Chapter 8

Existence and Uniqueness Theorems

8.1 Basic Results

In this chapter we are concerned with the first-order vector differential equation

$$x' = f(t, x). \tag{8.1}$$

We assume throughout this chapter that $f : D \to \mathbb{R}^n$ is continuous, where D is an open subset of $\mathbb{R} \times \mathbb{R}^n$.

Definition 8.1 We say x is a *solution* of (8.1) on an interval I provided $x : I \to \mathbb{R}^n$ is differentiable, $(t, x(t)) \in D$, for $t \in I$ and $x'(t) = f(t, x(t))$ for $t \in I$.

Note that if x is a solution of (8.1) on an interval I, then it follows from $x'(t) = f(t, x(t))$, for $t \in I$, that x is continuously differentiable on I.

In the next example we show that a certain nth-order scalar differential equation is equivalent to a vector equation of the form (8.1).

Example 8.2 Assume that D is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $F : D \to \mathbb{R}$ is continuous. We are concerned with the nth-order scalar differential equation

$$u^{(n)} = F(t, u, u', \cdots, u^{(n-1)}). \tag{8.2}$$

In this equation $t, u, u', \cdots, u^{(n)}$ denote variables. We say a scalar function $u : I \to \mathbb{R}$ is a solution of the nth-order scalar equation (8.2) on an interval I provided u is n times differentiable on I, $(t, u(t), u'(t), \cdots, u^{(n-1)}(t)) \in D$, for $t \in I$, and

$$u^{(n)}(t) = F(t, u(t), u'(t), \cdots, u^{(n-1)}(t)),$$

for $t \in I$. Note that if u is a solution of (8.2) on an interval I, then it follows that u is n times continuously differentiable on I. Now assume that u is a solution of the nth-order scalar equation (8.2) on an interval I. For

W.G. Kelley and A.C. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Universitext 278, DOI 10.1007/978-1-4419-5783-2_8,

$t \in I$, let

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} := \begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix}.$$

Then

$$\begin{aligned} x'(t) &= \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} u'(t) \\ u''(t) \\ \vdots \\ u^{(n)}(t) \end{pmatrix} \\ &= \begin{pmatrix} u'(t) \\ u''(t) \\ \vdots \\ F(t, u(t), u'(t), \cdots, u^{(n-1)}(t)) \end{pmatrix} \\ &= f(t, x(t)), \end{aligned}$$

if we define

$$f(t,x) = f(t, x_1, x_2, \cdots, x_n) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ F(t, x_1, x_2, \cdots, x_n) \end{pmatrix}, \tag{8.3}$$

for $(t, x) \in D$. Note that $f : D \to \mathbb{R}^n$ is continuous. Hence if u is a solution of the nth-order scalar equation (8.2) on an interval I, then

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} := \begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix},$$

$t \in I$, is a solution of a vector equation of the form (8.1) with $f(t, x)$ given by (8.3). Conversely, it can be shown that if x defined by

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

for $t \in I$ is a solution of a vector equation of the form (8.1) on an interval I with $f(t, x)$ given by (8.3); then $u(t) := x_1(t)$ defines a solution of (8.2) on the interval I. Because of this we say that the nth-order scalar equation (8.2) is equivalent to the vector equation (8.1) with f defined by (8.3). $\triangle$

Definition 8.3 Let $(t_0, x_0) \in D$. We say that x is a *solution of the IVP*

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{8.4}$$

on an interval I provided $t_0 \in I$, x is a solution of (8.1) on I, and $x(t_0) = x_0$.

Example 8.4 Note that if $D = \mathbb{R} \times \mathbb{R}^2$, then x defined by

$$x(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

for $t \in \mathbb{R}$ is a solution of the IVP

$$x' = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

on $I := \mathbb{R}$. △

Closely related to the IVP (8.4) is the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{8.5}$$

Definition 8.5 We say that $x : I \to \mathbb{R}^n$ is a *solution of the vector integral equation* (8.5) on an interval I provided $t_0 \in I$, x is continuous on I, $(t, x(t)) \in D$, for $t \in I$, and (8.5) is satisfied for $t \in I$.

The relationship between the IVP (8.4) and the integral equation (8.5) is given by the following lemma. Because of this result we say the IVP (8.4) and the integral equation (8.5) are equivalent.

Lemma 8.6 *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and $(t_0, x_0) \in D$; then x is a solution of the IVP* (8.4) *on an interval I iff x is a solution of the integral equation* (8.5) *on an interval I.*

Proof Assume that x is a solution of the IVP (8.4) on an interval I. Then $t_0 \in I$, x is differentiable on I (hence is continuous on I), $(t, x(t)) \in D$, for $t \in I$, $x(t_0) = x_0$, and

$$x'(t) = f(t, x(t)),$$

for $t \in I$. Integrating this last equation and using $x(t_0) = x_0$, we get

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

for $t \in I$. Thus we have shown that x is a solution of the integral equation (8.5) on the interval I.

Conversely assume x is a solution of the integral equation (8.5) on an interval I. Then $t_0 \in I$, x is continuous on I, $(t, x(t)) \in D$, for $t \in I$, and (8.5) is satisfied for $t \in I$. Since

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

for $t \in I$, $x(t)$ is differentiable on I, $x(t_0) = x_0$, and

$$x'(t) = f(t, x(t)),$$

for all $t \in I$. Hence we have shown that x is a solution of the IVP (8.4) on the interval I. $\square$

8.2 Lipschitz Condition and Picard-Lindelof Theorem

In this section we first defined what is meant by a vector function $f : D \to \mathbb{R}^n$ satisfies a *uniform Lipschitz condition with respect to* x on the open set $D \subset \mathbb{R} \times \mathbb{R}^n$. We then state and prove the important Picard-Lindelof theorem (Theorem 8.13), which is one of the main uniqueness-existence theorems for solutions of IVPs.

Definition 8.7 A vector function $f : D \to \mathbb{R}^n$ is said to satisfy a *Lipschitz condition with respect to* x on the open set $D \subset \mathbb{R} \times \mathbb{R}^n$ provided for each rectangle

$$Q := \{(t, x) : t_0 \le t \le t_0 + a, \|x - x_0\| \le b\} \subset D$$

there is a constant K_Q that may depend on the rectangle Q (and on the norm $\|\cdot\|$) such that

$$\|f(t, x) - f(t, y)\| \le K_Q \|x - y\|,$$

for all $(t, x), (t, y) \in Q$.

Definition 8.8 A vector function $f : D \to \mathbb{R}^n$ is said to satisfy a *uniform Lipschitz condition with respect to* x on D provided there is a constant K such that

$$\|f(t, x) - f(t, y)\| \le K \|x - y\|,$$

for all $(t, x), (t, y) \in D$. The constant K is called a *Lipschitz constant for* $f(t, x)$ *with respect to* x *on* D.

Definition 8.9 Assume the vector function $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R} \times \mathbb{R}^n$, is differentiable with respect to components of x. Then the *Jacobian matrix* $D_x f(t, x)$ of $f(t, x)$ with respect to x at (t, x) is defined by

$$D_x f(t, x) = \begin{pmatrix} \frac{\partial}{\partial x_1} f_1(t, x) & \cdots & \frac{\partial}{\partial x_n} f_1(t, x) \\ \frac{\partial}{\partial x_1} f_2(t, x) & \cdots & \frac{\partial}{\partial x_n} f_2(t, x) \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_1} f_n(t, x) & \cdots & \frac{\partial}{\partial x_n} f_n(t, x) \end{pmatrix}.$$

Example 8.10 If

$$f(t, x) = f(t, x_1, x_2) = \begin{pmatrix} t^2 x_1^3 x_2^4 + t^3 \\ 3t + x_1^2 + x_2^3 \end{pmatrix},$$

then

$$D_x f(t, x) = \begin{pmatrix} 3t^2 x_1^2 x_2^4 & 4t^2 x_1^3 x_2^3 \\ 2x_1 & 3x_2^2 \end{pmatrix}.$$

$\triangle$

Lemma 8.11 *Assume $D \subset \mathbb{R} \times \mathbb{R}^n$ such that for each fixed t, $D_t := \{x : (t,x) \in D\}$ is a convex set and $f : D \to \mathbb{R}^n$ is continuous. If the Jacobian matrix, $D_x f(t,x)$, of $f(t,x)$ with respect to x is continuous on D, then*

$$f(t,x) - f(t,y) = \int_0^1 D_x f(t, sx + (1-s)y)\, ds\, [x-y], \tag{8.6}$$

for all $(t,x), (t,y) \in D$.

Proof Let $(t,x), (t,y) \in D$; then D_t is convex implies that $(t, sx + (1-s)y) \in D$, for $0 \le s \le 1$. Now for $(t,x), (t,y) \in D$, and $0 \le s \le 1$, we can consider

$$\begin{aligned}
&\frac{d}{ds} f(t, sx + (1-s)y) \\
&= \frac{d}{ds} \begin{pmatrix} f_1(t, sx_1 + (1-s)y_1, \cdots, sx_n + (1-s)y_n) \\ f_2(t, sx_1 + (1-s)y_1, \cdots, sx_n + (1-s)y_n) \\ \cdots \\ f_n(t, sx_1 + (1-s)y_n)y_1, \cdots, sx_n + (1-s)y_n) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial x_1} f_1(\cdots)(x_1 - y_1) + \cdots + \frac{\partial}{\partial x_n} f_1(\cdots)(x_n - y_n) \\ \frac{\partial}{\partial x_1} f_2(\cdots)(x_1 - y_1) + \cdots + \frac{\partial}{\partial x_n} f_2(\cdots)(x_n - y_n) \\ \cdots \\ \frac{\partial}{\partial x_1} f_n(\cdots)(x_1 - y_1) + \cdots + \frac{\partial}{\partial x_n} f_n(\cdots)(x_n - y_n) \end{pmatrix} \\
&= D_x f(t, sx + (1-s)y)[x-y],
\end{aligned}$$

where the functions in the entries in the preceding matrix are evaluated at $(t, sx + (1-s)y)$. Integrating both sides with respect to s from $s = 0$ to $s = 1$ gives us the desired result (8.6). $\square$

Theorem 8.12 *Assume $D \subset \mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$, and the Jacobian matrix function $D_x f(t,x)$ is continuous on D. If for each fixed t, $D_t := \{x : (t,x) \in D\}$ is convex, then $f(t,x)$ satisfies a Lipschitz condition with respect to x on D.*

Proof Let $\|\cdot\|_1$ be the traffic norm (l_1 norm) defined in Example 2.47 and let $\|\cdot\|$ denote the corresponding matrix norm (see Definition 2.53). Assume that the rectangle

$$Q := \{(t,x) : |t - t_0| \le a,\ \|x - x_0\|_1 \le b\} \subset D.$$

Let

$$K := \max\{\|D_x f(t,x)\| : (t,x) \in Q\};$$

then using Lemma 8.11 and Theorem 2.54,

$$\begin{aligned}
\|f(t,x) - f(t,y)\|_1 &= \|\int_0^1 D_x f(t, sx + (1-s)y)\, ds\, [x-y]\|_1 \\
&\le \int_0^1 \|D_x f(t, sx + (1-s)y)\|\, ds \cdot \|x - y\|_1 \\
&\le K\|x - y\|_1,
\end{aligned}$$

for $(t, x), (t, y) \in Q$. Therefore, $f(t, x)$ satisfies a Lipschitz condition with respect to x on D. $\square$

Theorem 8.13 (Picard-Lindelof Theorem) *Assume that f is a continuous n-dimensional vector function on the rectangle*

$$Q := \{(t, x) : t_0 \leq t \leq t_0 + a, \|x - x_0\| \leq b\}$$

and assume that $f(t, x)$ satisfies a uniform Lipschitz condition with respect to x on Q. Let

$$M := \max\{\|f(t, x)\| : (t, x) \in Q\}$$

and

$$\alpha := \min\left\{a, \frac{b}{M}\right\}.$$

Then the initial value problem (8.4) *has a unique solution x on $[t_0, t_0 + \alpha]$. Furthermore,*

$$\|x(t) - x_0\| \leq b,$$

for $t \in [t_0, t_0 + \alpha]$.

Proof To prove the existence of a solution of the IVP (8.4) on $[t_0, t_0 + \alpha]$, it follows from Lemma 8.6 that it suffices to show that the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{8.7}$$

has a solution on $[t_0, t_0 + \alpha]$. We define the sequence of *Picard iterates* $\{x_k\}$ of the IVP (8.4) on $[t_0, t_0 + \alpha]$ as follows: Set

$$x_0(t) = x_0, \quad t \in [t_0, t_0 + \alpha],$$

and then let

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))\, ds, \quad t \in [t_0, t_0 + \alpha], \tag{8.8}$$

for $k = 0, 1, 2, \cdots$. We show by induction that each Picard iterate x_k is well defined on $[t_0, t_0 + \alpha]$, is continuous on $[t_0, t_0 + \alpha]$, and its graph is in Q. Obviously, $x_0(t)$ satisfies these conditions. Assume that x_k is well defined, x_k is continuous on $[t_0, t_0 + \alpha]$, and

$$\|x_k(t) - x_0\| \leq b, \quad \text{on} \quad [t_0, t_0 + \alpha].$$

It follows that

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))\, ds, \quad t \in [t_0, t_0 + \alpha]$$

is well defined and continuous on $[t_0, t_0+\alpha]$. Also,

$$\begin{aligned}\|x_{k+1}(t) - x_0\| &\leq \int_{t_0}^{t} |f(s, x_k(s))|\, ds \\ &\leq M(t-t_0) \\ &\leq M\alpha \\ &\leq b,\end{aligned}$$

for $t \in [t_0, t_0+\alpha]$ and the induction is complete.

Let K be a Lipschitz constant for $f(t,x)$ with respect to x on Q. We now prove by induction that

$$\|x_{k+1}(t) - x_k(t)\| \leq \frac{MK^k(t-t_0)^{k+1}}{(k+1)!}, \quad t \in [t_0, t_0+\alpha], \tag{8.9}$$

for $k = 0, 1, 2, \cdots$. We proved (8.9) when $k = 0$. Fix $k \geq 1$ and assume that (8.9) is true when k is replaced by $k-1$. Using the Lipschitz condition and the induction assumption, we get

$$\begin{aligned}\|x_{k+1}(t) - x_k(t)\| &= \|\int_{t_0}^{t} [f(s, x_k(s)) - f(s, x_{k-1}(s))]\, ds\| \\ &\leq \int_{t_0}^{t} \|f(s, x_k(s)) - f(s, x_{k-1}(s))\|\, ds \\ &\leq K\int_{t_0}^{t} \|x_k(s) - x_{k-1}(s)\|\, ds \\ &\leq MK^k \int_{t_0}^{t} \frac{(s-t_0)^k}{k!} ds \\ &= \frac{MK^k(t-t_0)^{k+1}}{(k+1)!},\end{aligned}$$

for $t \in [t_0, t_0+\alpha]$. Hence the proof of (8.9) is complete.

The sequence of partial sums for the infinite series

$$x_0(t) + \sum_{m=0}^{\infty} [x_{m+1}(t) - x_m(t)] \tag{8.10}$$

is

$$\{x_0(t) + \sum_{m=0}^{k-1} [x_{m+1}(t) - x_m(t)]\} = \{x_k(t)\}.$$

Hence we can show that the sequence of Picard iterates $\{x_k(t)\}$ converges uniformly on $[t_0, t_0+\alpha]$ by showing that the infinite series (8.10) converges uniformly on $[t_0, t_0+\alpha]$. Note that

$$\|x_{m+1}(t) - x_m(t)\| \leq \frac{M}{K}\frac{(K\alpha)^{m+1}}{(m+1)!},$$

for $t \in [t_0, t_0 + \alpha]$ and

$$\sum_{m=0}^{\infty} \frac{M}{K} \frac{(K\alpha)^{m+1}}{(m+1)!} \quad \text{converges.}$$

Hence from the Weierstrass M-test we get that the infinite series (8.10) converges uniformly on $[t_0, t_0+\alpha]$. Therefore, the sequence of Picard iterates $\{x_k(t)\}$ converges uniformly on $[t_0, t_0 + \alpha]$. Let

$$x(t) = \lim_{k\to\infty} x_k(t),$$

for $t \in [t_0, t_0 + \alpha]$. It follows that

$$\|x(t) - x_0\| \le b,$$

for $t \in [t_0, t_0 + \alpha]$. Since

$$\|f(t, x_k(t)) - f(t, x(t))\| \le K\|x_k(t) - x(t)\|$$

on $[t_0, t_0 + \alpha]$,

$$\lim_{k\to\infty} f(t, x_k(t)) = f(t, x(t))$$

uniformly on $[t_0, t_0 + \alpha]$. Taking the limit of both sides of (8.8), we get

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

for $t \in [t_0, t_0 + \alpha]$. It follows that x is a solution of the IVP (8.4).

To complete the proof it remains to prove the uniqueness of solutions of the IVP (8.4). To this end let y be a solution of the IVP (8.4) on $[t_0, t_0+\beta]$, where $0 < \beta \le \alpha$. It remains to show that $y = x$. Since y is a solution of the IVP (8.4) on $[t_0, t_0 + \beta]$, it follows from Lemma 8.6 that y is a solution of the integral equation

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

on $[t_0, t_0 + \beta]$. Similarly we can prove by mathematical induction that

$$\|y(t) - x_k(t)\| \le \frac{MK^k(t - t_0)^{k+1}}{(k+1)!}, \tag{8.11}$$

for $t \in [t_0, t_0 + \beta]$, $k = 0, 1, 2, \cdots$. It follows that

$$y(t) = \lim_{k\to\infty} x_k(t) = x(t),$$

for $t \in [t_0, t_0 + \beta]$. □

Corollary 8.14 *Assume the assumptions in the Picard-Lindelof theorem are satisfied and $\{x_k(t)\}$ is the sequence of Picard iterates defined in the proof of the Picard-Lindelof theorem. If x is the solution of the IVP* (8.4), *then*

$$\|x(t) - x_k(t)\| \le \frac{MK^k(t - t_0)^{k+1}}{(k+1)!}, \tag{8.12}$$

for $t \in [t_0, t_0 + \alpha]$, where K is a Lipschitz constant for $f(t,x)$ with respect to x on Q.

