma \in \Gamma}$ of functions from A to* $[0,1]$ *such that $f_\gamma(a) \geq g_\gamma(a)$ for all $a \in A$, $\gamma \in \Gamma$ and*

$$R(a,b) = \inf_{\gamma \in \Gamma} I_T(f_\gamma(a), g_\gamma(b)). \tag{5.3}$$

It is easy to see that if R is a $T_\mathbf{M}$-transitive fuzzy relation on A then each λ-cut of R is a transitive relation for $\lambda \in]0,1]$.

Negative S-transitivity is the dual concept of T-transitivity and vice versa. Therefore, only some main points are explained in detail. The others can be obtained by corresponding results on T-transitivity.

Definition 5.9. A fuzzy relation R on A is called *negatively* S*-transitive* if $R(a,b) \leq S(R(a,c), R(c,b))$ for all $a, b, c \in A$.

It is easily seen that a fuzzy relation R on A is negatively S-transitive if and only if its dual R^d is T-transitive, where (T, S, N) is still a De Morgan triplet.

Suppose that R is negatively S-transitive for a given S. Then R is negatively S'-transitive for any t-conorm S' such that $S' \geq S$. In particular, a negatively $S_\mathbf{M}$-transitive relation is negatively S'-transitive for any t-conorm S'.

If R is strongly S-complete and T-transitive on A then R is negatively S-transitive on A (for the proof see [27]).

5.3.1.2 Special Types of Fuzzy Relations

Fuzzy relations that are reflexive and T-transitive are called *fuzzy preorders* with respect to T, short *T-preorders*. Symmetric T-preorders are called *fuzzy equivalence relations* with respect to T, short *T-equivalences*. Note that the term *T-similarity relation* is also used for a T-equivalence relation. Similarity relations have been introduced and investigated by Zadeh [56] (see also [43, 44]).

The following result is a characterization and representation of T-equivalence relations (published under the term "indistinguishability" instead of equivalence, see [50]). Compare also with Theorem 5.1.

Theorem 5.2. [50] *Let R be a binary fuzzy relation on A. Then R is a T-equivalence relation on A if and only if there exists a family $\{h_\gamma\}_{\gamma\in\Gamma}$ of functions from A to $[0,1]$ so that for all $a,b \in A$*

$$R(a,b) = \inf_{\gamma\in\Gamma} I_T(\max\{h_\gamma(a), h_\gamma(b)\}, \min\{h_\gamma(a), h_\gamma(b)\}), \tag{5.4}$$

where I_T is the R-implication defined by T.

Equivalence classes of a T-equivalence relation consist of elements being close to each other, and formally are defined as follows. Let R be a T-equivalence relation on A. For any given $a \in A$, an *equivalence class* of a is a fuzzy set $R[a] : A \to [0,1]$ defined by $R[a](c) = R(a,c)$ for all $c \in A$. It may happen that $R[a] = R[b]$ for different elements $a, b \in A$. It is easy to verify that $R[a] = R[b]$ holds if and only if $R(a,b) = 1$. Each λ-cut of a fuzzy $T_{\mathbf{M}}$-equivalence relation is a crisp equivalence relation, as one can check it easily.

Strongly complete T-preorders are called *fuzzy weak orders* with respect to T, short *weak T-orders*.

Given a T-equivalence $E : X^2 \to [0,1]$, a binary fuzzy relation $L : X^2 \to [0,1]$ is called a *fuzzy order* with respect to T and E, short *T-E-order*, if it is T-transitive and additionally has the following two properties:

- *E-reflexivity:* $E(x,y) \le L(x,y)$ for all $x, y \in X$
- *T-E-antisymmetry:* $T(L(x,y), L(y,x)) \le E(x,y)$ for all $x, y \in X$

We are ready to state the first – score function-based – representation theorem of weak T-orders. For more details and proofs see [3].

Theorem 5.3. [3] *A binary fuzzy relation $R : X^2 \to [0,1]$ is a weak T-order if and only if there exists a non-empty domain Y, a T-equivalence $E : Y^2 \to [0,1]$, a strongly $S_{\mathbf{M}}$-complete T-E-order $L : Y^2 \to [0,1]$ and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:*

$$R(x,y) = L(f(x), f(y)). \tag{5.5}$$

This result is an extension of the following well-known classical theorem.

Theorem 5.4. *A binary relation $\lesssim$ on a non-empty domain X is a weak order if and only if there exists a linearly ordered non-empty set $(Y, \preceq)$ and a mapping $f : X \to Y$ such that $\lesssim$ can be represented in the following way for all $x, y \in X$:*

$$x \lesssim y \quad \textit{if and only if} \quad f(x) \preceq f(y). \tag{5.6}$$

The standard crisp case consists of the unit interval [0, 1] equipped with its natural linear order. Given a left-continuous t-norm T, the canonical fuzzification of the natural linear order on [0, 1] is the residual implication I_T [2,32,33]. The following proposition, therefore, provides us with a construction that can be considered as a straightforward counterpart of (5.6).

Proposition 5.1. *Given a function $f : X \to [0, 1]$, the binary fuzzy relation $R : X^2 \to [0, 1]$ defined by*

$$R(x, y) = I_T\big(f(x), f(y)\big) \tag{5.7}$$

is a weak T-order.

The function f is called a *score function*. Note that there are weak T-orders that cannot be represented by means of a single score function [3]. Therefore, a weak T-order $R : X^2 \to [0, 1]$ is called *representable* if there exists a function $f : X \to [0, 1]$, called *generating score function*, such that Eq. 5.7 holds. A representable weak $T_{\mathbf{M}}$-order is called *Gödel-representable* [12]. The following result is a unique characterization of representable fuzzy weak orders for continuous t-norms.

Theorem 5.5. [3] *Assume that T is continuous. Then a weak T-order R is representable if and only if the following function is a generating score function of R:*

$$\bar{f}(x) = \inf_{z \in X} R(z, x).$$

The following well-known theorem shows that fuzzy weak orders can be represented by more than one score function. Compare also with Theorems 5.1 and 5.2.

Theorem 5.6. [50] *Consider a binary fuzzy relation $R : X^2 \to [0, 1]$. Then the following two statements are equivalent:*

(i) *R is a T-preorder.*
(ii) *There exists a non-empty family of $X \to [0, 1]$ score functions $(f_i)_{i \in I}$ such that the following representation holds:*

$$R(x, y) = \inf_{i \in I} I_T(f_i(x), f_i(y)). \tag{5.8}$$

Now the following theorem provides us with a unique characterization of weak T-orders.

Theorem 5.7. [3] *Consider a binary fuzzy relation* $R : X^2 \to [0,1]$. *Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp weak order $\lesssim$ *and a non-empty family of* $X \to [0,1]$ *score functions* $(f_i)_{i \in I}$ *that are non-decreasing with respect to* $\lesssim$ *such that representation (5.8) holds.*

The following theorem finally characterizes weak T-orders as intersections of representable weak T-orders that are generated by score functions that are monotonic at the same time with respect to the same crisp linear order.

Theorem 5.8. [3] *Consider a binary fuzzy relation* $R : X^2 \to [0,1]$. *Then the following two statements are equivalent:*

(i) R is a weak T-order.
(ii) There exists a crisp linear order $\preceq$ *and a non-empty family of* $X \to [0,1]$ *score functions* $(f_i)_{i \in I}$ *that are non-decreasing with respect to* $\preceq$ *such that representation (5.8) holds.*

The interested reader can find further representation and construction results in [3]. This includes inclusion-based representations, and representations by decomposition into crisp linear orders and fuzzy equivalence relations, which also facilitates a pseudo-metric-based construction.

5.3.2 Additive Fuzzy Preference Structures: Bottom-Up Approach

5.3.2.1 Classical Preference Structures

Consider a set of alternatives A and suppose that a decision maker wants to judge these alternatives by pairwise comparison. Given two alternatives, the decision maker can act in one of the following three ways:

(i) He/she clearly prefers one to the other.
(ii) The two alternatives are indifferent to him/her.
(iii) He/she is unable to compare the two alternatives.

According to these cases, three binary relations can be defined in A: the *strict preference relation* P, the *indifference relation* I and the *incomparability relation* J. For any $(a,b) \in A^2$, we classify:

$$
\begin{aligned}
(a,b) \in P \quad &\Leftrightarrow \quad \text{he/she prefers } a \text{ to } b;\\
(a,b) \in I \quad &\Leftrightarrow \quad a \text{ and } b \text{ are indifferent to him/her;}\\
(a,b) \in J \quad &\Leftrightarrow \quad \text{he/she is unable to compare } a \text{ and } b.
\end{aligned}
$$

One easily verifies that the triplet (P, I, J) defined above satisfies the conditions formulated in the following definition of a preference structure. For a binary relation R in A, we denote its *converse* by R^t and its *complement* by co R.

Definition 5.10. [45] A preference structure on A is a triplet (P, I, J) of binary relations in A that satisfy:

(B1) P is irreflexive, I is reflexive and J is irreflexive.
(B2) P is asymmetrical, I is symmetrical and J is symmetrical.
(B3) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$.
(B4) $P \cup P^t \cup I \cup J = A^2$.

This definition is exhaustive: it lists all properties of the components P, I and J of a preference structure. The asymmetry of P can also be written as $P \cap P^t = \emptyset$. Condition (B4) is called the *completeness condition* and can be expressed equivalently (up to symmetry) in the following alternative ways: $\text{co}\,(P \cup I) = P^t \cup J$, $\text{co}\,(P \cup P^t) = I \cup J$, $\text{co}\,(P \cup P^t \cup I) = J$, $\text{co}\,(P \cup P^t \cup J) = I$ and $\text{co}\,(P^t \cup I \cup J) = P$.

It is possible to associate a single reflexive relation to any preference structure so that it completely characterizes this structure. A preference structure (P, I, J) on A is characterized by the reflexive binary relation $R = P \cup I$, its large preference relation, in the following way:

$$(P, I, J) = (R \cap \text{co } R^t, R \cap R^t, \text{co } R \cap \text{co } R^t).$$

Conversely, a triplet (P, I, J) constructed in this way from a reflexive binary relation R in A is a preference structure on A. The interpretation of the large preference relation is

$$(a, b) \in R \quad \Leftrightarrow \quad b \text{ is considered at most as good as } a.$$

The above definition of a preference structure can be written in the following minimal way, identifying a relation with its characteristic mapping [13]: I is reflexive and symmetrical, and for any $(a, b) \in A^2$:

$$P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1.$$

Thus, classical preference structures can also be considered as *Boolean preference structures*, employing 1 and 0 for describing presence or absence of strict preferences, indifferences and incomparabilities. Complement, intersection and union then correspond to Boolean negation, conjunction (i.e. minimum) and disjunction (i.e. maximum) on characteristic mappings.

5.3.2.2 The Quest for Fuzzy Preference Structures: The Axiomatic Approach

As preference structures are based on classical set theory and are therefore restricted to two-valued relations, they do not allow to express degrees of strict preference, indifference or incomparability. This is seen as an important drawback to their practical use, leading researchers already at an early stage to the theory of fuzzy sets, and in particular to the calculus of fuzzy relations. In that case, preference degrees are expressed on the continuous scale [0, 1] and operations from fuzzy logic are used for manipulating these degrees.

A *fuzzy preference structure (FPS)* on A is a triplet (P, I, J) of binary fuzzy relations in A satisfying:

(F1) P is irreflexive, I is reflexive and J is irreflexive.
(F2) P is T-asymmetrical, I and J are symmetrical.
(F3) $P \cap_T I = \emptyset$, $P \cap_T J = \emptyset$ and $I \cap_T J = \emptyset$.
(F4) a completeness condition, such as $\text{co}_N\,(P \cup_S I) = P^t \cup_S J$, $\text{co}_N\,(P \cup_S P^t \cup_S I) = J$ or $P \cup_S P^t \cup_S I \cup_S J = A^2$.

Invoking the **assignment principle:** *for any pair of alternatives* (a, b) *the decision maker is allowed to assign at least one of the degrees* $P(a, b)$, $P(b, a)$, $I(a, b)$ *and* $J(a, b)$ *freely in the unit interval*, shows that only a *nilpotent t-norm* T is acceptable, i.e. a ϕ'-transform of the Łukasiewicz t-norm: $T(x, y) := \phi'^{-1}(\max(\phi'(x) + \phi'(y) - 1, 0))$ [52]. For the sake of simplicity, we consider $\phi = \phi'$. Consequently, we will be working with a Łukasiewicz triplet $\left(T_\phi^\infty, S_\phi^\infty, N_\phi\right)$. The latter notation is used to indicate that the Łukasiewicz t-norm belongs to the Frank t-norm family $(T^s)_{s\in[0,\infty]}$ (which is also a family of copulas) and corresponds to the parameter value $s = \infty$ (note that the minimum operator and the algebraic product correspond to the parameter values $s = 0$ and $s = 1$, respectively). Moreover, in that case, the completeness conditions $\text{co}_\phi\left(P \cup_\phi^\infty I\right) = P^t \cup_\phi^\infty J$ and $\text{co}_\phi\left(P \cup_\phi^\infty P^t\right) = I \cup_\phi^\infty J$ become equivalent and turn out to be stronger than the other completeness conditions, with $P \cup_\phi^\infty P^t \cup_\phi^\infty I \cup_\phi^\infty J = A^2$ as weakest condition [52]. Restricting to the **strongest completeness condition(s)**, we then obtain the following definition.

Definition 5.11. Given a [0, 1]-automorphism ϕ, a ϕ-FPS (a ϕ-fuzzy preference structure) on A is a triplet of binary fuzzy relations (P, I, J) in A satisfying:

(F1) P is irreflexive, I is reflexive and J is irreflexive.
(F2) P is T_ϕ^∞-asymmetrical, I and J are symmetrical.
(F3) $P \cap_\phi^\infty I = \emptyset$, $P \cap_\phi^\infty J = \emptyset$ and $I \cap_\phi^\infty J = \emptyset$.
(F4) $\text{co}_\phi\left(P \cup_\phi^\infty I\right) = P^t \cup_\phi^\infty J$.

Moreover, a minimal formulation of this definition, similar to the classical one, exists: a triplet (P, I, J) of binary fuzzy relations in A is a ϕ-FPS on A if and only if I is reflexive and symmetrical, and for any $(a, b) \in A^2$:

$$\phi(P(a, b)) + \phi(P^t(a, b)) + \phi(I(a, b)) + \phi(J(a, b)) = 1.$$

In view of the above equality, fuzzy preference structures are also called *additive fuzzy preference structures*.

Axiomatic Constructions

Again choosing a continuous de Morgan triplet (T, S, N), we could transport the classical construction formalism to the fuzzy case and define, given a reflexive binary fuzzy relation R in A:

$$(P, I, J) = (R \cap_T \text{co}_N\ R^t, R \cap_T R^t, \text{co}_N\ R \cap_T \text{co}_N\ R^t).$$

At the same time, we want to keep R as the fuzzy large preference relation of the triplet (P, I, J), i.e. $R = P \cup_S I$ and $\text{co}_N R = P^t \cup_S J$. Fodor and Roubens observed that the latter is not possible in general, and proposed four axioms for defining fuzzy strict preference, indifference and incomparability relations [26, 27]. According to the first axiom, the *independence of irrelevant alternatives*, there exist three $[0,1]^2 \to [0,1]$ mappings p, i, j such that $P(a,b) = p(R(a,b), R(b,a))$, $I(a,b) = i(R(a,b), R(b,a))$ and $J(a,b) = j(R(a,b), R(b,a))$. The second and third axioms state that the mappings $p(x, N(y))$, $i(x, y)$ and $j(N(x), N(y))$ are increasing in both x and y, and that i and j are symmetrical. The fourth and main axiom requires that $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$, or explicitly, for any $(x, y) \in [0,1]^2$:

$$S(p(x,y), i(x,y)) = x,$$
$$S(p(x,y), j(x,y)) = N(y).$$

The latter axiom implies that $\text{co}_N (P \cup_S I) = P^t \cup_S J$, i.e. the first completeness condition.

Theorem 5.9. [26,27] *If (T, S, N) and (p, i, j) satisfy the above axioms, then there exists a $[0,1]$-automorphism ϕ such that*

$$(T, S, N) = \left(T_\phi^\infty, S_\phi^\infty, N_\phi\right)$$

and, for any $(x, y) \in [0,1]^2$:

$$\begin{aligned} T_\phi^\infty(x, N_\phi(y)) \leq &\quad p(x,y) &\leq T^0(x, N_\phi(y)), \\ T_\phi^\infty(x, y) \leq &\quad i(x,y) &\leq T^0(x, y), \\ T_\phi^\infty(N_\phi(x), N_\phi(y)) \leq &\quad j(x,y) &\leq T^0(N_\phi(x), N_\phi(y)). \end{aligned}$$

Moreover, for any reflexive binary fuzzy relation R in A, the triplet (P, I, J) of binary fuzzy relations in A defined by

$$\begin{aligned} P(a,b) &= p(R(a,b), R(b,a)), \\ I(a,b) &= i(R(a,b), R(b,a)), \\ J(a,b) &= j(R(a,b), R(b,a)) \end{aligned}$$

is a ϕ-FPS on A such that $R = P \cup_\phi^\infty I$ and $\text{co}_\phi\ R = P^t \cup_\phi^\infty J$.

Although in general the function i is of two variables, and there is no need to extend it for more than two arguments, it might be a t-norm. The following theorem states that the only construction methods of the above type based on continuous t-norms are the ones using two Frank t-norms with reciprocal parameters.

Theorem 5.10. [26, 27] *Consider a* $[0,1]$*-automorphism* ϕ *and two continuous t-norms* T_1 *and* T_2*. Define* p *and* i *by* $p(x,y) = T_1(x, N_\phi(y))$ *and* $i(x,y) = T_2(x,y)$*. Then* (p,i,j) *satisfies the above axioms if and only if there exists a parameter* $s \in [0,\infty]$ *such that* $T_1 = T_\phi^{1/s}$ *and* $T_2 = T_\phi^s$*. In this case, we have that* $j(x,y) = i(N_\phi(x), N_\phi(y))$*.*

Summarizing, we have that for any reflexive binary fuzzy relation R in A the triplets

$$(P_s, I_s, J_s) = \left(R \cap_\phi^{\frac{1}{s}} \mathrm{co}_\phi\ R^t, R \cap_\phi^s R^t, \mathrm{co}_\phi\ R \cap_\phi^s \mathrm{co}_\phi\ R^t\right),$$

with $s \in [0,\infty]$, are the only t-norm-based constructions of fuzzy preference structures that satisfy $R = P \cup_\phi^\infty I$ and $\mathrm{co}_\phi\ R = P^t \cup_\phi^\infty J$. Consequently, R is again called the *large preference relation*. Note that

$$\phi(R(a,b)) = \phi(P(a,b)) + \phi(I(a,b)).$$

In fact, in [27] it was only shown that ordinal sums of Frank t-norms should be used. For the sake of simplicity, only the ordinally irreducible ones were considered. However, we can prove that this is the only option.

Finally, we deal with the *reconstruction* of a ϕ-FPS from its large preference relation. As expected, an additional condition is required. A ϕ-FPS (P, I, J) on A is called:

(i) an (s,ϕ)-FPS, with $s \in \{0, 1, \infty\}$, if $P \cap_\phi^s P^t = I \cap_\phi^{\frac{1}{s}} J$;
(ii) an (s,ϕ)-FPS, with $s \in]0,1[\cup]1,\infty[$, if

$$s^{\phi\left(P \cap_\phi^s P^t\right)} + s^{-\phi\left(I \cap_\phi^{1/s} J\right)} = 2.$$

One can verify that the triplet (P_s, I_s, J_s) constructed above is an (s,ϕ)-FPS. Moreover, any (s,ϕ)-FPS can be reconstructed from its large preference relation by means of the corresponding construction. The characterizing condition of a $(0,\phi)$-FPS, respectively (∞,ϕ)-FPS, can also be written as $P \cap^0 P^t = \emptyset$, i.e. $\min(P(a,b), P(b,a)) = 0$ for any (a,b), respectively. $I \cap^0 J = \emptyset$, i.e. $\min(I(a,b), J(a,b)) = 0$ for any (a,b).

5.3.2.3 Additive Fuzzy Preference Structures and Indifference Generators

Now we reconsider the construction of additive fuzzy preference structures, not by rephrasing the conclusions resulting from an axiomatic study, but by starting from the minimal definition of an additive fuzzy preference structure. For the sake of brevity, we consider the case $\phi(x) = x$. For motivation and more details we refer to [1].

Definition 5.12. A triplet (p, i, j) of $[0, 1]^2 \to [0, 1]$ mappings is called a *generator triplet* compatible with a continuous t-conorm S and a strong negator N if and only if for any reflexive binary fuzzy relation R on a set of alternatives A it holds that the triplet (P, I, J) of binary fuzzy relations on A defined by:

$$P(a,b) = p(R(a,b), R(b,a)),$$

$$I(a,b) = i(R(a,b), R(b,a)),$$

$$J(a,b) = j(R(a,b), R(b,a))$$

is a FPS on A such that $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$.

The above conditions $P \cup_S I = R$ and $P^t \cup_S J = \text{co}_N R$ require the reconstructability of the fuzzy large preference relation R from the fuzzy preference structure it generates. The following theorem expresses that for that purpose only nilpotent t-conorms can be used.

Theorem 5.11. *If (p, i, j) is a generator triplet compatible with a continuous t-conorm S and a strong negator $N = N_\phi$, then $S = S^\infty_\psi$, i.e. S is nilpotent.*

Let us again consider the case $\psi(x) = x$. The above theorem implies that we can omit the specification "compatible with a continuous t-conorm S and strong negation N" and simply talk about generator triplets. The minimal definition of a fuzzy preference structure then immediately leads to the following proposition.

Proposition 5.2. *A triplet (p, i, j) is a generator triplet if and only if, for any $(x, y) \in [0, 1]^2$:*

(i) $i(1, 1) = 1$
(ii) $i(x, y) = i(y, x)$
(iii) $p(x, y) + p(y, x) + i(x, y) + j(x, y) = 1$
(iv) $p(x, y) + i(x, y) = x$

From this proposition it follows that a generator triplet is uniquely determined by, for instance, the generator i. Indeed, for any generator triplet (p, i, j) it holds that

$$p(x, y) = x - i(x, y),$$

$$j(x, y) = i(x, y) - (x + y - 1).$$

The fact that p and j take values in $[0, 1]$ implies that $T^\infty \leq i \leq T^0$. Moreover, from any symmetrical i such that $T^\infty \leq i \leq T^0$ a generator triplet can be built. It is therefore not surprising that additional properties of generator triplets (p, i, j) are completely determined by additional properties of i. In fact, in practice, it would be sufficient to talk about a single generator i. We could simply talk about *the generator* of the FPS. Note that the symmetry of i implies the symmetry of j.

Firstly, we try to characterize generator triplets fitting into the axiomatic framework of Fodor and Roubens.

Definition 5.13. A generator triplet (p, i, j) is called *monotone* if:

(i) p is increasing in the first and decreasing in the second argument.
(ii) i is increasing in both arguments.
(iii) j is decreasing in both arguments.

Inspired by the paper [37], we can show that monotone generator triplets are characterized by a 1-Lipschitz indifference generator, i.e. by a commutative quasi-copula.

Theorem 5.12. *A generator triplet (p, i, j) is monotone if and only if i is a commutative quasi-copula.*

The following theorem shows that when i is a symmetrical ordinal sum of Frank t-norms, $j(1-x, 1-y)$ is also a t-norm and $p(x, 1-y)$ is symmetrical. Note that by symmetrical ordinal sum we mean the following: if (a, b, T) is a summand, then also $(1-b, 1-a, T)$ is a summand.

The associativity of $p(x, 1-y)$, however, can only be guaranteed in case of an ordinally irreducible i, i.e. a Frank t-norm.

Theorem 5.13. *Consider a generator triplet (p, i, j) such that i is a t-norm, then the following statements are equivalent:*

(i) *The mapping $j(1-x, 1-y)$ is a t-norm.*
(ii) *The mapping $p(x, 1-y)$ is symmetrical.*
(iii) *i is a symmetrical ordinal sum of Frank t-norms.*

Theorem 5.14. *Consider a generator triplet (p, i, j) such that i is a t-norm, then the following statements are equivalent:*

(i) *The mapping $p(x, 1-y)$ is a t-norm.*
(ii) *i is a Frank t-norm.*

In the latter case, i.e. when i is a Frank t-norm, say $i = T^s$, $s \in [0, \infty]$, it holds that

$$p(x, y) = T^{1/s}(x, 1-y),$$

$$j(x, y) = T^s(1-x, 1-y).$$

This result closes the loop, and brings us back to the conclusions drawn from the axiomatic study of Fodor and Roubens expressed in Theorem 5.10.

5.4 Reciprocal Preference Relations

5.4.1 Reciprocal Relations

5.4.1.1 Definition

Another interesting class of $A^2 \to [0, 1]$ mappings is the class of *reciprocal relations* Q (also called *ipsodual relations* or *probabilistic relations* [20]) satisfying the condition $Q(a, b) + Q(b, a) = 1$, for any $(a, b) \in A^2$. For such relations, it holds in particular that $Q(a, a) = 1/2$. Many authors like viewing them as particular kinds of fuzzy relations, but we do not adhere to that view, as reciprocal relations are of an inherent bipolar nature. The usual operations (such as intersection, union and composition) on fuzzy relations simply make no sense on reciprocal relations. Not surprisingly then, as will be shown further on, also other notions of transitivity apply to them.

Reciprocity is intimately linked with completeness. Let R be a complete ($\{0, 1\}$-valued) relation on A, i.e. for any $(a, b) \in A^2$ it holds that $\max(R(a, b), R(b, a)) = 1$, then R has an equivalent $\{0, 1/2, 1\}$-valued reciprocal representation Q given by

$$Q(a, b) = \begin{cases} 1, & \text{if } R(a, b) = 1 \text{ and } R(b, a) = 0, \\ 1/2, & \text{if } R(a, b) = R(b, a) = 1, \\ 0, & \text{if } R(a, b) = 0 \text{ and } R(b, a) = 1, \end{cases}$$

or in a more compact arithmetic form:

$$Q(a, b) = \frac{1 + R(a, b) - R(b, a)}{2}. \qquad (5.9)$$

One easily verifies that R is transitive if and only if its reciprocal representation Q satisfies, for any $(a, b, c) \in A^3$:

$$(Q(a, b) \geq 1/2 \ \wedge \ Q(b, c) \geq 1/2) \ \Rightarrow \ Q(a, c) = \max(Q(a, b), Q(b, c)). \qquad (5.10)$$

Reciprocal relations generalize the above representation by taking values also in the intervals $]0, 1/2[$ and $]1/2, 1[$.

5.4.1.2 A Fuzzy Set Viewpoint

In the fuzzy set community (see e.g., [23, 30, 35]), the semantics attributed to a reciprocal relation Q is as follows:

$$Q(a,b) \in \begin{cases}]1/2,1], & \text{if } a \text{ is rather preferred to } b, \\ \{1/2\}, & \text{if } a \text{ and } b \text{ are indifferent,} \\ [0,1/2[, & \text{if } b \text{ is rather preferred to } a. \end{cases}$$

Hence, a reciprocal relation can be seen as a compact representation of an additive fuzzy preference structure in which the indifference relation I is a crisp relation, the incomparability relation J is empty and the strict preference relation P and its converse P^t are fuzzy relations complementing each other. Reciprocal relations are therefore closely related to fuzzy weak orders.

Note that, similarly as for a complete relation, a weakly complete fuzzy relation R on A can be transformed into a reciprocal representation $Q = P + 1/2I$, with P and I the strict preference and indifference components of the additive fuzzy preference structure (P, I, J) generated from R by means of the generator $i = T_{\mathbf{L}}$ [14, 52]:

$$\begin{aligned} P(a,b) &= T_{\mathbf{M}}(R(a,b), 1 - R(b,a)) = 1 - R(b,a), \\ I(a,b) &= T_{\mathbf{L}}(R(a,b), R(b,a)) = R(a,b) + R(b,a) - 1, \\ J(a,b) &= T_{\mathbf{L}}(1 - R(a,b), 1 - R(b,a)) = 0. \end{aligned}$$

Note that the corresponding expression for Q is formally the same as (5.9). This representation is not equivalent to the fuzzy relation R, as many weakly complete fuzzy relations R may have the same representation. Recall that, if the fuzzy relation R is also strongly complete, then the generator i used is immaterial.

5.4.1.3 A Frequentist View

However, the origin of reciprocal relations is not to be found in the fuzzy set community. For several decades, reciprocal relations are used as a convenient tool for expressing the results of the pairwise comparison of a set of alternatives in fields such as game theory [22], voting theory [42] and psychology [20]. A typical use is that where an individual is asked, in a controlled experimental set-up, to compare the same set of alternatives multiple times, where each time he can either prefer alternative a to b or b to a. The fraction of times a is preferred to b then yields $Q(a,b)$. In what follows, we will stay close to that frequentist view. However, we prefer to use the more neutral term reciprocal relation, rather than the term probabilistic relation.

5.4.2 The Cycle-Transitivity Framework

5.4.2.1 Stochastic Transitivity

Transitivity properties for reciprocal relations rather have the conditional flavor of (5.10). There exist various kinds of stochastic transitivity for reciprocal relations [5, 40]. For instance, a reciprocal relation Q on A is called *weakly stochastic transitive* if for any $(a,b,c) \in A^3$ it holds that $Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2$ implies $Q(a,c) \geq 1/2$. In [10], the following generalization of stochastic transitivity was proposed.

Definition 5.14. Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,1/2) \leq 1/2$. A reciprocal relation Q on A is called g-stochastic transitive if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2) \Rightarrow Q(a,c) \geq g(Q(a,b), Q(b,c)).$$

Note that the condition $g(1/2,1/2) \leq 1/2$ ensures that the reciprocal representation Q of any transitive complete relation R is always g-stochastic transitive. In other words, g-stochastic transitivity generalizes transitivity of complete relations. This definition includes the standard types of stochastic transitivity [40]:

(i) *Strong* stochastic transitivity when $g = \max$
(ii) *Moderate* stochastic transitivity when $g = \min$
(iii) *Weak* stochastic transitivity when $g = 1/2$

In [10], also a special type of stochastic transitivity was introduced.

Definition 5.15. Let g be an increasing $[1/2,1]^2 \to [0,1]$ mapping such that $g(1/2,1/2) = 1/2$ and $g(1/2,1) = g(1,1/2) = 1$. A reciprocal relation Q on A is called g-isostochastic transitive if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2) \Rightarrow Q(a,c) = g(Q(a,b), Q(b,c)).$$

The conditions imposed upon g again ensure that g-isostochastic transitivity generalizes transitivity of complete relations. Note that for a given mapping g, the property of g-isostochastic transitivity obviously is much more restrictive than the property of g-stochastic transitivity.

5.4.2.2 *FG*-Transitivity

The framework of FG-transitivity, developed by Switalski [47,51], formally generalizes g-stochastic transitivity in the sense that $Q(a,c)$ is bounded both from below and above by $[1/2,1]^2 \to [0,1]$ mappings.

Definition 5.16. Let F and G be two $[1/2,1]^2 \to [0,1]$ mappings such that $F(1/2,1/2) \leq 1/2 \leq G(1/2,1/2)$ and $G(1/2,1) = G(1,1/2) = G(1,1) = 1$ and $F \leq G$. A reciprocal relation Q on A is called FG-transitive if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2)$$
$$\Downarrow$$
$$F(Q(a,b), Q(b,c)) \leq Q(a,c) \leq G(Q(a,b), Q(b,c)).$$

5.4.2.3 Cycle-Transitivity

Similarly as the FG-transitivity framework, the cycle-transitivity framework involves two bounds. However, these bounds are not independent, and moreover, the arguments are subjected to a reordering before they are applied. More specifically, for a reciprocal relation Q, we define for all $(a,b,c) \in A^3$ the following quantities [10]:

$$\alpha_{abc} = \min(Q(a,b), Q(b,c), Q(c,a)),$$
$$\beta_{abc} = \text{median}(Q(a,b), Q(b,c), Q(c,a)),$$
$$\gamma_{abc} = \max(Q(a,b), Q(b,c), Q(c,a)).$$

Let us also denote $\Delta = \{(x,y,z) \in [0,1]^3 \mid x \leq y \leq z\}$. A function $U : \Delta \to \mathbb{R}$ is called an if it satisfies:

(i) $U(0,0,1) \geq 0$ and $U(0,1,1) \geq 1$;
(ii) for any $(\alpha, \beta, \gamma) \in \Delta$:

$$U(\alpha,\beta,\gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geq 1. \tag{5.11}$$

The function $L : \Delta \to \mathbb{R}$ defined by $L(\alpha,\beta,\gamma) = 1 - U(1-\gamma, 1-\beta, 1-\alpha)$ is called the *dual* of a given upper bound function U. Inequality (5.11) then simply expresses that $L \leq U$. Condition (i) again guarantees that cycle-transitivity generalizes transitivity of complete relations.

Definition 5.17. A reciprocal relation Q on A is called cycle-transitive w.r.t. an upper bound function U if for any $(a,b,c) \in A^3$ it holds that

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}), \tag{5.12}$$

where L is the dual lower bound function of U.

Due to the built-in duality, it holds that if (5.12) is true for some (a,b,c), then this is also the case for any permutation of (a,b,c). In practice, it is therefore sufficient to check (5.12) for a single permutation of any $(a,b,c) \in A^3$. Alternatively, due to the same duality, it is also sufficient to verify the right-hand inequality (or equivalently, the left-hand inequality) for two permutations of any $(a,b,c) \in A^3$ (not being cyclic permutations of one another), e.g., (a,b,c) and (c,b,a). Hence, (5.12) can be replaced by

$$\alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc}). \tag{5.13}$$

Note that a value of $U(\alpha, \beta, \gamma)$ equal to 2 is used to express that for the given values there is no restriction at all (as $\alpha + \beta + \gamma - 1$ is always bounded by 2).

Two upper bound functions U_1 and U_2 are called *equivalent* if for any $(\alpha, \beta, \gamma) \in \Delta$ it holds that $\alpha + \beta + \gamma - 1 \leq U_1(\alpha, \beta, \gamma)$ is equivalent to $\alpha + \beta + \gamma - 1 \leq U_2(\alpha, \beta, \gamma)$.

If it happens that in (5.11) the equality holds for all $(\alpha, \beta, \gamma) \in \Delta$, then the upper bound function U is said to be *self-dual*, since in that case it coincides with its dual lower bound function L. Consequently, also (5.12) and (5.13) can only hold with equality then. Furthermore, it then holds that $U(0, 0, 1) = 0$ and $U(0, 1, 1) = 1$.

The simplest example of a self-dual upper bound function is the median, i.e. $U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta$. Another example of a self-dual upper bound function is the function U_E defined by

$$U_E(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma.$$

Cycle-transitivity w.r.t. U_E of a reciprocal relation Q on A can also be expressed as

$$\alpha_{ijk}\beta_{ijk}\gamma_{ijk} = (1 - \alpha_{ijk})(1 - \beta_{ijk})(1 - \gamma_{ijk}).$$

It is then easy to see that cycle-transitivity w.r.t. U_E is equivalent to the notion of multiplicative transitivity [49]. Recall that a reciprocal relation Q on A is called *multiplicatively transitive* if for any $(a, b, c) \in A^3$ it holds that

$$\frac{Q(a,c)}{Q(c,a)} = \frac{Q(a,b)}{Q(b,a)} \cdot \frac{Q(b,c)}{Q(c,b)}.$$

The cycle-transitive formulation is more appropriate as it avoids division by zero.

5.4.2.4 Cycle-Transitivity Is a General Framework

Although C-transitivity is not intended to be applied to reciprocal relations, it can be formally cast quite nicely into the cycle-transitivity framework.

Proposition 5.3. [10] *Let C be a commutative conjunctor such that $C \leq T_{\mathbf{M}}$. A reciprocal relation Q on A is C-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_C defined by*

$$U_C(\alpha, \beta, \gamma) = \min(\alpha + \beta - C(\alpha, \beta), \beta + \gamma - C(\beta, \gamma), \gamma + \alpha - C(\gamma, \alpha)).$$

Moreover, if C is 1-Lipschitz, then U_C is given by

$$U_C(\alpha, \beta, \gamma) = \alpha + \beta - C(\alpha, \beta). \tag{5.14}$$

This proposition applies in particular to commutative quasi-copulas and copulas. In case of a copula, the expression in (5.14) is known as the dual of the copula. Consider the three basic t-norms/copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$:

(i) For $C = T_{\mathbf{M}}$, we immediately obtain as upper bound function the median

$$U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta.$$

(ii) For $C = T_{\mathbf{P}}$, we find

$$U_{\mathbf{P}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta.$$

(iii) For $C = T_{\mathbf{L}}$, we obtain

$$U_{\mathbf{L}}(\alpha, \beta, \gamma) = \begin{cases} \alpha + \beta, & \text{if } \alpha + \beta < 1, \\ 1, & \text{if } \alpha + \beta \geq 1. \end{cases}$$

An equivalent upper bound function is given by $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$.

Cycle-transitivity also incorporates stochastic transitivity, although the latter fits more naturally in the FG-transitivity framework. We list just one interesting proposition under mild conditions on the function g.

Proposition 5.4. *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping such that $g(1/2, x) \leq x$ for any $x \in [1/2, 1]$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma), & \text{if } \beta \geq 1/2 \wedge \alpha < 1/2, \\ 1/2, & \text{if } \alpha \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases} \tag{5.15}$$

A final simplification, eliminating the special case $\alpha = 1/2$ in (5.15), is obtained by requiring g to have as neutral element $1/2$, i.e. $g(1/2, x) = g(x, 1/2) = x$ for any $x \in [1/2, 1]$.

Proposition 5.5. *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$. A reciprocal relation Q on A is g-stochastic transitive if and only if it is cycle-transitive w.r.t. the upper bound U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma), & \text{if } \beta \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases} \tag{5.16}$$

This proposition implies in particular that strong stochastic transitivity ($g = \max$) is equivalent to cycle-transitivity w.r.t. the simplified upper bound function U'_{ss} defined by

$$U'_{ss}(\alpha,\beta,\gamma) = \begin{cases} \beta, & \text{if } \beta \geq 1/2, \\ 2, & \text{if } \beta < 1/2. \end{cases}$$

Note that g-stochastic transitivity w.r.t. a function $g \geq \max$ always implies strong stochastic transitivity. This means that any reciprocal relation that is cycle-transitive w.r.t. an upper bound function U_g of the form (5.16) is at least strongly stochastic transitive. It is obvious that $T_{\mathbf{M}}$-transitivity implies strong stochastic transitivity and that moderate stochastic transitivity implies $T_{\mathbf{L}}$-transitivity.

One particular form of stochastic transitivity deserves our attention. A probabilistic relation Q on A is called *partially stochastic transitive* [24] if for any $(a,b,c) \in A^3$ it holds that

$$(Q(a,b) > 1/2 \wedge Q(b,c) > 1/2) \Rightarrow Q(a,c) \geq \min(Q(a,b), Q(b,c)).$$

Clearly, it is a slight weakening of moderate stochastic transitivity. Interestingly, also this type of transitivity can be expressed elegantly in the cycle-transitivity framework [17] by means of a simple upper bound function.

Proposition 5.6. *Cycle-transitivity w.r.t. the upper bound function U_{ps} defined by*

$$U_{ps}(\alpha,\beta,\gamma) = \gamma$$

is equivalent to partial stochastic transitivity.

Finally, not surprisingly, isostochastic transitivity corresponds to cycle-transitivity w.r.t. particular self-dual upper bound functions [10]. An interesting way of constructing a self-dual upper bound function goes as follows.

Proposition 5.7. *Let g be a commutative, increasing $[1/2,1]^2 \to [1/2,1]$ mapping with neutral element $1/2$. It then holds that any $\Delta \to \mathbb{R}$ function U of the form*

$$U^s_g(\alpha,\beta,\gamma) = \begin{cases} \beta+\gamma-g(\beta,\gamma), & \textit{if } \beta \geq 1/2, \\ \alpha+\beta-1+g(1-\beta,1-\alpha), & \textit{if } \beta < 1/2, \end{cases}$$

is a self-dual upper bound function.

Note that the function g in Proposition 5.7 has the same properties as the function g in Proposition 5.5.

Proposition 5.8. *A reciprocal relation Q on A is cycle-transitive w.r.t. a self-dual upper bound function of type U^s_g if and only if it is g-isostochastic transitive.*

In particular, a reciprocal relation Q is $T_{\mathbf{M}}$-transitive if and only if

$$(Q(a,b) \geq 1/2 \wedge Q(b,c) \geq 1/2) \Rightarrow Q(a,c) = \max(Q(a,b), Q(b,c)),$$

for any $(a,b,c) \in A^3$. Note that this is formally the same as (5.10) with the difference that in the latter case Q was only $\{0,1/2,1\}$-valued.

If the function g is a commutative, associative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping with neutral element $1/2$, then the $[0, 1]^2 \to [0, 1]$ mapping S_g defined by

$$S_g(x, y) = 2g\left(\frac{1+x}{2}, \frac{1+y}{2}\right) - 1$$

is a t-conorm. For the self-dual upper bound function U_E, the associated t-conorm S_E is given by

$$S_E(x, y) = \frac{x+y}{1+xy},$$

which belongs to the parametric Hamacher t-conorm family, and is the co-copula of the Hamacher t-norm with parameter value 2 [37].

We have shown that the cycle-transitivity and FG-transitivity frameworks cannot easily be translated into one another, which underlines that these are two essentially different frameworks [6].

5.4.3 Comparison of Random Variables

5.4.3.1 Dice-Transitivity of Winning Probabilities

Consider three dice A, B and C which, instead of the usual numbers, carry the following integers on their faces:

$$A = \{1, 3, 4, 15, 16, 17\},\ B = \{2, 10, 11, 12, 13, 14\},\ C = \{5, 6, 7, 8, 9, 18\}.$$

Denoting by $\mathcal{P}(X, Y)$ the probability that dice X wins from dice Y, we have $\mathcal{P}(A, B) = 20/36$, $\mathcal{P}(B, C) = 25/36$ and $\mathcal{P}(C, A) = 21/36$. It is natural to say that dice X is strictly preferred to dice Y if $\mathcal{P}(X, Y) > 1/2$, which reflects that dice X wins from dice Y in the long run (or that X statistically wins from Y, denoted $X >_s Y$). Note that $\mathcal{P}(Y, X) = 1 - \mathcal{P}(X, Y)$ which implies that the relation $>_s$ is asymmetric. In the above example, it holds that $A >_s B$, $B >_s C$ and $C >_s A$: the relation $>_s$ is not transitive and forms a cycle. In other words, if we interpret the probabilities $\mathcal{P}(X, Y)$ as constituents of a reciprocal relation on the set of alternatives $\{A, B, C\}$, then this reciprocal relation is even not weakly stochastic transitive.

This example can be generalized as follows: we allow the dice to possess any number of faces (whether or not this can be materialized) and allow identical numbers on the faces of a single or multiple dice. In other words, a generalized dice can be identified with a multiset of integers. Given a collection of m such generalized dice, we can still build a reciprocal relation Q containing the *winning probabilities* for each pair of dice [19]. For any two such dice A and B, we define

$$Q(A, B) = \mathcal{P}\{A \text{ wins from } B\} + \frac{1}{2}\mathcal{P}\{A \text{ and } B \text{ end in a tie}\}.$$

The dice or integer multisets may be identified with independent discrete random variables that are uniformly distributed on these multisets (i.e. the probability of an integer is proportional to its number of occurrences); the reciprocal relation Q may be regarded as a quantitative description of the pairwise comparison of these random variables.

In the characterization of the transitivity of this reciprocal relation, a type of cycle-transitivity, which can neither be seen as a type of C-transitivity, nor as a type of FG-transitivity, has proven to play a predominant role. For obvious reasons, this new type of transitivity has been called dice-transitivity.

Definition 5.18. Cycle-transitivity w.r.t. the upper bound function U_D defined by

$$U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma,$$

is called *dice-transitivity*.

Dice-transitivity is closely related to $T_{\mathbf{P}}$-transitivity. However, it uses the quantities β and γ instead of the quantities α and β, and is therefore less restrictive. Dice-transitivity can be situated between $T_{\mathbf{L}}$-transitivity and $T_{\mathbf{P}}$-transitivity, and also between $T_{\mathbf{L}}$-transitivity and moderate stochastic transitivity.

Proposition 5.9. [19] *The reciprocal relation generated by a collection of generalized dice is dice-transitive.*

5.4.3.2 A Method for Comparing Random Variables

Many methods can be established for the comparison of the components (random variables, r.v.) of a random vector $(X_1, \ldots, X_n)$, as there exist many ways to extract useful information from the joint cumulative distribution function (c.d.f.) $F_{X_1,\ldots,X_n}$ that characterizes the random vector. A first simplification consists in comparing the r.v. two by two. It means that a method for comparing r.v. should only use the information contained in the bivariate c.d.f. F_{X_i,X_j}. Therefore, one can very well ignore the existence of a multivariate c.d.f. and just describe mutual dependencies between the r.v. by means of the bivariate c.d.f. Of course one should be aware that not all choices of bivariate c.d.f. are compatible with a multivariate c.d.f. The problem of characterizing those ensembles of bivariate c.d.f. that can be identified with the marginal bivariate c.d.f. of a single multivariate c.d.f. is known as the *compatibility problem* [41].

A second simplifying step often made is to bypass the information contained in the bivariate c.d.f. to devise a comparison method that entirely relies on the one-dimensional marginal c.d.f. In this case there is even not a compatibility problem, as for any set of univariate c.d.f. F_{X_i}, the product $F_{X_1} F_{X_2} \cdots F_{X_n}$ is a valid joint c.d.f., namely the one expressing the independence of the r.v. There are many ways to compare one-dimensional c.d.f., and by far the simplest one is the method that builds a partial order on the set of r.v. using the principle of first order stochastic dominance [49]. It states that a r.v. X is weakly preferred to a r.v. Y if for all $u \in \mathbb{R}$ it holds that $F_X(u) \leq F_Y(u)$. At the extreme end of the chain of simplifications are the methods that compare r.v. by means of a characteristic or a function of some characteristics derived from the one-dimensional marginal c.d.f. The simplest example is the weak order induced by the expected values of the r.v.

Proceeding along the line of thought of the previous section, a random vector $(X_1, X_2, \ldots, X_m)$ generates a reciprocal relation by means of the following recipe.

Definition 5.19. Given a random vector $(X_1, X_2, \ldots, X_m)$, the binary relation Q defined by

$$Q(X_i, X_j) = \mathcal{P}\{X_i > X_j\} + \frac{1}{2}\mathcal{P}\{X_i = X_j\}$$

is a reciprocal relation.

For two discrete r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k} p_{X_i,X_j}(k,k),$$

with p_{X_i,X_j} the joint probability mass function (p.m.f.) of (X_i, X_j). For two continuous r.v. X_i and X_j, $Q(X_i, X_j)$ can be computed as

$$Q(X_i, X_j) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{x} f_{X_i,X_j}(x,y)\,\mathrm{d}y,$$

with f_{X_i,X_j} the joint probability density function (p.d.f.) of (X_i, X_j).

For this pairwise comparison, one needs the two-dimensional marginal distributions. Sklar's theorem [41,46] tells us that if a joint cumulative distribution function F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a copula C_{ij} such that for all x, y:

$$F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y)).$$

If X_i and X_j are continuous, then C_{ij} is unique; otherwise, C_{ij} is uniquely determined on $\mathrm{Ran}(F_{X_i}) \times \mathrm{Ran}(F_{X_j})$.

As the above comparison method takes into account the bivariate marginal c.d.f. it takes into account the dependence of the components of the random vector. The information contained in the reciprocal relation is therefore much richer than if, for instance, we would have based the comparison of X_i and X_j solely on their expected values. Despite the fact that the dependence structure is entirely captured by the multivariate c.d.f., the pairwise comparison is only apt to take into account pairwise dependence, as only bivariate c.d.f. are involved. Indeed, the bivariate c.d.f. do not fully disclose the dependence structure; the r.v. may even be pairwise independent while not mutually independent.

Since the copulas C_{ij} that couple the univariate marginal c.d.f. into the bivariate marginal c.d.f. can be different from another, the analysis of the reciprocal relation and in particular the identification of its transitivity properties appear rather cumbersome. It is nonetheless possible to state in general, without making any assumptions on the bivariate c.d.f., that the probabilistic relation Q generated by an arbitrary random vector always shows some minimal form of transitivity.

Proposition 5.10. [8] *The reciprocal relation Q generated by a random vector is $T_{\mathbf{L}}$-transitive.*

5.4.3.3 Artificial Coupling of Random Variables

Our further interest is to study the situation where abstraction is made that the r.v. are components of a random vector, and all bivariate c.d.f. are enforced to depend in the same way upon the univariate c.d.f., in other words, we consider the situation of all copulas being the same, realizing that this might not be possible at all. In fact, this simplification is equivalent to considering instead of a random vector, a collection of r.v. and to artificially compare them, all in the same manner and based upon a same copula. The pairwise comparison then relies upon the knowledge of the one-dimensional marginal c.d.f. solely, as is the case in stochastic dominance methods. Our comparison method, however, is not equivalent to any known kind of stochastic dominance, but should rather be regarded as a graded variant of it (see also [7]).

The case $C = T_{\mathbf{P}}$ generalizes Proposition 5.9, and applies in particular to a collection of independent r.v. where all copulas effectively equal $T_{\mathbf{P}}$.

Proposition 5.11. [18, 19] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{P}}$ is dice-transitive,* i.e. *it is cycle-transitive w.r.t. the upper bound function given by $U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma$.*

We discuss next the case when using one of the extreme copulas to artificially couple the r.v. In case $C = T_{\mathbf{M}}$, the r.v. are coupled comonotonically. Note that this case is possible in reality.

Proposition 5.12. [16, 17] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{M}}$ is cycle-transitive w.r.t. to the upper bound function U given by $U(\alpha, \beta, \gamma) = \min(\beta + \gamma, 1)$. Cycle-transitivity w.r.t. the upper bound function U is equivalent to $T_{\mathbf{L}}$-transitivity.*

In case $C = T_{\mathbf{L}}$, the r.v. are coupled countermonotonically. This assumption can never represent a true dependence structure for more than two r.v., due to the compatibility problem.

Proposition 5.13. [16, 17] *The reciprocal relation Q generated by a collection of r.v. pairwise coupled by $T_{\mathbf{L}}$ is partially stochastic transitive,* i.e. *it is cycle-transitive w.r.t. to the upper bound function defined by $U_{ps}(\alpha, \beta, \gamma) = \max(\beta, \gamma) = \gamma$.*

The proofs of these propositions were first given for discrete uniformly distributed r.v. [16, 19]. It allowed for an interpretation of the values $Q(X_i, X_j)$ as winning probabilities in a hypothetical dice game, or equivalently, as a method for the pairwise comparison of ordered lists of numbers. Subsequently, we have shown that as far as transitivity is concerned, this situation is generic and therefore characterizes the type of transitivity observed in general [17, 18].

The above results can be seen as particular cases of a more general result.

Proposition 5.14. [8] *Let C be a commutative copula such that for any $n > 1$ and for any $0 \leq x_1 \leq \cdots \leq x_n \leq 1$ and $0 \leq y_1 \leq \cdots \leq y_n \leq 1$, it holds that*

$$\begin{aligned}
&\sum_i C(x_i, y_i) - \sum_i C(x_{n-2i}, y_{n-2i-1}) - \sum_i C(x_{n-2i-1}, y_{n-2i}) \\
&\quad \leq C\Bigg(x_n + \sum_i C(x_{n-2i-2}, y_{n-2i-1}) - \sum_i C(x_{n-2i}, y_{n-2i-1}), \\
&\qquad\qquad y_n + \sum_i C(x_{n-2i-1}, y_{n-2i-2}) - \sum_i C(x_{n-2i-1}, y_{n-2i})\Bigg), \qquad (5.17)
\end{aligned}$$

where the sums extend over all integer values that lead to meaningful indices of x and y. Then the reciprocal relation Q generated by a collection of random variables pairwise coupled by C is cycle-transitive w.r.t. to the upper bound function U^C defined by:

$$U^C(\alpha, \beta, \gamma) = \max(\beta + C(1-\beta, \gamma), \gamma + C(\beta, 1-\gamma)).$$

Inequality (5.17) is called the *twisted staircase condition* and appears to be quite complicated. Although its origin is well understood [8], it is not yet clear for which commutative copulas it holds. We strongly conjecture that it holds for all Frank copulas.

5.4.3.4 Comparison of Special Independent Random Variables

Dice-transitivity is the generic type of transitivity shared by the reciprocal relations generated by a collection of independent r.v. If one considers independent r.v. with densities all belonging to one of the one-parameter families in Table 5.1, then the corresponding reciprocal relation shows the corresponding type of cycle-transitivity listed in Table 5.2 [18].

Note that all upper bound functions in Table 5.2 are self-dual. More striking is that the two families of power-law distributions (one-parameter subfamilies of the two-parameter Beta and Pareto families) and the family of Gumbel distributions all yield the same type of transitivity as exponential distributions, namely cycle-transitivity w.r.t. the self-dual upper bound function U_E, or, in other words, multiplicative transitivity.

Table 5.1 Parametric families of continuous distributions

Name	Density function $f(x)$		
Exponential	$\lambda e^{-\lambda x}$	$\lambda > 0$	$x \in [0, \infty[$
Beta	$\lambda x^{(\lambda-1)}$	$\lambda > 0$	$x \in [0, 1]$
Pareto	$\lambda x^{-(\lambda+1)}$	$\lambda > 0$	$x \in [1, \infty[$
Gumbel	$\mu e^{-\mu(x-\lambda)} e^{-e^{-\mu(x-\lambda)}}$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, \infty[$
Uniform	$1/a$	$\lambda \in \mathbb{R},\ a > 0$	$x \in [\lambda, \lambda + a]$
Laplace	$(e^{-\|x-\lambda\|/\mu)})/(2\mu)$	$\lambda \in \mathbb{R},\ \mu > 0$	$x \in]-\infty, \infty[$
Normal	$(e^{-(x-\lambda)^2/2\sigma^2})/\sqrt{2\pi\sigma^2}$	$\lambda \in \mathbb{R}, \sigma > 0$	$x \in]-\infty, \infty[$

Table 5.2 Cycle-transitivity for the continuous distributions in Table 5.1

Name	Upper bound function $U(\alpha, \beta, \gamma)$
Exponential Beta Pareto Gumbel	$\alpha\beta + \alpha\gamma + \beta\gamma - 2\alpha\beta\gamma$
Uniform	$\begin{cases} \beta + \gamma - 1 + \frac{1}{2}\left[\max(\sqrt{2(1-\beta)} + \sqrt{2(1-\gamma)} - 1, 0)\right]^2 & \beta \geq 1/2 \\ \alpha + \beta - \frac{1}{2}\left[\max(\sqrt{2\alpha} + \sqrt{2\beta} - 1, 0)\right]^2, & \beta < 1/2 \end{cases}$
Laplace	$\begin{cases} \beta + \gamma - 1 + f^{-1}(f(1-\beta) + f(1-\gamma)), & \beta \geq 1/2 \\ \alpha + \beta - f^{-1}(f(\alpha) + f(\beta)), & \beta < 1/2 \end{cases}$ with $f^{-1}(x) = \frac{1}{2}\left(1 + \frac{x}{2}\right) e^{-x}$
Normal	$\begin{cases} \beta + \gamma - 1 + \Phi(\Phi^{-1}(1-\beta) + \Phi(1-\gamma)), & \beta \geq 1/2 \\ \alpha + \beta - \Phi(\Phi^{-1}(\alpha) + \Phi^{-1}(\beta)), & \beta < 1/2 \end{cases}$ with $\Phi(x) = (\sqrt{2\pi})^{-1} \int_{-\infty}^{x} e^{-t^2/2} dt$

In the cases of the unimodal uniform, Gumbel, Laplace and normal distributions we have fixed one of the two parameters in order to restrict the family to a one-parameter subfamily, mainly because with two free parameters, the formulae become utmost cumbersome. The one exception is the two-dimensional family of normal distributions. In [18], we have shown that the corresponding reciprocal relation is in that case moderately stochastic transitive.

5.4.4 *Mutual Ranking Probabilities in Posets*

Consider a finite poset $(P, \leq)$ with $P = \{x_1, \ldots, x_n\}$. A linear extension of P is an order-preserving permutation of its elements (hence, also a ranking of the elements compatible with the partial order). We denote by $p(x_i < x_j)$ the fraction of linear extensions of P in which x_i precedes x_j. If the space of all linear extensions of P is equipped with the uniform measure, the position of x in a linear extension can be regarded as a discrete random variable X with values in $\{1, \ldots, n\}$.

Since $p(x_i < x_j) = \mathcal{P}\{X_i < X_j\}$, the latter value is called a mutual rank probability. Note that P uniquely determines a random vector $X = (X_1, \dots, X_n)$ with multivariate distribution function $F_{X_1,\dots,X_n}$, whereas the mutual rank probabilities $p(x_i < x_j)$ are then computed from the bivariate marginal distributions F_{X_i,X_j}. Note that for general P, despite the fact that the multivariate distribution function $F_{X_1,\dots,X_n}$, or equivalently, the n-dimensional copula, can be very complex, certain pairwise couplings are trivial. Indeed, if in P it holds that $x_i < y_j$, then x_i precedes y_j in all linear extensions and X_i and X_j are comonotone, which means that X_i and X_j are coupled by (a discretization of) $T_{\mathbf{M}}$. For pairs of elements in P that are incomparable, the bivariate couplings can vary from pair to pair. The copulas are not all equal to $T_{\mathbf{L}}$, as can be seen already from the example where P is an antichain with three elements.

Definition 5.20. Given a poset $P = \{x_1, \dots, x_n\}$, consider the reciprocal relation Q_P defined by

$$Q_P(x_i, x_j) = \mathcal{P}\{X_i < X_j\} = p(x_i < x_j). \tag{5.18}$$

The problem of probabilistic transitivity in a finite poset P was raised by Fishburn [25]. It can be rephrased as follows: find the largest function $\delta : [0,1]^2 \to [0,1]$ such that for any finite poset and any x_i, x_j, x_k in it, it holds that

$$\delta(Q_P(x_i, x_j), Q_P(x_j, x_k)) \leq Q_P(x_i, x_k).$$

Fishburn has shown in particular that

$$\big(Q_P(x_i, x_j) \geq u \ \wedge \ Q_P(x_j, x_k) \geq u\big) \ \Rightarrow \ Q_P(x_i, x_k) \geq u$$

for $u \geq \rho \approx 0.78$.

A non-trivial lower bound for δ was obtained by Kahn and Yu [36] via geometric arguments. They have shown that $\delta^* \leq \delta$ with δ^* the conjunctor

$$\delta^*(u,v) = \begin{cases} 0, & \text{if } u + v < 1 \\ \min(u,v), & \text{if } u + v - 1 \geq \min(u^2, v^2) \\ \dfrac{(1-u)(1-v)}{(1-\sqrt{u+v-1})^2}, & \text{elsewhere} \end{cases}$$

Interestingly, the particular form of this function allows to state δ^*-transitivity also as

$$\begin{aligned} &Q_P(x_i, x_j) + Q_P(x_j, x_k) \geq 1 \\ &\quad \Rightarrow Q_P(x_i, x_k) \geq \delta^*(Q_P(x_i, x_j), Q_P(x_j, x_k)), \end{aligned}$$

which can be seen to be closely related to stochastic transitivity. Moreover, δ^*-transitivity can be positioned within the cycle-transitivity framework.

Proposition 5.15. [9] δ^*-Transitivity implies cycle-transitivity w.r.t. the upper bound function U_P defined by

$$U_P(\alpha, \beta, \gamma) = \alpha + \gamma - \alpha\gamma,$$

and hence also dice-transitivity.

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Chapter 6
Fuzzy Sets and Fuzzy Logic-Based Methods in Multicriteria Decision Analysis

Radko Mesiar and Lucia Vavríková

Abstract In this chapter, we discuss some fuzzy sets and fuzzy logic-based methods for multicriteria decision aid. Alternatives are identified with score vectors $\mathbf{x} \in [0,1]^n$, and thus they can be seen as fuzzy sets, too. After discussion of integral-based utility functions, we introduce a transformation of score $\mathbf{x}$ into fuzzy quantity $U(\mathbf{x})$. Orderings on fuzzy quantities induce orderings on alternatives. A special attention is paid to defuzzification-based orderings, especially to mean of maxima method. Our approach allows an easy incorporation of importance of criteria. Finally, a fuzzy logic-based construction method to build complete preference structures over set of alternatives is given.

Keywords Fuzzy set · Fuzzy quantity · Fuzzy utility · Dissimilarity · Defuzzification

6.1 Introduction

In this chapter we will deal with alternatives $\mathbf{x}$ from a set of alternatives $\mathcal{A}$. Each alternative $\mathbf{x} \subseteq \mathcal{A}$ is characterized by a score vector $(x_1, \ldots, x_n)$ and we will not distinguish $\mathbf{x}$ and $(x_1, \ldots, x_n)$. Score vector $(x_1, \ldots, x_n) \in [0,1]^n$ summarizes the information about the degrees of fulfilment of criteria $C_1, \ldots, C_n$ by the alternative $\mathbf{x}$. Here $x_i = 1$ means that $\mathbf{x}$ fully satisfies the criterion C_i, while $x_j = 0$ means that $\mathbf{x}$ is completely failed in the criterion C_j. We will not discuss any aspect of commensurability nor of vagueness of the degrees of satisfaction (for this item we recommend Chapter 5 of Bernard De Baets and Janos Fodor in this edited volume). Hence each alternative $\mathbf{x}$ can be seen as a fuzzy subset of the space $\mathcal{C} = \{C_1, \ldots, C_n\}$ of all criteria considered in our decision problems (here n is some fixed integer, the number of all considered criteria). The set $\mathcal{A}$ of discussed alternatives is then a subset of the set of all fuzzy subsets of $\mathcal{C}$. Alternatives from

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$\mathcal{A}$ will be denoted by **x**, **y**, **z**, etc. Note that we did not distinguish fuzzy subsets and their membership functions. Thus the same letter **x** is used for an alternative from $\mathcal{A}$, for its score vector and for the fuzzy subset of $\mathcal{C}$ with membership function $(x_1, \ldots, x_n) \in [0,1]^n$. We hope that this convention will not create any confusion. In some cases, score values from an interval I (e.g., $I = \mathbb{R}$) will be considered too, and then it will be mentioned explicitly.

We recall first some basic notions and notations from the fuzzy set theory [12, 46]. For a given non-empty set Ω (universe), a fuzzy set V (fuzzy subset V of Ω) is characterized by the membership function $\mu_V : \Omega \to [0,1]$. In this context, classical sets (subsets of Ω) are called crisp sets (crisp subsets of Ω) and they are characterized by the corresponding characteristic function. The height $\mathrm{hgt}(V)$ of a fuzzy set V is given by

$$\mathrm{hgt}(V) = \sup_{\omega \in \Omega} \mu_V(\omega).$$

Evidently, for non-empty crisp set V it holds $\mathrm{hgt}(V) = 1$. Vice versa, on any finite universe Ω, if $\mathrm{hgt}(V) = 1$ then there is a non-empty crisp set V' such that $\mu_{V'} \leq \mu_V$. For a given constant $\alpha \in [0,1]$, the corresponding α-cut $V^{(\alpha)}$ of a fuzzy set V is given by

$$V^{(\alpha)} = \{\omega \in \Omega \mid \mu_V(\omega) \geq \alpha\}.$$

Fuzzy set V is called normal if $V^{(1)} \neq \emptyset$, i.e., $\mu_V(\omega) = 1$ for some $\omega \in \Omega$. Fuzzy subsets of the real line $\mathbb{R} =]-\infty, \infty[$ (or any real interval I) are called fuzzy quantities. Moreover, a V is called convex whenever each α-cut $V^{(\alpha)}$ is a convex subset of $\mathbb{R}$, i.e., $V^{(\alpha)}$ is an interval for all $\alpha \in [0,1]$. Equivalently, convexity of a fuzzy quantity V can be characterized by the fulfilment of inequality

$$\mu_V(\lambda r + (1-\lambda)s) \geq \min(\mu_V(r), \mu_V(s)), \tag{6.1}$$

for all $r, s \in \mathbb{R}$ $(r, s \in I)$ and $\lambda \in [0,1]$. For more details see [10, 25].

The aim of this chapter is to discuss several methods for building (weak) orderings on $\mathcal{A}$ (in fact, on $[0,1]^n$), based on fuzzy set theory and fuzzy logic. The chapter will bring in the next section an overview of such methods which can be roughly seen as utility function based methods. In Section 6.3, methods based on some orderings of fuzzy quantities are introduced. A special case of MOM (mean of maxima) based defuzzification method of ordering fuzzy quantities is related to utility based decisions and it is investigated in Section 6.4. In Section 6.5, we exploit some fuzzy logic connectives to create a variety of complete preference structures on $[0,1]^n$, which need not be transitive, in general. Finally, some concluding remarks are included.

6.2 Fuzzy Set Based Utility Functions

Global evaluation of a fuzzy event **x** (measurable fuzzy subset of some measurable space $(\Omega, \mathcal{X})$) was first introduced by Zadeh [47] as a fuzzy probability measure $\mathcal{P}$, $\mathcal{P}(\mathbf{x}) = \int_\Omega \mathbf{x} \, \mathrm{d}P$, where P is some probability measure on $(\Omega, \mathcal{X})$, i.e., $\mathcal{P}$ is the expected value, $\mathcal{P}(\mathbf{x}) = E(\mathbf{x})$. In our case this means that

$$\mathcal{P}(\mathbf{x}) = \sum_{i=1}^{n} x_i p_i, \tag{6.2}$$

$p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, i.e., $\mathcal{P}$ is a normed additive utility function on $[0,1]^n$. The next step in this direction was done by Klement [19], when discussing normed utility functions $\mathcal{M}$ satisfying the valuation property $\mathcal{M}(\mathbf{x} \vee \mathbf{y}) + \mathcal{M}(\mathbf{x} \wedge \mathbf{y}) = \mathcal{M}(\mathbf{x}) + \mathcal{M}(\mathbf{y})$. Supposing the lower semicontinuity of $\mathcal{M}$, Klement's results for our case yield

$$\mathcal{M}(\mathbf{x}) = \sum_{i=1}^{n} F_i(x_i) p_i, \tag{6.3}$$

where $F_i : [0,1] \to [0,1]$ is a (restriction on $[0,1]$) distribution function of some random variable X_i acting on $]0,1[$, $F_i(u) = P(X_i < u)$.

Evidently, (6.3) generalizes (6.2). In fact, (6.2) corresponds to (6.3) in special case, when all random variables X_i are uniformly distributed over $]0,1[$. Utility function $\mathcal{M}$ given by (6.3) is a general version of additive (lower semicontinuous) utility function over $[0,1]^n$.

