# Chapter 7

# BVPs for Nonlinear Second-Order DEs

## 7.1 Contraction Mapping Theorem (CMT)

Our discussion of differential equations up to this point has focused on solutions of initial value problems. We have made good use of the fact that these problems have unique solutions under mild conditions. We have also encountered boundary value problems in a couple of chapters. These problems require the solution or its derivative to have prescribed values at two or more points. Such problems arise in a natural way when differential equations are used to model physical problems. An interesting complication arises in that BVPs may have many solutions or no solution at all. Consequently, we will present some results on the existence and uniqueness of solutions of BVPs. A useful device will be the contraction mapping theorem (Theorem 7.5), which is introduced by the definitions below. One of the goals of this chapter is to show how the contraction mapping theorem is commonly used in differential equations.

**Definition 7.1** A linear (vector) space $\mathbb{X}$ is called a normed linear space (NLS) provided there is a function $\|\cdot\|:\mathbb{X}\to\mathbb{R}$, called a norm, satisfying

(i) $\|x\|\ge 0$ for all $x\in\mathbb{X}$ and $\|x\|=0$ iff $x=0$,
(ii) $\|\lambda x\|=|\lambda|\|x\|$, for all $\lambda\in\mathbb{R}$ and $x\in\mathbb{X}$,
(iii) $\|x+y\|\le\|x\|+\|y\|$, for all $x,y\in\mathbb{X}$.

**Definition 7.2** We say that $T:\mathbb{X}\to\mathbb{X}$ is a *contraction mapping* on a NLS $\mathbb{X}$ provided there is a constant $\alpha\in(0,1)$ such that

$$\|Tx-Ty\|\le\alpha\|x-y\|,$$

for all $x,y\in\mathbb{X}$. We say $\overline{x}\in\mathbb{X}$ is a *fixed point* of $T$ provided $T\overline{x}=\overline{x}$.

**Definition 7.3** We say that $\{x_n\}\subset\mathbb{X}$ is a *Cauchy sequence* provided given any $\varepsilon>0$ there is a positive integer $N$ such that $\|x_n-x_m\|<\varepsilon$ for all $n,m\ge N$.

**Definition 7.4** We say $\mathbb{X}$ is a *Banach space* provided it is a NLS and every Cauchy sequence in $\mathbb{X}$ converges to an element in $\mathbb{X}$.

**Theorem 7.5** (Contraction Mapping Theorem) *If $T$ is a contraction mapping on a Banach space $\mathbb{X}$ with contraction constant $\alpha$, with $0<\alpha<1$, then*

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*$T$ has a unique fixed point $\overline{x}$ in $\mathbb{X}$. Also if $x_0 \in \mathbb{X}$ and we set $x_{n+1} = Tx_n$, for $n \geq 0$, then*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

*Furthermore,*

$$\|x_n - \overline{x}\| \leq \frac{\alpha^n}{1-\alpha}\|x_1 - x_0\|, \tag{7.1}$$

*for $n \geq 1$.*

***Proof*** Let $x_0 \in \mathbb{X}$ and define the sequence $\{x_n\}$ of "Picard iterates" by $x_{n+1} = Tx_n$, for $n \geq 0$. Consider

$$\begin{aligned} \|x_{m+1} - x_m\| &= \|Tx_m - Tx_{m-1}\| \\ &\leq \alpha\|x_m - x_{m-1}\| \\ &= \alpha\|Tx_{m-1} - Tx_{m-2}\| \\ &\leq \alpha^2\|x_{m-1} - x_{m-2}\| \\ &\cdots \\ &\leq \alpha^m\|x_1 - x_0\|. \end{aligned}$$

Next consider

$$\begin{aligned} & \|x_{n+k} - x_n\| \\ \leq\ & \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \cdots + \|x_{n+1} - x_n\| \\ \leq\ & \left(\alpha^{n+k-1} + \cdots + \alpha^n\right)\|x_1 - x_0\| \\ =\ & \alpha^n\left(1 + \alpha + \cdots + \alpha^{k-1}\right)\|x_1 - x_0\| \\ \leq\ & \alpha^n\left(1 + \alpha + \alpha^2 + \cdots\right)\|x_1 - x_0\| \\ =\ & \frac{\alpha^n}{1-\alpha}\|x_1 - x_0\|. \end{aligned}$$

This last inequality implies that $\{x_n\}$ is a Cauchy sequence. Hence there is an $\overline{x} \in \mathbb{X}$ such that

$$\lim_{n\to\infty} x_n = \overline{x}.$$

Letting $k \to \infty$ in the last inequality above, we get that (7.1) holds. Since $T$ is a contraction mapping on $\mathbb{X}$, $T$ is uniformly continuous on $\mathbb{X}$ and hence is continuous on $\mathbb{X}$. Therefore,

$$\overline{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\overline{x},$$

and so $\overline{x}$ is a fixed point of $T$. Assume $\overline{\overline{x}} \in \mathbb{X}$ is a fixed point of $T$. Then

$$\|\overline{x} - \overline{\overline{x}}\| = \|T\overline{x} - T\overline{\overline{x}}\| \leq \alpha\,\|\overline{x} - \overline{\overline{x}}\|,$$

which implies that

$$(1-\alpha)\|\overline{x} - \overline{\overline{x}}\| \leq 0,$$

which implies that $\overline{x} = \overline{\overline{x}}$, so $\overline{x}$ is the only fixed point of $T$. □

## 7.2 Application of the CMT to a Forced Equation

First we use the contraction mapping theorem to study the forced second-order self-adjoint equation

$$[p(t)x']' + q(t)x = f(t), \tag{7.2}$$

where $p, q, f$ are continuous on $[a, \infty)$ and $p(t) > 0$ on $[a, \infty)$. In the next theorem we will give conditions on $p, q, f$ that ensure that (7.2) has a zero tending solution.