Example 8.15 In this example we maximize the α in the Picard-Lindelof theorem by choosing the appropriate rectangle Q for the initial value problem

$$x' = x^2, \quad x(0) = 1. \tag{8.13}$$

If

$$Q = \{(t,x) : 0 \le t \le a, |x-1| \le b\},$$

then

$$M = \max\{|f(t,x)| = x^2 : (t,x) \in Q\} = (1+b)^2.$$

Hence

$$\alpha = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{(1+b)^2}\right\}.$$

Since we can choose a as large as we want, we desire to pick $b > 0$ so that $\frac{b}{(1+b)^2}$ is a maximum. Using calculus, we get $\alpha = \frac{1}{4}$. Hence by the Picard-Lindelof theorem we know that the solution of the IVP (8.13) exists on the interval $[0, \frac{1}{4}]$. The IVP (8.13) is so simple that we can solve this IVP to obtain $x(t) = \frac{1}{1-t}$. Hence the solution of the IVP (8.13) exists on $[0,1)$ [actually on $(-\infty, 1)$]. Note that $\alpha = \frac{1}{4}$ is not a very good estimate. $\triangle$

Example 8.16 Approximate the solution of the IVP

$$x' = \cos x, \quad x(0) = 0 \tag{8.14}$$

by finding the second Picard iterate $x_2(t)$ and use (8.12) to find how good an approximation you get.

First we find the first Picard iterate $x_1(t)$. From equation (8.8) with $k = 0$ we get

$$\begin{aligned} x_1(t) &= x_0 + \int_{t_0}^{t} \cos(x_0(s))\, ds \\ &= \int_0^t 1 ds \\ &= t. \end{aligned}$$

From equation (8.8) with $k = 1$ we get that the second Picard iterate $x_2(t)$ is given by

$$\begin{aligned} x_2(t) &= x_0 + \int_{t_0}^{t} \cos(x_1(s))\, ds \\ &= \int_0^t \cos s\, ds \\ &= \sin t. \end{aligned}$$

To see how good an approximation $x_2(t) = \sin t$ is for the solution $x(t)$ of the IVP (8.14), we get, applying (8.12), that

$$|x(t) - \sin t| \leq \frac{1}{6}t^3.$$

$\triangle$

Corollary 8.17 *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and the Jacobian matrix $D_x f(t,x)$ is also continuous on D. Then for any $(t_0, x_0) \in D$ the IVP* (8.4) *has a unique solution on an interval containing t_0 in its interior.*

Proof Let $(t_0, x_0) \in D$; then there are positive numbers a and b such that the rectangle

$$R := \{(t,x) : |t - t_0| \leq a, \ \|x - x_0\|_1 \leq b\} \subset D.$$

In the proof of Theorem 8.12 we proved that $f(t,x)$ satisfies a uniform Lipschitz condition with respect to x on R with Lipschitz constant

$$K := \max\{\|D_x f(t,x)\| : (t,x) \in R\},$$

where $\|\cdot\|$ is matrix norm corresponding to the traffic norm $\|\cdot\|_1$ (see Definition 2.53). Let $M = \max\{\|f(t,x)\| : (t,x) \in \mathbb{R}\}$ and let $\alpha := \min\{a, \frac{b}{M}\}$. Then by the Picard-Lindelof theorem (Theorem 8.13) in the case where we use the rectangle R instead of Q and we use the l_1 norm (traffic norm), the IVP (8.4) has a unique solution on $[t_0 - \alpha, t_0 + \alpha]$. $\square$

Corollary 8.18 *Assume A is a continuous $n \times n$ matrix function and h is a continuous $n \times 1$ vector function on an interval I. If $(t_0, x_0) \in I \times \mathbb{R}$, then the IVP*

$$x' = A(t)x + h(t), \quad x(t_0) = x_0$$

has a unique solution.

Proof Let

$$f(t,x) = A(t)x + h(t);$$

then

$$D_x f(t,x) = A(t),$$

and this result follows from Theorem 8.17. $\square$

In Theorem 8.65, we will show that under the hypotheses of Corollary 8.18 all solutions of $x' = A(t)x + h(t)$ exist on the whole interval I. Also, in Theorem 8.65 a bound on solutions will be given.

Corollary 8.19 *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$, the scalar function $F : D \to \mathbb{R}$ is continuous, and $F(t, x_1, x_2, \cdots, x_n)$ has continuous partial* *derivatives with respect to the variables $x_1, x_2, \cdots, x_n$ on D. Then for any $(t_0, u_0, u_1, \cdots, u_{n-1}) \in D$ the IVP*

$$u^{(n)} = F(t, u, u', \cdots, u^{(n-1)}), \tag{8.15}$$

$$u(t_0) = u_0, \quad u'(t_0) = u_1, \quad \cdots, \quad u^{(n-1)}(t_0) = u_{n-1} \tag{8.16}$$

has a unique solution on an interval containing t_0 in its interior.

Proof In Example 8.2 we proved that the differential equation (8.15) is equivalent to the vector equation $x' = f(t, x)$, where f is given by (8.3). Note that $f : D \to \mathbb{R} \times \mathbb{R}^n$ and the Jacobian matrix

$$D_x f(t,x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ F_{x_1}(t, x_1, \cdots, x_n) & & \cdots & & F_{x_n}(t, x_1, \cdots, x_n) \end{pmatrix}$$

are continuous on D. The initial condition $x(t_0) = x_0$ corresponds to

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} u(t_0) \\ u'(t_0) \\ \vdots \\ u^{(n-1)}(t_0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}$$

and the result follows from Corollary 8.17. □

Example 8.20 In this example we apply Corollary 8.19 to the second-order scalar equation

$$u'' = (\sin t)e^u + u^2 + (u')^2.$$

This equation is of the form $u'' = F(t, u, u')$, where

$$F(t, x_1, x_2) = (\sin t)e^{x_1} + x_1^2 + (x_2)^2.$$

Let $D := \mathbb{R}^3$; then D is an open set and $F(t, x_1, x_2)$ is continuous on D. Also, $F_{x_1}(t, x_1, x_2) = (\sin t)e^{x_1} + 2x_1$ and $F_{x_2}(t, x_1, x_2) = 2x_2$ are continuous on D. Hence by Corollary 8.19 we have that every IVP

$$u'' = (\sin t)e^u + u^2 + (u')^2,$$
$$u(t_0) = u_0, \quad u'(t_0) = u_1$$

has a unique solution on an open interval containing t_0. △

Example 8.21 In this example we apply Corollary 8.19 to the second-order scalar equation

$$u'' = u^{\frac{1}{3}} + 3u' + e^{2t}.$$

This equation is of the form $u'' = F(t, u, u')$, where

$$F(t, x_1, x_2) = x_1^{\frac{1}{3}} + 3x_2 + e^{2t}.$$

It follows that $F_{x_1}(t,x_1,x_2) = \frac{1}{3x_1^{\frac{2}{3}}}$ and $F_{x_2}(t,x_1,x_2) = 3$. If we let D be either the open set $\{(t,x_1,x_2) : t \in \mathbb{R},\ x_1 \in (0,\infty),\ x_2 \in \mathbb{R}\}$ or the open set $\{(t,x_1,x_2) : t \in \mathbb{R},\ x_1 \in (-\infty,0),\ x_2 \in \mathbb{R}\}$, then by Corollary 8.19 we have that for any $(t_0,u_0,u_1) \in D$ the IVP

$$u'' = u^{\frac{1}{3}} + 3u' + e^{2t}, \quad u(t_0) = u_0, \quad u'(t_0) = u_1 \tag{8.17}$$

has a unique solution on an open interval containing t_0. Note that if $u_0 = 0$, then Corollary 8.19 does not apply to the IVP (8.17). △

8.3 Equicontinuity and the Ascoli-Arzela Theorem

In this section we define what is meant by an equicontinuous family of functions and state and prove the very important Ascoli-Arzela theorem (Theorem 8.26). First we give some preliminary definitions.

Definition 8.22 We say that the sequence of vector functions $\{x_m(t)\}_{m=1}^{\infty}$ is *uniformly bounded on an interval* I provided there is a constant M such that

$$\|x_m(t)\| \leq M,$$

for $m = 1,2,3,\cdots$, and for all $t \in I$, where $\|\cdot\|$ is any norm on $\mathbb{R}^n$.

Example 8.23 The sequence of vector functions

$$x_m(t) = \begin{pmatrix} 2t^m \\ \sin(mt) \end{pmatrix},$$

$m = 1,2,3,\cdots$ is uniformly bounded on the interval $I := [0,1]$, since

$$\|x_m(t)\|_1 = 2|t^m| + |\sin(mt)| \leq M := 3,$$

for all $t \in I$, and for all $m = 1,2,3,\cdots$. △

Definition 8.24 We say that the family of vector functions $\{x_\alpha(t)\}$, for α in some index set A, is *equicontinuous on an interval* I provided given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|x_\alpha(t) - x_\alpha(\tau)\| < \epsilon,$$

for all $\alpha \in A$ and for all $t,\tau \in I$ with $|t-\tau| < \delta$.

We will use the following lemma in the proof of the Ascoli-Arzela theorem (Theorem 8.26).

Lemma 8.25 (Cantor Selection Theorem) *Let $\{f_k\}$ be a uniformly bounded sequence of vector functions on $E \subset \mathbb{R}^n$. Then if D is a countable subset of E, there is a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ that converges pointwise on D.*

Proof If D is finite the proof is easy. Assume D is countably infinite; then D can be written in the form

$$D = \{x_1, x_2, x_3, \cdots\}.$$

Since $\{f_k(x_1)\}$ is a bounded sequence of vectors, there is a convergent subsequence $\{f_{1k}(x_1)\}$. Next consider the sequence $\{f_{1k}(x_2)\}$. Since this is a bounded sequence of vectors, there is a convergent subsequence $\{f_{2k}(x_2)\}$. Continuing in this fashion, we get further subsequences such that

$$\begin{array}{ll} f_{11}(x),\ f_{12}(x),\ f_{13}(x),\ \cdots & \text{converges at } x_1 \\ f_{21}(x),\ f_{22}(x),\ f_{23}(x),\ \cdots & \text{converges at } x_1,\ x_2 \\ f_{31}(x),\ f_{32}(x),\ f_{33}(x),\ \cdots & \text{converges at } x_1,\ x_2,\ x_3 \\ \qquad\qquad \cdots \qquad \cdots & \cdots \end{array}$$

It follows that the diagonal sequence $\{f_{kk}\}$ is a subsequence of $\{f_k\}$ that converges pointwise on D. □

Theorem 8.26 (Ascoli-Arzela Theorem) *Let E be a compact subset of $\mathbb{R}^m$ and $\{f_k\}$ be a sequence of n-dimensional vector functions that is uniformly bounded and equicontinuous on E. Then there is a subsequence $\{f_{k_j}\}$ that converges uniformly on E.*

Proof In this proof we will use the same notation $\|\cdot\|$ for a norm on $\mathbb{R}^m$ and $\mathbb{R}^n$. If E is finite the result is obvious. Assume E is infinite and let

$$D = \{x_1, x_2, x_3, \cdots\}$$

be a countable dense subset of E. By the Cantor selection theorem (Theorem 8.25) there is a subsequence $\{f_{k_j}\}$ that converges pointwise on D. We claim that $\{f_{k_j}\}$ converges uniformly on E. To see this, let $\epsilon > 0$ be given. By the equicontinuity of the sequence $\{f_k\}$ on E there is a $\delta > 0$ such that

$$\|f_k(x) - f_k(y)\| < \frac{\epsilon}{3}, \tag{8.18}$$

when $\|x - y\| < \delta$, $x, y \in E$, $k \geq 1$. Define the ball about x_i with radius δ by

$$B(x_i) := \{x \in E : \|x - x_i\| < \delta\},$$

for $i = 1, 2, 3, \cdots$. Then $\{B(x_i)\}$ is an open covering of E. Since E is compact there is an integer J such that

$$\{B(x_i)\}_{i=1}^{J}$$

covers E. Since $\{f_{k_j}(x)\}$ converges pointwise on the finite set

$$\{x_1, x_2, \cdots, x_J\},$$

there is an integer K such that

$$\|f_{k_j}(x_i) - f_{k_m}(x_i)\| < \frac{\epsilon}{3} \tag{8.19}$$

when $k_j, k_m \geq K$, $1 \leq i \leq J$. Now assume $x \in E$; then

$$x \in B(x_{i_0}),$$

for some $1 \leq i_0 \leq J$. Using (8.18) and (8.19), we get

$$\begin{aligned} \|f_{k_j}(x) - f_{k_m}(x)\| &\leq \|f_{k_j}(x) - f_{k_j}(x_{i_0})\| \\ &+ \|f_{k_j}(x_{i_0}) - f_{k_m}(x_{i_0})\| + \|f_{k_m}(x_{i_0}) - f_{k_m}(x)\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

for $k_j, k_m \geq K$. □

8.4 Cauchy-Peano Theorem

In this section we use the Ascoli-Arzela theorem to prove the Cauchy-Peano theorem (8.27), which is a very important existence theorem.

Theorem 8.27 (Cauchy-Peano Theorem) *Assume $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and f is a continuous n-dimensional vector function on the rectangle*

$$Q := \{(t, x) : |t - t_0| \leq a, \|x - x_0\| \leq b\}.$$

Then the initial value problem (8.4) *has a solution x on $[t_0 - \alpha, t_0 + \alpha]$ with $\|x(t) - x_0\| \leq b$, for $t \in [t_0 - \alpha, t_0 + \alpha]$, where*

$$\alpha := \min\left\{a, \frac{b}{M}\right\}$$

and

$$M := \max\{\|f(t, x)\| : (t, x) \in Q\}.$$

Proof For m a positive integer, subdivide the interval $[t_0, t_0 + \alpha]$ into 2^m equal parts so that the interval $[t_0, t_0 + \alpha]$ has the partition points

$$t_0 < t_1 < t_2 \cdots < t_{2^m} = t_0 + \alpha.$$

So

$$t_j = t_0 + \frac{\alpha j}{2^m}, \quad 0 \leq j \leq 2^m.$$

For each positive integer m we define the function x_m (see Figure 1 for the scalar case) recursively with respect to the intervals $[t_j, t_{j+1}]$, $0 \leq j \leq 2^m$, as follows:

$$x_m(t) = x_0 + f(t_0, x_0)(t - t_0), \quad t_0 \leq t \leq t_1,$$

$x_1 = x_m(t_1)$, and, for $1 \leq j \leq 2^m - 1$,

$$x_m(t) = x_j + f(t_j, x_j)(t - t_j), \quad t_j \leq t \leq t_{j+1},$$

where

$$x_{j+1} = x_m(t_{j+1}).$$

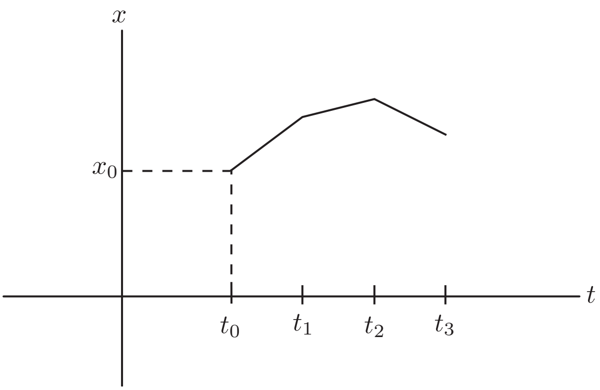

Figure 1. Approximate solution $x_m(t)$.

We show by finite mathematical induction with respect to j, $1 \le j \le 2^m$, that $x_m(t)$ is well defined on $[t_0, t_j]$, and

$$\|x(t) - x_0\| \le b, \quad \text{for} \quad t \in [t_0, t_j].$$

First, for $t \in [t_0, t_1]$,

$$x_m(t) = x_0 + f(t_0, x_0)(t - t_0)$$

is well defined and

$$\begin{aligned} \|x_m(t) - x_0\| &= \|f(t_0, x_0)\|(t - t_0) \\ &\le \frac{M\alpha}{2^m} \\ &\le b, \end{aligned}$$

for $t \in [t_0, t_1]$. Hence the graph of x_m on the first subinterval $[t_0, t_1]$ is in Q. In particular, $x_1 = x_m(t_1)$ is well defined with $(t_1, x_1) \in Q$.

Now assume $1 \le j \le 2^m - 1$ and that $x_m(t)$ is well defined on $[t_0, t_j]$ with

$$\|x_m(t) - x_0\| \le b, \quad \text{on} \quad [t_0, t_j].$$

Since $(t_j, x_j) = (t_j, x_m(t_j)) \in Q$,

$$x_m(t) = x_j + f(t_j, x_j)(t - t_j), \quad t_j \le t \le t_{j+1}$$

is well defined. Also,

$$\begin{aligned}
\|x_m(t)-x_0\| &= \|[x_m(t)-x_j]+[x_j-x_{j-1}]+\cdots+[x_1-x_0]\| \\
&\le \sum_{k=0}^{j-1}\|x_{k+1}-x_k\|+\|x_m(t)-x_j\| \\
&\le \sum_{k=0}^{j-1}\|f(t_k,x_k)(t_{k+1}-t_k)\|+\|f(t_j,x_j)(t-t_j)\| \\
&\le \sum_{k=0}^{j-1}\frac{\alpha}{2^m}\|f(t_k,x_k)\|+\|f(t_j,x_j)\|(t-t_j) \\
&\le \frac{\alpha j}{2^m}M+\frac{\alpha}{2^m}M \\
&\le \alpha M \\
&\le b,
\end{aligned}$$

for $t\in[t_j,t_{j+1}]$. Hence we have shown that $x_m(t)$ is well defined on $[t_0,t_0+\alpha]$ and

$$\|x_m(t)-x_0\|\le b$$

on $[t_0,t_0+\alpha]$. We will show that the sequence $\{x_m\}$ has a subsequence that converges uniformly on $[t_0,t_0+\alpha]$ to a vector function z and z is a solution of the IVP (8.4) on $[t_0,t_0+\alpha]$ whose graph on $[t_0,t_0+\alpha]$ is in Q.