Comonotone additive models are related to [8, 16]. In such a case, instead of probability measure P on $\mathcal{C}$ (p_i is the weight of criterion C_i) one needs to know a fuzzy measure M on $\mathcal{C}$ [18, 36], $M : 2^{\mathcal{C}} \to [0,1]$, M is non-decreasing and $M(\emptyset) = 0$, $M(\mathcal{C}) = 1$ (here $M(E)$ is the weight of group E of some criteria from $\mathcal{C}$), and then the corresponding utility function $Ch : [0,1]^n \to [0,1]$ is given by

$$Ch(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)}(M(A_i) - M(A_{i+1})), \tag{6.4}$$

where σ is a permutation of $(1,\ldots,n)$ such that $(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ is a non-decreasing permutation of $(x_1,\ldots,x_n)$, and $A_i = \{j \in \{1,\ldots,n\} \,|\, x_j \geq x_{\sigma(i)}\}$, with a convention $A_{n+1} = \emptyset$. A trivial generalization of (6.4) can follow the same line as the way relating (6.2) and (6.3), namely

$$Chg(\mathbf{x}) = \sum_{i=1}^{n} F_{\sigma(i)}(x_{\sigma(i)})(M(A_i) - M(A_{i+1})), \tag{6.5}$$

where σ is a permutation of $1,\ldots,n$ such that $(F_{\sigma(1)}(x_{\sigma(1)}),\ldots,F_{\sigma(n)}(x_{\sigma(n)}))$ is non-decreasing. Note that if fuzzy measure M is additive (i.e., it is a probability measure), then (6.2) coincides with (6.4), while (6.3) coincides with (6.5). For discussion of comonotone maxitive models we recommend [29].

Example 6.1. Let $n = 2$, $M(C_1) = M(C_2) = \frac{1}{3}$, $F_1(u) = u$, $F_2(u) = u^2$. Then (6.5) yields the utility function $Chg : [0,1]^2 \to [0,1]$ given by $Chg(x_1, x_2) = 2\left(x_1 \wedge x_2^2\right)/3 + \left(x_1 \vee x_2^2\right)/3 = \left(x_1 + x_2^2 + x_1 \wedge x_2^2\right)/3$.

The most general form of a normed utility function U, i.e., $U : [0,1]^n \to [0,1]$ is non-decreasing and $U(0) = 0$, $U(1) = 1$, represents aggregation operators discussed, e.g., in [5, 6, 22]. In fact, normed utility functions are exactly n-ary aggregation operators, and only expected properties of these functions restrict our possible choice to some well-known classes (this is, e.g., the case of formulas (6.2), (6.3) and (6.4), while the characterization of normed utility functions Chg introduced in (6.5) is related to the valuation property restricted to comonotone alternatives only). Among well-known classes of aggregation operators, we recall (weighted) quasi-arithmetic means characterized by the unanimity and the bisymmetry [1, 13], OWA operators [41] characterized by the anonymity and comonotone additivity, triangular norms, triangular conorms and uninorms characterized by the anonymity, associativity and neutral element [21, 43], quasi-copulas and related operators characterized by 1-Lipschitz property and neutral element [32], etc. A deep discussion and state-of-art overview of aggregation operators is the topic of a forthcoming monograph [17], and a lot of useful material on aggregation operators can be found in recent handbook [3] and monograph [37].

In any case when a normed utility function (an aggregation operator) $U:[0,1]^n \to [0,1]$ is exploited to build a preference structure on $\mathcal{A}$, this preference structure is evidently transitive and does not possess any incomparable pairs of alternatives, i.e., it is a weak ordering on $\mathcal{A}$ given by $\mathbf{x} \leq_U \mathbf{y}$ if and only if $U(\mathbf{x}) \leq U(\mathbf{y})$. However, we cannot avoid many possible ties in such a case. Among several ways of refining such kinds of preference structures (recall, e.g., Lorentz [24] approach to the problem how to refine the standard arithmetic mean), we focus now on a modification of a recent method based on limit approach we have introduced in [23]. The next result is based on an infinite sequence of $\mathcal{U} = (U_k)_{k\in N}$ of n-ary aggregation operators. Partial ordering $\precsim^{\mathcal{U}}$ induced by $\mathcal{U}$ is given as follows: $\mathbf{x} \precsim^{\mathcal{U}} \mathbf{y}$ if and only if there is a $k_0 \in N$ such that for all $k \in N, k \geq k_0$, it holds $U_k(\mathbf{x}) \leq U_k(\mathbf{y})$. Note that denoting by R_k the relation on $[0,1]^n \times [0,1]^n$ given by $R_k(\mathbf{x},\mathbf{y})$ if and only if $\mathbf{x} \leq_{U_k} \mathbf{y}, k \in N$ and R the relation given by $R(\mathbf{x},\mathbf{y})$ if and only if $\mathbf{x} \precsim^{\mathcal{U}} \mathbf{y}$ then $R = \liminf R_k$.

Proposition 6.1. *Let $\mathcal{U} = (U_k)_{k\in N}$ be a system of n-ary aggregation operators with pointwise limit U. Then $U : [0,1]^n \to [0,1]$ is an aggregation operator and the partial order $\precsim^{\mathcal{U}}$ on $[0,1]^n$ x $[0,1]^n$ is related to the weak order $\leq_U$ as follows:*

$$\boldsymbol{x} <_U \boldsymbol{y} \Rightarrow \boldsymbol{x} \prec^{\mathcal{U}} \boldsymbol{y},$$
$$\boldsymbol{x} \approx^{\mathcal{U}} \boldsymbol{y} \Rightarrow \boldsymbol{x} \approx_U \boldsymbol{y}.$$

Moreover, if $\precsim^{\mathcal{U}}$ does not admit incomparable pairs then it is a refinement of $\leq_U$. Note that the original roots of Proposition 6.1 can be found in [9] where the refinements of $\leq_{Min}$ and $\leq_{Max}$ weak orderings were discussed.

Example 6.2.

(i) Let $\mathcal{B} = (B_k)_{k\in N}$, where

$$B_k(x_1,\ldots,x_n) = k \log\left(\frac{\sum_{i=1}^n \exp\left(\frac{x_i}{k}\right)}{n}\right).$$

Then $\lim_{k\to\infty} B_k = M$ is the arithmetic mean. Note that for $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $B_k(\mathbf{x}) < B_k(\mathbf{y})$ if and only if

$$\frac{\sum_{i=1}^{n} \exp\left(\frac{x_i}{k}\right)}{n} < \frac{\sum_{i=1}^{n} \exp\left(\frac{y_i}{k}\right)}{n}.$$

By means of Taylor's series, we see that $B_k(\mathbf{x}) \leq B_k(\mathbf{y})$ if and only if

$$\frac{1}{k}\left(\sum_{i=1}^{n} x_i\right) + \frac{1}{2!k^2}\left(\sum_{i=1}^{n} x_i^2\right) + \frac{1}{3!k^3}\left(\sum_{i=1}^{n} x_i^3\right) + \cdots$$
$$\leq \frac{1}{k}\left(\sum_{i=1}^{n} y_i\right) + \frac{1}{2!k^2}\left(\sum_{i=1}^{n} y_i^2\right) + \frac{1}{3!k^3}\left(\sum_{i=1}^{n} y_i^3\right) + \cdots.$$

Then, for $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \precsim^{\mathcal{B}} \mathbf{y}$ if and only if $MOF_{\mathbf{x}} \leq MOF_{\mathbf{y}}$, where $MOF_{\mathbf{x}}$: N$\to$ $[0,1]$ is the moment function given by $MOF_{\mathbf{x}}(m) = \left(\frac{1}{n}\right)\sum_{i=1}^{n} x_i^m$, i.e., $MOF_{\mathbf{x}}(m)$ is the mth initial moment of a random variable described by the uniform sample $x = (x_1, \ldots, x_n)$. Observe that $\mathbf{x} \approx^{\mathcal{B}} y$ if and only if $\mathbf{x}$ is a permutation of $\mathbf{y}$.

(ii) Let $\mathcal{MAX} = (M_k)_{k\in N}$ be the system of root-power operators [11]

$$M_k(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^k\right)^{\frac{1}{k}}.$$

Then $\lim_{k\to+\infty} M_k = Max$.
For $\mathbf{x} \in [0,1]^n$, define the occurrence function $\sigma_{\mathbf{x}} : [0,1] \to N \cup \{0\}$, $\sigma_{\mathbf{x}}(u)$ =card$\{i \in \{1, \ldots, n\} \,|\, x_i = u\}$. Then $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if $\sigma_{\mathbf{x}} \circ \eta \leq_{Lex} \sigma_{\mathbf{y}} \circ \eta$, where $\eta : [0,1] \to [0,1]$ is given by $\eta(u) = 1 - u$.
Note that though $[0,1]$ is uncountable, the supports of both $\sigma_{\mathbf{x}} \circ \eta$ and $\sigma_{\mathbf{y}} \circ \eta$ are finite. The lexicographic relation $\sigma_{\mathbf{x}} \circ \eta <_{Lex} \sigma_{\mathbf{y}} \circ \eta$ means that there is $u \in [0,1]$ such that $\sigma_{\mathbf{x}}(1-u) <_{Lex} \sigma_{\mathbf{y}}(1-u)$ and for all $v \in [0, u[$ it holds $\sigma_{\mathbf{x}}(1-v) = \sigma_{\mathbf{y}}(1-v)$. Observe that on $[0,1]^n$, $\precsim^{\mathcal{MAX}} \equiv \leq_{LexiMax}$ is just the LexiMax preorder [11].

(iii) Starting from an arbitrary continuous Archimedean t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, see [21], also $t^k : [0,1] \to [0,\infty]$, $k \in N$, is an additive generator and it generates a continuous Archimedean t-norm T_k. Then $\lim_{k\to\infty} T_k = Min$ and for $\mathcal{MIN} = (T_k)_{k\in N}$, $\mathbf{x} \precsim^{\mathcal{MIN}} \mathbf{y}$ if and only if $Min(\mathbf{x}) = Min(\mathbf{y}) = 0$, or $\sigma_{\mathbf{x}}|[0,1[\leq_{Lex} \sigma_{\mathbf{y}}|[0,1[$.

Note that though we focus in this chapter on the comparison of alternatives described by score vectors with a fixed dimension (n is fixed), aggregation operator based approach allows to compare also the alternatives having score vectors with different dimension. Recall, e.g., classical comparison by means of the arithmetic mean. Interestingly, when applying Proposition 2.1 in such situation, one can obtain different partial orderings $\precsim^{\mathcal{U}_1}$ and $\precsim^{\mathcal{U}_2}$ of score vectors with non-fixed dimension, though for the fixed dimension n they coincide.

Example 6.3. Taking the system $\mathcal{MAX} = (M_k)_{k \in N}$ of general root-power operators defined for any arity n, we have seen in Example 2.2(ii) that $\precsim^{\mathcal{MAX}}$ on $[0,1]^n$ is the LexiMax preorder, whenever the arity n is fixed. However, when we have to compare alternatives $\mathbf{x} \in [0,1]^n$ and $\mathbf{y} \in [0,1]^m$ with $m \neq n$, then $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if

$$\frac{\sigma_{\mathbf{x}} \circ \eta}{n} \leq_{Lex} \frac{\sigma_{\mathbf{y}} \circ \eta}{m},$$

see [23]. Hence $\precsim^{\mathcal{MAX}}$ extends the standard LexiMax preorder in that sense that the score vector of $\mathbf{x}$ repeated m-times $\widetilde{\mathbf{x}} = \underbrace{(\mathbf{x}, \ldots, \mathbf{x})}_{m\text{-}times}$ and the score vector of $\mathbf{y}$ repeated n-times $\widetilde{\mathbf{y}} = \underbrace{(\mathbf{y}, \ldots, \mathbf{y})}_{n\text{-}times}$ are of the same $n.m$-arity and then we apply the standard LexiMax to compare $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$, $\mathbf{x} \precsim^{\mathcal{MAX}} \mathbf{y}$ if and only if $\widetilde{\mathbf{x}} \leq_{LexiMax} \widetilde{\mathbf{y}}$.

On the other hand, when modifying the Example 6.2(iii) by duality, we can start from an arbitrary continuous Archimedean t-conorm S with an additive generator $s : [0,1] \to [0,\infty]$, and work with S_k generated by s^k, $k \in N$. Take, for example $s(t) = t$, $t \in [0,1]$, i.e., $S_k(u_1, \ldots, u_n) = \min\left(1, \left(\sum_{i=1}^n u_i^k\right)^{\frac{1}{k}}\right)$. Let $\mathcal{S} = (S_k)_{k \in N}$. Then for $\mathbf{x}, \mathbf{y} \in [0,1[^n$ it holds $\mathbf{x} \precsim^{\mathcal{S}} \mathbf{y}$ if and only if $\mathbf{x} \leq_{LexiMax} \mathbf{y}$, i.e., $\precsim^{\mathcal{S}}$ and $\precsim^{\mathcal{MAX}}$ coincide on $[0,1[^n$ for each $n \in N$. However, if $\mathbf{x} \in [0,1[^n$ and $\mathbf{y} \in [0,1[^m$ with $n \neq m$, then $\mathbf{x} \precsim^{\mathcal{S}} \mathbf{y}$ if and only if $\mathbf{x}^* \leq_{LexiMax} \mathbf{y}^*$, where both $\mathbf{x}^* = (\mathbf{x}, \underbrace{0, \ldots, 0}_{m\text{-}times})$ and $\mathbf{y}^* = (\mathbf{y}, \underbrace{0, \ldots, 0}_{n\text{-}times})$ have the same dimension $n + m$.

Example 6.4. Put $\mathbf{x} = (0.8, 0.2)$ and $\mathbf{y} = (0.8, 0.3, 0.1)$. Then $\widetilde{\mathbf{x}} = (0.8, 0.2, 0.8, 0.2, 0.8, 0.2) >_{LexiMax} \widetilde{\mathbf{y}} = (0.8, 0.3, 0.1, 0.8, 0.3, 0.1)$ and $\mathbf{x}^* = (0.8, 0.2, 0, 0, 0) <_{LexiMax} \mathbf{y}^* = (0.8, 0.3, 0.1, 0, 0)$, i.e., $\mathbf{x} \succ^{\mathcal{MAX}} \mathbf{y}$ but $\mathbf{x} \prec^{\mathcal{S}} \mathbf{y}$.

6.3 Fuzzy Quantities Based Preference Structures Constructions

In several decision-making models, the choice of an appropriate alternative is transformed to the problem of comparison of some available quantitative information. Recall, e.g., the standard optimization problems arising from the minimal costs or the maximal profit, several ordering approaches such as leximin or discrimin, see [9], etc. The simplest situation occurs when each alternative is described by a single real value (say the costs), in which case from two alternatives we choose the cheaper one. From a mathematical point of view, we exploit here the standard ordering on the real line. Much more complex is the situation when alternatives are described by fuzzy reals [10,25]. In that case, there are several orderings known so far. For an exhaustive overview we recommend [39,40]. Because of greater flexibility and modelling power, we focus our attention to the last case, i.e., to the decision problems for alternatives characterized by fuzzy quantities.

On the set of alternatives $\mathcal{A}$, let each alternative $\mathbf{x}$ be described by the score vector $(x_1, \ldots, x_n)$, where n is the number of applied criteria and $x_1, \ldots, x_n \in I$ are the single score from some prescribed real interval I (usually, $I = [0,1]$ or $I = \mathbb{R}$). In the criterion i, the dissimilarity $D_i(x, y)$ of a score x and another score y, with $x, y \in I$, is described by the $D_i : I^2 \to \mathbb{R}$, such that

$$D_i(x, y) = K_i(f_i(x) - f_i(y)),$$

where $K_i : \mathbb{R} \to \mathbb{R}$ is a convex function with the unique minimum $K_i(0) = 0$ (shape function), and $f_i : I \to \mathbb{R}$ is a strictly monotone continuous function (scale transformation). Evidently, each D_i is then continuous. Observe that this approach to dissimilarity is based on the ideas of verbal fuzzy quantities as proposed and discussed in [26–28]. Note also that the concept of dissimilarity functions is closely related to the penalty functions proposed by Yager and Rybalov [44], compare also [4]. Finally, remark that the dissimilarity function D is related to some standard metric on the interval I whenever it is symmetric, i.e., if K is an even function. To be more precise, for any even shape function K, let L be the inverse function to $K|_{[0,\infty[}$. Then for the dissimilarity function $D(x, y) = K(f(x) - f(y))$ we have $L \circ D(x, y) = |f(x) - f(y)|$, i.e., $L \circ D$ is a metric. Typical examples of such dissimilarity functions are (on any real interval I):

- $D(x, y) = (f(x) - f(y))^2$
- $D(x, y) = |f(x) - f(y)|$
- $D(x, y) = 1 - \cos\left(\frac{\arctan x - \arctan y}{2}\right)$

As an example of a dissimilarity D which is not a transformed metric (only the symmetry is violated) we introduce functions D_c, with $c \in]0, \infty[$, $c \neq 1$, given by

$$D_c(x, y) = \begin{cases} c(y - x) & \text{if } x \leq y, \\ x - y & \text{else.} \end{cases} \tag{6.6}$$

The dissimilarity of score $(x_1, \ldots, x_n)$ and the unanimous score $(r, \ldots, r)$ is described by the real vector $(D_1(x_1, r), \ldots, D_n(x_n, r))$. The fuzzy utility function U, compare [14,45], assigns to each alternative $\mathbf{x}$ (with score $(x_1, \ldots, x_n)$) the fuzzy quantity $U(\mathbf{x})$ with membership function $\mu_{\mathbf{x}} : I \to [0, 1]$,

$$\mu_{\mathbf{x}}(r) = \frac{1}{1 + \sum_{i=1}^{n} D_i(x_i, r)}. \tag{6.7}$$

Proposition 6.2. *For each alternative $\boldsymbol{x} \in \mathcal{A}$, the fuzzy utility function value $U(\boldsymbol{x})$ with membership function given by (6.7) is a convex fuzzy quantity with continuous membership function.*

Proof. The continuity of $\mu_{\mathbf{x}}$ follows from the continuity of each dissimilarity function D_i. For arbitrary $r, s \in I$ and $\lambda \in [0,1]$, the convexity of $K_i, i = 1, \ldots, n$, and the strict monotonicity and the continuity of $f_i, i = 1, \ldots, n$, ensure the following facts: the function $g : I \to \mathbb{R}$, $g(r) = 1 + \sum_{i=1}^n D_i(x_i, r) \geq 1$, is continuous and convex, and thus $g(\lambda r + (1-\lambda)s) \leq \lambda g(r) + (1-\lambda)g(s) \leq \max(g(r), g(s))$, compare also [7]. As far as $\mu_\alpha = \frac{1}{g}$, the inequality (6.1) follows. □

Note that the convexity of all shapes K_i was crucial in the above proof, justifying the restriction of possible shapes to convex ones. Observe that applying the above-described procedures, introduced fuzzy utility function $U : I^n \to \mathcal{F}(\mathbb{R})$ assigns to n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ continuous convex quantity $U(\mathbf{x})$ with membership function $\mu_{\mathbf{x}}$ described by (6.7).

Example 6.5.

(i) Let $D_1 = \cdots = D_n = D$, $D(x,y) = (x-y)^2$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $U(\mathbf{x})$ has membership function

$$\mu_{\mathbf{x}}(r) = \frac{1}{n(\overline{x} - r)^2 + 1 - n\overline{x}^2 + \sum_{i=1}^n x_i^2} = \frac{1}{1 + n((\overline{x} - r)^2 + \sigma^2)},$$

which is symmetric w.r.t. point $r = \overline{x}$ (the arithmetic mean of $(x_1, \ldots, x_n)$, $\overline{x} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$), where the dispersion $\sigma^2 = \frac{1}{n}\sum_{i=1}^n x_i^2 - \overline{x}^2$ (i.e., $\mu_{\mathbf{x}}(\overline{x} - \epsilon) = \mu(\overline{x} + \epsilon)$ for all $\epsilon \in \mathbb{R}$). Moreover, $U(\mathbf{x})$ is unimodal fuzzy number with height $\frac{1}{1+n\sigma^2}$ attained in the point $r = \overline{x}$.

(ii) Let $D_1 = \cdots = D_n = D$, $D(x,y) = |x-y|$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then

$$\mu_{\mathbf{x}}(r) = \frac{1}{1 + \sum_{i=1}^n |x_i - r|}$$

is symmetric w.r.t. point $r = \overline{x}$ if and only if for the order statistics $x'_1, \ldots, x'_n$ of $(x_1, \ldots, x_n)$ it holds $x'_1 + x'_n = x'_2 + x'_{n-1} = \cdots = 2\overline{x}$. Moreover, $U(\mathbf{x})$ always attains its height in the median $\text{med}(x_1, \ldots, x_n)$ and it is unimodal if and only if n is odd, or if n is even, $n = 2k$, and $x'_k = x'_{k+1}$.

(iii) Let $D_1(x,y) = |x-y|$, $D_2(x,y) = (x-y)^2$ and $(x_1, x_2) \in \mathbb{R}^2$. Then

$$\mu_{(x_1,x_2)}(r) = \frac{1}{1 + |x_1 - r| + (x_2 - r)^2}$$

is symmetric w.r.t. point $r = \overline{x}$ if and only if $x_1 = x_2$. Moreover, it is unimodal and $U(x_1, x_2)$ attains its height in the point $r = \text{med}\left(x_1, x_2 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$.

Remark 6.1. Formula (6.7) arises from the global dissimilarity $\sum_{i=1}^n D_i(x_i, r) \in [0, \infty[$ and the decreasing bijection $\varphi : [0, \infty] \to [0,1]$ given by $\varphi(t) = \frac{1}{1+t}$. Concerning Proposition 6.2, the sum operator can be replaced by any other operator $H : [0, \infty[^n \to [0, \infty[$ such that $h(r) = H(D_1(x_1, r), \ldots, D_n(x_n, r))$ is a convex function. For example, we can take $H(t_1, \ldots, t_n) = \sum_{i=1}^n t_i^2$. Similarly, φ can be replaced by any other decreasing bijection $\eta : [0, \infty] \to [0,1]$, not violating the validity of Proposition 6.2. For example, we can take $\eta(t) = e^{-t}$.

On the set of all fuzzy quantities $\mathcal{F}$, respectively of all continuous convex fuzzy quantities $\mathcal{Q}$, there were introduced many types of orderings. For an exhaustive overview we recommend [10, 39, 40]. Based on any such ordering $\leq$, we can derive a preference relation $\precsim$ on the set $\mathcal{A}$ of all alternatives, $\mathbf{x} \precsim \mathbf{y}$ if and only if $U(\mathbf{x}) \leq U(\mathbf{y})$. Obviously, if $\leq$ is a fuzzy ordering, then $\precsim$ is a fuzzy ordering relation (for more details on fuzzy orderings and fuzzy preference structures see Chapter 5 in this edited volume). However, in this paper, we will deal with crisp preference relations (crisp orderings) only, i.e., only crisp orderings of fuzzy quantities will be taken into account. In such a case, we can even refine the derived weak ordering relation.

Definition 6.1. Let $\mathcal{A}$ be a set of alternatives and let $U : \mathcal{A} \to \mathcal{F}(\mathbb{R})$ be a fuzzy utility function given by (6.7). Let $\leq$ be a crisp ordering on the set of all continuous convex fuzzy quantities. Then we define a weak ordering relation $\precsim$ on $\mathcal{A}$ as follows: $\mathbf{x} \precsim \mathbf{y}$ whenever $U(\mathbf{x}) < U(\mathbf{y})$ or $U(\mathbf{x}) = U(\mathbf{y})$ and $\mathrm{hgt}(U(\mathbf{x})) \geq \mathrm{hgt}(U(\mathbf{x}))$.

It is evident that $\precsim$ given in the above definition is really a weak ordering relation on $\mathcal{A}$. However, it need not fit the Pareto principle, in general, i.e., for two alternatives $\mathbf{x}$ and $\mathbf{y}$ characterized by the respective scores $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_i \leq y_i$, $i = 1, \ldots, n$, we do not have necessarily $\mathbf{x} \precsim \mathbf{y}$.

A huge class of crisp orderings on fuzzy quantities (continuous, convex) is linked to defuzzification methods, see, e.g., [42], i.e., to mappings $\mathrm{DF} : \mathcal{F} \to \mathbb{R}$ or $\mathrm{DF} : \mathcal{Q} \to \mathbb{R}$. For a defuzzification DF, we simply have then $U(\mathbf{x}) \leq U(\mathbf{y})$ if and only if $\mathrm{DF}(U_{\mathbf{x}}) \leq \mathrm{DF}(U_{\mathbf{y}})$ (where the last inequality is the standard inequality among real numbers). Each such defuzzification method induces an operator $A_{\mathrm{DF}} : I^n \to \mathbb{R}$, $A_{\mathrm{DF}}(x_1, \ldots, x_n) = \mathrm{DF}(U_{\mathbf{x}})$. As already mentioned, the operator A_{DF} need not be monotone, in general, and thus the Pareto principle may fail.

Example 6.6. We continue Example 6.5. As a defuzzification method DF for a fuzzy quantity F with $(\mathrm{hgt}(F))$-cut $= [u, v]$ we take $\mathrm{DF}(F) = u + \frac{(v-u)^3}{1+(v-u)^2}$. Then:

(i) if $D_1 = \cdots = D_n = D$, $D(x, y) = (x - y)^2$, we obtain $A_{\mathrm{DF}}(x_1, \ldots, x_n) = \overline{x} = M(x_1, \ldots, x_n)$, where M is the standard arithmetic mean operator (acting on $\mathbb{R}$). Evidently, M is a monotone idempotent operator, i.e., the Pareto principle is satisfied.
(ii) if $D_1 = \cdots = D_n = D$, $D(x, y) = |x - y|$, then A_{DF} is an idempotent operator on $\mathbb{R}$ which is not monotone, and thus it violates the Pareto principle. For example $A_{\mathrm{DF}}(1, 3, 5, 6) = \frac{23}{5}$ but $A_{\mathrm{DF}}(2, 4, 5, 10) = \frac{9}{2}$.

Observe, however, that any defuzzification method DF compatible with the fuzzy maximum $\widetilde{\max}$ (i.e., $\mathrm{DF}(U_{\mathbf{x}}) \leq \mathrm{DF}(U_{\mathbf{y}})$ whenever $\widetilde{\max}\,(U_{\mathbf{x}}, U_{\mathbf{y}}) = U_{\mathbf{y}}$) yields a non-decreasing idempotent operator A_{DF}. Recall that

$$\mu_{\widetilde{\max}(U_{\mathbf{x}}, U_{\mathbf{y}})}(r) = \sup\left(\min(U_{\mathbf{x}}(t), U_{\mathbf{y}}(s)) \mid \max(t, s) = r\right),$$

see [12].

6.4 Mean of Maxima Defuzzification Approach

One of the simplest defuzzification methods is the MOM method [42], i.e., the centre of the (hgt(F))-cut.

Definition 6.2. For the MOM method, the operator $A_{\text{MOM}} = A : I^n \to I$ is given by

$$A(x_1, \ldots, x_n) = \frac{\inf\{r \mid \mu_{\mathbf{x}}(r) = \text{hgt}(U_{\mathbf{x}})\} + \sup\{r \mid \mu_{\mathbf{x}}(r) = \text{hgt}(U_{\mathbf{x}})\}}{2}. \quad (6.8)$$

For unanimous score $\mathbf{x} = (x, \ldots, x)$ it is obvious that for arbitrary dissimilarity functions $D_1, \ldots, D_n$, the membership function $\mu_{\mathbf{x}}(r) = \frac{1}{1+\sum_{i=1}^{n} D_i(x,r)}$ of $U(\mathbf{x})$ is normal and unimodal with the unique maximum $\mu_{\mathbf{x}}(x) = 1$, and hence the operator A is idempotent, $A(x, \ldots, x) = x$ for all $x \in I$. To show the monotonicity of A (and thus the fitting to the Pareto principle), we need first some lemmas.

Lemma 6.1. *Let $K : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for any $u, v, w \in \mathbb{R}$, $v \geq 0, w \geq 0$ it holds*

$$K(u+v) + K(u+w) \leq K(u) + K(u+v+w). \quad (6.9)$$

Proof. If $v = 0$ or $w = 0$, (6.9) trivially holds (even with equality). Suppose that $v > 0, w > 0$. Then $u + v = \frac{w}{v+w}u + \frac{v}{v+w}(u+v+w)$ and from the convexity of K it holds

$$K(u+v) \leq \frac{w}{v+w}K(u) + \frac{v}{v+w}K(u+v+w).$$

Similarly,

$$K(u+w) \leq \frac{v}{v+w}K(u) + \frac{w}{v+w}K(u+v+w).$$

Summation of the two last inequalities gives just the desired inequality (6.9). □

Lemma 6.2. *Let $K : \mathbb{R} \to \mathbb{R}$ be a convex function and let $f : I \to \mathbb{R}$ be a strictly monotone continuous function. Then for any $x, y, s, p \in I$, $x \leq y$ and $s \leq p$ it holds $K(f(y) - f(p)) - K(f(x) - f(p)) \leq K(f(y) - f(s)) - K(f(x) - f(s))$, i.e.,*

$$D(y, p) - D(x, p) \leq D(y, s) - D(x, s). \quad (6.10)$$

Proof. Applying Lemma 6.1, it is enough to put $f(x) - f(p) = u$, $f(p) - f(s) = v$ and $f(y) - f(x) = w$ if f is increasing; in the case when f is decreasing, it is enough to put $f(y) - f(s) = u$, $f(s) - f(p) = v$ and $f(x) - f(y) = w$. □

Theorem 6.1. *Let $D_i(x, y) = K_i(f_i(x) - f_i(y))$, $i = 1, \ldots, n$, $x, y \in I$, be given dissimilarity functions. Then the idempotent operator $A : I^n \to I$ given by (6.8) is monotone and thus an aggregation operator fitting the Pareto principle.*

Proof. For a given score $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\varepsilon > 0$, $i \in \{1, \ldots, n\}$, suppose that $\mathbf{y} = (x_1, \ldots, x_i + \varepsilon, \ldots, x_n) \in I^n$. Denote $P_{\mathbf{x}}(r) = \sum_{j=1}^{n} D_j(x_j, r)$, $r \in I$. Then $P_{\mathbf{y}}(r) = P_{\mathbf{x}}(r) + D_i(x_i + \varepsilon, r) - D_i(x_i, r)$. Take an arbitrary element $p \in I$ such that for all $r \in I$, $P_{\mathbf{x}}(r) \geq P_{\mathbf{x}}(p)$, i.e., p is an element minimizing $P_{\mathbf{x}}$ (existence of p follows from the continuity of all dissimilarities D_i and the convexity of all shapes K_i). Then for arbitrary $s \in I$, $s \leq p$ it holds $P_{\mathbf{x}}(s) \geq P_{\mathbf{x}}(p)$ and because of Lemma 6.2, inequality (6.10), also $D_i(x_i + \varepsilon, s) - D_i(x_i, s) \geq D_i(x_i + \varepsilon, p) - D_i(x_i, p)$. Summing the two last inequalities we obtain $P_{\mathbf{y}}(s) \geq P_{\mathbf{y}}(p)$. However, this means that there is a minimal element p' of $P_{\mathbf{y}}$ such that $p' \geq p$, and thus the centre of minimal elements of $P_{\mathbf{x}}$ is less or equal to the centre of minimal elements of $P_{\mathbf{y}}$. Observe that the set of minimal elements of $P_{\mathbf{x}}$ is the same as the set of maximal elements of $\mu_{\mathbf{x}} = \frac{1}{1+P_{\mathbf{x}}}$, and thus because of (6.8), $A(\mathbf{x}) \leq A(\mathbf{y})$. □

Note that if all dissimilarity functions D_i are equal, $D_1 = \cdots = D_n = D$, the concept of MOM-based aggregation operators coincides with the penalty based approach proposed by Yager and Rybalov [44] and further developed in [4], in which case the aggregation operator A is anonymous, i.e., for all $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and any permutation σ of $(1, \ldots, n)$ it holds $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Observe also that the convexity of function K_i involved in dissimilarities D_i is crucial to ensure that $\mu_{\mathbf{x}}$ is a quasi-convex membership function and that A is monotone. For example, for $K_1 = \cdots = K_n = K$,

$$K(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{else,} \end{cases} \qquad \text{and for any } f_i : I \to \mathbb{R},$$

the operator A derived by (6.8) is the modus operator which is not monotone. Indeed, $A(0, 0.2, 0.5, 1, 1) = 1$ while $A(0.5, 0.5, 0.5, 1, 1) = 0.5$. Note also that following the ideas of quantitative weights in aggregation discussed in [4], for any non-zero weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ we can derive the corresponding weighted aggregation operator $A_{\mathbf{w}}$ applying Definition 6.2 to the weighted dissimilarity functions $w_1 D_1, \ldots, w_n D_n$. For further generalization see [30, 31].

Remark 6.2. Definition 6.1 brings a refinement of the weak ordering $\leq_A$ on the set of alternatives $\mathcal{A}$ given by $\mathbf{x} \leq_A \mathbf{y}$ whenever $A(\mathbf{x}) \leq A(\mathbf{y})$, where A is an aggregation operator given by (6.8), see also Theorem 6.1. Indeed, for the weak ordering $\precsim$ introduced in Definition 6.1 and based on MOM fuzzification method and dissimilarity functions $D_1, \ldots, D_n$:

$$\mathbf{x} <_A \mathbf{y} \Rightarrow \mathbf{x} \prec \mathbf{y},$$
$$\mathbf{x} \approx \mathbf{y} \Rightarrow \mathbf{x} \approx_A \mathbf{y}.$$

Moreover, if $\mathbf{x} <_A \mathbf{y}$, then:

- if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) | \quad r \in I\right\} < \min\left\{\sum_{i=1}^{n} D_i(y_i, r) | \quad r \in I\right\}$ then $\mathbf{x} \prec \mathbf{y}$;

- if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) \middle| \quad r \in I\right\} > \min\left\{\sum_{i=1}^{n} D_i(y_i, r) \middle| \quad r \in I\right\}$ then $\mathbf{x} \succ \mathbf{y}$;
- and if $\min\left\{\sum_{i=1}^{n} D_i(x_i, r) \middle| \quad r \in I\right\} = \min\left\{\sum_{i=1}^{n} D_i(y_i, r) \middle| \quad r \in I\right\}$ then $\mathbf{x} \approx \mathbf{y}$.

Example 6.7. We continue the previous examples.

(i) For $D_1 = \ldots = D_n = D$, $D(x, y) = (x-y)^2$, the corresponding aggregation operator $A : \mathbb{R} \to \mathbb{R}$ is the standard arithmetic mean, $A = M$. The corresponding weighted aggregation operator $A_{\mathbf{w}}$ is the weighted arithmetic mean linked to the weighting vector $\mathbf{w}$,

$$A_{\mathbf{w}}(\mathbf{x}) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \quad .$$

Evidently $p = (p_1, \ldots, p_n)$ where $p_j = \frac{w_j}{\sum_{i=1}^{n} w_i}$, $j = 1, \ldots, n$, is a probability distribution on the set of criteria $\mathcal{C} = \{C_1, \ldots, C_n\}$ and then the weighted mean $A_{\mathbf{w}}$ is the expected value operator on the set of alternatives $\mathcal{A}$ (compare with formula (6.2)). Moreover, denoting the corresponding variance operator by $\sigma^2_{\mathbf{w}}$, we have the next probabilistic interpretation for the weak order $\precsim$ on $\mathcal{A}$ as given in Definition 6.1 for this case, namely $\mathbf{x} \precsim \mathbf{y}$ if and only if either $A_{\mathbf{w}}(\mathbf{x}) < A_{\mathbf{w}}(\mathbf{y})$ or $A_{\mathbf{w}}(\mathbf{x}) = A_{\mathbf{w}}(\mathbf{y})$ and $\sigma^2_{\mathbf{w}}(\mathbf{x}) \le \sigma^2_{\mathbf{w}}(\mathbf{y})$, i.e., $\precsim$ is just the $\left(A_{\mathbf{w}}, \sigma^2_{\mathbf{w}}\right)$ – lexicographical ordering as introduced in [23], $\mathbf{x} \precsim \mathbf{y}$ if and only if $\left(A_{\mathbf{w}}(\mathbf{x}), \sigma^2_{\mathbf{w}}(\mathbf{x})\right) \le_{Lex} \left(A_{\mathbf{w}}(\mathbf{y}), \sigma^2_{\mathbf{w}}(\mathbf{y})\right)$.
Modifying D into $D_f(x, y) = (f(x) - f(y))^2$, the quasi-arithmetic mean M_f is recovered, $A_f(\mathbf{x}) = M_f(\mathbf{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$. Finally, the corresponding weighted aggregation operator is the weighted quasi-arithmetic mean,

$$A_{f,\mathbf{w}}(\mathbf{x}) = f^{-1}\left(\frac{\sum_{i=1}^{n} w_i f(x_i)}{\sum_{i=1}^{n} w_i}\right).$$

In this case, the weak order $\precsim$ introduced in Definition 6.1 can be represented as follows: $\mathbf{x} \precsim \mathbf{y}$ if and only if $\left(A_{f,\mathbf{w}}(\mathbf{x}), \sigma^2_{\mathbf{w}}(f(\mathbf{x}))\right) \le_{Lex} \left(A_{f,\mathbf{w}}(\mathbf{y}), \sigma^2_{\mathbf{w}}(f(\mathbf{y}))\right)$.

(ii) For $D_1 = \cdots = D_n = D$, $D(x, y) = |x - y|$, Definition 6.2 leads to the standard median operator $A = \text{med}$,

$$A(\mathbf{x}) = \text{med}(x_1, \ldots, x_n) = \begin{cases} x'_{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \\ \frac{x'_{\frac{n}{2}} + x'_{\frac{n}{2}+1}}{2}, & \text{if } n \text{ is even,} \end{cases}$$

where $\left(x'_1, \ldots, x'_n\right)$ is a non-decreasing permutation of $(x_1, \ldots, x_n)$. Note that a modification of D into D_f, $D_f(x, y) = |f(x) - f(y)|$, has no influence on the resulting aggregation operator A in this case whenever n is odd. The corresponding weighted median $A_{\mathbf{w}} = \text{med}_{\mathbf{w}}$ corresponds to the weighted median proposed in [4, 44], and in the case of integer weights it is given by

$$\text{med}_{\mathbf{w}}(\mathbf{x}) = \text{med}(\underbrace{x_1,\ldots,x_1}_{w_1\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{w_n\text{-times}}).$$

Denote by $w_i^{'}$ the weight corresponding to $x_i^{'}$, and by $W_i^{'}$ the cumulative weight, i.e., $W_i^{'} = \sum_{j=1}^{i} w_j^{'}$. Define two functions: $C_{\mathbf{x},\mathbf{w}}, D_{\mathbf{x},\mathbf{w}} : \left[0, W_n^{'}\right] \to I$ by $C_{\mathbf{x},\mathbf{w}}(0) = 0$ and $C_{\mathbf{x},\mathbf{w}}(u) = x_i^{'}$ whenever $W_{i-1}^{'} < u \leq W_i^{'}$ with convention $W_0^{'} = 0$, and $D_{\mathbf{x},\mathbf{w}}(u) = C_{\mathbf{x},\mathbf{w}}\left(W_n^{'} - u\right)$. Denote $R_{\mathbf{x},\mathbf{w}} = \int_0^{W_n^{'}} |C_{\mathbf{x},\mathbf{w}}(u) - D_{\mathbf{x},\mathbf{w}}(u)|\, du$.

Then the weak order $\precsim$ given by Definition 6.1 and related to this framework can be represented in the following form: $\mathbf{x} \precsim \mathbf{y}$ if and only if $(\text{med}_{\mathbf{w}}(\mathbf{x}), R(\mathbf{x},\mathbf{w})) \leq_{Lex} (\text{med}_{\mathbf{w}}(\mathbf{y}), R(\mathbf{y},\mathbf{w}))$.

Note that in the case when $w_1 = \cdots = w_n = 1$ (i.e., when the standard median is considered), then $R(\mathbf{x}) = R(\mathbf{x},\mathbf{w}) = \sum_{i=1}^{n} |x_i - x_{n+1-i}|$ and we have a new refinement of the median weak order.

(iii) For $D_1(x,y) = |x-y|$ and $D_2(x,y) = (x-y)^2$ the aggregation operator A defined by (6.8) is given by $A(x_1,x_2) = \text{med}\left(x_1, x_2 - \frac{1}{2}, x_2 + \frac{1}{2}\right)$. The corresponding weighted aggregation operator $A_{\mathbf{w}}$ is given by

$$A_{\mathbf{w}}(x_1,x_2) = \text{med}\left(x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2}\right).$$

Finally, observe that if $D_1 = \cdots = D_n = D_c$, $c \in]0,\infty[$, $c \neq 1$ see formula (6.6), then the corresponding aggregation operator A_c is the order statistic corresponding to the $q = \frac{1}{1+c}.100\%$-quantile, and $A_1 = A$ from Example 6.7 (ii).

Remark 6.3. The operator A_{MOM} introduced in Definition 6.2, see (6.8), can be seen also as a solution of the minimization problem $A_{\text{MOM}}(\mathbf{x}) = r_{\mathbf{x}}$, where

$$\sum_{i=1}^{n} D_i(x_i, r_{\mathbf{x}}) = \min\left\{\sum_{i=1}^{n} D_i(x_i, r) | r \in I\right\}. \quad (6.11)$$

This approach leads to minimization problem of $\sum_{i=1}^{n} w_i D_i(x_i, r)$ when the weights $w_1,\ldots,w_n$ have to be incorporated. Moreover, if weights are input-dependent, i.e., when $w_i = w(x_i)$ for a given weighting function $w : I \to [0,\infty[$, then we have to minimize the expression $\sum_{i=1}^{n} w_i(x_i) D_i(x_i, r)$. However, the resulting operator $A_{\mathbf{w}} : I^n \to I$ need not fulfil Pareto principle, in general. For example, if $D_1 = \ldots = D_n = D$, $D(x,y) = (x-y)^2$ then

$$A_{\mathbf{w}}(\mathbf{x}) = \frac{\sum_{i=1}^{n} w_i(x_i) x_i}{\sum_{i=1}^{n} w_i(x_i)}$$

is a mixture operator [33–35]. For $I = [0,1]$ and $w = id_{[0,1]}$, $A_{\mathbf{w}}(0,1) = \frac{0+1}{0+1} = 1$ but $A_{\mathbf{w}}(0.5,1) = \frac{0.25+1}{0.5+1} = \frac{5}{6}$. For deeper discussion of mixture operators and related concepts we recommend [30, 31].

6.5 Fuzzy Logic-Based Construction of Preference Relations

Let $T : [0,1]^2 \to [0,1]$ be a left-continuous [21], i.e., T is associative, commutative, non-decreasing and $T(x,1) = x$ for all $x \in [0,1]$, and let $I_T : [0,1]^2 \to [0,1]$ be the adjoint residual implication,

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}.$$

Then $x \le y$ if and only if $I_T(x,y) \ge I_T(y,x)$. This fact allows to introduce preference relations on $\mathcal{A}$ as follows:

Definition 6.3. Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm and let $H : [0,1]^n \to [0,1]$ be an aggregation operator. Then the preference relation $R_{T,H} \subseteq \mathcal{A}^2$ is given by $(\mathbf{x},\mathbf{y}) \in R_{T,H}$, i.e., $\mathbf{x} \ge_{T,H} \mathbf{y}$ if and only if

$$H(I_T(x_1,y_1),\ldots,I_T(x_n,y_n)) \le H(I_T(y_1,x_1),\ldots,I_T(y_n,x_n)). \tag{6.12}$$

$R_{T,H}$ is a complete preference relation (i.e., there are no incomparable alternatives $\mathbf{x}$ and $\mathbf{y}$), but not a weak ordering, in general.

Note that if H has neutral element 1 or if H is strictly monotone then the decision about relation of $\mathbf{x}$ and $\mathbf{y}$ depends only on those score for which $x_i \ne y_i$, i.e., the discriminative approach to decision making as discussed in [11] is applied. Observe that though in some cases $\ge_{T,H}$ can be represented in the form of a transitive complete weak preference relation $\ge_Q$ (Q is an aggregation operator) and thus the preference relation $R_{T,H}$ is also transitive, in general this is not true. Note also that H need not be anonymous (symmetric).

Recall that the strongest t-norm $T_M : [0,1]^2 \to [0,1]$ is given by $T_M(x,y) = \min\{x,y\}$ and that the related implication $I_{T_M} : [0,1]^2 \to [0,1]$ which is called the Gödel implication is given by

$$I_{T_M}(x,y) = \begin{cases} 1 & \text{if} \quad x \le y, \\ y & \text{else.} \end{cases}$$

An important class of continuous t-norm is the class of continuous Archimedean t-norms $T : [0,1]^2 \to [0,1]$ characterized by additive generators $t : [0,1] \to [0,\infty]$, which are continuous, strictly decreasing and $t(1) = 0$, in such a way that for all $(x,y) \in [0,1]^2$ it holds

$$T(x,y) = t^{-1}(\min\{t(0), t(x) + t(y)\}). \tag{6.13}$$

Then the corresponding residual implication $I_T : [0,1]^2 \to [0,1]$ is given by

$$I_T(x,y) = t^{-1}(\max\{0, t(y) - t(x)\}). \tag{6.14}$$

If $t(0)$ is finite then T is called a nilpotent t-norm. Prototypical example of nilpotent t-norm is the Łukasiewicz t-norm $T_L : [0,1]^2 \to [0,1]$ given by $T_L(x,y) = \max\{0, x+y-1\}$ with an additive generator $t_L : [0,1] \to [0,\infty]$, $t_{T_L}(x) = 1-x$. The corresponding Łukasiewicz implication $I_{T_L} : [0,1]^2 \to [0,1]$ is given by $I_{T_L}(x,y) = \min\{1, 1-x+y\}$.

If $t(0) = \infty$ then the corresponding t-norm T is called a strict t-norm. A prototypical example of a strict t-norm is the product t-norm $T_P : [0,1]^2 \to [0,1]$, $T_P(x,y) = xy$, where adjoint residual implication $I_{T_P} : [0,1]^2 \to [0,1]$ is called the Goguen implication and it is given by

$$I_{T_P}(x,y) = \begin{cases} 1 & \text{if} \quad x \le y, \\ \frac{y}{x} & \text{else.} \end{cases}$$

For more details about t-norms we recommend monographs [2, 20, 21].

Example 6.8.

(i) For any nilpotent t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, see [21], and the quasi-arithmetic mean M_t generated by t, see [5], $\ge_{T,M_t}$ is exactly $\ge_{M_t}$, and $\mathbf{x} = (x_1,\ldots,x_n) \ge_{T,M_t} \mathbf{y} = (y_1,\ldots,y_n)$ if and only if $\sum_{i=1}^n t(x_i) \le \sum_{i=1}^n t(y_i)$. Thus the transitivity of $\ge_{T,M_t}$ is obvious.
Note that considering $T = T_L$ we have $M_t = M$ the standard arithmetic mean and $t : [0,1] \to [0,\infty]$ is given by $t(x) = 1-x$. Then $\sum_{i=1}^n t(x_i) = n - \sum_{i=1}^{n} x_i \le n - \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} t(y_i)$ if and only if $\frac{1}{n}\sum_{i=1}^n x_i = M(\mathbf{x}) \ge M(\mathbf{y}) = \frac{1}{n}\sum_{i=1}^{n} y_i$.

(ii) Similarly for any strict t-norm T with an additive generator $t : [0,1] \to [0,\infty]$, $\ge_{T,M_t}$ is transitive, but there is no aggregation operator Q such that $\ge_{T,M_t} \equiv \ge_Q$. Observe that now $\mathbf{x} \ge_{T,M_t} \mathbf{y}$ if and only if $\sum_{x_i \ne y_i} t(x_i) \le \sum_{x_i \ne y_i} t(y_i)$.
In the particular case $T = T_P$, $M_t = G$ is the geometric mean and $t : [0,1] \to [0,\infty]$ is given by $t(x) = -\log x$. Then $G(\mathbf{x}) > G(\mathbf{y})$ implies $\mathbf{x} >_{T,M_t} \mathbf{y}$ and $G(\mathbf{x}) = G(\mathbf{y}) > 0$ implies $\mathbf{x} \approx_{T,M_t} \mathbf{y}$. In the case when $G(\mathbf{x}) = G(\mathbf{y}) = 0$, we should apply Discri-G comparison, i.e., we omit coordinates i where $x_i = y_i$ $(= 0)$ and then we apply the geometric mean to the remaining scores for both alternatives. Summarizing, we see that $\le_{T,M_t}$ is now the Discri-M_t weak order. Due to the isomorphism of strict t-norms with the product t-norm T_P, the same conclusion is true also in general, i.e., for any strict t-norm T with an additive generator t, the weak ordering $\le_{T,M_t}$ is just the Dicsri-M_t ordering as introduced in [12].

(iii) For any nilpotent t-norm T with an additive generator t, $\geq_{T,T_M}$ is not transitive. Observe that $\mathbf{x} \geq_{T,T_M} \mathbf{y}$ if and only if

$$\min\left\{t^{-1}(\max(t(y_1)-t(x_1),0)),\ldots,t^{-1}(\max(t(y_n)-t(x_n),0))\right\}$$
$$\leq \min\left\{t^{-1}(\max(t(x_1)-t(y_1),0)),\ldots,t^{-1}(\max(t(x_n)-t(y_n),0))\right\}.$$

For $T = T_L$ (Łukasiewicz t-norm), we have $\mathbf{x} \geq_{T_L,T_M} \mathbf{y}$ if and only if $\max\{x_i - y_i | i = 1,\ldots,n\} \geq \max\{y_i - x_i | i = 1,\ldots,n\}$. Let $\mathbf{x} = (1,0,0.5)$, $\mathbf{y} = (0,0.2,0.6)$ and $\mathbf{z} = (0.5,0.6,0)$. Then $\mathbf{x} >_{T_L,\,T_M} \mathbf{y} >_{T_L,T_M} \mathbf{z} >_{T_L,T_M} \mathbf{x}$, visualizing the non-transitivity of $\geq_{T_L,T_M}$.

In some cases, the approach introduced in Definition 6.3 can be related to the bipolar aggregation B of the vector $\mathbf{x}-\mathbf{y}$, and then $\mathbf{x} \geq_{T,H} \mathbf{y}$ if and only if $B(\mathbf{x}-\mathbf{y}) \geq 0$.

Example 6.9. Consider $H = Ch$ the Choquet integral based on a fuzzy measure M, see (6.4), and let $T = T_L$ be the Łukasiewicz t-norm. Then for any $\mathbf{x},\mathbf{y} \in \mathcal{A}$, $H(I_T(x_1,y_1),\ldots,I_T(x_n,y_n)) = Ch(\min(1,1-x_i+y_i))$. Considering the [3,6,18, 37,38], Ši with respect to M, it holds $Ch(\min(1,1-\mathbf{x}+\mathbf{y})) = 1-\check{S}i(\max(0,\mathbf{x}-\mathbf{y}))$, and thus $\mathbf{x} \geq_{Ch,T_L} \mathbf{y}$ if and only if $1-\check{S}i(\max(0,\mathbf{x}-\mathbf{y})) \leq 1-\check{S}i(\max(0,\mathbf{y}-\mathbf{x}))$.

However, due to the properties of the Šipoš integral, the latest inequality is equivalent to the inequality $\check{S}i(\mathbf{x},\mathbf{y}) \geq 0$, i.e., $\mathbf{x} \geq_{Ch,T_L} \mathbf{y}$ if and only if $\check{S}i(\mathbf{x},\mathbf{y}) \geq 0$.

6.6 Concluding Remarks

We have introduced and discussed several methods of multicriteria decision making based on the fuzzy set theory and on the fuzzy logic. In some cases, well-established methods were rediscovered, such as the comparison of alternatives by means of the aggregation of the corresponding score vectors, or by means of the discriminative aggregation (i.e., aggregating only those score values where both compared alternatives differ). Definition 6.1 has brought as a corollary a refinement of aggregation based comparison for those cases when Theorem 6.1 applies (note that there our approach covers a big part of aggregation functions known from multicriteria decision-making problems). Based on fuzzy logic ideas, also non-transitive preference structure on the set of alternatives $\mathcal{A}$ was proposed. Approaches discussed in Sections 6.3 and 6.4 enable an easy implementation of weights/importance of single criteria into the corresponding processing. Evidently, our overview of fuzzy set/logic based methods is only a small part of numerous decision-making methods proposed in the framework of multicriteria characterization of alternatives. We have chosen only some of recently proposed methods, and in some particular cases this is the first place where the discussed methods appear. In some cases, presented approaches allow to compare also alternatives described by score vectors with different dimension (i.e., not each criterion was evaluated for all compared alternatives). As a by-product, two different extensions of well-known LexiMax (LexiMin) methods were introduced. Among several new approaches with promising potential for multicriteria decision making, but still to be developed, recall the level-dependent fuzzy measures and the related fuzzy integrals, such as the Choquet integral with respect to the level-dependent fuzzy measures proposed and discussed in [15], or the extended Sugeno integral representing the comonotone maxitive utility functions proposed in [29].

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Chapter 7
Argumentation Theory and Decision Aiding

Wassila Ouerdane, Nicolas Maudet, and Alexis Tsoukiàs

Abstract The purpose of this chapter is to examine the existent and potential contribution of argumentation theory to decision aiding, more specifically to multi-criteria decision aiding. On the one hand, decision aiding provides a general framework that can be adapted to different contexts of decision making and a formal theory about preferences. On the other hand, argumentation theory is a growing field of Artificial Intelligence, which is interested in non-monotonic logics. It is the process of collecting arguments in order to justify and explain conclusions. The chapter is decomposed into three successive frames, starting from general considerations regarding decision theory and Artificial Intelligence, moving on to the specific contribution of argumentation to decision-support systems, to finally focus on multi-criteria decision aiding.

Keywords Argumentation theory · Multiple criteria · Decision analysis · Decision-support systems

7.1 Introduction

Decision-support systems aim at helping the user to shape a problem situation, formulate a problem and possibly try to establish a viable solution to it. Under such a perspective decision aiding can be seen as the construction of the reasons for which an action is considered a "solution to a problem" rather than the solution itself [131]. Indeed the problem of decisions *accountability* is almost as important as the decision itself. Decision support can therefore be seen as an activity aiming to construct arguments through which a decision maker will convince first herself and then other actors involved in a problem situation that "that action" is the best one (we are not

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going to discuss the rationality hypotheses about "best" here). Decision theory and multiple criteria decision analysis have focused on such issues for a long time, but more on how this "best solution" should be established and less on how a decision maker should be convinced about that (for exceptions on that see [16, 25]).

On the other hand, in the field of Artificial Intelligence, argumentation has been put forward as a very general approach allowing to support different kinds of decision making [95, 101]. Typically, one will construct for each possible decision (alternative) a set of positive arguments and a set of negative arguments. Adopting an argumentation-based approach in a decision problem would have some obvious benefits, as "on the one hand, the user will be provided with a 'good' choice and with the reasons underlying this recommendation, in a format that is easy to grasp. On the other hand, argumentation-based decision making is more akin with the way humans deliberate and finally make a choice [2]." Moreover, the arguments allow us to take into account the non-monotonic aspect of a decision process and the problem of incomplete information. Aspects that are, sometimes, poorly controlled in decision theory.

The aim through this chapter is to introduce the reader to some contribution of argumentation theory to decision aiding. The chapter is organised as follows: in the next section, we introduce and discuss two main concepts that follow more or less directly from Simon's criticisms to classical models of rationality. In particular, we shall see what requirements this puts on decision-support approaches. Firstly, decisions result from a complex process which is hardly captured by classical mathematical languages. Secondly, these languages are not necessarily appropriate to handle preferential information as it is stated by the client (because it may, for instance, involve generic, ambiguous or incomplete statements). The section which follows (Section 7.3) advances that argumentation theory is a good candidate to handle some of the challenging requirements that came out from the previous discussion. To justify this claim, we first offer a brief introduction to argumentation theory (in an AI oriented perspective). We then review a number of approaches that indeed use argumentative techniques to support decision making. This section is intended to offer a broad (even though not exhaustive) overview of the range of applicability of argumentation. In fact, the use of argumentation in decision-support systems has been greatly encouraged. Such systems have the aim to assist people in decision making. The need to introduce arguments in such systems has emerged from the demand to justify and to explain the choices and the recommendations provided by them. Section 7.4 focuses more specifically on approaches adopting (more or less explicitly) a multiple criteria decision analysis perspective. The final section presents some advances on the use of argumentation in a decision-aiding process.

7.2 Decision Theory and AI

The story we are going to tell in this chapter results from a long history that we can trace back to Simon's criticisms to traditional "economic" models of rationality (and thus of rational behaviour and decision making, see [126]). Focussing on how real decision makers behave within real organisations Simon argued that several postulates of "classic rationality": well-defined problems, full availability of information, full availability of computing resources, were utopian and unrealistic. Instead decisions (following Simon) are based upon a "bounded rationality" principle which is subjective, local and procedural. A decision is thus "rational" now, under that available information, with that given computing capability, within that precise context because it subjectively satisfies the decision maker.

These ideas can be found at the origin of research conducted both in new directions of Decision Analysis and in Artificial Intelligence (see more in [132]). We are not going to explore the whole contribution of Simon, we are going to emphasise two specific innovations Simon directly or indirectly introduced:

- The concept of decision process
- The subjective handling of preferences

7.2.1 Decision Process and Decision Aiding

Decisions are not just an "act of choice", they are the result of a "decision process", a set of cognitive activities enabling to go from a "problem" (a state of the world perceived as unsatisfactory) to its "solution" (a state of the world perceived as satisfactory, if any exists). Even if we consider at the place of a human decision maker an automatic device (such as a robot or other device with some sort of autonomous behaviour) we can observe, describe and analyse the process through which a "decision" is reached. However, it is clear that is not a process only about solving a problem: a decision process implies also understanding and shaping a decision problem.

In fact, research conducted in what is known as "Problem Structuring Methodologies" [48,111,122] emphasised that decision aiding is not just to offer a solution to well-established mathematically formulated problem, but to be able to support the whole decision process, representing the problem situation, formulating a problem and possibly constructing a reasonable recommendation. Thus, to the concept of decision process, we can associate the concept of "decision aiding process" where the cognitive artefacts representing the modelling of different elements of the decision process are described and analysed. A decision aiding context implies the existence of at least two distinct actors (the decision maker and the analyst) both playing different roles; at least two objects, the client's concern and the analyst's methodological knowledge, money, time, etc. [113, 115, 132].

A formal model of decision-aiding process, that meets these needs, is the one described by Tsoukiàs in [131]. The ultimate objective of this process is to come up with a consensus between the decision maker and the analyst. Four cognitive artefacts summarise the overall process:

- A representation of a problem situation: the first artefact consists in offering a representation of the problem situation for which the decision maker has asked the analyst to intervene.
- A problem formulation: given a representation of the problem situation, the analyst may provide the decision maker with one or more problem formulation. The idea is that a problem formulation translates the client's concern, using the decision-support language, into a "formal problem".
- An evaluation model: For a given problem formulation, the analyst may construct an evaluation model, that is to organise the available information in such a way that it will be possible to obtain a formal answer to a problem statement.
- A final recommendation: The evaluation model will provide an output which is still expressed in terms of the decision-support language. The final recommendation is the final deliverable which translates the output into the decision-maker's language.

The above process is a dialogue between the analyst and the decision maker where the preference statements of the former are elaborated using some methodology by the latter, the result expected to be a contextual and subjective model of the decision-maker's values as perceived, modelled and manipulated by the analyst. Such a process is expected to be understood and validated by the client. In a "human-to-human" interaction the above dialogue is handled through typical human interaction, possibly supported by standard protocols (as in the case of constructing a value or a utility function or assessing importance parameters in majority based procedures). In any case we can fix some explicit formal rules on how such a process can be conducted. Consider now the case where the analyst part is played by a device collecting information about some user's preferences (a typical case being recommender systems). We need to be able to structure the dialogue on a formal basis in order to be able to control and assess what the device concludes as far as the user preference models are concerned and what type of recommendations (if any) is going to reach.

In both the above cases (human-to-human and human-to-machine) we need on the one hand some formal theory about preferences (and this is basically provided by decision analysis), and on the other hand some formal language enabling to represent the dialogue, to explain it, to communicate its results, to convince the user/decision maker that what is happening is both theoretically sound and operationally reasonable. Under such a perspective we consider that *Argumentation theory* provides a useful framework within which to develop such a dialogue.

7.2.2 Preferences and Decision Aiding

Decision support is based on the elaboration of preferential information. The basic idea in decision-aiding methodology is that, given a decision problem, we collect preferential information from the decision maker such that his/her system of values is either faithfully represented or critically constructed and thus we are able to build a model which, when applied, should turn a recommendation for action to the decision maker. Then the fundamental step in decision aiding is the modelling and the representation of the decision maker's preferences on the set of actions [26]. Furthermore, handling the preferences of a decision maker in a decision aiding process implies going through the following steps:

1. *Preference Learning* Acquire from the decision maker preferential information under form of preference statements on a set of "alternatives" A. Such statements can be on single attribute comparisons or assessments (I prefer red shoes to brown shoes; red shoes are nice) or multi-attribute ones (I prefer shoe x to shoe y; x is a nice shoe, x and y being vectors of information on a set of attributes). Possibly such statements can carry some further quantitative information or take a more complex form: my preference of x over y is stronger than the one of z over w or twice stronger, etc. Problems arising here include what to ask, how to ask, what rationality hypotheses to do about the decision maker, what degrees of freedom allow to the decision-maker's replies, how much the interaction protocol influence the decision maker (see more in [15,22,37,59,61,62,64,119,120,147]).
2. *Preference Modelling* Transform the preference statements in models. These can take the form of binary relations on the set of actions A, on the set $A \times A$, on the set $A \times P \cup P \times A$ (P being a set of reference points) or of functions [46,72, 89,110,112]. In this latter case we can talk about "measuring the preferences" on some appropriate scale. Once again the models may concern single or multiple attributes. An attribute to which we are able to associate a preference model is denoted as a criterion. There is a very standard theory on how single attribute (or uni-dimensional) preference models can be defined and these concern the well-known concepts of total order, weak order, semi-order, interval order, partial order, etc. It is less studied (mainly in conjoint measurement theory) in the case of multi-attribute preference models. We call representation theorems the results providing necessary and sufficient conditions for numerical representations of particular types of preference models. The typical problem is to fit the preference statements in one such representation theorem (if any)
3. *Preference Aggregation* In case we have several attributes on which we constructed preference models we may consider the problem of aggregating such preferences in one single model [116,117,138]. It is the typical problem of both social choice and multi-attribute utility theory. There exist several procedures and methods proposed for this purpose. In case we have global preference statements and/or a multi-attribute preference model we may consider the inverse problem: obtain preferences on single attributes compatible with such global statements and/or model.
4. *Exploiting Preferences* Constructing a preference model (either directly or through preference aggregation) does not necessarily imply that we can get an "operational result". That is we do not necessarily have an "order" such that we can identify a subset of maximal elements or at least a partial ranking, etc. It might be that it is necessary to make some further manipulation in order to get such a result. A simple case is to have an acyclic binary relation, but not

transitive, and to complete it through transitive closure. There are many procedures suggested for these types of problems [27, 137].

However, handling the preferential information provided by the decision maker may seem a simple task but in reality presents two majors problems.

1. From a formal point of view preferences are binary relations applied on a set of objects (alternatives, lotteries, combinations of values in a multi-attribute space, etc.). However, not always the decision maker is able to provide the information under such a form. She may be able to state that she likes red shoes, but not that these are necessarily better than brown shoes, or that she dislikes black shoes, but not more than that. She may have a target of shoes in mind but not necessarily these are the maximal elements of a ranking of available shoes in the market. In other terms the preference statements a decision maker may make do not necessarily fit the formal language traditionally used for representing preferences.
2. The way through which preferences are expressed depends on the process though which they are acquired and on the model expected to be used in order to elaborate them. However, we do not know a priori what type of method we should use. We also do not know what information is lost or neglected when we make a certain type of question. If we ask somebody what music he likes to hear we do not consider the option of silence. If we ask somebody how much is ready to pay more in order to increase safety of a certain device we implicitly assume that costs and safety can compensate one the other (but perhaps we never asked if this makes sense: worse, had we make the question, it is not sure this would be understood).

Thus, we can observe that in practice conventional mathematical languages used in decision theory do not necessarily fit with such requirements, therefore, it is necessary to look for languages explicitly allowing to take them into account (see for instance [130]; an alternative idea, the so called "Decision Rule approach", has been developed by Greco et al., see [57, 58]). Such idea was emphasised by Doyle and Thomason in [40] who suggest that it is essential to formalise the decision-making process more generally than classical decision theory does (where actions and outcomes are assumed to be fixed to start with, for example). Indeed, if you are to send a robot to complete a mission on a remote area, it is of course crucial to cater for the possibility that some *information may be missing*, that *preferences may change*, that *goals can be revised*, and so on; but also to provide *explanations and reasons* as to why this particular action has been eventually chosen. In short, many elements of the practice of decision analysis need to be incorporated in a model. But this means that the formal language used in classical decision theory is maybe not enough expressive, not enough flexible. One distinctive feature of AI approaches is precisely that they usually base their representation of agents' preferences on cognitive attitudes, like goals or beliefs (see [36]), which are expressive and intelligible to the human user. Moving to this type of representations allows to represent and reason about the underlying reasons motivating a particular preference statement: for instance, it becomes possible to identify conflicting goals or unforeseen consequences of certain actions to be chosen.

Regarding the issues of expressiveness and ability to deal with contradiction that we emphasised here, *argumentation* seems a good candidate. Indeed, the AI literature offers a corresponding argument-based approach to decision making [2, 7, 23, 31, 45, 55, 69, 91]. It appears that such approaches have much to offer to decision models, because they allow a great expressivity in the specification of agents' preferences, because they naturally cater for partial specification of preferences, and because they make explicit many aspects that are usually somewhat hidden in decision models.

7.3 Argumentation for Decision Support

In this section our first aim is to provide an overview of argumentation theory. As briefly mentioned before, it may offer several advantages to multi-criteria decision analysis, such as the justification and the explanation of the result, the expressive nature of the language used, or the possibility to handle incomplete or even contradictory information. Thus, after a brief introduction to argumentation theory, we present some decision-support systems that use diverse elements of this theory. The purpose is to show the different areas involving both decision and argumentation.

After that, we propose to discuss, in more detail, some approaches that have focused to what may be an argument for an action (a decision), and this discussion will be from an MCDA point of view.

7.3.1 Argumentation Theory

Under the classical logical reasoning (propositional, predicate, etc.), we can infer that a conclusion is true despite the additions in the set of proposition which allowed us to reach this conclusion. That is what we call *monotonicity*. In other words, no additional information can cause conclusions to be modified or withdrawn. There are no rules which allow to draw conclusions which may be faulty, but are nonetheless better than indecision. This is obvious if our reasoning concerns a mathematical demonstration (indeed classic formal logic has been developed mainly for such a purpose [148]). It is far less obvious if we are concerned by more general reasoning languages where conclusions are not necessarily definite truths.