**Theorem 7.6** *Assume*

(i) $p(t) > 0,\ q(t) \geq 0,$ *for all* $t \in [a, \infty)$
(ii) $\int_a^\infty \frac{1}{p(t)} dt < \infty$
(iii) $\int_a^\infty q(t)P(t)\, dt < \infty,$ *where* $P(t) := \int_t^\infty \frac{1}{p(s)} ds$
(iv) $\int_a^\infty f(t)\, dt < \infty$

*hold; then equation* (7.2) *has a solution that converges to zero as* $t \to \infty$.

***Proof*** We will use the contraction mapping theorem to prove this theorem. Since $\int_a^\infty q(t)P(t)\, dt < \infty$, we can choose $b \in [a, \infty)$ sufficiently large so that

$$\alpha := \int_b^\infty q(\tau)P(\tau)\, d\tau < 1. \tag{7.3}$$

Let $\mathbb{X}$ be the Banach space (see Exercise 7.2) of all continuous functions $x : [b, \infty) \to \mathbb{R}$ that converge to zero as $t \to \infty$ with the norm $\|\cdot\|$ defined by

$$||x|| = \max_{t \in [b,\infty)} |x(t)|.$$

Define an operator $T$ on $\mathbb{X}$ by

$$Tx(t) = K(t) + P(t) \int_b^t q(\tau)x(\tau)\, d\tau + \int_t^\infty q(\tau)x(\tau)P(\tau)\, d\tau,$$

for $t \in [b, \infty)$, where

$$K(t) := \int_t^\infty \frac{F(s)}{p(s)} ds, \quad F(t) := \int_t^\infty f(s)\, ds, \quad \text{and} \quad P(t) := \int_t^\infty \frac{1}{p(s)} ds,$$

for $t \in [b, \infty)$. Clearly, $Tx$ is continuous on $[b, \infty)$. Hence to show $T : \mathbb{X} \to \mathbb{X}$ it remains to show that $\lim_{t\to\infty} Tx(t) = 0$. To show this it suffices to show that if

$$y(t) := P(t) \int_b^t q(\tau)x(\tau)\, d\tau,$$

then $\lim_{t\to\infty} y(t) = 0$. To see this let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} x(t) = 0$, there is a $c \geq b$ such that

$$|x(t)| < \epsilon \quad \text{for} \quad t \in [c, \infty).$$

Since $\lim_{t\to\infty} P(t) = 0$, there is a $d \geq c$ such that

$$P(t)\int_b^c q(\tau)|x(\tau)|\, d\tau < \epsilon,$$

for $t \geq d$. Consider for $t \geq d$

$$\begin{aligned}
|y(t)| &\leq P(t)\int_b^t q(\tau)|x(\tau)|\, d\tau \\
&= P(t)\int_b^c q(\tau)|x(\tau)|\, d\tau + P(t)\int_c^t q(\tau)|x(\tau)|\, d\tau \\
&\leq \epsilon + \epsilon P(t)\int_c^t q(\tau)\, d\tau \\
&\leq \epsilon + \epsilon\int_c^t q(\tau)P(\tau)\, d\tau \\
&\leq \epsilon + \epsilon\alpha \leq 2\epsilon.
\end{aligned}$$

Hence $\lim_{t\to\infty} y(t) = 0$ and so $T : \mathbb{X} \to \mathbb{X}$.

Next we show that $T$ is a contraction mapping on $X$. Let $x, y \in X$, $t \geq b$, and consider

$$\begin{aligned}
& |Tx(t) - Ty(t)| \\
\leq\ & P(t)\int_b^t q(\tau)|x(\tau) - y(\tau)|\, d\tau + \int_t^\infty q(\tau)P(\tau)|x(\tau) - y(\tau)|\, d\tau \\
\leq\ & \left\{P(t)\int_b^t q(\tau)\, d\tau + \int_t^\infty q(\tau)P(\tau)\, d\tau\right\}\|x - y\| \\
\leq\ & \left\{\int_b^t q(\tau)P(\tau)\, d\tau + \int_t^\infty q(\tau)P(\tau)\, d\tau\right\}\|x - y\| \\
=\ & \int_b^\infty q(\tau)P(\tau)\, d\tau||x - y|| \\
=\ & \alpha||x - y||,
\end{aligned}$$

where we have used the fact that $P$ is decreasing and we have also used (7.3). Therefore, $T$ is a contraction mapping on $X$. Hence by the contraction mapping theorem (Theorem 7.5), $T$ has a unique fixed point $x \in X$. This implies that

$$x(t) = K(t) + P(t)\int_b^t q(\tau)x(\tau)\, d\tau + \int_t^\infty q(\tau)P(\tau)x(\tau)\, d\tau, \qquad (7.4)$$

for all $t \geq b$. Since $x \in \mathbb{X}$,

$$\lim_{t\to\infty} x(t) = 0.$$

It remains to show that x given by (7.4) is a solution of (7.2). Differentiating both sides of (7.4), we get

$$\begin{aligned} x'(t) &= K'(t) + P(t)q(t)x(t) + P'(t)\int_b^t q(\tau)x(\tau)\, d\tau - q(t)P(t)x(t) \\ &= -\frac{F(t)}{p(t)} - \frac{1}{p(t)}\int_b^t q(\tau)x(\tau)\, d\tau, \end{aligned}$$

and so

$$p(t)x'(t) = -F(t) - \int_b^t q(\tau)x(\tau)\, d\tau,$$

and therefore

$$(p(t)x'(t))' = f(t) - q(t)x(t).$$

So we have $x$ is a solution of the forced equation (7.2). □

## 7.3 Applications of the CMT to BVPs

We begin our study of the nonlinear BVP

$$x'' = f(t, x), \quad x(a) = A, \quad x(b) = B, \tag{7.5}$$

where we assume $a < b$ and $A, B \in \mathbb{R}$, by using the contraction mapping theorem to establish the existence of a unique solution for a class of functions $f$.