We now claim that for all $t,\tau\in[t_0,t_0+\alpha]$

$$\|x_m(t)-x_m(\tau)\|\le M|t-\tau|. \tag{8.20}$$

We will prove this only for the case $t_{k-1}<\tau\le t_k<t_l<t\le t_{l+1}$ as the other cases are similar. For this case

$$\begin{aligned}
&\|x_m(t)-x_m(\tau)\| \\
= &\ \|[x_m(t)-x_m(t_l)]+\sum_{j=k}^{l-1}[x_m(t_{j+1})-x_m(t_j)]+[x_m(t_k)-x_m(\tau)]\| \\
\le &\ \|x_m(t)-x_m(t_l)\|+\sum_{j=k}^{l-1}\|x_m(t_{j+1})-x_m(t_j)\|+\|x_m(t_k)-x_m(\tau)\| \\
\le &\ \|f(t_l,x_l)\|(t-t_l)+\sum_{j=k}^{l-1}\|f(t_j,x_j)\|(t_{j+1}-t_j) \\
+ &\ \|f(t_{k-1},x_{k-1})\|(t_k-\tau) \\
\le &\ M(t-\tau).
\end{aligned}$$

Hence (8.20) holds for all $t,\tau\in[t_0,t_0+\alpha]$.

Since f is continuous on the compact set Q, f is uniformly continuous on Q. Hence given any $\epsilon>0$ there is a $\delta>0$ such that

$$\|f(t,x)-f(\tau,y)\|<\epsilon, \tag{8.21}$$

for all $(t,x),(\tau,y) \in Q$ with $|t-\tau| < \delta$, $\|x-y\| < \delta$. Pick m sufficiently large so that

$$\max\left\{\frac{\alpha}{2^m}, \frac{M\alpha}{2^m}\right\} < \delta.$$

Then, for $t_j \le t \le t_{j+1}$, $0 \le j \le 2^m - 1$, since

$$\|x_m(t) - x_j\| \le \frac{M\alpha}{2^m} < \delta$$

and

$$|t_j - t| \le \frac{\alpha}{2^m} < \delta,$$

we get from (8.21)

$$\|f(t, x_m(t)) - f(t_j, x_j)\| < \epsilon,$$

for $t_j \le t \le t_{j+1}$, $0 \le j \le 2^m - 1$. Since

$$x_m'(t) = f(t_j, x_j),$$

for $t_j < t < t_{j+1}$, $0 \le j \le 2^m - 1$, we get that

$$\|f(t, x_m(t)) - x_m'(t)\| < \epsilon,$$

for $t_j < t < t_{j+1}$, $0 \le j \le 2^m - 1$. Hence if

$$g_m(t) := x_m'(t) - f(t, x_m(t)),$$

for $t \in [t_0, t_0 + \alpha]$, where $x_m'(t)$ exists, then we have shown that

$$\lim_{m\to\infty} g_m(t) = 0$$

uniformly on $[t_0, t_0 + \alpha]$, except for a countable number of points.

Fix $t \in [t_0, t_0 + \alpha]$; then there is a j such that $t_j \le t \le t_{j+1}$. Then

$$\begin{aligned} x_m(t) - x_0 &= x_m(t) - x_m(t_0) \\ &= [x_m(t) - x_m(t_j)] + \sum_{k=1}^{j} [x_m(t_k) - x_m(t_{k-1})] \\ &= \int_{t_j}^{t} x_m'(s)\, ds + \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} x_m'(s)\, ds \\ &= \int_{t_0}^{t} x_m'(s)\, ds \end{aligned}$$

for $t \in [t_0, t_0 + \alpha]$. Hence

$$x_m(t) = x_0 + \int_{t_0}^{t} [f(s, x_m(s)) + g_m(s)]\, ds, \tag{8.22}$$

for $t \in [t_0, t_0 + \alpha]$. Since

$$\|x_m(t)\| \le \|x_m(t) - x_0\| + \|x_0\| \le \|x_0\| + b,$$

the sequence $\{x_m(t)\}$ is uniformly bounded on $[t_0, t_0+\alpha]$. Since the sequence $\{x_m(t)\}$ is uniformly bounded and by (8.20) equicontinuous on $[t_0, t_0 + \alpha]$, we get from the Ascoli-Arzela theorem that the sequence $\{x_m(t)\}$ has a uniformly convergent subsequence $\{x_{m_k}(t)\}$ on $[t_0, t_0 + \alpha]$. Let

$$z(t) := \lim_{k\to\infty} x_{m_k}(t),$$

for $t \in [t_0, t_0 + \alpha]$. It follows that

$$\lim_{k\to\infty} f(t, x_{m_k}(t)) = f(t, z(t))$$

uniformly on $[t_0, t_0 + \alpha]$. Replacing m in equation (8.22) by $\{m_k\}$ and letting $k \to \infty$, we get that $z(t)$ is a solution of the integral equation

$$z(t) = x_0 + \int_{t_0}^{t} f(s, z(s))\, ds$$

on $[t_0, t_0+\alpha]$. It follows that $z(t)$ is a solution of the IVP (8.4) with $\|z(t) - x_0\| \le b$, for $t \in [t_0, t_0 + \alpha]$. Similarly, we can show that the IVP (8.4) has a solution $v(t)$ on $[t_0 - \alpha, t_0]$ with $\|v(t) - x_0\| \le b$, for $t \in [t_0 - \alpha, t_0]$. It follows that

$$x(t) := \begin{cases} v(t), & t \in [t_0 - \alpha, t_0], \\ z(t), & t \in [t_0, t_0 + \alpha] \end{cases}$$

is a solution of the IVP (8.4) with

$$\|x(t) - x_0\| \le b$$

on $[t_0 - \alpha, t_0 + \alpha]$.

□

Under the hypotheses of the Cauchy-Peano theorem we get that IVPs have solutions, but they need not be unique. To see how bad things can be, we remark that in Hartman [**19**], pages 18–23, an example is given of a scalar equation $x' = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is continuous, where for every IVP (8.4) there is more than one solution on $[t_0, t_0+\epsilon]$ and $[t_0-\epsilon, t_0]$ for arbitrary $\epsilon > 0$.

Theorem 8.28 *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and K is a compact subset of D. Then there is an $\alpha > 0$ such that for all $(t_0, x_0) \in K$ the IVP (8.4) has a solution on $[t_0 - \alpha, t_0 + \alpha]$.*

Proof For $(t, x), (t, y) \in \mathbb{R} \times \mathbb{R}^n$, define the distance from (t, x) to (t, y) by

$$d[(t, x), (\tau, y)] = \max\{|t - \tau|, \|x - y\|\}.$$

If the boundary of D, $\partial D \ne \emptyset$, set $\rho = d(K, \partial D) > 0$. In this case define

$$K_\rho = \{(t, x) : d[(t, x), K] \le \frac{\rho}{2}\}.$$

If $\partial D = \emptyset$, then let

$$K_\rho = \{(t, x) : d[(t, x), K] \le 1\}.$$

Then $K_\rho \subset D$ and K_ρ is compact. Let

$$M := \max\{\|f(t, x)\| : (t, x) \in K_\rho\}.$$

Let $(t_0, x_0) \in K$; then if $\delta := \min\{1, \frac{\rho}{2}\}$,

$$Q := \{(t,x) : |t - t_0| \leq \delta, \|x - x_0\| \leq \delta\} \subset K_\rho.$$

Note that

$$\|f(t,x)\| \leq M, \quad (t,x) \in Q.$$

Hence by the Cauchy-Peano theorem (Theorem 8.27) the IVP (8.4) has a solution on $[t_0 - \alpha, t_0 + \alpha]$, where $\alpha := \min\{\frac{\rho}{2}, \frac{\rho}{2M}\}$ if $\partial D \neq \emptyset$ and $\alpha := \min\{1, \frac{1}{M}\}$ if $\partial D = \emptyset$. □

8.5 Extendability of Solutions

In this section we will be concerned with proving that each solution of $x' = f(t,x)$ can be extended to a maximal interval of existence. First we define what we mean by the extension of a solution.

Definition 8.29 Assume x is a solution of $x' = f(t,x)$ on an interval I. We say that a solution y on an interval J is an *extension* of x provided $J \supset I$ and $y(t) = x(t)$, for $t \in I$.

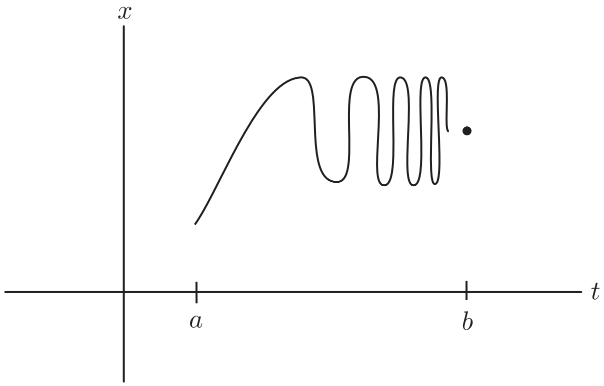

Figure 2. Impossible solution of scalar equation $x' = f(t,x)$ on $[a,b)$.

The next theorem gives conditions where a solution of $x' = f(t,x)$ on a half open interval $[a,b)$ can be extended to a solution on the closed interval $[a,b]$. This result implies that there is no solution to the scalar equation $x' = f(t,x)$ of the type shown in Figure 2.

Theorem 8.30 *Assume that f is continuous on $D \subset \mathbb{R} \times \mathbb{R}^n$ and that x is a solution of $x' = f(t,x)$ on the half-open interval $[a,b)$. Assume there is an increasing sequence $\{t_k\}$ with limit b and $\lim_{k\to\infty} x(t_k) = x_0$. Further assume there are constants $M > 0$, $\alpha > 0$, $\beta > 0$ such that $\|f(t,x)\| \leq M$ on $D \cap \{(t,x) : 0 < b - t \leq \alpha, \|x - x_0\| \leq \beta\}$. Furthermore, if $f(b, x_0)$ can be defined so that f is continuous on $D \cup \{(b, x_0)\}$, then x can be extended to be a solution on $[a,b]$.*

Proof Pick an integer N sufficiently large so that

$$0 < b - t_k \le \alpha, \quad \|x(t_k) - x_0\| < \frac{\beta}{2}, \quad \text{and} \quad 0 < b - t_k \le \frac{\beta}{2M},$$

for all $k \ge N$. We claim

$$\|x(t) - x_0\| < \beta, \quad \text{for} \quad t_N < t < b.$$

Assume not; then there is a first point $\tau > t_N$ such that

$$\|x(\tau) - x_0\| = \beta.$$

But then

$$\begin{aligned} \frac{\beta}{2} &= \beta - \frac{\beta}{2} \\ &< \|x(\tau) - x_0\| - \|x(t_N) - x_0\| \\ &\le \|x(\tau) - x(t_N)\| \\ &= \|\int_{t_N}^{\tau} x'(s)\, ds\| \\ &= \|\int_{t_N}^{\tau} f(s, x(s))\, ds\| \\ &\le M|\tau - t_N| \\ &\le M \cdot \frac{\beta}{2M} = \frac{\beta}{2}, \end{aligned}$$

which is a contradiction and hence our claim holds.

Note that for $t, \tau \in [t_N, b)$,

$$\|x(t) - x(\tau)\| = \|\int_{\tau}^{t} x'(s)\, ds\| = \|\int_{\tau}^{t} f(s, x(s))\, ds\| \le M|t - \tau|.$$

It follows that the Cauchy criterion for $\lim_{t\to b-} x(t)$ is satisfied and hence

$$\lim_{t\to b-} x(t) = x_0$$

exists. Define

$$x(b) = x_0 = \lim_{t\to b-} x(t).$$

Now assume that $f(b, x_0)$ is defined so that f is continuous on $D \cup \{(b, x_0)\}$, then

$$\begin{aligned} \lim_{t\to b-} x'(t) &= \lim_{t\to b-} f(t, x(t)) \\ &= f(b, x_0). \end{aligned}$$

Hence, using the mean value theorem,

$$\begin{aligned} x'(b) &= \lim_{t\to b-} \frac{x(b)-x(t)}{b-t} \\ &= \lim_{t\to b-} x'(\xi_t), \quad t<\xi_t<b, \\ &= f(b,x_0) \\ &= f(b,x(b)). \end{aligned}$$

Therefore, x is a solution on $[a,b]$.

□

Definition 8.31 Assume D is an open subset of $\mathbb{R}\times\mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and x is a solution of $x' = f(t,x)$ on (a,b). Then we say (a,b) is a *right maximal interval of existence* for x provided there does not exist a $b_1 > b$ and a solution y such that y is a solution on (a,b_1) and $y(t) = x(t)$ for $t \in (a,b)$. *Left maximal interval of existence* for x is defined in the obvious way. Finally, we say (a,b) is a *maximal interval of existence* for x provided it is both a right and left maximal interval of existence for x.

Definition 8.32 Assume D is an open subset of $\mathbb{R}\times\mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and x is a solution of $x' = f(t,x)$ on (a,b). We say $x(t)$ *approaches the boundary* of D, denoted ∂D, as $t \to b-$, write $x(t) \to \partial D$ as $t \to b-$, in case either
(i) $b = \infty$
    or
(ii) $b < \infty$ and for each compact subset $K \subset D$ there is a $t_K \in (a,b)$ such that $(t,x(t)) \notin K$ for $t_K < t < b$.
Similarly (see Exercise 8.20), we can define $x(t)$ approaches the boundary of D as $t \to a+$, write $x(t) \to \partial D$ as $t \to a+$.

Theorem 8.33 (Extension Theorem) *Assume D is an open subset of $\mathbb{R}\times\mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous, and x is a solution of $x' = f(t,x)$ on (a,b), $-\infty \le a < b \le \infty$. Then x can be extended to a maximal interval of existence (α,ω), $-\infty \le \alpha < \omega \le \infty$. Furthermore, $x(t) \to \partial D$ as $t \to \omega-$ and $x(t) \to \partial D$ as $t \to \alpha+$.*

Proof We will just show that x can be extended to a right maximal interval of existence (a,ω) and $x(t) \to \partial D$ as $t \to \omega-$.

Let $\{K_k\}$ be a sequence of open sets such that the closure of K_k, $\bar{K}_k$, is compact, $\bar{K}_k \subset K_{k+1}$, and $\cup_{k=1}^{\infty} K_k = D$ (see Exercise 8.21).

If $b = \infty$, we are done, so assume $b < \infty$. We consider two cases:

Case 1. Assume for all $k \ge 1$ there is a τ_k such that for $t \in (\tau_k, b)$ we have that

$$(t,x(t)) \notin \bar{K}_k.$$

Assume there is a $b_1 > b$ and a solution y that is an extension of x to the interval (a, b_1). Fix $t_0 \in (a, b)$; then

$$A := \{(t, y(t)) : t_0 \leq t \leq b\}$$

is a compact subset of D. Pick an interger k_0 sufficiently large so that

$$\{(t, x(t)) : t_0 \leq t < b\} \subset A \subset K_{k_0}.$$

This is a contradiction and hence (a, b) is a right maximal interval of existence for x. Also, it is easy to see that $x(t) \to \partial D$ as $t \to b-$.

Case 2. There is an integer m_0 and an increasing sequence $\{t_k\}$ with limit b such that $(t_k, x(t_k)) \in \bar{K}_{m_0}$, for all $k \geq 1$. Since $\bar{K}_{m_0}$ is compact, there is a subsequence $(t_{k_j}, x(t_{k_j}))$ such that

$$\lim_{j \to \infty} (t_{k_j}, x(t_{k_j})) = (b, x_0)$$

exists and $(b, x_0) \in \bar{K}_{m_0} \subset D$. By Theorem 8.30 we get that we can extend the solution x to $(a, b]$ by defining $x(b) = x_0$. By Theorem 8.28 for each $k \geq 1$ there is a $\delta_k > 0$ such that for all $(t_1, x_1) \in \overline{K}_k$ the IVP

$$x' = f(t, x), \quad x(t_1) = x_1$$

has a solution on $[t_1 - \delta_k, t_1 + \delta_k]$. Hence the IVP $x' = f(t, x)$, $x(b) = x_0$, has a solution y on $[b, b + \delta_{m_0}]$ and so if we extend the definition of x by

$$x(t) = \begin{cases} x(t), & \text{on } (a, b], \\ y(t), & \text{on } (b, b + \delta_{m_0}], \end{cases}$$

then x is a solution on $(a, b + \delta_{m_0}]$. If the point $(b + \delta_{m_0}, x(b + \delta_{m_0})) \in \overline{K}_{m_0}$, then we repeat the process using a solution of the IVP

$$x' = f(t, x), \quad x(b + \delta_{m_0}) = x(b + \delta_{m_0}),$$

to get an extension of the solution x to $(a, b + 2\delta_{m_0}]$, which we also denote by x. Since $\overline{K}_{m_0}$ is compact, there is a first integer $j(m_0)$ such that $(b_1, x(b_1)) \notin \overline{K}_{m_0}$, where

$$b_1 = b + j(m_0)\delta_{m_0}.$$

However, $(b_1, x(b_1)) \in D$ and hence there is an $m_1 > m_0$ such that

$$(b_1, x(b_1)) \in K_{m_1}.$$

Therefore, the extension procedure can be repeated using $\overline{K}_{m_1}$ and the associated δ_{m_1}. Then there is a first integer $j(m_2)$ such that $(b_2, x(b_2)) \notin \overline{K}_{m_1}$, where

$$b_2 = b_1 + j(m_1)\delta_{m_1}.$$

Continuing in this fashion, we get an infinite sequence $\{b_k\}$. We then define

$$\omega = \lim_{k \to \infty} b_k.$$

We claim we have that the solution x has been extended to be a solution of $x' = f(t, x)$ on the interval (a, ω). To see this, let $\tau \in (a, \omega)$; then there is a b_{k_0} such that $\tau < b_{k_0} < \omega$ and x is a solution on $(a, b_{k_0}]$. Since $\tau \in (a, \omega)$

is arbitrary, x is a solution on the interval (a, ω). If $\omega = \infty$, we are done. Assume that $\omega < \infty$. Note that the interval (a, ω) is right maximal, since $(b_k, x(b_k)) \notin \overline{K}_{m_{k-1}}$, for each $k \geq 1$ (Why?).