For instance, if we look at our daily life reasoning, we can observe that this reasoning is not necessarily monotonic. Indeed, we can change our minds and move from one to another conclusion on the simple fact that new information is available or not. Besides, we are often faced with decision situations where we are far from knowing with certainty all data and information necessary to make this decision. We build our conclusion on the basis of available information at that moment and we reserve the right to change it at any time. Indeed, we do not have the time or mental capacity to collect, evaluate and process all the potentially relevant information before deciding what to do or think. In such cases monotonicity in reasoning is not very useful. In the sense that it does not offer ways to face this type of reasoning. Another example is where we take into account beliefs. Indeed, a human reasoning is not based solely on facts or action but also on its own beliefs. In this case, classical logic offers no theory about how to deal with beliefs. For instance, which beliefs to prefer given that certain things are known in a particular case.

These limitations of classical logic caused a number of Artificial Intelligence researchers to explore the area of *non-monotonic* logics. The emergence of these logics was initially developed by McCarthy [81], McDermott and Doyle [82], and Reiter [105]. Part of the original motivation was to provide a formal framework within which to model phenomena such as defeasible inference and defeasible knowledge representation, i.e., to provide a formal account of the fact that reasoners can reach conclusions tentatively, reserving the right to retract them in the light of further information. A familiar example in the literature of this kind of reasoning is the one of Reiter [106]:

Example 7.1. ([405])

- Birds fly
- Tweety is a bird
- Therefore, Tweety flies

The problem with this example concerns the interpretation of the first premise "Birds fly". To infer a valid conclusion, a possible interpretation can be: "for all x, if x is a bird, then x flies". But what is if Tweety is a penguin, a type of bird that does not fly? If we add this new information, the conclusion becomes false, but the second premise is true, therefore to maintain the deduction valid, the first premise should be false. However, this interpretation is problematic, because the first premise, in reality, still seems be true. As Reiter said:

> a more natural reading of this premise is one that allows for possible exceptions and allows for the possibility that Tweety could be an exceptional type of bird with respect to the property of flying, that is, 'Normally, birds fly' or 'typically the birds fly' or 'if x is a typical bird, then we can assume by *default* that x flies'.

The *default* refers to the fact that we should consider that Tweety flies until we can say or prove that it is atypical.

Much interest has been brought to non-monotonic reasoning from researchers in Artificial Intelligence, in particular, from those interested in model human intelligence in computational terms. The challenge has been to formalise non-monotonic inference, to describe it in terms of a precisely defined logical system which could then be used to develop computer programs that replicate everyday reasoning. Different non-monotonic reasoning formalism emerged, within AI, such as: default logic [105], autoepistemic logic [83], etc. In this chapter we are interested by one kind of these reasonings which is *argumentation theory*.

Indeed, argumentation provides an alternative way to mechanise such kind of reasoning. Specifically, argument-based frameworks view this problem as a process in which arguments for and against conclusions are constructed and compared. Non-monotonicity arises from the fact that new premises may enable the construction of new arguments to support new conclusion, or stronger counter-arguments against existing conclusions. Thus, argumentation is a reasoning model based on the construction and the evaluation of interacting arguments. Those arguments are intended to support, explain or attack statements that can be decisions, opinions, preferences, values, etc.

The important motivations that brought argumentation into use in AI drove from the issues of reasoning and explanation in the presence of incomplete and uncertain information. In the 1960s and 1970s Perelman and Toulmin were the most influential writers on argumentation. Perelman tried to find a description of techniques of argumentation used by people to obtain the approval of others for their opinions. Perelman and Olbrechts-Tyteca called this "new rhetoric" [94]. Toulmin, on the other hand, developed his theory (starting in 1950s) in order to explain how argumentation occurs in the natural process of an everyday argumentation. He called his theory "The uses of argument" [129]. Early studies using argumentation inspired methods in AI contexts can be found in the work of Birnbaum et al. [21] in which a structural model of argument embracing notions of support and attack within a graph-theoretic base. Moreover, the need of some model of argument for common sense reasoning can be traced to Jon Doyle's work on truth maintenance systems [39]. Doyle offered a method for representing beliefs together with the justifications for such belief, as well as procedures for dealing with the incorporation of new information.

In most AI oriented approaches argumentation is viewed as taking place against the background of an inconsistent knowledge base, where the knowledge base is a set of propositions represented in some formal logic (classical or non-monotonic). Argumentation in this conception is a method for deducing justified conclusion from an inconsistent knowledge base. Which conclusions are justified depends on attack and defeat relations among the arguments which can be constructed from the knowledge base. Instantiation of Dung's [41] abstract argumentation framework are typically models of this kind. In such a framework, an argumentation system is a pair of a set of argument and a relation among the arguments, called an attack relation.

However, in the decision-making context, it is not always possible to assume the existence of a fixed knowledge base to start the process. This point has been emphasised by Gordon and Walton [56], who state:

> in decision-making processes, we cannot assume the existence of a knowledge base as input into the process. Problems for which all the relevant information and knowledge have been previously represented in formal logic are rare.

Indeed, we are often faced with decision situations where we are far from knowing with certainty all data and information necessary to make this decision. We build our conclusion on the basis of available information at that moment and we reserve the right to change it at any time. As a consequence, argumentation can be seen as:

> a kind of process for making justified, practical decisions [...] The goal of the process is to clarify and decide the issues, and produce a justification of the decision which can withstand a critical evaluation by a particular audience. [56]

On the other hand, argumentation systems formalise non-monotonic reasoning in terms of the dialectical interaction between arguments and counterarguments. They tell us how arguments can be constructed, when arguments are in conflict, how conflicting arguments can be compared, and which arguments survive the competition between all conflicting arguments. Thus, an argumentation process can be described as a succession of different steps. Prakken and Sartor [100] suggest to distinguish the following layers in an argumentation process:

- **Logical Layer** It is concerned with the language in which information can be expressed, and with the rules for constructing arguments in that language. In other terms, it defines what arguments are, i.e., how pieces of information can be combined to provide basic support for a claim. There are many ways to address the form of an argument: as trees of inferences [68], as sequences of inferences (deductions) [134], or as simple premise–conclusion pairs. The different forms of arguments depend on the language and on the rules for constructing them [4,18,32,101,141]. The choice between the different options depends on the context and the objective sought through the use of argumentation. A general form is the one of *Argument Schemes* [142]. These are forms of arguments that capture stereotypical patterns of humans reasoning, especially defeasible ones [87, 143]. The first attempt to give an account of scheme was in the work of Aristotle. Indeed, he has introduced schemes in a common form of argumentation called *topics* (places) in *Topics* [10], *On Sophistical Refutations* and *Rhetoric* [9]. After that, argument schemes have been employed by Perelman and Olbrecht-Tyteca [94] in *The New Rhetoric*, as tools for analysing and evaluating argument used in everyday and legal discourse. More recently there has been considerable interest in schemes in computer science, especially in AI, where they are increasingly being recognised, in fields like multi-agent system, for their usefulness to refine the reasoning capabilities of artificial agents [102, 136]. For special use in Artificial Intelligence systems, Pollock's OSCAR identified some 10 schemes [96]. In addition, Reed and Walton [103] present some examples of application of argument schemes.
- **Dialectical Layer** It focuses on conflicting arguments and introduces notions such as counter-argument, attack, rebuttal, etc. Pollock [96] drew an important distinction between two kinds of arguments that can attack and defeat another argument, calling them *rebutting defeaters* and *undercutting defeaters*. A rebutting attack concerns arguments that have contradictory conclusions. An undercutting defeater has a different claim. It attacks the inferential link between the conclusion and the premise rather than attacking the conclusion. Moreover, recent studies have proposed to represent another kind of relation between argument, namely a positive relation, called *support relation* [4, 68, 135]. Indeed, an argument can defeat another argument, but it can also support another one. This new relation is completely independent of the attack one (i.e., the support relation is

not defined in terms of the defeat relation, and vice versa). This suggests a notion of bipolarity, i.e., the existence of two independent kinds of information which have a diametrically opposed nature and which represent contrasting forces [30]. Another way to challenge an argument is to use the concept of *Critical Questions* [63]. The critical questions are associated to an argument scheme. They represent attacks, challenges or criticisms which, if not answered adequately, falsify the argument fitting the scheme. In other terms, asking such question throws doubt on the structural link between the premises and the conclusion. They can be applied when a user is confronted with the problem of replying to that argument or evaluating that argument and whether to accept it [136, 144, 145].

- **Procedural Layer** It regulates how an actual dispute can be conducted, i.e., how parties can introduce or challenge new information and state new argument. In other words, this level defines the possible speech acts, and the discourse rules governing them. In fact, arguments are embedded in a procedural context, in that they can be seen as having been put forward on one side or the other of an issue during a dialogue between human and/or artificial agents. In other terms, one way to define argumentation logics is in the dialectical form of *dialogue games* (or dialogue systems). Such games model interaction between two or more players, where arguments in favour and against a proposition are exchanged according to certain rules and conditions [29]. The information provided by a dialogue for constructing and evaluating argument is richer than just a set of sentences. Indeed, the context can tell us whether some party has questioned or conceded a statement, or whether a decision has been taken accepting or rejecting a claim [5, 80, 98]. An influential classification of dialogue type is that of Walton and Krabbe [146]. Indeed, the authors have identified a number of distinct dialogue types used in human communication: Persuasion, Negotiation, Inquiry, Information-Seeking, Deliberation, and Eristic Dialogues.

Finally, recent research has shown that argumentation can be integrated in a growing number of applications. As examples we quote: *legal reasoning* [100], *handling inconsistency in knowledge bases* [3, 20], *knowledge engineering* [28], *clustering* [53], *multi-agent systems* [6, 90, 93], *decision making* [8, 23, 92].

In this chapter, we are interested in presenting the use of argumentation for multiple criteria decision aiding. Thus, the rest of this document is devoted to such a purpose.

7.3.2 Argumentation-Based Decision-Support Systems

Computer based systems are being increasingly used to assist people in decision making. Such systems are known as decision-support systems. The need to introduce arguments in such systems has emerged from the demand to justify and to explain the choices and the recommendation provided by them. Other needs have motivated the use of argumentation, such as dealing with incomplete information, qualitative information and uncertainty [8, 23, 45, 91]. In what follows we present a range of decision systems involving the use of argumentation. This section does not pretend to list all the existing systems but simply to give an overview of the different domains where argumentation has proven to be useful for supporting decision making. As we shall see, these different application domains may involve very different type of decision makers, from experts (medical domains) to potential buyers (recommender systems) or simple citizens (public debate); and even largely autonomous pieces of software that should act on behalf of a user (multi-agent). Of course the contexts of these applications vary a lot, from mediated human interactions to human–computer interaction. Our ambition is not to discuss the technical specificities of each of these, but merely to illustrate the wide range of application of argumentation techniques.

Supporting an Expert Decision Medical applications using argumentation have been numerous. We just cite three examples here. Atkinson et al. in [14] describe how to use argumentation in a system for reasoning about the medical treatment of a patient. The focus of the paper is the *Drama* (Deliberative Reasoning with ArguMents about Actions) agent which deals with a number of information sources (e.g., medical knowledge) in argumentation terms, and comes to a decision based on an evaluation of the competing arguments. Glasspool et al. in [50] present a software application (REACT, for Risks, Events, Action and their Consequences over Time), based on argumentation logic, which provides support for clinicians and patients engaged in a medical planning. The approach may provide a general aid for clinicians and patients in visualising, customising, evaluating and communicating about care plans. Shankar et al. present the system WOZ [121] as an explanation framework of a clinical decision-support system based on Toulmin's argument structure to define pieces of explanatory information. Initially, WOZ was developed as a part of the EON architecture [86] – a set of software components with which developers can build robust guideline-based decision-support systems. After that, an extension of WOZ was realised in order to build the explanation function of ATHENA DSS, a decision-support system for managing primary hypertension [52].

Mediating Public Debate Atkinson [12] presents one particular system – the PARMENIDES (PErsuasive ArguMENt In DEmocracieS) developed by Atkinson et al. [13]. It is designed to encourage public participation and debate regarding the Government's justifications for proposed actions. The idea is to enable members of the public to submit their opinions about the Government's justification of a particular action. Morge [84] presents a computer-supported collaborative argumentation for the public debate. The framework is based on the Analytic Hierarchy Process (AHP, [118]). The aim is to provide a tool to help the users to build an argumentation schema and to express preferences about it. The "Risk Agora" has been proposed as a system to support deliberations over the potential health and environmental risks of new chemicals and substances, and the appropriate regulation of these substances [78, 79, 104]. The framework is grounded in a philosophy of scientific inquiry and discourse, and uses a model of dialectical argumentation. The system is intended to formally model and represent debates in the risk domain.

Acting as a Collaborative Assistant George Ferguson et al. [42] implemented a mixed-initiative planning system for solving routing problems in transportation domains. By mixed-initiative, they refer to the fact that the computer acts as a collaborating assistant to the human, anticipating need, performing the tasks it is well suited for, and leaving the remaining task to the human. The unifying model of interaction was implemented as a form of dialogue. Both the system and human are participants in a dialogue. The ZENO system was developed to support decision making in urban planning [54, 55, 71, 104]. The system was designed to be used in a mediation system, an advanced kind of electronic discussion forum with special support for arguments, negotiation and other structured forms of group decision making. The argumentation model used by ZENO is a formal version of IBIS system (Informal Issue-Based Information) [109] modified for the urban-planning model. The modification allows the expression of preferences. Karacapilidis and Papadias describe an advanced Group Decision Support System [75] for cooperative and non-cooperative argumentative discourse, named HERMES System [68–70]. The system can be used for distributed, asynchronous or synchronous collaboration, allowing agents to communicate without constraints of time and space. Moreover, it supports defeasible and qualitative reasoning in the presence of ill-structured information. HERMES system is a variant of the informal IBIS model of argumentation [76, 109].

Recommending Novice Web Users Recommender systems are aimed at helping users with the problem of information overload by facilitating access to relevant items [77]. They are programs that create a model of the user's preferences or the users task with the purpose of facilitating access to items that the user may find useful. While in many situations the user explicitly posts a request for recommendations in the form of a query, many recommender systems attempt to anticipate the user's need and are capable of proactively providing assistance. These systems adopt two different approaches to help predict information needs. The fist one, called user modelling, is based on the use of the user model or user profile which is constructed by observing the behaviour of the user. The second approach is based on task modelling, and the recommendations are based on the context in which the user is immersed. Consequently, two principles paradigms for computing recommendations have emerged, content-based and collaborative filtering [51]. Advanced recommender systems tend to combine collaborative and content-based filtering, trying to mitigate the drawbacks of either approach and exploiting synergetic effect. ArgueNet system is an approach towards the integration of user support systems such as critics and recommender systems with a defeasible argumentation framework [31,34,35]. Critiquing and recommendation systems have evolved in the last years as specialised tools to assist users in a set of computer-mediated tasks by providing guidelines or hints [51,74,77,108]. ArgueNet provides solutions to problems encountered with existing recommender systems. Indeed, they are unable to perform qualitative inference on the recommendations they offer and are incapable of dealing with defeasible nature of user's preferences (see [34]). In this context, defeasible argumentation frameworks [97, 139, 140] have evolved to become a sound setting to formalise qualitative reasoning. The basic idea in this system is to model the preference associated with the active user and a pool of users by means of facts, strict rules and defeasible rules, named a DeLP program [49]. These preferences are combined with additional background information and used by an argumentation framework to prioritise potential recommendations, thus enhancing the final results provided to the active user. An application where such argumentation framework is used is the one proposed by Chesñevar et al. [33], where the authors present an argumentative approach to providing proactive assistance for language usage assessment on the basis of usage indices, which are good indicators of the suitability of an expression on the basis of the Web Corpus [73].

Autonomous Decision Making In recent years, argumentation has been promoted as a primary technique to support autonomous decision making for agents acting in multi-agent environment. Kakas and Moraitis [67] present an argumentation based framework to support the decision making of an agent within a modular architecture for agents. The proposed framework is dynamic as it allows the agent to adapt his decisions in a changing environment. In addition, abduction was integrated within this framework in order to enable the agent to operate within an environment where the available information may be incomplete. Parsons and Jennings [92] summarise their work on mixed-initiative decision making which extends both classical decision theory and a symbolic theory of decision making based on argumentation to a multi-agent domain. One focus of this work is the development of multi-agent systems which deal with real-world problems, an example being the diagnosis of faults in electricity distribution networks. Sillince [124] has investigated conflict resolution within a computational framework for argumentation. The author analysed how agents attempt to make claims using tactical rules (such as fairness and commitment). The system does not require truth propagation or consistency maintenance. Indeed, agents may support inconsistent beliefs until another agent is able to attack their beliefs with a strong argument. Parsons et al. in [93] try to link agents and argumentation using multi-context systems [60]. In this approach agents are able to deal with conflicting information, making it possible for two or more agents to engage into dialogue to resolve conflicts between them. Sycara [127, 128] developed PERSUADER, a framework for intelligent computer-supported conflict resolution through negotiation and mediation. She advocates persuasive argumentation as a mechanism for group problem solving of agents who are not fully cooperative. Construction of arguments is performed by integrating case-based reasoning, graph search and approximate estimation of agent's utilities. The paper of Sierra et al. [123] describes a general framework for negotiation in which agents exchange proposals backed by arguments which summarise the reasons why the proposals should be accepted. The framework is inspired by the work of the authors in the domain of business process management and is explained using examples from that domain.

7.4 Arguing Over Actions: A Multiple Criteria Point of View

The ultimate aim of a multi-criteria decision analysis study is to build a possible recommendation that will be considered useful by the users in the decision process where they are involved. Such recommendations are based on formal preference models [89]. Different steps (which can be implicit in a decision process) are required in order to obtain a recommendation [117]: formulate and structure the problem, build an evaluation model which allow us to obtain a formal answer to a given problem and construct a recommendation which translates the output of the process into the client's language. To reach a recommendation, multi-criteria decision analysis uses different tools for learning and aggregating preferences [25, 43, 138].

In an argumentation context, in general, the whole decision process will be made explicit in terms of different steps: construct arguments in favour and against each alternative; evaluate the strength of each argument [3, 20, 99, 125]; and compare pairs of choices on the basis of the quality of their arguments. The comparison can be based on different aggregation procedures of arguments (e.g., [24]).

Very broadly speaking, on the one hand, we have an evaluation process and on the other hand an argumentation process. The first is devoted to construct the necessary mechanisms to achieve "the best solution" on the basis of different points of view and preferences. The second one also leads to the "best solution" but in such a manner that will provide the explanation and the justification for this choice. Both processes appear to borrow two different ways to reach a solution, but in substance are very complementary.

In this section, we present a set of approaches that attempted to combine both evaluation and argumentation (or explanation). Before doing that, we start by discussing in general the notions of arguments and criteria. We then provide some examples of argument schemes proposed in the literature to account for decision making, and more generally to decision-aiding processes.

7.4.1 Arguments, Criteria and Actions

When facing a decision problem, the first step may be to identify the different objects submitted to the decision-aiding process. These objects can be potential decisions, projects, feasible solutions, items, units, alternatives, candidates, etc. and will be called the *actions*. In decision analysis, the concept of *criterion* is a tool constructed for evaluating and comparing the actions according to a point of view which must be (as far as possible) well defined. Thus, a criterion plays an important role in the process of action evaluation. Indeed, the construction of the set of criteria is a central activity in the decision-aiding process. It can be either the result of a direct process (creating from dimensions through direct questioning of the client) or of an indirect process (establishing criteria "explaining" global preferences expressed by the client on examples or already known cases [26, 59, 64]). A criterion can be regarded as a point of view against which it is possible to compare different alternatives. For such a comparison we need the user's preferences either explicitly stated (through a binary ordering relation) or implicitly associated to "values" (how much it is?) and "utilities" (measures or preferences). Therefore, a criterion models the decision-maker's preferences [26, 132]. On the other hand, the evaluation of an action can be the result of the construction of positive and negative reasons for that action. In argumentation theory, such reasons are formally represented by the mean of arguments. Thus, we can have both arguments in favour of and against an action. Those arguments are intended to support, explain or attack the action.

Consequently, we have two evaluation tools, but two different practices. An argument is designed more to justify the consequences of an action. The criterion, in turn, is built for purposes of preference representation. Indeed, the structure (or more precisely the premises) of an argument provides explicit evidence that will be used to support (or not) a precise action. The criterion, however, does not seem to have this feature. It certainly helps to model the decision-maker's preferences, which then can be used to justify why we can be in favour of an action. The problem is that this information is not explicit and visible for the decision maker. It is not easy to guess what is the model (or reasoning) that helped to promote an action rather than other.

A further difference between an argument and a criterion concerns the way by which the actions are compared. Decision analysis allows to identify models of preferences which can be used to compare and choose actions, either on the basis of an *intrinsic evaluation* (the evaluation of an action is based on its comparison to some pre-established norms) or a *pairwise comparison* (the choice is defined with respect to the comparison of the actions among themselves). In argumentation, however, the evaluation is rather intrinsic and the pairwise comparison of actions only comes as a by-product of the construction of arguments pro/con each alternative. One may argue that, in decision analysis, it is always possible to retrieve pairwise comparison on the basis of intrinsic valuations. But this is more than a simple technicality. The hypothesis done in almost all Multiple Criteria Decision Analysis methods (see [72, 116]) is that criteria represent complete preferences (all alternatives being comparable to all the other ones). This is empirically falsifiable as well as other hypotheses (for instance transitivity of preferences).

Finally, a basic requirement on the criteria set is separability: each criterion alone should be able to discriminate the actions, regardless of how these behave on the other criteria (further conditions can apply, that we shall not discuss here; for more details the reader is referred to [72, 117, 138]). With arguments, it is not possible to provide such a result on the set of action on the basis of a single argument. Each argument constructed concerns a particular action.

To summarise, the concept of criterion is devoted to model the decision-maker's preferences, and an argument is designed, in general, to explain and justify conclusions. From our point of view, argumentation can be seen as a way to make explicit the reasons justifying each preference ranking among actions. That is, if the decision making were to ask the question "why did you say that you preferred a over b?", we may give those reasons.

Under such a perspective, what can be the structure of such reasons? In other terms, what is the structure of an argument for an action? In fact, argumentation is usually conceived as a process for handling (potentially conflicting) *beliefs*. In AI, many systems have been proposed that allow to capture the defeasible nature of this kind of reasoning. Thus, the basic building block (the argument) can typically be defined as a premise/conclusion pair, whereby you state that this conclusion should be reached under these premises. What is discussed here is the truth-value of the conclusion, so an argument supporting a conclusion basically asserts some evidence to believe that this conclusion holds. When it comes to decision making though, this rather crude argument scheme needs to be refined. Indeed, as it has been recognised for a long-time now, a significant difference exists between argumentation for beliefs and argumentation for actions [46, 47]. This is better explained by means of a simple example, borrowed to Amgoud [1].

Example 7.2. This example is about having a surgery or not, knowing the patient has colonic polyps. The knowledge base contains the following information: "having a surgery has side effects", "not having a surgery avoids having side-effects", "when having a cancer, having a surgery avoids loss of life", "if a patient has cancer and has no surgery, the patient would lose his life", "the patient has colonic polyps, having colonic polyps may lead to cancer".

- The first argument: "the patient has a colonic polyps" and "having colonic polyps may lead to cancer", is considered as an epistemic argument believing that the patient may have cancer. While,
- The second argument: "the patient may have cancer", "when having a cancer, having a surgery avoids loss of life", is a practical argument for having a surgery. This argument is in favour of (or supports) the option "having a surgery".

In what follows, we address some of the approaches that have contributed to improve our understanding on what makes the argumentation about actions crucially different from mere epistemic argumentation. The idea is to understand how to judge or evaluate an action in order to conclude that we have an argument in its favour. Moreover, we propose to review these works, taking a decision theory perspective, more precisely a multi-criteria decision analysis perspective. Thus, the intuitive reading for an argument for an action is that action a will have "good consequences". Then, we must first somehow *valuate* the outcome of the action. In decision models, this would typically be done by using an ordered scale defining the different values that can be used to assess the action (for instance, marks from 0 to 20 for students). After that, what counts as a positive or negative outcome is specific to each decision maker, and depends on its (subjective) preferences. That is, we have to classify the outcome of the actions. In decision models, one approach is that the decision maker uses an evaluation scale and specifies a frontier, that is, a *neutral point* (or zone), thus inducing a *bipolar scale*. This will in turn allow us to determine what counts as an argument pro, or against, the action.

The concept of "bipolarity" in scales measuring value is not really new in the literature. Rescher [107] has been the first to introduce this concept. Roy [114] has introduced the concept of concordance/discordance in multiple criteria decision analysis (through the outranking procedures) and Tsoukiàs and Vincke [133] used a specific logic formalism in order to extend preference models under the presence of positive and negative reasons, among others. In this work, the concept of bipolarity refers to the existence of two independent types of information, positive and negative. The first provides support to the action and the second allows to express a disagreement against this action. Furthermore, in argumentation theory, several studies have emphasised the possibility of having this bipolarity (positive vs. negative) between arguments [4, 135]. Thus, we can construct arguments in favour of a conclusion and arguments against that conclusion.

Consequently, considering such aspects of multi-criteria evaluation, what are the benefits provided by these approaches and what are their limits in this type of context? To guide our discussion, for each approach we will try to provide answers to the following questions:

- How is the notion of criterion (point of view) captured in this model?
- How are pro/con arguments constructed?
  - How are the user's preferences represented?
  - What is the scale used to evaluate outcomes?
  - Is there an explicit reference to a preference model?

In our view, these issues include the major necessary basic elements to build an argument in favour of an action, by taking into account the different aspects of a multi-criteria evaluation.

7.4.2 Argument Schemes for Action

Our aim in what follows is to present and discuss different approaches that have attempted to define an argument for an action. we will use the following example to illustrate the different approaches.

Example 7.3. We want to select a candidate for a given position, and we have a number of candidates applying for it. We need to evaluate the outcome of each possible action, that is, how good is the situation induced by accepting each given candidate. For instance, a desired consequence is to have a strong enough candidate as far as academic level is concerned. Let us suppose that this is assessed by using a bipolar scale referring to marks, where 12 stands for our neutral point. Then, we could say that according to "marks", we have an argument in favour of accepting this candidate if its mark is more than 12.

Fox and Parsons [46] is one of the first works that tried to advocate an argumentative approach to decision making, building on Fox's earlier work [42]. They recognise and clearly state what makes argumentation for action different from argumentation for beliefs, and put forward argument schemes as shown in Table 7.1.

This argument can be represented as follows:

$$\begin{array}{llll} A \rightarrow C & : G & : + & e_1 \\ C & : G' & : + & v_1 \\ A & : (e_1, v1) & : + & ev_1 \end{array}$$

Table 7.1 Fox and Parsons' argument scheme

We should perform A (A has positive expected value)
Whose effects will lead to the condition C
Which has a positive value

where in the general formulae $< St : G : Sg >$: St (Sentence) represents the claim, G (Grounds) are the formulae used to justify the argument, and Sg (a signe) is a number or a symbol which indicates the confidence warranted in the conclusion. As explained by Fox and Parsons, the advantage of this representation is that it makes explicit three inference steps:

- e_1: that C will indeed result from action A
- v_1: that C has some positive value, and eventually
- ev_1: that A has a positive expected value

Clearly, steps (v_1) and (ev_1) require additional information in order to be able to assign values to situations, and to decide whether the action has indeed a positive expected value. The valuation of the condition is subjective (depending on the agent's preference), and represented here by "labelling the proposition describing C with a sign drawn from a dictionary", which can be qualitative or not and plays the role of *a scale*. Interestingly, different values can be assigned to C from different *points of view*. However, it is not clear how we can handle these different points of view in order to reach a conclusion. For instance, one can ask if these points of view are predefined.

We can apply this approach to Example 7.3, then we can say, for instance, opting for a given candidate (say a) could lead to an outcome where the chosen candidate has a mark of 14. This would be captured by the first epistemic step e_1 of the scheme, where ga stands for the justification of this step. Together with the two following steps, this could be represented with this scheme as follows:

$$\begin{array}{llll} chose_a \rightarrow mark = 14 & : \ ga & : + \ e_1 & \\ mark = 14 & : \ va & : + \ v_1 & (case\ 1) \\ chose_a & : \ (e_1, v_1) & : + ev_1 & \end{array}$$

The second step (v_1) means that the condition $mark = 14$ is positively evaluated by our agent (noted by symbol +) (it then counts as a positive argument), where va is the justification for this value assignment. Although this aspect is not deeply explored in the paper, a very interesting feature of this approach is that it makes explicit the grounds allowing to assign this value to this condition: what may count as obvious candidates to justify this value assignment, if we take the view of the multi-criteria decision approach, would be *the user's preferences* ("I consider that the mark is good beyond 12"), as well as *the preference model* used ("a mark is good (or positive) as long as it is beyond the limit previously stated").

We could also directly encode within this scheme that opting for a given candidate would lead to an outcome where the condition that the chosen candidate has a mark over 12 is satisfied, a fact that we consider positive. This could be represented as follows:

$$\begin{array}{llll} chose_a \rightarrow mark \geq 12 & : ga & : +\, e_1 & \\ mark \geq 12 & : va & : +\, v_1 & \end{array} \quad (case\ 2)$$

meaning that the condition $mark \geq 12$ is positively evaluated by our agent (noted by symbol +) (it then counts as a positive argument), where *va* is the justification for this value assignment. In this case, the nature of this justification is less clear, for it leads to support the agent's preferences.

These two alternative ways of representing argument schemes about actions seem somewhat unsatisfactory. On the one hand, choosing to directly represent the neutral action (here 12) drawn from the agent's preferences (case 2) drops the relation linking an action and its consequences. On the other hand, not representing it (case 1) assumes it is somehow encoded within a "value assignment" mechanism. Finally, this approach does not really acknowledge that actions themselves can be evaluated against a number of meaningful, predefined, dimensions: in fact, each condition induces a new dimension against which the action can be evaluated.

One of the most convincing proposals recently put forward to account for argument-based decision making is the one by Atkinson et al. [12,14]. They propose an extension of the "sufficient condition" argument scheme proposed by Walton [142], called argument scheme for practical reasoning.

The need for practical reasoning has emerged from the recent growth of interest in software agent technologies (see [149]), which puts action at the centre of the stage. Indeed, for software agents to have the capability of interacting with their environment they also need to be equipped with an ability to reason about what actions are the best to execute in given situations.

To define the scheme of Table 7.2, the authors have taken Walton's notion of a goal and separated it into three distinct elements: *states* (a set of propositions about the world to which they can assign a truth value), *goals* (propositional formulae on this set of propositions) and *values* (functions on goals). Therefore, unlike the previous approach, the notion of value is used here in a different sense. Atkinson explains [11] that values should not be confused with goals as "they provide the actual reasons for which an agent wishes to achieve a goal".

A given action can induce different state of affairs that may satisfy many goals, hence affecting different values. Indeed, a function *value* maps goals to pairs $\langle v, sign \rangle$ where $v \in V$, and *sign* belongs to the scale $\{+, -, =\}$. Thus, the valuation of the consequences of an action is based on a scale, related to v, which expresses

Table 7.2 Atkinson's argument scheme

In the circumstances R
We should perform A
Whose effects will result in state of affairs S
Which will realise a goal G
Which will promote some value V

the fact the value is promoted or demoted. So, unlike the previous one, this approach addresses explicitly action's consequences, and states actually desired by the agent (preferences). We believe this distinction remains important even if there is no discrepancy between observed and inferred states [19]. For instance, using our running example, we could have

$$value(mark \geq 12) = \langle academic\_level, + \rangle$$

meaning that the value (criterion) academic level is promoted when the mark is over 12.

In this approach, *values* seem to play the role of *criteria* in the sense that we can assess the action's consequences according to a point of view (here v). The particularity of a criterion is its ability to model the agent's desires. In this approach such desires are specified through the goals. However, the declarative nature of goals allows for more flexible classifications than what we typically have in decision models. For instance, it is possible to easily express that

$$value(age \geq 18 \wedge age \leq 32) = \langle youth, + \rangle$$

the value "youth" is only promoted when the *age* falls between 18 and 32. It is also important to note that this approach does not cater for an explicit representation of all the justifications of the value assignment (this only relies on the logical satisfaction: a goal reached or not, which justifies the value assignment). In this case, it is not possible to represent or indeed challenge the preference structured used. We also refer to Bench-Capon and Prakken [19] for a detailed discussion related to this scheme.

In Amgoud et al. [2], the authors proposed an approach explicitly linking argumentation to multi-criteria decision making. They see an argument as a 4-tuple $\langle S, x, c, g \rangle$ where

- $S \subseteq \mathcal{K}$: the support of the argument
- $x \in \mathcal{X}$: the conclusion of the argument
- $c \in \mathcal{C}$: is the *criterion* which is evaluated for x
- $g \in \mathcal{G}$: represents the way in which c is satisfied by x (goals)

where: $\mathcal{K}$ represents a knowledge base gathering the available information about the world; $\mathcal{X}$ is the set of all possible decisions; $\mathcal{C}$ is a base containing the different criteria and $\mathcal{G}$ is the set of all goals.

It is required that S is consistent when we add the fact that the action x has taken place. Here, in a way that is reminiscent of the previous approach, each goal g is explicitly associated to a criterion by means of a propositional formula $g \rightarrow c$, although the possibility of having goals referring to different criteria is also mentioned. More precisely, the goal g refers to the satisfaction of criterion c. Indeed, each criterion can be translated into a set of consequences. In turn, the consequences are associated with the satisfactory level of the corresponding criterion. This satisfaction is measured on the basis of a bipolar scale which has a neutral point that separates the positive and the negative values. Therefore, in this approach, unlike in [11], the use of (bipolar) scale is explicitly mentioned: the goals will fall either on the negative or on the positive side, which represents two subsets of consequences. In addition, this approach also allows for a quantitative measure of how good are the attained goals.

To apply this approach to Example 7.3, we may specify that the knowledge base has several layers

$$G_2^+ = \{mark \geq 14\};\ G_1^+ = \{14 > mark \geq 12\};\ G_1^- = \{mark < 12\}$$

which means that the marks are considered as "good" from 12, and even "very good" from 14, while it is insufficient when it is below 12. This comes together with formulae of the form

$$mark \geq 14 \rightarrow academic\_level$$

which explicitly states that the goal G_2^+ affects the criterion "academic level". Now each decision will have some consequences, which will in turn fulfil some goals or not. An argument is in favour of this decision if this later satisfies positively a criterion. In other terms it satisfies positive goals. However, an argument is against a decision if the decision satisfies insufficiently a given criterion. So, it satisfies negative goals. Thus, it is possible to identify arguments pro and cons a given decision x, by simply scanning the knowledge base and checking which positive (resp. negative) goals are satisfied by the occurrence of a given decision x. For instance, in our example of choosing a candidate a having a mark $= 14$, we have an argument in favour of $chose_a$ because it satisfies the positive goal G_2^+.

In conclusion we can notice that this approach seems to be the first tentative work that investigates the interest and the question raised by the introduction of argumentation capabilities in multiple criteria decision making.

To conclude, what we have seen along this section, is that each approach is rather marginally different from the other ones, but they share the fact that a decision process can be represented by explicit and distinct steps. Therefore, these approaches allow to focus on the different aspects of this process. Specifically, Fox and Parsons [46] are the only ones to explicitly represent the justification of a value assignment; however, they do not fully explore this avenue; and hardwire the possibility of having different criteria. Atkinson [11] makes this latter distinction clear, but on the other hand, does not cater for an explicit representation of all the justifications of the value assignment (this only relies on the logical satisfaction: a goal is reached or not, which justifies the value assignment). In this case, it is not possible to represent or indeed challenge the preference structures used. Amgoud et al. [2] also rely on the logical satisfaction of goals to justify the value assignment, but the goals are ordered in a way that indeed allows to refine the preference structure, to express various degrees of satisfaction of a goal. Still, this is directly encoded in the knowledge base and cannot be discussed in the process. Also, by using a bipolar scale, they constrain the syntax of goals and prevent themselves from using the full expressivity provided by the logic.

There are, on the other hand, many similarities between these approaches. First, the evaluation is made possible by an explicit representation of the consequences of the action. By relying on logic to represent such states of affairs, it is more expressive than the ordered scale that is usually used in decision models. One further possibility that is offered by this representation is that some action evaluation may be implicit or partial, whereas in decision models you would require each action to be evaluated on each criterion. The third, perhaps most striking similarity, is that they all rely on a method of *intrinsic evaluation*, and use more or less explicitly a neutral (or fictive) action.

However, if we consider the context of decision-aiding process, such approaches do not necessarily meet the expectations of such a field. Indeed, most approaches do not refer explicitly to the criterion which is the main tool to assess and compare alternatives according to a well-defined point of view. This concept does not only evaluate actions but reflects the decision-maker's preferences. Moreover, unlike in decision analysis, where several different problem statements are allowed (such as choosing, rejecting, ranking, classifying, etc.), the different argumentation-based approaches [68, 85] assume only one kind of decision problem, namely "choosing", where the aim is to select the best solution. Other approaches [12, 46] rely on the intrinsic evaluation of the consequences of an action, while many decision problems involve the relative evaluation of actions. Furthermore, they focus much more on the construction of arguments for and against an action and do not care about the construction of the final recommendation. Finally, several approaches [2, 24] used aggregation procedures based only on the number or the strength of arguments, while in decision analysis there exists a range of aggregation procedures. Regarding the latter, one can ask the question of how to justify the use of a procedure rather than another. Indeed, argument schemes can also be designed to make explicit aggregation techniques that can be used on the basis of preferential information.

7.4.3 Argument Schemes for the Decision-Aiding Process

Decision aiding has to be understood as a process, where several different versions of the cognitive artefacts may be established. Such different versions are due, essentially, to the fact that the client does not know how to express clearly, at the beginning of the process, what is his problem and what are his preferences. Thus, as the model is constructed, the decision maker revises and updates his preferences and/or objectives. Thus, it is necessary to have a language that enables to:

- capture the feedback loops present in such process,
- account for the inconsistencies which may appear during the process,
- account for irreducible uncertainties, possibly of qualitative nature and
- consider the necessary revisions and updates that may occur along such processes.

Under such a perspective, argumentation, as we have seen throughout this chapter, seems to be a good alternative in order to reply to such needs. A first work that tried to introduce argumentation within a decision-aiding process is the one of Dimopoulos et al. [38]. The aim of the authors was, on the one hand, to design autonomous agents able to undertake decision-aiding tasks and on the other hand to show why such a theory could be helpful for automatic decision purposes in autonomous agents. Moreover, they claimed to be able to provide a way allowing the revision and the update of the cognitive artefacts of decision-aiding process. To do that, they use different elements from Kakas et al. [66,67]. They establish:

- a number of object level rules showing the relations between problem formulation and evaluations models,
- the default context priority rules which help in applying the object level ones and
- the specific context rules which will give priority to the exceptional conditions rules.

The idea is to show how argumentation theory can be used in order to model the decision-aiding process, besides being a formalism enabling to take in account the defeasible character of the outcomes of such a process. It is clear that this approach represents a first step towards using argumentation in decision-aiding process. However, some features remain not clear or unsatisfactory. For instance, a decision-aiding process is an interaction between an analyst and a decision maker, and in this framework it is not very clear how we can model this interaction, even through an automatic system.

In a recent paper, Ouerdane et al. [88] advocated the use of argument schemes to handle the various stages of decision-aiding processes. Following this approach, a hierarchy of schemes can be designed, allowing to make explicit many of the assumptions that remain otherwise hidden in the process, for instance: justification and revision. The idea is to specify in argumentative terms the steps involved in a multi-criteria evaluation process. To do that, they make use of the notion of argument scheme already introduced before. Thus, a hierarchical structure of argument schemes allows to decompose the process into several distinct steps – and for each of them the underlying premises are made explicit, which allows in turn to identify how these steps can be dialectically defeated.

7.5 Conclusion

This chapter explores the links between decision aiding and argumentation theory. We did a brief introduction to argumentation (in an AI perspective), and discussed how and why it results to be useful in different contexts of decision aiding. We have also discussed several existing approaches to argument-based decision making involving (or at least referring to) more specifically MCDA techniques. In particular, we have seen that argument schemes:

- can be employed to explicitly state what justifies a chosen course of action. They can be based on various notions: underlying motivations, goals, or direct comparison of alternatives based on user's preference statement. Note that by relying on underlying goals, we must then chose a specific criterion to be able to compare two possible states of the world (do I prefer a situation where many secondary goals are satisfied vs. one in which only, but prominent, is?). There are of course many possible options here (see [24]) that we shall not discuss further. From our brief review, it came out that different approaches make explicit different steps of the process.
- are of primary importance: by explicitly representing the inference steps of an argument, we also define what counts as valid "critical question", that is how arguments will interact with each other (how they can be attacked and so on).
- more prospectively, argument schemes can also be designed to make explicit aggregation procedures that can be used on the basis of preferential information. For instance, a user may want to challenge the use of a weighted majority principle. Even more than that, we have seen that in a real decision-aiding process it is possible to modify problem formulations, or other statements.

So far, research has largely focused on the benefits of adopting an argument-based approach in that it allows to ground preferential information on underlying motivational attitudes. We want to stress here that we believe it also has much to contribute when it comes to capture the decision-aiding process. We conclude adding one more research perspective concerning interleaving argument structures and preferential information. This highly interesting avenue of research is taken by a growing number of researchers [17, 65], which only provides a further and final example of the vivid activity blossoming at the interface of MCDA and argumentation.

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Chapter 8
Problem Structuring and Multiple Criteria Decision Analysis

Valerie Belton and Theodor Stewart

Abstract This chapter addresses two complementary themes in relation to problem structuring and MCDA. The first and primary theme highlights the nature and importance of problem structuring *for* MCDA and then reviews suggested ways for *how* this process may be approached. The integrated use of problem structuring methods (PSMs) and MCDA is one such approach; this potential is explored in greater depth and illustrated by four short case studies. In reflecting on these and other experiences we conclude with a brief discussion of the complementary theme that MCDA can also be viewed as creating a problem structure within which many other standard tools of OR may be applied, and could therefore also be viewed as a PSM.

Keywords Problem structuring · Decision methodology · Case studies

8.1 Introduction

As the field of MCDA began to develop as a distinctive area of activity in the late 1960s and 1970s [6, 23, 42, 46, 53, 95] the initial focus was primarily on developing methods with relatively little attention to methodology or process, and to a large extent that emphasis remains strong. As the field became more established in the 1980s, consideration of both philosophical and methodological aspects of the use of MCDA started to grow [74, 84, 102] and increasing attention was paid to the structuring of MCDA models [13, 17, 98, 99]. In parallel with this, growing interest

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in the UK in so-called "soft" methods for operational research began to attract the attention of MCDA practitioners [7, 100] who recognized the potential of these approaches, in particular cognitive mapping [32], to support the MCDA process. The importance of problem structuring for MCDA is now widely recognised: the Manifesto for a New Era of MCDA by Bouyssou et al. [11] stressed the importance of understanding decision processes and broadening the reach of MCDA; Keeney [51] highlighted the need to pay attention to understanding values – in the sense of "what matters" to decision makers; issues relating to problem structuring in general and to value elicitation in particular were the focus of the article by Wright and Goodwin [101] and the associated comments [2, 5, 16, 27, 28, 40, 73, 88, 92, 97]; and the book by Belton and Stewart [8] is the first compendium of MCDA methods to afford significant attention to problem structuring.

We chose to quote Keeney [51, p9], of many possible authors, to reflect the concerns of many with regard to MCDA when he wrote:

> Invariably, existing methodologies are applied to decision problems once they are structured . . . such methodologies are not very helpful for the ill-defined decision problems where one is in a major quandary about what to do or even what can possibly be done.

He went on to articulate what could be interpreted as a need for problem structuring in the statement [51, p9]:

> What is missing in most decision making methodologies is a philosophical approach and methodological help to understand and articulate values and to use them to identify decision opportunities and to create alternatives.

The importance of good problem structuring in any context is widely acknowledged. Dewey [30] wrote "It is a familiar and significant saying that a problem well put is half solved. To find out what the problem and problems are which a problematic situation presents to be inquired into, is to be well along in inquiry. To mistake the problem involved is to cause subsequent inquiry to be irrelevant or go astray." The final sentence identifies what the statistician Kimball [55] labels as an error of the third kind – or solving the wrong problem – a concept which Mitroff and Featheringham [64] translate into domain of organisational problem solving and one which is widely recognised. In Belton and Stewart [8] we stress the importance of problem structuring both as a means of establishing the potential for MCDA and as an integral part of the MCDA process, as illustrated in Fig. 8.1.

The aim of this chapter is to provide an overview of current thinking and practice with regard to problem structuring for MCDA. We begin with a brief discussion of the nature of problems in general, of multicriteria problems in particular and what we are seeking to achieve in structuring a problem for multicriteria analysis. In Section 8.3 we outline the key literature which explores and offers suggestions on how this task might be approached in practice. Following on from this, in Section 8.4, we discuss the potential to provide integrated support, from problem structuring to evaluation, through the combined use of MCDA and one of the "problem structuring methods" (PSMs) described by Rosenhead and Mingers [83]. Section 8.5 reviews some of the practicalities of problem structuring for MCDA before a selection of case studies, illustrating the use of different processes and

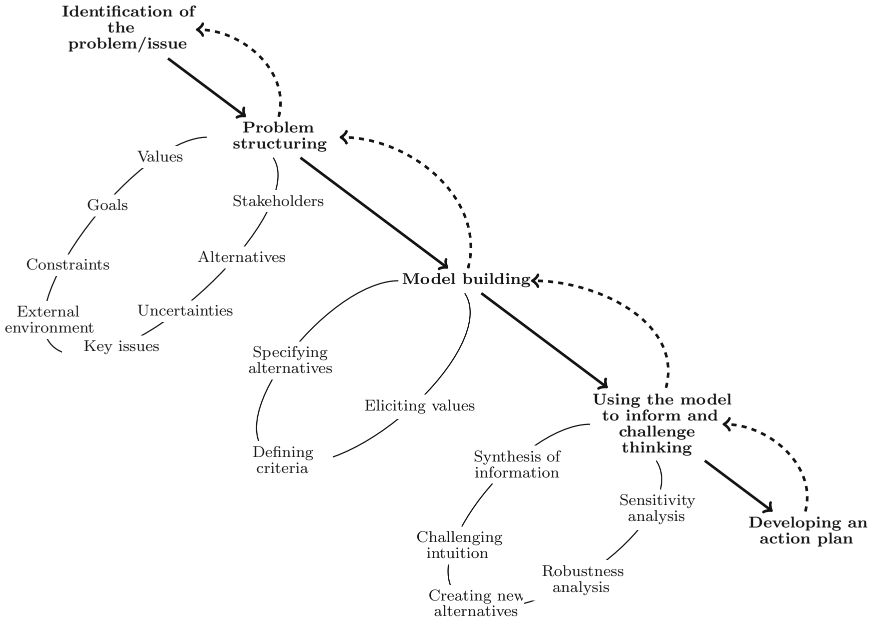

Fig. 8.1 The process of MCDA (from [8])

methods, is presented in Section 8.6. We conclude in Section 8.7 with reflection on the extent to which the MCDA process exhibits the characteristics of PSMs and might itself provide a framework for problem structuring in some situations.

8.2 The Nature of Problems and Problem Structuring for MCDA

Much has been written about the nature of problems and it is not our intention to delve too deeply into these issues here. Pidd [77], building on Ackoff [1], defines puzzles, problems and messes in terms of whether or not their formulation and solution are agreed or arguable. In the case of puzzles both formulation and solution are agreed (but not necessarily obvious); in the case of problems the formulation is agreed but the solution is arguable; and in the case of messes both are arguable. A mess is a complex and dynamic system of interacting problems, differently perceived by many different stakeholders. Other authors have used different labels to describe such situations, for example: Schon's [91] swamp, in contrast to the high ground; Rittel and Webber's [79] wicked as opposed to tame problems; or Dewey's [30] problematic situation.

We describe MCDA as a collection of formal approaches to help individuals or groups explore "decisions that matter" in a way which takes explicit account of multiple, usually conflicting, criteria [8] and will refer to a decision which merits such consideration as a “multicriteria problem”. A multicriteria problem could never be described as a puzzle in the above sense, as differing value systems prioritise different criteria, leading to a preference for different outcomes, thus the “solution” is almost always potentially arguable (a rare exception being when all parties agree on the relevant criteria and there is an option which dominates all others, performing at least as well or better than all others in all respects) . A structured multicriteria problem, one for which alternatives and relevant criteria have been defined, fits well with Ackoff’s definition of a problem. However, as the above quotation of Keeney highlights, multicriteria problems are not “given”, they must be revealed from a situation which, to a greater or lesser degree, is messy. The main focus of this chapter is on ways of facilitating this process, on problem structuring *for* MCDA, but before going on to discuss *how* this might be done, in the remainder of this section we briefly review *what* it is we want to achieve and why it is not straightforward.

Whilst recognising that any investigation or intervention is necessarily a process (Roy [84] in French and translated into English in [86]), as emphasised in Fig. 8.1, the starting point for multicriteria *analysis* is a well-framed problem in which the following components are clearly stated:

- The set of alternatives or decision space from which a choice (decision) has to be made
- The set of criteria against which the alternatives are to be evaluated
- The model, or method, to be used to effect that evaluation

Much of the literature on MCDA, in particular that focused on methods of analysis, takes a well-structured problem as a starting point. The effect of this is to convey, perhaps unintentionally, the erroneous impression that arriving at this point is a relatively trivial task. It is also our experience in practice that some decision makers who seek to engage in an MCDA process do so in the belief that they have a clear understanding of relevant criteria and the options open to them. More often than not, it is not so simple. Thus, the role for problem structuring for MCDA may be to provide a rich representation of a problematic situation in order to enable effective multicriteria analysis or it may be to problematise a decision which is initially simplistically presented. In both cases, the aim is to ensure that the multicriteria problem is appropriately framed and to avoid errors of the third kind; to achieve this, attention should be paid to the following, inter-related questions:

Who are the relevant stakeholders? In any decision, whether personal or organisational, there are likely to be multiple stakeholders – clients, decision makers, those affected by a decision, those who have to implement it. Who are they? Should they be involved in the process? What are their views, should they be taken into account and if so, how?

Are there key uncertainties or constraints and how should these be managed? There are inevitably internal or external uncertainties of some form and it is important to assess whether these should be explicitly incorporated in some way in the multicriteria model, explored through sensitivity or scenario analysis, or are not judged to be a significant concern.

What is the appropriate frame? Different frames may emerge for a number of reasons, for example: different stakeholder perspectives, or worldviews; the output of a process of creative thinking; or the consequence of critical reflection on an issue. Differently framed decisions can surface very different alternatives and criteria, potentially leading to very different outcomes. Consider the simple situation of a family with two teenage children owning two cars. One of these has just been written off and thus the family is faced with the question of what to do. The problem could be framed in many different ways, for example:

- What would be the best model of new car to replace the one which has been written off? This might surface fundamental objectives related to economy, safety, environmental impact, suitability for the users' needs, etc., and a list of candidate replacement cars.
- How many cars do the family need and what should these be? Similar objectives to those identified above might emerge, alongside other considerations which arise if the number of cars is changed, such as meeting the needs of different family members, and the list of potential options becomes much broader.
- How can the family use this opportunity to minimise its travel-related carbon footprint? This frame may not change the nature of the fundamental objectives, but by according greater importance to particular ones may encourage the creation of very different options, such as dual-fuel cars, acquisition of bicycles, even consideration of a house move which may lead on to thinking about possible job changes.

Not only do the different frames outlined above suggest problems of varying levels of complexity, but associated with that is the need to identify and adopt a commensurate approach to dealing with the issue. This is referred to by Russo and Schoemaker [87] as the metadecision and calls for attention to: process; method; and extent and nature of stakeholder involvement.

The approaches to problem structuring to be discussed in the following sections both encourage and support fuller consideration of the above issues, with the aim of achieving a shared view (i.e. shared by those stakeholders participating in the process) of the problem and designing an MCDA model which is characterised by:

- A comprehensive specification of options / alternatives
- A set of criteria which is preferentially independent, complete, concise, well defined and operationally meaningful
- Identification and incorporation of all relevant stakeholder perspectives
- An appreciation of critical uncertainties and how these will be explored

We have previously defined and find helpful, both in teaching and practising MCDA, the mnemonic CAUSE (Criteria, Alternatives, Uncertainties, Stakeholders and External/Environmental factors) as a framework to prompt consideration of these aspects. However, SUECA (a Portuguese card game) better reflects the order in which we consider the elements!

8.3 How Has Problem Structuring for MCDA Been Approached?

Attention to problem structuring for MCDA has developed in a number of ways. One direction has been to put increased emphasis on problem structuring within the existing MCDA framework. Keeney's [51] work on value focused thinking (VFT) exemplifies this. Once a decision problem or opportunity has been recognised, VFT emphasises the stages of surfacing and understanding the decision makers' values, and associated objectives and then using these as the basis for creative generation of alternatives prior to evaluation of selected alternatives and selection of a preferred one. Understanding the decision frame, defined by the decision context and associated fundamental objectives, is key to VFT and the set of alternatives for consideration should only be established, with an emphasis on creative design of good candidates, once the frame is clear. Keeney stresses the importance of ensuring that these three components (frame, objectives and alternatives) are coherently specified. As we saw earlier, a slight modification of the frame can lead to a differentiated, albeit overlapping, set of objectives and alternatives and it may make no sense to evaluate alternatives generated in one frame against the objectives for another.

Value focused thinking distinguishes the hierarchy of fundamental objectives relevant to a decision context (which are ends in themselves and capture "what matters" to those facing the problem, thus are necessarily subjective) from a network of means-end objectives (which is more objective and indicates how to achieve the fundamental objectives). However, it is important to recognise that fundamental objectives in relation to one decision situation may become means in a higher level context. The reader is referred to Keeney [51] for a full description of VFT and to Keeney and McDaniels [52] and Keeney [49, 50] for further applications.

Keeney contrasted value focused thinking with alternative focused thinking, starting from a specified set of alternatives and using these as the stimulus to identify values; whilst he did not actually advocate that these should be seen as competing approaches, some authors present them as potentially being so [60, 101]. Wright and Goodwin [101], recalling March's "Technology of Foolishness" [62], point to difficulties in identifying values which are potentially relevant to decision making with regard to previously unexperienced circumstances; and argue the importance of experience, real or simulated, in surfacing values. In their commentaries on this article, several authors point to the inter-related nature of the components of a problem and the need to explore these interactions and employ them in the process of learning about the issue.

The framework proposed by Corner et al. [24] and labelled "dynamic decision problem structuring" is one which seeks to do just that. This approach makes explicit and actively encourages a continuing process of iteration between value focused thinking and alternative focused thinking, as illustrated in Fig. 8.2. Consideration of values prompts creative thinking about possible alternatives, which in turn surface new values, and so on. The iterative process encourages decision makers to reflect on and learn about their values and the problem context. It may lead to a reframing

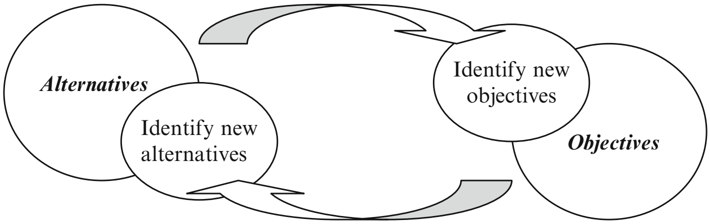

Fig. 8.2 An illustration of the process of "dynamic decision problem structuring" suggested by Corner et al. [24]

of the issue from a simple choice such as the selection of a new model of car, as outlined in the first example frame described earlier, to the more complex consideration of lifestyle captured in the third frame.

For the approach to be effective, it would seem to be important not to conclude the process prematurely. To some extent, this is a process which informally underpins much MCDA in practice, and is in part implicit in adopting an approach to structuring value hierarchies which combines top-down and bottom-up thinking [17]. Applications of the dynamic decision problem structuring process are described by Henig and Katz [43] and Henig and Weintraub [44].

Brugha's [14] proposed framework for MCDA, which is founded on principles of nomology, also recognises the importance of fully accessing a decision maker's relevant constructs in a comprehensive and convincing manner and advocates the iterative use of value (criteria) focused thinking and alternative focused thinking. In common with others [36, 45, 90] he suggests that the core concepts of Kelly's Personal Construct Theory [54], which also underpins cognitive mapping [31], are very relevant in problem structuring for MCDA.

A second stream of development in problem structuring for MCDA has been research directed towards integration of one of the problem structuring methods described by Rosenhead and Mingers [83] with a multicriteria approach. The majority of published applications have combined cognitive / causal mapping [15, 34, 35] with multi-attribute value analysis (for example, [3, 9, 39]) and this combination of methods is being used increasingly in the field of environmental management [63]. Neves et al. [69] use Checkland's Soft Systems Methodology (SSM) [18, 19, 21] to structure thinking and analysis of new initiatives to improve energy efficiency. Interestingly, the authors define the process of analysis of options, based on multiple perspectives, as a part of the SSM model. They do not go as far as discussing the actual analysis of initiatives but suggest that, as the aim would be a broad categorisation of options, ELECTRE-TRI [68] would be an appropriate methodology. Losa and Belton [61] describe the integrated use of conflict analysis and multi-attribute value analysis to help understand a situation of organisational conflict. Daellenbach [26] and Daellenbach and Nilakant [27] also discuss the potential for SSM to support problem structuring for MCDA.

A more recent approach, stimulated both by practical experience and theoretical considerations with regard to the linking of the two distinct methodologies of cognitive/causal mapping and multi-attribute value analysis, has been the development of Reasoning Maps [65–67]. Earlier applications combining the use of these two approaches (see examples referenced above) proceeded by first working with the decision makers, possibly in conjunction with other stakeholders, to develop a detailed causal map capturing participants' broad perspectives on the issue under consideration. Whilst this provides a rich description of the issue and ensures a sound basis for MCDA, the complexity and level of detail of the map (which may contain tens or hundreds of concepts) typically exceed that appropriate for multicriteria analysis. This necessitates a stage of transition between the causal map and MCDA model structure, a process facilitated by the analyst in negotiation with participants. Reasoning Maps seek to integrate these two phases by developing a focused causal map which enables the qualitative analysis of alternatives within the structure of the map, removing the need for transition to a simplified multicriteria model structure.

In the next section we explore the links between the problem structuring methods outlined by Rosenhead and Mingers [83] and MCDA in greater depth.

8.4 Problem Structuring Methods and the Potential for Integration with MCDA

The UK "school" of problem structuring methods (PSMs) began to emerge in the 1970s in reaction to a perceived failure of traditional, optimisation-based methods of OR to address messy problems [81]. In their Editorial to the first of two special issues of the Journal of the Operational Research Society devoted to PSMs, Shaw et al. [93] describe PSMs as:

> ... a collection of participatory modelling approaches that aim to support a diverse collection of actors in addressing a problematic situation of shared concern. The situation is normally characterised by high levels of complexity and uncertainty, where differing perspectives, conflicting priorities and prominent intangibles are the norm rather than the exception..

PSMs have been characterised in a number of ways and we aim to provide a richer sense of the methods by presenting a selection of these. Perhaps the classical characterisation of PSMs is Rosenhead and Mingers' contrast with the traditional OR paradigm in which they highlight the following aspects:

- Non-optimising, seeking solutions which are acceptable on separate dimensions without trade-offs rather than formulating the problem in terms of a single, quantifiable objective
- Reduced data demands, achieved by greater integration of hard and soft data with social judgements, thereby seeking to avoid problems of availability, reliability and credibility
- Simplicity and transparency aimed at clarifying the terms of conflict
- Conceptualising people as active subjects rather than treating them as passive objects

- Facilitating planning from the bottom up in contrast to an autocratic, hierarchically implemented process
- Accepting uncertainty and the need to address this through qualitative analyses and aiming to keep options open, rather than pre-taking decisions on the basis of expected probabilities

(based on Rosenhead and Mingers [83], p11).

Daellenbach [25, p533] provides a more process oriented description, which gives a sense of the nature of the methods in practice, as follows:

- Focusing on structuring a problem situation, rather than on solving a problem
- Aiming to facilitate a dialogue between stakeholders in order to achieve greater shared perception of the problem situation, rather than to provide a decision aid to the decision maker
- Initially considering 'What' questions, such as: "what is the nature of the issue?"; "what are appropriate objectives given the differing worldviews of stakeholders?"; "which changes are systemically desirable and culturally feasible?" and only then "how could these changes be best achieved?"
- Seeking to elicit resolution of the problem through debate and negotiation between the stakeholders, rather than from the analyst
- Seeing the role of the "analyst" as facilitator and resource person who relies on the technical subject expertise of the stakeholders.

An additional process requirement specified by Rosenhead and Mingers [83, p14–15] is that the process should be iterative, moving between "analysis of judgemental inputs and the application of judgement to analytic outputs" [81, p162] and that it should support partial commitment in the sense that whilst participants are satisfied that there has been incremental progress with respect to their concerns, there is no requirement for "... commitment to a comprehensive solution of all the interacting strands that make up the problematic situation" [81, p162].

The descriptions so far have highlighted the nature of the contexts in which PSMs are used and characteristics of the associated processes, but have said very little about the methods at the level of what is in the facilitator's toolbag. Each approach has its own, specific approach(es) to facilitate the capture, structure and analysis of relevant material, but share an emphasis on the use of visual and qualitative representations of an issue.

Rosenhead and Mingers [83] focus on five principal methods for problem structuring, namely Strategic Options Development and An-lysis (SODA) [34], Soft Systems Methodology (SSM) [18], Strategic Choice Approach (SCA) [41], Robustness Analysis [80] and Drama Theory [10], all of which have their roots in the UK communities of Operational Research or Systems Thinking. The first three of these approaches – SODA, SSM and SCA – are the most generally applicable, in the sense that they can be used to surface ideas and structure thinking with respect to any broadly defined issue, and, as a consequence, the most widely known and applied [33]. Robustness Analysis has a particular focus on consideration of uncertainty about the future and Drama Theory on the tensions underlying the potential for cooperation or conflict between multiple parties. A participative approach

Table 8.1 Problem structuring methods and the link to MCDA

Method	Key features	Potential link to MCDA
SODA	Beginning with a process of idea generation, seeks to capture and structure the complexity of an issue reflected by multiple perspectives	Can be used flexibly with MCDA, as a precursor or in an integrated manner. Incorporates simple, holistic preferencing
SSM	Uses rich pictures, CATWOE, root definitions and conceptual models to explore the issue from a number of different perspectives	Can be used flexibly with MCDA, as a precursor or in an integrated manner
SCA	Four modes – Shaping, Designing, Comparing, Choosing. Focuses on key uncertainties (about related areas, environment and values) and analysis of interconnected decision options	Parallels MCDA – shaping and designing highlight key choices and comparing evaluates these using a simple form of multicriteria evaluation
Robustness Analysis	Focuses on identifying options which perform well in all possible futures	Complementary to MCDA – focus on different aspects of an issue
Drama Theory	Appropriate in multi-party contexts, where the outcome is dependent on the inter-dependent actions of the parties – seeks to identify stable options	Drama theory requires possible futures to be ranked according to preference, which is done holistically

to MCDA, together with System Dynamics and Viable Systems Modelling are also briefly outlined, as "near neighbours" of PSMs [83, Chapter 12]. Other approaches which it is felt might lay claim to share the same characteristics [83, p xv] are Ackoff's Idealized Planning, Mason and Mitroff's Strategic Assumption Surfacing and Testing and Saaty's Analytic Hierarchy Process. The key features of the five principal approaches and the potential for integration with MCDA are summarised in Table 8.1.

In exploring how PSMs and MCDA might be combined it may be helpful to distinguish between process and modelling. Although all PSMs stress the importance of a participative process, many considerations are general in nature and applicable to many forms of process consultancy, including a participative approach to MCDA [37, 76, 89]. SODA [32, 35, 38] is the only PSM which pays explicit attention to process, distinguishing different modes of working. In the first of these, known as SODA I, individual cognitive maps are developed in 1 to 1 interviews with participants; these maps are then merged to create a group map which provides the starting point for a facilitated workshop. In SODA II participants are jointly involved in creating a shared model in a facilitated workshop, either using a manual Oval Mapping process, or a direct entry multi-user system. An intervention using MCDA, in isolation or in combination with mapping or another PSM, might adopt any of these three processes. The question then becomes one of how the modelling

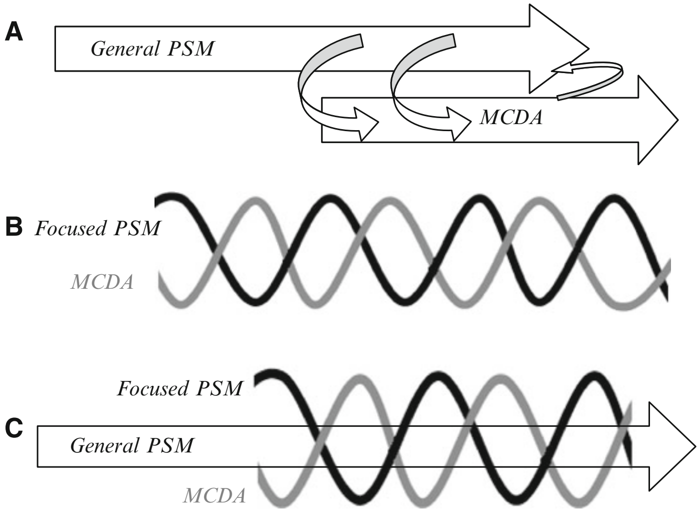

Fig. 8.3 Combining problem structuring and multicriteria modelling

methods which define the different approaches to problem structuring and MCDA can be effectively combined. The diagrammatic representation of the MCDA process in Fig. 8.1 suggests a natural way to do so, with the problem structuring phase supported by one of the more general PSMs and providing a rich description of the problem from which an appropriate multicriteria model may be derived. This way of working, illustrated in Fig. 8.3A, provides the foundation for an Honours year class, entitled "Problem Structuring to Evaluation", which one of the authors has taught, together with Fran Ackermann since 2003. Whilst there is rich potential for interaction and iteration between models, in reality this is dependent on practical circumstances, in particular constraints on time and resource. Figure 8.3B illustrates a way of working which is perhaps more likely if using MCDA with one of the more focused PSMs (see, for example [61]); however, such an intervention might be further enhanced if initiated and supported by the use of a general PSM, as shown in Fig. 8.3C.

These models for integration of PSMs and MCDA resemble those presented by Pidd [78] and Brown et al. [12] for mixing "soft" and "hard" OR. These are further discussed by Kotiadis and Mingers [56] who explore philosophical issues relating to paradigm commensurability, cognitive issues regarding inclination and ability to work across paradigms, and practical challenges associated with mixing hard and soft methods. Rather than "soft" and "hard" we are concerned here with PSMs and MCDA and our intention is not to suggest that MCDA should be seen as a "hard" methodology (it is our view that the philosophical positions of the actors engaged in an MCDA decision process rather than the adopted method define the paradigm), nevertheless, we believe that the issues raised are ones with which the MCDA community should seek to engage.

8.5 Implementing Problem Structuring for MCDA

In the next section, we illustrate the use of different problem structuring processes and methods in conjunction with MCDA through a number of case studies. It is useful, however, to reflect first on some general practicalities in facilitating the process of problem structuring for MCDA.