**Theorem 7.7** *Assume $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to $x$ on $[a, b] \times \mathbb{R}$ with Lipschitz constant $K$; that is,*

$$|f(t, x) - f(t, y)| \leq K|x - y|,$$

*for all $(t, x)$, $(t, y) \in [a, b] \times \mathbb{R}$. If*

$$b - a < \frac{2\sqrt{2}}{\sqrt{K}}, \tag{7.6}$$

*then the BVP (7.5) has a unique solution.*

***Proof*** Let $\mathbb{X}$ be the Banach space of continuous functions on $[a, b]$ with norm (called the max norm) defined by

$$\|x\| = \max\{|x(t)| : a \leq t \leq b\}.$$

Note that $x$ is a solution of the BVP (7.5) iff $x$ is a solution of the linear, nonhomogeneous BVP

$$x'' = h(t) := f(t, x(t)), \quad x(a) = A, \quad x(b) = B. \tag{7.7}$$

But the BVP (7.7) has a solution iff the integral equation

$$x(t) = z(t) + \int_a^b G(t, s)f(s, x(s))\, ds$$

has a solution, where $z$ is the solution of the BVP

$$z'' = 0, \quad z(a) = A, \quad z(b) = B$$

and $G$ is the Green's function for the BVP

$$x'' = 0, \quad x(a) = 0, \quad x(b) = 0.$$

Define $T : \mathbb{X} \to \mathbb{X}$ by

$$Tx(t) = z(t) + \int_a^b G(t,s) f(s, x(s))\, ds,$$

for $t \in [a,b]$. Hence the BVP (7.5) has a unique solution iff the operator $T$ has a unique fixed point.

We will use the contraction mapping theorem (Theorem 7.5) to show that $T$ has a unique fixed point in $\mathbb{X}$. Let $x, y \in \mathbb{X}$ and consider

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_a^b |G(t,s)||f(s,x(s)) - f(s,y(s))|\, ds \\ &\leq \int_a^b |G(t,s)| K |x(s) - y(s)|\, ds \\ &\leq K \int_a^b |G(t,s)|\, ds\, \|x - y\| \\ &\leq K \frac{(b-a)^2}{8}\, \|x - y\|, \end{aligned}$$

for $t \in [a,b]$, where for the last inequality we used Theorem 5.109. It follows that

$$\|Tx - Ty\| \leq \alpha \|x - y\|,$$

where from (7.6)

$$\alpha := K \frac{(b-a)^2}{8} < 1.$$

Hence $T$ is a contraction mapping on $\mathbb{X}$, and by the contraction mapping theorem (Theorem 7.5) we get the desired conclusion. □

In the proof of Theorem 7.7 we used the contraction mapping theorem to prove the existence of the solution of the BVP (7.7). In this proof the operator $T$ was defined by

$$Tx(t) = z(t) + \int_a^b G(t,s) f(s, x(s))\, ds,$$

for $t \in [a,b]$. Hence, in this case, if $x_0$ is any continuous function on $[a,b]$, then recall that the Picard iterates are defined by

$$x_{n+1}(t) = Tx_n(t),$$

for $t \in [a,b]$, $n = 0, 1, 2, \cdots$. Hence

$$x_{n+1}(t) = z(t) + \int_a^b G(t,s) f(s, x_n(s))\, ds,$$

for $t \in [a, b]$, $n = 0, 1, 2, \cdots$. It follows from this that the Picard iterate $x_{n+1}$ solves the BVP

$$x''_{n+1} = f(t, x_n(t)),$$
$$x(a) = A, \quad x(b) = B,$$

which is usually easy to solve.

**Example 7.8** Consider the BVP

$$x'' = -1 - \sin\, x, \quad x(0) = 0, \quad x(1) = 0. \tag{7.8}$$

Here $f(t, x) = -1 - \sin\, x$ and so

$$|f_x(t, x)| = |\cos\, x| \leq K := 1.$$

Since

$$b - a = 1 < \frac{2\sqrt{2}}{\sqrt{K}} = 2\sqrt{2},$$

we have by Theorem 7.7 that the BVP (7.8) has a unique solution. If we take as our initial approximation $x_0(t) \equiv 0$, then the first Picard iterate $x_1$ solves the BVP

$$x''_1 = f(t, x_0(t)) = f(t, 0) = -1,$$
$$x_1(0) = 0, \quad x_1(1) = 0.$$

Solving this BVP, we get that the first Picard iterate is given by

$$x_1(t) = -\frac{1}{2}t(t-1), \quad t \in [0, 1].$$

Next we want to use (7.1) to see how good an approximation $x_1$ is to the actual solution $x$ of the BVP (7.8). Recall in the proof of Theorem 7.7 the norm $\|\cdot\|$ was the max norm and the contraction constant $\alpha$ is given by

$$\alpha = K\frac{(b-a)^2}{8} = \frac{1}{8}.$$

Hence by (7.1)

$$\begin{aligned} \|x - x_1\| &\leq \frac{\alpha}{1-\alpha}\|x_1 - x_0\| \\ &= \frac{1}{7}\|x_1\| = \frac{1}{56}. \end{aligned}$$

It follows that

$$|x(t) - x_1(t)| \leq \frac{1}{56}, \quad t \in [0, 1],$$

and we see that $x_1(t) = \frac{1}{2}t(1-t)$ is a very good approximation of the actual solution $x$ of the BVP (7.8). △

Next we see if we use a slightly more complicated norm than in the proof of Theorem 7.7 we can prove the following theorem, which is a better theorem in the sense that it allows the length of the interval $[a, b]$ to be larger. However, since the norm is more complicated it makes the application of the inequality (7.1) more complicated.