We claim that

$$x(t) \to \partial D \quad \text{as} \quad t \to \omega - .$$

To see this, assume not; then there is a compact set $H \subset D$ and a strictly increasing sequence $\{\tau_k\}$ with limit ω such that

$$(\tau_k, x(\tau_k)) \in H, \quad \text{for} \quad k \geq 1.$$

But by the definition of the sets $\{K_m\}$, there is a $\overline{m}$ such that $H \subset K_{\overline{m}}$ which leads to a contradiction. □

Theorem 8.34 (Extended Cauchy-Peano Theorem) *Assume that D is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is continuous. Let*

$$Q := \{(t, x) : |t - t_0| \leq a, \ \|x - x_0\| \leq b\} \subset D.$$

Let

$$\alpha = \min \left\{ a, \frac{b}{M} \right\},$$

where

$$M = \max\{\|f(t, x)\| : (t, x) \in Q\}.$$

Then **every** *solution of the IVP* (8.4) *exists on* $[t_0 - \alpha, t_0 + \alpha]$.

Proof Assume x is a solution of the IVP (8.4) on $[t_0, t_0 + \epsilon]$, where $0 < \epsilon < \alpha$. Let $[t_0, \omega)$ be a right maximal interval of existence for x. Since $x(t) \to \partial D$ as $t \to \omega-$ and $Q \subset D$ is compact, there is a $\beta > t_0$ such that

$$x(t) \notin Q,$$

for $\beta \leq t < \omega$. Let t_1 be the first value of $t > t_0$ such that $(t_1, x(t_1)) \in \partial Q$. If $t_1 = t_0 + a$, we are done. So assume

$$\|x(t_1) - x_0\| = b.$$

Note that

$$\begin{aligned} b &= \|x(t_1) - x_0\| \\ &= \left\| \int_{t_0}^{t_1} x'(s)\, ds \right\| \\ &= \left\| \int_{t_0}^{t_1} f(s, x(s))\, ds \right\| \\ &\leq \int_{t_0}^{t_1} \|f(s, x(s))\|\, ds \\ &\leq M(t_1 - t_0). \end{aligned}$$

Solving this inequality for t_1, we get

$$t_1 \geq t_0 + \frac{b}{M} \geq t_0 + \alpha.$$

The other cases of the proof are left to the reader. □

Corollary 8.35 *Assume that* $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and bounded; then every solution of* $x' = f(t, x)$ *has the maximal interval of existence* $(-\infty, \infty)$.

Proof Since f is bounded on $\mathbb{R} \times \mathbb{R}^n$, there is a $M > 0$ such that

$$\|f(t, x)\| \leq M,$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Assume x_1 is a solution of $x' = f(t, x)$ with maximal interval of existence (α_1, ω_1), $-\infty \leq \alpha_1 < \omega_1 \leq \infty$. Let $t_0 \in (\alpha_1, \omega_1)$ and let

$$x_0 := x_1(t_0).$$

Then x_1 is a solution of the IVP (8.4). For arbitrary $a > 0$, $b > 0$ we have by Theorem 8.34 that x_1 is a solution on $[t_0 - \alpha, t_0 + \alpha]$, where

$$\alpha = \min\left\{a, \frac{b}{M}\right\}.$$

Since M is fixed, we can make α as large as we want by taking a and b sufficiently large, and the proof is complete. □

Theorem 8.36 (Uniqueness Theorem) *Assume* $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ *and the* n*-dimensional vector function* f *is continuous on the rectangle*

$$Q := \{(t, x) : t_0 \leq t \leq t_0 + a, \|x - x_0\| \leq b\}.$$

If the dot product

$$[f(t, x_1) - f(t, x_2)] \cdot [x_1 - x_2] \leq 0,$$

for all (t, x_1), $(t, x_2) \in Q$, *then the IVP* (8.4) *has a unique solution in* Q.

Proof By the Cauchy-Peano theorem (Theorem 8.27), the IVP (8.4) has a solution. It remains to prove the uniqueness. Assume that $x_1(t)$ and $x_2(t)$ satisfy the IVP (8.4) on the interval $[t_0, t_0 + \epsilon]$ for some $\epsilon > 0$ and their graphs are in Q for $t \in [t_0, t_0 + \epsilon]$. Let

$$h(t) := \|x_1(t) - x_2(t)\|^2, \quad t \in [t_0, t_0 + \epsilon],$$

where $\|\cdot\|$ is the Euclidean norm. Note that $h(t) \geq 0$ on $[t_0, t_0 + \epsilon]$ and $h(t_0) = 0$. Also, since $h(t)$ is given by the dot product,

$$h(t) = [x_1(t) - x_2(t)] \cdot [x_1(t) - x_2(t)],$$

we get

$$\begin{aligned} h'(t) &= 2[x_1'(t) - x_2'(t)] \cdot [x_1(t) - x_2(t)] \\ &= 2[f(t, x_1(t)) - f(t, x_2(t))] \cdot [x_1(t) - x_2(t)] \\ &\leq 0 \end{aligned}$$

on $[t_0, t_0 + \epsilon]$. Thus $h(t)$ is nonincreasing on $[t_0, t_0 + \epsilon]$. Since $h(t_0) = 0$ and $h(t) \geq 0$ we get that $h(t) = 0$ for $t \in [t_0, t_0 + \epsilon]$. Hence

$$x_1(t) \equiv x_2(t)$$

on $[t_0, t_0 + \epsilon]$. Hence the IVP (8.4) has only one solution. $\square$

The following corollary follows from Theorem 8.36, and its proof is Exercise 8.22.

Corollary 8.37 *Assume $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and the scalar function f is continuous on the planar rectangle*

$$Q := \{(t, x) : t_0 \leq t \leq t_0 + a, |x - x_0| \leq b\}.$$

If for each fixed $t \in [t_0, t_0 + a]$, $f(t, x)$ is nonincreasing in x, then the IVP (8.4) *has a unique solution in Q.*

In the next example we give an application of Corollary 8.37, where the Picard-Lindelof theorem (Theorem 8.13) does not apply.

Example 8.38 Consider the IVP

$$x' = f(t, x), \quad x(0) = 0, \tag{8.23}$$

where

$$f(t, x) := \begin{cases} 0, & t = 0, \ -\infty < x < \infty, \\ 2t, & 0 < t \leq 1, \ -\infty < x < 0, \\ 2t - \frac{4x}{t}, & 0 < t \leq 1, \ 0 \leq x \leq t^2, \\ -2t, & 0 < t \leq 1, \ t^2 < x < \infty. \end{cases}$$

It is easy to see that f is continuous on $[0, 1] \times \mathbb{R}$ and for each fixed $t \in [0, 1]$ $f(t, x)$ is nonincreasing with respect to x. Hence by Corollary 8.37 the IVP (8.23) has a unique solution. See Exercise 8.23 for more results concerning this example.

$\triangle$

8.6 Basic Convergence Theorem

In this section we are concerned with proving the basic convergence theorem (see Hartman's book [**19**]). We will see that many results depend on this basic convergence theorem; for example, the continuous dependence of solutions on initial conditions, initial points, and parameters.

Theorem 8.39 (Basic Convergence Theorem) *Assume that $\{f_k\}$ is a sequence of continuous n-dimensional vector functions on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ and assume that*

$$\lim_{k \to \infty} f_k(t, x) = f(t, x)$$

uniformly on each compact subset of D. For each integer $k \geq 1$ let x_k be a solution of the IVP $x' = f_k(t, x)$, $x(t_k) = x_{0k}$, with $(t_k, x_{0k}) \in D$, $k \geq 1$, and $\lim_{k \to \infty}(t_k, x_{0k}) = (t_0, x_0) \in D$ and let the solution x_k have maximal interval of existence (α_k, ω_k), $k \geq 1$. Then there is a solution x of the *limit IVP* (8.4) *with maximal interval of existence* (α, ω) *and a subsequence* $\{x_{k_j}(t)\}$ *of* $\{x_k(t)\}$ *such that given any compact interval* $[\tau_1, \tau_2] \subset (\alpha, \omega)$

$$\lim_{j\to\infty} x_{k_j}(t) = x(t)$$

uniformly on $[\tau_1, \tau_2]$ *in the sense that there is an integer* $J = J(\tau_1, \tau_2)$ *such that for all* $j \geq J$, $[\tau_1, \tau_2] \subset (\alpha_{k_j}, \omega_{k_j})$ *and*

$$\lim_{j\to\infty, j\geq J} x_{k_j}(t) = x(t)$$

uniformly on $[\tau_1, \tau_2]$. *Furthermore,*

$$\limsup_{j\to\infty} \alpha_{k_j} \leq \alpha < \omega \leq \liminf_{j\to\infty} \omega_{k_j}.$$

Proof We will only prove that there is a solution of the limit IVP (8.4) with right maximal interval of existence $[t_0, \omega)$ and a subsequence $\{x_{k_j}(t)\}$ of $\{x_k(t)\}$ such that for each $\tau \in (t_0, \omega)$ there is an integer $J = J(\tau)$ such that $[t_0, \tau] \subset (\alpha_{k_j}, \omega_{k_j})$, for $j \geq J$ and

$$\lim_{j\to\infty, j\geq J} x_{k_j}(t) = x(t)$$

uniformly on $[t_0, \tau]$.

Let $\{K_k\}_{k=1}^{\infty}$ be a sequence of open subsets of D such that $\overline{K}_k$ is compact, $\overline{K}_k \subset K_{k+1}$, and $D = \cup_{k=1}^{\infty} K_k$. For each $k \geq 1$, if $\partial D \neq \emptyset$ let

$$H_k := \left\{(t, x) \in D : d((t, x), \overline{K}_k) \leq \frac{\rho_k}{2}\right\},$$

where $\rho_k := d(\partial D, \overline{K}_k)$, and if $\partial D = \emptyset$ let

$$H_k := \left\{(t, x) \in D : d((t, x), \overline{K}_k) \leq 1\right\}.$$

Note that H_k is compact and $\overline{K_k} \subset H_k \subset D$, for each $k \geq 1$. Let $f_0(t, x) := f(t, x)$ in the remainder of this proof. Since

$$\lim_{k\to\infty} f_k(t, x) = f(t, x)$$

uniformly on each compact subset of D, $f_0(t, x) = f(t, x)$ is continuous on D and for each $k \geq 1$ there is an $M_k > 0$ such that

$$\|f_m(t, x)\| \leq M_k \quad \text{on} \quad H_k,$$

for all $m \geq 0$. For each $k \geq 1$ there is a $\delta_k > 0$ such that for all $m \geq 0$ and for all $(\tau, y) \in \overline{K}_k$ every solution z of the IVP

$$x' = f_m(t, x), \quad x(\tau) = y$$

exists on $[\tau - \delta_k, \tau + \delta_k]$ and satisfies $(t, z(t)) \in H_k$, for $t \in [\tau - \delta_k, \tau + \delta_k]$. Since $(t_0, x_0) \in D$, there is an integer $m_1 \geq 1$ such that $(t_0, x_0) \in K_{m_1}$. Let

$$\epsilon_k := \frac{\delta_k}{3}, \quad \text{for} \quad k \geq 1.$$

Since

$$\lim_{k\to\infty} (t_k, x_{0k}) = (t_0, x_0),$$

there is an integer N such that

$$(t_k, x_{0k}) \in K_{m_1} \quad \text{and} \quad |t_k - t_0| < \epsilon_{m_1},$$

for all $k \geq N$. Then for $k \geq N$,

$$[t_0, t_0 + \epsilon_{m_1}] \subset (\alpha_k, \omega_k)$$

and $(t, x_k(t)) \in H_{m_1}$, for $k \geq N$. This implies that the sequence of functions $\{x_k\}_{k=N}^{\infty}$ is uniformly bounded on $[t_0, t_0 + \epsilon_{m_1}]$ and since

$$\begin{aligned} \|x_k(\tau_2) - x_k(\tau_1)\| &= \|\int_{\tau_1}^{\tau_2} x_k'(s)\, ds\| \\ &= \|\int_{\tau_1}^{\tau_2} f_k(s, x_k(s))\, ds\| \\ &\leq M_{m_1}|\tau_2 - \tau_1|, \end{aligned}$$

for all $\tau_1, \tau_2 \in [t_0, t_0 + \epsilon_{m_1}]$, the sequence of functions $\{x_k\}_{k=N}^{\infty}$ is equicontinuous on $[t_0, t_0+\epsilon_{m_1}]$. By the Ascoli-Arzela theorem (Theorem 8.26) there is a subsequence $\{k_1(j)\}_{j=1}^{\infty}$ of the sequence $\{k\}_{k=N}^{\infty}$ such that

$$\lim_{j\to\infty} x_{k_1(j)}(t) = x(t)$$

uniformly on $[t_0, t_0+\epsilon_{m_1}]$. This implies that x is a solution of the limit IVP (8.4) on $[t_0, t_0 + \epsilon_{m_1}]$.

Note that

$$\lim_{j\to\infty} (t_0 + \epsilon_{m_1}, x_{k_1(j)}(t_0 + \epsilon_{m_1})) = (t_0 + \epsilon_{m_1}, x(t_0 + \epsilon_{m_1})) \in H_{m_1} \subset D.$$

If $(t_0 + \epsilon_{m_1}, x(t_0 + \epsilon_{m_1})) \in K_{m_1}$, then repeat the process and obtain a subsequence $\{k_2(j)\}_{j=1}^{\infty}$ of $\{k_1(j)\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} x_{k_2(j)}(t)$$

exists uniformly on $[t_1 + \epsilon_{m_1}, t_1 + 2\epsilon_{m_1}]$ and call the limit function x as before. Then x is a solution of the IVP

$$y' = f(t, y), \quad y(t_0 + \epsilon_{m_1}) = x(t_0 + \epsilon_{m_1}).$$

It follows that

$$\lim_{j\to\infty} x_{k_2(j)}(t) = x(t)$$

uniformly on $[t_0, t_0 + 2\epsilon_{m_1}]$. Continuing in this manner, there is a first integer $j(m_1)$ such that an appropriate subsequence converges uniformly to an extended x on $[t_0, t_0 + j(m_1)\epsilon_{m_1}]$ and

$$(t_0 + j(m_1)\epsilon_{m_1}, x(t_0 + j(m_1)\epsilon_{m_1})) \notin K_{m_1}.$$

Pick $m_2 > m_1$ so that

$$(t_0 + j(m_1)\epsilon_{m_1}, x(t_0 + j(m_1)\epsilon_{m_1})) \in K_{m_2}.$$

We then continue this process in the obvious manner to get the desired result. $\square$

8.7 Continuity of Solutions with Respect to ICs

In this section we are concerned with the smoothness of solutions with respect to initial conditions, initial points, and parameters. Two very important scalar equations that contain a parameter λ are Legendre's equation

$$(1-t^2)u'' - 2tu' + \lambda(\lambda+1)u = 0$$

and Bessel's equation

$$t^2u'' + tu' + (t^2 - \lambda^2)u = 0.$$

Theorem 8.40 (Continuity of Solutions with Respect to Initial Conditions and Parameters) *Assume that D is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and Λ is an open subset of $\mathbb{R}^m$ and f is continuous on $D \times \Lambda$ with the property that for each $(t_0, x_0, \lambda_0) \in D \times \Lambda$, the IVP*

$$x' = f(t, x, \lambda_0), \quad x(t_0) = x_0 \tag{8.24}$$

has a unique solution denoted by $x(t; t_0, x_0, \lambda_0)$. Then x is a continuous function on the set $\alpha < t < \omega$, $(t_0, x_0, \lambda_0) \in D \times \Lambda$.

Proof Assume

$$\lim_{k\to\infty} (t_{0k}, x_{0k}, \lambda_k) = (t_0, x_0, \lambda_0) \in D \times \Lambda.$$

Define

$$f_k(t, x) = f(t, x, \lambda_k),$$

for $(t, x) \in D$, $k \geq 1$. Then

$$\lim_{k\to\infty} f_k(t, x) = f(t, x, \lambda_0)$$

uniformly on compact subsets of D. Let $x_k(t) = x(t; t_{0k}, x_{0k}, \lambda_k)$; then x_k is the solution of the IVP

$$x' = f_k(t, x), \quad x(t_{0k}) = x_{0k}.$$

Let (α_k, ω_k) be the maximal interval of existence for x_k, for $k \geq 1$ and let $x(t; t_0, x_0, \lambda_0)$ be the solution of the limit IVP (8.24) with maximal interval of existence (α, ω). Then by the basic convergence theorem (Theorem 8.39),

$$\begin{aligned} \lim_{k\to\infty} x_k(t) &= \lim_{k\to\infty} x(t; t_{0k}, x_{0k}, \lambda_k)) \\ &= x(t; t_0, x_0, \lambda_0) \end{aligned}$$

uniformly on compact subintervals of (α, β) (see Exercise 8.28). The continuity of x with respect to its four arguments follows from this. $\Box$

Theorem 8.41 (Integral Means) *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is continuous. Then there is a sequence of vector functions $\{g_k(t, x)\}$, called* integral means*, such that $g_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, along with its first-order partial derivatives with respect to components of x,* *are continuous and g_k satisfies a uniform Lipschitz condition with respect to x on $\mathbb{R} \times \mathbb{R}^n$, for each $k = 1, 2, 3, \cdots$. Furthermore,*

$$\lim_{k\to\infty} g_k(t,x) = f(t,x)$$

uniformly on each compact subset of D.