We have seen that the problem structuring phase of MCDA needs to be directed towards obtaining a shared view (i.e. shared by those stakeholders participating in the process) of all of the issues summarised in the CAUSE (or SUECA) checklist mentioned earlier. In order to apply any of the methodologies of MCDA, the output from the problem structuring phase needs at very least to include a clear statement of the alternatives or decision space to be considered, and of the criteria to be used in evaluating or comparing elements of the decision space.

In applying problem structuring in the MCDA context, it will become evident from the case studies in the next section that at least three phases of the process may be identified.

"Brainstorming" or "Idea Generation" in groups or by one-on-one interviews between analyst and different stakeholders or their representatives, to get as many issues on to the table as possible. At this stage the process is divergent, and the facilitator/ analyst needs to encourage unconstrained lateral thinking. The present authors have made extensive use of "post-it" sessions (or related methods such as oval mapping) for group brainstorming, the practical implementation of which is described in Section 3.3 of Belton and Stewart [8]. Computer systems for such brainstorming are, however, also available, for example Group Explorer or *Thinktank* (`http://www.groupsystems.com/`).

Representation of issues (jointly or separately for different stakeholders) in a form which facilitates clustering of the generated concepts into categories which eventually may be associated with all the "CAUSE" issues. The present authors have made extensive use of causal mapping for this step and this is illustrated in the case studies.

Apart from providing a succinct summary of issues, such maps can be analysed in order to identify, inter alia:

- Nodes which have outgoing influence or causal arcs, but no incoming arcs: these suggest driving forces which may be external constraints or action alternatives.
- Nodes which have incoming influence or causal arcs, but no outgoing arcs: these suggest extrinsic goals or consequences, which may be associated with ultimate performance measures or criteria of evaluation.
- Closed loops of cause–effect relationships (arcs): action may need to be taken to break such loops, especially of "vicious" rather than "virtuous" cycles.

Critical evaluation of the emerging structure: The structure which emerges from the previous step does still need to be subjected to critical evaluation by analysts and stakeholders (either in further workshops or through individual written submissions) before moving to the more conventional analytical processes of MCDA. In particular, as an agreed decision space and criteria emerge from the process, the facilitator needs to ensure a continuing and critical reflection on these outputs and their impact on the choice of MCDA methodology to be adopted at the analytical phase.

In our experience, key questions which should be posed include:

- *Should the decision space be represented by a discrete set of alternatives, or as a continuous set of possibilities? Have alternatives been comprehensively specified?* All too often, in reported applications, it seems that the specification of discrete or continuous sets is dictated more by the authors' preferred methodology than by the demands of the problem at hand.
- *What are the fundamental points of motivation for the set of criteria used? Are the criteria complete, while at the same time being non-redundant, preferentially independent and operationally meaningful and well-defined?* Such motivation is as critically important to the practice of MCDA as the criteria themselves. Once again, it seems that in many reported applications, the criteria are simply listed with at most a brief statement that the criteria were agreed by all participants.
- *Which uncertainties are critical to assessing performance of different alternatives and how will consideration of these be incorporated in the analysis?*
- *Are other worldviews or problem frames still possible?*

Whilst problem structuring methods can assist in achieving the critical reflection, they do not guarantee it.

It must be emphasised, however, that there is not a discrete point of movement from "structuring" to "analysis". The analytical phase may well raise additional unresolved questions which demand a restructuring as illustrated in Fig. 8.1. For example:

- The process of evaluating alternatives in terms of identified criteria may generate unexpected conflicts between stakeholders, signifying criteria which are either inadequately defined operationally or incomplete in some sense.
- The absence of alternatives which perform at a satisfactory level on all criteria may leave a feeling of unease or dissatisfaction with the results of the MCDA, leading to a need to seek further alternatives.

We now turn to an illustration of the problem structuring process by means of four case studies.

8.6 Case Studies in Problem Structuring for MCDA

In this section we describe four case studies. The first, concerned with the allocation of fishing rights in the Western Cape of South Africa, provides a brief overview of cognitive/causal mapping and describes its use with a number of community groups as a precursor to the development of a multicriteria model. The second study, focusing on a critical funding decision for an SME, starts with mapping individuals' ideas in one-to-one interviews and uses this information as the starting point for a group workshop. In this case the use of a very simple multicriteria evaluation serves to crystallise the ideas emerging from the map, but remains strongly linked to and dependent on it. The third case summarises SSM and outlines its use with another SME undergoing organisational growth and culture change. In this situation the SSM process provides a broad understanding of the company's activity and highlights a specific issue which is explored in detail using a multi-attribute value analysis. The fourth intervention also makes use of mapping to structure a value tree, but sets the scene for the final section of the paper which considers MCDA *as* problem structuring.

Case Study 1: Fisheries Rights Allocation in Western Cape (South Africa) – from Group Mapping Sessions to a Shared Multi-attribute Value Tree

This case has been reported in [47, 48, 94]. The background was that many fisheries in the Western Cape province of South Africa are under stress, with decreasing stocks and increasing poaching of threatened species. Fishing has, however, been the traditional occupation of many poverty stricken communities, providing a source both of food and of financial income. Before the constitutional changes in 1994, fishing quotas were largely allocated to large commercial companies, and traditional communities were marginalised. With the new constitution, the government pledged to ensure that the allocation of fishing rights would grant greater representation to formerly disadvantaged groups. With the new policies, however, there arose a number of substantial conflicts, between goals of transformation and socio-economic upliftment, of preservation of threatened stocks, and of sustainable economic growth.

Three community groups were selected for purposes of analysing and evaluating potential strategies for fisheries rights allocation. These groups were judged to be representative of the broader fishing community in the province, and included a relatively rural fishing community (Hawston/ Hermanus), located along a stretch of coast about 100–150 km east of Cape Town, and two rather more urbanised communities within the boundaries of the City of Cape Town (Kalk Bay and Ocean View).

Workshops were conducted with representatives in each community. These started with opportunities for free expression of concern, after which a more formalised brainstorming session using "post-its" was employed in order to capture perceptions of problems, goals and potential courses of action. Results from the post-it sessions were summarised in the form of causal maps as a basis for further interaction. In other words, a relatively simple "brainstorming" was structured by the project team into a causal map which could be reflected back to all groups involved (including officials from the relevant government department).

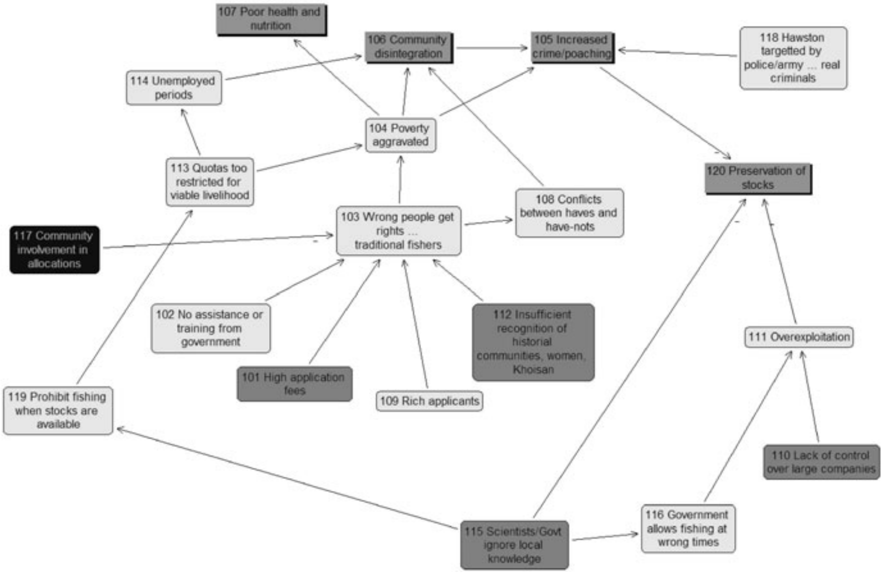

Fig. 8.4 Causal map from Hawston workshops

As illustration, the causal map extracted from the Hawston/Hermanus community is shown in Fig. 8.4. A central theme is probably that of concept 103, namely that the "wrong" people were receiving rights, rather than the traditional fishing communities. A number of reasons for this feature are identified, suggesting possible courses of action and external forces. More importantly to the interests of MCDA, a small number of concepts appear near the top of the map, and which are at the end of chains of links. These concepts can then be interpreted (at least tentatively) as the primary objectives of the rights allocation process as perceived by the Hawston/Hermanus community, namely (a) reduction of poverty and unemployment in the community, (b) countering of gangsterism and crime, and (c) preservation of stocks.

Although, different structures emerged from the three community groups, it was evident that there was also a high degree of commonality, especially as regards the objectives identified and the key driving forces. The emerging structure from the three causal maps, especially as regards the objectives identified from concepts at the ends of chains of links, almost naturally fell into a value tree (hierarchy of objectives). In summary, the upper level of the value tree consisted of the following criteria (which had been made more operationally meaningful than the more abstract fundamental criteria such as "community cohesiveness"):

1. Previous involvement in the industry
2. Knowledge and skill in the fishery
3. Status as a "historically disadvantage person" (HDP)
4. History of compliance with previous quotas and conservation regulations

The value tree was presented back to the communities for their consensus. At final workshops with community representatives from two of the communities (Hawston and Ocean View), measures of the relative importance of the above four primary criteria were obtained by providing delegates with stickers to place against each. Once again a high level of consistency of views between these two communities emerged.

Interaction with the relevant government department, the Marine and Coastal Management (MCM) directorate of the Department of Environmental Affairs and Tourism, was carried out separately from that with community representatives. The process was similar, but also included a deconstruction of effective criteria that were in operation in previous rights allocations (even if not explicitly stated at the time). Similar structures evolved, and the objectives identified with MCM did include those identified by the communities. However, additional criteria emerged, notably those related to concerns for financial stability of the industry and regional economic development. Finally, an aggregate value tree incorporating all concerns could be tabled as a basis for future rights allocation decisions. This tree is presented in Fig. 8.5.

The problem structuring exercises with the community and MCM can still be described as "soft OR". In contentious problems such as the fisheries rights allocations

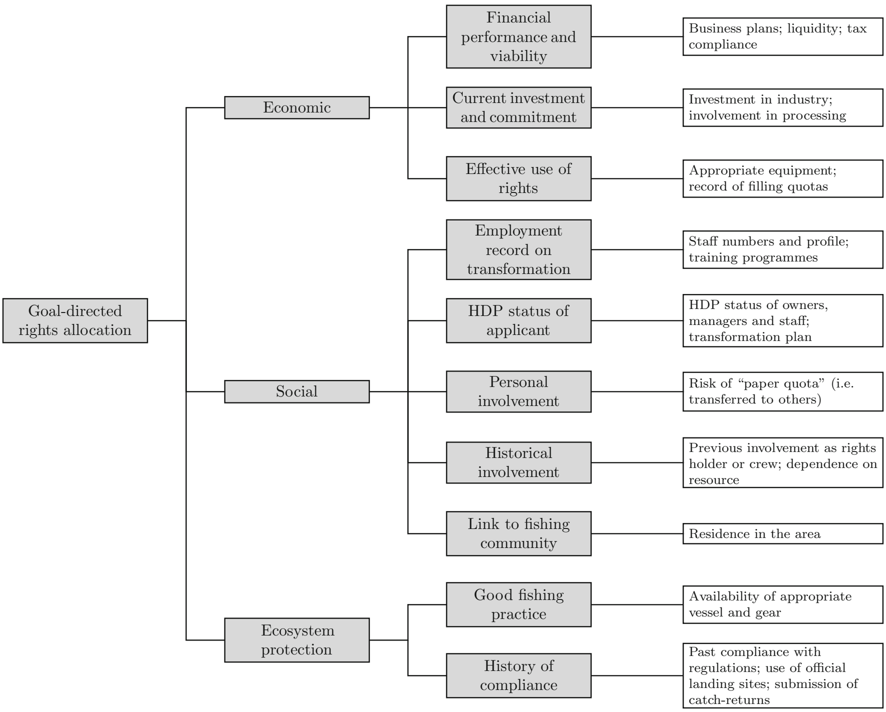

Fig. 8.5 Overall value tree for fisheries rights allocation

under discussion here, such identification and documentation of community views are critical. The process cannot, however, stop there! Eventually, a decision process for the final identification of those who will or will not be allocated rights has to be put in place. This process needs to be well documented, in a manner which facilitates auditing and public defence. In practice, this requires some degree of quantification. It is at this point that a formal additive value function model was proposed as a basis for "scoring" applicants. From the value tree, it was possible to develop a simple spreadsheet-based decision support system both to capture relevant information from applicants for rights and to implement the simple value function model.

Case Study 2: To Venture for the Venture Capital or not? From Individual Mapping to Group Workshop

This intervention was to support Visual Thinking, a small UK-based software company in the early development stage of growth, in thinking through a major decision regarding funding. The company designed, developed and marketed simulation software, also providing consultancy, training and technical support. The six employees were all highly educated, welcomed challenges and were ready to try new ideas, furthermore, the company structure was very flat and the culture open and informal: it fitted De Geus' [29] description of a learning organisation. The CEO, who was the majority shareholder, had been seeking venture capital funding to enable the company to grow and expand internationally and this intervention was brought about when an initial offer was received. The offer was less favourable than had been initially hoped for and the aim of the intervention was to help the company decide, in the short space of time available, whether to accept the offer or to continue to look for other options. The offer of venture capital was largely premised on the appointment of a senior marketing manager, who would pursue a particular strategy, and a larger board with a non-executive chairman who supported that strategy. The decision represented a real dilemma for the company. Key issues were related to: the achievable level of growth with and without the funding; the loss of control, together with organisational and cultural changes that would potentially result if the funding was accepted; and the costs and uncertainties associated with the proposed marketing strategy.

Although there was a clear decision, with well-defined options, facing the company – i.e. to accept or reject the offer – it was felt that much benefit could be derived from spending time to fully explore the issues surrounding the decision, in order to ensure that all employees had the opportunity to contribute their views, to understand each other's views and the implications of the decision. Although the culture was an open one, employees had different levels of knowledge about the issue and different perspectives as a consequence of their position and length of time in the company. It was decided to take an approach which started with one-to-one interviews of all employees, during which their views would be mapped; the material from these interviews would then be synthesised and form the basis for a one day workshop which we expected to lead to an evaluation of options open to the company.

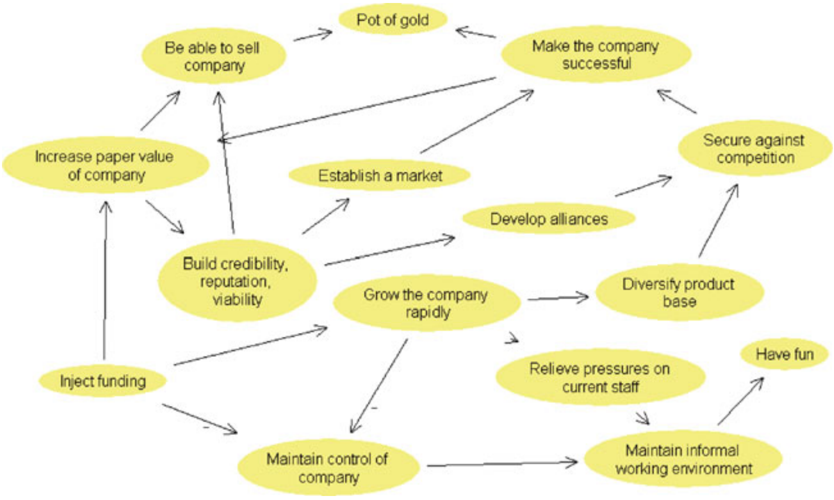

Fig. 8.6 Synthesized map which provided the starting point for the visual thinking workshop

The individual interviews revealed a strong consistency in employee values and desire to see the company grow in order to realise, eventually, a "pot of gold". However, different perspectives on how this might be achieved were apparent and the interview process led one employee to realise that they had more substantial concerns about developments that would ensue from accepting the venture capital than they had previously thought. The key concepts from the individual maps were consolidated into one map (Fig. 8.6) by the team of facilitators before the workshop and, as this was of a manageable size, it was manually recreated step-by-step in front of the group of employees. As the map unfolded, participants were able to comment on the concepts and links and to further develop these.

When new ideas were no longer being generated the focus was shifted towards creative thinking about options. Participants were asked to suggest potential futures that they had not or would not normally consider and for each of these options to collectively identify three positive and three negative consequences for the company; as new ideas surfaced these were added to the map. By this time the map had grown to include a very large number of concepts and it was judged appropriate to refocus discussion on the decision. The group decided to evaluate four potential strategies: the one associated with accepting the venture capital; the status quo (i.e. continue to grow at the current rate); and two intermediate options which reflected less aggressive approaches to growth. The map was used to identify and prioritise company goals and those judged to be most significant for the decision (in terms intrinsic importance and the extent to which they differentiated selected options) were transferred to another whiteboard to create a values/consequences table which formed the basis for a very simple evaluation using a five-point qualitative scale. This was represented visually as a profile graph and further discussion ensued, but it was felt that no further analysis was necessary (nor was it possible, given the impending deadline to respond to the offer).

It was the CEO who had to make the final decision, but he wanted to be able to do so in the knowledge that the whole company was behind it. The workshop enabled this through the development of greater group cohesiveness, a shared understanding of the issues and a common sense of purpose. When the CEO announced that his decision was to accept the offer everyone was in support, including the person who had had doubts. The intervention was part of a wider research project, which involved follow up interviews to capture participants' reflections on the process. For the CEO the workshop did not prompt new ideas, but it did provide a framework to think about the decision, allowed him (and others) to see the links between issues and to assess their importance and consequence. For the employees, in addition to this understanding, it generated new ideas about how they could see their roles developing. It was interesting that the participants did not feel that any specific stage of the workshop was in its own right particularly helpful, rather that it worked well as a whole. It was felt that the final evaluation crystallised the ideas that had been built up during the day, but could not have been done without the detailed discussion that developed the map – the specified goals were not completely independent, nor were they comprehensive or clearly defined, nevertheless, the exercise was judged to be useful. It may have been possible, given more time, to develop a more "robust" or "credible" model; however, or it could be said that in this instance the detailed map legitimized the sparse, simple multicriteria model, or perhaps that the sum of the models was requisite [74].

Case Study 3: Meeting Customer Needs: From SSM to Multi-attribute Value Analysis via CAUSE

This small case study is based on a project which was carried out by a group of students as part of the class mentioned earlier. Their client was the Managing Director (MD) of King Communications and Security Ltd., a Scotland-based SME providing integrated security and telecommunications solutions to business. The MD has been in post for less than a year and is seeking to improve customer service at a time of substantial business growth.

Checkland and Poulter [20], who provide a very accessible explanation of the process of SSM, describe it as an "... action-oriented process of inquiry into problematic situations in the everyday world; users learn their way from finding out about the situation to defining/taking action to improve it. The learning emerges via an organised process in which the real situation is explored, using as intellectual devices – which serve to provide structure to discussion – models of purposeful activity built to encapsulate pure, stated worldviews." The four elements of the SSM learning cycle are shown at the bottom left of Fig. 8.7, these are: a process of finding out about a problematical situation; exploration of the situation through the building of a number of purposeful activity models relevant to the situation, each corresponding to a clearly defined worldview; use of the models to prompt questions and structure discussion about the real situation with a view to identifying changes

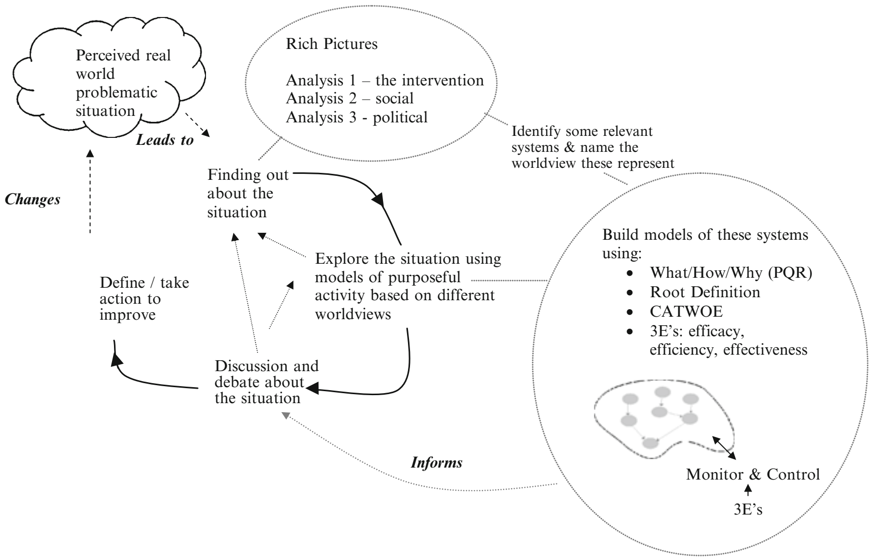

Fig. 8.7 SSM process and tools

that are systemically desirable and culturally feasible; and action to improve the situation. Figure 8.7 also illustrates the range of tools, or techniques which might be used to support each element of the SSM process; there is no compulsion to use all of these, or to do so in any particular order.

Rich pictures are probably the most widely known device of SSM; their purpose is visually to represent the main features of a problem situation – the structures, processes, stakeholders, relationships, culture, conflicts, issues, etc. Figure 8.8 shows the initial rich picture drawn with KCS Ltd. The picture shows current KCS clients on the right and (below) the nature of the business they bring and associated products. The left hand side of the picture shows the company and its operations. We see the organisational structure – the management team of father and son, the office staff, and three teams, each with their own leader, focusing on different elements of the business (service & maintenance, installation and nursing jobs). Also shown (at the top of the picture, towards the right) are the two systems currently used to manage customer information and jobs (Job Master and Merlin). Reviewing the rich picture at this point seemed to suggest a gap between the resources available and the customer (jobs). It emerged that this was a key issue for the company as there was no distinct system for allocating jobs to the staff, compromising the efficiency of the company's operations. The middle of the picture, which depicts the system for prioritising and allocating work, was then defined as a result of evaluating systems currently in place and ideas in the pipeline. This issue became the focus of the further analysis.

The rich picture ensured that the client and consultants had a shared understanding of the company's organisation and focus; gaps in the picture began to highlight

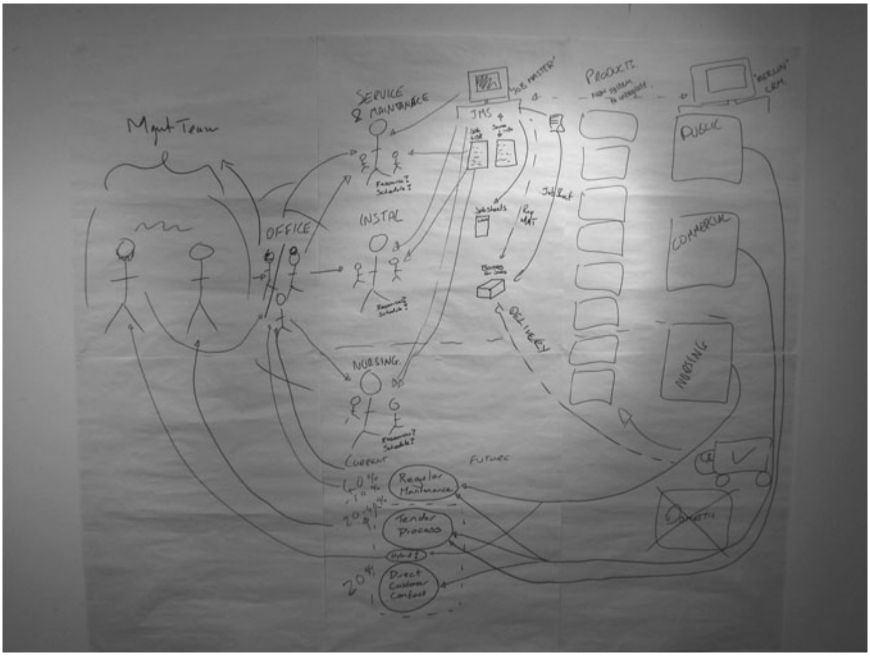

Fig. 8.8 Rich picture for King Communications and Security Ltd

some of the issues and uncertainties faced. This was complemented by a stakeholder analysis, which sought to identify and map all stakeholders against the two dimensions of interest in the issue and their power to influence outcomes (positively or negatively). See Eden and Ackermann [35, p121–125] for a more detailed discussion of this type of analysis. This power-interest grid groups stakeholders in four categories: *players* (high interest and power) need to be managed closely; *subjects* (high interest, low power) should be kept informed; *context setters* (high power but low interest) need to be kept satisfied; and the *crowd* (low interest and power) who should be monitored.

Following on from this, the PQR formula (do P by Q in order to contribute to the achievement of R) was used to develop a description of the organisation as a purposeful activity system, leading to an associated root definition (a structured description of the system from a particular perspective), as follows:

> To design, install and maintain customer systems (*do P*) through re-sell and adding value of products (*by Q*) to provide a customer solution, in order to maximise profits through meeting demand, in order to make a living (*in order to R*).

The stakeholder analysis and PQR formula surfaced more or less the same issues as would the use of CATWOE, a checklist which surfaces different perspectives from which the system under consideration can be viewed. Although CATWOE did not form part of the actual intervention, we include an outline here for illustration:

Customers Those organisations that employ KCS to deliver solutions and maintain systems: KCS suppliers

Actors All staff of KCS

Transformation ptCustomer demand for security/telecoms systems? Installed systems which satisfy customer demand

Worldview That KCS is a viable business model that will generate profits for its owners

Owners King family, including the Managing Director

Environment Regulatory bodies, competitors

Finally, we consider the performance measures – the 3 E's – efficacy, efficiency and effectiveness, against which the corresponding activity system would be evaluated.

Efficacy Does the transformation produce the intended outcome – i.e. are systems actually being installed?

Efficiency Is the transformation achieved with minimum use of resources – e.g. are components procured as cheaply as possible, are systems designed, installed and maintained using the appropriate number and level of staff, etc.

Effectiveness Does the transformation help to achieve higher level aims – e.g. are customers satisfied with the service received and systems installed? Is new demand created? Are profits generated?

From these initial considerations it emerged that an issue of particular concern to the MD was the company's ability to meet customer demand in a timely manner, given recently generated growth in demand, shortage of skilled staff and a changing organisational culture. In particular, the system in place to schedule and manage projects was considered to be no longer fit-for-purpose. If time had been available, the group might have developed a more focused root definition and purposeful activity model to represent the project management process, but given the limited nature of the intervention and the time available decided to use the CAUSE framework, as a stepping stone to a multicriteria analysis of options for such a system. The initial analysis was done using multiattribute value analysis, and ELECTRE III was used to validate some aspects of this. The value tree used and high-level value profiles of the alternatives considered are shown in Fig. 8.9.

In this particular intervention the analysis using SSM served to provide a very general framework to facilitate high level understanding of an issue, from which a focused MCDA problem was identified and structured. However, the flexible nature of SSM permits many different ways in which it might be used in an integrated way with MCDA. For example, Daellenbach and Nilakant [27] write: "The reason for exploring several root definitions is to discover and contrast the implications of each different worldview, to gain a deeper and more varied understanding of the conflicts between them, and discover opportunities for new choices. If there is more than one decision maker or active stakeholder, the debate tends to bring about a shared consensus on values and decisions".

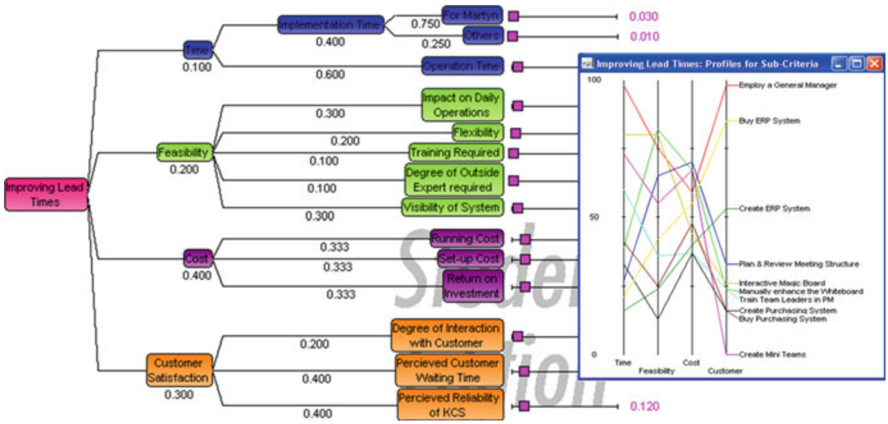

Fig. 8.9 Multicriteria model for King Communications and Security Ltd

Case Study 4: Research Project Evaluation: Using MCDA to Guide Problem Structuring

In this case study we describe the use of MCDA to guide the problem structuring process (see also Section 8.7). The South African National Research Foundation (NRF) is a statutory organisation, tasked *inter alia* with distributing state research funds to university departments and other research groupings. Our involvement related to applications for funding within a number of so-called focus area programmes. Applications could be made by "rated researchers" (those who had submitted themselves to a process of peer review of their research, and had received a "rating"), or by unrated researchers who might be given funding for up to 4 years in order to raise their research output to a rateable level.

The background to our involvement was an increasing concern that the ranking of project proposals and subsequent funding decisions should be goal-directed, equitable and transparent. We were tasked, however, primarily to develop a process for ranking of project proposals.

Although researchers themselves are important stakeholders, the project described here involved only the management of the NRF, with only the final results reported back to researchers for final agreement at the end of the process. Nevertheless, even within the NRF management there existed a diverse range of interests, between representatives of different areas of research (ranging from engineering to sociology), and between those emphasising the fundamental benefits of research and those emphasising the need to use state-funded research primarily for purposes of training researchers.

Once again we started with a workshop in which participants first made opening statements after which a "post-it" session addressed the question as to what issues need to be taken into consideration when comparing and evaluating research proposals. In this case, the "post-its" were immediately grouped into clusters of related

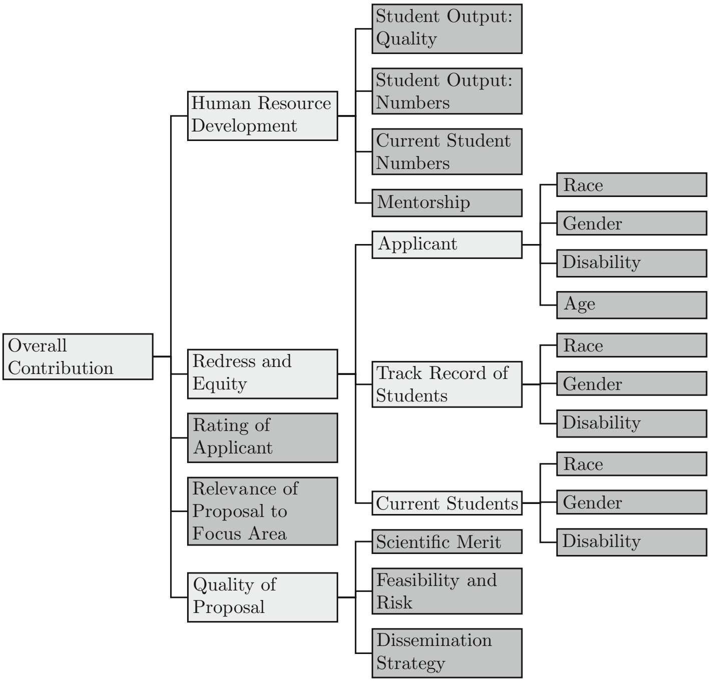

Fig. 8.10 NRF value tree

concepts. There did not seem to be a need for causal mapping, as the "alternatives" were defined by the applicants, and the stakeholders and environment were self-evident. The MCDA perspective led to using emergent criteria as the rationale for clustering. In between the first and second workshops a value tree could thus be structured and circulated to participants, and was broadly accepted in the form displayed in Fig. 8.10.

The real structuring challenge lay in creating operationally meaningful and transparent definitions of levels of performance for each of the lowest level criteria. This step was achieved by establishing small work groups, each of which was commissioned to develop clear verbal descriptions of 3–5 levels of performance for a small number of criteria. These were then reported back to the full workshop for comment and criticism before being reworked.

The discipline of constructing these performance descriptions was perhaps the most critical phase of problem structuring. As groups struggled to reach consensus on the definitions, it became clear that certain criteria were differently understood by many participants (e.g., contributions to corrective action addressing historical disadvantages and their links to rate of student output), while others were fundamentally non-measurable (e.g., mentorship of other staff in the applicants own institution).

The value tree was thus simplified to some extent, with the following amendments arising from the attempted construction of performance measures:

- *Relevance to focus area* became simply a "hurdle"; projects deemed not to be relevant were excluded, but relevance otherwise was not to be used a criterion for ranking.
- Three criteria were omitted as not operationally measurable, namely *student quality, mentorship* and *dissemination strategy*.
- Performance in terms of redress and equity was to be described by integrated scenario descriptions involving race and gender only for students, and race, gender and age for the applicant. Disability issues were omitted, with a decision to create an opportunity for those with disabilities to apply for supplementary grants.

With these modifications, a broadly acceptable structuring of the remaining criteria was achieved. As an example, we display in Fig. 8.11 a set of scenarios agreed to define performance levels for one of the identified criteria (namely *numbers of current postgraduate students*).

At this stage, it became relatively simple to construct an additive value function to provide a means for ranking proposals within focus areas. The clearly defined operational levels for performance on each criterion (as illustrated in Fig. 8.11) facilitated both within-criterion value scoring (as the categories of performance were well specified) and elicitation of swing weights (as the ranges on each criterion were specified). When the results were conveyed to researchers in a series of presentations round the country, it was in fact the value tree plus the full value scoring that was presented. Although some comments were received leading to minor modifications, there were no major objections to the scoring system from researchers.

1.2	**Current postgraduate students** This criterion addresses the number of doctoral and masters students currently registered under the supervision of the applicant, i.e. students who registered for their respective degrees in the last three years, including the present year (i.e. 2005, 2006 and 2007). Only full-time students should be taken into account that are in their 1st, 2nd or 3rd year of doctoral study (i.e. 2005, 2006 and 2007) or in their 1st or 2nd year of masters study. For part-time students: 1st, 2nd, 3rd, 4th, 5th year of doctoral, 1st, 2nd or 3rd year masters.			
SCENARIOS	(a)	The number of doctoral students registered under supervision of the applicant in the last three years is 50% or more than the average number (of doctoral students) per supervisor in the discipline over the same period AND the number of registered doctoral students should at least be four.	200	Only students in their first three years of registration for a full-time doctoral degree will be counted. Part-time students in their first five years of registration will be taken into account.
	(b)	The total number of doctoral and masters students registered under supervision of the applicant in the last three years is 25% or more than the average number (of doctoral and masters students) per supervisor in the discipline over the same period AND the number of registered doctoral students should at least be two.	150	Only students in their first three years of registration for a full-time doctoral degree and students in their first two years of registration for a full-time masters degree will be counted. In the case of part-time study, doctoral students in their first five years of registration and masters students in their first three years of registration will be taken into account.
	(c)	The total number of doctoral and masters students registered under supervision of the applicant in the last three years is about average per supervisor in the discipline over the same period .	100	As above
	(d)	The total number of doctoral and masters students registered under supervision of the applicant in the last three years is 25% or more below average per supervisor in the discipline over the same period.	50	As above
	(e)	No doctoral or masters students registered in the last three years.	0	
	(f)	Score for applicants who are new entrants or museum researchers	100	Applicants who fall into this category who have students which meet the requirements for scenarios (a) or (b) should be counted in these scenarios.

Fig. 8.11 Example of the use of scenarios to define performance levels

8.7 MCDA as Problem Structuring

There are strong parallels between PSMs and MCDA. Other writers have often stressed the importance of MCDA as a process rather than as a problem solving tool. For example:

> The decision unfolds through a process of learning, understanding, information processing, assessing and defining the problem and its circumstances. The emphasis must be on the process, not on the act or outcome of making a decision ... Zeleny [102]

> ...decision analysis helps to provide a structure to thinking, a language for expressing concerns of the group and a way of combining different perspectives. Phillips [75]

Distinguishing features of a PSM as seen by various authors have been commented upon earlier. Fundamentally, these views have stressed the purpose of a PSM as being that of providing alternative views and framings for the situation at hand, and a "rich picture" in which hard and soft issues are comprehensively identified. The end result is a structuring of the mess into a problem amenable to analytic solution.

We may characterize the MCDA approach as providing:

- Identification of a complete, relevant and operational set of criteria
- Evaluation and/or comparison of alternatives in terms of each criterion
- Aggregation of preferences across criteria

These features of MCDA provide in effect a *pro forma* structuring template, to guide representation of the "mess" into a problem defined in terms of criteria, alternatives, uncertainties, stakeholders and environment. Some illustrations of this view of the MCDA process include:

- Keeney's Value-Focused Thinking [51] which provides structured guidelines for identification of criteria, by means of consideration of distinctions between fundamental or means-ends objectives.
- The construction of performance measures for each criterion (which may be seen by some technocratically as an analytical step) is in fact an important structuring process, by means of which stakeholder conflicts, ambiguities and uncertainties are revealed.
- Even the seemingly hard analytical step of evaluation and comparison of alternatives is itself tentative and exploratory in good MCDA practice, leading to improved perception of available alternatives and of differing world views.
- Roy's [85] differentiation between "decision making" and "decision aiding", according to which the analyst works with a client to co-construct the problem in a mutual learning process, utilising model components which retain a degree of ambiguity, such as partial preference structures. This constructivist perspective of decision was further developed by Landry [57], Landry et al. [58, 59], Banville et al. [4], Norese [70], Ostanello [71], Ostanello and Tsoukiàs [72] and more recently by Tsoukiàs [96].

Table 8.2 Consideration of MCDA in the light of Rosenhead and Minger's characteristics of problem structuring methods

Problem Structuring Methods	MCDA
Non-optimising – seeks alternative solutions without trade-offs	Value tradeoffs may emerge, but MCDA starts with incommensurate criteria
Reduced data demands	MCDA can (does) operate with subjective judgments
Simplicity and transparency, to clarify conflict	These are the characterizing features of the MCDA process
People as active subjects	Absolutely! The need for MCDA to incorporate subjective judgment necessitates the involvement of problem owners
Accepts uncertainty – keeps options open	MCDA can take account of both internal and external uncertainties. The inputs and value judgments are not viewed as rigid or precise in MCDA
Facilitates bottom-up planning	MCDA has been used with grass-roots stakeholders as well as top management

It is useful to compare the MCDA process as described above with the characteristics of problem structuring methods as stated by Rosenhead and Mingers [83, p11]. Such a comparison is suggested in Table 8.2.

Of course, MCDA does not end with structuring. On occasions, a good structuring may make the solution self-evident, but in most cases the analytical or convergent phase of MCDA will follow the structuring. The advantage of using the MCDA framework as a template for structuring is that the transition from divergent structuring to convergent analysis is essentially seamless. The general feature of the case studies above was that the analytical models (in these cases multiattribute value functions, but the method is not central) flowed out of the structuring as a natural consequence.

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Chapter 9
Robust Ordinal Regression

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Abstract Within disaggregation–aggregation approach, *ordinal regression* aims at inducing parameters of a preference model, for example, parameters of a value function, which represent some holistic preference comparisons of alternatives given by the Decision Maker (DM). Usually, from among many sets of parameters of a preference model representing the preference information given by the DM, only one specific set is selected and used to work out a recommendation. For example, while there exist many value functions representing the holistic preference information given by the DM, only one value function is typically used to recommend the best choice, sorting, or ranking of alternatives. Since the selection of one from among many sets of parameters of the preference model compatible with the preference information given by the DM is rather arbitrary, *robust ordinal regression* proposes taking into account all the sets of parameters compatible with the preference information, in order to give a recommendation in terms of necessary and possible consequences of applying all the compatible preference models on the considered set of alternatives. In this chapter, we present the basic principle of robust ordinal

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regression, and the main multiple criteria decision methods to which it has been applied. In particular, *UTA*^{GMS} and *GRIP* methods are described, dealing with choice and ranking problems, then *UTADIS*^{GMS}, dealing with sorting (ordinal classification) problems. Next, we present robust ordinal regression applied to Choquet integral for choice, sorting, and ranking problems, with the aim of representing interactions between criteria. This is followed by a characterization of robust ordinal regression applied to outranking methods and to multiple criteria group decisions. Finally, we describe an interactive multiobjective optimization methodology based on robust ordinal regression, and an evolutionary multiobjective optimization method, called *NEMO*, which is also using the principle of robust ordinal regression.

Keywords Robust ordinal regression · Multiple criteria · Choice, sorting and ranking · Additive value functions · Choquet integral · Outranking methods · Multiple criteria group decisions · Interactive multiobjective optimization · Evolutionary multiobjective optimization

9.1 Introduction

In Multiple Criteria Decision Analysis (MCDA) (for a recent state of the art see [14]), an alternative a, belonging to a finite set of alternatives $A = \{a, b, \ldots\}$ $(|A| = m)$, is evaluated on the basis of a family of n criteria $F = \{g_1, g_2, \ldots, g_i, \ldots, g_n\}$, with $g_i\colon A \rightarrow \mathbb{R}$. For example, in a decision problem regarding a recruitment of new employees, the alternatives are the candidates and the criteria can be a certain number of characteristics useful to give a comprehensive evaluation of the candidates, such as educational degree, professional experience, age, and interview assessment. From here on, we will use the term criterion g_i, or criterion i interchangeably $(i = 1, 2, \ldots, n)$. For the sake of simplicity, but without loss of generality, we suppose that the evaluations on criteria are increasing with respect to preference, i.e., the more the better, defining a marginal weak preference relation as follows:

$$\text{“}a \text{ is at least as good as } b\text{” with respect to criterion } i \Leftrightarrow g_i(a) \geq g_i(b).$$

The purpose of Multi-Attribute Utility Theory (MAUT) [13, 42] is to represent the preferences of the Decision Maker (DM) on a set of alternatives, A, by an overall value function $U(g_1(\cdot), \ldots, g_n(\cdot))\colon \mathbb{R}^n \rightarrow \mathbb{R}$, such that:

- a is indifferent to $b \Leftrightarrow U(\underline{g}(a)) = U(\underline{g}(b))$;
- a is preferred to $b \Leftrightarrow U(\underline{g}(a)) > U(\underline{g}(b))$,

where for simplicity of notation, we used $U(\underline{g}(a))$, instead of $U((g_1(a), \ldots, g_n(a)))$.

The principal value function aggregation model is the multiple attribute additive utility [42]:

$$U(\underline{g}(a)) = u_1(g_1(a)) + u_2(g_2(a)) + \cdots + u_n(g_n(a)) \quad \text{with} \quad a \in A,$$

where u_i are nondecreasing marginal value functions, $i = 1, 2, \ldots, n$.

Even if multiple attribute additive utility is the most well-known aggregation model, some critics have been advanced to it because it does not permit to represent interactions between the considered criteria. For example, in evaluating a car one can consider criteria such as maximum speed, acceleration, and price. In this case, very often there is a negative interaction (redundancy) between maximum speed and acceleration of cars: in fact, a car with a high maximum speed has, usually, also a good acceleration and thus, even if these two criteria can be very important for a person who likes sport cars, their comprehensive importance is smaller than the importance of the two criteria considered separately. In the same decision problem, very often there is a positive interaction (synergy) between maximum speed and price of cars: in fact, a car with a high maximum speed has, usually, also a high price, and thus a car with a high maximum speed and not so high price is very much appreciated. So, the comprehensive importance of these two criteria is greater than the importance of the two criteria considered separately. To handle the interactions between criteria one can consider *nonadditive integrals*, such as Choquet integral [11] and Sugeno integral [61] (for a comprehensive survey on the use of nonadditive integrals in MCDA, see [21, 25]).

Another interesting decision model permitting representation of interactions between criteria is the Dominance-based Rough Set Approach (DRSA) [29, 59]. In DRSA, the DM's preference model is a set of decision rules, i.e., easily understandable "*if..., then...*" statements, such as "if the maximum speed is at least 200 km/h and the price is not greater than $50,000, then the car is attractive." In general, we shall call the decision models, which, differently from multiple attribute additive utility, permit to represent the interaction between criteria *nonadditive decision models*.

Each decision model requires the specification of some parameters. For example, using MAUT, the parameters are related to the formulation of the marginal value functions $u_i(g_i(a))$, $i = 1, 2, \ldots, n$, while using nonadditive integrals, the parameters are related to so-called fuzzy measures, which permit to model the importance not only of each criterion $g_i \in F$, but also of any subset of criteria $R \subseteq F$. Within MCDA, many methods have been proposed to determine the parameters characterizing the considered decision model in a direct way, i.e., asking them directly to the DM, or in an indirect way, i.e., inducing the values of such parameters from some holistic preference comparisons of alternatives given by the DM. In general, this is a difficult task for several reasons. For example, it is acknowledged that the DM's preference information is often incomplete because the DM is not fully aware of the multiple criteria approach adopted, or because the preference structure is not well defined in DM's mind [43, 62].

Recently, MCDA methods based on indirect preference information and on the *disaggregation approach* [40] are considered more interesting, because they require a relatively smaller cognitive effort from the DM than methods based on direct preference information. In these methods, the DM provides some holistic preference comparisons on a set of reference alternatives A^R, and from this information the parameters of a decision model are induced using a methodology called *ordinal regression*. Then, a consistent decision model is taken into consideration to evaluate the alternatives from set A (*aggregation approach*.) Typically, ordinal regression has been applied to MAUT models, such that in these cases we speak of *additive ordinal regression*. For example, additive ordinal regression is applied by the well-known method called *UTA* [39]. The principle of ordinal regression has also been applied to some nonadditive decision models. In this case, we speak of *nonadditive ordinal regression* exemplified by some *UTA*-like methods involving the Choquet integral [1, 47], and by the DRSA methodology [29, 59].

Usually, from among many sets of parameters of a preference model representing the preference information given by the DM, only one specific set is selected and used to work out a recommendation. For example, while there exist many value functions representing the holistic preference information given by the DM, only one value function is typically used to recommend the best choice, sorting, or ranking of alternatives. Since the selection of one from among many sets of parameters compatible with the preference information given by the DM is rather arbitrary, *robust ordinal regression* proposes taking into account all the sets of parameters compatible with the preference information, in order to give a recommendation in terms of necessary and possible consequences of applying all the compatible preference models on the considered set of alternatives.

The first method of robust ordinal regression is a recent generalization of the *UTA* method, called *UTA*^{GMS} [34]. The *UTA*^{GMS} is a multiple criteria method, which, instead of considering only one additive value function *compatible* with the preference information provided by the DM, as *UTA* does, takes into consideration the whole set of compatible additive value functions.

In particular, the *UTA*^{GMS} method requires from the DM a set of pairwise comparisons on a set of reference alternatives $A^R \subseteq A$ as preference information.

Then, using linear programming, one obtains two relations in set A: the *necessary* weak preference relation, which holds for any two alternatives $a, b \in A$ if and only if all compatible value functions give to a a value greater than the value provided to b, and the *possible* weak preference relation, which holds for this pair if and only if at least one compatible value function gives to a a value greater than the value given to b.

More recently, an extension of *UTA*^{GMS} has been proposed: the *GRIP* method [18]. The *GRIP* method builds a set of additive value functions, taking into account not only a set of pairwise comparisons of reference alternatives, but also the intensities of preference among reference alternatives.

This kind of preference information is often required in other well-known MCDA methods such as *MACBETH* [6] and *AHP* [54, 55].

Both UTA^{GMS} and *GRIP* apply the robust ordinal regression to the MAUT models, so we can say that these methods apply the *additive robust ordinal regression.*

Finally, *nonadditive robust ordinal regression* has been proposed, applying the basic ideas of robust ordinal regression to a value function expressed as Choquet integral in order to represent positive and negative interactions between criteria. More precisely, the disaggregation–aggregation approach used in this context has been inspired by UTA^{GMS} and *GRIP* methods, but in addition to preference information required by these methods, it includes some preference information on the sign and intensity of interaction between couples of criteria.

The chapter is organized as follows. Section 9.2 is devoted to a presentation of a general scheme of the constructive learning interactive procedure. It provides a brief reminder on learning of one compatible additive piecewise-linear value function for multiple criteria ranking problems using the *UTA* method. In Section 9.3, the *GRIP* method is presented, which is presently the most general of all *UTA*-like methods. Section 9.4, makes a comparison of *GRIP* to its main competitors in the field of MCDA. First *GRIP* is compared to *AHP* method, which requires pairwise comparisons of alternatives and criteria, and yields a priority ranking of solutions. Then *GRIP* is compared to *MACBETH* method, which also takes into account a preference order of alternatives and intensity of preference for pairs of alternatives. The preference information used in *GRIP* does not need, however, to be complete: the DM is asked to provide comparisons of only those ordered pairs of selected alternatives on particular criteria for which his/her judgment is sufficiently certain. This is an important advantage comparing to methods which, instead, require comparison of all possible pairs of alternatives on all the considered criteria. Section 9.5 presents robust ordinal regression applied to sorting problems. Section 9.6 presents the concept of "most representative" value function. Section 9.7 deals with nonadditive robust ordinal regression considering an application of robust ordinal regression methodology to a decision model formulated in terms of Choquet integral. Section 9.8 describes an interactive multiobjective optimization method based on robust ordinal regression. Section 9.9 presents *NEMO*, being an evolutionary multiobjective optimization method based on robust ordinal regression. Section 9.10 shows how robust ordinal regression can deal with outranking methods. Section 9.11 deals with robust ordinal regression applied to multiple criteria group decisions. Section 9.12 presents a didactic example relative to an interactive application of the robust ordinal regression to a multiple objective optimization problem. In Section 9.13, some conclusions and further research directions are provided.

9.2 Ordinal Regression for Multiple Criteria Ranking Problems

The preference information may be either direct or indirect, depending upon whether it specifies directly values of some parameters used in the preference model (e.g., trade-off weights, aspiration levels, discrimination thresholds, etc.) or, whether it specifies some examples of holistic judgments from which compatible values of the preference model parameters are induced. Eliciting direct preference information from the DM can be counterproductive in real-world decision-making situations because of a high cognitive effort required. Consequently, asking directly the DM to provide values for the parameters seems to make the DM uncomfortable. Eliciting indirect preference is less demanding of the cognitive effort. Indirect preference information is mainly used in the ordinal regression paradigm. According to this paradigm, a holistic preference information on a subset of some reference or training alternatives is known first and then a preference model compatible with the information is built and applied to the whole set of alternatives in order to rank them.

The ordinal regression paradigm is concordant with the posterior rationality postulated by March in [46]. It has been known for at least 50 years in the field of multidimensional analysis. It is also concordant with the induction principle used in machine learning. This paradigm has been applied within the two main MCDA approaches mentioned above: those using a value function as preference model [39, 51, 58, 60], and those using an outranking relation as preference model [44, 49, 50]. This paradigm has also been used since mid-1990s in MCDA methods involving a new, third family of preference models – a set of dominance decision rules induced from rough approximations of holistic preference relations [28, 29, 31, 59].

Recently, the ordinal regression paradigm has been revisited with the aim of considering the whole set of value functions compatible with the preference information provided by the DM, instead of a single compatible value function used, for example, in *UTA*-like methods [39, 58]. This extension has been implemented in a method called UTA^{GMS} [34], further generalized in another method called *GRIP* [18]. UTA^{GMS} and *GRIP* are not revealing to the DM only one compatible value function, but they are using the whole set of compatible (general, not piecewise-linear only) additive value functions to set up a necessary weak preference relation and a possible weak preference relation in the whole set of considered alternatives.

9.2.1 Concepts: Definitions and Notation

We are considering an MCDA problem where a finite set of alternatives $A = \{x, \ldots, y, \ldots, w, \ldots, z\}$ ($|A| = m$), is evaluated on a family $F = \{g_1, g_2, \ldots, g_n\}$ of n criteria. Let $I = \{1, 2, \ldots, n\}$ denote the set of criteria indices. We assume, without loss of generality, that the greater $g_i(x)$, the better alternative x on criterion g_i, for all $i \in I$, $x \in A$. A DM is willing to rank the alternatives of A from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information provided by the DM and on the way of exploiting this information. The family of criteria F is supposed to satisfy consistency conditions, i.e., completeness (all relevant criteria are considered), monotonicity (the better the evaluation of an alternative on the considered criteria, the more it is preferable to another), and nonredundancy (no superfluous criteria are considered) [53].

Such a decision-making problem statement is called *multiple criteria ranking problem*. It is known that the only information coming out from the formulation of this problem is the dominance ranking. Let us recall that in the dominance ranking, alternative $x \in A$ is preferred to alternative $y \in A$, $x \succ y$, if and only if $g_i(x) \geq g_i(y)$ for all $i \in I$, with at least one strict inequality. Moreover, x is indifferent to y, $x \sim y$, if and only if $g_i(x) = g_i(y)$ for all $i \in I$. Hence, for any two alternatives $x, y \in A$, one of the four situations may arise in the dominance ranking: $x \succ y$, $y \succ x$, $x \sim y$ and $x?y$, where the last one means that x and y are incomparable. Usually, the dominance ranking is very poor, i.e., the most frequent situation is $x?y$.

In order to enrich the dominance ranking, the DM has to provide preference information, which is used to construct an aggregation model making the alternatives more comparable. Such an aggregation model is called preference model. It induces a preference structure on set A, whose proper exploitation permits to work out a ranking proposed to the DM.

In what follows, the evaluation of each alternative $x \in A$ on each criterion $g_i \in F$ will be denoted either by $g_i(x)$ or x_i. Let G_i denote the value set (scale) of criterion g_i, $i \in I$. Consequently,

$$G = \prod_{i \in I} G_i$$

represents the evaluation space, and $x \in G$ denotes a profile of an alternative in such a space. We consider a weak preference relation $\succsim$ on A which means, for each pair of vectors, $x, y \in G$,

$$x \succsim y \Leftrightarrow \text{“}x \text{ is at least as good as } y\text{”}.$$

This weak preference relation can be decomposed into its asymmetric and symmetric parts, as follows,

(1) $x \succ y \equiv [x \succsim y$ and not $y \succsim x] \Leftrightarrow$ “x is preferred to y”, and
(2) $x \sim y \equiv [x \succsim y$ and $y \succsim x] \Leftrightarrow$ “x is indifferent to y”.

From a pragmatic point of view, it is reasonable to assume that $G_i \subseteq \mathbb{R}$, for $i = 1, \ldots, n$. More specifically, we will assume that the evaluation scale on each criterion g_i is bounded, such that $G_i = [\alpha_i, \beta_i]$, where α_i, β_i, $\alpha_i < \beta_i$ are the worst and the best (finite) evaluations, respectively. Thus, $g_i : A \to G_i$, $i \in I$. Therefore, each alternative $x \in A$ is associated with an evaluation vector denoted by $\underline{g}(x) = (x_1, x_2, \ldots, x_n) \in G$.

9.2.2 The UTA Method

In this section, we recall the principle of the ordinal regression *via* linear programming, as proposed in the original *UTA* method, see [39].

9.2.2.1 Preference Information

The preference information is given in the form of a complete preorder on a subset of reference alternatives $A^R \subseteq A$ (where $|A^R| = p$), called *reference preorder*. The reference alternatives are usually those contained in set A for which the DM is able to express holistic preferences. Let $A^R = \{a, b, c, \ldots\}$ be the set of reference alternatives.

9.2.2.2 Additive Model

The additive value function is defined on A such that for each $\underline{g}(x) \in G$,

$$U(\underline{g}(x)) = \sum_{i \in I} u_i(g_i(x_i)), \tag{9.1}$$

where, u_i are nondecreasing marginal value functions, $u_i : G_i \to \mathbb{R}, i \in I$. For the sake of simplicity, we shall write (1) as follows,

$$U(x) = \sum_{i \in I} u_i(x_i) \quad \text{or} \quad U(x) = \sum_{i=1}^{n} u_i(x_i). \tag{9.2}$$

In the *UTA* method, the marginal value functions u_i are assumed to be piecewise-linear functions. The ranges $[\alpha_i, \beta_i]$ are divided into $\gamma_i \geq 1$ equal sub-intervals $\left[x_i^0, x_i^1\right], \left[x_i^1, x_i^2\right], \ldots, \left[x_i^{\gamma_i - 1}, x_i^{\gamma_i}\right]$, where $x_i^j = \alpha_i + \frac{j}{\gamma_i}(\beta_i - \alpha_i)$, $j = 0, \ldots, \gamma_i$, and $i \in I$. The marginal value of an alternative $x \in A$ is obtained by linear interpolation,

$$u_i(x) = u_i\left(x_i^j\right) + \frac{x_i - x_i^j}{x_i^{j+1} - x_i^j}\left(u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right)\right), \; x_i \in \left[x_i^j, x_i^{j+1}\right]. \tag{9.3}$$

The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e., $u_i\left(x_i^0\right) = u_i(\alpha_i), u_i\left(x_i^1\right), u_i\left(x_i^2\right), \cdots, u_i\left(x_i^{\gamma_i}\right) = u_i(\beta_i)$.

In what follows, the principle of the *UTA* method is described as it was recently presented in [58]. Therefore, a value function $U(x) = \sum_{i=1}^n u_i(x_i)$ is compatible if it satisfies the following set of constraints.

$$\left.\begin{array}{l} U(a) > U(b) \Leftrightarrow \; a \succ b \\ U(a) = U(b) \Leftrightarrow \; a \sim b \end{array}\right\} \quad \forall \; a, b \in A^R$$
$$\begin{array}{l} u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right) \geq 0, \; i = 1, ..., n, \; j = 0, ..., \gamma_i - 1 \\ u_i(\alpha_i) = 0, \; i = 1, ..., n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1 \end{array} \tag{9.4}$$

9.2.2.3 Checking for Compatible Value Functions Through Linear Programming

To verify if a compatible value function $U(x) = \sum_{i=1}^{n} u_i(x_i)$ restoring the reference preorder $\succsim$ on A^R exists, one can solve the following linear programming problem, where $u_i\left(x_i^j\right), i = 1, \ldots, n,\ j = 1, \ldots, \gamma_i$, are unknown, and $\sigma^+(a), \sigma^-(a)$ $(a \in A^R)$ are auxiliary variables:

$$\begin{array}{l}
\text{Min } Z = \displaystyle\sum_{a \in A^R}^{m} \left(\sigma^+(a) + \sigma^-(a)\right) \\
\text{s.t.} \\
\quad \left.\begin{array}{l}
U(a) + \sigma^+(a) - \sigma^-(a) \geq \\
\qquad U(b) + \sigma^+(b) - \sigma^-(b) + \varepsilon \;\Leftrightarrow\; a \succ b \\
U(a) + \sigma^+(a) - \sigma^-(a) = \\
\qquad U(b) + \sigma^+(b) - \sigma^-(b) \;\Leftrightarrow\; a \sim b
\end{array}\right\} \forall a, b \in A^R \\
\quad u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right) \geq 0,\ i = 1, \ldots, n,\ j = 0, \ldots, \gamma_i - 1 \\
\quad u_i(\alpha_i) = 0,\ i = 1, \ldots, n \\
\quad \displaystyle\sum_{i=1}^{n} u_i(\beta_i) = 1 \\
\quad \sigma^+(a),\ \sigma^-(a) \geq 0,\ \forall a \in A^R,
\end{array} \tag{9.5}$$

where ε is an arbitrarily small positive value so that $U(a) + \sigma^+(a) - \sigma^-(a) > U(b) + \sigma^+(b) - \sigma^-(b)$ in case of $a \succ b$.

If the optimal value of the objective function of program (9.5) is equal to zero ($Z^* = 0$), then there exists at least one value function $U(x) = \sum_{i=1}^{n} u_i(x_i)$ satisfying (9.4), i.e., compatible with the reference preorder on A^R. In other words, this means that the corresponding polyhedron (9.4) of feasible solutions for $u_i\left(x_i^j\right),\ i = 1, \ldots, n,\ j = 1, \ldots, \gamma_i$, is not empty.

Let us remark that the transition from the preorder $\succsim$ to the marginal value function exploits the ordinal character of the criterion scale G_i. Notice, however, that the scale of the marginal value function is a conjoint interval scale. More precisely, for the considered additive value function $\sum_{i=1}^{n} u_i(x_i)$, the admissible transformations on the marginal value functions $u_i(x_i)$ have the form $u_i^*(x_i) = k \times u_i(x_i) + h_i$, $h_i \in \mathbb{R},\ i = 1, \ldots, n, k > 0$, such that for all $[x_1, \ldots, x_n], [y_1, \ldots, y_n] \in \prod_{i=1}^{n} G_i$

$$\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \Leftrightarrow \sum_{i=1}^{n} u_i^*(x_i) \geq \sum_{i=1}^{n} u_i^*(y_i).$$

An alternative way of representing the same preference model is:

$$U(x) = \sum_{i \in I} w_i \hat{u}_i(x), \tag{9.6}$$

where $\hat{u}(\alpha_i) = 0$, $\hat{u}(\beta_i) = 1$, $w_i \geq 0 \ \forall i \in I$, and $\sum_{i \in I} w_i = 1$. Note that the correspondence between (9.6) and (2) is such that $w_i = u_i(\beta_i)$, $\forall i \in I$. Due to the cardinal character of the marginal value function scale, the parameters w_i can be interpreted as trade-off weights among marginal value functions $\hat{u}_i(x)$. We will use, however, the preference model (2) with normalization constraints bounding $U(x)$ to the interval [0, 1].

When the optimal value of the objective function of the program (9.5) is greater than zero ($Z^* > 0$), then there is no value function $U(x) = \sum_{i \in I} u_i\ (x_i)$ compatible with the reference preorder on A^R. In such a case, three possible moves can be considered:

- Increasing the number of linear pieces γ_i for one or several marginal value function u_i could make it possible to find an additive value function compatible with the reference preorder on A^R.
- Revising the reference preorder on A^R could lead to find an additive value function compatible with the new preorder.
- Searching over the relaxed domain $Z \leq Z^* + \eta$ could lead to an additive value function giving a preorder on A^R sufficiently close to the reference preorder (in the sense of Kendall's τ).

9.3 Robust Ordinal Regression for Multiple Criteria Ranking Problems

Recently, two new methods, *UTA*^{GMS} [34] and *GRIP* [18], have generalized the ordinal regression approach of the *UTA* method in several aspects:

- Taking into account all additive value functions (1) compatible with the preference information, while *UTA* is using only one such function.
- Considering marginal value functions of (1) as general nondecreasing functions, and not piecewise-linear, as in *UTA*.
- Asking the DM for a ranking of reference alternatives, which is not necessarily complete (just pairwise comparisons).
- Taking into account additional preference information about intensity of preference, expressed both comprehensively and with respect to a single criterion.
- Avoiding the use of the exogenous, and not neutral for the result, parameter ε in the modeling of strict preference between alternatives.

UTA^{GMS} and *GRIP* produce two rankings on the set of alternatives A, such that for any pair of alternatives $a, b \in A$:

- In the *necessary* ranking, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for *all* value functions compatible with the preference information.
- In the *possible* ranking, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for *at least one* value function compatible with the preference information.

The necessary ranking can be considered as *robust* with respect to the preference information. Such robustness of the necessary ranking refers to the fact that any pair of alternatives compares in the same way whatever the additive value function compatible with the preference information. Indeed, when no preference information is given, the necessary ranking boils down to the dominance relation, and the possible ranking is a complete relation. It allows for taking into account the incomparability between alternatives. Every new pairwise comparison of reference alternatives, for which the dominance relation does not hold, is enriching the necessary ranking and it is impoverishing the possible ranking, so that they converge with the growth of the preference information.

Moreover, such an approach gives space for interactivity with the DM. Presentation of the necessary ranking, resulting from a preference information provided by the DM, is a good support for generating reactions from part of the DM. Namely, (s)he could wish to enrich the ranking or to contradict a part of it. Such a reaction can be integrated in the preference information considered in the next iteration.

The idea of considering the whole set of compatible value functions was originally introduced in UTA^{GMS}. *GRIP* (*G*eneralized *R*egression with *I*ntensities of *P*reference) can be seen as an extension of UTA^{GMS} permitting to take into account additional preference information in the form of comparisons of intensities of preference between some pairs of reference alternatives. For alternatives $x, y, w, z \in A$, these comparisons are expressed in two possible ways (not exclusive): (i) comprehensively, on all criteria, like "x is preferred to y at least as much as w is preferred to z"; and, (ii) partially, on any criterion, like "x is preferred to y at least as much as w is preferred to z, on criterion $g_i \in F$". Although UTA^{GMS} was historically the first method among the two, as *GRIP* incorporates and extends UTA^{GMS}, in the following we shall present only *GRIP*.

9.3.1 The Preference Information Provided by the Decision Maker

The DM is expected to provide the following preference information:

- A partial preorder $\succsim$ on A^R whose meaning is: for $x, y \in A^R$

  $$x \succsim y \Leftrightarrow x \text{ is at least as good as } y.$$

  Moreover, $\succ$ (preference) is the asymmetric part of $\succsim$ and $\sim$ (indifference) is the symmetric part given by $\succsim \cap \succsim^{-1}$. ($\succsim^{-1}$ is the inverse of $\succsim$, i.e., for all $x, y \in A^R$, $x \succsim^{-1} y \Leftrightarrow y \succsim x$).
- A partial preorder $\succsim^*$ on $A^R \times A^R$, whose meaning is: for $x, y, w, z \in A^R$,

  $$(x, y) \succsim^* (w, z) \Leftrightarrow \begin{array}{l} x \text{ is preferred to } y \text{ at least as much as } w \\ \text{is preferred to } z. \end{array}$$

Also in this case, $\succ^*$ is the asymmetric part of $\succsim^*$ and $\sim^*$ is the symmetric part given by $\succsim^* \cap \succsim^{*^{-1}}$ ($\succsim^{*^{-1}}$ is the inverse of $\succsim^*$, i.e., for all $x, y, w, z \in A^R$, $(x, y) \succsim^{*^{-1}} (w, z) \Leftrightarrow (w, z) \succsim^* (x, y)$).
- A partial preorder $\succsim_i^*$ on $A^R \times A^R$, whose meaning is: for $x, y, w, z \in A^R$, $(x, y) \succsim_i^* (w, z) \Leftrightarrow x$ is preferred to y at least as much as w is preferred to z on criterion g_i, $i \in I$.

In the following, we also consider the weak preference relation $\succsim_i$ being a complete preorder whose meaning is: for all $x, y \in A$,

$$x \succsim_i y \quad \Leftrightarrow \quad x \text{ is at least as good as } y \text{ on criterion } g_i, \quad i \in I.$$

Weak preference relations $\succsim_i$, $i \in I$, are not provided by the DM, but it is obtained directly from the evaluation of alternatives x and y on criterion g_i, i.e., $x \succsim_i y \Leftrightarrow g_i(x) \geq g_i(y)$.

9.3.2 Possible and Necessary Rankings

While the preference information provided by the DM is rather similar to that of *UTA*, the output of *GRIP* is quite different. In *GRIP*, the preference information has the form of a partial preorder in a set of reference alternatives $A^R \subseteq A$ (i.e., a set of pairwise comparisons of reference alternatives), augmented by information about intensities of preferences.