**Theorem 7.9** *Assume $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to $x$ on $[a,b] \times \mathbb{R}$ with Lipschitz constant $K$; that is,*

$$|f(t,x) - f(t,y)| \le K|x-y|,$$

*for all $(t,x)$, $(t,y) \in [a,b] \times \mathbb{R}$. If*

$$b - a < \frac{\pi}{\sqrt{K}}, \tag{7.9}$$

*then the BVP (7.5) has a unique solution.*

***Proof*** Let $\mathbb{X}$ be the Banach space (see Exercise 7.3) of continuous functions on $[a,b]$ with norm defined by

$$\|x\| = \max\left\{\frac{|x(t)|}{\rho(t)} : a \le t \le b\right\},$$

where $\rho$ is a positive continuous function on $[a,b]$ that we will choose later. Note that $x$ is a solution of the BVP (7.5) iff $x$ is a solution of the linear, nonhomogeneous BVP

$$x'' = h(t) := f(t, x(t)), \quad x(a) = A, \quad x(b) = B. \tag{7.10}$$

But the BVP (7.10) has a solution iff the integral equation

$$x(t) = z(t) + \int_a^b G(t,s) f(s, x(s))\, ds$$

has a solution, where $z$ is the solution of the BVP

$$z'' = 0, \quad z(a) = A, \quad z(b) = B,$$

and $G$ is the Green's function for the BVP

$$x'' = 0, \quad x(a) = 0, \quad x(b) = 0.$$

Define $T : \mathbb{X} \to \mathbb{X}$ by

$$Tx(t) = z(t) + \int_a^b G(t,s) f(s, x(s))\, ds,$$

for $t \in [a,b]$. Hence the BVP (7.5) has a unique solution iff the operator $T$ has a unique fixed point.

We will use the contraction mapping theorem (Theorem 7.5) to show that $T$ has a unique fixed point in $\mathbb{X}$. Let $x, y \in \mathbb{X}$ and consider

$$\begin{aligned}
\frac{|Tx(t) - Ty(t)|}{\rho(t)} &\leq \frac{1}{\rho(t)} \int_a^b |G(t,s)||f(s,x(s)) - f(s,y(s))| \, ds \\
&\leq \frac{1}{\rho(t)} \int_a^b |G(t,s)|K|x(s) - y(s)| \, ds \\
&= \frac{1}{\rho(t)} \int_a^b |G(t,s)|K\rho(s)\frac{|x(s) - y(s)|}{\rho(s)} ds \\
&\leq \frac{K}{\rho(t)} \int_a^b |G(t,s)|\rho(s) \, ds \, \|x - y\|,
\end{aligned}$$

for $t \in [a,b]$. If we could choose a positive continuous function $\rho$ such that there is a constant $\alpha$ for which

$$K\frac{1}{\rho(t)} \int_a^b |G(t,s)|\rho(s) \, ds \leq \alpha < 1, \tag{7.11}$$

for $a \leq t \leq b$, then it would follow that $T$ is a contraction mapping on $\mathbb{X}$, and by the contraction mapping theorem (Theorem 7.5) we would get the desired conclusion. Hence it remains to show that we can pick a positive continuous function $\rho$ such that (7.11) holds.

Since $b - a < \pi/\sqrt{K}$, the function defined by

$$\rho(t) = \sin(\sqrt{K/\alpha}(t - c))$$

is positive on the interval $[a,b]$, for $\alpha < 1$ near 1 and $c < a$ near $a$. Now $\rho$ satisfies

$$\rho'' + \frac{K}{\alpha}\rho = 0,$$

so

$$\begin{aligned}
\rho(t) &= \frac{\rho(b)(t-a) + \rho(a)(b-t)}{b-a} + \int_a^b G(t,s)\left(-\frac{K}{\alpha}\rho(s)\right) \, ds \\
&> \int_a^b G(t,s)\left(-\frac{K}{\alpha}\rho(s)\right) \, ds.
\end{aligned}$$

Since the Green's function $G(t,s) \leq 0$, we have

$$\rho(t) > \int_a^b |G(t,s)|[\frac{K}{\alpha}\rho(s)] \, ds.$$

Then (7.11) holds, and the proof is complete. □

The next example shows that Theorem 7.9 is sharp.

**Example 7.10** Consider the BVP

$$x'' + Kx = 0, \quad x(a) = 0, \quad x(b) = B,$$

where $K$ is a positive constant.