Proof Let $\{K_k\}_{k=1}^\infty$ be a sequence of open subsets of D such that $\overline{K}_k$ is compact, $\overline{K}_k \subset K_{k+1}$, and $D = \cup_{k=1}^\infty K_k$. By the Tietze-Urysohn extension theorem [**44**], for each $k \geq 1$ there is a continuous vector function h_k on all of $\mathbb{R} \times \mathbb{R}^n$ such that

$$h_k\,|_{\overline{K}_k} = f,$$

and

$$\max\{\|h_k(t,x)\| : (t,x) \in \mathbb{R} \times \mathbb{R}^n\} = \max\{\|f(t,x)\| : (t,x) \in \overline{K}_k\} =: M_k.$$

Let $\{\delta_k\}$ be a strictly decreasing sequence of positive numbers with limit 0. Then for each $k \geq 1$, define the integral mean g_k by

$$g_k(t,x) := \frac{1}{(2\delta_k)^n} \int_{x_1-\delta_k}^{x_1+\delta_k} \cdots \int_{x_n-\delta_k}^{x_n+\delta_k} h_k(t, y_1, y_2, \cdots, y_n)\, dy_n \cdots dy_1,$$

where $x = (x_1, x_2, \cdots, x_n)$, for $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. We claim that the sequence $\{g_k(t,x)\}$ satisfies the following:

(i) g_k is continuous on $\mathbb{R} \times \mathbb{R}^n$, for each $k \geq 1$,
(ii) $\|g_k(t,x)\| \leq M_k$ on $\mathbb{R} \times \mathbb{R}^n$, for each $k \geq 1$,
(iii) g_k has continuous first-order partial derivatives with respect to components of x on $\mathbb{R} \times \mathbb{R}^n$, for each $k \geq 1$,
(iv) $\|\frac{\partial g_k}{\partial x_i}\| \leq \frac{M_k}{\delta_k}$ on $\mathbb{R} \times \mathbb{R}^n$, for $1 \leq i \leq n$ and for each $k \geq 1$,
(v) $\lim_{k\to\infty} g_k(t,x) = f(t,x)$ uniformly on each compact subset of D.

We will only complete the proof for the scalar case ($n = 1$). In this case

$$g_k(t,x) = \frac{1}{2\delta_k} \int_{x-\delta_k}^{x+\delta_k} h_k(t,y)\, dy,$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}$. We claim that (i) holds for $n = 1$. To see this, fix $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$. We will show that $g_k(t,x)$ is continuous at (t_0, x_0). Let $\epsilon > 0$ be given. Since $h_k(t,x)$ is continuous on the compact set

$$Q := \{(t,x) : |t - t_0| \leq \delta_k, |x - x_0| \leq 2\delta_k\},$$

$h_k(t,x)$ is uniformly continuous on Q. Hence there is a $\delta \in (0, \delta_k)$ such that

$$|h_k(t_2, x_2) - h_k(t_1, x_1)| < \frac{\epsilon}{2}, \tag{8.25}$$

for all $(t_1, x_1), (t_2, x_2) \in Q$ with $|t_1 - t_2| \leq \delta$, $|x_1 - x_2| \leq \delta$.

For $|t-t_0|<\delta$, $|x-x_0|<\delta$, consider

$$\begin{aligned}
& |g_k(t,x)-g_k(t_0,x_0)| \\
\le\ & |g_k(t,x)-g_k(t_0,x)|+|g_k(t_0,x)-g_k(t_0,x_0)| \\
\le\ & \frac{1}{2\delta_k}\int_{x-\delta_k}^{x+\delta_k}|h_k(t,y)-h_k(t_0,y)|\,dy \\
+\ & \frac{1}{2\delta_k}\left|\int_{x-\delta_k}^{x+\delta_k}h_k(t_0,y)\,dy-\int_{x_0-\delta_k}^{x_0+\delta_k}h_k(t_0,y)\,dy\right| \\
\le\ & \frac{\epsilon}{2}+\frac{1}{2\delta_k}\left|\int_{x_0+\delta_k}^{x+\delta_k}h_k(t_0,y)\,dy-\int_{x_0-\delta_k}^{x-\delta_k}h_k(t_0,y)\,dy\right| \\
\le\ & \frac{\epsilon}{2}+\frac{1}{2\delta_k}\left|\int_{x_0+\delta_k}^{x+\delta_k}|h_k(t_0,y)|\,dy\right|+\frac{1}{2\delta_k}\left|\int_{x_0-\delta_k}^{x-\delta_k}|h_k(t_0,y)|\,dy\right| \\
\le\ & \frac{\epsilon}{2}+\frac{M_k|x-x_0|}{\delta_k} \\
<\ & \frac{\epsilon}{2}+\frac{\delta M_k}{\delta_k},
\end{aligned}$$

where we have used (8.25). Hence, if we further assume $\delta<\frac{\epsilon\delta_k}{2M_k}$, then we get that

$$|g_k(t,x)-g_k(t_0,x_0)|<\epsilon,$$

if $|t-t_0|<\delta$, $|x-x_0|<\delta$. Therefore, g_k is continuous at (t_0,x_0). Since $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}$ and $k\ge 1$ are arbitrary, we get that (i) holds for $n=1$.

To see that (ii) holds for $n=1$, consider

$$\begin{aligned}
|g_k(t,x)| &= \frac{1}{2\delta_k}\left|\int_{x-\delta_k}^{x+\delta_k}h_k(t,y)\,dy\right| \\
&\le \frac{1}{2\delta_k}\int_{x-\delta_k}^{x+\delta_k}|h_k(t,y)|\,dy \\
&\le \frac{1}{2\delta_k}\int_{x-\delta_k}^{x+\delta_k}M_k\,dy \\
&= M_k,
\end{aligned}$$

for all $(t,x)\in\mathbb{R}\times\mathbb{R}$ and for all $k\ge 1$. To see that (iii) and (iv) hold for $n=1$, note that

$$\frac{\partial g_k}{\partial x}(t,x)=\frac{1}{2\delta_k}[h_k(t,x+\delta_k)-h_k(t,x-\delta_k)]$$

is continuous on $\mathbb{R}\times\mathbb{R}$. Furthermore,

$$\begin{aligned}
\left|\frac{\partial g_k}{\partial x}(t,x)\right| &\le \frac{1}{2\delta_k}[|h_k(t,x+\delta_k)|+|h_k(t,x-\delta_k)|] \\
&\le \frac{M_k}{\delta_k},
\end{aligned}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$ and $k \geq 1$. This last inequality implies that $g_k(t,x)$ satisfies a uniform Lipschitz condition with respect to x on $\mathbb{R} \times \mathbb{R}$.

Finally we show that (v) holds for $n = 1$. Let H be a compact subset of D. Then there is an integer $m_0 \geq 1$ such that $H \subset K_{m_0}$. Let $\epsilon > 0$ be given. Fix $(t_0, x_0) \in H$. Let

$$\rho := d(H, \partial K_{m_0}).$$

Since f is uniformly continuous on the compact set $\overline{K}_{m_0}$, there is an $\eta > 0$ such that

$$|f(t_2,x_2) - f(t_1,x_1)| < \epsilon, \tag{8.26}$$

if $(t_1,x_1),(t_2,x_2) \in \overline{K}_{m_0}$ with $|t_1 - t_2| < \eta$, $|x_1 - x_2| < \eta$. Pick an integer $N \geq m_0$ sufficiently large so that $\delta_k \leq \min\{\rho, \eta\}$, for all $k \geq N$. Then for all $k \geq N$,

$$\{(t,x) : |t - t_0| \leq \delta_k, |x - x_0| \leq \delta_k\} \subset \overline{K}_{m_0}.$$

Then for all $k \geq N$,

$$\begin{aligned} & |g_k(t_0,x_0) - f(t_0,x_0)| \\ = & \left| \frac{1}{2\delta_k} \int_{x_0-\delta_k}^{x_0+\delta_k} h_k(t_0,y)\, dy - f(t_0,x_0) \right| \\ = & \left| \frac{1}{2\delta_k} \int_{x_0-\delta_k}^{x_0+\delta_k} h_k(t_0,y)\, dy - \frac{1}{2\delta_k} \int_{x_0-\delta_k}^{x_0+\delta_k} f(t_0,x_0)\, dy \right| \\ \leq & \frac{1}{2\delta_k} \int_{x_0-\delta_k}^{x_0+\delta_k} |f(t_0,y) - f(t_0,x_0)|\, dy \\ < & \epsilon, \end{aligned}$$

where we have used (8.26) in the last step. Since $(t_0, x_0) \in H$ is arbitrary, we get

$$|g_k(t,x) - f(t,x)| < \epsilon,$$

for all $(t,x) \in H$, for all $k \geq N$. Since H is an arbitrary compact subset of D, we get

$$\lim_{k\to\infty} g_k(t,x) = f(t,x)$$

uniformly on compact subsets of D. □

8.8 Kneser's Theorem

In this section we will prove Kneser's theorem.

Theorem 8.42 (Kneser's Theorem) *Assume D is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is continuous. If the compact interval $[t_0, c]$ is a subset of the maximal interval of existence for all solutions of the IVP*(8.4)*, then the cross-sectional set*

$S_c := \{y \in \mathbb{R}^n : y = x(c),$ *where x is a solution of the IVP* (8.4) *on* $[t_0, c]\}$

is a compact, connected subset of $\mathbb{R}^n$.

Proof To show that S_c is compact we will show that S_c is closed and bounded. First we show that S_c is a closed set. Let $\{y_k\} \subset S_c$ with

$$\lim_{k \to \infty} y_k = y_0,$$

where $y_0 \in \mathbb{R}^n$. Then there are solutions x_k of the IVP (8.4) on $[t_0, c]$ with $x_k(c) = y_k$, $k = 1, 2, 3, \cdots$. By the basic convergence theorem (Theorem 8.39) there is a subsequence $\{x_{k_j}\}$ such that

$$\lim_{j \to \infty} x_{k_j}(t) = x(t)$$

uniformly on $[t_0, c]$, where x is a solution of the IVP (8.4) on $[t_0, c]$. But this implies that

$$y_0 = \lim_{j \to \infty} y_{k_j} = \lim_{j \to \infty} x_{k_j}(c) = x(c) \in S_c$$

and hence S_c is closed.

To see that S_c is bounded, assume not; then there is a sequence of points $\{y_k\}$ in S_c such that

$$\lim_{k \to \infty} \|y_k\| = \infty.$$

In this case there is a sequence of solutions $\{z_k\}$ of the IVP (8.4) such that $z_k(c) = y_k$, $k \geq 1$. But, by the basic convergence theorem (Theorem 8.39), there is a subsequence of solutions $\{z_{k_j}\}$ such that

$$\lim_{j \to \infty} z_{k_j}(t) = z(t)$$

uniformly on $[t_0, c]$, where z is a solution of the IVP (8.4) on $[t_0, c]$. But this implies that

$$\lim_{j \to \infty} z_{k_j}(c) = z(c),$$

which contradicts

$$\lim_{j \to \infty} \|z_{k_j}(c)\| = \lim_{j \to \infty} \|y_{k_j}\| = \infty.$$

Hence we have proved that S_c is a compact set.

We now show that S_c is connected. Assume not; then, since S_c is compact, there are disjoint, nonempty, compact sets A_1, A_2 such that

$$A_1 \cup A_2 = S_c.$$

Let

$$\delta := d(A_1, A_2) > 0.$$

Then let

$$h(x) := d(x, A_1) - d(x, A_2),$$

for $x \in \mathbb{R}^n$. Then $h : \mathbb{R}^n \to \mathbb{R}$ is continuous,

$$h(x) \leq -\delta, \quad \text{for} \quad x \in A_1,$$

and

$$h(x) \geq \delta, \quad \text{for} \quad x \in A_2.$$

In particular, we have

$$h(x) \neq 0$$

for all $x \in S_c$. We will contradict this fact at the end of this proof. Since $f : D \to \mathbb{R}^n$ is continuous, we have by Theorem 8.41 that there is a sequence of vector functions $\{g_k\}$ such that $g_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and continuous and g_k satisfies a uniform Lipschitz condition with respect to x on $\mathbb{R} \times \mathbb{R}^n$, for each $k = 1, 2, 3, \cdots$. Furthermore,

$$\lim_{k\to\infty} g_k(t,x) = f(t,x)$$

uniformly on each compact subset of D. Since $A_i \neq \emptyset$ for $i = 1, 2$ there are solutions x_i, $i = 1, 2$ of the IVP (8.4) with

$$x_i(c) \in A_i.$$

Now define for $i = 1, 2$

$$g_{ik}(t,x) := \begin{cases} g_k(t,x) + f(t_0, x_i(t_0)) - g_k(t_0, x_i(t_0)), & t < t_0, \\ g_k(t,x) + f(t, x_i(t)) - g_k(t, x_i(t)), & t_0 \le t \le c, \\ g_k(t,x) + f(c, x_i(c)) - g_k(c, x_i(c)), & c < t, \end{cases}$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $k = 1, 2, 3, \cdots$. It follows that g_{ik} is continuous and bounded on $\mathbb{R} \times \mathbb{R}^n$ and satisfies a uniform Lipschitz condition with respect to x on $\mathbb{R} \times \mathbb{R}^n$, for each $i = 1, 2$ and $k = 1, 2, 3, \cdots$. Furthermore,

$$\lim_{k\to\infty} g_{ik}(t,x) = f(t,x)$$

uniformly on each compact subset of D, for $i = 1, 2$. From the Picard-Lindelof theorem (Theorem 8.13) we get that each of the IVPs

$$x' = g_{ik}(t,x), \quad x(\tau) = \xi$$

has a unique solution and since each g_{ik} is bounded on $\mathbb{R} \times \mathbb{R}^n$, the maximal interval of existence of each of these solutions is $(-\infty, \infty)$. Note that the unique solution of the IVP

$$x' = g_{ik}(t,x), \quad x(t_0) = x_0$$

on $[t_0, c]$ is x_i for $i = 1, 2$.

Let

$$p_k(t,x,\lambda) := \lambda g_{1k}(t,x) + (1-\lambda) g_{2k}(t,x),$$

for $(t,x,\lambda) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, $k = 1, 2, 3, \cdots$. Note that for each fixed $\lambda \in \mathbb{R}$, p_k is bounded and continuous and satisfies a uniform Lipschitz condition with respect to x. Furthermore, for each fixed λ,

$$\lim_{k\to\infty} p_k(t,x,\lambda) = f(t,x)$$

uniformly on each compact subset of D. Let for each fixed $\lambda \in \mathbb{R}$, $x_k(\cdot, \lambda)$ be the unique solution of the IVP

$$x' = p_k(t,x,\lambda), \quad x(t_0) = x_0.$$

It follows that $q_k : \mathbb{R} \to \mathbb{R}$, defined by

$$q_k(\lambda) = h(x_k(c,\lambda)),$$

is, by Theorem 8.40, continuous. Since

$$q_k(0) = h(x_k(c, 0)) = h(x_2(c)) \geq \delta,$$

and

$$q_k(1) = h(x_k(c, 1)) = h(x_1(c)) \leq -\delta,$$

there is, by the intermediate value theorem, a λ_k with $0 < \lambda_k < 1$ such that

$$q_k(\lambda_k) = h(x_k(c, \lambda_k)) = 0,$$

for $k = 1, 2, 3, \cdots$. Since the sequence $\{\lambda_k\} \subset [0, 1]$, there is a convergent subsequence $\{\lambda_{k_j}\}$. Let

$$\lambda_0 = \lim_{j\to\infty} \lambda_{k_j}.$$

It follows that the sequence $x_{k_j}(t, \lambda_{k_j})$ has a subsequence $x_{k_{j_i}}(t, \lambda_{k_{j_i}})$ that converges uniformly on compact subsets of $(-\infty, \infty)$ to $\overline{x}(t)$, where $\overline{x}$ is a solution of the limit IVP

$$x' = f(t, x), \quad x(t_0) = x_0.$$

It follows that

$$\lim_{i\to\infty} x_{k_{j_i}}(c, \lambda_{k_{j_i}}) = \overline{x}(c) \in S_c.$$

Hence

$$\lim_{i\to\infty} h(x_{k_{j_i}}(c, \lambda_{k_{j_i}})) = h(\overline{x}(c)) = 0,$$

which is a contradiction.

□

8.9 Differentiating Solutions with Respect to ICs

In this section we will be concerned with differentiating solutions with respect to initial conditions, initial points, and parameters.

Theorem 8.43 (Differentiation with Respect to Initial Conditions and Initial Points) *Assume f is a continuous n dimensional vector function on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ and f has continuous partial derivatives with respect to the components of x. Then the IVP* (8.4)*, where $(t_0, x_0) \in D$, has a unique solution, which we denote by $x(t; t_0, x_0)$, with maximal interval of existence (α, ω). Then $x(t; t_0, x_0)$ has continuous partial derivatives with respect to the components x_{0j}, $j = 1, 2, \cdots, n$ of x_0 and with respect to t_0 on (α, ω). Furthermore, $z(t) := \frac{\partial x(t;t_0,x_0)}{\partial x_{0j}}$ defines the unique solution of the IVP*

$$z' = J(t)z, \quad z(t_0) = e_j,$$

where $J(t)$ is the Jacobian matrix of f with respect to x along $x(t; t_0, x_0)$, that is,

$$J(t) = D_x f(t, x(t; t_0, x_0))$$

and

$$e_j = \begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{nj} \end{pmatrix},$$

where δ_{ij} *is the Kronecker delta function, for* $t \in (\alpha, \omega)$. *Also,* $w(t) := \frac{\partial x(t;t_0,x_0)}{\partial t_0}$ *defines the unique solution of the IVP*

$$w' = J(t)w, \quad w(t_0) = -f(t_0, x_0),$$

for $t \in (\alpha, \omega)$.