A value function is called *compatible* if it is able to restore the partial preorder $\succsim$ on A^R, as well as the given relation of intensity of preference among ordered pairs of reference alternatives. Each compatible value function induces, moreover, a complete preorder on the whole set A. In particular, for any two alternatives $x, y \in A$, a compatible value function orders x and y in one of the following ways: $x \succ y$, $y \succ x$, $x \sim y$. With respect to $x, y \in A$, it is thus reasonable to ask the following two questions:

- Are x and y ordered in the same way by *all* compatible value functions?
- Is there *at least one* compatible value function ordering x at least as good as y (or y at least as good as x)?

Having answers to these questions for all pairs of alternatives $(x, y) \in A \times A$, one gets a *necessary weak preference relation* $\succsim^N$ (partial preorder), whose semantics is $U(x) \geq U(y)$ for all compatible value functions, and a *possible weak preference relation* $\succsim^P$ in A (strongly complete and negatively transitive relation), whose semantics is $U(x) \geq U(y)$ for at least one compatible value function.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible value function. Observe also that in this case, for any $x, y \in A^R$,

$$x \succsim y \Rightarrow x \succsim^N y$$

and

$$x \succ y \Rightarrow \text{ not}\left(y \succsim^P x\right).$$

In fact, if $x \succsim y$, then for any compatible value function, $U(x) \geq U(y)$ and, therefore, $x \succsim^N y$. Moreover, if $x \succ y$, then for any compatible value function, $U(x) > U(y)$ and, consequently, there is no compatible value function such that $U(y) \geq U(x)$, which means not$(y \succsim^P x)$.

9.3.3 Linear Programming Constraints

In this section, we present a set of constraints that interprets the preference information in terms of conditions on the compatible value functions.

The value function $U : A \rightarrow [0,1]$ should satisfy the following constraints corresponding to DM's preference information,

(a) $U(w) > U(z)$ if $w \succ z$
(b) $U(w) = U(z)$ if $w \sim z$
(c) $U(w) - U(z) > U(x) - U(y)$ if $(w,z) \succ^* (x,y)$
(d) $U(w) - U(z) = U(x) - U(y)$ if $(w,z) \sim^* (x,y)$
(e) $u_i(w) \geq u_i(z)$ if $w \succsim_i z, i \in I$
(f) $u_i(w) - u_i(z) > u_i(x) - u_i(y)$ if $(w,z) \succ_i^* (x,y), i \in I$
(g) $u_i(w) - u_i(z) = u_i(x) - u_i(y)$ if $(w,z) \sim_i^* (x,y), i \in I$

Let us remark that within *UTA*-like methods, constraint (a) is written as $U(w) \geq U(z) + \varepsilon$, where $\varepsilon > 0$ is a threshold exogenously introduced. Analogously, constraints (c) and (f) should be written as,

$$U(w) - U(z) \geq U(x) - U(y) + \varepsilon$$

and

$$u_i(w) - u_i(z) \geq u_i(x) - u_i(y) + \varepsilon.$$

However, we would like to avoid the use of any exogenous parameter and, therefore, instead of setting an arbitrary value of ε, we consider it as an auxiliary variable, and we test the feasibility of constraints (a), (c), and (f) (see Section 9.3.4). This permits to take into account all possible value functions, even those which satisfy the constraints for having a very small threshold ε. This is safer also from the viewpoint of "objectivity" of the selected methodology. In fact, the value of ε is not meaningful in itself and it is useful only because it permits to discriminate preference from indifference.

Moreover, the following normalization constraints should also be taken into account:

(h) $u_i\left(x_i^*\right) = 0$, where x_i^* is such that $x_i^* = \min\{g_i(x) : x \in A\}$
(i) $\sum_{i \in I} u_i\left(y_i^*\right) = 1$, where y_i^* is such that $y_i^* = \max\{g_i(x) : x \in A\}$

If the constraints from (a) to (i) are fulfilled, then the partial preorders $\succsim$ and $\succsim^*$ on A^R and $A^R \times A^R$ can be extended on A and $A \times A$, respectively.

9.3.4 Computational Issues

In order to conclude the truth or falsity of binary relations $\succsim^N$, $\succsim^P$, $\succsim^{*N}$, $\succsim^{*P}$, $\succsim_i^{*N}$ and $\succsim_i^{*P}$, we have to take into account that, for all $x, y, w, z \in A$ and $i \in I$:

(1) $x \succsim^N y \Leftrightarrow \inf\Big\{U(x) - U(y)\Big\} \geq 0$

(2) $x \succsim^P y \Leftrightarrow \inf\Big\{U(y) - U(x)\Big\} \leq 0$

(3) $(x, y) \succsim^{*N} (w, z) \Leftrightarrow \inf\Big\{\Big(U(x) - U(y)\Big) - \Big(U(w) - U(z)\Big)\Big\} \geq 0$

(4) $(x, y) \succsim^{*P} (w, z) \Leftrightarrow \inf\Big\{\Big(U(w) - U(z)\Big) - \Big(U(x) - U(y)\Big)\Big\} \leq 0$

(5) $(x, y) \succsim_i^{*N} (w, z) \Leftrightarrow \inf\Big\{\Big(u_i(x_i) - u_i(y_i)\Big) - \Big(u_i(w_i) - u_i(z_i)\Big)\Big\} \geq 0$

(6) $(x, y) \succsim_i^{*P} (w, z) \Leftrightarrow \inf\Big\{\Big(u_i(w_i) - u_i(z_i)\Big) - \Big(u_i(x_i) - u_i(y_i)\Big)\Big\} \leq 0$

with the infimum computed on the set of value functions satisfying constraints from (a) to (i). Let us remark, however, that the linear programming is not able to handle strict inequalities such as the above (a), (c), and (f). Moreover, linear programming permits to compute the minimum or the maximum of an objective function and not an infimum. Nevertheless, reformulating properly the above properties (1) to (6), a result presented in [47] permits to use linear programming for testing the truth of binary relations, $\succsim^N$, $\succsim^P$, $\succsim^{*N}$, $\succsim^{*P}$, $\succsim_i^{*N}$ and $\succsim_i^{*P}$.

In order to use such a result, constraints (a), (c) and (f) have to be reformulated as follows:

(a') $U(x) \geq U(y) + \varepsilon$ if $x \succ y$
(c') $U(x) - U(y) \geq U(w) - U(z) + \varepsilon$ if $(x, y) \succ^* (w, z)$
(f') $u_i(x) - u_i(y) \geq u_i(w) - u_i(z) + \varepsilon$ if $(x, y) \succ_i^* (w, z)$

with $\varepsilon > 0$.

Then, properties $(1) - (6)$ have to be reformulated such that the search of the infimum is replaced by computing the maximum value of ε on the set of value functions satisfying constraints from (a) to (i), with constraints (a), (c) and (f) transformed to (a'), (c') and (f'), plus constraints specific for each point:

(1′) $x \succsim^P y \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \max\ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $U(x) \geq U(y)$

(2′) $x \succsim^N y \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \max\ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $U(y) \geq U(x) + \varepsilon$

(3′) $(x, y) \succsim^{*P} (w, z) \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \max\ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $\Big(U(x) - U(y)\Big) - \Big(U(w) - U(z)\Big) \geq 0$

(4′) $(x, y) \succsim^{*^N} (w, z) \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \max \ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $\Big(U(w) - U(z)\Big) - \Big(U(x) - U(y)\Big) \geq \varepsilon$

(5′) $(x, y) \succsim_i^{*^P} (w, z) \Leftrightarrow \varepsilon^* > 0$,
where $\varepsilon^* = \max \ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $\Big(u_i(x_i) - u_i(y_i)\Big) - \Big(u_i(w_i) - u_i(z_i)\Big) \geq 0$

(6′) $(x, y) \succsim_i^{*^N} (w, z) \Leftrightarrow \varepsilon^* \leq 0$,
where $\varepsilon^* = \max \ \varepsilon$, subject to the constraints (a'), (b), (c'), (d), (e), (f'), plus the constraint $\Big(u_i(w_i) - u_i(z_i)\Big) - \Big(u_i(x_i) - u_i(y_i)\Big) \geq \varepsilon$.

9.4 Comparison of *GRIP* with other MCDA Methods

9.4.1 Comparison of GRIP with the AHP

In *AHP* (*A*nalytical *H*ierarchy *P*rocess) [54, 55], criteria should be pairwise compared with respect to their importance. Alternatives are also pairwise compared on particular criteria with respect to intensity of preference. The following nine point scale is used:

1 – Equal importance-preference
3 – Moderate importance-preference
5 – Strong importance-preference
7 – Very strong or demonstrated importance-preference
9 – Extreme importance-preference

2, 4, 6, and 8 are intermediate values between the two adjacent judgements. The ratio of importance of criterion g_i over criterion g_j is the inverse of the ratio of importance of g_j over g_i. Analogously, the intensity of preference of alternative x over alternative y is the inverse of the intensity of preference of y over x. The above scale is a ratio scale. Therefore, the difference of importance is read as the ratio of weights w_i and w_j, corresponding to criteria g_i and g_j, and the intensity of preference is read as the ratio of the attractiveness of x and the attractiveness of y, with respect to the considered criterion g_i. In terms of value functions, the intensity of preference can be interpreted as the ratio $\frac{u_i(g_i(x))}{u_i(g_i(y))}$. Thus, the problem is how to obtain values of w_i and w_j from ratio $\frac{w_i}{w_j}$, and values of $u_i(g_i(x))$ and $u_i(g_i(y))$ from ratio $\frac{u_i(g_i(x))}{u_i(g_i(y))}$.

In *AHP* it is proposed that these values are supplied by the principal eigenvectors of the matrices composed of the ratios $\frac{w_i}{w_j}$ and $\frac{u_i(g_i(x))}{u_i(g_i(y))}$. The marginal value functions $u_i(g_i(x))$ are then aggregated by means of a weighted-sum using the weights w_i.

Comparing *AHP* with *GRIP*, we can say that with respect to single criteria the type of questions addressed to the DM is the same: express intensity of preference in qualitative-ordinal terms (equal, moderate, strong, very strong, extreme). However, differently from *GRIP*, this intensity of preference is translated in *AHP* into quantitative terms (the scale from 1 to 9) in a quite arbitrary way. In *GRIP*, instead, the marginal value functions are just a numerical representation of the original qualitative-ordinal information, and no intermediate transformation in quantitative terms is exogenously imposed.

Other differences between *AHP* and *GRIP* are related to the following aspects.

1. In *GRIP*, the value functions $u_i(g_i(x))$ depend mainly on comprehensive preferences involving jointly all the criteria, while this is not the case in *AHP*.
2. In *AHP*, the weights w_i of criteria g_i are calculated on the basis of pairwise comparisons of criteria with respect to their importance; in *GRIP*, this is not the case, because the value functions $u_i(g_i(x))$ are expressed on the same scale and thus they can be summed up without any further weighting.
3. In *AHP*, all unordered pairs of alternatives must be compared from the viewpoint of the intensity of preference with respect to each particular criterion. Therefore, if m is the number of alternatives, and n the number of criteria, then the DM has to answer $n \times \frac{m\times(m-1)}{2}$ questions. Moreover, the DM has to answer questions relative to $\frac{n\times(n-1)}{2}$ pairwise comparisons of considered criteria with respect to their importance. This is not the case in *GRIP*, which accepts partial information about preferences in terms of pairwise comparison of some reference alternatives. Finally, in *GRIP* there is no question about comparison of relative importance of criteria.

As far as point 2 is concerned, observe that the weights w_i used in *AHP* represent trade-offs between evaluations on different criteria. For this reason it is doubtful that if they could be inferred from answers to questions concerning comparison of importance. Therefore, *AHP* has a problem with meaningfulness of its output with respect to its input, and this is not the case of *GRIP*.

9.4.2 Comparison of GRIP with MACBETH

MACBETH (*M*easuring *A*ttractiveness by a *C*ategorical *B*ased *E*valuation *T*ec*H*nique) is a method for MCDA [5,6], which builds a value function from qualitative judgements obtained from DMs about differences of values quantifying the relative attractiveness of alternatives or criteria.

When using *MACBETH*, the DM is asked to provide the following preference information about every two alternatives from set A:

- First, through an (ordinal) judgement on their relative attractiveness.
- Second, (if the two alternatives are not considered to be equally attractive), through a qualitative judgement about the difference of attractiveness between these two alternatives.

Seven semantic categories of difference of attractiveness are considered in *MACBETH*: null, very weak, weak, moderate, strong, very strong, extreme.

The main idea of *MACBETH* is to build an interval scale from the preference information provided by the DM. It is, however, necessary that the above categories correspond to disjoint intervals (represented in terms of the real numbers). The bounds for such intervals should not be arbitrarily fixed a priori, but they should be calculated simultaneously with the numerical values of all particular alternatives from A, so as to ensure compatibility between these values [5]. Linear programming models are used for these calculations. In case of inconsistent judgments, *MACBETH* provides the DM with information permitting to eliminate such inconsistency.

When comparing *MACBETH* with *GRIP* the following aspects should be considered:

- Both deal with qualitative judgements.
- Both need a set of comparisons of alternatives or pairs of alternatives to work out a numerical representation of preferences, however, *MACBETH* depends on the definition of two characteristic levels on the original scale, "neutral" and "good," to obtain the numerical representation of preferences.
- *GRIP* adopts the disaggregation–aggregation approach and, therefore, it considers also comprehensive preferences relative to comparisons involving jointly all the criteria, which is not the case of *MACBETH*.
- *GRIP* is, however, more general than *MACBETH* since it can take into account the same kind of qualitative judgments as *MACBETH* (the difference of attractiveness between pairs of alternatives) and the intensity of preferences of the type "x is preferred to y at least as much as z is preferred to w".

As for the last item, it should be noticed that the intensity of preference considered in *MACBETH* and the intensity coming from comparisons of the type "x is preferred to y at least as strongly as w is preferred to z" (i.e., the quaternary relation $\succsim^*$) are substantially the same. In fact, the intensities of preference are equivalence classes of the preorder generated by $\succsim^*$. This means that all the pairs (x, y) and (w, z), such that x is preferred to y with the same intensity as w is preferred to z, belong to the same semantic category of difference of attractiveness considered in *MACBETH*. To be more precise, the structure of intensity of preference considered in *MACBETH* is a particular case of the structure of intensity of preference represented by $\succsim^*$ in *GRIP*. Still more precisely, *GRIP* has the same structure of intensity as *MACBETH* when $\succsim^*$ is a complete preorder. When this does not occur, *MACBETH* cannot be used while *GRIP* can naturally deal with this situation.

Comparison of *GRIP* and *MACBETH* could be summarized in the following points:

1. *GRIP* is using preference information relative to: (a) comprehensive preference on a subset of reference alternatives with respect to all criteria, (b) partial intensity of preference on some single criteria, and (c) comprehensive intensity of preference with respect to all criteria, while *MACBETH* requires preference information on all pairs of alternatives with respect to each one of the considered criteria.

2. Information about partial intensity of preference is of the same nature in *GRIP* and *MACBETH* (equivalence classes of relation $\succsim_i^*$ correspond to qualitative judgements of *MACBETH*), but in *GRIP* it may not be complete.
3. *GRIP* is a "disaggregation–aggregation" approach while *MACBETH* makes use of the "aggregation" approach and, therefore, it needs weights to aggregate evaluations on the criteria.
4. *GRIP* works with all compatible value functions, while *MACBETH* builds a single interval scale for each criterion, even if many such scales would be compatible with preference information.

9.5 Robust Ordinal Regression for Multiple Criteria Sorting Problems

Robust ordinal regression has been proposed also for sorting problems [32, 35, 45]. In the following, we present the new $UTADIS^{GMS}$ method [32, 35]. $UTADIS^{GMS}$ considers an additive value function

$$U(a) = \sum_{i=1}^{n} u_i(g_i(a))$$

as a preference model ($a \in A$). Let us remember that sorting procedures consider a set of p predefined preference ordered classes $C_1, C_2, \ldots, C_p$, where $C_{h+1} \gg C_h$ ($\gg$ a complete order on the set of classes), $h = 1, \ldots, p-1$. The aim of a sorting procedure is to assign each alternative to one class or to a set of contiguous classes. The robust ordinal regression uses a value function U to decide the assignments in such a way that if $U(a) > U(b)$, then a is assigned to a class not worse than b.

We suppose the DM provides preference information in form of possibly imprecise assignment examples on a reference set A^*, i.e., for all $a^* \in A^*$ the DM defines a desired assignment $a^* \to [C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$, where $[C_{L^{DM}(a^*)}, C_{R^{DM}(a^*)}]$ is an interval of contiguous classes $C_{L^{DM}(a^*)}$, $C_{L^{DM}(a^*)+1}$, ..., $C_{R^{DM}(a^*)}$. Each such alternative is called a reference alternative. $A^* \subseteq A$ is called the set of reference alternatives. An assignment example is said to be precise if $L^{DM}(a^*) = R^{DM}(a^*)$, and imprecise, otherwise.

Given a value function U, a set of assignment examples is said to be *consistent with* U iff

$$\forall a^*, b^* \in A^*, \quad U(a^*) \geq U(b^*) \;\Rightarrow\; R^{DM}(a^*) \geq L^{DM}(b^*) \tag{9.7}$$

which is equivalent to

$$\forall a^*, b^* \in A^*, \quad L^{DM}(a^*) > R^{DM}(b^*) \Rightarrow U(a^*) > U(b^*) \tag{9.8}$$

On the basis of (9.8), we can state that, formally, a general additive compatible value function is an additive value function $U(a) = \sum_{i=1}^{n} u_i(a)$ satisfying the following set of constraints:

$$\left.\begin{array}{l} U(a^*) > U(b^*) \;\Leftrightarrow\; L^{DM}(a^*) > R^{DM}(b^*) \quad \forall a^*, b^* \in A^* \\ u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\ i = 1,...,n,\ j = 2,...,m \\ u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\ i = 1,...,n, \\ u_i(\alpha_i) = 0,\ i = 1,...,n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1, \end{array}\right\} (E^{A^*})$$

where α_i and β_i are, respectively, the worst and the best evaluations on each criterion g_i, and τ_i is the permutation on the set of indices of alternatives from A^* that reorders them according to the increasing evaluation on criterion g_i, i.e.,

$$g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \ldots \leq g_i(a_{\tau_i(m-1)}) \leq g_i(a_{\tau_i(m)}).$$

Let us observe that the set of constraints (E^{A^*}) is equivalent to

$$\left.\begin{array}{l} U(a^*) \geq U(b^*) + \varepsilon \Leftrightarrow L^{DM}(a^*) > R^{DM}(b^*) \quad \forall a^*, b^* \in A^* \\ u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\ i = 1,...,n,\ j = 2,...,m \\ u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\ i = 1,...,n, \\ u_i(\alpha_i) = 0,\ i = 1,...,n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1, \end{array}\right\} (E^{A^*})'$$

with $\varepsilon > 0$. Thus, to verify that the set of all compatible value functions $\mathcal{U}_{\mathcal{A}^*}$ is not empty, it is sufficient to verify that $\varepsilon^* > 0$, where $\varepsilon^* = \max\ \varepsilon$, subject to set of constraints $(E^{A^*})'$.

Taking into account a single value function $U \in \mathcal{U}_{\mathcal{A}^*}$ and its associated assignment examples relative to the reference set A^*, an alternative $a \in A$ can be assigned to an interval of classes $[C_{L^U(a)}, C_{R^U(a)}]$, in the following way:

$$L^U(a) = Max\left(\{1\} \cup \left\{L^{DM}(a^*) : U(a^*) \leq U(a),\ a^* \in A^R\right\}\right), \tag{9.9}$$

$$R^U(a) = Min\left(\{p\} \cup \left\{R^{DM}(a^*) : U(a^*) \geq U(a),\ a^* \in A^R\right\}\right). \tag{9.10}$$

For each nonreference alternative $a \in A \setminus A^*$ the indices satisfy the following condition:

$$L^U(a) \leq R^U(a). \tag{9.11}$$

In order to take into account the whole set of value functions one can proceed as follows. Given a set A^* of assignment examples and a corresponding set $\mathcal{U}_{\mathcal{A}^*}$ of compatible value functions, for each $a \in A$, we define the possible assignment $C_P(a)$ as the set of indices of classes C_h for which there exist at least one value function $U \in \mathcal{U}_{\mathcal{A}^*}$ assigning a to C_h, and the necessary assignment $C_N(a)$ as set of indices of classes C_h for which all value functions $U \in \mathcal{U}$ assign a to C_h, that is:

$$C_P(a) = \left\{h \in H : \exists U \in \mathcal{U}_{\mathcal{A}^*} \text{ for which } h \in \left[L^U(a), R^U(a)\right]\right\} \tag{9.12}$$

$$C_N(a) = \left\{h \in H : \forall U \in \mathcal{U}_{\mathcal{A}^*} \text{ it holds } h \in \left[L^U(a), R^U(a)\right]\right\} \tag{9.13}$$

To compute the possible and necessary assignments $C_P(a)$ and $C_N(a)$, we can consider the following indices:

- minimum possible class:

$$L_P^{\mathcal{U}}(a) = Max\left(\{1\} \cup \left\{L^{DM}(a^*) : \forall U \in \mathcal{U}_{\mathcal{A}^*}, U(a^*) \le U(a),\ a^* \in A^*\right\}\right) \tag{9.14}$$

- minimum necessary class:

$$L_N^{\mathcal{U}}(a) = Max\left(\{1\} \cup \left\{L^{DM}(a^*) : \exists U \in \mathcal{U}_{\mathcal{A}^*} \text{ for which } U(a^*) \le U(a),\ a^* \in A^*\right\}\right) \tag{9.15}$$

- maximum necessary class:

$$R_N^{\mathcal{U}}(a) = Min\left(\{p\} \cup \left\{R^{DM}(a^*) : \exists U \in \mathcal{U}_{\mathcal{A}^*} \text{ for which } U(a) \le U(a^*),\ a^* \in A^*\right\}\right) \tag{9.16}$$

- maximum possible class:

$$R_P^{\mathcal{U}}(a) = Min\left(\{p\} \cup \left\{R^{DM}(a^*) : \forall U \in \mathcal{U}_{\mathcal{A}^*}, U(a) \le U(a^*),\ a^* \in A^*\right\}\right) \tag{9.17}$$

Using indices $L_P^{\mathcal{U}}(a)$, $L_N^{\mathcal{U}}(a)$, $R_N^{\mathcal{U}}(a)$ and $R_P^{\mathcal{U}}(a)$, the possible and necessary assignments $C_P(a)$ and $C_N(a)$ can be expressed as follows:

$$C_P(a) = \left[L_P^{\mathcal{U}}(a), R_P^{\mathcal{U}}(a)\right]$$

and, if $L_N^{\mathcal{U}}(a) \le R_N^{\mathcal{U}}(a)$, then

$$C_N(a) = \left[L_N^{\mathcal{U}}(a), R_N^{\mathcal{U}}(a)\right]$$

while, if $L_N^{\mathcal{U}}(a) > R_N^{\mathcal{U}}(a)$, then

$$C_N(a) = \emptyset.$$

As in the methods *UTA*^{GMS} and *GRIP*, on the basis of all compatible value functions $\mathcal{U}_{\mathcal{A}^*}$, we can define two binary relations on the set of alternatives A:

- *Necessary* weak preference relation $\succsim^N$, in case $U(a) \ge U(b)$ for all compatible value functions
- *Possible* weak preference relation $\succsim^P$, in case $U(a) \ge U(b)$ for at least one compatible value function

Using necessary weak preference relation $\succsim^N$ and possible weak preference relation $\succsim^P$ we can redefine indices $L_P^{\mathcal{U}}(a)$, $L_N^{\mathcal{U}}(a)$, $R_N^{\mathcal{U}}(a)$ and $R_P^{\mathcal{U}}(a)$ as follows:

- minimum possible class:

$$L_P^{\mathcal{U}}(a) = Max\left(\{1\} \cup \left\{L^{DM}(a^*) : a \succsim^N a^*,\ a^* \in A^*\right\}\right), \tag{9.18}$$

- minimum necessary class:

$$L_N^{\mathcal{U}}(a) = Max\left(\{1\} \cup \left\{L^{DM}(a^*) : a \succsim^P a^*,\ a^* \in A^*\right\}\right), \tag{9.19}$$

- maximum necessary class:

$$R_N^{\mathcal{U}}(a) = Min\left(\{p\} \cup \left\{R^{DM}(a^*) : a^* \succsim^P a,\ a^* \in A^*\right\}\right), \tag{9.20}$$

- maximum possible class:

$$R_P^{\mathcal{U}}(a) = Min\left(\{p\} \cup \left\{R^{DM}(a^*) : a^* \succsim^N a,\ a^* \in A^*\right\}\right). \tag{9.21}$$

Thus, using necessary weak preference relation $\succsim^N$ and possible weak preference relation $\succsim^P$, it is possible to deal quite simply with the sorting problem.

Therefore, on the basis of the above observations, the following example-based sorting procedure can be proposed:

1. Ask the DM for an exemplary sorting.
2. Verify that the set of compatible value functions $\mathcal{U}_{A^*}$ is not empty.
3. Calculate the necessary and the possible weak preference relations $a \succsim^N a^*$, $a \succsim^P a^*$, $a^* \succsim^N a$ and $a^* \succsim^P a$, with $a^* \in A^*$ and $a \in A$.
4. Calculate for each $a \in A$ the indices $L_P^{\mathcal{U}}(a)$, $L_N^{\mathcal{U}}(a)$, $R_N^{\mathcal{U}}(a)$ and $R_P^{\mathcal{U}}(a)$ using (9.18), (9.19), (9.20) and (9.21).
5. Assign to each $a \in A$ its possible assignment $C_P(a) = \left[L_P^{\mathcal{U}}(a), R_P^{\mathcal{U}}(a)\right]$.
6. Assign to each $a \in A$ its necessary assignment, which is $C_N(a) = \left[L_N^{\mathcal{U}}(a), R_N^{\mathcal{U}}(a)\right]$ in case $L_N^{\mathcal{U}}(a) \leq R_N^{\mathcal{U}}(a)$, and $C_N(a) = \emptyset$ otherwise.

In [16], one can find a proposal how to handle within *UTADIS*^{GMS} an additional preference information about intensity of preference.

9.6 The Most Representative Value Function

The robust ordinal regression builds a set of additive value functions compatible with preference information provided by the DM and results in two rankings, necessary and possible. Such rankings answer to robustness concerns, since they are in general "more robust" than a ranking made by an arbitrarily chosen compatible value function. However, in practice, for some decision-making situations, a score is needed to assign to the different alternatives, and despite the interest of the rankings provided, some users would like to see, and they indeed need to know, the "most representative" value function among all the compatible ones. This allows assigning a score to each alternative.

Recently, a methodology to identify the "most representative" value function in *GRIP*, without loosing the advantage of taking into account all compatible value functions, has been proposed in [17]. The idea is to select among compatible value functions that one which better highlights the necessary ranking maximizing the difference of values between alternatives for which there is a preference in the necessary ranking. As secondary objective, one can consider minimizing the difference of values between alternatives for which there is no preference in the necessary ranking. This comprehensive "most representative" value function can be determined via the following procedure:

1. Determine the necessary and the possible rankings in the considered set of alternatives.
2. For all pairs of alternatives (a, b), such that a is necessarily preferred to b, add the following constraints to the linear programming constraints of *GRIP*: $U(a) \geq U(b) + \varepsilon$.
3. Maximize the objective function ε.
4. Add the constraint $\varepsilon = \varepsilon^*$, with $\varepsilon^* = \max \varepsilon$ from the previous point, to the linear programming constraints of robust ordinal regression.
5. For all pairs of alternatives (a, b), such that neither a is necessarily preferred to b nor b is necessarily preferred to a, add the following constraints to the linear programming constraints of *GRIP* and to the constraints considered in above point 4): $U(a) - U(b) \leq \delta$ and $U(b) - U(a) \leq \delta$.
6. Minimize the objective function δ.

This procedure maximizes the minimal difference between values of alternatives for which the necessary preference holds. If there is more than one such value function, the above procedure selects the most representative compatible value function giving the greatest minimal difference between values of alternatives for which the necessary preference holds, and the smallest maximal difference between values of alternatives for which the possible preference holds.

Notice that the concept of the "most representative" value function thus defined is still based on the necessary and possible preference relations, which remain crucial for *GRIP*, and, in a sense, it gives the most faithful representation of this necessary and possible preference relations.

In [27] the concept of the "most representative" value function has been extended to robust ordinal regression applied to sorting problems within $UTADIS^{GMS}$.

The idea is to select among all compatible value functions that one which better highlights the possible sorting considered as the most stable part of the robust sorting obtained by $UTADIS^{GMS}$. In consequence, the selected value function is that one which maximizes the difference of values between alternatives for which the intervals of possible sorting are disjoint. As secondary objective, to tie-breaking, one can wish to maximize the minimal difference between values of alternatives a and b such that for any compatible value function U a is assigned to a class not worse than the class of b and for at least one compatible value function a is assigned to a class which is better than the class of b. In case there is still more than one such value function, the "most representative" function minimizes the maximal difference between values of alternatives a and b being in the same class for all compatible value functions U or such that the order of classes is not univocal in the sense that for some compatible value functions U a is assigned to a class better than b and for other compatible value function b is assigned to a class better than a.

The following three-stage procedure for determining the most representative value function can be proposed:

1. Determine the possible sorting $C_P(a)$ and the necessary sorting $C_N(a)$ for each considered alternative $a \in A$.
2. For all pairs of alternatives (a, b), such that $L_P^{\mathcal{U}}(a) > R_P^{\mathcal{U}}(b)$, add the following constraint to the linear programming constraints of *UTADIS*^{GMS}, E^{A^R}:

$$U(a) \geq U(b) + \varepsilon.$$

3. Maximize the objective function ε subject to the set of linear constraints from point 2.
4. Add the constraint $\varepsilon = \varepsilon^*$, with $\varepsilon^* = \max \varepsilon$ from the previous point, to the linear programming constraints of *UTADIS*^{GMS}, E^{A^R}.
5. For all pairs of alternatives (a, b), such that for any compatible value function U a is assigned to a class not worse than the class of b and for at least one compatible value function a is assigned to a class which is better than the class of b, add the following constraint to the linear programming constraints from point 4:

$$U(a) \geq U(b) + \gamma.$$

6. Maximize the objective function γ subject to the set of linear constraints from point 5.
7. Add the constraint $\gamma = \gamma^*$, with $\gamma^* = \max \gamma$ from the previous point, to the linear programming constraints from point 5.
8. For all pairs of alternatives (a, b), such that they are in the same class for all compatible value functions U, or such that the order of classes is not univocal, add the following constraints to the linear programming constraints from point 7:

$$U(a) - U(b) \leq \delta \text{ and } U(b) - U(a) \leq \delta.$$

9. Minimize the objective function δ subject to the set of linear constraints from point 8.

Notice that the concept of the "most representative" value function thus defined is based on the possible assignments and supplies the most faithful representation of the recommendation given by *UTADIS*^{GMS}. Therefore, it can play a significant role in supporting the DM to understand the results of the robust sorting. Moreover, the most representative value function U^R chosen according to the above principles, can be used along with the assignment examples supplied at the beginning by the DM to drive an autonomous example-based sorting procedure. In such a way the most representative assignment for each alternative $a \in A$ can be determined.

9.7 Nonadditive Robust Ordinal Regression

To take into account interactions between criteria, robust ordinal regression has been applied to Choquet integral [2].

Let 2^G be the power set of G (i.e., the set of all the subsets of the set of criteria G); a fuzzy measure on G is defined as a set function $\mu : 2^G \to [0, 1]$ which satisfies the following properties:

(1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions)
(2a) $\forall\, T \subseteq R \subseteq G,\;\; \mu(T) \leq \mu(R)$ (monotonicity condition)

In the framework of multiple criteria decision problems, a fuzzy measure $\mu(R)$ is related to the importance weight given by the DM to every subset of criteria R that can be evaluated by the Shapley value [56], defined later in this section.

Let $x \in A$ and μ be a fuzzy measure on G, then the *Choquet integral* [11] is defined by:

$$C_\mu(x) = \sum_{i=1}^{n} \left[\left(g_{(i)}(x)\right) - \left(g_{(i-1)}\left(x\right)\right)\right] \mu\left(A_i\right), \tag{9.22}$$

where $_{(\cdot)}$ stands for a permutation of the indices of evaluations of criteria such that:

$$g_{(1)}\left(x\right) \leq g_{(2)}\left(x\right) \leq g_{(3)}\left(x\right) \leq \ldots \leq g_{(n)}\left(x\right),$$

with $A_i = \{(i), \ldots, (n)\}$, $i = 1, \ldots, n$, and $g_{(0)} = 0$.

One of the main drawbacks of the Choquet integral is the necessity to elicit and give an adequate interpretation of $2^{|G|} - 2$ parameters. In order to reduce the number of parameters to be computed and to eliminate a too strict description of the interactions among criteria, which is not realistic in many applications, one can consider the concept of fuzzy k-additive measure [22].

Given a partial preorder $\succeq$ on A^R, a set of fuzzy measures $\boldsymbol{\mu}$ is called compatible if the Choquet integral, calculated with respect to it, restores the DM's ranking on A^R, i.e.,

$$a \succeq b \;\; \Leftrightarrow \;\; C_{\boldsymbol{\mu}}(a) \geq C_{\boldsymbol{\mu}}(b) \;\;\; \forall a, b \in A^R.$$

The procedure proposed is composed of three successive phases:

(I) Elicitation of preference information on a reference set $A^R \subseteq A$ of alternatives
(II) Evaluation of all the *compatible* fuzzy measures to establish the preference relations $a \succeq^P b$ and $a \succeq^N b$ for every ordered pair of alternatives $(a, b) \in A \times A$
(III) Exploitation of the results obtained to detect possible DM's inconsistencies or to revise the preference model obtained

In the phase of elicitation of preference information, the DM is asked to provide the following preference information:

(a) A partial preorder $\succeq$ on A^R, i.e., for $a, b \in A^R$:

$$a \succeq b \;\Leftrightarrow\; a \text{ is at least as good as } b.$$

(b) A partial preorder $\succeq^*$ on $A^R \times A^R$, i.e., for $a, b, c, d \in A^R$

$$(a, b) \succeq^* (c, d) \Leftrightarrow a \text{ is preferred to } b \text{ at least as much as } c \text{ is preferred to } d.$$

(c) A partial preorder $\rhd$ on G, for $i, j \in G$, whose definition is:

$$i \rhd j \;\Leftrightarrow\; \text{criterion } i \text{ is more important than criterion } j.$$

(d) A partial preorder $\rhd^*$ on $G \times G$, whose definition is: for $i, j, l, k \in G$ $(i, j) \rhd^* (l, k)$ $\Leftrightarrow$ the difference of importance between criteria i and j is at least as much as difference of importance between criteria l and k.
(e) A sign (positive or negative) of interaction of couples of criteria.
(f) A partial preorder $\rhd_{\text{Int}}$ on $G \times G$, whose definition is: for $i, j, l, k \in G$,

$$(i, j) \rhd_{\text{Int}} (l, k) \;\Leftrightarrow$$

intensity of interaction between criteria i and j is at least as strong as intensity of interaction between criteria l and k.
(g) A partial preorder $\rhd^*_{\text{Int}}$ on G^4, whose definition is: for $i, j, l, k, r, s, t, w \in G$,

$$[(i, j), (l, k)] \rhd^*_{\text{Int}} [(r, s), (t, w)] \;\Leftrightarrow$$

difference of intensity of interaction between criteria i and j, and intensity of interaction between criteria l and k is at least as strong as difference of intensity of interaction between criteria r and s, and intensity of interaction between criteria t and w. In this phase, the DM compares the intensity of interaction for pairs of criteria, both redundant or synergic.

The preference information of type (b), (d), (f) and (g) can be provided by the DM using a semantic scale in a similar way to the approaches of *MACBETH* [6], *AHP* [54] and *GRIP* [18]. More precisely, given an ordinal scale such as "null," "small," "medium," "large," and "extreme," the DM can give information of the type: "the preference of alternative a over alternative b is large" or "the difference of importance between criteria g_i and g_j is medium" or "the synergy between criteria g_i and g_j is small".

In Phase II, the set of all *compatible* fuzzy measures is determined as those fuzzy measures satisfying a system of linear constraints representing all the preference information given by the DM in Phase I, plus the monotonicity and boundary conditions of fuzzy measures.

In Phase III, the obtained preference model, i.e., the system of linear constraints determining the set of all compatible fuzzy measures, is used to determine the *necessary* preference relation $\succsim^N$ and the *possible* preference relation $\succsim^P$ on A, as follows:

$$x \succsim^N y \;\Leftrightarrow\; C_\mu(x) \geq C_\mu(y)$$

for all compatible sets of fuzzy measures $\boldsymbol{\mu}$, with $x, y \in A$, and

$$a \succsim^P b \;\Leftrightarrow\; C_\mu(x) \geq C_\mu(y)$$

for at least one compatible set of fuzzy measures $\boldsymbol{\mu}$, with $x, y \in A$.

In [3], nonadditive robust ordinal regression has been proposed to deal with sorting problems. In simple words, the methodology follows the principle of *UTADIS*^{GMS}, but considering the Choquet integral, instead of an additive value function. In [4], nonadditive robust ordinal regression has been extended in turn to deal with some generalizations of Choquet integral, such as bipolar Choquet integral [23, 24, 30] and the level dependent Choquet integral [26].

9.8 Robust Ordinal Regression in Interactive Multiobjective Optimization

Classical ordinal regression methods have been applied in Multiobjective Optimization (MOO) in [38] and in [57], where an additive value function interactively built using the *UTA* method is optimized within the feasible region. In the same spirit, robust ordinal regression has been applied to MOO problems in [15], as explained below. We assume that the Pareto optimal set of an MOO problem is generated prior to an interactive exploration of this set. Instead of the whole and exact Pareto optimal set of a MOO problem, one can also consider a proper representation of this set, or its approximation. In any case, an interactive exploration of this set should lead the DM to a conviction that either there is no satisfactory solution to the considered problem, or there is at least one such solution. We will focus our attention on the interactive exploration, and the proposed interactive procedure will be valid for any finite set of solutions to be explored. Let us denote this set by A. Note that such set A can be computed using evolutionary multiobjective optimization. For a recent state of the art of interactive and evolutionary approaches to MOO, see [8].

In the course of the interactive procedure, the preference information provided by the DM concerns a small subset of A, called reference or training sample, and denoted by A^R. The preference information is transformed by an ordinal regression method into a DM's preference model. We propose to use at this stage the *GRIP* method, thus the preference model is a set of general additive value functions compatible with the preference information. A compatible value function compares the solutions from the reference sample in the same way as the DM. The obtained preference model is then applied on the whole set A, which results in possible and necessary rankings of solutions. These rankings are used to select a new sample of reference solutions, which is presented to the DM, and the procedure cycles until a satisfactory solution is selected from the sample or the DM comes to conclusion that there is no satisfactory solution for the current problem setting.

The proposed interactive procedure is composed of the following steps:

- ***Step* 1.** Select a representative reference sample A^R of solutions from set A.
- ***Step* 2.** Present the sample A^R to the DM.
- ***Step* 3.** If the DM is satisfied with at least one solution from the sample, then this is the satisfactory solution and the procedure stops. The procedure also stops in this step if the DM concludes that there is no satisfactory solution for the current problem setting. Otherwise continue.
- ***Step* 4.** Ask the DM to provide information about his/her preferences on set A^R in the following terms:
  - Pairwise comparison of some solutions from A^R
  - Comparison of intensities of comprehensive preferences between some pairs of solutions from A^R
  - Comparison of intensities of preferences on single criteria between some pairs of solutions from A^R
- ***Step* 5.** Use the *GRIP* method to build a set of additive monotonically nondecreasing value functions compatible with the preference information obtained from the DM in *Step* 4.
- ***Step* 6.** Apply the set of compatible value functions built in *Step* 5 on the whole set A, and present the possible and necessary rankings (see Section 9.4.2) resulting from this application to the DM.
- ***Step* 7.** Taking into account the possible and necessary rankings, let the DM select a new reference sample of solutions $A^R \subseteq A$, and go to *Step* 2.

In *Step* 4, the information provided by the DM may lead to a set of constraints, which define an empty polyhedron of the compatible value functions. In this case, the DM gets information about which items of his/her preference information make the polyhedron empty, so as to enable revision in the next round. This point is explained in detail in [18, 34]. Moreover, information provided by the DM in *Step* 4 cannot be considered as irreversible. Indeed, the DM can retract to one of previous iterations and continue from this point. This feature is concordant with the spirit of a learning oriented conception of multiobjective interactive optimization, i.e., it confirms the idea that the interactive procedure permits the DM to learn about his/her preferences and about the "shape" of the Pareto optimal set (see [7]).

Notice that the proposed approach allows to elicit incrementally preference information from the DM. In *Step* 7, the "new" reference sample A^R is not necessarily different from the previously considered, however, the preference information elicited from the DM in the next iteration is richer than previously, due to the learning effect. This permits to build and refine progressively the preference model: in fact, each new item of information provided in *Step* 4 restricts the set of compatible value functions and defines the DM's preferences more and more precisely.

Let us also observe that information obtained from the DM in *Step* 4 and information given to the DM in *Step* 6 is composed of very simple and easy to understand statements: preference comparisons in *Step* 4, and possible and necessary rankings in *Step* 6 (i.e., a necessary ranking that holds for all compatible value functions, and a possible ranking that holds for at least one compatible value function; see Section 9.4.2). Thus, the nature of information exchanged with the DM during the interaction is purely ordinal. Indeed, monotonically increasing transformations of evaluation scales of considered criteria have no influence on the final result.

Finally, observe that a very important characteristic of our method from the point of view of learning is that the DM can observe the impact of information provided in *Step* 4 in terms of possible and necessary rankings of solutions from set A.

9.9 Robust Ordinal Regression in Evolutionary Interactive Multiobjective Optimization

Most of the research in evolutionary multiobjective optimization (EMO) attempts to approximate the complete Pareto optimal front by a set of well-distributed representatives of Pareto optimal solutions. The underlying reasoning is that in the absence of any preference information, all Pareto optimal solutions have to be considered equivalent.

On the other hand, in most practical applications, the DM is eventually interested in only a single solution. In order to come up with a single solution, it is necessary to involve the DM. This is the underlying idea of another multiobjective optimization paradigm: interactive multiobjective optimization (IMO). IMO deals with the identification of the most preferred solution by means of a systematic dialogue with the DM. Only recently, the scientific community has discovered the great potential of combining the two paradigms (for a recent survey, see [41]). From the point of view of EMO, involving the DM in an interactive manner will allow to focus the search on the area of the Pareto front which is most relevant to the DM. This, in turn, may allow to find more appropriate solutions faster. In particular, in the case of many objectives, EMO has difficulties, because the number of Pareto-optimal solutions becomes huge, and Pareto-optimality is not sufficiently discriminative to guide the search into better regions. Integrating user preferences promises to alleviate these problems, allowing to converge faster to the preferred region of the Pareto-optimal front.

Robust ordinal regression has been applied to EMO in a methodology called *NEMO* (*N*ecessary preference-based *E*volutionary *M*ultiobjective *O*ptimization) presented in [9, 10]. *NEMO* combines *NSGA*-II [12], a widely used EMO technique, with the IMO methodology based on robust ordinal regression presented in Section 9.4. The *NEMO* methodology takes into account the information about necessary preferences, given by the robust ordinal regression, in order to focus the search on the most promising parts of the Pareto optimal front. More specifically, robust ordinal regression based on information obtained through interaction with the DM determines the set of compatible value functions, and an EMO procedure searches for all nondominated solutions taking into account all compatible value functions in parallel.

We believe that the integration of robust ordinal regression into EMO is particularly promising for two reasons:

1. The preference information required by robust ordinal regression is very basic and easy to provide by the DM. All that the DM is asked for is to compare two nondominated solutions, and to reveal whether one is preferred over the other.
2. The resulting set of compatible value functions reveals implicitly an appropriate scaling of the criteria, an issue that is largely ignored by the EMO community so far.

A crucial step in *NSGA*-II, is the ranking of solutions (individuals) in a current population according to two criteria.

The primary criterion is the so-called dominance-based ranking. This criterion ranks individuals by iteratively determining the nondominated solutions in the population (nondominated front), assigning those individuals the next best rank, and removing them from the population. The result is a partial ordering, favoring individuals closer to the Pareto optimal front.

According to the secondary criterion, individuals which have the same dominance-rank (primary criterion) are sorted with respect to the crowding distance, which is defined as the sum of distances between a solution and its neighbors on either side in each dimension of the objective space. Individuals with a large crowding distance are preferred, as they are in a less crowded region of the objective space, which is concordant with the goal of preserving diversity in the population.

NEMO combines the robust ordinal regression with *NSGA*-II in three different variants:

- *NEMO*-0: a single compatible value function is used to rank solutions in a population. For example, one can consider the value function obtained by the *UTA* method.
- *NEMO*-I: the whole set of compatible value functions is considered and the dominance relation used in *NSGA*-II to rank solutions is replaced by the necessary preference relation of robust ordinal regression.
- *NEMO*-II: the whole set of compatible value functions is also considered, but differently from *NEMO*-I, the solutions in the population are ranked according to a score calculated as the max–min difference of values between a given solution and other solutions in the population, for the whole set of compatible value functions.

In *NEMO*-0, *NEMO*-I, and *NEMO*-II, the following types of value functions are considered:

- Linear value function, i.e.,

$$U(g(a)) = \lambda_1(g_1(a)) + \lambda_2(g_2(a)) + \ldots + \lambda_n(g_n(a)) \quad \text{with} \quad a \in A,$$

  $\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0, \quad \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$
- Piecewise-linear value function, as in the *UTA* method (see Section 9.3)

- General additive value function, as in the UTA^{GMS} and *GRIP* methods (see Section 9.4)

Observe that the *NEMO*-I variant with a linear value function corresponds to the method proposed in [37].

In the following, we present *NEMO*-I with a general additive value function, which was the first variant proposed in [9]. The modifications of *NEMO*-I with respect to *NSGA*-II are the following:

1. *NEMO*-I replaces the dominance-based ranking procedure by the necessary ranking procedure. The necessary ranking procedure works analogously to the dominance-based ranking procedure, but taking into account the preference information by the DM through the necessary preference relations. More precisely, the procedure first puts in the best rank all solutions, which are not preferred by any other solution in the population, then removes them from the population and creates the second best rank composed of solutions, which are not preferred by any other solution in the reduced population, and so on.
2. *NEMO*-I replaces the crowding-distance by a distance calculated in the space of marginal values, taking into account the multidimensional scaling given by the "the most representative" value function among the whole set of compatible value functions (see Section 9.7). More precisely, the crowding distance is calculated according to the procedure used in *NSGA*-II with the only difference that in calculating the average side-length of the cuboid the distance is measured in terms of marginal values of the "most representative" value function.

Preferences are elicited by asking the DM to compare pairs of nondominated solutions, and specify a preference relation between them.

The overall *NEMO*-I algorithm is outlined in Algorithm 1. Although the general procedure is rather straightforward, there are several issues that need to be considered:

Algorithm 1: Basic *NEMO*-I

```
Generate initial solutions randomly
Elicit DM's preferences {Present to the DM a pair of nondominated solutions and ask for a
preference comparison}
Determine necessary ranking {Replaces dominance ranking in NSGA-II}
Determine secondary ranking {Order solutions within the same rank, based on the crowding
distance measured in terms of the "most representative value function"}
repeat
   Mating selection and offspring generation
   if Time to ask the DM then
      Elicit DM's preferences
   end if
   Determine necessary ranking
   Determine secondary ranking
   Environmental selection
until Stopping criterion met
Return all preferred solutions according to necessary ranking
```

1. How many pairs of solutions are shown to the DM, and when? In [9], one pairwise comparison of nondominated solutions was asked every k generations, i.e., every k generations, *NEMO*-I is stopped, and the user is asked to provide preference information about one given pair of individuals. Preliminary experiments show that $k = 20$ in 300 generation runs gives satisfactory results.
2. Which pairs of solutions should be presented to the DM for comparison? In [9], each pair of solutions was picked randomly from among the best solutions not related by the necessary preference relation, i.e., from solutions having the best rank. This avoids that the DM can specify inconsistent information, inverting the necessary preference relation (including dominance) between two solutions. To speed up convergence, it would be reasonable, however, to pick pairs of solutions having the best rank and being close with respect to the overall value but diversified on respective marginal values, for "the most representative" value function.

An important remark about the *NEMO* methodology regards its approximation power. In fact, *NSGA*-II can identify all nondominated solutions, even improper ones, i.e., nondominated points that allow unbounded trade-off between objective functions [20], in problems where the nondominated frontier has discontinuities or it is nonconvex. From this point of view, *NEMO* methodology maintains this good property. More precisely, considering linear value functions in *NEMO*-0 or *NEMO*-II, one cannot deal with improper solutions and discontinuous or nonconvex frontier, because there can be no linear value function giving the best value to some efficient solutions. *NEMO*-I can find, however, all nondominated points because it compares pairs of solutions and, therefore, there can be linear compatible value functions for which the considered nondominated solution, possibly improper, is preferred to other nondominated solutions, even in case of discontinuities of nonconvexity. Using a general additive value function in *NEMO*-0, *NEMO*-I, or *NEMO*-II, improper efficient points, discontinuous or nonconvex nondominated frontiers can be dealt without any difficulty. To explain this ability, remark that:

(a) The class of value functions, which can be expressed as additive value functions is very large, including, for instance, value functions of the form

$$U(g(a)) = u_1(g_1(a))^{\lambda_1} \times u_2(g_2(a))^{\lambda_2} \times \ldots \times u_n(g_n(a))^{\lambda_n}$$

with $a \in A, \lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0$, whose logarithm takes the form

$$U^*[g(a)] = log[U(g(a))] = \lambda_1 \times log[u_1(g_1(a))] + \lambda_2 \times log[u_2(g_2(a))] + \ldots + \lambda_n \times log[u_n(g_n(a))].$$

(b) Marginal value functions $u_i(g_i(a)), i = 1, \ldots, n$, can be constant in some parts of their domains

Remark (a) explains why *NEMO*-I and *NEMO*-II are able to deal with discontinuous and nonconvex nondominated frontiers, while remark (b) explains why *NEMO*-I and *NEMO*-II are able to deal with improper points.

Piecewise linear value functions have a behavior, which is intermediate between the linear value functions and general additive value functions. One can say, in general, that the greater the number of linear pieces assumed for each marginal value function, the more similar the final results are to the case of general additive value functions. This means that increasing the number of linear pieces, one improves the capacity of dealing with improper solutions, discontinuities and nonconvexities. However, the more flexible the value function model, the more preference information, and thus more interactions with the DM, is required to focus the search on the most preferred region of the Pareto optimal front.

9.10 Robust Ordinal Regression for Outranking Methods

Outranking relation is a noncompensatory preference model used in the *ELECTRE* family of MCDA methods [52]. Its construction involves two concepts known as concordance and discordance. Outranking relation, usually denoted by S, is a binary relation on a set A of alternatives. For an ordered pair of alternatives $(a, b) \in A$, aSb means "a is at least as good as b." The assertion aSb is considered to be true if the coalition of criteria being in favor of this statement is "strong enough" comparing to the rest of criteria, and if among the criteria opposing to this statement, there is no one for which a is "significantly worse" than b. The first condition is called concordance test, and the second, non-discordance test.

Let us denote by k_i the weight assigned to criterion g_i, $i = 1, \ldots, n$; it represents a relative importance of criterion g_i within family F of n criteria. The indifference, preference and veto thresholds on criterion g_i are denoted by q_i, p_i and v_i, respectively. For consistency, $v_i > p_i > q_i \geq 0$, $i = 1, \ldots, n$. In all formulae that follow, we suppose, without loss of generality, that all these thresholds are constant, that preferences are increasing with evaluations on particular criteria, and that criteria are identified by their indices.

The concordance test involves calculation of concordance index $C(a, b)$. It represents the strength of the coalition of criteria being in favor of aSb. This coalition is composed of two subsets of criteria:

- Subset of criteria being clearly in favor of aSb, i.e., such that $g_i(a) \geq g_i(b) - q_i$.
- Subset of criteria that do not oppose to aSb, while being in an ambiguous position with respect to this assertion; these are those criteria for which a weak preference relation bQa holds; i.e., such that $g_i(b) - p_i \leq g_i(a) < g_i(b) - q_i$.

Consequently, the concordance index is defined as

$$C(a, b) = \frac{\sum_{i=1}^{n} \phi_i(a, b) \times k_i}{\sum_{i=1}^{n} k_i}, \tag{9.23}$$

where, for $i = 1, \ldots, n$,

$$\phi_i(a,b) = \begin{cases} 1, & \text{if } g_i(a) \geq g_i(b) - q_i, \\ \dfrac{g_i(a) - [g_i(b) - p_i]}{p_i - q_i}, & \text{if } g_i(b) - p_i \leq g_i(a) < g_i(b) - q_i, \\ 0, & \text{if } g_i(a) < g_i(b) - p_i. \end{cases} \tag{9.24}$$

$\phi_i(a,b)$ is a marginal concordance index, indicating to what extent criterion g_i contributes to the concordance index $C(a,b)$. As defined by (9.24), $\phi_i(a,b)$ is a piecewise linear function, nondecreasing with respect to $g_i(a) - g_i(b)$.

Remark that $C(a,b) \in [0,1]$, where $C(a,b) = 0$ if $g_i(a) \leq g_i(b) - p_i$, $i = 1, \ldots, n$ (b is strictly preferred to a on all criteria), and $C(a,b) = 1$ if $g_i(a) \geq g_i(b) - q_i$, $i = 1, \ldots, n$ (a outranks b on all criteria).

The result of the concordance test for a pair $(a,b) \in A$ is positive if $C(a,b) \geq \lambda$, where $\lambda \in [0.5, 1]$ is a cutting level, which has to be fixed by the DM.

Once the result of the concordance test has been positive, one can pass to the non-discordance test. Its result is positive for the pair $(a,b) \in A$ unless "a is significantly worse than b" on at least one criterion, i.e., if $g_i(b) - g_i(a) < v_i$ for $i = 1, \ldots, n$.

It follows from above that the outranking relation for a pair $(a,b) \in A$ is true, and denoted by aSb if both the concordance test and the non-discordance test are positive. On the other hand, the outranking relation for a pair $(a,b) \in A$ is false, and denoted by aS^cb, either if the concordance test or the non-discordance test is negative.

Knowing S or S^c for all ordered pairs $(a,b) \in A$, one can proceed to exploitation of the outranking relation in set A, which is specific for the choice, or sorting or ranking problem, as described in [19].

Experience indicates that elicitation of preference information necessary for construction of the outranking relation is not an easy task for a DM. In particular, the inter-criteria preference information concerning the weights of criteria and the veto thresholds are difficult to be expressed directly.

For this reason, some disaggregation–aggregation procedures have been proposed in the past to assist the elicitation of the weights of criteria and all the thresholds required to construct the outranking relation [48–50]. The most general proposal, however, has been presented in [33,36]. It permits to asses the whole set of outranking relations compatible with some exemplary pairwise comparisons of few real or fictitious reference alternatives, using a robust ordinal regression approach. Below, we briefly sketch this proposal.

We assume that the preference information provided by the DM is a set of pairwise comparisons of some reference alternatives. The set of reference alternatives is denoted by A^R, and it is usually, although not necessarily, a subset of set A. The comparison of a pair of alternatives $(a,b) \in A^R$ states the truth or falsity of the outranking relation, denoted by aSb or aS^cb, respectively. It is worth stressing that the DM does not need to provide all pairwise comparisons of reference alternatives, so this comparison can be confined to a small subset of pairs.

We also assume that the intra-criterion preference information concerning indifference and preference thresholds $p_i > q_i \geq 0$, $i = 1, \ldots, n$, is given. The last assumption is not unrealistic because these thresholds are relatively easy to provide by an analyst who is usually aware what is the precision of criteria, and how much difference is nonsignificant or relevant.

In order to simplify calculations of the ordinal regression, we assume that the weights of criteria sum up to one, i.e., $\sum_{i=1}^{n} k_i = 1$. Thus, (9.23) becomes

$$C(a,b) = \sum_{i=1}^{n} \phi_i(a,b) \times k_i = \sum_{i=1}^{n} \psi_i(a,b), \tag{9.25}$$

where the marginal concordance index $\psi_i(a,b) = \phi_i(a,b) \times k_i$ is a monotone nondecreasing function with respect to $g_i(a) - g_i(b)$, such that $\psi_i(a,b) \geq 0$ for all $(a,b) \in A^R \times A^R$, $i = 1, \ldots, n$, $\psi_i(a,b) = 0$ for all $g_i(b) - g_i(a) \geq p_i$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \psi_i(a,b) = 1$ in case $g_i(a) - g_i(b) \geq -q_i$ for all $i = 1, \ldots, n$.

The ordinal regression constraints defining the set of concordance indices $C(a,b)$, cutting levels λ and veto thresholds v_i, $i = 1, \ldots, n$, compatible with the pairwise comparisons provided by the DM have the following form:

$$\left.\begin{array}{l}
C(a,b) = \sum_{i=1}^{n} \psi_i(a,b) \geq \lambda \ \text{ and } \ g_i(b) - g_i(a) \leq v_i - \varepsilon, i = 1, \ldots, n, \\
\quad \text{if } aSb, \ \text{ for } (a,b) \in A^R \times A^R, \\
C(a,b) = \sum_{i=1}^{n} \psi_i(a,b) \leq \lambda - \varepsilon + M_0(a,b) \text{ and } g_i(b) - g_i(a) \leq v_i - \delta M_i(a,b), \\
M_i(a,b) \in \{0,1\}, \ \sum_{i=0}^{n} M_i(a,b) \leq n, \ i = 1, \ldots, n, \\
\quad \text{if } aS^c b, \ \text{ for } (a,b) \in A^R \times A^R, \\
1 \geq \lambda \geq 0.5, \quad v_i \geq p_i, \ i = 1, \ldots, n, \\
\psi_i(a,b) \geq 0, \ \text{ for all } (a,b) \in A^R \times A^R, \ i = 1, \ldots, n, \\
\psi_i(a,b) = 0 \ \text{ if } \ g_i(b) - g_i(a) \geq p_i, \ \text{ for all } (a,b) \in A^R \times A^R, \ i = 1, \ldots, n, \\
\sum_{i=1}^{n} \psi_i(a,b) = 1 \ \text{ if } \ g_i(a) - g_i(b) \geq -q_i \ \text{ for all } (a,b) \in A^R \times A^R, \ i = 1, \ldots, n, \\
\psi_i(a,b) \geq \psi_i(c,d) \ \text{ if } \ g_i(a) - g_i(b) \geq g_i(c) - g_i(d), \\
\quad \text{for all } a,b,c,d \in A^R, \ i = 1, \ldots, n,
\end{array}\right\} E(A^R)$$

where ε is a small positive value and δ is a big positive value. Remark that $E(A^R)$ are constraints of a 0-1 mixed linear program.

Given a pair of alternatives $(x,y) \in A$, x *necessarily* outranks y, which is denoted by $xS^N y$, if and only if $d(x,y) \geq 0$, where

$$d(x,y) = Min\left\{\sum_{i=1}^{n} \psi_i(x,y) - \lambda\right\},$$

subject to constraints $E(A^R)$, plus constraints $\psi_i(x,y) \geq 0$, $\psi_i(x,y) = 0$ if $g_i(y) - g_i(x) \geq p_i$, $\psi_i(a,b) \geq \psi_i(c,d)$ if $g_i(a) - g_i(b) \geq g_i(c) - g_i(d)$, for all $a,b,c,d \in A^R \cup \{x,y\}$, $i = 1,\ldots,n$, $\sum_{i=1}^n \psi_i(x,y) = 1$ if $g_i(x) - g_i(y) \geq -q_i$ for all $i = 1,\ldots,n$, and $g_i(y) - g_i(x) \leq v_i$, $i = 1,\ldots,n$.

$d(x,y) \geq 0$ means that for all compatible outranking models x outranks y. Obviously, for all $(x,y) \in A^R$, $xSy \Rightarrow xS^N y$.

Analogously, given a pair of alternatives $(x,y) \in A$, x *possibly* outranks y, which is denoted by $xS^P y$, if and only if $D(x,y) \geq 0$, where

$$D(x,y) = Max\left\{\sum_{i=1}^{n} \psi_i(x,y) - \lambda\right\},$$

subject to constraints $E(A^R)$, plus constraints $\psi_i(x,y) \geq 0$, $\psi_i(x,y) = 0$ if $g_i(y) - g_i(x) \geq p_i$, $\psi_i(a,b) \geq \psi_i(c,d)$ if $g_i(a) - g_i(b) \geq g_i(c) - g_i(d)$, for all $a,b,c,d \in A^R \cup \{x,y\}$, $i = 1,\ldots,n$, $\sum_{i=1}^n \psi_i(x,y) = 1$ if $g_i(x) - g_i(y) \geq -q_i$ for all $i = 1,\ldots,n$, and $g_i(y) - g_i(x) \leq v_i$, $i = 1,\ldots,n$.

$D(x,y) \geq 0$ means that for at least one compatible outranking model x outranks y.

Moreover, for any pair of alternatives $(x,y) \in A$:

$$xS^N y \Leftrightarrow not(xS^{cP} y) \quad \text{and} \quad xS^P y \Leftrightarrow not(xS^{cN} y)$$

so, only $xS^N y$ and $xS^P y$ are to be checked.

The necessary and the possible outranking relations are to be exploited as usual outranking relations in the context of choice, sorting, and ranking problems.

9.11 Robust Ordinal Regression for Multiple Criteria Group Decisions

The robust ordinal regression can be adapted to the case of group decisions [36]. In this case, several DMs cooperate in a decision problem to make a collective decision. DMs share the same "description" of the decision problem (the same set of alternatives, family of criteria and performance matrix). Each DM provides his/her own preference information, composed of pairwise comparisons of some reference alternatives. The collective preference model accounts for the preference expressed by each DM.

Let us denote the set of DMs by $\mathcal{D} = \{d_1,\ldots,d_p\}$.

In case of ranking and choice problems, for each DM $d_h \in \mathcal{D}' \subseteq \mathcal{D}$, we consider all compatible value functions. Four situations are interesting for a pair $(a,b) \in A$:

- $a \succsim^{N,N}_{\mathcal{D}'} b$: $a \succsim^N b$ for *all* $d_h \in \mathcal{D}'$
- $a \succsim^{N,P}_{\mathcal{D}'} b$: $a \succsim^N b$ for *at least one* $d_h \in \mathcal{D}'$

- $a \succsim^{P,N}_{\mathcal{D}'} b : a \succsim^{P} b$ for *all* $d_h \in \mathcal{D}'$
- $a \succsim^{P,P}_{\mathcal{D}'} b : a \succsim^{P} b$ for *at least one* $d_h \in \mathcal{D}'$

In case of sorting problems, for each DM $d_r \in \mathcal{D}' \subseteq \mathcal{D}$, we consider the set of all compatible value functions $\mathcal{U}^{d_r}_{A^R}$. Given a set A^R of assignment examples, for each $a \in A$ and for each DM $d_r \in \mathcal{D}'$, we define his/her possible and necessary assignments as

$$C^{d_r}_P(a) = \left\{h \in H : \exists U \in \mathcal{U}^{d_r}_{A^R} \text{ assigning } a \text{ to } C_h\right\}, \tag{9.26}$$

$$C^{d_r}_N(a) = \left\{h \in H : \forall U \in \mathcal{U}^{d_r}_{A^R} \text{ assigning } a \text{ to } C_h\right\}. \tag{9.27}$$

Moreover, for each subset of DMs $D' \subseteq \mathcal{D}$, we define the following assignments:

$$C^{D'}_{P,P}(a) = \bigcup_{d_r \in \mathcal{D}'} C^{d_r}_P(a), \tag{9.28}$$

$$C^{D'}_{N,P}(a) = \bigcup_{d_r \in \mathcal{D}'} C^{d_r}_N(a), \tag{9.29}$$

$$C^{D'}_{P,N}(a) = \bigcap_{d_r \in \mathcal{D}'} C^{d_r}_P(a), \tag{9.30}$$

$$C^{D'}_{N,N}(a) = \bigcap_{d_r \in \mathcal{D}'} C^{d_r}_N(a). \tag{9.31}$$

Possible and necessary assignments $C^{d_r}_P(a)$ and $C^{d_r}_N(a)$ are calculated for each decision maker $d_r \in \mathcal{D}$ using $UTADIS^{GMS}$, and then the four assignments $C^{D'}_{P,P}(a)$, $C^{D'}_{N,P}(a)$, $C^{D'}_{P,N}(a)$ and $C^{D'}_{P,P}(a)$ can be calculated for all subsets of decision makers $\mathcal{D}' \subseteq \mathcal{D}$.

In case of application of robust ordinal regression to outranking methods, for each DM $d_h \in D' \subseteq D$, we consider all compatible outranking models. Four situations are interesting for a pair $(x, y) \in A$:

- $x\, S^{N,N}_{D'}\, y : xS^N y$ for all $d_h \in D'$
- $x\, S^{N,P}_{D'}\, y : xS^N y$ for at least one $d_h \in D'$
- $x\, S^{P,N}_{D'}\, y : xS^P y$ for all $d_h \in D'$
- $x\, S^{P,P}_{D'}\, y : xS^P y$ for at least one $d_h \in D'$.

9.12 An Illustrative Example

In this section, we present a didactic example proposed in [15], illustrating how robust ordinal regression can support the DM to specify his/her preferences in a multiobjective optimization problem. In this didactic example, we shall imagine an interaction with a fictitious DM so as to exemplify and illustrate the type of interaction proposed in our methodology.

We consider an MOO problem involving five objectives that are to be maximized. Let us consider a subset A of the Pareto frontier of the MOO problem consisting of

Table 9.1 The set A of Pareto optimal solutions for the illustrative MOO problem

s_1	=	(14.5, 147, 4, 1014, 5.25)
s_2	=	(13.25, 199.125, 4, 1014, 4)
s_3	=	(15.75, 164.375, 16.5, 838.25, 5.25)
s_4	=	(12, 181.75, 16.5, 838.25, 4)
s_5	=	(12, 164.375, 54, 838.25, 4)
s_6	=	(13.25, 199.125, 29, 662.5, 5.25)
s_7	=	(13.25, 147, 41.5, 662.5, 5.25)
s_8	=	(17, 216.5, 16.5, 486.75, 1.5)
s_9	=	(17, 147, 41.5, 486.75, 5.25)
s_{10}	=	(15.75, 216.5, 41.5, 662.5, 1.5)
s_{11}	=	(15.75, 164.375, 41.5, 311, 6.5)
s_{12}	=	(13.25, 181.75, 41.5, 311, 4)
s_{13}	=	(12, 199.125, 41.5, 311, 2.75)
s_{14}	=	(17, 147, 16.5, 662.5, 5.25)
s_{15}	=	(15.75, 199.125, 16.5, 311, 6.5)
s_{16}	=	(13.25, 164.375, 54, 311, 4)
s_{17}	=	(17, 181.75, 16.5, 486.75, 5.25)
s_{18}	=	(14.5, 164.375, 41.5, 838.25, 4)
s_{19}	=	(15.75, 181.75, 41.5, 135.25, 5.25)
s_{20}	=	(15.75, 181.75, 41.5, 311, 2.75)

20 solutions (see Table 9.1). Note that this set A is to be computed using MOO or EMO algorithms (see [8]). Let us suppose that the reference sample A^R of solutions from set A is the following: $A^R = \{s_1, s_2, s_4, s_5, s_8, s_{10}\}$. For the sake of simplicity, we shall consider the set A^R constant across iterations (although the interaction scheme permits A^R to evolve during the process). For the same reason, we will suppose that the DM expresses preference information only in terms of pairwise comparisons of solutions in A^R (intensity of preference will not be expressed in the preference information).