By Theorem 7.9, if $b - a < \frac{\pi}{\sqrt{K}}$ the preceding BVP has a unique solution. Note, however, that if $b - a = \frac{\pi}{\sqrt{K}}$, then the given BVP does not have a solution if $B \neq 0$ and has infinitely many solutions if $B = 0$. △

**Theorem 7.11** *Assume $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the uniform Lipschitz condition with respect to $x$ and $x'$,*

$$|f(t, x, x') - f(t, y, y')| \leq K|x - y| + L|x' - y'|,$$

*for $(t, x, x'), (t, y, y') \in [a, b] \times \mathbb{R}^2$, where $L \geq 0, K > 0$ are constants. If*

$$K\frac{(b-a)^2}{8} + L\frac{b-a}{2} < 1, \tag{7.12}$$

*then the BVP*

$$x'' = f(t, x, x'), \quad x(a) = A, \quad x(b) = B \tag{7.13}$$

*has a unique solution.*

***Proof*** Let $\mathbb{X}$ be the Banach space of continuously differentiable functions on $[a, b]$ with norm $\|\cdot\|$ defined by

$$\|x\| = \max\{K|x(t)| + L|x'(t)| : a \leq t \leq b\}.$$

Note that $x$ is a solution of the BVP (7.13) iff $x$ is a solution of the BVP

$$x'' = h(t) := f(t, x(t), x'(t)), \quad x(a) = A, \quad x(b) = B. \tag{7.14}$$

But the BVP (7.14) has a solution iff the integral equation

$$x(t) = z(t) + \int_a^b G(t, s) f(s, x(s), x'(s))\, ds$$

has a solution, where $z$ is the solution of the BVP

$$z'' = 0, \quad z(a) = A, \quad z(b) = B,$$

and $G$ is the Green's function for the BVP

$$x'' = 0, \quad x(a) = 0, \quad x(b) = 0.$$

Define $T : \mathbb{X} \to \mathbb{X}$ by

$$Tx(t) = z(t) + \int_a^b G(t, s) f(s, x(s), x'(s))\, ds,$$

for $t \in [a, b]$. Then the BVP (7.13) has a unique solution iff the operator $T$ has a unique fixed point. We will use the contraction mapping theorem (Theorem 7.5) to show that $T$ has a unique fixed point. Let $x, y \in \mathbb{X}$ and consider

$$
\begin{aligned}
& K|Tx(t) - Ty(t)| \\
\leq\ & K \int_a^b |G(t,s)||f(s, x(s), x'(s)) - f(s, y(s), y'(s))|\, ds \\
\leq\ & K \int_a^b |G(t,s)|\, [K|x(s) - y(s)| + L|x'(s) - y'(s)|]\, ds \\
\leq\ & K \int_a^b |G(t,s)|\, ds\, \|x - y\| \\
\leq\ & K \frac{(b-a)^2}{8}\, \|x - y\|,
\end{aligned}
$$

for $t \in [a, b]$, where we have used Theorem 5.109. Similarly,

$$
\begin{aligned}
& L|(Tx)'(t) - (Ty)'(t)| \\
\leq\ & L \int_a^b |G_t(t,s)||f(s, x(s), x'(s)) - f(s, y(s), y'(s))|\, ds \\
\leq\ & L \int_a^b |G_t(t,s)|\, [K|x(s) - y(s)| + L|x'(s) - y'(s)|]\, ds \\
\leq\ & L \int_a^b |G_t(t,s)|\, ds\, \|x - y\| \\
\leq\ & L \frac{(b-a)}{2}\, \|x - y\|,
\end{aligned}
$$

for $t \in [a, b]$, where we have used Theorem 5.109. It follows that

$$\|Tx - Ty\| \leq \alpha \|x - y\|,$$

where

$$\alpha := K \frac{(b-a)^2}{8} + L \frac{b-a}{2} < 1$$

by (7.12). From the contraction mapping theorem (Theorem 7.5) we get that $T$ has a unique fixed point in $\mathbb{X}$. This implies that the BVP (7.13) has a unique solution. □

**Theorem 7.12** *Assume $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there are nonnegative continuous functions $p$, $q$ on $[a, b]$ such that*

$$|f(t, x, x') - f(t, y, y')| \leq p(t)|x - y| + q(t)|x' - y'|,$$

*for $(t, x, x'), (t, y, y') \in [a, b] \times \mathbb{R}^2$. Let $u$ be the solution of the IVP*

$$u'' + q(t)u' + p(t)u = 0, \quad u(a) = 0, \quad u'(a) = 1; \tag{7.15}$$

*then if $u'(t) > 0$ on $[a, b]$ it follows that the right-focal BVP*

$$x'' = f(t, x, x'), \quad x(a) = A, \quad x'(b) = m, \tag{7.16}$$

*where $A$ and $m$ are constants, has a unique solution.*

***Proof*** Let $\mathbb{X}$ be the Banach space of continuously differentiable functions on $[a, b]$ with norm $\|\cdot\|$ defined by

$$\|x\| = \max\left\{\max_{t\in[a,b]} \frac{|x(t)|}{v(t)}, \max_{t\in[a,b]} \frac{|x'(t)|}{w(t)}\right\},$$

where $v, w$ are positive continuous functions that we will choose later in this proof. Then $\mathbb{X}$ is a Banach space. Define $T : \mathbb{X} \to \mathbb{X}$ by

$$Tx(t) = z(t) + \int_a^b G(t,s)f(s,x(s),x'(s))\, ds,$$

for $t \in [a, b]$, where $z$ is the solution of the BVP

$$z'' = 0, \quad z(a) = A, \quad z'(b) = m,$$

and $G$ is the Green's function for the BVP

$$x'' = 0, \quad x(a) = 0, \quad x'(b) = 0.$$

Then the BVP (7.16) has a unique solution iff the operator $T$ has a unique fixed point. We will use the contraction mapping theorem (Theorem 7.5) to show that $T$ has a unique fixed point. Let $x, y \in \mathbb{X}$ and consider