Proof Let $\tau \in (\alpha, \omega)$. Then choose an interval $[t_1, t_2] \subset (\alpha, \omega)$ such that $\tau,\ t_0 \in (t_1, t_2)$. Since

$$\{(t, x(t; t_0, x_0)) : t \in [t_1, t_2]\} \subset D$$

is compact and D is open, there exists a $\delta > 0$ such that

$$Q_\delta := \{(\overline{t}, \overline{x}) : |\overline{t} - t| \le \delta, \|\overline{x} - x(t; t_0, x_0))\| \le \delta, \text{ for } t \in [t_1, t_2]\}$$

is contained in D. Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of nonzero real numbers with $\lim_{k\to\infty} \delta_k = 0$. Let $x(t) := x(t; t_0, x_0)$ and let x_k be the solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0 + \delta_k e_j,$$

where $1 \le j \le n$ is fixed. It follows from the basic convergence theorem (Theorem 8.39) that if (α_k, ω_k) is the maximal interval of existence of x_k, then there is an integer k_0 so that for all $k \ge k_0$, $[t_1, t_2] \subset (\alpha_k, \beta_k)$ and

$$\|x(t) - x_k(t)\| < \delta$$

on $[t_1, t_2]$. Without loss of generality we can assume that the preceding properties of $\{x_k\}$ hold for all $k \ge 1$. Since, for $t_1 < t < t_2$, $0 \le s \le 1$,

$$\begin{aligned} &\|[sx_k(t) + (1-s)x(t)] - x(t)\| \\ &= s\|x_k(t) - x(t)\| \le s\delta \le \delta, \end{aligned}$$

it follows that

$$(t, sx_k(t) + (1-s)x(t)) \in Q_\delta,$$

for $t_1 < t < t_2$, $0 \le s \le 1$. By Lemma 8.6,

$$\begin{aligned} x_k'(t) - x'(t) &= f(t, x_k(t)) - f(t, x(t)) \\ &= \int_0^1 D_x f(t, sx_k(t) + (1-s)x(t))\, ds\, [x_k(t) - x(t)], \end{aligned}$$

for $t_1 < t < t_2$. Let

$$z_k(t) := \frac{1}{\delta_k}[x_k(t) - x(t)];$$

then z_k is a solution of the IVP

$$z' = \int_0^1 D_x f(t, sx_k(t) + (1-s)x(t))\, ds\ z, \quad x(t_0) = e_j,$$

where e_j is given in the statement of this theorem. Define $h_k : (t_1, t_2) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$h_k(t, z) := \int_0^1 D_x f(t, s x_k(t) + (1-s)x(t))\, ds\; z,$$

for $(t, x) \in (t_1, t_2) \times \mathbb{R}^n$, $k \geq 1$. Then $h_k : (t_1, t_2) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and since

$$\lim_{k \to \infty} x_k(t) = x(t)$$

uniformly on $[t_1, t_2]$, we get that

$$\lim_{k \to \infty} h_k(t, z) = D_x f(t, x(t)) z$$

uniformly on each compact subset of $(t_1, t_2) \times \mathbb{R}^n$. By the basic convergence theorem (Theorem 8.39),

$$\lim_{k \to \infty} z_k(t) = z(t)$$

uniformly on each compact subinterval of (t_1, t_2), where z is the solution of the limit IVP

$$z' = D_x f(t, x(t)) z = D_x f(t, x(t; t_0, x_0)) z, \quad z(t_0) = e_j. \tag{8.27}$$

In particular,

$$\lim_{k \to \infty} \frac{1}{\delta_k} \left[x_k(\tau) - x(\tau)\right] = z(\tau),$$

where z is the solution of the IVP (8.27). Since this is true for any sequence $\{\delta_k\}$ of nonzero numbers with $\lim_{k \to \infty} \delta_k = 0$, we get that

$$\frac{\partial x}{\partial x_{0j}}(\tau, t_0, x_0)$$

exists and equals $z(\tau)$. Since $\tau \in (\alpha, \omega)$ was arbitrary, we get that for each $t \in (\alpha, \omega)$,

$$\frac{\partial x}{\partial x_{0j}}(t, t_0, x_0)$$

exists and equals $z(t)$. The rest of this proof is left to the reader. $\square$

Definition 8.44 The equation $z' = J(t)z$ in Theorem 8.43 is called the *variational equation of $x' = f(t, x)$ along the solution $x(t; t_0, x_0)$.*

Example 8.45 Find $\frac{\partial x}{\partial x_0}(t; 0, 1)$ and $\frac{\partial x}{\partial t_0}(t; 0, 1)$ for the differential equation

$$x' = x - x^2. \tag{8.28}$$

By inspection,

$$x(t, 0, 1) \equiv 1.$$

Here $f(t, x) = x - x^2$, so the variational equation along $x(t; 0, 1)$ is

$$z' = (1 - 2x(t; 0, 1)) z,$$

which simplifies to

$$z' = -z.$$

By Theorem 8.43, $z(t) := \frac{\partial x}{\partial x_0}(t; 0, 1)$ solves the IVP

$$z' = -z, \quad z(0) = 1,$$

and $w(t) := \frac{\partial x}{\partial t_0}(t; 0, 1)$ solves the IVP

$$w' = -w, \quad w(0) = -f(0, 1) = 0.$$

It follows that

$$z(t) = \frac{\partial x}{\partial x_0}(t; 0, 1) = e^{-t}$$

and

$$w(t) = \frac{\partial x}{\partial t_0}(t; 0, 1) = 0.$$

It then follows that

$$x(t; 0, 1 + h) \approx he^{-t} + 1,$$

for h close to zero and

$$x(t; h, 1) \approx 1,$$

for h close to zero. Since the equation (8.28) can be solved (see Exercise 8.32) by separating variables and using partial fractions, it can be shown that

$$x(t; t_0, x_0) = \frac{x_0 e^{t-t_0}}{1 - x_0 + x_0 e^{t-t_0}}.$$

It is easy to see (see Exercise 8.32) from this that the expressions for $\frac{\partial x}{\partial x_0}(t; 0, 1)$ and $\frac{\partial x}{\partial t_0}(t; 0, 1)$ given previously are correct. △

Corollary 8.46 *Assume that A is a continuous $n \times n$ matrix function and h is a continuous $n \times 1$ vector function on an interval I. Further assume $t_0 \in I$, $x_0 \in \mathbb{R}^n$, and let $x(\cdot; t_0, x_0)$ denote the unique solution of the IVP*

$$x' = A(t)x + h(t), \quad x(t_0) = x_0.$$

Then $z(t) := \frac{\partial x}{\partial x_{0j}}(t; t_0, x_0)$ defines the unique solution of the IVP

$$z' = A(t)z, \quad z(t_0) = e_j,$$

on I. Also $w(t) := \frac{\partial x}{\partial t_0}(t; t_0, x_0)$ defines the unique solution of the IVP

$$w' = A(t)x, \quad w(t_0) = -A(t_0)x_0 - h(t_0).$$

Proof If $f(t, x) := A(t)x + h(t)$, then

$$D_x f(t, x) = A(t)$$

and the conclusions in this corollary follow from Theorem 8.43. □

Example 8.47 Assume that $f(t, u, u', \cdots, u^{(n-1)})$ is continuous and real valued on $(a, b) \times \mathbb{R}^n$ and has continuous first-order partial derivatives with respect to each of the variables $u, u', \cdots, u^{(n-1)}$ on $(a, b) \times \mathbb{R}^n$. Then the IVP

$$u^{(n)} = f(t, u, u', \cdots, u^{(n-1)}), \;\; u(t_0) = y_{01}, \;\; \cdots, u^{(n-1)}(t_0) = y_{0n}, \quad (8.29)$$

where $t_0 \in (a,b)$, $y_{0k} \in \mathbb{R}$, $1 \le k \le n$, is equivalent to the vector IVP

$$y' = h(t,y), \quad y(t_0) = y_0,$$

where

$$h(t,y) := \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(t, y_1, \cdots, y_n) \end{pmatrix}, \quad y_0 := \begin{pmatrix} y_{01} \\ y_{02} \\ \vdots \\ y_{0n} \end{pmatrix}.$$

Then

$$y(t;t_0,y_0) = \begin{pmatrix} u(t;t_0,y_0) \\ u'(t;t_0,y_0) \\ \vdots \\ u^{(n-1)}(t;t_0,y_0) \end{pmatrix},$$

where $u(\cdot;t_0,y_0)$ is the solution of the IVP (8.29). From Theorem 8.43,

$$z(t) := \frac{\partial y}{\partial y_{0j}}(t;t_0,y_0)$$

is the solution of the IVP

$$z' = J(t)z, \quad z(t_0) = e_j,$$

where J is the matrix function

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ f_u & f_{u'} & & \cdots & f_{u^{(n-1)}} \end{pmatrix}$$

evaluated at $(t, y(t;t_0,x_0))$. Since

$$\frac{\partial u}{\partial y_{0j}}(t;t_0,y_0)$$

is the first component of

$$z(t) = \frac{\partial y}{\partial y_{0j}}(t;t_0,y_0),$$

we get that

$$v(t) := \frac{\partial u}{\partial y_{0j}}(t;t_0,y_0)$$

solves the IVP

$$\begin{aligned} v^{(n)} &= \sum_{k=0}^{n-1} f_{u^{(k)}}(t, u(t;t_0,y_0), \cdots, u^{(n-1)}(t;t_0,y_0))v^{(k)}, \\ v^{(i-1)} &= \delta_{ij}, \quad i = 1, 2, \cdots, n, \end{aligned}$$

where δ_{ij} is the Kronecker delta function. The nth-order linear differential equation

$$v^{(n)} = \sum_{k=0}^{n-1} f_{u^{(k)}}(t, u(t;t_0,y_0), u'(t;t_0,y_0), \cdots, u^{(n-1)}(t;t_0,y_0))v^{(k)} \quad (8.30)$$

is called the variational equation of $u^{(n)} = f(t,u,u',\cdots,u^{(n-1)})$ along $u(t;t_0,y_0)$. Similarly,

$$w(t) := \frac{\partial u}{\partial t_0}(t;t_0,y_0)$$

is the solution of the variational equation satisfying the initial conditions

$$w(t_0) = -y_{02},\ \cdots,\ w^{(n-2)}(t_0) = -y_{0n},\ w^{(n-1)}(t_0) = -f(t_0,y_{01},\cdots,y_{0n}).$$

△

Example 8.48 If $u(t;t_0,c_1,c_2)$ denotes the solution of the IVP

$$u'' = u - u^3, \quad u(t_0) = c_1,\ u'(t_0) = c_2,$$

find

$$\frac{\partial u}{\partial c_2}(t;0,0,0)$$

and use your answer to approximate $u(t;0,0,h)$ when h is small.

We know that

$$v(t) := \frac{\partial u}{\partial c_2}(t;0,0,0)$$

gives the solution of the IVP

$$v'' = f_u(t,u(t;0,0,0),u'(t;0,0,0))v + f_{u'}(t,u(t;0,0,0),u'(t;0,0,0))v',$$

$$v(0) = 0, \quad v'(0) = 1.$$

Since in this example $f(t,u,u') = u - u^3$, we get that $f_u(t,u,u') = 1 - 3u^2$ and $f_{u'}(t,u,u') = 0$. Since by inspection $u(t;0,0,0) \equiv 0$, we get that

$$f_u(t,u(t;0,0,0),u'(t;0,0,0)) = f_u(t,0,0) = 1$$

and

$$f_{u'}(t,u(t;0,0,0),u'(t;0,0,0)) = f_{u'}(t,0,0) = 0.$$

Hence v solves the IVP

$$v'' = v, \quad v(0) = 0, \quad v'(0) = 1,$$

which implies that $v(t) = \sinh t$. Therefore,

$$\frac{\partial u}{\partial c_2}(t;0,0,0) = \sinh t.$$

It follows that

$$u(t;0,0,h) \approx h \sinh t,$$

for small h. Similarly, it can be shown that

$$\frac{\partial u}{\partial c_1}(t,0,0,0) = \cosh t$$

and

$$\frac{\partial u}{\partial t_0}(t,0,0,0) = 0.$$

It then follows that

$$u(t;0,h,0) \approx h\cosh t,$$

and

$$u(t;h,0,0) \approx 0,$$

for h close to zero. △

Theorem 8.49 (Differentiating Solutions with Respect to Parameters) *Assume $f(t,x,\lambda)$ is continuous and has continuous first-order partial derivatives with respect to components of x and λ on an open subset $D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. For each $(t_0,x_0,\lambda_0) \in D$, let $x(t;t_0,x_0,\lambda_0)$ denote the solution of the IVP*

$$x' = f(t,x,\lambda_0), \quad x(t_0) = x_0, \tag{8.31}$$

with maximal interval of existence (α,ω). Then $x(t;t_0,x_0,\lambda_0)$ has continuous first-order partial derivatives with respect to components of $\lambda_0 = (\lambda_{01},\cdots,\lambda_{0m})$ and $z(t) := \frac{\partial x}{\partial \lambda_{0k}}(t;t_0,x_0,\lambda_0)$, $1 \le k \le m$, is the solution of the IVP

$$z' = J(t;t_0,x_0,\lambda_0)z + g_k(t), \quad z(t_0) = 0,$$

on (α,ω), where

$$J(t;t_0,x_0,\lambda_0) := D_x f(t,x(t;t_0,x_0,\lambda_0),\lambda_0)$$

and

$$g_k(t) = \begin{pmatrix} \frac{\partial f_1}{\partial \lambda_{0k}}(t,x(t;t_0,x_0,\lambda_0),\lambda_0) \\ \frac{\partial f_2}{\partial \lambda_{0k}}(t,x(t;t_0,x_0,\lambda_0),\lambda_0) \\ \vdots \\ \frac{\partial f_n}{\partial \lambda_{0k}}(t,x(t;t_0,x_0,\lambda_0),\lambda_0) \end{pmatrix},$$

for $t \in (\alpha,\omega)$.

Proof Let

$$y := \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}, \quad h(t,y) := \begin{pmatrix} f_1(t,x,\lambda) \\ \vdots \\ f_n(t,x,\lambda) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$y(t;t_0,x_0,\lambda_0):=\begin{pmatrix} x_1(t;t_0,x_0,\lambda_0)\\ \vdots \\ x_n(t;t_0,x_0,\lambda_0)\\ \lambda_{01}\\ \vdots \\ \lambda_{0m}\end{pmatrix}$$

is the unique solution of the IVP

$$y'=h(t,y),\quad y(t_0)=\begin{pmatrix} x_{01}\\ \vdots \\ x_{0n}\\ \lambda_{01}\\ \vdots \\ \lambda_{0m}\end{pmatrix},$$

with maximal interval of existence (α,ω). It follows from Theorem 8.43 that

$$\tilde{z}(t):=\frac{\partial y\ (t;t_0,x_0,\lambda_0)}{\partial\lambda_{0k}}$$

exists, is continuous on (α,ω), and is the solution of the IVP

$$\tilde{z}'=J(t,y(t;t_0,x_0,\lambda_0))\tilde{z},\quad \tilde{z}(t_0)=e_{n+k},$$

where e_{n+k} is the unit vector in $\mathbb{R}^{n+m}$ whose $n+k$ component is 1 and

$$J(t,y(t;t_0,x_0,\lambda_0))=\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \lambda_{01}} & \cdots & \frac{\partial f_1}{\partial \lambda_{0m}}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \lambda_{01}} & \cdots & \frac{\partial f_n}{\partial \lambda_{0m}}\\ 0 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 0 & 0 & \cdots & 0\end{pmatrix},$$

where the entries of this last matrix are evaluated at $(t,x(t;t_0,x_0,\lambda_0),\lambda_0)$. This implies $\tilde{z}_j'(t)=0$, $n+1\le j\le n+m$. This then implies that

$$\tilde{z}_j(t)\equiv 0,\quad n+1\le j\le n+m,\text{ but } j\ne n+k,$$

and

$$\tilde{z}_{n+k}(t)\equiv 1,$$

for $t \in (\alpha, \omega)$. Hence

$$\tilde{z}(t) = \begin{pmatrix} z(t) \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the $n + k$ component of $\tilde{z}(t)$ is 1. It follows that $z(t) = \frac{\partial x(t;t_0,x_0,\lambda_0)}{\partial \lambda_{0k}}$ is the solution of the IVP

$$z' = J(t; t_0, x_0, \lambda_0)z + g_k(t), \quad z(t_0) = 0,$$

on (α, ω), where

$$J(t; t_0, x_0, \lambda_0) = D_x f(t, x(t; t_0, x_0, \lambda_0), \lambda_0),$$

and

$$g_k(t) = \begin{pmatrix} \frac{\partial f_1}{\partial \lambda_{0k}}(t, x(t; t_0, x_0, \lambda_0), \lambda_0) \\ \frac{\partial f_2}{\partial \lambda_{0k}}(t, x(t; t_0, x_0, \lambda_0), \lambda_0) \\ \vdots \\ \frac{\partial f_n}{\partial \lambda_{0k}}(t, x(t; t_0, x_0, \lambda_0), \lambda_0) \end{pmatrix},$$

for $t \in (\alpha, \omega)$. □

Similar to Example 8.47, we get the following result.

Example 8.50 Assume that $f(t, u, u', \cdots, u^{(n-1)}, \lambda_1, \cdots, \lambda_m)$ is continuous on $(a, b) \times \mathbb{R}^n \times \mathbb{R}^m$ and has continuous first-order partial derivatives with respect to each of the variables $u, u', \cdots, u^{(n-1)}, \lambda_1, \cdots, \lambda_m$. Then the unique solution $u(t; t_0, y_{01}, y_{02}, \cdots, y_{0n}, \lambda_{01}, \cdots, \lambda_{0m}) = u(t; t_0, y_0, \lambda_0)$ of the initial value problem

$$\begin{aligned} &u^{(n)} = f(t, u, u', \cdots, u^{(n-1)}, \lambda_{01}, \cdots, \lambda_{0m}), \\ &u(t_0) = y_{01},\ u'(t_0) = y_{02},\ \cdots,\ u^{(n-1)}(t_0) = y_{0n}, \end{aligned}$$

where $(t_0, y_0, \lambda_0) \in (a, b) \times \mathbb{R}^n \times \mathbb{R}^m$ has continuous first-order partial derivatives with respect to $\lambda_{01}, \lambda_{02} \cdots, \lambda_{0m}$ and

$$v(t) := \frac{\partial u}{\partial \lambda_{0j}}(t; t_0, y_0, \lambda_0),$$

$1 \le j \le m$, is the solution of the IVP

$$v^{(n)} = \sum_{k=0}^{n-1} f_{u^{(k)}}(t, u(t; t_0, y_0, \lambda_0), \cdots, u^{(n-1)}(t; t_0, y_0, \lambda_0), \lambda_0) v^{(k)}$$
$$+ f_{\lambda_j}(t, u(t; t_0, y_0, \cdots, u^{n-1}(t; t_0, y_0, \lambda_0), \lambda_0), \cdots, u^{(n-1)}(t; t_0, y_0, \lambda_0), \lambda_0),$$
$$v^{(i-1)}(t_0) = 0, \quad i = 1, \cdots, n.$$

△

Example 8.51 Let $u(t; t_0, a, b, \lambda)$, $\lambda > 0$, denote the solution of the IVP

$$u'' = -\lambda u, \quad u(t_0) = a, \quad u'(t_0) = b.$$

Then by Example 8.50 we get that

$$v(t) = \frac{\partial u}{\partial \lambda}(t; 0, 1, 0, \lambda)$$

is the solution of the IVP

$$v'' = -\lambda v - \cos(\sqrt{\lambda} t), \quad v(0) = 0, \quad v'(0) = 0.$$

Solving this IVP, we get

$$v(t) = \frac{\partial u}{\partial \lambda}(t; 0, 1, 0, \lambda) = -\frac{1}{2\sqrt{\lambda}} t \sin(\sqrt{\lambda} t).$$

Since

$$u(t; 0, 1, 0, \lambda) = \cos(\sqrt{\lambda} t),$$

we can check our answer by just differentiating with respect to λ. △

8.10 Maximum and Minimum Solutions

Definition 8.52 Assume that ϕ is a continuous real-valued function on an open set $D \subset \mathbb{R} \times \mathbb{R}$ and let $(t_0, u_0) \in D$. Then a solution u_M of the IVP

$$u' = \phi(t, u), \quad u(t_0) = u_0, \tag{8.32}$$

with maximal interval of existence (α_M, ω_M) is called a *maximum solution* of the IVP (8.32) in case for any other solution v of the IVP (8.32) on an interval I,

$$v(t) \le u_M(t), \quad \text{on} \quad I \cap (\alpha_M, \omega_M).$$

In a similar way we can also define a *minimum solution* of the IVP (8.32).