The DM does not see any satisfactory solution in the reference sample A^R (s_1, s_2, s_4 and s_5 have too weak evaluations on the first criterion, while s_8 and s_{10} have the worst evaluation in A on the last criterion), and wishes to find a satisfactory solution in A. Obviously, solutions in A are not comparable unless preference information is expressed by the DM. In this perspective, he/she provides a first pairwise comparison: $s_1 \succ s_2$.

Considering the provided preference information, we can compute the necessary and possible rankings on set A. The DM decided to consider the necessary ranking only, as it has more readable graphical representation than the possible ranking at the stage of relatively poor preference information. The partial preorder of the necessary ranking is depicted in Fig. 9.1 and shows the comparisons that hold for all additive value functions compatible with the information provided by the DM (i.e., $s_1 \succ s_2$). It should be observed that the computed partial preorder contains the preference information provided by the DM (dashed arrow), but also additional comparisons that result from the initial information (continuous arrows); for instance, $s_3 \succ^N s_4$ holds as $U(s_3) > U(s_4)$ holds for all compatible value functions.

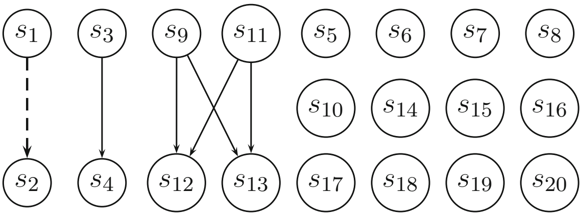
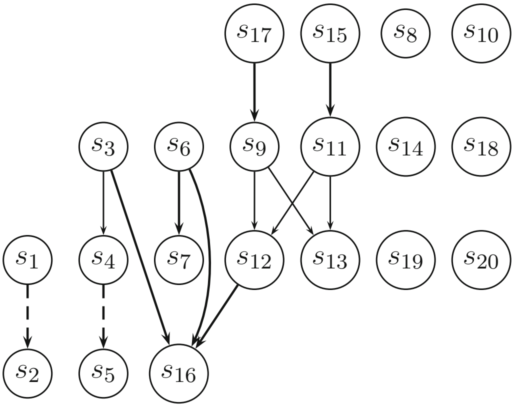


Fig. 9.1 Necessary partial ranking at the first iteration

Fig. 9.2 Necessary partial ranking at the second iteration



Analyzing this first result, the DM observes that the necessary ranking is still very poor, which makes it difficult to discriminate among the solutions in A. He/she reacts by stating that s_4 is preferred to s_5. Considering this new piece of preference information, the necessary ranking is computed again and shown in Fig. 9.2. At this second iteration, it should be observed that the resulting necessary ranking has been enriched as compared to the first iteration (bold arrows), narrowing the set of "best choices," i.e., solutions that are not preferred by any other solution in the necessary ranking: $\{s_1, s_3, s_6, s_8, s_{10}, s_{14}, s_{15}, s_{17}, s_{18}, s_{19}, s_{20}\}$.

The DM believes that this necessary ranking is still insufficiently decisive and adds a new pairwise comparison: s_8 is preferred to s_{10}. Once again, the necessary ranking is computed and shown in Fig. 9.3.

At this stage, the set of possible "best choices" has been narrowed down to a limited number of solutions, among which s_{14} and s_{17} are judged satisfactory by the DM. In fact, these two solutions have a very good performance on the first criterion without "dramatically" bad evaluation on the other criteria.

The current example stops at this step, but the DM could then decide to provide further preference information to enrich the necessary ranking. He/she could also compute new Pareto optimal solutions "close" to s_{14} and s_{17} to focus the search in this area. In this example, we have shown that the proposed interactive process

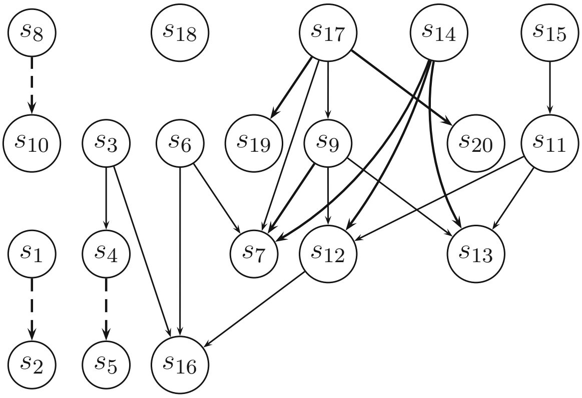


Fig. 9.3 Necessary partial ranking at the third iteration

supports the DM in choosing most satisfactory solutions, without imposing any strong cognitive effort, as the only information required is a holistic information.

9.13 Conclusions and Further Research Directions

In this chapter we presented the basic principle of robust ordinal regression, which is to take into account all the sets of parameters of a preference model compatible with the preference information given by the DM. We recalled the main multiple criteria decision methods to which it has been applied, in particular *UTA*^{GMS} and *GRIP* dealing with choice and ranking problems, and *UTADIS*^{GMS} dealing with sorting (ordinal classification) problems. We presented also robust ordinal regression applied to Choquet integral for choice, ranking, and sorting problems, with the aim of representing interactions between criteria. Moreover, we described an interactive multiobjective optimization methodology based on robust ordinal regression, and an evolutionary multiobjective optimization methodology, called *NEMO*, which is also using the principle of robust ordinal regression. In order to show that robust ordinal regression is a general paradigm, independent of the type of preference model involved, we described the robust ordinal regression methodology for outranking methods, and for multiple criteria group decisions. Finally, we presented an exemplary application of robust ordinal regression methodology. Future research will be related to the development of a user friendly software and to specialization of robust ordinal regression methodology to specific real-life problems, such us environmental management, financial planning, and bankruptcy risk evaluation.

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Chapter 10
Stochastic Multicriteria Acceptability Analysis (SMAA)

Risto Lahdelma and Pekka Salminen

Abstract Stochastic multicriteria acceptability analysis (SMAA) is a family of methods for aiding multicriteria group decision making in problems with uncertain, imprecise or partially missing information. These methods are based on exploring the weight space in order to describe the preferences that make each alternative the most preferred one, or that would give a certain rank for a specific alternative. The main results of the analysis are rank acceptability indices, central weight vectors and confidence factors for different alternatives. The rank acceptability indices describe the variety of different preferences resulting in a certain rank for an alternative, the central weight vectors represent the typical preferences favouring each alternative, and the confidence factors measure whether the criteria measurements are sufficiently accurate for making an informed decision. A general approach for applying SMAA in real-life decision problems is to use it repetitively with more and more accurate information until the information is sufficient for making a decision. Between the analyses, information can be added by making more accurate criteria measurements, or assessing the DMs' preferences more accurately in terms of various preference parameters.

Keywords SMAA · Stochastic multicriteria acceptability analysis · MCDA · Decision support

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10.1 Introduction

10.1.1 Aims and Goals of SMAA Methods

In real-life decision problems, most of the associated information is to some degree uncertain or imprecise and sometimes relevant information can even be missing [40]. Stochastic multicriteria acceptability analysis (SMAA, pronounced /sma:/) is a family of multicriteria decision-aiding (MCDA) methods for problems where the uncertainty is so significant that it should be considered explicitly.

The central component of any MCDA method is the *decision model* that combines (aggregates) the criteria measurements with decision-makers' (DMs) preferences in order to evaluate the alternatives. Different MCDA methods use different decision models, such as value/utility functions, outranking relations or reference point models. SMAA can be applied with any decision model, and it can also be used in different problem settings: for choosing one or a few "best" alternatives, ranking the alternatives, and classifying the alternatives into different categories.

SMAA is based on simulating different value combinations for uncertain parameters, and computing statistics about how the alternatives are evaluated. Depending on the problem setting, this can mean computing how often each alternative becomes most preferred, how often it receives a particular rank or obtains a particular classification.

SMAA was initially developed for a public real-life MCDA problem where we could not obtain weight information from a large number of political DMs. With missing weight information, it was not possible to apply the traditional MCDA approach where the "best" (most preferred) alternative is identified based on criteria measurements and DMs' preferences (Fig. 10.1). This led to the idea of *inverse*

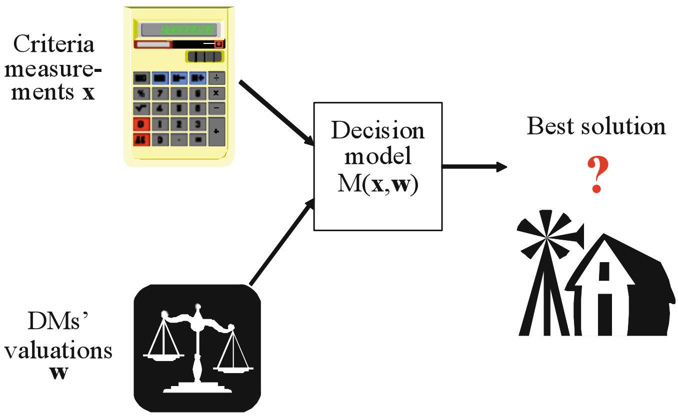

Fig. 10.1 Traditional approach: decision model determines the "best" solution based on criteria measurements and DMs' preferences

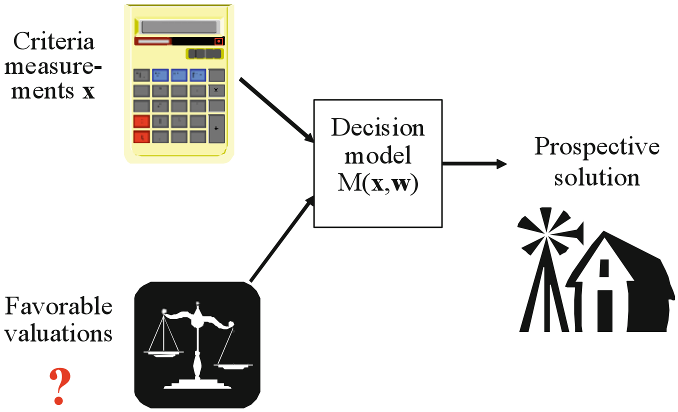

Fig. 10.2 Inverse approach: identify preferences that are favourable for each alternative solution

weight space analysis. Instead of the traditional approach, we applied simulation with randomized weights in order to reveal what kinds of weights make each alternative solution most preferred (Fig. 10.2).

The results from inverse analysis are often descriptive, i.e., they characterize what kinds of preferences correspond to each alternative. Inverse analysis cannot in general determine a unique best solution. However, often some inferior solutions can be eliminated, because they do not correspond to any possible preferences. Also, it is possible to identify widely acceptable solutions that are favoured by a large variety of different preferences.

Although SMAA was originally developed for situations with absence of preference information, it is not a pure inverse analysis method. In the majority of real-life decision making processes, some kind of preference information is available, and also the criteria measurements can be accurate up to a certain degree. This means that the real-life cases fall between the extremes shown in Figs. 10.1 and 10.2. In SMAA, different kinds of uncertain criteria and preference information are modelled using suitable probability distributions. The true strength of SMAA is that in such cases it is able to handle flexibly the whole range of uncertain, imprecise or partially missing information.

Typically, a real-life decision process will start with very vague and uncertain criteria and preference information, and more accurate information is obtained during the process. In this kind of process SMAA can be used iteratively after each round of information collection, until the information is accurate enough for making the decision. SMAA can help to determine if the information is accurate enough for making the decision, and also pinpoint which parts of the information still need to be made more accurate. This can (1) protect from making wrong decisions due to insufficient information and also (2) cause significant savings in information collection if less accurate information can be deemed sufficient for decision making.

10.1.2 Variants of SMAA

A number of different variants of SMAA methods exist. In the original SMAA method by Lahdelma et al. [29] inverse weight space analysis was performed based on an additive utility or value function and stochastic criteria data to identify for each alternative the weights that made it most preferred. SMAA-2 [34] generalized the analysis to apply a general utility or value function, to include various kinds of preference information and to consider holistically all ranks. SMAA-3 [35] is based on *pseudocriteria* as in the ELECTRE III decision aid (see, e.g., [47, 55, 56, 71]). SMAA-D [36] applies, instead of a value function, the efficiency score of Data Envelopment Analysis (DEA). The SMAA-O method [32] extended SMAA-2 for treating mixed ordinal and cardinal criteria in a comparable manner.

Recent developments of SMAA include versions based on different kinds of decision models. SMAA-P [38] is based on piecewise linear prospect theory where alternatives are evaluated with respect to gains and losses from reference points. SMAA-DS is based on Dempster–Shafer theory of evidence trying to represent absence of information in a more consistent way [43]. SMAA-A methods [10, 11, 33] compare the alternatives by applying reference points and Wierzbicki's achievement scalarizing functions. SMAA-III [64] is based on the full ELECTRE III outranking process with uncertain criteria, weights and thresholds. SMAA-NC [73] is a nominal classification method that classifies alternatives into unordered pre-defined classes. SMAA-OC [39] is an ordinal classification (sorting) method that classifies alternatives into ordered pre-defined classes. SMAA-TRI [67] is an ordinal classification method based on ELECTRE-TRI with uncertain criteria, thresholds and weights.

For a survey on different SMAA methods, see Tervonen and Figueira [66].

10.1.3 Related Research

The idea of weight space analysis was first presented by Charnetski [7] and Charnetski and Soland [8]. They introduced for multicriteria problems the comparative hypervolume criterion, which is based on computing for each alternative the volume of the multidimensional weight space that makes the alternative the most preferred. This method can handle preference information in the form of linear constraints for the weights, but is restricted to deterministic criteria measurements and an additive utility function. Rietveld [53] and Rietveld and Ouwersloot [54] presented similar methods for problems with ordinal criteria and ordinal preference information. The Qualiflex method [1, 50] approaches similar problems by testing how each possible ranking of alternatives is supported by different criteria. Bana e Costa [2, 3] introduced the Overall Compromise Criterion method for identifying alternatives generating the least conflict between several DMs. This method can handle partial preference information in the form of arbitrary weight distributions. The SMAA method was initially developed based on the last-mentioned method.

The above-mentioned methods tried to analyse the weight space using partly analytical methods. Those methods require either some simplifying assumptions about decision model or otherwise the decision problem must be very small to allow sufficiently fast solution.

The dramatically increased performance of computers has allowed solving complex decision problems with uncertainty efficiently using numerical methods. In particular, the Monte-Carlo simulation technique applied in SMAA allows efficient analysis of arbitrarily detailed models without being forced to make simplifying assumptions. Related simulation approaches for analysing multicriteria problems with uncertainty include, e.g., those by Stewart [59–61], Butler et al. [6], Jia et al. [21], Durbach and Stewart [13], and García et al. [15].

In SMAA, uncertain information is modelled explicitly by probability distributions. Other techniques for modelling such information have also been suggested, such as fuzzy set theory [74] and rough sets [51]. In the context of multicriteria decision making, Fuzzy Set Theory has been applied, e.g., by Hipel [17] to aggregate together subjective criteria data from multiple DMs. Rough sets have been applied to multicriteria choice and ranking problems by Greco et al. [16]. From the stated assumptions, the stochastic model is accurate and comprehensive while other techniques are approximations. Approximate methods may be preferable when they allow a simpler problem representation, are easier to understand and explain to the users, are easier to implement or are computationally less expensive.

With the aid of modern computers, the stochastic model of SMAA can be used successfully in applications where approximate techniques have traditionally been applied. As a special case, the stochastic model allows an equally simple problem representation in form of independent, uniform or normal distributions for criteria. When required, it is possible to use more complex dependent distributions with a marginal effect on computational efficiency.

10.2 SMAA Approach

In the following, we first define the problem representation in SMAA as a stochastic MCDA model. Then we illustrate the inverse weight space analysis using graphical examples. Following this, we describe the generic simulation approach for the SMAA computations. After that we describe the basic SMAA-2 method based on utility or value functions.

10.2.1 Problem Representation

In SMAA we consider discrete multicriteria decision problems consisting of a set of m alternatives $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ that are evaluated in terms of n criteria. SMAA can be used in different problem settings. The DMs may, e.g., want to choose from the set one or more of their most preferred alternatives, produce a partial or complete ranking of the alternatives or classify the alternatives into pre-defined categories.

We assume that the DMs jointly accept some decision model $M(\mathbf{x}, \mathbf{w})$ that is suitable for the problem setting. Here $\mathbf{x} = [x_{ij}]$ is a matrix of criteria measurements with i referring to the alternative and j to the criterion. $\mathbf{w} = [w_j]$ is a vector of preference parameters representing the DMs' subjective preferences. The preference parameters and their interpretation depend on the preference model. Typically $\mathbf{w}$ will contain importance weights for the criteria, and possibly other preference parameters, such as

- various shape parameters (e.g., risk aversion coefficients) in non-linear utility and value function models;
- reference points or aspiration levels in prospect theory and reference point methods; and
- indifference, preference and veto thresholds in outranking models.

With deterministic (precise) $\mathbf{x}$ and $\mathbf{w}$, the decision model will produce precise results according to the problem setting. This means that under perfect information about the criteria measurements and consensus about precise values for the preference parameters, the decision-making problem is trivially solved by applying the decision model and accepting the recommended solution. However, in real-life problems, both criteria and preference information are incomplete and potentially conflicting. To explicitly represent the incompleteness of the information, we extend the deterministic MCDA problem into a *stochastic MCDA problem*.

In a stochastic MCDA problem incomplete criteria and preference information are represented by suitable (joint) probability distributions $f_X(\mathbf{x})$ and $f_W(\mathbf{w})$. As we will see later, probability distributions allow very flexible modelling of different kinds of inaccurate, uncertain, imprecise or partially missing information. Because all information is represented uniformly, this allows using efficient simulation techniques for analysing the problem and deriving results about prospective solutions and their robustness based on statistical methods.

10.2.2 Inverse Weight Space Analysis

The idea of inverse weight space analysis is to describe the preferences that make each alternative most preferred, or give it a particular rank of classification. We illustrate weight space analysis assuming a linear value function. However, weight space analysis works the same way with arbitrarily shaped utility functions and also with other decision models that are based on some kind of weights.

A linear utility function defines the overall utility of an alternative as a weighted sum of *partial utilities*. A linear utility function has thus the form

$$u(\mathbf{x}_i, \mathbf{w}) = w_1 \cdot u_{i1} + w_2 \cdot u_{i2} + \cdots + w_n \cdot u_{in}. \tag{10.1}$$

The partial utilities u_{ij} are computed from the actual criteria measurements x_{ij} through linear scaling so that the worst value is mapped to 0 and the best value becomes 1. w_j are the importance weights (tradeoffs) for criteria. Typically the weights should be non-negative and normalized so that their sum is 1. In the absence or weight information, we assume that any non-negative and normalized weights are equally possible.

Consider as an example deterministic the 2-criterion problem defined in Table 10.1. The first step is to normalize the criteria measurements into partial utilities u_{ij}. Normalization can be done in different ways. The best and worst values can, e.g., be identified among the alternatives or some theoretical ideal and anti-ideal values can be used. In Table 10.1 we have normalized the criteria simply by dividing the measurements for Criterion 1 by 10 and Criterion 2 by 100. The normalized problem is illustrated in Fig. 10.3.

Without weight information the most preferred alternative cannot be determined. However, we can easily see that alternatives $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are *efficient*, i.e., any one of them could be most preferred, subject to suitable weights. Alternative $\mathbf{x}_4$ is *inefficient*, because it cannot be the most preferred one with any feasible

Table 10.1 Problem with four alternatives and two criteria to be maximized

	Criteria measurements		Partial utilities	
Alternative	Criterion 1	Criterion 2	u_{i1}	u_{i2}
$\mathbf{x}_1$	6	80	0.6	0.8
$\mathbf{x}_2$	9	30	0.9	0.3
$\mathbf{x}_3$	1	90	0.1	0.9
$\mathbf{x}_4$	5	40	0.5	0.4

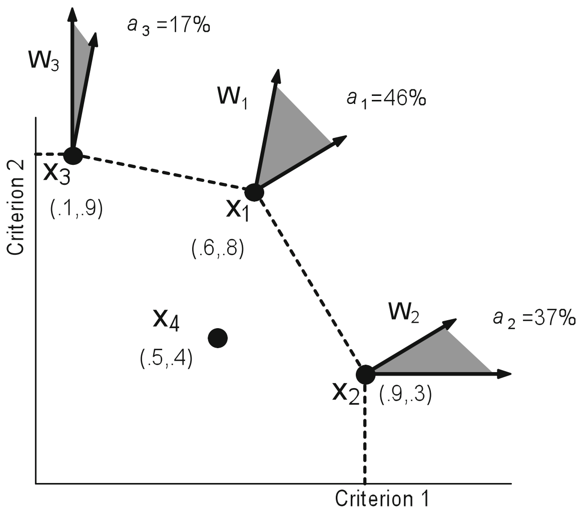

Fig. 10.3 Favourable weights and acceptability indices in deterministic 2-criterion case with linear utility function

combination of weights. The figure illustrates as grey sectors the sets of favourable weights W_i that make the corresponding alternative $\mathbf{x}_i$ most preferred one. In SMAA we characterize the sets of *favourable weights* by two measures: their relative size and midpoint (centre of gravity). The relative size of set W_i in comparison to the set of feasible weights W is the *acceptability index* a_i. The acceptability index of each efficient alternative is illustrated in the figure and it measures the variety of weights that make alternative $\mathbf{x}_i$ most preferred. The midpoint of W_i is the *central weight vector* and it describes typical weights that favour alternative $\mathbf{x}_i$.

Besides the weights that make an alternative $\mathbf{x}_i$ most preferred, we can also identify the sets of *favourable rank weights* W_i^r that give a particular rank r for alternative $\mathbf{x}_i$. Figure 10.4 illustrates the favourable rank weights for the alternatives. We can see that alternative $\mathbf{x}_2$ will always obtain either rank 1 or 2 and never rank 3 or 4, while $\mathbf{x}_4$ can only receive either one of the two last ranks. The relative size of set W_i^r is the *rank acceptability index*, and it measures the variety of weights that give rank r to alternative $\mathbf{x}_i$. We can see that the second and third rank favourable weights are not necessarily connected sets. Therefore the midpoints of the favourable rank weights are meaningful only for the first rank.

The weight space analysis is simple to perform graphically or analytically in 2-criterion problems with linear (and the more general additive) utility functions and deterministic criteria measurements. However, in higher dimensions, with more complex decision models, and with stochastic criteria, numerical methods for the analysis are required.

Different visualization techniques are also needed with more complex models. The rank acceptability indices can be presented as a 3D column chart as in Fig. 10.5.

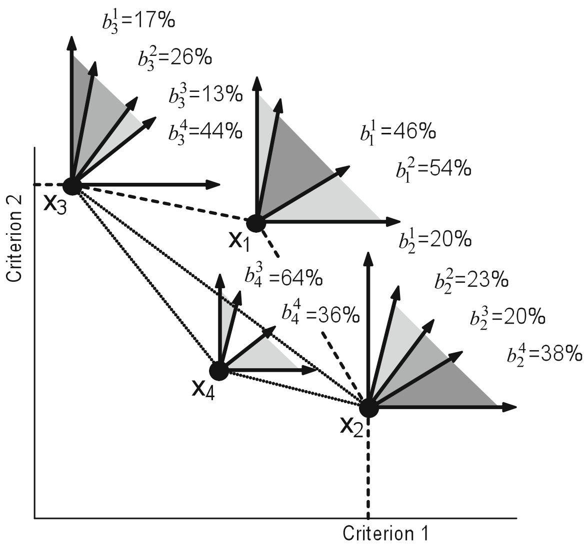

Fig. 10.4 Favourable rank weights and rank acceptability indices in deterministic 2-criterion case with linear utility function

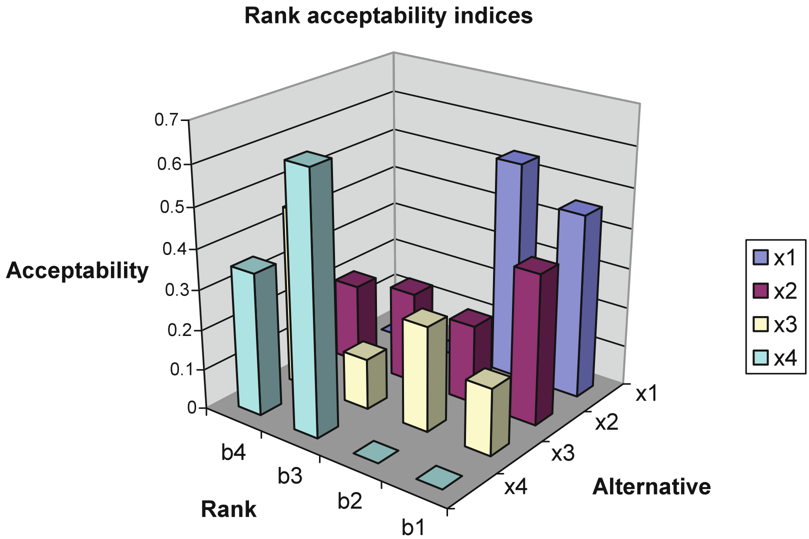

Fig. 10.5 Rank acceptability indices in deterministic 2-criterion problem

Sorting the alternatives into lexicographical order according to their rank acceptability indices makes the chart easy to read. The most acceptable alternatives can be easily identified as the ones with tall columns for the best ranks and low columns for the worst ranks.

10.2.3 Generic Simulation Approach

The generic simulation scheme for analyzing stochastic MCDA problems with different variants of SMAA is presented in Algorithm 1.

Algorithm 1: Generic SMAA simulation

Assume a decision model $M(\mathbf{x}, \mathbf{w})$ for ranking or classifying the alternatives using precise information (criteria matrix $\mathbf{x}$ and preference parameter vector $\mathbf{w}$)
{Use Monte-Carlo simulation to treat stochastic criteria and weights:}
repeat
    Draw $< \mathbf{x},\mathbf{w} >$ from their distributions
    Rank, sort or classify the alternatives using $M(\mathbf{x}, \mathbf{w})$
    Update statistics about alternatives
until Repeated K times
Compute results based on the collected statistics

The simulation scheme is simple to implement and with modern computers very efficient. Complex distributions could in principle slow down the computation, but we have never experienced this in real-life MCDA applications. The statistical results from the Monte-Carlo simulation are approximate, but in MCDA applications it is not necessary to compute results with greater accuracy than a couple of decimal places. Criteria and preference information is typically uncertain, and it is meaningless to compute any derived results with greater accuracy than the input data. The computational accuracy of the main results depends on the square root of the number of iterations, i.e., increasing the number of iterations by a factor of 100 will increase the accuracy by one decimal place. See Section 10.4.1 for details on this. Another remarkable property of Monte-Carlo simulation is that the accuracy of the results does not depend on the dimensions (number of alternatives and criteria) of the problem.

10.2.4 The SMAA-2 Method

In SMAA-2 the decision model is a general utility or value function. SMAA-2 can be used as aid in several different problem settings, such as choosing a single "best" alternative, choosing a set of best alternatives or ranking the alternatives.

The main results of the analysis are *rank acceptability indices*, *central weight vectors* and *confidence factors* for different alternatives. The rank acceptability indices describe the variety of different preferences resulting in a certain rank for an alternative; the central weight vectors represent the typical preferences favouring each alternative; and the confidence factors measure whether the criteria data are sufficiently accurate for making an informed decision.

With deterministic criteria measurements and preference parameters, the utility function $u(\mathbf{x}_i, \mathbf{w})$ evaluates how good each alternative is by a real number from 0 to 1. The preference parameters and their interpretation depend on the shape of the utility function. Often an additive utility function is used, i.e., the utility of each alternative is expressed as a weighted sum of partial utilities:

$$u(\mathbf{x}_i, \mathbf{w}) = w_1 u_1(x_{i1}) + w_2 u_2(x_{i2}) + \cdots + w_n u_n(x_{in}). \tag{10.2}$$

For conditions that allow additive decomposition of the utility function, see [26]. Here the preference parameters are importance weights for criteria $\mathbf{w} = [w_1, w_2, \ldots, w_n]$. The weights are non-negative and normalized so that their sum is 1. The partial utility functions $u_j(x_{ij})$ map the original criteria measurements into partial utilities in the range [0,1]. The partial utility functions are typically monotonic mappings. If they are linear, then the overall utility function is linear.

Uncertain or imprecise criteria measurements $\mathbf{x}$ are represented by a matrix of stochastic variables $[x_{ij}]$ with distribution $f_X(\mathbf{x})$. Similarly, the DMs' unknown or partially known preferences are represented by a weight distribution with density function $f_W(\mathbf{w})$. Unknown preferences are represented by a uniform distribution in the *set of feasible weights*

$$W = \{\mathbf{w} | w_j \geq 0 \text{ and } w_1 + w_2 + \cdots + w_n = 1\}. \tag{10.3}$$

The simulation scheme presented in the previous section is then applied. During each iteration, criteria measurements and weights are drawn from their distributions and the utility function is used to rank the alternatives. Then statistics about the ranking is updated. (Observe that this approach differs from traditional utility function methods that compute the expected utility). Based on the ranking, the following statistics is collected:

B_{ir} The number of times alternative $\mathbf{x}_i$ obtained rank r
C_{ik} The number of times alternative $\mathbf{x}_i$ was more preferred than $\mathbf{x}_k$
W_{ij} Sum of the weights that made alternative $\mathbf{x}_i$ most preferred

Based on the collected statistics, estimates for the descriptive measures of SMAA-2 are computed.

The primary descriptive measure of SMAA-2 is the *rank acceptability index*b_i^r, which measures the variety of different preferences that grant alternative $\mathbf{x}_i$ rank r. It is the share of all feasible weights that make the alternative acceptable for a particular rank, and it is most conveniently expressed in percent. The rank acceptability index is estimated from the simulation results as

$$b_i^r \approx \mathrm{B}_{ir}/K. \tag{10.4}$$

The most acceptable (best) alternatives are those with high acceptabilities for the best ranks. Evidently, the rank acceptability indices are in the range [0,1] where 0 indicates that the alternative will never obtain a given rank and 1 indicates that it will obtain the given rank always with any choice of weights. Graphical examination of the rank acceptability indices (see Fig. 10.5) is useful for comparing how different varieties of weights support each rank for each alternative. Alternatives with high acceptabilities for the best ranks are taken as candidates for the most acceptable solution. On the other hand, alternatives with large acceptabilities for the worst ranks should be avoided when searching for compromises – even if they would have fairly high acceptabilities for the best ranks.

The first rank acceptability index is called the *acceptability index* a_i. The acceptability index is particularly interesting, because it is non-zero for efficient alternatives and zero for inefficient alternatives. The acceptability index not only identifies the efficient alternatives, but also measures the strength of the efficiency considering the uncertainty in criteria and DMs' preferences. The acceptability index can thus be used for classifying the alternatives into stochastically efficient ones ($a_i > 0$) and inefficient or weakly efficient ones (a_i zero or near-zero). The acceptability index can also be interpreted as the number of DMs voting for an alternative, assuming that the applied weight distribution represents the DMs' preferences accurately. However, in practice, this assumption may not be valid. The acceptability index should thus not in general be used for ranking the alternatives strictly.

SMAA-2 defines also for each alternative a *holistic acceptability index* which aims to measure the overall acceptability of the alternative. The holistic acceptability index is defined as a weighted sum of the rank acceptabilities

$$a_i^h = \sum_{r=1}^{m} \alpha_r b_i^r, \tag{10.5}$$

where suitable *meta-weights* $1 = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$ are used to model that the best ranks are preferred over the worst ranks. The holistic acceptability index is thus in the interval [0,1] and it is always greater than or equal to the acceptability index. Different ways to choose the meta-weights are discussed in [34].

The *pairwise winning index* c_{ik} is the probability for alternative $\mathbf{x}_i$ being more preferred than $\mathbf{x}_k$, considering the uncertainty in criteria and preferences [44]. The pairwise winning index is estimated from the simulation results as

$$c_{ik} \approx \mathrm{C}_{ik}/K. \tag{10.6}$$

The pairwise winning indices are useful when comparing the mutual performance of two alternatives. This information can be used, e.g., when it is necessary to eliminate some alternatives from further and more detailed analyses.

The *central weight vector* $\mathbf{w}_i^c$ is the expected centre of gravity of the favourable first rank weights of an alternative. The central weight vector represents the preferences of a "typical" DM supporting this alternative. The central weight vectors of different alternatives can be presented to the DMs in order to help them understand how different weights correspond to different choices with the assumed preference model. The central weight vector is estimated from the simulation results as

$$w_{ij}^c \approx \mathrm{W}_{ij}/B_{ii},\ j = 1, \ldots, n. \tag{10.7}$$

A second simulation is needed to compute additional information, such as confidence factors and cross confidence factors. This computation is presented in Algorithm 2. The following statistics is collected:

P_{ik} The number of times alternative $\mathbf{x}_i$ was most preferred using weights $\mathbf{w}_k^c$.

The *confidence factor* p_i^c is the probability for an alternative to obtain the first rank when its central weight vector is chosen. The confidence factor is estimated from the simulation results as

$$p_i^c \approx \mathrm{P}_{ii}/K. \tag{10.8}$$

Algorithm 2: Computation of confidence factors and cross confidence factors in SMAA-2

```
repeat
  Draw x from its distribution
  for central weight vector w_i^c of each alternative do
    Rank the alternatives using u(x_i, w_i^c)
    Update statistics about alternatives
  end for
until Repeated K times
```

The confidence factors measure whether the criteria data are accurate enough to discern the efficient alternatives. A low confidence factor indicates that even when applying the central weight vector, the alternative cannot reliably be considered the most preferred one. The confidence factor can also be used together with the acceptability index for eliminating weakly efficient alternatives. If the acceptability index is very low (near-zero, $\ll 1/m$) and the confidence factor is low (less than, say, 5%), we can argue that such an alternative is very unlikely the most preferred by any DM. In contrast, a very high confidence factor (over 95%) indicates that with suitable preferences, the alternative is almost certainly the most preferred one.

Observe that confidence factors can be calculated similarly for any given weight vector. The confidence factor can be interpreted as the proportion of the stochastic criterion space that makes the alternative most preferred with the given weight vector.

The *cross confidence factors* p_{ik}^c are based on computing confidence factors for alternatives using each other's central weight vectors. The cross confidence factors are estimated from the simulation results as

$$p_{ik}^c \approx \mathrm{P}_{ik}/K. \tag{10.9}$$

The cross confidence factor is the probability for an alternative to obtain the first rank (considering the uncertainty in the criteria measurements) when central weight vector of the target alternative is chosen. Therefore, the non-zero cross confidence factors identify which alternative's $\mathbf{x}_i$ compete about the first rank with a given alternative's $\mathbf{x}_k$ central weight vector and how strongly they do it. Observe that the target alternative has to be efficient; otherwise its central weight vector is undefined. In principle, the cross confidence factors can of course be computed using arbitrary weight vectors, e.g., weight vectors specified by the DMs.

10.3 Modelling Uncertain Information

Real-life MCDA problems often involve a considerable amount of uncertainty. Uncertainties arise for a number of different reasons. For example, French [14] identifies 10 different sources of uncertainty. Belton and Stewart [4] suggest that for purposes of multicriteria decision aid, differentiating between internal and external uncertainty could be useful.

In metrology [62], measurements are treated as stochastic quantities that are approximations or estimations of reality. Uncertainty is classified into statistical (Type A) and subjective (Type B) uncertainty. The use of probability distributions to represent statistical uncertainty is mathematically justified. With subjective uncertainty the "true" distribution is not known, and any applied distribution is an approximation. In some situations this approach can be justified. For example, if the uncertainty is small, then different distributions produce almost identical results in the overall analysis, it does not matter which distribution to use. However, when uncertainty is substantial, assuming one distribution instead of another may cause different results. In such cases, the analyst and DMs should be aware of the uncertainty of the uncertainty modelling, and take this into account when drawing conclusions based on the analysis results. To select suitable distributions, it is necessary to understand how the uncertainty of different parameters is formed.

In the following sections we describe different ways to represent uncertain information in criteria and in preferences.

10.3.1 Representing Uncertain Criteria

10.3.1.1 Cardinal Criteria

Uncertainty of a measurement can be expressed, e.g., as a confidence interval at given confidence level, as the standard deviation of the measurement (so called *standard uncertainty* in Metrology), or as a specific probability distribution around the expected value. In SMAA the different representations are transformed into suitable probability distributions to be used in the simulation. The most commonly used distributions have been simple parametric distributions such as uniform and normal distributions, but any other distributions can be easily handled in the simulation.

In the absence of information about the true distribution of an uncertain measurement, it is necessary to assume some distribution. The uniform distribution is a choice to represent criteria measurements for which a confidence interval has been defined. The normal distribution is likewise a convenient choice when the measurement is expressed by its mean and standard deviation. Our experiences with several real-life MCDA problems indicate that, e.g., uniform and normal distributions produce virtually identical recommendations.

When the uncertainties of different measurements are independent, then the joint distribution or the criteria measurements $f_X(\mathbf{x})$ is the product of independent distributions $f_{X_{ij}}(\mathbf{x}_{ij})$ for each measurement. In some real-life problems, the uncertainties of the criteria measurements may be dependent. In particular, when a common uncertain external factor affects simultaneously different measurements, the uncertainties of the measurements will be dependent. In these situations it is necessary to represent the uncertainties by a multidimensional joint distribution $f_X(\mathbf{x})$. Again, simple parametric representations can be applied. For example, the multivariate Gaussian (Normal) distribution is convenient if information about the mean of individual measurements and their covariance (or correlation) is available [30]. Another approach is convenient when an external simulation model is available to compute criteria measurements, their uncertainties and dependencies. In that case the simulated discrete sample can be used directly in SMAA [31].

10.3.1.2 Ordinal Criteria

An ordinal criterion is measured by ranking the alternatives according to the criterion. The ranking can be complete or partial. A complete ranking defines for each pair of alternatives either that one of them is better than the other, or that the alternatives are equally good. A complete ranking is defined simply by assigning 1 to the best alternative(s), 2 to the second best, etc. Equally good alternatives are assigned the same rank and in that case we should actually talk about rank levels rather than ranks. A partial ranking can for some pairs of alternatives leave unspecified whether one is better than the other or if they are equally good. A partial ranking is represented by a directed acyclic graph between the alternatives.

In SMAA-O, ordinal criteria are treated by simulating corresponding (unknown) cardinal criteria values. The best alternative(s) corresponds to cardinal value 1 and the worst alternative(s) corresponds to 0. In the case of a complete ranking, the intermediate ranks correspond to values between 0 and 1. These values are generated randomly from a uniform distribution, sorted into descending order, and assigned for the intermediate ranks. Equally good alternatives on the same rank level are assigned the same value. In the case of a partial ranking, sorting can be implemented efficiently using a variant of the topological sorting algorithm [22]. Figure 10.6 illustrates random cardinal values generated for a complete ranking with four different ranks.

The process described converts ordinal criteria into stochastic cardinal criteria. If the problem contains both ordinal and cardinal criteria, the cardinal criteria are drawn from their distributions. Additional information about the ordinal scales can also be taken into account. If we know, for example, that the first interval is larger than the second interval, we simply discard generated mapping not satisfying this condition.

As stated previously, the SMAA methods can be used with any form of value function, as long as all the DMs jointly accept it. However, in SMAA-O, we can relax this requirement if the value function is additive. If the DMs' partial value functions are unknown, we can, in principle, simulate them in the same way as we

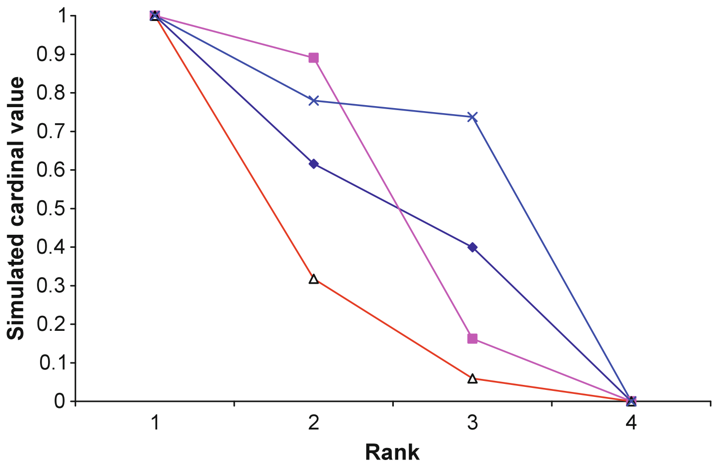

Fig. 10.6 Examples of simulated cardinal values for four ranks

converted ordinal criteria into cardinal. However, such additional simulation is not necessary for ordinal criteria, because we can interpret the simulated cardinal values directly as partial values on a linear scale. Therefore, if the DMs accept an additive value function, it is not necessary for the DMs to agree on a common shape for the partial value functions for the ordinal criteria. Nor is it necessary to know these shapes for the ordinal criteria.

10.3.2 Incomplete Preference Information

Preference information can be incomplete in many different ways. In the following we consider different cases of incomplete weight information. However, the same techniques can be used for other preference parameters, as well.

10.3.2.1 Missing Weight Information

In the extreme case, no weight information is available. However, even in that case, we can assume something about the weights. Normally the MCDA problem can be formulated so that all DMs want to maximize the outcome for each criterion. Therefore, we can assume that the weights should be non-negative. Also, practically all decision models are insensitive to scaling all weights by a positive factor. Therefore, we can, without loss of generality, assume that the weights are normalized somehow, e.g., so that their sum equals 1. This means that the feasible weight space is an $(n-1)$-dimensional simplex (see Eq. 10.3). Figure 10.7 illustrates the feasible weight space in the 3-criterion case as a triangle with corners (1,0,0), (0,1,0) and (0,0,1). We therefore represent missing weight information by a uniform weight distribution in the feasible weight space. Because the uniform distribution carries the least amount of information, it is well justified to use it to represent missing weight information.

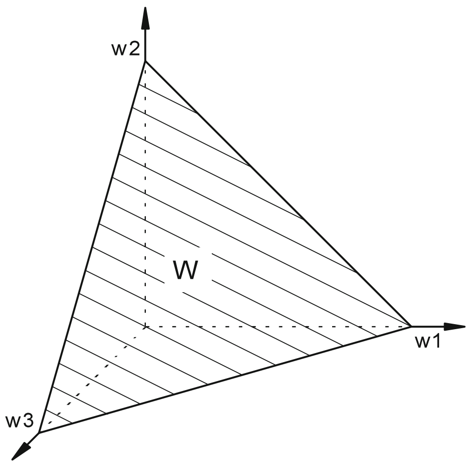

Fig. 10.7 Feasible weight space in the 3-criterion case

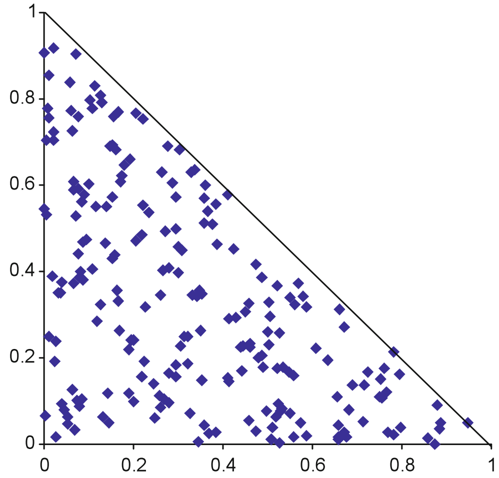

Fig. 10.8 Uniform distribution in the weight space in the 3-criterion case (projected into two dimensions)

Generation of uniformly distributed normalized weights requires a special technique [69]. First we generate $n-1$ independent uniformly distributed random numbers in the interval [0,1] and sort these along with 0 and 1 into ascending order to get $0 = r_0 \leq r_1 \leq \cdots \leq r_n = 1$. Then we compute the weights as the intervals $w_1 = r_1 - r_0, w_2 = r_2 - r_1, \ldots, w_n = r_n - r_{n-1}$. Obviously the resulting weights will be non-negative and normalized. That the resulting joint distribution indeed is uniform is proved in [9].

Without going into details of the proof, we make a few observations. In the 2-criterion case only a single uniformly distributed random number is generated and the weights are determined as $w_1 = r_1$ and $w_2 = 1 - r_1$. Now the weight vectors follow a uniform distribution along the line segment from (0,1) to (1,0) and also the marginal distribution of each weight is uniform. However, when $n > 2$ a uniform joint distribution in the $(n-1)$-dimensional simplex does not correspond to uniform marginal distributions for the individual weights. Figure 10.8 illustrates the generated weights in the three-dimensional case. It is easy to see that the marginal density function of w_1 is proportional to the height of the triangle. Because the area under the density function must equal 1, the actual marginal density function is $2(1-w_1)$. Due to symmetry the marginal distributions of all weights are identical. In the general n-dimensional case the marginal density of weight w_j is $(n-1)(1-w_j)^{n-2}$.

10.3.2.2 Intervals for Weights

Weight intervals can be expressed as $w_j \in \left[w_j^{\min}, w_j^{\max}\right]$. Weight intervals may result from DMs' preference statements of type "the importance weight for criterion j is between $w_j^{\min}$ and $w_j^{\max}$". Weight intervals can also be computed to include precise weights or weight intervals of multiple DMs. The intervals can be represented as a distribution by restricting the uniform weight distribution with

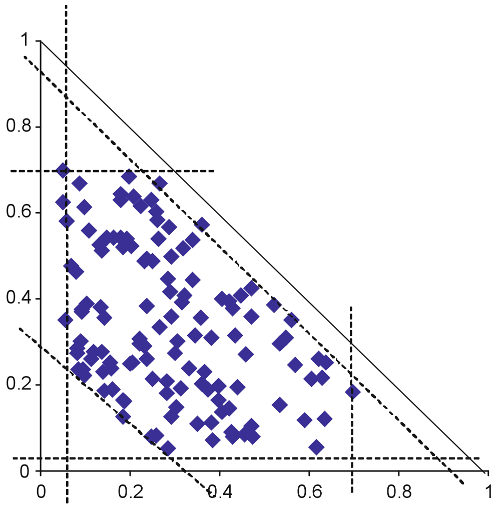

Fig. 10.9 Uniformly distributed weights with interval constraints for weights

linear inequality constraints based on the intervals. Generation of weights from the restricted distributioncan be implemented easily by modifying the above procedure to reject weights that do not satisfy the interval constraints. Upper bounds can be treated efficiently using the rejection technique, but in high-dimensional cases lower bounds may cause a high rejection rate. A special scaling technique can be applied to treat lower bounds efficiently [69]. Figure 10.9 illustrates the resulting weight distribution. The interval constraints are parallel to the sides of the triangle.

10.3.2.3 Intervals for Trade-Off Ratios of Criteria

Intervals for trade-off ratios of criteria can be expressed as $w_j/w_k \in \left[w_{jk}^{\min}, w_{jk}^{\max}\right]$. Such intervals may result from preference statements like "criterion j is from $w_{jk}^{\min}$ to $w_{jk}^{\max}$ times more important than criterion k". These intervals can also be determined to include the preferences of multiple DMs. The intervals can be represented as a distribution by restricting the uniform weight distribution with linear constraints based on the intervals. The rejection technique applies also for generating weights from this distribution. Constraints for trade-off ratios run through the corners of the triangle as illustrated in Fig. 10.10.

10.3.2.4 Ordinal Preference Information

Ordinal preference information can be expressed as linear constraints $w_1 \geq w_2 \geq \cdots \geq w_n$. This is consistent with the DMs' preference statement that "the criterion 1 is most important, 2 is second, etc". It is also possible to allow unspecified

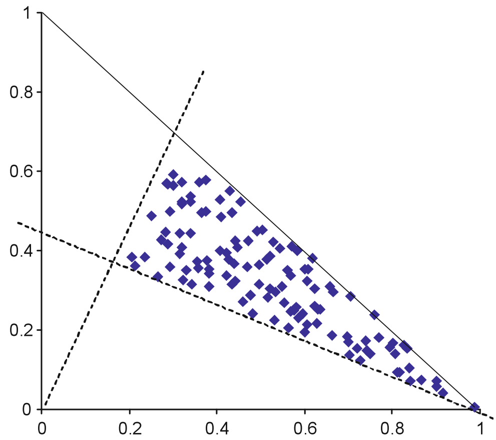

Fig. 10.10 Uniformly distributed weights with two constraints for trade-off ratios

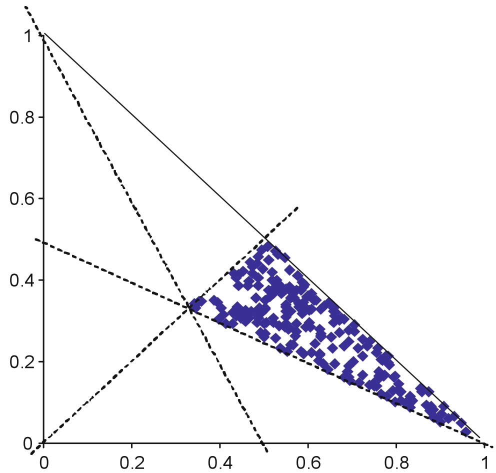

Fig. 10.11 Uniformly distributed weights with importance order

importance ranking for some criteria (w_j ? w_k) or equal importance ($w_j = w_k$). In the general case, ordinal preference information means that the DMs' preference statements correspond to a partial importance ranking of the criteria. Multiple DMs may either agree on a common partial ranking, or they can provide their own rankings, which can then be combined into a partial ranking that is consistent with each DM's preferences. Without any consensus, the combined order may result into absent preference information.

The rejection technique applies also for generating weights that satisfy a (partial) ranking, but equality constraints must be treated by using the same weight for the associated criteria thus reducing the dimensionality of the weight space. A more efficient technique for generating ranked weights is based on sorting uniformly distributed weights into consistent order with a minimal number of adjacent swaps. Figure 10.11 illustrates how the ordinal constraints form medians for the triangular weight space.

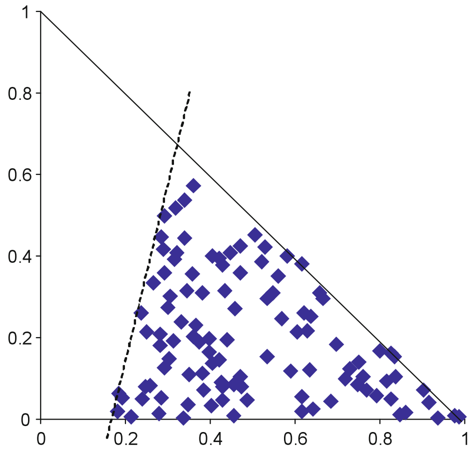

Fig. 10.12 With an additive utility/value function holistic preference statements correspond to general linear constraints in the weight space

10.3.2.5 Implicit Weight Information

Implicit weight information can result, e.g., from DMs' holistic preference statements "alternative $\mathbf{x}_i$ is more preferred than $\mathbf{x}_k$". This can be expressed as $\mathbf{x}_i \succ \mathbf{x}_k$, and it means that the weights and other possible preference parameters must be such that the decision model ranks the alternatives consistent with the preference statements. In the case of an additive utility or value model, such holistic constraints result into general linear constraints in the weight space, as illustrated in Fig. 10.12. In the general case, with non-additive utility/value function model, outranking models, etc., holistic constraints correspond to non-linear constraints in the weight space. The rejection technique can be used to generate the weights with arbitrary decision models.

10.3.2.6 Non-uniform Distributions

Non-uniform distributions can also be used easily in the simulation scheme. The rejection technique works equally with non-uniform distributions to treat different kinds of preference information. However, some additional evidence or justification is necessary to motivate applying non-uniform distribution. For example, if a DM has expressed precise weights, but we understand that some amount of uncertainty or impreciseness must be present, it may be justified to apply a distribution with decreasing density around the expressed weights. Examples of such distributions are, e.g., triangular distributions and (truncated) normal distributions.

10.3.2.7 Combining Preference Information

The problem with combining preference information from different sources is that there is no universally acceptable way to do it. If a number of DMs have provided different weights, a commonly used technique is to compute some kind of average weights, and use them in the decision model. Such average weights may result into a solution that nobody prefers. The averaging process will also eliminate valuable information about potential conflicts.

Different sources may provide information about either consistent or conflicting preferences. Using distributions to represent preference information allows using a wide range of different techniques combine preference information. Instead of hiding the conflict by computing some kind of averages, it may sometimes be better to represent the conflict by a distribution that preserves the information.

10.4 Implementation Techniques

The efficient implementation and computational efficiency of SMAA methods have been described in [69]. The results show that sufficient accuracy for purposes of real-life decision aiding can be obtained very quickly on a standard personal computer.

10.4.1 Accuracy of the SMAA Computations

The accuracy of the results can be calculated by considering the Monte-Carlo simulations as point estimators for rank acceptability indices b_i^r, pairwise winning indices c_{ik}, and confidence factors p_i^c. According to the central limit theorem, the stochastic estimates are normally distributed if the number of iterations is large enough (>25) [49]. In practical SMAA computations the number of iterations is typically from 10,000 up to a million.

To achieve precision d with 95% confidence for b_i^r and c_{ik}, the number of Monte-Carlo iterations needed is [49]:

$$K = \frac{1.96^2}{4d^2}. \tag{10.10}$$

For example, an error limit of 0.01 can be accomplished with 95% confidence by performing 9,604 Monte-Carlo iterations. Increasing the number of iterations by a factor of 100 will increase the accuracy by one decimal place. The accuracy of p_i^c depends on the accuracy of the central weight vectors and the criteria distribution in a complex manner. In theory, an arbitrarily small error in a central weight vector may cause an arbitrarily large error in a confidence factor. If we disregard this error source for the confidence factors, then the same accuracy analysis applies for the confidence factors as for the rank acceptability indices, and the same number of iterations is sufficient.

The accuracy of central weights w_i^c does not depend on the total number of Monte-Carlo iterations, but rather on the number of iterations that contribute to the computation of that central weight vector. To achieve an accuracy of d with 95% confidence for w_i^c, the required number of iterations is

$$K = \frac{1.96^2}{a_i \cdot 4d^2}. \tag{10.11}$$

Thus, alternatives with small acceptability indices require more iterations to compute their central weight vectors with a given accuracy. In practice, we are normally not interested in central weight vectors for alternatives with extremely low acceptability indices.

10.4.2 Efficiency of Computations

The SMAA computations can be implemented efficiently and with sufficient accuracy using Monte-Carlo simulation. With cardinal criteria, the computation time is nearly proportional to $n \cdot m$ and with ordinal criteria it is proportional to $n \cdot m \cdot \log(m)$.

In a group decision-making process, it is common that new preference information is received and old information is adjusted as the process evolves. When new information is added to the model, the SMAA computations must be repeated. Empirical efficiency tests have shown that the required time for computing a typical decision-making problem with 10 alternatives and 8 criteria with a personal computer is less than a second [69]. Thus, the effect of modified preference information on the results can be investigated interactively by the decision makers.

The efficiency and low time complexity of the Monte-Carlo implementation allow using SMAA also for solving continuous multicriteria decision problems by transforming them into large sets of discrete alternatives.

10.5 Applications

In the following, we describe briefly the real-life applications of SMAA listed in Table 10.2. The majority of these applications contain environmental aspects. Such problems are often characterized by multiple DMs and large uncertainty in both criteria measurements and preferences.

Helsinki General Cargo Harbour EIA [19, 28] SMAA was originally developed for this real-life problem, since during the process it turned out that no weight information is available from the 81 DMs consisting of the members of the City Council of Helsinki. The problem was to evaluate 25 alternatives: 24 different ways to carry out the development project and one for not carrying out the project. The alternative actions were evaluated based on 11 criteria. Two types of cardinal scales were used for representing the criteria: continuous linear scales for those criteria that could be

Table 10.2 SMAA applications

Application	Methods	Publications
Helsinki general cargo harbour EIA	SMAA-1, SMAA-2	[19, 28]
General plan of Kirkkonummi	SMAA-3	[18, 27]
Technology competition for cleaning polluted soil	SMAA-2	[20]
EIA for locating South-Karelian waste treatment facility	SMAA-O	[41]
Waste storage of Pietarsaari multi-fuel power plant	SMAA-O	[37]
Choosing land-fill reparation method at Huuna	SMAA-2, SMAA-O	[42]
Land use planning in Oulu region	SMAA-2, SMAA-O	unpublished
Strategic planning of an electricity retailer	SMAA-2, multivariate criteria	[46]
Management of university courses	SMAA-DS	[55]
Elevator planning	SMAA-2, SMAA-O	[68]
Locating a kindergarten in Madrid	SMAA-III, SMAA-TRI	[63]
Nanomaterial risk assessment	SMAA-TRI	[70]
Airport hub for centralizing cargo in Morocco	SMAA-2, SMAA-O	[48]

represented by one specific measurable factor and linear value functions constructed by experts for the remaining more complex criteria. The SMAA analysis resulted in four highly acceptable alternatives including the “not carry out the project”.

General Plan of Kirkkonummi [18, 27] The goal was to determine the implementation order of the general plan of Kirkkonummi, Finland. The SMAA-3 method was developed during this real application. Again, no weight information was available from the DMs consisting of 13 members of the Municipal board of Kirkkonummi. Instead of using the original SMAA, we decided to combine the pseudo criteria model of ELECTRE III with the weight space analysis of SMAA. The ELECTRE model was used without discordance and the exploitation of the outranking relation was done by Min in Favor choice procedure [52]. The problem consisted of seven different parts of a general plan that were evaluated based on five criteria. The criteria were measured similarly to the Helsinki harbour case. The SMAA-3 analysis resulted in a partial implementation order of the alternatives.

Technology Competition for Cleaning Polluted Soil [20] An industrial area in Toukolanranta, Helsinki was replanned for residential use, and therefore the area had to be cleaned before building. The SMAA-2 method was used in choosing three finalists for test-cleaning a small part of the region considered. The winner of this test got the contract for cleaning the whole area. The nine competitors were evaluated based on five criteria. Initially, each of the eight experts forming the competition board defined their own preferences for each criterion, and individual rankings were formed using SMART and ELECTRE III methods. Because the results were very conflicting, the SMAA-2 method was then applied for analysing the acceptability of each candidate. All criteria except costs were measured with linear value functions defined by the competition board. Instead of having no weight information in the SMAA-2 analysis, the uniform distribution for the weights was defined between the DMs' observed minimum and maximum weights from the earlier individual ranking phase. SMAA-2 analysis revealed unambiguously the three best candidates to continue the competition.

EIA for Locating South-Karelian Waste Treatment Facility [41] A new solid waste management area was to be built near Lappeenranta in South-Eastern Finland to replace outdated landfills in the region. The DMs in this problem were the board members of Etelä-Karjalan Jätehuolto Oy (South-Karelian Solid Waste Treatment Ltd). However, several other decisions had to be made regarding new permits required before a waste treatment area can be established. Four different potential regions for the new area were evaluated based on 17 ordinal criteria. The reason for using ordinal criteria only was that the criteria applied could not be measured on natural cardinal scales, and the efforts required for defining linear value functions for each criterion in consensus were considered too demanding. The SMAA-O analysis of the problems was first carried out with two different ways: using the full range of ranks from 1 to 4, and for the shared ranks, using only given ranks. No remarkable differences were observed between these two analyses. One alternative received clearly highest first acceptability index in both analyses (80%). After this, the experts were able to state for some criteria that some rank interval is more or less significant than another. A new SMAA-O analysis with these new constraints was carried out. This raised the first rank acceptability index of the most acceptable alternative up to 85%. The DMs' choice was another promising alternative with a first rank acceptability of 15%.

Waste Storage of Pietarsaari Multi-fuel Power Plant [37] Siting the storage area for the by-products of a biofuel-based combined heat and power plant was considered in Pietarsaari, on the West-Coast of Finland. Four different siting areas were evaluated based on 11 criteria. The analysis was done with the SMAA-O method in a similar way than in the previous landfill-siting problem. The result of the analysis could reveal only one unacceptable alternative thus leaving the choice between the three remaining acceptable alternatives to the DMs. This time the DMs' however, decided to choose the alternative with the highest first rank acceptability index.

Choosing Landfill Reparation Method at Huuna [42] A former industrial waste landfill in Huuna, Tervakoski was to be repaired soon, since it was considered as a so-called risk landfill. Initially six different alternative options for dealing with the landfill were considered. The evaluation was done with 13 criteria. During the process, a new seventh alternative was formed. The difference to earlier SMAA applications was in this case that both cardinal and ordinal criteria measurements were used. In addition, some criteria values were re-evaluated during the decision process. For the first time, the (partial) importance order of the criteria was used as preference information in SMAA. In fact, two different partial importance orders were considered, since additional experts did not agree with the importance order defined by the other participants. These additional analyses, however, did not change the results of the analysis: the newly formed alternative received a first rank acceptability of more than 90%.

Land-Use Planning in Oulu Region (2000–2002) A former industrial area in Pateniemi, Oulu, Finland was polluted due to past industrial activities. This project involved integration of MCDA methods with a geographical information system (GIS) to compare different alternatives to clean a polluted area and zone it for residential, recreational and commercial activities. The results of this study have not been published.

Strategic Planning of an Electricity Retailer [46] Liberalization of the electricity market, unbundling of vertically integrated businesses, scarcity of natural resources and increasing emphasis on the environmental effects of the energy sector have created for energy companies a new business environment, where complex, interacting decision problems must be solved in co-ordination [45]. These decision problems involve a large number of variables, multiple criteria, and they are stochastic by nature. In this application, a combination of efficient simulation and optimization methods were applied to analyse the effect of different strategic choices, concerning the pricing policy, risk attitude and environmental inclination. A total of 81 different alternatives were evaluated in terms of 4 criteria. The alternatives were compared using the SMAA-2 method and different ways to represent the uncertain criteria (independent normal distributions, a multivariate Gaussian (joint) distribution and a discrete sample obtained from simulation).

Management of University Courses [65] The Department of Information Technology at the University of Turku collects course feedback. After each course, students fill in a form where they rate different aspects of the course. During the years 2003–2004, 10 courses were selected for modelling collective preferences of the students. The purpose was to reveal possible needs of improvement of the selected courses. The courses were evaluated based on six criteria from lecturing skills to course material. The course criteria measurements were modelled as Gaussian distributed. The SMAA-DS method was used for generating recommendations for the department. The analysis divided the courses in to three classes: (1) good, (2) small improvements needed and (3) large improvements needed. The personnel of the department obtained valuable support for developing course organization.

Elevator Planning [68] Modern elevator systems in high-rise buildings consist of groups of elevators with centralized control. The goal in elevator planning is to configure a suitable elevator group to be built. The elevator group must satisfy specific minimum requirements for a number of standard performance criteria. In addition, it is desirable to optimize the configuration in terms of other criteria related to the performance, cost and service level of the elevator group. Different stakeholders involved in the planning phase emphasize different criteria. Most of the criteria measurements are by nature uncertain. Some criteria can be estimated by using analytical models, while others, especially those related to the service level in different traffic patterns, require simulations. In this application, 10 feasible elevator group configurations for a 20-floor building were compared. The problem contained criteria of mixed type. Some criteria were represented by multivariate Gaussian distribution, others by deterministic values and ordinal (ranking) information. SMAA was used to identify the configurations that best satisfy the goals of different stakeholders.

Locating a Kindergarten in Madrid [63] The largest private university of Madrid, San Pablo CEU, needed to build a kindergarten for children of the personnel. The problem is to choose between seven different locations evaluated in terms of five criteria. The criteria measurements, as well as the preferences, contained large uncertainties. Therefore, the problem was analysed by using the SMAA-III method that allows modelling uncertainties through joint probability distributions. The results were also cross-validated by applying a modified version of the SMAA-3 method. Both methods produced essentially the same results. Based on the analysis, the DMs were able to identify their most preferred alternative.

Nanomaterial Risk Assessment [70] Nanotechnology is a rapidly growing research field with an increasing impact on our everyday lives. Although nanomaterials are used in common consumer products, the lack of information about human health and environmental risks may hamper the full-scale implementation of this technology. To guide scientists and engineers in nanomaterial research and application as well as to promote the safe handling and use of these materials, SMAA-TRI was used to construct a generic risk-assessment model for assessing nanomaterial safety by assigning them into five ordered risk classes. The use of SMAA-TRI allowed to include imprecise expert judgements in the model. Although the current knowledge about nanomaterial risks is very partial, the model allowed to provide reasonable recommendations about which nanomaterials may need more precise measurements and testing to be safely deployed in consumer products.

Airport Hub for Centralizing Cargo in Morocco [48] The main objective of this study was to provide support for the decision on the location of a centralized air cargo hub at one of the existing airports in Morocco. The hub will serve the multimodal transport of goods between four continents. The decision process was undertaken by the National Airports Authority of Morocco (ONDA) in cooperation with the Civil Aeronautical Department (DAC) and the Air Bases Department (DBA). Nine alternative locations were compared in terms of six criteria (some of them assessed on ordinal scales). For this reason, the SMAA-O method that allows mixed ordinal and cardinal criteria was applied. The results indicated that two of the alternatives were superior and the choice between them depends on how the different criteria are weighted. Among these, ONDA chose the more widely acceptable alternative and started negotiations with investors about building the hub.

SMAA has been widely applied also in problems in the forest sector, see, e.g., Kangas et al. [24, 25]. For descriptive use of SMAA, see Durbach [12].

10.6 Discussion and Future Research

In SMAA uniform distributions are used to represent absence of information both in criteria measurements and preferences. Ordinal criteria are transformed into cardinal measurements by simulating consistent ordinal to cardinal mappings. The simulation process is equivalent to treating the absence of interval information of ordinal scales as uniform joint distributions. Similarly, absence of weight information is treated as a uniform joint distribution in the feasible weight space.

Representing absence of information through uniform distributions is not absolutely correct in theoretical sense. However, we justify this approach by the fact that uniform distributions carry the least amount of information (maximal entropy) among all distributions. Therefore, it makes sense to use them to represent absence of information.

In particular, when no preference information is available, the scaling of criteria values may greatly influence the results given by the method. When weight information is expressed, the DMs must also consider the scaling and understand how their weights affect the overall utilities. Scaling is generally done by using ideal and anti-ideal criteria values. The choice to be made here is whether these values are taken from the set of alternatives at hand, or whether some information outside the problem is taken into account when defining these ideals [72].

Although SMAA can be used with arbitrarily shaped utility functions, in real-life applications, simple forms, such as linear or some concave shapes, are the most likely ones. Assessing the precise preference structure of DMs can be difficult and time-consuming in practice. Prospect theory models [23] are a promising alternative to utility functions due to their descriptive power observed in several studies [57]. SMAA can also be used with these decision models [38].