$$\begin{aligned}
&\frac{|Tx(t) - Ty(t)|}{v(t)} \\
\leq\ & \frac{1}{v(t)} \int_a^b |G(t,s)||f(s,x(s),x'(s)) - f(s,y(s),y'(s))|\, ds \\
\leq\ & \frac{1}{v(t)} \int_a^b |G(t,s)|\,[p(s)|x(s) - y(s)| + q(s)|x'(s) - y'(s)|]\, ds \\
\leq\ & \frac{1}{v(t)} \int_a^b |G(t,s)|\left[p(s)v(s)\frac{|x(s) - y(s)|}{v(s)} + q(s)w(s)\frac{|x'(s) - y'(s)|}{w(s)}\right] ds \\
\leq\ & \frac{1}{v(t)} \int_a^b |G(t,s)|\,[p(s)v(s) + q(s)w(s)]\, ds\, \|x - y\|,
\end{aligned}$$

for $t \in [a, b]$. Similarly,

$$\begin{aligned} & \frac{|(Tx)'(t) - (Ty)'(t)|}{w(t)} \\ \leq \quad & \frac{1}{w(t)} \int_a^b |G_t(t,s)||f(s, x(s), x'(s)) - f(s, y(s), y'(s))| \, ds \\ \leq \quad & \frac{1}{w(t)} \int_a^b |G_t(t,s)| \, [p(s)|x(s) - y(s)| + q(s)|x'(s) - y'(s)|] \, ds \\ \leq \quad & \frac{1}{w(t)} \int_a^b |G_t(t,s)|[p(s)v(s) \frac{|x(s) - y(s)|}{v(s)} \\ & + q(s)w(s) \frac{|x'(s) - y'(s)|}{w(s)}] ds \\ \leq \quad & \frac{1}{w(t)} \int_a^b |G_t(t,s)| \, [p(s)v(s) + q(s)w(s)] \, ds \, \|x - y\|, \end{aligned}$$

for $t \in [a, b]$. Thus we see that we want to find positive continuous functions $v, w$ and a constant $\alpha$ such that

$$\frac{1}{v(t)} \int_a^b |G(t,s)| \, [p(s)v(s) + q(s)w(s)] \, ds \leq \alpha < 1, \tag{7.17}$$

and

$$\frac{1}{w(t)} \int_a^b |G_t(t,s)| \, [p(s)v(s) + q(s)w(s)] \, ds \leq \alpha < 1, \tag{7.18}$$

for $t \in [a, b]$. If we can do this, then $T$ is a contraction mapping on $\mathbb{X}$ and the conclusion of this theorem follows from the contraction mapping theorem (Theorem 7.5). By hypothesis the solution $u$ of the IVP (7.15) satisfies $u'(t) > 0$ on $[a, b]$. Let $v$ be the solution of the IVP

$$v'' = -\frac{1}{\alpha}[p(t)v + q(t)v'], \quad v(a) = \epsilon, \quad v'(a) = 1,$$

where we pick $\epsilon > 0$ sufficient close to zero and $\alpha < 1$ sufficiently close to one so that

$$v(t) > 0, \quad v'(t) > 0, \quad \text{on} \quad [a, b].$$

Let $w(t) := v'(t)$. If $B := v'(b) > 0$, it follows that $v$ solves the BVP

$$v'' = -\frac{1}{\alpha}[p(t)v + q(t)v'], \quad v(a) = \epsilon, \quad v'(b) = B. \tag{7.19}$$

Let $z_1$ be the solution of the BVP

$$z_1'' = 0, \quad z_1(a) = \epsilon > 0, \quad z_1'(b) = B > 0.$$

It follows that $z_1(t) > 0$ and $z_1'(t) > 0$ on $[a,b]$. Since $v$ solves the BVP (7.19),

$$\begin{aligned} v(t) &= z_1(t) + \frac{-1}{\alpha}\int_a^b G(t,s)\,[p(s)v(s)+q(s)v'(s)]\,ds \qquad (7.20)\\ &> \frac{-1}{\alpha}\int_a^b G(t,s)\,[p(s)v(s)+q(s)v'(s)]\,ds\\ &= \frac{1}{\alpha}\int_a^b |G(t,s)|\,[p(s)v(s)+q(s)w(s)]\,ds, \end{aligned}$$

for $t \in [a,b]$. From this we get the desired inequality (7.17), for $t \in [a,b]$. Differentiating both sides of (7.20), we get

$$\begin{aligned} w(t) &= v'(t) = z_1'(t) + \frac{-1}{\alpha}\int_a^b G_t(t,s)\,[p(s)v(s)+q(s)v'(s)]\,ds\\ &> \frac{-1}{\alpha}\int_a^b G_t(t,s)\,[p(s)v(s)+q(s)v'(s)]\,ds\\ &= \frac{1}{\alpha}\int_a^b |G_t(t,s)|\,[p(s)v(s)+q(s)w(s)]\,ds, \end{aligned}$$

for $t \in [a,b]$, but this gives us the other desired inequality (7.18), for $t \in [a,b]$. □

Now we deduce some useful information about the linear differential equation with constant coefficients

$$u'' + Lu' + Ku = 0. \qquad (7.21)$$

**Definition 7.13** Assume that $u$ is the solution of the IVP (7.21),

$$u(0) = 0, \quad u'(0) = 1.$$

We define $\rho(K,L)$ as follows: If $u'(t) > 0$ on $[0,\infty)$, then $\rho(K,L) := \infty$. Otherwise $\rho(K,L)$ is the first point to the right of 0 where $u'(t)$ is zero.

In Exercise 7.14 you are asked to show that if $u$ is a nontrivial solution of (7.21) with $u(0) = 0$, then if $u'(t) \neq 0$ on $[0,\infty)$, then $\rho(K,L) := \infty$. Otherwise $\rho(K,L)$ is the first point to the right of 0 where $u'(t)$ is zero. We will use this fact in the proof of the next theorem.