Theorem 8.53 *Assume that ϕ is a continuous real-valued function on an open set $D \subset \mathbb{R} \times \mathbb{R}$. Then the IVP* (8.32) *has both a maximum and minimum solution and these solutions are unique.*

Proof For each $n \ge 1$, let u_n be a solution of the IVP

$$u' = f(t, u) + \frac{1}{n}, \quad u(t_0) = u_0,$$

with maximal interval of existence (α_n, ω_n). Since

$$\lim_{n\to\infty}\left(f(t,u)+\frac{1}{n}\right)=f(t,u)$$

uniformly on each compact subset of D, we have by Theorem 8.39 that there is a solution u of the IVP (8.32) with maximal interval of existence (α_u, ω_u) and a subsequence $\{u_{n_j}(t)\}$ that converges to $u(t)$ in the sense of Theorem 8.39.

Similarly, for each integer $n \geq 1$, let v_n be a solution of the IVP

$$v' = f(t,v) - \frac{1}{n}, \quad v(t_0) = u_0,$$

with maximal interval of existence (α_n^*, ω_n^*). Since

$$\lim_{n\to\infty}\left(f(t,v)-\frac{1}{n}\right)=f(t,v)$$

uniformly on each compact subset of D, we have by Theorem 8.39 that there is a solution v of the IVP (8.32) with maximal interval of existence (α_v, β_v) and a subsequence $v_{n_j}(t)$ that converges to $v(t)$ in the sense of Theorem 8.39. Define

$$u_M(t) = \begin{cases} u(t), & \text{on} \quad [t_0, \omega_u) \\ v(t), & \text{on} \quad (\alpha_v, t_0), \end{cases}$$

and

$$u_m(t) = \begin{cases} v(t), & \text{on} \quad [t_0, \omega_v) \\ u(t), & \text{on} \quad (\alpha_u, t_0). \end{cases}$$

We claim that u_M and u_m are maximum and minimum solutions of the IVP (8.32), respectively. To see that u_M is a maximum solution on $[t_0, \omega_u)$, assume not, then there is a solution z of the IVP (8.32) on an interval I and there is a $t_1 > t_0$, $t_1 \in I \cap [t_0, \omega_u)$ such that

$$z(t_1) > u_M(t_1) = u(t_1).$$

Since

$$\lim_{j\to\infty} u_{n_j}(t_1) = u(t_1),$$

we can pick J sufficiently large so that

$$u_{n_J}(t_1) < z(t_1).$$

Since $z(t_0) = u_0 = u_{n_J}(t_0)$, we can pick $t_2 \in [t_0, t_1)$ such that

$$z(t_2) = u_{n_J}(t_2)$$

and

$$z(t) > u_{n_J}(t)$$

on $(t_2, t_1]$. This implies that $z'(t_2) \geq u'_{n_J}(t_2)$. But

$$\begin{aligned} u'_{n_J}(t_2) &= f(t_2, u_{n_J}(t_2)) + \frac{1}{n_J} \\ &> f(t_2, u_{n_J}(t_2)) \\ &= f(t_2, z(t_2)) \\ &= z'(t_2), \end{aligned}$$

which gives us our contradiction. There are three other cases that are similar and so will be omitted (see Exercise 8.42). It is easy to see that the maximal and minimal solutions of the IVP (8.32) are unique. Note that since the maximal solution of the IVP (8.32) is unique, we have that any sequence $\{u_n\}$ of solutions of the IVPs

$$u' = f(t, u) + \frac{1}{n}, \quad u(t_0) = u_0$$

converges to u_M on $[t_0, \omega_u)$ in the sense of Theorem 8.39. □

In the remainder of this section we use the following notation.

Definition 8.54 Assume u is defined in a neighborhood of t_0. Then

$$D^+u(t_0) := \limsup_{h \to 0+} \frac{u(t_0 + h) - u(t_0)}{h},$$

$$D_+u(t_0) := \liminf_{h \to 0+} \frac{u(t_0 + h) - u(t_0)}{h},$$

$$D^-u(t_0) := \limsup_{h \to 0-} \frac{u(t_0 + h) - u(t_0)}{h},$$

$$D_-u(t_0) := \liminf_{h \to 0-} \frac{u(t_0 + h) - u(t_0)}{h}.$$

Theorem 8.55 *Assume that D is an open subset of $\mathbb{R}^2$ and $\phi : D \to \mathbb{R}$ is continuous. Assume that $(t_0, u_0) \in D$, $v : [t_0, t_0 + a] \to \mathbb{R}$ is continuous, $(t, v(t)) \in D$, for $t_0 \leq t \leq t_0 + a$ with $D^+v(t) \leq \phi(t, v(t))$, and $v(t_0) \leq u_0$; then*

$$v(t) \leq u_M(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_M, \omega_M),$$

where u_M is the maximum solution of the IVP (8.32)*, with maximal interval of existence (α_M, ω_M).*

Proof Let v and u_M be as in the statement of this theorem. We now prove that

$$v(t) \leq u_M(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_M, \omega_M).$$

Assume not; then there is a $t_1 > t_0$ in $[t_0, t_0 + a] \cap (\alpha_M, \omega_M)$, such that

$$v(t_1) > u_M(t_1).$$

By the proof of Theorem 8.53 we know that if $\{u_n\}$ is a sequence of solutions of the IVPs

$$u' = \phi(t,u) + \frac{1}{n}, \quad u(t_0) = x_0,$$

respectively, then

$$\lim_{n\to\infty} u_n(t) = u_M(t)$$

uniformly on compact subintervals of $[t_0, \omega)$. Hence we can pick a positive integer N such that the maximal interval of existence of u_N contains $[t_0, t_1]$ and

$$u_N(t_1) < v(t_1).$$

Choose $t_2 \in [t_0, t_1)$ such that $u_N(t_2) = v(t_2)$ and

$$v(t) > u_N(t), \quad t \in (t_2, t_1].$$

Let

$$z(t) := v(t) - u_N(t).$$

For $h > 0$, sufficiently small,

$$\frac{z(t_2+h) - z(t_2)}{h} > 0,$$

and so

$$D^+ z(t_2) \geq 0.$$

But

$$\begin{aligned} D^+ z(t_2) &= D^+ v(t_2) - u_N'(t_2) \\ &\leq \phi(t_2, v(t_2)) - \phi(t_2, u_N(t_2)) - \frac{1}{N} \\ &= -\frac{1}{N} < 0, \end{aligned}$$

which is a contradiction. □

Similarly, we can prove the following three theorems:

Theorem 8.56 *Assume that D is an open subset of $\mathbb{R}^2$ and $\phi : D \to \mathbb{R}$ is continuous. Assume that $(t_0, u_0) \in D$, $v : [t_0, t_0 + a] \to \mathbb{R}$ is continuous, $(t, v(t)) \in D$, for $t_0 \leq t \leq t_0 + a$, with $D_+ v(t) \geq \phi(t, v(t))$, and $v(t_0) \geq u_0$; then*

$$v(t) \geq u_m(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_m, \omega_m),$$

where u_m is the minimum solution of the IVP (8.32)*, with maximal interval of existence (α_m, ω_m).*

Theorem 8.57 *Assume that D is an open subset of $\mathbb{R}^2$ and $\phi : D \to \mathbb{R}$ is continuous. Assume that $(t_0, u_0) \in D$, $v : [t_0 - a, t_0] \to \mathbb{R}$ is continuous, $(t, v(t)) \in D$, for $t_0 - a \leq t \leq t_0$, with $D^- v(t) \geq \phi(t, v(t))$ and $v(t_0) \leq u_0$; then*

$$v(t) \leq u_M(t), \quad t \in [t_0 - a, t_0] \cap (\alpha_M, \omega_M),$$

where u_M is the maximum solution of the IVP (8.32)*, with maximal interval of existence (α_M, ω_M).*

Theorem 8.58 *Assume that D is an open subset of $\mathbb{R}^2$ and $\phi : D \to \mathbb{R}$ is continuous. Assume that $(t_0, u_0) \in D$, $v : [t_0 - a, t_0] \to \mathbb{R}$ is continuous, $(t, v(t)) \in D$, for $t_0 - a \le t \le t_0$, with $D_- v(t) \le \phi(t, v(t))$, and $v(t_0) \ge u_0$; then*

$$v(t) \ge u_m(t), \quad t \in [t_0 - a, t_0] \cap (\alpha_m, \omega_m),$$

where u_m is the minimum solution of the IVP (8.32)*, with maximal interval of existence (α_m, ω_m).*

We will leave the proof of the following important comparison theorem as an exercise (Exercise 8.46).

Corollary 8.59 *Let D be an open subset of $\mathbb{R}^2$ and assume that $\Psi : D \to \mathbb{R}$ and $\phi : D \to \mathbb{R}$ are continuous with*

$$\Psi(t, u) \le \phi(t, u), \quad (t, u) \in D.$$

If u_M is the maximum solution of the IVP (8.32)*, with maximal interval of existence (α_M, ω_M), then if v is a solution of $v' = \Psi(t, v)$ with $v(t_0) \le u_0$, then $v(t) \le u_M(t)$ on $[t_0, t_0 + a] \cap (\alpha_M, \omega_M)$.*

We can now use Theorems 8.55 and 8.56 to prove the following corollary.

Corollary 8.60 *Assume that D is an open subset of $\mathbb{R}^2$ and $\phi : D \to \mathbb{R}$ is continuous. Assume that $(t_0, u_0) \in D$, and there is a continuously differentiable vector function $x : [t_0, t_0 + a] \to \mathbb{R}^n$ such that $(t, \|x(t)\|) \in D$ for $t_0 \le t \le t_0 + a$ with $\|x'(t)\| \le \phi(t, \|x(t)\|)$; then*

$$\|x(t)\| \le u_M(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_M, \omega_M),$$

where u_M is the maximum solution of the IVP

$$u' = \phi(t, u), \quad u(t_0) = \|x(t_0)\|,$$

with maximal interval of existence (α_M, ω_M). Similarly, if u_m is the minimum solution of the IVP

$$u' = -\phi(t, u), \quad u(t_0) = \|x(t_0)\|,$$

then

$$\|x(t)\| \ge u_m(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_m, \omega_m).$$

Proof Note that for $h > 0$, sufficiently small,

$$\frac{\|x(t+h)\| - \|x(t)\|}{h} \le \frac{\|x(t+h) - x(t)\|}{h} = \left\| \frac{x(t+h) - x(t)}{h} \right\|,$$

implies that

$$D^+ \|x(t)\| \le \|x'(t)\| \le \phi(t, \|x(t)\|),$$

for $t \in [t_0, t_0 + a]$. Hence by Theorem 8.55, we get that

$$\|x(t)\| \le u_M(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_M, \omega_M).$$

Next, for $h > 0$, sufficiently small,

$$\|x(t+h) - x(t)\| \geq \|x(t)\| - \|x(t+h)\| = -\{\|x(t+h)\| - \|x(t)\|\}.$$

Hence for $h > 0$, sufficiently small,

$$\frac{\|x(t+h)\| - \|x(t)\|}{h} \geq -\frac{\|x(t+h) - x(t)\|}{h} = -\left\|\frac{x(t+h) - x(t)}{h}\right\|,$$

and consequently

$$D_+\|x(t)\| \geq -\|x'(t)\| \geq -\phi(t, \|x(t)\|),$$

for $t \in [t_0, t_0 + a]$. It then follows from Theorem 8.56 that

$$\|x(t)\| \geq u_m(t), \quad t \in [t_0, t_0 + a] \cap (\alpha_m, \omega_m).$$

□

Theorem 8.61 (Generalized Gronwall's Inequality) *Assume $\phi : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R}$ is continuous and for each fixed $t \in [t_0, t_0 + a]$, $\phi(t, u)$ is nondecreasing with respect to u. Assume that the maximum solution u_M of the IVP* (8.32) *exists on $[t_0, t_0 + a]$. Further assume that $v : [t_0, t_0 + a] \to \mathbb{R}$ is continuous and satisfies*

$$v(t) \leq u_0 + \int_{t_0}^{t} \phi(s, v(s))\, ds$$

on $[t_0, t_0 + a]$. Then

$$v(t) \leq u_M(t), \quad t \in [t_0, t_0 + a].$$

Proof Let

$$z(t) := u_0 + \int_{t_0}^{t} \phi(s, v(s))\, ds,$$

for $t \in [t_0, t_0 + a]$. Then $v(t) \leq z(t)$ on $[t_0, t_0 + a]$ and

$$z'(t) = \phi(t, v(t)) \leq \phi(t, z(t)), \quad t \in [t_0, t_0 + a].$$

It follows from Theorem 8.55 that

$$z(t) \leq u_M(t), \quad t \in [t_0, t_0 + a].$$

Since $v(t) \leq z(t)$ on $[t_0, t_0 + a]$, we get the desired result. □

As a consequence to this last theorem, we get the well-known Gronwall's inequality as a corollary.

Corollary 8.62 (Gronwall's Inequality) *Let u, v be nonnegative, continuous functions on $[a, b]$, $C \geq 0$ be a constant, and assume that*

$$v(t) \leq C + \int_a^t v(s) u(s)\, ds,$$

for $t \in [a, b]$. Then

$$v(t) \leq C e^{\int_a^t u(s) ds}, \quad t \in [a, b].$$

In particular, if $C = 0$, then $v(t) \equiv 0$.

Proof This result follows from Theorem 8.61, where we let $\phi(t,w) := u(t)w$, for $(t,w) \in [a,b] \times \mathbb{R}$ and note that the maximum solution (unique solution) of the IVP

$$w' = \phi(t,w) = u(t)w, \quad w(a) = C$$

is given by

$$w_M(t) = Ce^{\int_a^t u(s)ds}, \quad t \in [a,b].$$

$\square$

Theorem 8.63 (Extendability Theorem) *Assume $\phi : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R}$ is continuous and u_M is the maximum solution of the scalar IVP (8.32), where $u_0 \geq 0$, and assume that u_M exists on $[t_0, t_0 + a]$. If the vector function $f : [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and*

$$\|f(t,x)\| \leq \phi(t, \|x\|), \quad (t,x) \in [t_0, t_0 + a] \times \mathbb{R}^n,$$

then any solution x of the vector IVP

$$x' = f(t,x), \quad x(t_0) = x_0,$$

where $\|x_0\| \leq u_0$, exists on $[t_0, t_0 + a]$ and $\|x(t)\| \leq u_M(t)$ on $[t_0, t_0 + a]$.

Proof Fix x_0 so that $\|x_0\| \leq u_0$. By the extension theorem (Theorem 8.33) and the extended Cauchy–Peano Theorem (Theorem 8.34) there is an $\alpha > 0$ such that all solutions of this IVP exist on $[t_0, t_0 + \alpha]$. Let x be one of these solutions with right maximal interval of existence $[t_0, \omega_x)$. Then

$$\begin{aligned} D^+\|x(t)\| &= \limsup_{h \to 0+} \frac{\|x(t+h)\| - \|x(t)\|}{h} \\ &\leq \limsup_{h \to 0+} \left\| \frac{x(t+h) - x(t)}{h} \right\| \\ &= \|x'(t)\| \leq \phi(t, \|x(t)\|). \end{aligned}$$

Hence from Theorem 8.55

$$\|x(t)\| \leq u_M(t), \quad t \in [t_0, \omega_x).$$

If $\omega_x \leq t_0 + a$, then by the extended Cauchy-Peano theorem (Theorem 8.34),

$$\lim_{t \to \omega_x} \|x(t)\| = \infty,$$

which is a contradiction. Hence we must have

$$\|x(t)\| \leq u_M(t), \quad t \in [t_0, t_0 + a],$$

and, in particular, the solution x exists on all of $[t_0, t_0 + a]$. $\square$

Corollary 8.64 *Assume that* $\Psi : [0,\infty) \to (0,\infty)$ *is continuous and there is a* $y_0 \in [0,\infty)$ *such that*

$$\int_{y_0}^{\infty} \frac{dv}{\Psi(v)} = \infty.$$

If $f : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and*

$$\|f(t,x)\| \le \Psi(\|x\|), \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^n,$$

then for all $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| \le y_0$, *all solutions of the vector IVP*

$$x' = f(t,x), \quad x(t_0) = x_0$$

exist on $[t_0,\infty)$.