Future research needs comparisons with different types of preference models. These include, for example, the value function based versions, ELECTRE based versions (e.g., Roy [55, 56]), PROMETHEE (e.g., Brans and Vincke [5]) and reference point methods. All of these procedures have been successfully applied in real decision making with explicit weights. It would be interesting to study these models in the SMAA sense: how useful information is handled through each technique, how the results differ, and what the DMs' actual comments are. Another important research topic is the treatment of absence of information in the analysis towards, for example, the direction of Dempster–Shafer evidence theory [58].

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Chapter 11
Multiple Criteria Approaches to Group Decision and Negotiation

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Abstract Collective decision making, the processes of group decision and negotiation, and the differences between them are explained and illustrated. Then the applicability of techniques of multiple criteria decision analysis (MCDA) to problems of group decision and negotiation (GDN) is discussed. A review of systems for Group Decision Support and Negotiation Support then highlights the contributions of MCDA techniques. The roles of decision makers and others in GDN are discussed, and overall progress in GDN is reviewed. Finally, some suggestions for worthwhile future contributions from MCDA are put forward.

Keywords Collective decisions · Group decision · Negotiation · Multiple criteria decision analysis · Multiple-party multiple-objective decisions · Multilateral · Bilateral

11.1 Introduction: Group Decision and Negotiation

The ability to reach informed and appropriate collective decisions is probably a prerequisite for civilization, and is certainly crucial for individuals and organizations today. Formal procedures for reaching a decision are often recommended, reflecting the belief that collective decision making can be "improved" by a systematic

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approach. Group Decision and Negotiation (GDN) is an academic and professional field that aims to assist groups, or individuals within groups, in interacting and collaborating to reach a collective decision. The broad aims of the field are to provide procedures to ensure that collective decisions are as good as possible, and to study the nature of the structural, strategic, tactical, social, and psychological issues faced by individuals as they narrow in on a collective choice. GDN combines approaches from operations research, computer science, psychology, political economy, systems engineering, social choice theory, game theory, system dynamics, and many other fields, including Multiple Criteria Decision Analysis (MCDA). In fact, the recently published *Handbook of Group Decision and Negotiation* [51] contains a full chapter on the contributions of MCDA [84].

The field of GDN boasts a large and growing research literature. Within the Web of Science database, for instance, a search on the keywords "group decision" and "group negotiation" identifies about 20,000 papers, scattered over more than 100 research areas including management science, engineering, psychology, neuroscience, political science, and many others [81].

The amount of research is not surprising, given that collective decision making is among the most important processes carried out by corporations, governmental and nongovernmental organizations (NGOs), and individuals around the world. For example, negotiations conducted by the United Nations among national governments, regional organizations, NGOs, and other groups, cover a broad range of issues including international law, international security, economic and social development, and human rights [78]. Meetings aimed at making a group decision are ubiquitous in corporate and governmental organizations; they often follow explicit procedures in order to obtain information that is as accurate and complete as practicable, thereby reaching appropriate decisions judiciously and quickly.

What is group decision and what is negotiation? Is it useful to distinguish between them? A **Group Decision** is a decision problem shared by two or more concerned parties who must make a choice for which all parties will bear some responsibility. A **Negotiation** is a process in which two or more independent, concerned parties may make a collective choice, or may make no choice at all. Generally speaking, group decision is a generic process and negotiation is a specific one and, as discussed by Walton and MacKersie [91], negotiation often has a distributive dimension that group decision lacks. The points of difference between group decisions and negotiations are reflected in the possible outcomes, the process, the numbers of participants, the existence of common ground, and the types of participation. The details are given next.

Difference in outcome: The possibility of disagreement is the major distinction between group decision and negotiation. In a group decision process a decision must be made, whereas in negotiation each party has the option of "walking away." Because of this fundamental fact, negotiating parties are advised to take into account what Fisher et al. [38] and Raiffa [76] call their BATNA, or Best Alternative To Negotiated Agreement. A party that prefers not to have responsibility for a particular choice need not agree to it.

It may seem that group decision could be made similar to negotiation by allowing the group the option of deferring a decision, or doing nothing. But this similarity is superficial as a group may decide to do nothing because of the numbers and influence of those members who prefer that option; other members may disagree, but remain in the group. The fundamental property of negotiation is that all parties *agree* with the collective choice. In group decision, by contrast, it is common for the parties to disagree on what is the best choice, but to select an alternative that achieves a minimum level of support within the group.

Difference in process: One well-known group decision procedure is voting. In a properly conducted election, all options are known at the time of voting, and the members of the group indicate preferences, which are combined according to some standard systems to obtain a group choice [21]. Voting has some special properties that make it very useful for some decisions but not for others. For instance, most voting systems give all voters equal weight in the final decision; tinkering with the "one person, one vote" property usually causes a voting system to lose many of its other appealing properties [37]. Moreover, provided there are three or more voters and three or more alternatives, every voting system is vulnerable to strategy, and therefore voting results do not provide a reliable reading of "group preference" [4]. Voting is widely used for surveying opinion (a "straw vote") but rarely considered sufficient for a group decision in a corporate context, in part because the act of voting provides no forum for information search or exchange, for the development of preferences, or for the modification of the information or preferences of others. In negotiation, of course, the opportunity for persuasion is central, and indeed is often the point of the process.

Difference in numbers of participants: Among the less important differences between group decisions and negotiations is the tendency for group decisions to involve larger numbers of parties. A group decision or negotiation that involves two parties is called *bilateral*; if it involves more than two parties, it is called *multilateral*. Almost all group decisions are multilateral, at least in the formal sense of the number of parties at the table. By contrast, bilateral negotiations are at least as common as multilateral.

Difference in common ground: Group decisions are usually made by a group, i.e., by parties who have something in common; typically, they are all employees of the same corporation, and can therefore be assumed to have an interest in the quality of the decision, insofar as it contributes to the success of their common enterprise. The group decision ideal is that a "meeting" of individuals with a common interest in a good decision, but different information and perspectives, can be an informative and even creative process that identifies the choice most consistent with the common interest. Good decisions are certainly the key to success for a business, and it is widely held that even an expert individual is rarely as successful in decision making as a process that canvasses multiple points of view [88]. Negotiations, on the other hand, often involve parties whose relationship is partially or even entirely adversarial, and who begin with evaluations of options that differ substantially and even (in the bilateral case) diametrically.

Difference in types of participation: Another small but significant difference is that parties in negotiations are often represented by negotiators, who may be compensated for their efforts. One reason is that many negotiations, especially those involving large organizations–labor and management, for example – are rarely limited in the time and resources they use up. Another reason may be the view that tactics and style are important determinants of success in negotiation, causing each party to want professionalism on its side.

Despite their differences, group decision and negotiation are often studied together, or in parallel, mainly because of the collective decision aspect that they share. For example, group decision and negotiation can and should include searches for new alternatives, efforts to repackage existing options to form new alternatives, and detailed assessments and evaluations of alternatives. Many procedures studied in the field of Group Decision and Negotiation are designed to assist in these endeavors. Of course, GDN has its own distinctive problems and issues. For instance, the decision environment in GDN is usually ill-structured and dynamic. Moreover, the vague or conflicting perceptions of decision makers often make it difficult to pin down exactly which problem each one understands the group to be facing. For example, [68] discussed problem structuring in the GDN context; the ability to find a shared vision of a group problem can be crucial in a context of uncertainty, inconsistent perceptions, and diverging interests. For these and other reasons, it can be extremely difficult to apply standard tools to understand and analyze practical GDN problems [47].

The development of GDN as a field has been motivated by the need for better approaches to collective decision problems. Tools from other fields have often been successful in GDN, even though they may fit only a few GDN problems, and often not exactly. However, as will be illustrated later, there remain aspects of collective decision making that are not accounted for in GDN. For many of these, adaptation is essential to comprehend the nature of the difficulties, which is often to only way to achieve a good solution.

11.2 Multiple Criteria Decision Analysis in Group Decision and Negotiation

Multiple criteria decision analysis (MCDA) is a set of techniques and principles designed to help a decision maker (DM) compare and evaluate alternatives according to two or more criteria (objectives), which are usually conflicting [10]. Most MCDA procedures are designed to elicit a DM's preferences, both over the level of performance of alternatives according to a particular criterion and over the relative importance of criteria, or to combine these preferences according to a procedure that helps the DM choose the best alternative, rank the alternatives, or sort the full set of alternatives into a few (ordered) groups of approximately equally preferred alternatives [80].

Traditionally, MCDA takes the set of alternatives and the set of criteria as given, and focuses on preference elicitation and aggregation. Recently, though, there has been more attention to problem construction, and greater effort is now recommended to search out new alternatives, identify objectives more appropriately, and select criteria that reflect the DM's real interests and objectives. This trend began with Keeney's ideas on value-focused thinking [49], which suggest a systematic method that provides an excellent approach to this aspect of MCDA.

MCDA procedures are designed to be applied to data measuring the performance of each alternative according to each criterion, the so-called *performance matrix*, plus input from the DM that describes the DM's preferences. Procedures for choosing, ranking, or sorting are different, but clearly they have much in common.

There have been many applications of MCDA to collective decision problems. For example, many participative decision-aiding frameworks implemented in environmental contexts, including Marchi et al. [65], Norese [71], Strager and Rosenberg [87], Mustajoki et al. [69], and Adrianto et al. [1], use MCDA methodologies. But applying an MCDA technique to a collective decision problem poses an unavoidable theoretical problem: Collective preferences may not exist. In other words, the "DM's" preferences are an essential input to any MCDA method, and each individual in the group may have well-defined preferences, but the notion that collective preferences are determined by individual preferences is naive.

The existence of individual preferences does not imply the existence of a collective preference with properties similar to those of the individual preferences, as is illustrated by the well-known Condorcet Paradox [4]. In this example, three individuals, 1, 2, and 3, are asked to consider three alternatives, A, B, and C. As shown in Table 11.1, Person 1 prefers A to B to C; Person 2 prefers B to C to A; and Person 3 prefers C to A to B. It is obvious that two people prefer A to B, two people prefer B to C, and two people prefer C to A.

The existence of a collective preference therefore presents a dilemma: We must either accept that group preference can be intransitive, or we must make one person a dictator. The first option is to accept that even though A is preferred to B and B to C, it does not follow that A is preferred to C, as transitivity would require–in this case, in fact, the opposite preference holds. The second option is to accept that a collective preference is transitive only because one person is more important than the other two combined. If the collective ordering in the Condorcet example A to B to C, for instance, then person 1 is more important than 2 and 3 combined, in that 1's preference of A to C outweighs 2's and 3's preferences of C to A. Simply put, person 1 a dictator, so the group preference is the same as 1's preference, and 2 and 3 simply don't matter. In any case, we must accept that even when individual preferences are well-behaved, a well-behaved group preference that reflects all individual preferences may not exist.

Table 11.1 Condorcet paradox: Preference orderings of 1, 2, and 3 over A, B, and C

Individual	Preference order
Person 1	$A > B > C$
Person 2	$B > C > A$
Person 3	$C > A > B$

The lesson of the Condorcet Paradox is relevant to the application of MCDA procedures to a “DM” who is really a group, because MCDA procedures require the DM to have a preference with respect to performance on criteria, and with respect to criteria. A useful group preference may exist in some cases, but in others a transitive non-dictatorial group preference may simply not be available. Of course there are situations in which group preference is well-defined and has the expected properties, or in which, even absent a sensible group preference, MCDA procedures give sensible results. Moreover, from a practical point of view, the fact that a decision has a shaky theoretical basis does not imply that it is necessarily bad. Nonetheless, the Condorcet Paradox shows that, when the DM is replaced by a group, MCDA procedures may be inapplicable in principle.

This concern applies whenever MCDA procedures are applied to GDN as if a group of individuals with an interest in a decision were an individual DM. In some instances, the group–say, the planning committee for a new building or a retail expansion–has the collective power to make a decision. The group is asked to answer, collectively, the same questions that would be used to elicit the preferences of a DM in accordance with the MCDA procedure selected. The inferred “collective preference” is then used to make a decision, or as input to a decision. But, as noted above, such an application of an MCDA procedure may be difficult to justify–even though it often works well in practice.

A variant of this idea for applying MCDA procedures is based on a consultation meeting or process involving the “stakeholders” affected by a decision to be made, for example, by a government agency or a corporation. In some instances, the stakeholders have some control over certain aspects of the decision [7, 41]. But the process may have other goals, such as making the stakeholders aware of the full range of issues entering into the decision, or even making a decision that is politically acceptable. Inasmuch as the stakeholders have the power to make a decision, the techniques of GDN are certainly applicable. The possibility of other roles in a group decision process is discussed in detail below.

Another “pitfall” for applications of MCDA techniques to GDN problems is the identification of individuals with criteria. This strategy is simple, and quickly solves the problem of identifying the criteria, but it is generally not helpful. One major objective of MCDA techniques is to trade criteria off against each other in a controlled way and, in extreme cases, to drop or combine criteria. Procedures advising that some group members be dropped and that others be combined, or that explain how to trade one person off against another, are rarely credible or persuasive in a group setting. MCDA procedures are designed for multiple criteria; they succeed by helping the DM break the problem down into comparisons of performance on each criterion, and then into weighing the relative importance of the criteria. If the “DM” is a group, it will be more difficult to elicit the DM’s attitude and judgment, but to ignore actual criteria on which alternatives can be measured and compared is to throw away the proven features of MCDA methods [7].

In the end, of course, a group decision process may take on many “political” features, and it may be inevitable that the interests of some are sacrificed in favor of the interests of others [44]. And, while not recommended, the identification of individuals with criteria can produce some useful insights into the process–for instance, Nurmi [72] provides an assessment of the vulnerability of MCDA to many well-known voting paradoxes. But in general we recommend that criteria and individuals be treated as separate entities.

We now turn to a general description of the application of MCDA methods to group decision and negotiation. While we classify this work into the categories *MCDA and group decision support* and *MCDA and negotiation*, we will make note of ideas, techniques, and systems that fit into both categories.

11.3 MCDA and Group Decision Support (GDS)

To understand Group Decision and Group Decision Support, it is useful to distinguish among the possible roles of an individual (or a group of individuals with a common viewpoint) in a group decision process. In addition, there are a few studies of actor typologies within decision-aiding processes, including [5, 57].

A *DM* is a member of the decision-making group; together, the DMs have control of the decision process, including data collection, data aggregation and assessment, and final implementation. A *stakeholder* is an individual (often a representative of a group) who is significantly affected by the final decision, but does not necessarily have any control over it. The main difference between DMs and stakeholders is that although stakeholders have preferences over the resolution of a decision problem, their primary concern is not with the full scope of the resolution, and they may have no significant influence on the decision process. Another role is that of *expert*, an individual with special knowledge of the decision problem, but no interest (in the sense of being disinterested–of having nothing to gain or lose) in its resolution.

MCDA provides useful terminology to describe these roles. A stakeholder, typically, is concerned about only a few of the criteria. The expert, because of both special knowledge and disinterest, is often called upon to provide an "unbiased" assessment of the performance of alternatives according to one or several criteria, or of the relative weights of the criteria themselves.

Group Decision Support aims to provide formal assistance to a group as it moves toward a decision by encouraging focused communication about the possible alternatives, the choice of criteria, the measurement of performance, the weights to be given to criteria, and the overall evaluation of alternatives. Basic techniques for group decision support have been available for many years. They include

- *Brainstorming*, originally suggested by Osborn [75], is a group creativity technique designed to generate a large number of ideas for the solution to a problem.
- *Nominal group technique* [48], an approach for use among groups of many sizes, aiming to make a decision quickly, as by a vote, but taking everyone's opinions taken into account (as opposed to traditional plurality voting, where the largest group prevails).
- *Delphi method* [79], a systematic interactive method to obtain and integrate knowledge from a panel of independent experts.

- *Voting* [23], which can be carried out according to various voting procedures and aggregation rules.

Note, for example, that Delphi is designed exclusively to enable a group of experts to produce a consistent judgment, whereas voting on alternatives or criteria makes sense for DMs only.

It is clear that GDS has a role for MCDA, as group decision making must involve assessments of the relevance of performance levels on criteria, and of the relative importance of criteria. One of the first applications of MCDA methodologies to group decision support was Bui's [12] discussion of the analysis, design, and implementation of group decision support systems from an MCDA viewpoint. Bui's system utilized several MCDA methods for individual preference elicitation and preference aggregation to support a group decision process.

As Bui and Jarke [13] suggested, MCDA can provide a systematic framework for tackling three important tasks in GDS: organization of the whole decision process, preference representation for different DMs, and preference aggregation. The combination of MCDA and GDS has generated many important research products. These methods can be divided, roughly, into two categories.

- *GDS based on procedures:* These MCDA-based methods focus on the design of effective procedures through which the DMs can interact in a way that brings out important information, generates new ideas, minimizes disagreement, and leads to a final choice. Procedure-based MCDA-GDS methods are very close to negotiation in some sense and have been extensively applied to both GDS and multilateral negotiation problems. Clearly, the aforementioned "soft-thinking" approaches, such as brainstorming, the nominal group technique, and the Delphi method already incorporate some multiple-criteria or multiple-objective ideas for GDS. Other procedure-based MCDA-GDS methods that have been developed, refined, and applied to group decision problems are summarized next.
  - MEDIATOR is a negotiation support system based on evolutionary systems design and database-centered implementation [46], but it can be used usefully as a group decision support system.
  - Outranking Methods, a family of popular MCDA methodologies that originated in Europe in the mid-1960s [80]. Both ELECTRE and PROMETHEE have been applied to support group decisions, for example, in [24, 28] and [64].
  - Preference disaggregation approaches, which analyze a DM's global assessment in order to identify the criterion aggregation model that underlies the preferences, have been extended to a group decision context, for example, in [66].
  - SCDAS (Selection Committee Decision Analysis and Support) is a system designed to support a group of decision makers working together on selecting the best alternative from a given finite set of alternatives. The framework utilizes aspiration-led and quasi-satisficing paradigms for eliciting user preference, and an achievement function for ranking alternatives [58].
  - A topologically based approach to measuring the distance between DMs that was used by [14] to formalize the problem of reaching consensus.

– Several techniques based on AHP (Analytic Hierarchy Process) [83] have been applied to GDS problems in various contexts. These include [2, 29, 56], and [77].
– JUDGES, a descriptive GDS for the cooperative ranking of alternatives [14].
– An integrated framework of Drama Theory and MCDA, developed in [61], exploit their potential for synergy, with a view to providing more effective unilateral or multilateral decision support.
– The idea of jointly improving directions is at the basis of a series of research initiatives aimed at reaching Pareto optimality in group decisions over multiple continuous issues, for example, in [30, 31], and [32].

An early book on MCDA-GDS that summarizes many GDS approaches is [43]. The impacts of three procedural factors on information sharing and quality of group decision are examined in [42].

- *GDS based on optimization and aggregation:* These approaches aim to generate the optimal group decision by designing and employing optimization models. Representative methods include

  – Various techniques based on fuzzy logic, such as [18, 73] and [99], have been developed to incorporate multiple experts' ratings into GDS.
  – The Dempster-Shafer evidential reasoning approach [85] has been applied to GDS, for example, by [8, 9, 27] and [97], to effectively aggregate different DMs' knowledge.
  – Ordered Weighted Average (OWA) operators, initially proposed by Yager [96], constitute convenient ways to average information from multiple sources or different DMs for GDS [3, 60] and [94].
  – A few optimization aggregation procedures have been designed to integrate preference data in multiple formats, such as fuzzy logic, interval relations, and probability, from different DMs, by [39, 62] and [95].

  Many other optimization aggregation approaches, including [45, 55], and [92], to name a few, are applicable to GDS problems.

Group Decision Support Systems (GDSS) are computer systems specifically designed to guide and assist a group of DMs. Many GDSSs require a special computer installation, a special room, or specialized expertise to operate the system, especially if facilitation of the decision process is going on at the same time. The Co-oP system [12], one of the earliest GDSSs, incorporated MCDA methods such as AHP and outranking techniques to encourage cooperative multiple criteria group decision making. Another example is Web-HIPRE [70], a Java applet-based MCDA system that provides a common platform for individual and GDSS. Virtually all GDSSs encourage communication and contribution by each member of the group; many of them aim at the creative generation of additional alternatives and the creation or confirmation of a group identity and role, and some also include substantial recording and surveying capabilities, and may be designed for meetings that are distributed in space and even in time ("non-synchronous"). One well-known GDSS for electronic meetings is Meetingworks [67].

Even broader than GDSS are Group Collaboration Systems (GCS), which support collaborative processes of strategy-making, process engineering, or product design and development [26]. The most ambitious of the GCS can support collaboration across several organizations. GCS are a natural extension of GDSS, and are often classed as Group Decision Support, as they may be used to develop a decision on a specific problem, or a set of linked decisions. Well-known group support systems, often called groupware, include Decision Explorer [6] and ThinkLets [11].

Of course, it is difficult to choose among systems using only their specifications. Davey and Olson [25] compared GDSS using laboratory methods. Other comparative research includes [20, 22, 40], and [74].

11.4 MCDA and Negotiations

As an area of study, negotiation is much more diffuse than Group Decision. The reason is that negotiations can be conducted under fixed rules only if all parties agree, since any party has the option of "walking away" from the process–and can be expected to do so, if it perceives a strategic advantage. Advice on negotiation can be found by following up on advertisements in popular magazines or in popular trade books such as *Getting to Yes* [38]. Among the more serious general academic studies of negotiation is Raiffa et al.'s *The Art and Science of Negotiation* [76]. These two works are generally credited with popularizing the concept of a negotiator's BATNA, or Best Alternative To Negotiated Agreement. To a rational, informed negotiator, the BATNA is a "hard" floor, to which any agreement must be superior.

One important subdivision of negotiation is bilateral (two parties) or multilateral (many parties). Negotiations are interest-based or positional – or often both. In interest-based negotiations, which may be bilateral or multilateral, the possible alternatives are not specified in advance, and the parties typically have some common preferences. A win–win solution is possible; efforts to generate new alternatives or creatively recombine old ones are often repaid with a solution that is better for both sides. On the other hand, positional (or zero-sum, or fixed-pie) negotiations are generally bilateral; if so, the two sides' evaluations of the alternatives are diametrically opposite. There is little room for creativity or even new information, and bargaining proceeds mostly by threatening and holding out. Positional negotiations are widely understood to be the most intractable of negotiations, because there is no possible win–win outcome. Of course, most practical negotiations are partly positional; for example, if there is a fixed list of alternatives, and the parties' rankings of their desirability are the same, then the bargaining must be positional.

Many useful approaches to the study of negotiation, especially at a theoretical level, have come from traditional social sciences like economics and political science, and from the study of mathematical models of negotiation. The examination of many game-theory models of negotiation, such as the Stahl [86]–Rubinstein [82] alternating-offer bargaining model, reveals that in interest-based negotiations, negotiators are most likely to achieve a win–win outcome if they work to learn about the values of others, while slowly revealing their own values. In fact, this idea presumes multiple-criteria outcomes, where the relative values of criteria (and even the evaluations of performance on a criterion) may be different for different parties. In this context, there is scope to find outcomes that are preferred to the status quo for both parties, or *Pareto-superior* to it. Bargaining is efficient (in the economic sense) if it achieves a *Pareto-optimal* outcome, one to which no other outcome is Pareto-superior. Political science has contributed some practical studies, often at the diplomatic level, of multilateral negotiation procedures (like the "rolling text" now commonly used by the United Nations) and of negotiation ripeness [98].

We now proceed to a brief survey of MCDA-related work on negotiation. As already noted, both negotiation and group decision are problems of collective decision making, so many procedures have some applicability to both. For example, many of the procedure-based GDS approaches reviewed above can be applied to negotiation; in fact, some were originally developed to support negotiation. Hence, we do not repeat our discussion of these methods, and focus on procedures that are applicable mainly or exclusively to negotiation support.

One important and rapidly developing area is e-negotiation, which refers to negotiations using computers–usually the Internet–as the medium of communication. Negotiators may be humans, who may be distant in time and space, electronic negotiating agents, or robots. For a general account of the development of systems, see [50]. The development of the World Wide Web has given great impetus to e-negotiation and its role in e-marketplaces, especially in personalizing and customizing processes. Clearly, electronic negotiating agents can be used only in a context in which they can evaluate offers; these agents are now well developed, and e-negotiation systems often offer human negotiators a "wizard" to assist them in evaluating offers. Note that preferences must be input to a negotiating agent or wizard, either explicitly or based on inferences from choices on some test set of cases. Many ideas for these systems have been imported directly from MCDA. For example, Vetschera [90] examined whether DMs' preferences embedded in e-negotiation models are actually reflected in the behavior of negotiators, or in negotiation outcomes.

Other computer-related work involves multiagent modeling of negotiation, an area of study in which autonomous agents carry out a sequence of negotiations, based on some ideas from complexity theory and agent-based models. The parameters of the agents are set using ideas from MCDA. Some relevant work includes [19, 33, 59], and [93].

An MCDA approach to positional negotiation has been developed using a novel case-based distance method [15], in which a case set provided by each negotiator is input to a program to generate criterion weights for a weighted distance representation of the negotiator's preference. Then negotiation support using these distances helps the negotiators to identify and reach an efficient compromise.

Finally, negotiation involves many essentially strategic choices–negotiators must choose courses of action (what to offer, whether to accept an offer) that determine how well their goals and objectives are met. Strategic advice is important, particularly in negotiation preparation, and several systems that assist participants in strategic conflict are applicable. One is the Graph Model for Conflict Resolution, implementedin the Decision Support System GMCR II [34–36, 52], and [53]. Because preferences are its input, the Graph Model presumes the availability of an MCDA system to evaluate a DM's preferences for various possible outcomes, which can be interpreted as "packages" of features. Thus, the Graph Model goes one step further than MCDA, assisting a DM at planning negotiation strategy and at responding to unexpected developments during a negotiation.

11.5 Examples

We now use two practical examples to illustrate real-world processes that fall between negotiation and group decision, showing that strategic choice and negotiation preparation play an important part in the determining outcomes. The first example includes the application of MCDA concepts and methods for group decision; the second demonstrates analysis and support for a multilateral negotiation using the Graph Model for Conflict Resolution and the associated decision support system GMCR II [34, 53]). The intention here is to raise open issues and suggest directions for development.

Example 11.1. **Ralgreen Brownfield Redevelopment**

Brownfields are abandoned, idle, or underutilized commercial or industrial properties, with potential for redevelopment, where past activities caused, or may have caused, environmental contamination [89]. In many countries, interest in brownfield redevelopment (BR) increased rapidly in the 1990s as it became clear that revitalization of urban areas was the only way to relieve expansion pressure on the greenfields surrounding urban centers. BR often involves multiple DMs and stakeholders, as illustrated in the story of the Ralgreen BR project that follows.

The Ralgreen community is located in Kitchener, Ontario, Canada. It now contains 101 residential units, including semi-detached and row housing and low-rise apartments. Until the late 1940s, the Ralgreen area was part of a family farm just outside the city. Around 1948, the property owners and the City of Kitchener had agreed that a pond on the property could be infilled with organic materials that included cinders and ash from the City's incinerators. The land was used for agricultural purposes until 1965, when the property was sold to a developer who built the Ralgreen subdivision.

Beginning in 1996, Ralgreen residents complained to the City of Kitchener about geotechnical issues: settlement and displacement of structures; seepage of liquids into basements; methane; and mould. By 1997, investigations by consultants for the Ralgreen residents had linked the problems to the infilled pond. In 2000, a mediated settlement was reached by residents and the City of Kitchener. The subdivision would be cleaned up in accordance with the guidelines of the Ontario Ministry of Environment. Figure 11.1 shows the DMs and stakeholders involved in this issue just prior to the agreement in 2000.

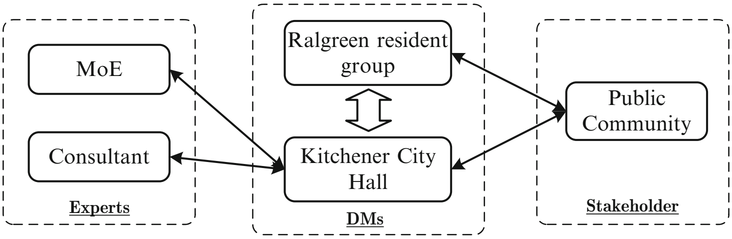

Fig. 11.1 DMs and stakeholders in Ralgreen BR

The two key DMs in the Ralgreen BR were the City of Kitchener (City Hall) and the Ralgreen residents' group. Some but not all of their interests were in direct conflict [17]. Other participants included in the negotiation were Ontario Ministry of Environment (MoE), a consultant employed by City Hall playing the role of expert, and environmental groups in the broader community, which can be considered stakeholders. The arrows in Fig. 11.1 show the interactions among the DMs and other participants. For instance, the consultants communicated with City Hall suggesting for possible clean-up plans and evaluating them according to several criteria.

The four clean-up plans finalized and presented to Kitchener City Council in 2001 are listed below.

- A^1: Demolition of 20 houses, structural renovation of nine houses, full-scale waste removal and building/lot resale to neighborhood density.
- A^2: Demolition of 14 houses, structural renovation of 15 houses, full-scale waste removal and building/lot resale to neighborhood density.
- A^3: Demolition of 18 houses, partial waste excavation, landfill encapsulation, site-specific risk assessment, and parkland construction.
- A^4: Demolition of 14 houses, structural renovation of 15 houses, stratified removal of waste to 1.5 m below surface, with clean soil fill.

A two-stage decision procedure was conducted including initial screening with selection of an option by the group of stakeholders. Options were evaluated according to several criteria [16]. The two stages are described next.

Initial Screening: A screening procedure was used to identify and remove inferior alternatives, so that further comparisons involved only feasible and promising options. Accordingly, each alternative was assessed for feasibility and evaluated, in accordance with MoE guidelines, based on the following criteria:

- Construction and field implementation (CFI)
- Consistency with the mediated settlement and legal acceptability (CML)
- Compatibility of land reuse with residential setting (CLR)

As shown in Table 11.2, A^4 failed the CML and CLR tests and was therefore rejected. The remaining three alternatives were carried forward for further investigation.

Table 11.2 Satisfaction of screening criteria

Alternatives	Criteria		
	CFI	CML	CLR
A^1	✓	✓	✓
A^2	✓	✓	✓
A^3	✓	✓	✓
A^4	✓	×	×

Further Investigation: In 2002, after several meetings and public consultations, Kitchener City Council adopted alternative A^1. In a study of this project [16], the authors reanalyzed the decision process by applying MCDA tools to A^1, A^2, and A^3, the three alternatives that survived the screening process. Criteria were selected for the evaluation of these alternatives, as follows: C_1: protection of human health and the environment; C_2: acceptability to the community; C_3: operational and maintenance costs; C_4: expected property tax returns; C_5: effect on property values; C_6: demolition time; C_7: amount of waste to be excavated.

The analysis was conducted using the Analytic Hierarchy Process (AHP) [83]. After the establishment of alternatives and criteria, analysis steps included comparisons of all possible pairs of alternatives on all criteria, followed by evaluations, which produced an overall score for each alternative. For details on the AHP analysis, refer [16]. Alternative A^1 had the highest score, confirming its choice in the actual event. In 2005, soil reports confirmed the success of the Ralgreen redevelopment, and the project was considered to be complete.

Example 11.2. **Elmira Aquifer Pollution**

The town of Elmira, with a population of about 7,500, is located within a rich agricultural region about 100 km west of Toronto in Ontario, Canada. Until 1989, Elmira's municipal water supply was drawn from the aquifer underlying the town. In late 1989, the Ontario Ministry of Environment (MoE) discovered that this underground aquifer had been contaminated by the carcinogen N-nitroso demethylamine (NDMA). Blame fell on the pesticide and rubber products plant of Uniroyal Chemical Ltd. (Uniroyal), which was located in Elmira, had a history of environmental problems, and was associated with NDMA-producing processes. MoE issued a Control Order under the Environmental Protection Act of Ontario, requiring that Uniroyal implement a long-term collection and treatment system, undertake studies to assess the need for a cleanup, and execute any necessary cleanup under MoE supervision. Uniroyal immediately exercised its right to appeal. At the same time various interest groups formed and attempted to influence the process through lobbying and other means. The Regional Municipality of Waterloo and the Township of Woolwich (Local Government) adopted common positions in the dispute and, encouraged by MoE, hired independent consultants and procured extensive legal advice at substantial cost. Negotiations among MoE, Uniroyal and Local Government commenced in mid-1991. MoE's objective was to carry out its mandate as effectively as possible; Uniroyal wanted the Control Order rescinded or at least modified; Local Government wanted to protect its citizens and its industrial base.

DMs	Options	Distinct States								
		7	3	4	8	5	1	2	6	9
MoE	1. Modify (Modify the control order for Uniroyal)	N	N	Y	Y	N	N	Y	Y	---
Uniroyal	2. Delay (Lengthen the appeal process)	N	N	N	N	Y	Y	Y	Y	---
	3. Accept (Accept current control order)	Y	Y	Y	Y	N	N	N	N	---
	4. Abandon (Abandon Elmira operation)	N	N	N	N	N	N	N	N	Y
Local Government	5. Insist (Insist that the original control should be applied)	Y	N	N	Y	Y	N	N	Y	---

Fig. 11.2 Feasible states in the Elmira conflict model

This rather typical environmental conflict was studied by Kilgour et al. [54], who assessed what negotiation support could have been provided by the Graph Model for Conflict Resolution and the DSS GMCR II [34, 53]). The application of GMCR II takes into account not only multilateral negotiation, but also the possible benefits of coalition formation. Essentially, one asks how well each DM can do on his or her own, and then whether some group of DMs would benefit by cooperating in a coalition.

The left column of Fig. 11.2 lists the three DMs in the basic Elmira graph model, followed by the options or courses of action each one controlled. The right portion of Fig. 11.2 shows the nine feasible states, listed in descending order of MoE's preference. Each column on the right represents a state: "Y" indicates that an option is selected by the DM controlling it, "N" means that it is not selected, and "-" means either Y or N. For example, state 8 is the scenario in which MoE modifies the control order, Uniroyal accepts this modification, and Local Government continues to insist on the original control order. In state 9, Uniroyal abandons its Elmira facility, so all other options are irrelevant–the resulting states are considered indistinguishable.

GMCR II contains a procedure called Option Prioritization to input state preferences for each DM. (For example, Fig. 11.3 shows MoE's ranking in descending order of preference, with ties allowed.) The hierarchal preference statements used by GMCR II to order the feasible states are provided in Fig. 11.3; statements are ordered from most to least important. (Numbers in the left column refer to options in Fig. 11.2.) Notice that MoE most by prefers that Uniroyal not abandon its Elmira plant, indicated by the initial statement "−4," which implies that states with N opposite option 4 precede those with Y beside option 4. Next in MoE's order of priority is that Uniroyal accept the current control order (indicated "3"), so among states with the same status relative to the highest priority statement (−4), states with Y beside option 3 are preferred to those with N. As illustrated in Fig. 11.3, Option Prioritization can handle an if and only if (iff) preference statement. In fact, this procedure accommodates any statement in "first-order logic," and ranks states according to the truth or falsity of these statements, using an algorithm that assumes transitivity of preferences. The algorithm for producing a preference ranking based on these priorities is similar to the discrimination method

	Preference Statements	Explanation
More Preferred	−4	MoE most prefers that Uniroyal not abandon its Elmira plant.
↑	3	Next, MoE would like Uniroyal to accept the current control order.
	−2	MoE then prefers that Uniroyal not delay the appeal process.
↓	−1	MoE would not like to modify the control order.
Less Preferred	5 IFF -1	MoE prefers that Local Government insists that the original control order be applied (5), if and only if (iff) it does not modify the control order (-1) itself.

Note that "-" represents the option is not chosen.

Fig. 11.3 Option prioritizing for MoE

DMs	Options	Status Quo	Transitional Non-cooperative Equilibrium	Cooperative Equilibrium
MoE	1. Modify	N	N ⟶	Y
Uniroyal	2. Delay	Y	Y ⟶	N
	3. Accept	N	N ⟶	Y
	4. Abandon	N	N	N
Local Government	5. Insist	N ⟶	Y	Y
State Number		1	5	8

Fig. 11.4 Evolution of the Elmira conflict

of MacCrimmon [63]. The ranking of states (entered in a similar way) for Uniroyal is $1 \succ 4 \succ 8 \succ 5 \succ 9 \succ 3 \succ 7 \succ 2 \succ 6$, and for Local Government it is $7 \succ 3 \succ 5 \succ 1 \succ 8 \succ 6 \succ 4 \succ 2 \succ 9$.

Once a graph model has been constructed by defining DMs, options, allowable transitions, and relative preferences, GMCR II carries out a stability analysis to determine which states are stable for each DM according to a rich range of solution concepts describing potential human behavior under conflict. States that are stable for all DMs according to a particular mode of behavior constitute a possible equilibrium or compromise resolution. Figure 11.4 shows how choices in the Elmira model evolved from the status quo state, state 1, via a transitional noncooperative equilibrium to the final cooperative equilibrium. At the status quo state, MoE is refusing to modify its control order, Uniroyal is delaying the negotiation process and Local Government has not taken a position. As shown, Local Government caused the conflict to move from state 1 to 5 by insisting that MoE implements the original control order. Later, MoE and Uniroyal formed a coalition in which MoE modified the control order and Uniroyal accepted the revision, moving the state of the conflict from 5 to 8, as depicted in Fig. 11.4. Keep in mind that both MoE and Uniroyal prefer state 8 to state 5 and, hence, it was in their joint interest to form a coalition and thereby achieve a result they both preferred. Local Government did not benefit, however. These strategic and coalitional events model very well the actual historical evolution of the Elmira dispute.

11.6 Conclusions

The need for more and better collective decisions, to address problems such as global warming, probably guarantees that systems for the support of group decision and negotiation have a strong future. Formal procedures have proven successful in facilitating better collective decisions, and have become crucial to many individuals and organizations. As an academic and professional field, GDN has demonstrated its ability to assist groups, or individuals within groups, in interacting and collaborating to reach a collective decision. It seems very likely that in the future more collective decision making and collaboration will take place at a distance, probably using the internet. If so, the recently developed systems for e-meetings and e-negotiations have a particularly strong future.

GDN combines approaches from operations research, computer science, psychology, political economy, system engineering, social choice theory, game theory, system dynamics, and many other fields. Multiple Criteria Decision Analysis (MCDA) has played, and is playing, an important role. While ideas and techniques from MCDA are directly applicable to GDN only rarely (exceptions include systems for the support of two negotiators in a multi-issue positional negotiation, like the case-based distance approach of [15]), it is clear that many successful systems for the support of negotiators, or the support of group decisions, have borrowed and adapted ideas and techniques from MCDA. Since collective decisions are important, this flow can be expected to continue. To some degree, ideas from GDN will find application in broader areas of MCDA. One example is Nurmi's study [72] of whether MCDA is vulnerable to voting paradoxes.

Ideas that could improve collective decision making are likely to receive a good trial in GDN, simply because the problems are ubiquitous and the issues are often crucial. Some of the open issues that we have suggested above include different decision roles, strategy, and coalition formation. New methods for addressing these issues will be helpful, though we should not forget the successes that existing GDN methods have achieved. Any procedures that will help organizations and individuals search out useful information, exchange it efficiently, and use it to reach decisions judiciously and quickly, are sure to be in demand for a long time.

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Chapter 12
Recent Developments in Evolutionary Multi-Objective Optimization

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Abstract By now evolutionary multi-objective optimization (EMO) is an established and a growing field of research and application with numerous texts and edited books, commercial software, freely downloadable codes, a biannual conference series running successfully since 2001, special sessions and workshops held at all major evolutionary computing conferences, and full-time researchers from universities and industries from all around the globe. In this chapter, we discuss the principles of EMO through an illustration of one specific algorithm and an application to an interesting real-world bi-objective optimization problem. Thereafter, we provide a list of recent research and application developments of EMO to paint a picture of some salient advancements in EMO research. Some of these descriptions include hybrid EMO algorithms with mathematical optimization and multiple criterion decision-making procedures, handling of a large number of objectives, handling of uncertainties in decision variables and parameters, solution of different problem-solving tasks better by converting them into multi-objective problems, runtime analysis of EMO algorithms, and others. The development and application of EMO to multi-objective optimization problems and their continued extensions to solve other related problems has elevated the EMO research to a level which may now undoubtedly be termed as an active field of research with a wide range of theoretical and practical research and application opportunities.

Keywords Evolutionary optimization · Multi-objective optimization · Evolutionary multi-objective optimization · EMO

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12.1 Introduction

Since the middle of Nineties, evolutionary multi-objective optimization (EMO) has become a popular and useful field of research and application. In a recent survey announced during the World Congress on Computational Intelligence (WCCI) in Vancouver 2006, EMO has been judged as one of the three fastest growing fields of research and application among all computational intelligence topics. Evolutionary optimization (EO) algorithms use a population-based approach in which more than one solution participates in an iteration and evolves a new population of solutions in each iteration. The reasons for their popularity are many. Some of them are: (i) EOs do not require any derivative information, (ii) EOs are relatively simple to implement, and (iii) EOs are flexible and have a widespread applicability. For solving single-objective optimization problems or in other tasks focusing on finding a single optimal solution, the use of a population of solutions in each iteration may at first seem like an overkill but they help provide an implicit parallel search ability, thereby making EOs computationally efficient [48, 53], in solving multi-objective optimization problems an EO procedure is a perfect match [19].

Multi-objective optimization problems, by nature, give rise to a set of Pareto-optimal solutions which need further processing to arrive at a single preferred solution. To achieve the first task, it becomes quite a natural proposition to use an EO, because the use of a population in an iteration helps an EO to simultaneously find multiple nondominated solutions, which portrays a trade-off among objectives, in a single run of the algorithm.

In this chapter, we begin with a brief description of the principles of an EMO in solving multi-objective optimization problems and then illustrate its working through a specific EMO procedure, which has been popularly and extensively used over the past 5–6 years. Besides this specific algorithm, there exist a number of other equally efficient EMO algorithms which we do not describe here for brevity. Instead, in this chapter, we discuss a number of recent advancements of EMO research and application which are driving the researchers and practitioners ahead. Fortunately, researchers have utilized the EMO's principle of solving multi-objective optimization problems in handling various other problem-solving tasks. The diversity of EMO's research is bringing researchers and practitioners together with different backgrounds including computer scientists, mathematicians, economists, and engineers. The topics we discuss here amply demonstrate why and how EMO researchers from different backgrounds must and should collaborate in solving complex problem-solving tasks which have become the need of the hour in most branches of science, engineering, and commerce.

12.2 Evolutionary Multi-objective Optimization (EMO)

A multi-objective optimization problem involves a number of objective functions which are to be either minimized or maximized subject to a number of constraints and variable bounds:

$$\left.\begin{array}{lll} \text{Minimize/Maximize } f_m(\mathbf{x}), & & m = 1, 2, \ldots, M; \\ \text{subject to } g_j(\mathbf{x}) \geq 0, & & j = 1, 2, \ldots, J; \\ h_k(\mathbf{x}) = 0, & & k = 1, 2, \ldots, K; \\ x_i^{(L)} \leq x_i \leq x_i^{(U)}, & & i = 1, 2, \ldots, n. \end{array}\right\} \quad (12.1)$$

A solution $\mathbf{x} \in \mathbf{R}^n$ is a vector of n decision variables: $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. The solutions satisfying the constraints and variable bounds constitute a S in the decision variable space $\mathbf{R}^n$. One of the striking differences between single-objective and multi-objective optimization is that in multi-objective optimization the objective function vectors belong to a multidimensional objective space $\mathbf{R}^M$. The objective function vectors constitute a feasible set Z in the objective space. For each solution $\mathbf{x}$ in S, there exists a point $\mathbf{z} \in Z$, denoted by $\mathbf{f}(\mathbf{x}) = \mathbf{z} = (z_1, z_2, \ldots, z_M)^T$. To make the descriptions clear, we refer a decision variable vector as a solution and the corresponding objective vector as a point.

The optimal solutions in multi-objective optimization can be defined from a mathematical concept of *partial ordering*. In the parlance of multi-objective optimization, the term *domination* is used for this purpose. In this section, we restrict ourselves to discuss unconstrained (without any equality, inequality, or bound constraints) optimization problems. The domination between two solutions is defined as follows [19, 72]:

Definition 12.1. A solution $\mathbf{x}^{(1)}$ is said to dominate the another solution $\mathbf{x}^{(2)}$, if both the following conditions are true:

1. The solution $\mathbf{x}^{(1)}$ is no worse than $\mathbf{x}^{(2)}$ in all objectives. Thus, the solutions are compared based on their objective function values (or location of the corresponding points ($\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$) in the objective function set Z).
2. The solution $\mathbf{x}^{(1)}$ is strictly better than $\mathbf{x}^{(2)}$ in at least one objective.

For a given set of solutions (or corresponding points in the objective function set Z, for example, those shown in Fig. 12.1a), a pair-wise comparison can be made using the above definition and whether one point dominates another point can also be established.

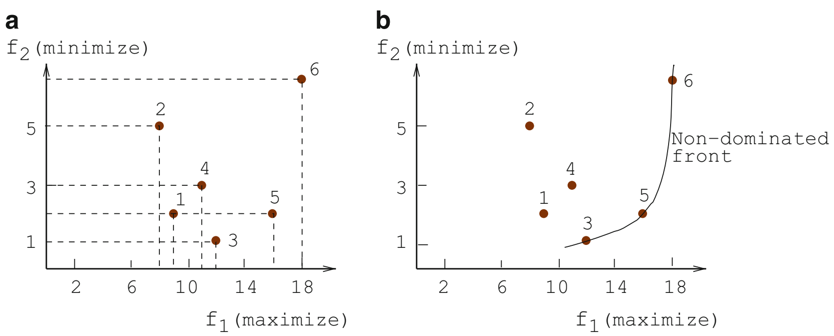

Fig. 12.1 A set of points and the first non-dominated front are shown

All points which are not dominated by any other member of the set are called the nondominated points of class one, or simply the nondominated points. For the set of six points shown in the figure, they are points 3, 5, and 6. One property of any two such points is that a gain in an objective from one point to the other happens only due to a sacrifice in at least one other objective. This *trade-off* property between the non-dominated points makes the practitioners interested in finding a wide variety of them before making a final choice. These points make up a front when viewed together on the objective space; hence the non-dominated points are often visualized to represent a *non-dominated front*. The theoretical computational effort needed to select the points of the non-dominated front from a set of N points is $O(N \log N)$ for two and three objectives, and $O(N \log^{M-2} N)$ for $M > 3$ objectives [65], but for a moderate number of objectives, the procedure need not be particularly computationally effective in practice.

With the above concept, it is now easier to define the *Pareto-optimal solutions* in a multi-objective optimization problem. If the given set of points for the above task contain all points in the decision variable space, the points lying on the non-domination front, by definition, do not get dominated by any other point in the objective space; hence are Pareto-optimal points (together they constitute the Pareto-optimal front) and the corresponding pre-images (decision variable vectors) are called Pareto-optimal solutions. However, more mathematically elegant definitions of Pareto-optimality (including the ones for continuous search space problems) exist in the multi-objective optimization literature [55, 72].

12.2.1 EMO Principles

In the context of multi-objective optimization, the extremist principle of finding the optimum solution cannot be applied to one objective alone, when the rest of the objectives are also important. This clearly suggests two ideal goals of multi-objective optimization:

Convergence: Find a (finite) set of solutions which lie on the Pareto-optimal front, and
Diversity: Find a set of solutions which are diverse enough to represent the entire range of the Pareto-optimal front.

EMO algorithms attempt to follow both the above principles, similar to a posteriori MCDM method. Figure 12.2 shows schematically the principles followed in an EMO procedure.

Since EMO procedures are heuristic based, they may not guarantee finding the exact Pareto-optimal points, as a theoretically provable optimization method would do for tractable (e.g., linear or convex) problems. But EMO procedures have essential operators to constantly improve the evolving nondominated points (from the point of view of convergence and diversity mentioned above) similar to the way most natural and artificial evolving systems continuously improve their solutions.

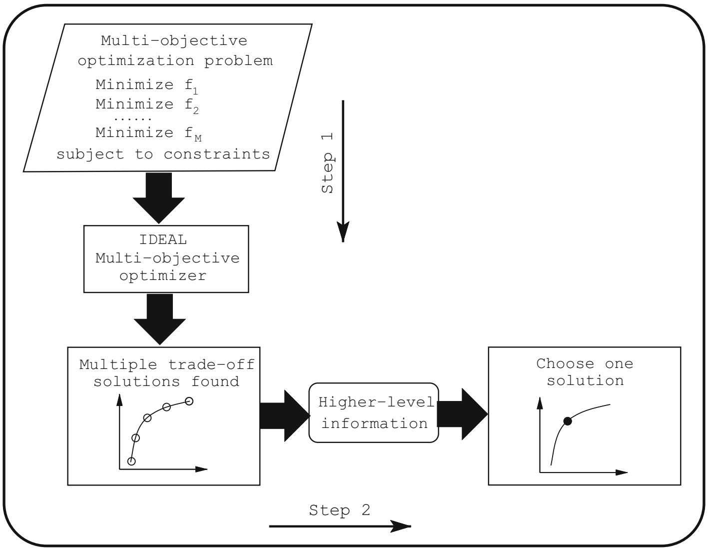

Fig. 12.2 Schematic of a two-step multi-criteria optimization and decision-making procedure

To this effect, a recent study [32] has demonstrated that a particular EMO procedure, starting from random non-optimal solutions, can progress towards the theoretical Karush-Kuhn-Tucker (KKT) points with iterations in real-valued multi-objective optimization problems. The main difference and advantage of using an EMO compared to a posteriori MCDM procedures is that multiple trade-off solutions can be found in a single run of an EMO algorithm, whereas most a posteriori MCDM methodologies would require multiple independent runs.

In Step 1 of the EMO-based multi-objective optimization and decision-making procedure (the task shown vertically downwards in Fig. 12.2), multiple trade-off, nondominated points are found. Thereafter, in Step 2 (the task shown horizontally, towards the right), higher-level information is used to choose one of the obtained trade-off points.

12.2.2 A Posteriori MCDM Methods and EMO

In the "a posteriori" MCDM approaches (also known as "generating MCDM methods"), the task of finding multiple Pareto-optimal solutions is achieved by executing many independent single-objective optimizations, each time finding a single Pareto-optimal solution [72]. A parametric scalarizing approach (such as the weighted-sum approach, ϵ-constraint approach, and others) can be used to convert multiple objectives into a parametric single-objective function. By simply varying the parameters (weight vector or ϵ-vector) and optimizing the scalarized function,

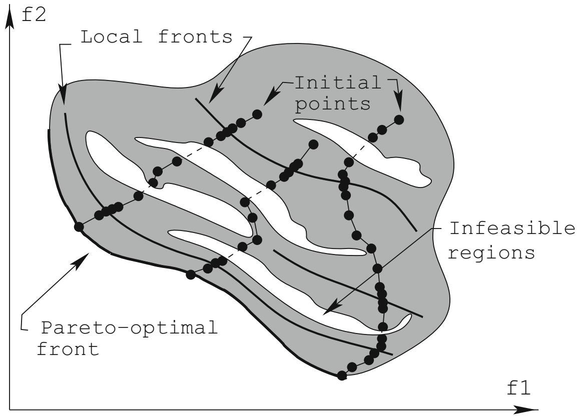

Fig. 12.3 A posteriori MCDM methodology employing independent single-objective optimizations

different Pareto-optimal solutions can be found. In contrast, in an EMO, multiple Pareto-optimal solutions are attempted to be found in a single run of the algorithm by emphasizing multiple non-dominated and isolated solutions in each iteration of the algorithm without the use of any scalarization of objectives.

Consider Fig. 12.3, in which we sketch how multiple independent parametric single-objective optimizations (through a posteriori MCDM method) may find different Pareto-optimal solutions.

It is worth highlighting here that the Pareto-optimal front corresponds to global optimal solutions of several problems each formed with a different scalarization of objectives. During the course of an optimization task, algorithms must overcome a number of difficulties, such as infeasible regions, local optimal solutions, flat or non-improving regions of objective landscapes, isolation of optimum, etc., to finally converge to the global optimal solution. Moreover, due to practical limitations, an optimization task must also be completed in a reasonable computational time. All these difficulties in a problem require that an optimization algorithm strikes a good balance between exploring new search directions and exploiting the extent of search in currently-best search direction. When multiple runs of an algorithm need to be performed independently to find a set of Pareto-optimal solutions, the above balancing act must be performed in every single run. Since runs are performed independently from one another, no information about the success or failure of previous runs is utilized to speed up the overall process. In difficult multi-objective optimization problems, such memory-less, a posteriori methods may demand a large overall computational overhead to find a set of Pareto-optimal solutions [85]. Moreover, despite the issue of global convergence, independent runs may not guarantee achieving a good distribution among obtained points by an easy variation of scalarization parameters.

EMO, as mentioned earlier, constitutes an inherent parallel search. When a particular population member overcomes certain difficulties and makes a progress towards the Pareto-optimal front, its variable values and their combination must reflect this fact. When a recombination takes place between this solution and another population member, such valuable information of variable value combinations gets shared through variable exchanges and blending, thereby making the overall task of finding multiple trade-off solutions a parallelly processed task.

12.3 A Brief History of EMO Methodologies

During the early years, EA researchers realized the need of solving multi-objective optimization problems in practice and mainly resorted to using weighted-sum approaches to convert multiple objectives into a single goal [40, 78].

However, the first implementation of a real multi-objective evolutionary algorithm (vector-evaluated GA or VEGA) was suggested by David Schaffer in the year 1984 [84]. Schaffer modified the simple three-operator genetic algorithm [53] (with selection, crossover, and mutation) by performing independent selection cycles according to each objective. The selection method is repeated for each individual objective to fill up a portion of the mating pool. Then the entire population is thoroughly shuffled to apply crossover and mutation operators. This is performed to achieve the mating of individuals of different subpopulation groups. The algorithm worked efficiently for some generations but in some cases suffered from its bias towards some individuals or regions (mostly individual objective champions). This does not fulfil the second goal of EMO, discussed earlier.

Ironically, no significant study was performed for almost a decade after the pioneering work of Schaffer, until a revolutionary 10-line sketch of a new non-dominated sorting procedure suggested by David E. Goldberg in his seminal book on GAs [48]. Since an EA needs a fitness function for reproduction, the trick was to find a single metric from a number of objective functions. Goldberg's suggestion was to use the concept of *domination* to assign more copies to non-dominated individuals in a population. Since diversity is the other concern, he also suggested the use of a *niching* strategy [49] among solutions of a non-dominated class. Getting this clue, at least three independent groups of researchers developed different versions of multi-objective evolutionary algorithms during 1993–1994 [43, 54, 87]. These algorithms differ in the way a fitness assignment scheme is introduced to each individual.

These EMO methodologies gave a good head-start to the research and application of EMO, but suffered from the fact that they did not use an elite-preservation mechanism in their procedures. Inclusion of elitism in an EO provides a monotonically non-degrading performance [79]. The second generation EMO algorithms implemented an elite-preserving operator in different ways and gave birth to elitist EMO procedures, such as NSGA-II [21], Strength Pareto EA (SPEA) [94], Pareto-archived ES (PAES) [60], and others. Since these EMO algorithms are state-of-the-art and commonly used procedures, we describe one of these algorithms in detail.

12.4 Elitist EMO: NSGA-II

The NSGA-II procedure [21] is one of the popularly used EMO procedures which attempt to find multiple Pareto-optimal solutions in a multi-objective optimization problem and has the following three features:

1. It uses an elitist principle
2. It uses an explicit diversity-preserving mechanism and
3. It emphasizes non-dominated solutions

At any generation t, the offspring population (say, Q_t) is first created by using the parent population (say, P_t) and the usual genetic operators. Thereafter, the two populations are combined together to form a new population (say, R_t) of size $2N$. Then, the population R_t is classified into different non-dominated classes. Thereafter, the new population is filled by points of different non-dominated fronts, one at a time. The filling starts with the first non-dominated front (of class one) and continues with points of the second non-dominated front, and so on. Since the overall population size of R_t is $2N$, not all fronts can be accommodated in N slots available for the new population. All fronts which could not be accommodated are deleted. When the last allowed front is being considered, there may exist more points in the front than the remaining slots in the new population. This scenario is illustrated in Fig. 12.4. Instead of arbitrarily discarding some members from the last front, the points which will make the diversity of the selected points the highest are chosen.

The crowded-sorting of the points of the last front which could not be accommodated fully is achieved in the descending order of their *crowding distance values* and

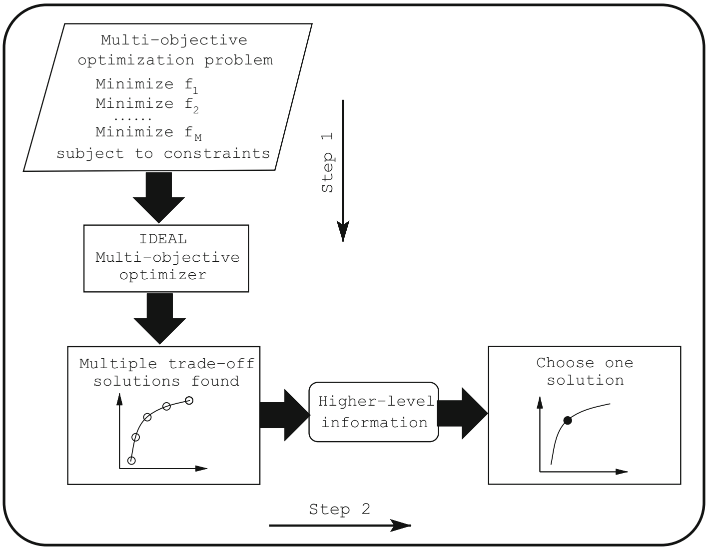

Fig. 12.4 Schematic of the NSGA-II procedure

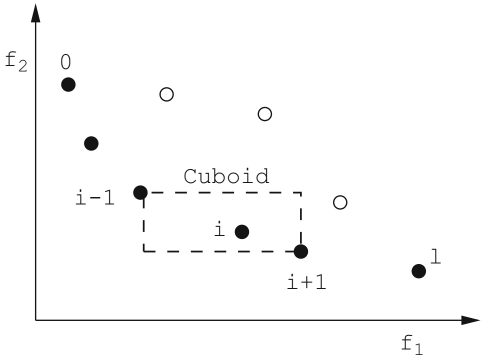

Fig. 12.5 The crowding distance calculation

points from the top of the ordered list are chosen. The crowding distance d_i of point i is a measure of the objective space around i which is not occupied by any other solution in the population. Here, we simply calculate this quantity d_i by estimating the perimeter of the cuboid (Fig. 12.5) formed by using the nearest neighbors in the objective space as the vertices (we call this the *crowding distance*).

12.4.1 Sample Results

Here, we show results from several runs of the NSGA-II algorithm on two test problems. The first problem (ZDT2) is two-objective, 30-variable problem with a concave Pareto-optimal front:

$$\text{ZDT2}: \begin{cases} \text{Minimize} & f_1(\mathbf{x}) = x_1, \\ \text{Minimize} & f_2(\mathbf{x}) = s(\mathbf{x})\left[1 - (f_1(\mathbf{x})/s(\mathbf{x}))^2\right], \\ \text{where} & s(\mathbf{x}) = 1 + \frac{9}{29}\sum_{i=2}^{30} x_i \\ & 0 \le x_1 \le 1, \\ & -1 \le x_i \le 1, \quad i = 2, 3, \ldots, 30. \end{cases} \tag{12.2}$$

The second problem (KUR), with three variables, has a disconnected Pareto-optimal front:

$$\text{KUR}: \begin{cases} \text{Minimize} & f_1(\mathbf{x}) = \sum_{i=1}^{2}\left[-10\exp\left(-0.2\sqrt{x_i^2 + x_{i+1}^2}\right)\right], \\ \text{Minimize} & f_2(\mathbf{x}) = \sum_{i=1}^{3}\left[|x_i|^{0.8} + 5\sin\left(x_i^3\right)\right], \\ & -5 \le x_i \le 5, \quad i = 1, 2, 3. \end{cases} \tag{12.3}$$

NSGA-II is run with a population size of 100 for 250 generations. The variables are used as real numbers and an SBX recombination operator [20] with $p_c = 0.9$, distribution index of $\eta_c = 10$, a polynomial mutation operator [19] with $p_m = 1/n$ (n is the number of variables), and distribution index of $\eta_m = 20$ are used.

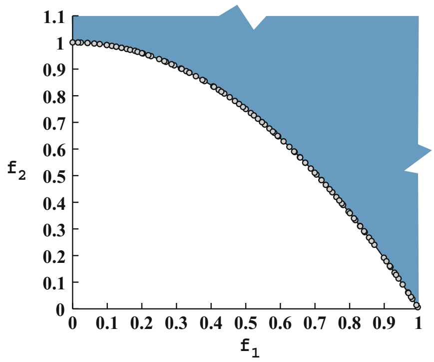

Fig. 12.6 NSGA-II on ZDT2

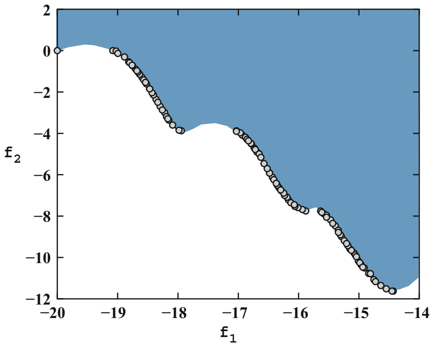

Fig. 12.7 NSGA-II on KUR

Figures 12.6 and 12.7 show that NSGA-II converges to the Pareto-optimal front and maintains a good spread of solutions on both test problems.

There also exist other competent EMOs, such as strength Pareto evolutionary algorithm (SPEA) and its improved version SPEA2 [93], Pareto-archived evolution strategy (PAES) and its improved versions PESA and PESA2 [16], multi-objective messy GA (MOMGA) [89], multi-objective-GA [12], neighbourhood constraint GA [69], ARMOGA [80], and others. Besides, there exists other EA-based methodologies, such as particle swarm EMO [19,73], ant-based EMO [50,71], and differential evolution-based EMO [1].

12.4.2 Constraint Handling in EMO

The constraint handling method modifies the binary tournament selection, where two solutions are picked from the population and the better solution is chosen. In the presence of constraints, each solution can be either feasible or infeasible. Thus, there may be at most three situations: (i) both solutions are feasible, (ii) one is feasible and other is not, and (iii) both are infeasible. We consider each case by simply redefining the domination principle as follows (we call it the *constrained-domination* condition for any two solutions $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$):

Definition 12.2. A solution $\mathbf{x}^{(i)}$ is said to 'constrained-dominate' a solution $\mathbf{x}^{(j)}$ (or $\mathbf{x}^{(i)} \preceq_c \mathbf{x}^{(j)}$), if any of the following conditions are true:

1. Solution $\mathbf{x}^{(i)}$ is feasible and solution $\mathbf{x}^{(j)}$ is not.
2. Solutions $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ are both infeasible, but solution $\mathbf{x}^{(i)}$ has a smaller constraint violation, which can be computed by adding the normalized violation of all constraints:

$$\mathrm{CV}(\mathbf{x}) = \sum_{j=1}^{J} \max\left(0, -\bar{g}_j(\mathbf{x})\right) + \sum_{k=1}^{K} \mathrm{abs}(\bar{h}_k(\mathbf{x})).$$

   The normalization of a constraint $g_j(\mathbf{x}) \geq g_{j,r}$ can be achieved as $\bar{g}_j(\mathbf{x}) \geq 0$, where $\bar{g}_j(\mathbf{x}) = g_j(\mathbf{x})/g_{j,r} - 1$.
3. Solutions $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ are feasible and solution $\mathbf{x}^{(i)}$ dominates solution $\mathbf{x}^{(j)}$ in the usual sense (Definition 12.1).

The above change in the definition requires a minimal change in the NSGA-II procedure described earlier. Figure 12.8 shows the nondominated fronts on a six-member population due to the introduction of two constraints (the minimization problem is described as CONSTR elsewhere [19]). In the absence of the constraints, the nondominated fronts (shown by dashed lines) would have been `((1,3,5), (2,6), (4))`, but in their presence, the new fronts are `((4,5), (6), (2), (1), (3))`.

The first nondominated front consists of the "best" (i.e., nondominated and feasible) points from the population and any feasible point lies on a better nondominated front than an infeasible point.

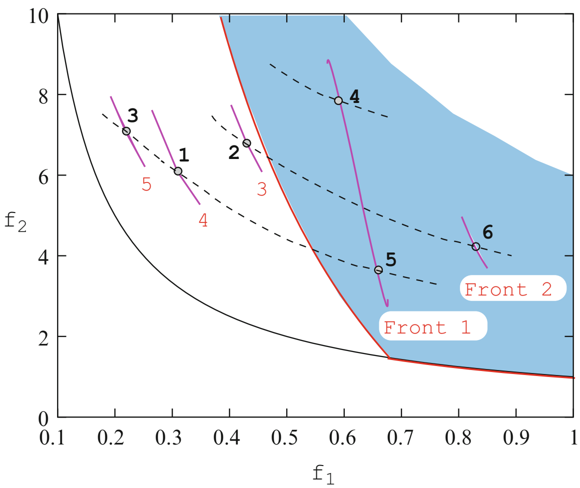

Fig. 12.8 Non-constrained-domination fronts

12.5 Applications of EMO

Since the early development of EMO algorithms in 1993, they have been applied to many challenging real-world optimization problems. Descriptions of some of these studies can be found in books [13, 19, 77], dedicated conference proceedings [15, 41, 76, 91], and domain-specific books, journals and proceedings. In this section, we describe one case study which clearly demonstrates the EMO philosophy which we described in Section 12.2.1.

12.5.1 Spacecraft Trajectory Design

Coverstone-Carroll et al. [17] proposed a multi-objective optimization technique using the original non-dominated sorting algorithm (NSGA) [87] to find multiple trade-off solutions in a spacecraft trajectory optimization problem. To evaluate a solution (trajectory), the SEPTOP (Solar Electric Propulsion Trajectory Optimization) software [81] is called, and the delivered payload mass and the total time of flight are calculated. The multi-objective optimization problem has eight decision variables controlling the trajectory, three objective functions: (i) maximize the delivered payload at destination, (ii) maximize the negative of the time of flight, and (iii) maximize the total number of heliocentric revolutions in the trajectory, and three constraints limiting the SEPTOP convergence error and minimum and maximum bounds on heliocentric revolutions.

On the Earth–Mars rendezvous mission, the study found interesting trade-off solutions [17]. Using a population of size 150, the NSGA was run for 30 generations. The obtained nondominated solutions are shown in Fig. 12.9 for two of the three objectives and some selected solutions are shown in Fig. 12.10.

It is clear that there exist short-time flights with smaller delivered payloads (solution marked 44 with 1.12 years of flight and delivering 685.28 kg load) and long-time flights with larger delivered payloads (solution marked 36 with close to 3.5 years of flight and delivering about 900 kg load).

While solution 44 can deliver a mass of 685.28 kg and requires about 1.12 years, solution 72 can deliver almost 862 kg with a travel time of about 3 years. In these figures, each continuous part of a trajectory represents a *thrusting* arc and each dashed part of a trajectory represents a *coasting* arc. It is interesting to note that only a small improvement in delivered mass occurs in the solutions between 73 and 72 with a sacrifice in flight time of about 1 year.