**Theorem 7.14** *Assume $K, L \geq 0$; then*

$$\rho(K,L) = \begin{cases} \frac{2}{L}, & \text{if } L^2 - 4K = 0,\ L > 0,\\ \frac{2}{\sqrt{L^2-4K}}\operatorname{arccosh}\left(\frac{L}{2\sqrt{K}}\right), & \text{if } L^2 - 4K > 0,\ K > 0,\\ \frac{2}{\sqrt{4K-L^2}}\arccos\left(\frac{L}{2\sqrt{K}}\right), & \text{if } L^2 - 4K < 0,\\ \infty, & \text{if } L \geq 0,\ K = 0. \end{cases}$$

***Proof*** We will only consider the case $L^2 - 4K > 0$, $K > 0$. The other cases are contained in Exercise 7.15. In the case $L^2 - 4K > 0$, a general solution of (7.21) is

$$u(t) = Ae^{-\frac{L}{2}t} \cosh\left(\frac{\sqrt{L^2-4K}}{2}t\right) + Be^{-\frac{L}{2}t} \sinh\left(\frac{\sqrt{L^2-4K}}{2}t\right).$$

The boundary condition $u(0) = 0$ gives us that $A = 0$ and hence

$$u(t) = Be^{-\frac{L}{2}t} \sinh\left(\frac{\sqrt{L^2-4K}}{2}t\right).$$

Differentiating, we get

$$\begin{aligned} u'(t) &= -\frac{L}{2}Be^{-\frac{L}{2}t} \sinh\left(\frac{\sqrt{L^2-4K}}{2}t\right) \\ &+ \quad B\frac{\sqrt{L^2-4K}}{2}e^{-\frac{L}{2}t} \cosh\left(\frac{\sqrt{L^2-4K}}{2}t\right). \end{aligned}$$

Setting $u'(t) = 0$, dividing by $Be^{-\frac{L}{2}t}$, and simplifying, we get

$$\frac{L}{2} \sinh\left(\frac{\sqrt{L^2-4K}}{2}t\right) = \frac{\sqrt{L^2-4K}}{2} \cosh\left(\frac{\sqrt{L^2-4K}}{2}t\right).$$

Squaring both sides, we get

$$\frac{L^2}{4} \sinh^2\left(\frac{\sqrt{L^2-4K}}{2}t\right) = \frac{L^2-4K}{4} \cosh^2\left(\frac{\sqrt{L^2-4K}}{2}t\right).$$

Using the identity $\cosh^2\theta - \sinh^2\theta = 1$ and simplifying, we get

$$\cosh^2\left(\frac{\sqrt{L^2-4K}}{2}t\right) = \frac{L^2}{4K}.$$

Solving for $t$, we get the desired result

$$t = \frac{2}{\sqrt{L^2-4K}} \operatorname{arccosh}\left(\frac{L}{2\sqrt{K}}\right).$$

□

**Theorem 7.15** *Assume $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there are constants $K$, $L$ such that*

$$|f(t,x,x') - f(t,y,y')| \le K|x-y| + L|x'-y'|,$$

*for $(t,x,x'), (t,y,y') \in [a,b] \times \mathbb{R}^2$. If*

$$b - a < \rho(K,L),$$

*then it follows that the right-focal BVP*

$$x'' = f(t,x,x'), \quad x(a) = A, \quad x'(b) = m, \tag{7.22}$$

*where $A$ and $m$ are constants, has a unique solution.*

***Proof*** By Theorem 7.14, the solution of the IVP

$$v'' + Lv' + Kv = 0, \quad v(0) = 0, \quad v'(0) = 1$$

satisfies $v'(t) > 0$ on $[0, d]$, if $d < \rho(K, L)$. But this implies that $u(t) := v(t-a)$ is the solution of the IVP

$$u'' + Lu' + Ku = 0, \quad u(a) = 0, \quad u'(a) = 1$$

and satisfies $u'(t) > 0$ on $[a, b]$ if $b - a < \rho(K, L)$. It then follows from Theorem 7.12 that the BVP (7.22) has a unique solution. □

In the next example we give an interesting application of Theorem 7.15 (see [**3**]).

**Example 7.16** We are interested in how tall a vertical thin rod can be before it starts to bend (see Figure 1). Assume that a rod has length $T$, weight $W$, and constant of flexual rigidity $B > 0$. Assume that the rod is constrained to remain vertical at the bottom end of the rod, which is located at the origin in the $st$-plane shown in Figure 1. Further, assume that the rod has uniform material and constant cross section. If we let $x(t) = \frac{ds}{dt}(t)$, then it turns out that $x$ satisfies the differential equation

$$x'' + \frac{W}{B}\frac{T-t}{T}x = 0. \tag{7.23}$$

If the rod does not bend, then the BVP (7.23),

$$x(0) = 0, \quad x'(T) = 0$$

has only the trivial solution. If the rod bends, then this BVP has a nontrivial solution (so solutions of this BVP are not unique). We now apply Theorem 7.15 to this BVP. Here $f(t, x, x') = -\frac{W}{B}\frac{T-t}{T}x$ and so

$$|f_x(t, x, x')| = \frac{W}{B}\frac{T-t}{T} \le K := \frac{W}{B},$$

and

$$|f_{x'}(t, x, x')| = 0 =: L.$$

Since

$$L^2 - 4K = -4\frac{W}{B} < 0,$$

we have by Theorem 7.14,

$$\rho(K, L) = \frac{2}{\sqrt{4K - L^2}} \arccos\left(\frac{L}{2\sqrt{K}}\right) = \frac{\pi}{2}\sqrt{\frac{B}{W}}.$$