Proof Let u be a solution of the IVP

$$u' = \Psi(u), \quad u(t_0) = y_0,$$

with right maximal interval of existence $[t_0,\omega_u)$. If $\omega_u < \infty$, then

$$\lim_{t\to\omega_u-} |u(t)| = \infty.$$

Since $u'(t) = \Psi(u(t)) > 0$, we must have

$$\lim_{t\to\omega_u-} u(t) = \infty.$$

For $t \ge t_0$, $u'(t) = \Psi(u(t))$ implies that

$$\frac{u'(t)}{\Psi(u(t))} = 1.$$

Integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} \frac{u'(s)}{\Psi(u(s))} ds = t - t_0.$$

Letting $v = u(s)$, we get that

$$\int_{y_0}^{u(t)} \frac{dv}{\Psi(v)} = t - t_0.$$

Letting $t \to \omega_u-$, we get the contradiction

$$\infty = \int_{y_0}^{\infty} \frac{dv}{\Psi(v)} = \omega_u - t_0 < \infty.$$

Hence we must have $\omega_u = \infty$ and so u is a solution on $[t_0,\infty)$. It then follows from Theorem 8.63 that all solutions of the vector IVP

$$x' = f(t,x), \quad x(t_0) = x_0$$

exist on $[t_0,\infty)$. □

Theorem 8.65 *Assume A is a continuous $n \times n$ matrix function and h is a continuous $n \times 1$ vector function on I. Then the IVP*

$$x' = A(t)x + h(t), \quad x(t_0) = x_0, \tag{8.33}$$

where $(t_0, x_0) \in I \times \mathbb{R}^n$, has a unique solution x and this solution exists on all of I. Furthermore,

$$\|x(t)\| \leq \left\{ \|x_0\| + \left| \int_{t_0}^{t} \|h(s)\| \, ds \right| \right\} e^{\left| \int_{t_0}^{t} \|A(s)\| ds \right|}, \tag{8.34}$$

for $t \in I$.

Proof By Corollary 8.18 we have already proved the existence and uniqueness of the solution of the IVP (8.33), so it only remains to show that the solution of this IVP exists on all of I and that the inequality (8.34) holds. We will just prove that the solution of this IVP exists on I to the right of t_0 and the inequality (8.34) holds on I to the right of t_0. Assume $t_1 > t_0$ and $t_1 \in I$. Let $f(t,x) := A(t)x + h(t)$, for $(t,x) \in I \times \mathbb{R}^n$ and note that

$$\begin{aligned} \|A(t)x + h(t)\| &\leq \|A(t)\|\|x\| + \|h(t)\| \\ &\leq M_1\|x\| + M_2, \end{aligned}$$

for $(t,x) \in [t_0, t_1] \times \mathbb{R}^n$, where M_1, M_2 are suitably chosen positive constants. Since the IVP

$$u' = M_1 u + M_2, \quad u(t_0) = \|x_0\|,$$

has a unique solution that exists on $[t_0, t_1]$, we get from Theorem 8.63 that the solution of the IVP (8.33) exists on $[t_0, t_1]$. Since this holds, for any $t_1 > t_0$ such that $t_1 \in I$ we get that the solution of the IVP (8.33) exists on I to the right of t_0.

Next we show that the solution x of the IVP (8.33) satisfies (8.34) on I to the right of t_0. Since x solves the IVP (8.33) on I, we have that

$$x(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + h(s)] \, ds,$$

for $t \in I$. It follows that for $t \in I$, $t \geq t_0$

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|A(s)\|\|x(s)\| \, ds + \int_{t_0}^{t} \|h(s)\| \, ds \\ &= \left\{ \|x_0\| + \int_{t_0}^{t} \|h(s)\| \, ds \right\} + \int_{t_0}^{t} \|A(s)\|\|x(s)\| \, ds. \end{aligned}$$

Let $t_1 \in I$ and assume that $t_1 > t_0$. Then for $t \in [t_0, t_1]$,

$$\|x(t)\| \leq \left\{ \|x_0\| + \int_{t_0}^{t_1} \|h(s)\| \, ds \right\} + \int_{t_0}^{t} \|A(s)\|\|x(s)\| \, ds.$$

Using Gronwall's inequality, we get that

$$\|x(t)\| \le \left\{\|x_0\| + \int_{t_0}^{t_1} \|h(s)\|\, ds\right\} e^{\int_{t_0}^{t} \|A(s)\| ds},$$

for $t \in [t_0, t_1]$. Letting $t = t_1$, we get

$$\|x(t_1)\| \le \left\{\|x_0\| + \int_{t_0}^{t_1} \|h(s)\|\, ds\right\} e^{\int_{t_0}^{t_1} \|A(s)\| ds}.$$

Since $t_1 \in I$, $t_1 > t_0$ is arbitrary, we get the desired result. The remainder of this proof is left as an exercise (see Exercise 8.49). □

8.11 Exercises

8.1 Assume I is an open interval, $p_k : I \to \mathbb{R}$ is continuous for $0 \le k \le n$, $p_n(t) \ne 0$ for $t \in I$, and $h : I \to \mathbb{R}$ is continuous. Show that the nth-order linear equation

$$p_n(t)u^{(n)} + p_{n-1}(t)u^{(n-1)} + \cdots + p_0(t)u = h(t)$$

is equivalent to a vector equation of the form (8.1) with $D := I \times \mathbb{R}^n$. Give what you think would be the appropriate definition of u is a solution of this nth-order equation on I.

8.2 Find the Jacobian matrix of $f(t, x)$ with respect to x for each of the following:

(a) $f(t,x) = \begin{pmatrix} x_1^2 e^{2x_1x_2} + t^2 \\ 5x_1x_2^3 \end{pmatrix}$ (b) $f(t,x) = \begin{pmatrix} \sin(x_1^2x_2^3) \\ \cos(t^2x_1x_2) \end{pmatrix}$

(c) $f(t,x) = \begin{pmatrix} x_1^2t^2 + x_2^2 + x_3^2 \\ x_1x_2x_3 \\ 4x_1^2x_3^4 \end{pmatrix}$ (d) $f(t,x) = \begin{pmatrix} e^{x_1^2x_2^2x_3^2} \\ x_1^3 + x_3^2 \\ 4x_1x_2x_3 + t \end{pmatrix}$

8.3 Show that $f(t,x) = \frac{x^2}{1+t^2}$ satisfies a Lipschitz condition with respect to x on $\mathbb{R} \times \mathbb{R}$, but does not satisfy a uniform Lipshitz condition with respect to x on $\mathbb{R} \times \mathbb{R}$.

8.4 Show that $f(t,x) = e^{3t} + 3|x|^p$, where $0 < p < 1$, does not satisfy a uniform Lipschitz condition with respect to x on $\mathbb{R}^2$.

8.5 Maximize the α in the Picard-Lindelof theorem by choosing the appropriate rectangle Q concerning the solution of the IVP

$$x' = x^3, \quad x(0) = 2.$$

Then solve this IVP to get the maximal interval of existence of the solution of this IVP.

8.6 Maximize the α in the Picard-Lindelof theorem by choosing the appropriate rectangle Q concerning the solution of the IVP

$$x' = 5 + x^2, \quad x(1) = 2.$$

8.7 Maximize the α in the Picard-Lindelof theorem by choosing the appropriate rectangle Q concerning the solution of the IVP

$$x' = (x+1)^2, \quad x(1) = 1.$$

Then solve this IVP to get the maximal interval of existence of the solution of this IVP.

8.8 Maximize the α in the Picard-Lindelof theorem by choosing the appropriate rectangle Q concerning the solution of the IVP

$$x' = t + x^2, \quad x(0) = 1.$$

8.9 Approximate the solution of the IVP

$$x' = \sin\left(\frac{\pi}{2}x\right), \quad x(0) = 1$$

by finding the second Picard iterate $x_2(t)$ and use (8.12) to find how good an approximation you get.

8.10 Approximate the solution of the IVP

$$x' = 2x^2 - x^3, \quad x(0) = 1$$

by finding the second Picard iterate $x_2(t)$.

8.11 Approximate the solution of the IVP

$$x' = 2x - x^2, \quad x(0) = 1$$

by finding the second Picard iterate $x_2(t)$.

8.12 Approximate the solution of the IVP

$$x' = \frac{x}{1+x^2}, \quad x(0) = 1$$

by finding the second Picard iterate $x_2(t)$ and use (8.12) to find how good an approximation you get.

8.13 Approximate the solution of the IVP

$$x' = \frac{1}{1+x^2}, \quad x(0) = 0$$

by finding the second Picard iterate $x_2(t)$ and use (8.12) to find how good an approximation you get.

8.14 Using Corollary 8.19, what can you say about solutions of IVPs for each of the following?

(i) $x'' = \sin(tx') + (x-2)^{\frac{2}{3}}$
(ii) $x''' = t^2 + x + (x')^2 + (x'')^3$

8.15 Show that the sequence of functions $\{x_n(t) := t^n\}, \quad 0 \le t \le 1$, satisfies all the hypotheses of the Ascoli-Arzela theorem (Theorem 8.26) except the fact that this sequence is equicontinuous. Show that the conclusion of the Ascoli-Arzela theorem (Theorem 8.26) for this sequence does not hold.

8.16 Can the Ascoli-Arzela theorem (Theorem 8.26) be applied to the sequence of functions $\{x_n(t) := \sin(nt)\}_{n=1}^{\infty}, \quad 0 \le t \le \pi$?

8.17 Verify that the sequence of functions $\{x_n(t) := \frac{1}{n}\sin(nt)\}_{n=1}^{\infty}$, is equicontinuous on $\mathbb{R}$.

8.18 Assume $g : [0,\infty) \to \mathbb{R}$ is continuous. Show that if $g'(3) = 0$, then $\{g_n(t) := g(nt) : n \in \mathbb{N}\}$ is not equicontinuous on $[0,\infty)$.

8.19 Assume that $\{x_n\}$ is an equicontinuous sequence of real-valued functions on $[a,b]$, which converges pointwise to x on $[a,b]$. Further assume there is a constant $p \ge 1$ such that for each $n \in \mathbb{N}$, $\int_a^b |x_n(t)|^p dt$ exists, and

$$\lim_{n\to\infty} \int_a^b |x_n(t) - x(t)|^p dt = 0.$$

Show that the sequence $\{x_n\}$ converges uniformly to x on $[a,b]$.

8.20 Write out the definition of $x(t) \to \partial D$ as $t \to a+$ mentioned in Definition 8.32.

8.21 At the beginning of the proof of Theorem 8.33, show how you can define the sequence of open sets $\{K_k\}$ such that the closure of K_k, $\bar{K}_k$, is compact, $\bar{K}_k \subset K_{k+1}$, and $\cup_{k=1}^{\infty} K_k = D$.

8.22 Use Theorem 8.36 to prove Corollary 8.37.

8.23 Find constants α and β so that $x(t) = \alpha t^{\beta}$ is a solution of the IVP (8.23) in Example 8.38. Show that the the sequence of Picard iterates $\{x_k(t)\}$ [with $x_0(t) \equiv 0$] for the IVP (8.23) does not even have a subsequence that converges to the solution of this IVP. Show directly by the definition of a Lipschitz condition that the $f(t,x)$ in Example 8.38 does not satisfy a Lipschitz condition with respect to x on $[0,1] \times \mathbb{R}$.

8.24 Show that $x_1(t) := 0$ and $x_2(t) := \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ define solutions of the IVP

$$x' = x^{\frac{1}{3}}, \quad x(0) = 0.$$

Even though solutions of IVPs are not unique, show that the sequence of Picard iterates $\{x_k(t)\}$ [with $x_0(t) \equiv 0$] converges to a solution of this IVP.

8.25 For each constant $x_0 \neq 0$, find the maximal interval of existence for the solution of the IVP $x' = x^3$, $x(0) = x_0$. Show directly that the conclusions of Theorem 8.33 concerning this solution hold.

8.26 Show that the IVP $x'' = -6x(x')^3$, $x(-1) = -1$, $x'(-1) = \frac{1}{3}$, has a unique solution x and find the maximal interval of existence of x. *H*int: Look for a solution of the given IVP of the form $x(t) = \alpha t^{\beta}$, where α and β are constants.

8.27 Show that the IVP

$$x' = -x^{\frac{1}{3}} - t^2 \arctan x, \quad x(0) = 0$$

has a unique solution. What is the unique solution of this IVP? Does the Picard-Lindelof theorem apply?

8.28 Show that if $\{x_k\}_{k=1}^{\infty}$ is a sequence of n-dimensional vector functions on $[a,b]$ satisfying the property that every subsequence has a subsequence that converges uniformly on $[a,b]$ to the same function x, then $\lim_{k\to\infty} x_k(t) = x(t)$ uniformly on $[a,b]$.

8.29 If $f(x) = |x|$ for $x \in \mathbb{R}$ and if $\delta > 0$ is a constant, find a formula for the integral mean $g_\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$g_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy,$$

for $x \in \mathbb{R}$. Use your answer to show that g_δ is continuously differentiable on $\mathbb{R}$. Then show directly that $\lim_{\delta\to 0+} g_\delta(x) = f(x)$ uniformly $\mathbb{R}$.

8.30 Given that

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0, \end{cases}$$

and $\delta > 0$, find the integral mean

$$g_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy,$$

for $x \in \mathbb{R}$.

8.31 Find the cross-sectional set S_6 in Kneser's theorem (Theorem 8.42) for each of the following IVPs:

(i) $x' = x^{\frac{2}{3}}, \quad x(0) = 0$
(ii) $x' = x^2, \quad x(0) = \frac{1}{8}$

8.32 Find a formula for the solution $x(t; t_0, x_0)$ of the initial value problem

$$x' = x - x^2, \quad x(t_0) = x_0.$$

Use your answer to find $z(t) = \frac{\partial x}{\partial x_0}(t; 0, 1)$ and $w(t) = \frac{\partial x}{\partial t_0}(t; 0, 1)$. Compare your answers to the results given in Example 8.45.

8.33 Let $x(t; a, b)$ denote the solution of the IVP

$$x' = 8 - 6x + x^2, \quad x(a) = b.$$

Without solving this IVP, find

$$\frac{\partial x}{\partial b}(t; 0, 2) \quad \text{and} \quad \frac{\partial x}{\partial a}(t; 0, 2).$$

Use your answers to approximate $x(t; h, 2)$, when h is close to zero and $x(t; 0, k)$, when k is close to 2.

8.34 Let $x(t; a, b)$ denote the solution of the IVP

$$x' = 1 + x^2, \quad x(a) = b.$$

Use Theorem 8.43 to find

$$\frac{\partial x}{\partial b}(t;0,0) \quad \text{and} \quad \frac{\partial x}{\partial a}(t;0,0).$$

Then check your answers by solving the preceding IVP and then finding these partial derivatives.

8.35 Let $x(t;a,b)$ denote the solution of the IVP

$$x' = \arctan x, \quad x(a) = b.$$

Use Theorem 8.43 to find

$$\frac{\partial x}{\partial b}(t;0,0) \quad \text{and} \quad \frac{\partial x}{\partial a}(t;0,0).$$

8.36 For the differential equation $u'' = u - u^3$, find

$$\frac{\partial u}{\partial y_{01}}(t;0,1,0), \quad \frac{\partial u}{\partial y_{02}}(t;0,1,0), \quad \text{and} \quad \frac{\partial u}{\partial t_0}(t;0,1,0),$$

and use your answers to approximate

$$u(t;0,1+h,0), \quad u(t;0,1,h), \quad \text{and} \quad u(t;h,1,0),$$

respectively, for h close to zero.

8.37 Let $u(t;t_0,a,b)$ denote the solution of the IVP

$$u'' = 4 - u^2, \quad u(t_0) = a, \quad u'(t_0) = b.$$

Using a result in Example 8.47, find $v(t) = \frac{\partial u}{\partial b}(t;0,2,0)$.

8.38 Let $x_k(t,t_0)$, $0 \le k \le n-1$, be the normalized solutions (see Definition 6.18) of $L_n x = 0$ (see Definition 6.1) at $t = t_0$. Derive formulas for $\frac{\partial x_k}{\partial t_0}(t,t_0)$, $0 \le k \le n-1$, in terms of the coefficients of $L_n x = 0$ and the normalized solutions $x_k(t,t_0)$, $0 \le k \le n-1$.

8.39 For each of the following find the normalized solutions $x_k(t,t_0)$, $0 \le k \le n-1$, of the given differential equation at $t = t_0$, and check, by calculating $\frac{\partial x_k}{\partial t_0}(t,t_0)$, $0 \le k \le n-1$, that the formulas that you got in Exercise 8.38 are satisfied.

(i) $x'' + x = 0$
(ii) $x''' - x'' = 0$

8.40 Let $u(t;t_0,a,b,\lambda)$, $\lambda > 0$, denote the solution of the IVP

$$u'' = \lambda u, \quad u(t_0) = a, \quad u'(t_0) = b.$$

Use Example 8.50 to calculate

$$\frac{\partial u}{\partial \lambda}(t;0,0,1,\lambda).$$

Check your answer by finding $u(t;0,0,1,\lambda)$ and differentiating with respect to λ.

8.41 Let $u(t; t_0, a, b, \lambda)$ denote the solution of the IVP

$$u'' + 4u = \lambda, \quad u(t_0) = a, \quad u'(t_0) = b.$$

Use Example 8.50 to find

$$\frac{\partial}{\partial \lambda} u(t; 0, 0, 2, 0).$$

Check your answer by actually finding $u(t; t_0, a, b, \lambda)$ and then taking the partial derivative with respect to λ.

8.42 Near the end of the proof of Theorem 8.53 one of the four cases was considered. Prove one of the remaining three cases.

8.43 Find the maximum and minimum solutions of the IVP

$$x' = x^{\frac{2}{3}}, \quad x(0) = 0,$$

and give their maximal intervals of existence.

8.44 Use Theorem 8.55 to show that if $v : [a, b] \to \mathbb{R}$ is continuous and if $D^+ v(t) \leq 0$ on $[a, b]$, then $v(t) \leq v(a)$ for $t \in [a, b]$.

8.45 Prove Theorem 8.56.

8.46 Prove Corollary 8.59.

8.47 Show that every solution of the IVP $x' = f(x)$, $x(0) = x_0$, where

$$f(x) = \begin{cases} \frac{x}{\sqrt{\|x\|}}, & x \neq 0 \\ 0, & x = 0, \end{cases} \qquad x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

exists on $[0, \infty)$ and find a bound on all such solutions.

8.48 Use Theorem 8.63 to find a lower bound for the right end point of the right maximal interval of existence for the solution of the IVP

$$\begin{aligned} x_1' &= x_1^2 - 2x_1 x_2, \\ x_2' &= x_1 + x_2^2, \end{aligned}$$
$$x_1(0) = 1, \quad x_2(0) = 0.$$

8.49 Do the part of the proof of Theorem 8.65, where $t \in I$ and $t < t_0$.