The multiplicity in trade-off solutions, as depicted in Fig. 12.10, is what we envisaged in discovering in a multi-objective optimization problem by using a posteriori procedure, such as a generating method or using an EMO procedure vis-a-vis a priori approach in which a single scalarized problem is solved with a single preferred parameter setting to find a single Pareto-optimal solution. This aspect was also discussed in Fig. 12.2. Once a set of solutions with a good trade-off among objectives is obtained, one can analyze them for choosing a particular solution. For example,

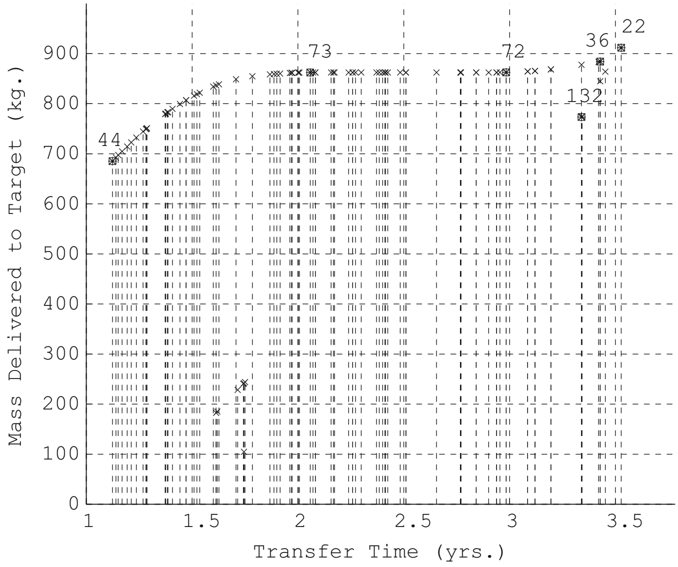

Fig. 12.9 Obtained nondominated solutions using NSGA

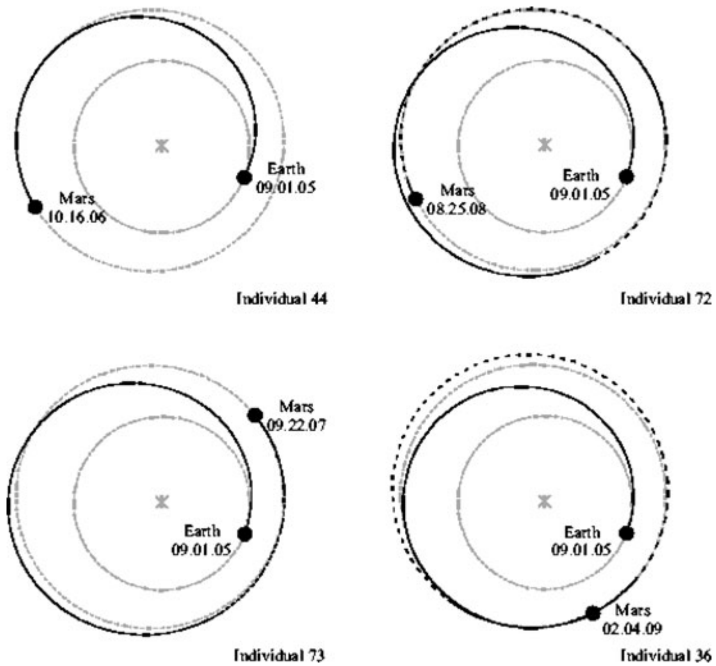

Fig. 12.10 Four trade-off trajectories

in this problem context, it makes sense to not choose a solution between points 73 and 72 due to poor trade-off between the objectives in this range, a matter which is only revealed after a representative set of trade-off solutions are found. On the other hand, choosing a solution within points 44 and 73 is worthwhile, but which particular solution to choose depends on other mission-related issues. But by first finding a wide range of possible solutions and revealing the shape of front in a computationally quicker manner, EMO can help a decision maker in narrowing down the choices and in allowing to make a better decision (e.g., in the above example, focussing to choose a solution with a transfer time less than 2 years). Without the knowledge of such a wide variety of trade-off solutions, proper decision making may be a difficult task. The use of a priori approach to find a single solution using for example, the ϵ-constraint method with a particular ϵ vector, the decision maker will always wonder what solution would have been derived if a different ϵ vector was chosen. For example, if $\epsilon_1 = 2.5$ years is chosen and mass delivered to the target is maximized, a solution in between points 73 and 72 will be found. As discussed earlier, this part of the Pareto-optimal front does not provide the best trade-offs between objectives that this problem can offer. A lack of knowledge of good trade-off regions before a decision is made may allow the decision maker to settle for a solution which, although optimal, may not be a good compromised solution. The EMO procedure allows a flexible and a pragmatic procedure for finding a well-diversified set of solutions simultaneously so as to enable picking a particular region for further analysis or a particular solution for implementation.

12.6 Salient Recent Developments of EMO

An interesting aspect regarding research and application of EMO is that soon after a number of efficient EMO methodologies had been suggested and applied in various interesting problem areas, researchers did not waste any time to look for opportunities to make the field broader and more useful by diversifying EMO applications to various other problem-solving tasks. In this section, we describe a number of such salient recent developments of EMO.

12.6.1 Hybrid EMO Algorithms

Search operators used in EMO are heuristic-based. Thus, these methodologies are not guaranteed to find Pareto-optimal solutions with a finite number of solution evaluations in an arbitrary problem. In single-objective EA research, hybridization of EAs is common for ensuring convergence to an optimal solution, it is not surprising that studies on developing hybrid EMOs are now being pursued to ensure finding of true Pareto-optimal solutions by hybridizing them with mathematically convergent ideas.

EMO methodologies provide adequate emphasis to currently non-dominated and isolated solutions so that population members progress towards the Pareto-optimal front iteratively. To make the overall procedure faster and to perform the task with a more theoretical emphasis, EMO methodologies are combined with mathematical optimization techniques having local convergence properties. A simple-minded approach would be to start the process with an EMO and the solutions obtained from EMO can be improved by optimizing a composite objective derived from multiple objectives to ensure a good spread by using a local search technique [22]. Another approach would be to use a local search technique as a mutation-like operator in an EMO so that all population members are at least guaranteed to be local optimal solutions [22, 86]. To save computational time, instead of performing the local search for every solution in a generation, a mutation can be performed only after a few generations. Some recent studies [56, 82, 86] have demonstrated the usefulness of such hybrid EMOs for a guaranteed convergence.

Although these studies have concentrated on ensuring convergence to the Pareto-optimal front, some emphasis should now be placed in providing an adequate diversity among obtained solutions, particularly when a continuous Pareto-optimal front is represented by a finite set of points. Some ideas of maximizing hypervolume measure [39] or maintenance of uniform distance between points are proposed for this purpose, but how such diversity-maintenance techniques would be integrated with convergence-ensuring principles in a synergistic way would be interesting and useful future research. Some relevant studies in this direction exist [4, 56, 66].

12.6.2 Multi-objectivization

Interestingly, the act of finding multiple trade-off solutions using an EMO procedure has found its application outside the realm of solving multi-objective optimization problems. The concept of finding near-optimal trade-off solutions is applied to solve other kinds of optimization problems as well. For example, the EMO concept is used to solve constrained single-objective optimization problems by converting the task into a two-objective optimization task of additionally minimizing an aggregate constraint violation [14]. This eliminates the need to specify a penalty parameter while using a penalty based constraint handling procedure. If viewed this way, the usual penalty function approach used in classical optimization studies is a special weighted-sum approach to the bi-objective optimization problem of minimizing the objective function and minimizing the constraint violation, for which the weight vector is a function of the penalty parameter. A well-known difficulty in genetic programming studies, called *bloating*, arises due to the continual increase in the size of evolved “genetic programs” with iteration. The reduction of bloating by minimizing the size of a program as an additional objective helped find high-performing solutions with a smaller size of the code [3, 57]. In clustering algorithms, minimizing the intra-cluster distance and maximizing inter-cluster distance simultaneously in a bi-objective formulation of a is found to yield better solutions than the usual single-objective minimization of the ratio of the intra-cluster distance to the inter-cluster distance [51]. An EMO is used to solve minimum spanning tree problem better than a single-objective EA [75]. A recent edited book [62] describes many such interesting applications in which EMO methodologies have helped solve problems which are otherwise (or traditionally) not treated as multi-objective optimization problems.

12.6.3 Uncertainty-based EMO

A major surge in EMO research has taken place in handling uncertainties among decision variables and problem parameters in multi-objective optimization. Practice is full of uncertainties and almost no parameter, dimension, or property can be guaranteed to be fixed at a value it is aimed at. In such scenarios, evaluation of a solution is not precise, and the resulting objective and constraint function values becomes probabilistic quantities. Optimization algorithms are usually designed to handle such stochasticities by using crude methods, such as the Monte Carlo simulation of stochasticities in uncertain variables and parameters and by sophisticated stochastic programming methods involving nested optimization techniques [24]. When these effects are taken care of during the optimization process, the resulting solution is usually different from the optimum solution of the problem and is known as a "robust" solution. Such an optimization procedure will then find a solution which may not be the true global optimum solution, but one which is less sensitive to uncertainties in decision variables and problem parameters. In the context of multi-objective optimization, a consideration of uncertainties for multiple objective functions will result in a robust frontier which may be different from the globally Pareto-optimal front. Each and every point on the robust frontier is then guaranteed to be less sensitive to uncertainties in decision variables and problem parameters. Some such studies in EMO are [2, 23].

When the evaluation of constraints under uncertainties in decision variables and problem parameters are considered, deterministic constraints become stochastic (they are also known as "chance constraints") and involves a *reliability index* (R) to handle the constraints. A constraint $g(\mathbf{x}) \geq 0$ then becomes $\text{Prob}(g(\mathbf{x}) \geq 0) \geq R$. In order to find left side of the above chance constraint, a separate optimization methodology [18], is needed, thereby making the overall algorithm a bi-level optimization procedure. Approximate single-loop algorithms exist [34] and recently one such methodology has been integrated with an EMO [24] and shown to find a "reliable" frontier corresponding a specified reliability index, instead of the Pareto-optimal frontier, in problems having uncertainty in decision variables and problem parameters. More such methodologies are needed, as uncertainties is an integral part of practical problem-solving and multi-objective optimization researchers must look for better and faster algorithms to handle them.

12.6.4 EMO and Decision Making

Searching for a set of Pareto-optimal solutions by using an EMO fulfils only one aspect of multi-objective optimization, as choosing a particular solution for an implementation is the remaining decision-making task which is equally important. For many years, EMO researchers have postponed the decision-making aspect and concentrated on developing efficient algorithms for finding multiple trade-off solutions. Having pursued that part somewhat, now for the past couple of years or so, EMO researchers are putting efforts to design combined algorithms for optimization and decision making. In the view of the author, the decision-making task can be considered from two main considerations in an EMO framework:

1. **Generic consideration:** There are some aspects which most practical users would like to use in narrowing down their choice. We have discussed above the importance of finding robust and reliable solutions in the presence of uncertainties in decision variables and/or problem parameters. In such scenarios, an EMO methodology can straightway find a robust or a reliable frontier [23, 24] and no subjective preference from any decision maker may be necessary. Similarly, if a problem resorts to a Pareto-optimal front having *knee* points, such points are often the choice of decision makers. Knee points demand a large sacrifice in at least one objective to achieve a small gain in another thereby making it discouraging to move out from a knee point [7]. Other such generic choices are related to Pareto-optimal points depicting certain pre-specified relationship between objectives, Pareto-optimal points having multiplicity (say, at least two or more solutions in the decision variable space mapping to identical objective values), Pareto-optimal solutions which do not lie close to variable boundaries, Pareto-optimal points having certain mathematical properties, such as all Lagrange multipliers having more or less identical magnitude – a condition often desired to make an equal importance to all constraints, and others. These considerations are motivated from the fundamental and practical aspects of optimization and may be applied to most multi-objective problem-solving tasks, without any consent of a decision maker. These considerations may narrow down the set of non-dominated points. A further subjective consideration (discussed below) may then be used to pick a preferred solution.
2. **Subjective consideration:** In this category, any problem-specific information can be used to narrow down the choices and the process may even lead to a single preferred solution at the end. Most decision-making procedures use some preference information (utility functions, reference points [90], reference directions [63], marginal rate of return, and a host of other considerations [72]) to select a subset of Pareto-optimal solutions. A recent book [8] is dedicated to the discussion of many such multi-criteria decision analysis (MCDA) tools and collaborative suggestions of using EMO with such MCDA tools. Some hybrid EMO and MCDA algorithms are suggested in the recent past [25, 26, 31, 70, 88].

Many other generic and subjective considerations are needed and it is interesting that EMO and MCDM researchers are collaborating on developing such complete algorithms for multi-objective optimization [8].

12.6.5 EMO for Handling a Large Number of Objectives

Soon after the development of efficient EMO methodologies, researchers were interested in exploring whether existing EMO methodologies are adequate to handle a large number of objectives (say, ten or more). An earlier study [58] with eight objectives revealed somewhat negative results. But the author in his book [19] and recent other studies [59] have clearly explained the reason for this behavior of EMO algorithms. EMO methodologies work by emphasizing non-dominated solutions in a population. Unfortunately, as the number of objectives increase, most population members in a randomly created population tend to become non-dominated to each other. For example, in a three-objective scenario, about 10% members in a population of size 200 are nondominated, whereas in a 10-objective problem scenario, as high as 90% members in a population of size 200 are nondominated. Thus, in a large-objective problem, an EMO algorithm runs out of room to introduce new population members into a generation, thereby causing a stagnation in the performance of an EMO algorithm. It has been argued that to make EMO procedures efficient, an exponentially large population size (with respect to number of objectives) is needed. This makes an EMO procedure slow and computationally less attractive.

However, practically speaking, even if an algorithm can find tens of thousands of Pareto-optimal solutions for a multi-objective optimization problem, besides simply getting an idea of the nature and shape of the front, they are simply too many to be useful for any decision-making purposes. Keeping these views in mind, EMO researchers have taken two different approaches in dealing with large-objective problems.

12.6.5.1 Finding a Partial Set

Instead of finding the complete Pareto-optimal front in a problem having a large number of objectives, EMO procedures can be used to find only a part of the Pareto-optimal front. This can be achieved by indicating preference information by various means. Ideas, such as reference point-based EMO [31,70], "light beam search" [26], biased sharing approaches [6], cone dominance [33], etc. are suggested for this purpose. Each of these studies have shown that up to 10, and 20-objective problems, although finding the complete frontier is a difficulty, finding a partial frontier corresponding to certain preference information is not that difficult a proposition. Despite the dimension of the partial frontier being identical to that of the complete Pareto-optimal frontier, the closeness of target points in representing the desired partial frontier helps make only a small fraction of an EMO population to be nondominated, thereby making rooms for new and hopefully better solutions to be found and stored.

The computational efficiency and accuracy observed in some EMO implementations have led a distributed EMO study [33] in which each processor in a distributed computing environment receives a unique cone for defining domination. The cones are designed carefully so that at the end of such a distributed computing EMO procedure, solutions are found to exist in various parts of the complete Pareto-optimal front. A collection of these solutions together is then able to provide a good representation of the entire original Pareto-optimal front.

12.6.5.2 Identifying and Eliminating Redundant Objectives

Many practical optimization problems can easily list a large of number of objectives (often more than ten), as many different criteria or goals are often of interest to practitioners. In most instances, it is not entirely sure whether the chosen objectives are all in conflict to each other or not. For example, minimization of weight and minimization of cost of a component or a system are often mistaken to have an identical optimal solution, but may lead to a range of trade-off optimal solutions. Practitioners do not take any chance and tend to include all (or as many as possible) objectives into the optimization problem formulation. There is another fact which is more worrisome. Two apparently conflicting objectives may show a good trade-off when evaluated with respect to some randomly created solutions. But if these two objectives are evaluated for solutions close to their optima, they tend to show a good correlation. That is, although objectives can exhibit conflicting behavior for random solutions, near their Pareto-optimal front, the conflict vanishes and optimum of one becomes close to the optimum of the other.

Thinking of the existence of such problems in practice, recent studies [29, 83] have performed linear and non-linear principal component analysis (PCA) to a set of EMO-produced solutions. Objectives causing positively correlated relationship between each other on the obtained NSGA-II solutions are identified and are declared as redundant. The EMO procedure is then restarted with non-redundant objectives. This combined EMO-PCA procedure is continued until no further reduction in the number of objectives is possible. The procedure has handled practical problems involving five and more objectives and has shown to reduce the choice of real conflicting objectives to a few. On test problems, the proposed approach has shown to reduce an initial 50-objective problem to the correct three-objective Pareto-optimal front by eliminating 47 redundant objectives. Another study [9] used an exact and a heuristic-based conflict identification approach on a given set of Pareto-optimal solutions. For a given error measure, an effort is made to identify a minimal subset of objectives which do not alter the original dominance structure on a set of Pareto-optimal solutions. This idea has recently been introduced within an EMO [10], but a continual reduction of objectives through a successive application of the above procedure would be interesting.

This is a promising area of EMO research and definitely more computationally faster objective-reduction techniques are needed for the purpose. In this direction, the use of alternative definitions of domination is important. One such idea redefined the definition of domination: a solution is said to dominate another solution, if the former solution is better than latter in more objectives. This certainly excludes finding the entire Pareto-optimal front and helps an EMO to converge near the intermediate and central part of the Pareto-optimal front. Another EMO study used a fuzzy dominance [38] relation (instead of Pareto-dominance), in which superiority of one solution over another in any objective is defined in a fuzzy manner. Many other such definitions are possible and can be implemented based on the problem context.

12.6.6 Knowledge Extraction Through EMO

One striking difference between a single-objective optimization and multi-objective optimization is the cardinality of the solution set. In the latter, multiple solutions are the outcome and each solution is theoretically an optimal solution corresponding to a particular trade-off among the objectives. Thus, if an EMO procedure can find solutions close to the true Pareto-optimal set, what we have in our hand are a number of high-performing solutions trading-off the conflicting objectives considered in the study. Since they are all near-optimal, these solutions can be analyzed for finding properties which are common to them. Such a procedure can then become a systematic approach in deciphering important and hidden properties which optimal and high-performing solutions must have for that problem. In a number of practical problem-solving tasks, the so-called *innovization* procedure is shown to find important knowledge about high-performing solutions [30]. Such useful properties are expected to exist in practical problems, as they follow certain scientific and engineering principles at the core, but finding them through a systematic scientific procedure had not been paid much attention in the past. The principle of first searching for multiple trade-off and high-performing solutions using a multi-objective optimization procedure and then analyzing them to discover useful knowledge certainly remains a viable way forward. The current efforts to automate the knowledge extraction procedure through a sophisticated data-mining task should make the overall approach more appealing and useful in practice.

12.6.7 Dynamic EMO

Dynamic optimization involves objectives, constraints, or problem parameters which change over time. This means that as an algorithm is approaching the optimum of the current problem, the problem definition has changed and now the algorithm must solve a new problem. This is not equivalent to another optimization task in which a new and different optimization problem must be solved afresh.

Often, in such dynamic optimization problems, an algorithm is usually not expected to find the optimum, instead it is best expected to track the changing optimum with iteration. The performance of a dynamic optimizer then depends on how close it is able to track the true optimum (which is changing with iteration or time). Thus, practically speaking, optimization algorithms may hope to handle problems which do not change significantly with time. From the algorithm's point of view, since in these problems the problem is not expected to change too much from one time instance to another and some good solutions to the current problem are already at hand in a population, researchers fancied solving such dynamic optimization problems using evolutionary algorithms [5].

A recent study [28] proposed the following procedure for dynamic optimization involving single or multiple objectives. Let $\mathcal{P}(t)$ be a problem which changes with time t (from $t = 0$ to $t = T$). Despite the continual change in the problem, we assume that the problem is fixed for a time period τ, which is not known a priori and the aim of the (offline) dynamic optimization study is to identify a suitable value of τ for an accurate as well computationally faster approach. For this purpose, an optimization algorithm with τ as a fixed time period is run from $t = 0$ to $t = T$ with the problem assumed fixed for every τ time period. A measure $\Gamma(\tau)$ determines the performance of the algorithm and is compared with a pre-specified and expected value Γ_L. If $\Gamma(\tau) \geq \Gamma_L$, for the entire time domain of the execution of the procedure, we declare τ to be a permissible length of stasis. Then, we try with a reduced value of τ and check if a smaller length of statis is also acceptable. If not, we increase τ to allow the optimization problem to remain static for a longer time so that the chosen algorithm can now have more iterations (time) to perform better. Such a procedure will eventually come up with a time period τ^* which would be the smallest time of statis allowed for the optimization algorithm to work based on chosen performance requirement. Based on this study, a number of test problems and a hydrothermal power dispatch problem have been recently tackled [28].

In the case of dynamic multi-objective problem-solving tasks, there is an additional difficulty which is worth mentioning here. Not only does an EMO algorithm needs to find or track the changing Pareto-optimal fronts, in a real-world implementation, it must also make an immediate decision about which solution to implement from the current front before the problem changes to a new one. Decision-making analysis is considered to be time-consuming involving execution of analysis tools, higher-level considerations, and sometimes group discussions. If dynamic EMO is to be applied in practice, *automated* procedures for making decisions must be developed. Although it is not clear how to generalize such an automated decision-making procedure in different problems, problem-specific tools are certainly possible and certainly a worthwhile and fertile area for research.

12.6.8 Quality Estimates for EMO

When algorithms are developed and test problems with known Pareto-optimal fronts are available, an important task is to have performance measures with which the

EMO algorithms can be evaluated. Thus, a major focus of EMO research has been spent to develop different performance measures. Since the focus in an EMO task is multifaceted – convergence to the Pareto-optimal front and diversity of solutions along the entire front – it is also expected that one performance measure to evaluate EMO algorithms will be unsatisfactory. In the early years of EMO research, three different sets of performance measures were used:

1. Metrics evaluating convergence to the known Pareto-optimal front (such as error ratio, distance from reference set, etc.)
2. Metrics evaluating spread of solutions on the known Pareto-optimal front (such as spread, spacing, etc.) and
3. Metrics evaluating certain combinations of convergence and spread of solutions (such as hypervolume, coverage, R-metric, etc.)

Some of these metrics are described in texts [13, 19]. A detailed study [61] comparing most existing performance metrics based on out-performance relations has recommended the use of the S-metric (or the hypervolume metric) and R-metric suggested by [52]. A recent study has argued that a single unary performance measure or any finite combination of them (e.g., any of the first two metrics described above in the enumerated list or both together) cannot adequately determine whether one set is better than another [95]. That study also concluded that binary performance metrics (indicating usually two different values when a set of solutions A is compared against B and B is compared against A), such as epsilon indicator, binary hypervolume indicator, utility indicators R1 to R3, etc., are better measures for multi-objective optimization. The flip side is that the chosen binary metric must be computed $K(K-1)$ times when comparing K different sets to make a fair comparison, thereby making the use of binary metrics computationally expensive in practice. Importantly, these performance measures have allowed researchers to use them directly as fitness measures within indicator-based EAs (IBEAs) [92]. In addition, of [42, 44] provide further information about location and inter-dependencies among obtained solutions.

12.6.9 Exact EMO with Run-time Analysis

Since the suggestion of efficient EMO algorithms, they have been increasingly applied in a wide variety of problem domains to obtain trade-off frontiers. Simultaneously, some researchers have also devoted their efforts in developing exact EMO algorithms with a theoretical complexity estimate in solving certain discrete multi-objective optimization problems. The first such study [68] suggested a pseudo-Boolean multi-objective optimization problem – a two-objective LOTZ (Leading Ones Trailing Zeroes) – and a couple of EMO methodologies – a simple evolutionary multi-objective optimizer (SEMO) and an improved version fair evolutionary multi-objective optimizer (FEMO). The study then estimated the worst-case computational effort needed to find all Pareto-optimal solutions of the problem LOTZ. This study spurred a number of improved EMO algorithms with run-time estimates and resulted in many other interesting test problems [46, 47, 64, 67]. Although these test problems may not resemble common practical problems, the working principles of suggested EMO algorithms to handle specific problem structures bring in a plethora of insights about the working of multi-objective optimization, particularly in comprehensively finding all (not just one, or a few) Pareto-optimal solutions.

12.6.10 EMO with Meta-models

The practice of optimization algorithms is often limited by the computational overheads associated with evaluating solutions. Certain problems involving expensive computations, such as numerical solution of partial differential equations describing the physics of the problem, finite difference computations involving an analysis of a solution, computational fluid dynamics simulation to study the performance of a solution over a changing environment, etc. In some such problems, evaluation of each solution to compute constraints and objective functions may take a few hours to a complete day or two. In such scenarios, even if an optimization algorithm needs 100 solutions to get anywhere close to a good and feasible solution, the application needs an easy 3–6 months of continuous computational time. In most practical purposes, this is considered a "luxury" in an industrial set-up. Optimization researchers are constantly at their toes in coming up with approximate yet faster algorithms.

A little thought brings out an interesting fact about how optimization algorithms work. The initial iterations deal with solutions which may not be close to optimal solutions. Therefore, these solutions need not be evaluated with high precision. Meta-models for objective functions and constraints have been developed for this purpose. Two different approaches are mostly followed. In one approach, a sample of solutions are used to generate a meta-model (approximate model of the original objectives and constraints) and then efforts have been made to find the optimum of the meta-model, assuming that the optimal solutions of both the meta-model and the original problem are similar to each other [35, 45]. In the other approach, a successive meta-modelling approach is used in which the algorithm starts to solve the first meta-model obtained from a sample of the entire search space [27, 37, 74]. As the solutions start to focus near the optimum region of the meta-model, a new and more accurate meta-model is generated in the region dictated by the solutions of the previous optimization. A coarse-to-fine-grained meta-modelling technique based on artificial neural networks is shown to reduce the computational effort by about 30–80% on different problems [74]. Other successful meta-modeling implementations for multi-objective optimization based on Kriging and response surface methodologies exist [36, 37].

12.7 Conclusions

The research and application in evolutionary multi-objective optimization (EMO) is now at least over 15 years old and has resulted in a number of efficient algorithms for finding a set of well-diversified, near Pareto-optimal solutions. EMO algorithms are now regularly being applied to different problems involving most branches of science, engineering, and commerce.

This chapter started with discussing principles of EMO and illustrated the principle by depicting one efficient and popularly used EMO algorithm. Results from an interplanetary spacecraft trajectory optimization problem reveal the importance of principles followed in EMO algorithms. Thereafter, we made a brief description of a specific constraint handling procedure used in EMO studies.

However, the highlight of this chapter is the description of some of the current research and application activities involving EMO. One critical area of current research lies in collaborative EMO-MCDM algorithms for achieving a complete multi-objective optimization task of finding a set of trade-off solutions and finally arriving at a single preferred solution. Another direction taken by the researchers is to address guaranteed convergence and diversity of EMO algorithms through hybridizing them with mathematical and numerical optimization techniques as local search algorithms. Interestingly, EMO researchers have discovered its potential in solving traditionally hard optimization problems, but not necessarily multi-objective in nature, in a convenient manner using EMO algorithms. The so-called multi-objectivization studies are attracting researchers from various fields to develop and apply EMO algorithms in many innovative ways. A considerable research and application interest has also been put in addressing practical aspects into existing EMO algorithms. Towards this direction, handling uncertainty in decision variables and parameters, meeting an overall desired system reliability in obtained solutions, handling dynamically changing problems (on-line optimization), and handling a large number of objectives have been discussed in this paper. Besides the practical aspects, EMO has also attracted mathematically oriented theoreticians to develop EMO algorithms and design suitable problems for coming up with a computational complexity analysis. There are many other research directions which could not even mention due to space restrictions.

It is clear that the field of EMO research and application, in a short span of about 15 years, now has efficient algorithms and numerous interesting and useful applications, and has been able to attract theoretically and practically oriented researchers to come together and make collaborative activities. The practical importance of EMO's working principle, the flexibility of evolutionary optimization which lies at the core of EMO algorithms, and demonstrated diversification of EMO's principle to a wide variety of different problem-solving tasks are the main cornerstones for their success so far. The scope of research and application in EMO and using EMO are enormous and open-ended. This chapter remains an open invitation to everyone who is interested in any type of problem-solving tasks to take a look at what has been done in EMO and to explore how one can contribute in collaborating with EMO to address problem-solving tasks which are still in need of a better solution procedure.

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Chapter 13
Multiple Criteria Decision Analysis and Geographic Information Systems

Jacek Malczewski

Abstract This chapter focuses on a review of Geographic Information System-based Multicriteria Decision Analysis (GIS-MCDA). These two distinctive areas of research can benefit from each other. On the one hand, GIS techniques and procedures have an important role to play in analyzing spatial decision problems. Indeed, GIS is often recognized as a spatial decision support system. On the other hand, MCDA provides a rich collection of techniques and procedures for structuring decision problems, designing, evaluating, and prioritizing alternative decisions. At the most rudimentary level, GIS-MCDA can be thought of as a process that transforms and combines geographical (spatial) data and value judgments (the decision maker's preferences) to obtain information for decision making. It is in the context of the synergetic capabilities of GIS and MCDA that one can see the benefit for advancing theoretical and applied research on GIS-MCDA. The chapter is structured into seven sections. The introductory section outlines the synergetic capabilities of GIS and MCDA. Subsequently, the chapter provides an introduction to the basic concepts of GIS, a historical perspective of GIS-MCDA, and a survey of the GIS-MCDA literature. The following section focuses on the MCDA functions in GIS-based analysis and Multicriteria Spatial Decision Support System (MC-SDSS). The concluding section presents the challenges and prospects for advancing GIS-MCDA.

Keywords Geographic information systems · Spatial decision support

13.1 Introduction

Spatial decision problems typically involve a set of decision alternatives and multiple, conflicting and incommensurate evaluation criteria. The alternatives are often evaluated by a number of individuals (decision makers, managers, stakeholders, interest groups). The individuals are usually characterized by unique preferences with

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respect to the relative importance of criteria on the basis of which the alternatives are evaluated. The critical aspect of spatial decision analysis is that it involves evaluation of the spatial defined decision alternative and the decision maker's preferences. This implies that the results of the analysis depend not only on the geographic pattern of decision alternatives but also on the value judgments involved in the decision-making process. Accordingly, many spatial decision problems give rise to GIS-MCDA. At the most fundamental level, GIS-MCDA is a process that combines and transforms geographic data (input maps) and the decision maker's preferences into a resultant decision (output map). The GIS-MCDA procedures involve the utilization of geographic data, the decision maker's preferences, and the manipulation of the data and preferences according to specified decision rules.

GIS provides a set of methods and procedures for processing the geographic data to obtain information for decision making. However, GIS has very limited capabilities of storing and analyzing data on the decision maker's preferences. These capabilities can be enhanced by integrating MCDA and GIS. MCDA provides a methodology for guiding the decision maker(s) through the critical process of clarifying evaluation criteria (attributes and/or objectives), and of defining values that are relevant to the decision situation. The major advantage of incorporating MCDA into GIS is that a decision maker can introduce value judgments (i.e., preferences with respect to decision criteria and/or alternatives) into GIS-based decision making. The integration of MCDA into GIS can enhance a decision maker's confidence in the likely outcomes of adopting a specific strategy relative to his/her values. MCDA can help decision makers to understand the results of GIS-based decision-making procedures, including trade-offs among policy objectives, and then use the results in a systematic and defensible way to develop policy recommendations.

13.2 GIS: Basic Concepts

13.2.1 Definition of GIS

GIS is often defined with a reference to two aspects of the system: *technology* and/or *problem-solving* [43,76,109]. The system is conventionally seen as a set of tools for the input, the storage and retrieval, the manipulation and analysis, and the output of spatial data. Data input refers to the process of identifying and gathering the data required for a specific application. The process involves acquisition, reformatting, georeferencing, compiling, and documenting the data. The data storage and management component of a GIS includes those functions needed to store and retrieve data from the database. Most GIS systems are database oriented (see Section 13.2.2). The distinguishing feature of a GIS is its capability of performing an integrated analysis of spatial and attribute data. The data are manipulated and analyzed to obtain information useful for a particular application. There is an enormously wide range of analytical operations available to the GIS users (see Section 13.2.3). The data output component of a GIS provides a way to see the data/information in the form of maps, tables, diagrams, etc. The technology-oriented approach ignores the problem-solving aspects of GIS. It can be argued however that GIS contains a set of procedures to support decision-making activities. Indeed, GIS can be thought of "as a decision support system involving the integration of spatially referenced data in a problem solving environment" [43]. In this context, GIS should be considered as a special-purpose digital database in which a common geographic coordinate system is the primary means of storing and accessing data, and analyzing the data to obtain information for decision making and that an ultimate aim of GIS is to provide support for making decisions.

13.2.2 GIS Data Models

GIS utilizes two basic types of data: *spatial data* and *attribute data* [76]. The former describes the absolute and relative locations of spatial (geographic) entities (e.g., building, parcel of land, street, river, lake, state, country, etc.). The attributes refer to the properties of spatial entities. These properties can be quantitative and/or qualitative in nature. Attribute data are often referred to as tabular data.

Spatial data are typically arranged in a GIS using one of two models: *vector* and *raster* (Fig. 13.1). Entities in vector format are represented by strings of coordinates.

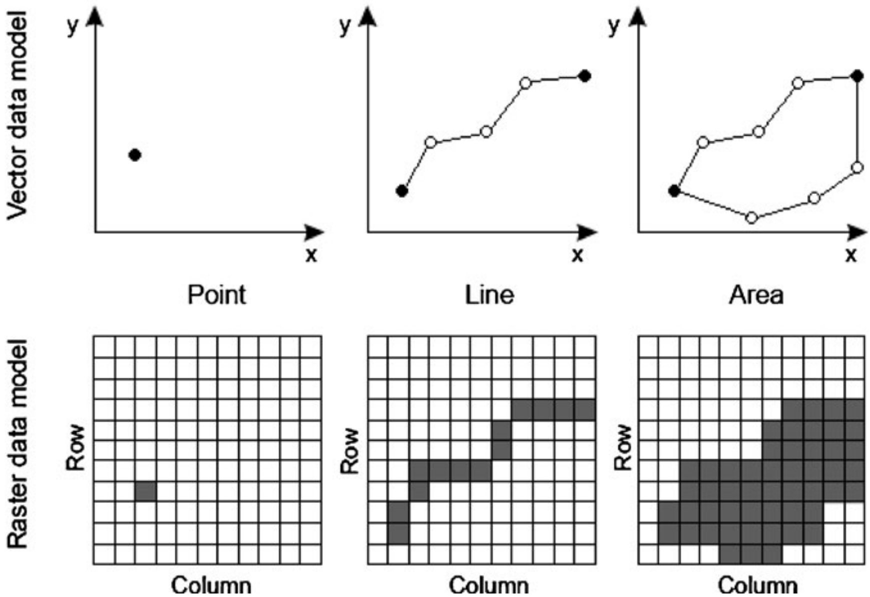

Fig. 13.1 Vector and raster data models in GIS

A point is one coordinate; that is, points on a map are stored in the computer with their coordinates. Points can be connected to form a line which is represented as a number of coordinates along its length. Chains can be connected back to the starting point to enclose polygons or areas. A polygon is represented as a set of coordinates at its corners. Each object (point, line and polygon) has associated attribute data. For example, a point which represents a village or town may have a database entry for its name, size, population, household income, etc. A line which represents a road may have a database entry for its route number, traffic capacity, emergency route, etc. A polygon which represents an administrative unit may have a database entry for the various socio-economic, environmental, and population attributes. Each of these spatial objects may have an identifier which is a key to an attached database containing the attributes (tabular data) of the object. In the vector representation, the various objects have a definite spatial relation called topology.

Data in a raster model are stored in a two-dimensional matrix of uniform grid cells (pixels or rasters) on a regular grid (Fig. 13.1). Each cell is supposedly homogeneous; that is, the map is incapable of providing information at any resolution finer than the individual cell. Areas are made up of contiguous pixels with the same value. Lines are made by connecting cells into a one-pixel thick line. Points are single cells. All spatial objects have location information inherent to where they lie in the grid. The map shows exactly one attribute value (e.g., land use, elevation, political division) for each cell. The size of the grid can vary from sub-meter to many kilometers. The spatial resolution of the data is determined by the grid size. The higher the level of resolution, the greater the detail one can distinguish on a raster map.

13.2.3 GIS Analytical Operations

GIS analytical operations or functions allow the user to combine different spatial data (maps) to produce a new data set (map). There is an enormously wide range of the GIS analytical operations available and a number of classifications of those operations have been suggested [23,68,76]. Two broad categories of GIS operations can be distinguished: *fundamental* (or *basic*) and *advanced functions*. This distinction is based on the extent to which those functions can be used in a variety of spatial analyses including spatial decision analysis. The functions considered to be useful for a wide range of applications are referred to as fundamental ones. They are more generic than the advanced functions in the sense that they are available in a wide variety of GIS for different data structures. The fundamental functions include: measurement, (re)classification, scalar and overlay operations, neighborhood operations, and connectivity operations. Many popular GIS packages, such as ArcGIS – ArcView [54,55], IDRISI [52], GRASS [180], MapInfo [120], and Maptitude/TransCAD [27] have the capability to perform most, if not all, of the basic analytical functions.

The fundamental functions are invariably low-level geometric operations and could be thought of as simple tools that build relationships among and between spatial objects. To be useful for spatial decision making, GIS should also provide the capabilities of statistical and mathematical manipulation of data based on theoretical models. These capabilities are referred to as GIS advanced functions or "compound" operations [153]. MCDA provides an example of GIS advanced functions. Most GIS systems have limited capabilities of performing MCDA. Notable exceptions include IDRISI [52, 53], Common GIS [61], ILWIS [80], and TNT-GIS [128]. It should be noted however that most GIS packages have the capabilities of performing cartographic modeling and map algebra operations that can be used for building simple MCDA models such as conjunctive/disjunctive screening and weighed summation [116, 118].

13.3 Brief History of GIS-MCDA

The evolution of GIS-MCDA has been a function of the development of information technology (including geographic information technology) and the evolving perspectives of planning/decision making. The modern era in GIS can be divided into three time periods: (i) the GIS research frontier period in the 1950s–1970s which can be referred to as the innovation stage, (ii) the development of general-purpose GISystems in the 1980s or the integration stage, and (iii) the proliferation stage which is characterized by the development of the user-oriented GIS technology in the last 20 years or so [60, 186]. The progression in the GIS development corresponds to the likewise evolving perspectives of planning/decision making. To this end, the primary focus of the planning/decision making has been shifted over time from the scientific, system approaches, through political perspectives to the public participatory and collective design approaches [22].

Interdisciplinary interest in GIS-MCDA can be traced from its roots in OR/MS and landscape architecture. Two advances within these fields were of particular importance for establishing the GIS-MCDA paradigm: (i) the development of *mathematical programming* as a method for system analysis in OR/MS, and (ii) the development of *overlay modeling* (cartographic modeling) as a method for land use/suitability analysis in landscape architecture and planning.

13.3.1 Innovation: GIS and OR/MS

One of the precursors of today's GIS-MCDA was the introduction of systems analysis and optimization methods, first in OR/MS and then in a number of disciplines including regional science, urban and regional planning, and geography [31]. Although the foundations of systems thinking were developed in the 1940s, it was not until a considerable increase in accessibility to computer-based mathematical programming software in the 1960s, that systems thinking became a practical proposition for decision making and planning [31]. This development coincided with the advances in computer technology that allowed the development of automated systems for storing, manipulating and displaying geographic data. The first systems we now call GIS were emerging in the 1960s, just as computers were becoming accessible to large government and academic institutions [40]. During the 1970s the single-objective approaches to system optimization were increasingly questioned. The criticism was part of a broader critique of the positivist paradigm that led to the adoption of a political perspective on planning and decision making. This perspective recognized that planning deals with socio-political systems that are composed of interest groups with conflicting values and preferences and therefore must include considerations of public participation, negotiation, compromise, consensus building, and conflict management and resolution [41]. The development of MCDA was one of the responses to the criticism of the classical system analysis and single-criterion (single-objective) approaches to spatial decision making and planning problems [39, 134]. Planners and regional scientists were among the first to advance the idea of combining the multi-objective mathematical programming techniques with GIS/computer-assisted mapping. Diamond and Wright [48] provide an example of an earlier work on integrating multi-objective programming and GIS techniques. This area of research has recently been extended to GIS-based approaches for solving spatial problems with artificial intelligence (AI) techniques [49]. During the past decade or so, many advanced paradigms integrating individual components of AI and GIS have emerged. Prominent research areas in developing hybrid systems include the integration of GIS and AI approaches such as evolutionary (genetic) algorithms and simulated annealing [2, 10, 50, 72, 130, 166, 194, 195, 201].

13.3.2 Integration: Cartographic Modeling and MCDA

A second and quite distinct history of GIS-MCDA stems from landscape architecture and land-use planning. Arguably, the transparent map overlay approach to land-use suitability analysis pioneered by McHarg [125] and his associates has had a greater influence on the development GIS-MCDA than any other single event in GIS history. McHarg analyzed land-use suitability decision problems by representing each evaluation criterion as a transparent map with the darkest gradations of tones representing areas with the greatest value, and the lightest tones associated with the least significant value. All of the transparent criterion maps were then superimposed upon one another to identify the most suitable land for development. In the 1970s, McHarg's approach has been used in several computer-assisted mapping and GIS applications [77, 111, 129, 132, 178]. Today, the method forms the basis of many land-use suitability approaches in GIS-MCDA [32, 49, 53, 76, 117].

McHarg's method has been refined and advanced by the introduction of map algebra techniques (cartographic modeling) into computer-assisted mapping [176]. This development was an important step towards integrating GIS and MCDA. Map algebra techniques include fundamental methods of MCDA such as Boolean screening and weighted linear combination [53]. This development has coincided with a rapid change in availability of computer technology. As computing power increased and hardware prices plummeted in the 1980s, GIS became a viable technology for state and municipal planning agencies and academic departments. This stimulated the development of low-cost GIS systems such as the Map Analysis Package (MAP) [176], MAP II [144], GRASS [180], SPANS GIS [179], and IDRISI [52]. The systems have been designed around the concept of map algebra and cartographic modeling. IDRISI [52, 53] and SPANS [179] had modules specifically designed for MCDA.

13.3.3 Proliferation: The User-oriented GIS-MCDA

Although the advent of desktop computing and cartographic modeling in the 1980s was instrumental in stimulating the development of GIS-MCDA, it was not until the 1990s that GIS-MCDA established itself as an identifiable subfield of research within the GIS literature [16, 29, 36, 66, 82, 104, 105, 115, 118, 145, 173]. The proliferation of GIS-MCDA can be illustrated by examining the increasing number of relevant publications in refereed journals. Figure 13.2 shows that there has been an exponential growth of the number of refereed publications on GIS-MCDA in the post-1990 period [18]. The rapid increase in the volume of GIS-MCDA research can be attributed to two main factors. First, during the 1990s, increasingly powerful personal computer-based GIS software was developed, refined, and utilized in applications. GIS has gradually been regarded as a routine software application within

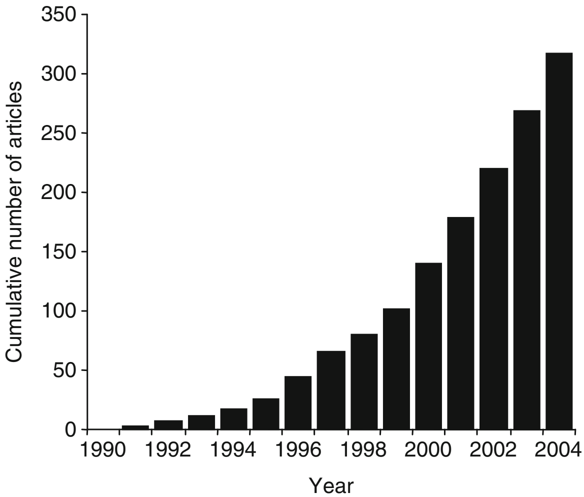

Fig. 13.2 Cumulative numbers of the GIS-MCDA articles published in refereed journals, 1990–2004 [996]

the grasp of lay individuals. At the same time, better awareness of the value of digital spatial data and GIS-based solutions to planning, decision making and management problems have produced a large market for GIS. The technological progress has been accompanied by an explosion of digital data available to the private and public sector organizations. This development has been stimulated by the availability of Web-based mapping (e.g., Google Maps and MSN Virtual Earth) and Web-based spatial decision support [78, 86, 100, 157, 163, 199, 201].

Second, the increasing accessibility of GIS to general public has resulted in a greater recognition of the importance of decision analysis and support within the broader field of GIScience (see the National Center for Geographic Information and Analysis initiatives on 'Spatial Decision Support Systems', 'Collaborative Spatial Decision Making', 'GIS and Society', and 'Empowerment, Marginalization and Public Participation GIS' at http://www.ncgia.ucsb.edu/ncgia.html [134]). Together these factors gave impetus to considerable progress in the quantity and quality of research on integrating GIS and MCDA. It can be argued that over the last decade or so GIS-MCDA study has generated a literature large enough for it to be regarded as a legitimate subfield of research within GIScience [118].

13.4 A Survey of the GIS-MCDA Literature

There is now a well-established body of literature on GIS-MCDA [16, 28, 29, 33, 48, 51, 81, 85, 87, 88, 104, 115, 118, 147]. One of the most remarkable features of the GIS-MCDA methods is the wide range of decision and management situations in which they have been applied. Major application areas include: environmental planning and management [18, 20, 37, 63, 71, 74, 127, 145, 150, 162, 175, 182, 204], transportation planning and management [7, 21, 38, 85, 89, 110, 167, 203], urban and regional planning [14, 46, 59, 65, 87, 138, 147, 172, 200], waste management [24, 29, 34, 99, 112, 170, 177], hydrology and water resource [64, 106, 113, 123, 136, 143], agriculture and forestry [3, 30, 95, 97, 97, 98, 103, 124, 131, 151, 160, 181], geology and natural hazard [8, 12, 26, 47, 70, 152, 191], and real estate and industrial facility management [28, 94, 96, 135, 149, 183]. Malczewski [118] surveyed the GIS-MCDA literature with a comprehensive review of 319 refereed articles published from 1990 through 2004 (a list of these articles can be found at *http://publish.uwo.ca/ jmalczew/gis-mcda.htm*).

13.4.1 Taxonomy of GIS-MCDA

Two classification schemes for the GIS-MCDA literature were developed [118]. First, all articles were classified based on the GIS components of the GIS-MCDA methods. This classification involved the following considerations: (i) the geographical data models, (ii) the spatial dimension of the evaluation criteria, and (iii) the spatial definition of decision alternatives. Second, the articles were classified according to the elements of the MCDA methods. This taxonomy was based on (i) the nature of evaluation criteria, (ii) the number of individuals involved in the decision-making process, and (iii) the nature of uncertainties.

13.4.2 GIS Components of GIS-MCDA

Of the 319 papers reviewed by Malczewski [118], 152 (47.6%) articles reported the raster-data-based research [1, 37, 44, 51, 118, 145, 196] and 150 (47.0%) articles discussed research involving the vector-based GIS-MCDA [28, 56, 57, 81, 181]. There were 17 articles which did not provide any information on the geographical data model. It is important to note that some of the works reported in the GIS-MCDA articles have been based on a combination of the raster and vector data models.

The raster- and vector-based GIS-MCDA approaches can further be subdivided into two categories depending on the nature of criteria. *Explicitly spatial* criteria are present in the decision problems that involve spatial characteristics as criteria (e.g., size, shape, contiguity, and compactness of a site or area). Many decision problems involve criteria which are *implicitly spatial* [75]. A criterion is said to be implicitly spatial, if spatial data (e.g., distance, proximity, accessibility, elevation, slope, etc.) are needed to compute the level of achievement of that criterion. It should be noted that these two categories are not mutually exclusive (see Table 13.1). Indeed, majority of the studies (almost 70%) involved both explicitly and implicitly spatial criteria [6, 99, 108, 123, 161, 190]. Of the 152 raster-based GIS-MCDA articles, 12 (7.9%) and 45 (29.6%) articles have reported research that involved explicitly and implicitly spatial criteria, respectively. Examples of the former category include: [24, 37, 45, 51, 161]. References [23, 62, 156] provide examples of the raster-based implicitly spatial criteria. Similar classification for the vector-based GIS-MCDA showed that there were 20 (13.3%) articles reporting the use of explicitly spatial criteria [112, 188, 193] and 12 (8.0%) studies involved implicitly spatial criteria [97, 181].

The raster- and vector-based GIS-MCDA approaches can also be categorized according to the nature of decision alternatives [118]. A spatial decision alternative

Table 13.1 Classification of the GIS-MCDA articles according to the GIS data model, the spatial dimension of the evaluation criteria (EC), and the spatial definition of decision alternatives (DA)

		Explicitly spatial		Implicitly spatial		Explicitly/implicitly spatial	
		EC	DA	EC	DA	EC	DA
Data model	Raster	12	57	45	41	95	54
	Vector	20	58	12	49	118	43
	Unspecified	2	8	7	4	8	5
	Total	34	123	64	94	221	102

consists of at least two elements: action (what to do?) and location (where to do it?). The spatial component of a decision alternative can be specified explicitly or implicitly. Examples of explicitly spatial alternatives include: alternative sites for locating facilities [4, 99], alternative location-allocation patterns [9, 42, 119], and alternative patterns of land use-suitability [6, 24, 51]. In many decision situations the spatial component of an alternative decision is not present explicitly. However, there may be a spatial implication associated with implementing an alternative decision. In such a case, the alternative is referred to as an implicitly spatial alternative [75]. Spatially distributed impacts can emerge, for example, through the implementation of a particular solution to minimize flood risks in which favorable impacts are produced at one location while negative consequences result at another [95, 181].

Table 13.1 shows that the articles reporting on the use of explicitly spatial alternatives accounted for 57% or 37.5% of all the raster-based GIS-MCDA [37, 99, 158]. The implicitly spatial alternatives were used in 41 (27.0%) articles on the raster-based GIS-MCDA [26, 95, 191]. The GIS-MCDA database contained 58 articles (38.7%) which have been categorized as the vector-based GIS-MCDA and explicitly spatial alternatives categories [112, 188, 193]. There were 49 (32.7%) articles in the vector-based-implicitly-spatial-alternative category [97, 131, 181].

13.4.3 MCDA Components of GIS-MCDA

Criterion is a generic term including both the concept of *attribute* and *objective* [79]. Accordingly, GIS-MCDA can be classified into two categories: multi-attribute decision analysis (GIS-MADA) and multi-objective decision analysis (GIS-MODA). The results of the survey [118] indicate that a majority of the articles falls into the GIS-MADA category (Table 13.2). The GIS-MADA approaches account for about

Table 13.2 Classification of the GIS-MCDA articles according to the multicriteria decision rule; multiattribute decision analysis (MADA) and multiobjective decision analysis (MODA). Some articles presented more than one combination rule

	Decision rules	♯	%
MADA	Weighted summation/ overlay	143	39.4
	Ideal/reference point	35	9.6
	Analytical Hierarchy Process (AHP)	34	9.4
	Outranking methods (ELECTRE, PROMETHEE)	17	4.7
	Other	30	8.3
	Total (GIS-MADA)	259	71.3
MODA	Multi-objective programming algorithms (linear-integer programming)	57	15.7
	Heuristic search/evolutionary/genetic algorithms	29	8.0
	Goal programming/reference point algorithms	9	2.5
	Other	9	2.5
	Total (GIS-MODA)	104	28.7
	Total	363	100.0

Table 13.3 Classification of GIS-MCDA papers according to the type of multicriteria decision method for individual decision maker

Multicriteria analysis type	Type of uncertainty: Deterministic		Probabilistic		Fuzzy		Total	
Multiattribute	119	(43.9)	17	(6.3)	37	(13.7)	173	(63.8)
Multiobjective	89	(32.8)	5	(1.8)	4	(1.5)	98	(36.2)
Total	208	(76.8)	22	(8.1)	41	(15.1)	271	(100.0)

Note: percentages of the total are given in brackets

Table 13.4 Classification of GIS-MCDA papers according to the type of multicriteria decision methods for group decision making

Multicriteria analysis type	Type of uncertainty: Deterministic		Probabilistic		Fuzzy		Total	
Multiattribute	50	(71.4)	5	(7.2)	8	(11.4)	63	(90.0)
Multiobjective	5	(7.2)	1	(1.4)	1	(1.4)	7	(10.0)
Total	55	(78.6)	6	(8.6)	9	(12.8)	70	(100.0)

Note: percentages of the total are given in brackets

70% of the total [16,56,81,90,103,114,145]. Approximately 30% of the approaches fall into the GIS-MODA category [1, 2, 6, 10, 65, 97, 107, 192, 194]. The GIS-MADA and GIS-MODA approaches can be further subdivided into two categories: individual and group decision making. A majority of the GIS-MCDA articles represented the individual decision maker's approaches (Tables 13.3 and 13.4). These approaches were found in about 64% of the GIS-MADA articles [16, 29, 96, 145] and 36% of the GIS-MODA papers [1, 6, 38, 97, 99, 192]. The group/participatory approaches were presented in 70 articles. They were found in 63 (90%) articles on GIS-MADA (e.g., 59, 60, 106, 118, 141, 155). There were only seven papers in the GIS-MODA-group decision-making category [18, 19, 161].

The GIS-MCDA studies can be categorized according to the amount of information about the decision situation that is available to the decision maker/ analyst. To this end, three categories of decision problems can be distinguished: *deterministic*, *probabilistic*, and *fuzzy* decision problems. Many analysts deliberately choose to model spatial decisions as occurring under a condition of certainty because of insufficient data or because the uncertainty is so remote that it can be disregarded as a factor [79, 115]. Consequently, a majority of the GIS-MCDA articles fall into the deterministic category (Tables 13.3 and 13.4). The deterministic approaches were presented in approximately 77% of the total [24, 29, 38, 85, 114, 177]. Of the 78 articles on decision problems under condition of uncertainty 36% fall into the probabilistic decision analysis category [102, 139, 161, 185] and 64% of the articles were found to represent the fuzzy decision making [13, 16, 90, 113, 123, 133, 152, 164, 167, 174, 189].

13.5 Functions of MCDA in GIS

The integration of MCDA into GIS enhances the analytical and decision support capabilities of conventional GIS software. These capabilities are related to the major components of MCDA including: decision problem structuring, value scaling, criterion weighting, combining criteria with decision rules, and performing sensitivity analysis.

13.5.1 Decision Problem Structuring

Two main approaches to structuring decision problems can be distinguished; namely the alternative-focused approach and the value-focused approach [101]. While the former centers on generating decision alternatives, the latter uses value judgments as the fundamental element of the decision analysis. The general principle for structuring decision problems is that decision alternatives should be generated so that the values specified for the decision are best achieved [101].

GIS-MCDA has been dominated by alternative-focused approaches. Malczewski's review [996] found that only about 30% of decision problems are structured around value-focused approaches [20, 93, 94], and the remainder are either alternative-focused or something else [9,44,48,56,114,187]. Giupponi et al. [64] give an example of the concept and process of hierarchical structuring for spatial decision problems in GIS-MCDA.

13.5.2 Value Scaling

Most MCDA procedures require that the variety of scales on which criteria are measured must be transformed to comparable units. Linear scale transformations are the simplest and most frequently used GIS-based methods for transforming (or standardizing) input data into criterion values [29, 51, 81, 90, 96, 137, 145, 157]. Unlike conventional GIS-based standardization procedures, MCDA-oriented value scaling approaches involve the construction of a value (utility) function which is a formal representation of a human's judgment [101]. The value function converts different levels of an attribute into relevant and representative value scores. Examples of the value function approaches in the GIS-MCDA procedures can be found in refs. [64, 160].

Jiang and Eastman [64, 160] suggest that the concept of fuzzy measures provides a more realistic basis for developing a generalized value scaling approach in the evaluation process. This approach can be seen as one of recasting values into statements of set memberships as shown in refs. [30, 103]. According to Jiang and Eastman [90] fuzzy measures provide substantial advantages over conventional methods of standardizing criteria in GIS-MCDA procedures by offering a theoretical basis for criterion standardization and bridging a conceptual gap betweenthe Boolean procedure and continuous scaling in weighted linear combination [113, 152].

13.5.3 Criterion Weighting

Spatial decision problems typically involve criteria (objectives and attributes) that are of different importance to different decision makers. The pairwise comparison weighting procedure is the most popular weighting approach used in the GIS-MCDA literature. According to Malczewski's survey [118], this method was used in almost one third of the GIS-MCDA research. Examples of pairwise comparisons for criterion weighting in GIS-MCDA can be found in [16, 35, 58, 90, 103, 114, 171, 177, 197] among others. Some researchers have argued that trade-off approaches provide the strongest theoretical base for criterion weighting [77, 100]. However, this method has seldom been used in GIS-MCDA studies [160].

13.5.4 Decision Rules

At the most fundamental level, a decision rule is a procedure for ordering the alternatives considered in a decision process [79]. It dictates how best to order alternatives or to decide which alternative is preferred over another. Although a considerable number of decision rules are proposed in the MCDA literature [39, 79], the use of the combination rules in the GIS-based procedures is limited to a few well-known approaches, such as weighted summation, ideal/reference point, and outranking methods [118]. Weighted summation and related procedures (such as AHP) have been by far the most popular form of GIS-based multi-attribute analysis accounting for almost 50% of the total (Table 13.2). The weighted summation methods are typically used in conjunction with Boolean operations [29, 51, 103, 116] and many studies have used weighted summation along with linear transformations for standardizing criteria, and pairwise comparisons for deriving criterion weights [30, 53, 103, 171]. Ideal/reference methods comprise the second most often used GIS-MCDA decision rule [29, 81, 114, 141, 145, 169, 198]. A considerable number of studies use outranking methods to evaluate criteria [32, 91, 92, 122, 160].

GIS-based multi-objective methods can be grouped into three main categories; namely multi-objective linear-integer programming, goal programming/reference point algorithms, and heuristic search/evolutionary algorithms (Table 13.2). The multi-objective models are often implemented by converting them into single-objective problems and solving the problems using standard linear-integer programming methods [39]. This approach is most often used in GIS-based multi-objective research [65, 188]. Another group of analysis methods includes those in which goal programming/reference point algorithms are used [6, 165]. These traditional approaches have some limitations including restricted applicability for very large and very complex problems. However, much progress has been made, especially in recent years, in implementing GIS-based artificial intelligence algorithms for solving large, complex spatial decision problems [2, 10, 19, 142, 195].

13.5.5 Sensitivity Analysis

Sensitivity analysis aims at examining how robust the ordering of alternatives, or the choice of a single alternative, is to relatively small changes in the components of MCDA (i.e., the problem structure, criterion values, criterion weights, and decision rules). Malczewski's survey [118] found that only about 16% of the articles published since 1990 have used some form of sensitivity analysis as part of the GIS-based decision-making procedures. Sensitivity analysis is most often performed on the criterion weights to test the robustness and veracity of a decision solution subject to changing the weights for a predetermined set of criteria across alternatives and reevaluating the alternative ranks [58]. This allows the effect of changes in criteria weights on the rank order of alternatives to be analyzed systematically. Examples of sensitivity analysis for GIS-MCDA can be found in refs. [35, 58, 64] among others.

13.6 Multicriteria Spatial Decision Support Systems (MC-SDSS)

Many researchers in OR/MS recognize that MCDA is at the core of decision support and problem solving [101]. The same realization has permeated into published research in GIScience, as revealed by the increased number of papers using MCDA techniques to address spatial decision support [11, 67, 84, 85].

13.6.1 Components of MC-SDSS

A number of frameworks for conceptualizing MC-SDSS have been proposed over the last 2 decades or so [17, 48, 53, 84, 121]. In general, the MC-SDSS frameworks focus on the integration of GIS capabilities and MCDA techniques. The way the two components are integrated depends on the philosophy behind the design strategy (e.g., a system for supporting a single-user versus that for group decision-making), the types of decision problems (e.g., environmental versus urban/transportation planning decision problems), and the MCDM models incorporated into the MC-SDSS (e.g., multi-objective versus multi-attribute decision models) [121]. Despite these differences, the basic structure, upon which most of the MC-SDSS designers agree, is composed of three main elements:

(i) *Geographical Data Management and Analysis Toolbox* should contain a set of tools that are available in full-fledged GIS systems [76, 109]. It typically includes techniques and methods for exploratory data analysis, generating criterion maps, and alternative spatial patterns [48, 115].
(ii) *MCDA Toolbox* includes tools for generating value structure, preference modeling, and multicriteria decision rules for evaluating a set of alternative decision and for choosing the best (compromise) alternative, and performing sensitivity analysis [38, 53, 115].
(iii) *User Interface* includes all the mechanisms by which commands, requests, and data are entered into MC-SDSS, as well as all the methods by which results and information are output by the system.

These three components constitute the software portion of an MC-SDSS. In addition, the *decision-maker* or *user* is considered to be a part of the system. The unique contributions of MC-SDSS are derived from the interaction between the computer and the user. In this context, the use of visualization techniques in spatial decision analysis is of critical importance for the computer–user communication [5, 109, 119, 154]. Specifically, the system should have the capability of displaying the input data and the results of multicriteria analysis in the decision space and criterion outcome space simultaneously [38].

13.6.2 Integrating GIS and MCDA

MC-SDSS can be classified according to: (i) the extent of integration, and (ii) the direction of integration of GIS and MCDA. Malczewski's survey [118] identified three categories of MC-SDSS based on the extent of integration: (i) loose-coupling, (ii) tight-coupling, and (iii) full integration [69, 81, 96, 115]. In the loose coupling approach, two systems (GIS and multicriteria modeling system) exchange files such that a system uses data from the other system as the input data. A tight-coupling strategy is based on a single data or model manager and a common user interface. Thus, the two systems share not only the communication files but also a common user-interface. A more complete integration can be achieved by creating user-specified routines using generic programming languages. The routines then can be added to the existing set of commands or routines of the GIS package. This coupling strategy is referred to as a full integration approach.

MC-SDSS can also be classified in terms of the direction of integration. Four categories of approaches can be identified: (i) one-direction integration with GIS as principle software, (ii) one-direction integration with MCDA system as principle software, (iii) bi-directional integration, and (iv) dynamic integration [96]. One-direction integration provides mechanism for importing/exporting information via a single flow that originates either in the GIS or MCDA software. This type of integration can be based on GIS or MCDA as the principle software. In the bidirectional integration approach the flow of data/information can originate and end in the GIS and MCDA modules. While bidirectional integration involves one-time flow of information, dynamic integration allows for a flexible moving of information back and forth between the GIS and MCDA modules according to the user's needs [96].

Of the 319 articles reviewed by Malczewski [118], each of the loose-coupling and tight-coupling approaches account for approximately 30% of all strategies for designing MC-SDSS [73,81,153]. About 10% of MC-SDSS can be classified as the full integrated decision support systems [5, 53, 61, 124]. At the same time, almost half of the research has used GIS as the principle software for integrating MCDA and GIS [81, 96, 140]. The MCDA as the principle software strategy for integrating MCDA and GIS was used in about 12% of the research [6, 97, 126, 142]. The loose-coupling one-directional integration with GIS as the principal software is the largest category of MC-SDSS [52, 81, 155]. It accounts for more than 30% of the total.

13.7 Conclusions: Challenges and Prospects

The last 20 years or so have evidenced remarkable progress in the quantity and quality of research in integrating GIS and MCDA. The GIS community have recognized the great benefits to be gained by incorporating MCDA into a suite of GIS capabilities [116, 173]. On the other hand, the GIS capabilities have been identified as an essential component of MCDA by the MCDA researchers and practitioners [65, 166, 188]. Indeed, Wallenius and his colleagues [184] have documented the integration of GIS and MCDA as one of the significant accomplishments in expanding MCDA into new application domains. The efforts to integrate MCDA into GIS were instrumental for developing the paradigm of spatial decision support [11,52,84,115]. This development has been paralleled by the evolution of GIS from a 'close'-expert-oriented to an 'open'-user-oriented technology, which in turn has stimulated a movement in the GIScience community towards using the technology to increase the democratization of decision-making process via public participation. By their nature, MCDA tools allow the integration of multiple views on decision problems. They can improve the communication and understanding of a decision problem among multiple decision makers and facilitate numerous ways of building consensus and reaching policy compromises. Consequently, MC-SDSS have the potential to improve collaborative decision-making processes by providing flexible problem-solving approaches where those involved in collaborative tasks can explore, understand, and redefine a decision problem [934, 962, 981]. The integration of MCDA into GIS can support collaborative work by providing a means of structuring group decision-making problems and organizing communication within a group setting, as MCDA offers a framework for handling debates on the identification of components of a decision problem, organizing the elements into a hierarchical structure, understanding the relationships between components of the problem, and stimulating communication among participants.

Malczewski [117] suggests that it is in the context of the debate on the interrelationship between GIS and society [148] that one can see the potential for advancing the role of GIS-MCDA in the participatory GIScience. Specifically, GIS-MCDA should be constructed with two perspectives in mind: the techno-positivist perspective on GIS, and the sociopolitical, participatory GIS perspective. It is expected that the trend towards advancing public participatory approach to the GIS-MCDA system design and application development will be of critical importance for a successful use of the GIS-MCDA approaches in the real-world decision situations [84]. Ascough et al. [11] suggest that the development of GIS-MCDA must absorb new trends in geographical information technology including the Web-based applications [78,86,155,159,163,199,202] and location-based services [156].

Recent developments in spatial analysis show that geo-computation (computational intelligence) offers new opportunities for GIS-MCDA [187, 189, 190, 195, 201]. Geo-computational tools can potentially help in modeling and describing complex systems for inference and decision making. An integration of MCDA and geo-computation can enhance the GIS-MCDA capabilities of handling larger and more diverse spatial datasets. Another significant trend has been associated with developing map-centered exploratory approaches to GIS-MCDA [5, 9, 83]. The main purpose of these approaches is to provide the decision maker with insights into the nature of spatial decision problems not readily obtained by conventional methods (such as tabular displays). The power of map-centered exploratory analysis comes from the confidence in the GIS-based MCDA procedures that grows as decision makers see the procedures confirm their understanding of the decision problem at hand.

The GIS-MCDA research has tended to concentrate on the technical questions of integrating MCDA into GIS. As a consequence, our understanding of the benefits of such integration is limited by the lack of research on conceptual and operational validation of the use of MCDA in solving real-world spatial problems. Very little empirical research has been undertaken to appreciate the dynamics of spatial decision making [84]. There are also other, more general, concerns surrounding the use of MCDA in GIS that require careful consideration. More attention should be paid to the theoretical foundations and operational validation of the GIS-MCDA methods. Some MCDA procedures lack a proper scientific foundation and some methods involve a set of stringent assumptions, which are difficult to validate in real-world situations [15]. These problems have, to a large extent, been ignored by the GIS-MCDA community. If the primary purpose of GIS-MCDA is to process and synthesize large spatial datasets and value judgments, and to examine the implications of those value judgments for planning and policy making, then more careful consideration must be given to the assumptions underlying the MCDA procedures.

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Index