Thus if $T < \frac{\pi}{2}\sqrt{\frac{B}{W}}$, then the rod is stable (does not bend). It was pointed out in [**3**] that it has been shown that if $T < 2.80\sqrt{\frac{B}{W}}$, then the rod is stable. Note that the upper bound on $T$ is larger if $B$ is larger and smaller if $W$ is larger, which is what we expect intuitively. △

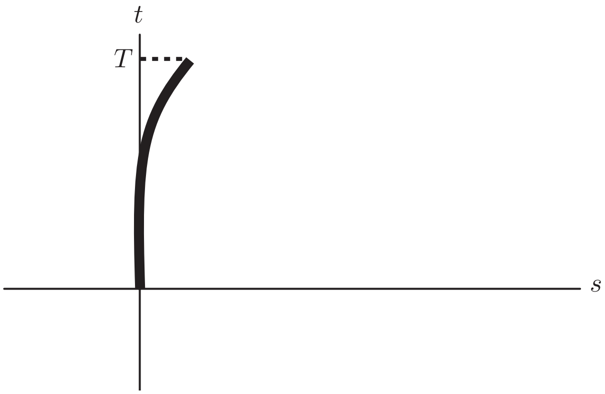


Figure 1. Vertical rod.

## 7.4 Lower and Upper Solutions

In this section, we will take a different approach to the study of BVPs, namely we will define functions called upper and lower solutions that, not only imply the existence of a solution of a BVP but also provide bounds on the location of the solution. The following result is fundamental:

**Theorem 7.17** *Assume that $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded. Then the BVP*

$$x'' = f(t, x, x'),$$

$$x(a) = A, \quad x(b) = B,$$

*where $A$ and $B$ are constants, has a solution.*

***Proof*** We can give a straightforward and more intuitive proof if we assume (as in Chapter 3) that $f$ has continuous partials with respect to $x$ and $x'$. Then IVP's have unique solutions that are continuous functions of their initial values. For a proof of the more general result, see Hartman [**18**].

Let $|f(t, x, x')| \leq P$, for $(t, x, x') \in [a, b] \times \mathbb{R}^2$. Define $\phi(t, A, v)$ to be the solution of $x'' = f(t, x, x')$ satisfying the initial conditions $x(a) = A$, $x'(a) = v$. Then $\phi$ is continuous.

We first show that for each $v$, $\phi(t, A, v)$ exists on $[a, b]$. Since $|\phi''| \leq P$, integration yields

$$|\phi'(t, A, v) - \phi'(a, A, v)| \leq P|t - a|,$$

so that

$$|\phi'(t, A, v)| \leq P(b - a) + |v|,$$

and $\phi'$ is bounded on the intersection of its maximal interval of existence with $[a, b]$. By integrating again, we get that

$$|\phi(t, A, v)| \leq |A| + P(b - a)^2 + |v|(b - a),$$

so $\phi$ is bounded on the same set. It follows from Theorem 3.1 that $\phi(t, A, v)$ exists for $a \le t \le b$.

To complete the proof, we need to show that there is a $w$ so that $\phi(b, A, w) = B$. As in the preceding paragraph, we have

$$\phi'(a, A, v) - \phi'(t, A, v) \le P(b - a),$$

so

$$\phi'(t, A, v) \ge v - P(b - a),$$

for $a \le t \le b$. Choose

$$v > \frac{B - A}{b - a} + P(b - a).$$

Then

$$\phi'(t, A, v) > \frac{B - A}{b - a},$$

and by integrating both sides of the last inequality from $a$ to $b$, we have

$$\phi(b, A, v) > B.$$

Similarly, we can find a value of $v$ for which $\phi(b, A, v) < B$. The intermediate value theorem implies the existence of an intermediate value $w$ so that $\phi(b, A, w) = B$. $\square$

The method outlined in the previous proof of obtaining a solution of the BVP by varying the derivative of solutions at the initial point is called the *shooting method* and is one of the best ways to find numerical solutions of BVPs. Otherwise, Theorem 7.17 appears to be a result of little utility since one rarely deals with differential equations of the form $x'' = f(t, x, x')$ in which the function $f$ is bounded. However, we shall see that this theorem is very useful indeed. A given equation with an unbounded, continuous right-hand side can be modified to obtain an equation in which the right-hand side is bounded and continuous and then Theorem 7.17 is applied to get a solution of the modified equation. It often happens that the solution of the modified equation is also a solution of the original equation.

First we define lower and upper solutions of $x'' = f(t, x, x')$.

**Definition 7.18** We say that $\alpha$ is a *lower solution* of $x'' = f(t, x, x')$ on an interval $I$ provided

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)),$$

for $t \in I$. Similarly, we say $\beta$ is an *upper solution* of $x'' = f(t, x, x')$ on an interval $I$ provided

$$\beta''(t) \le f(t, \beta(t), \beta'(t)),$$

for $t \in I$.

In two of the main results (Theorem 7.20 and Theorem 7.34) in this chapter, we assume that

$$\alpha(t) \le \beta(t), \quad t \in [a, b],$$

which is the motivation for calling $\alpha$ a lower solution and $\beta$ an upper solution.

A very simple motivating example for what follows is given in the next example.

**Example 7.19** Consider the differential equation

$$x'' = 0. \tag{7.24}$$

Note that a lower solution $\alpha$ of (7.24) satisfies $\alpha''(t) \geq 0$ and hence is concave upward, while an upper solution $\beta$ of (7.24) satisfies $\beta''(t) \leq 0$ and hence is concave downward. To be specific, note that $\alpha(t) := t^2$ and $\beta(t) := -t^2 + 3$ are lower and upper solutions, respectively, of (7.24) satisfying $\alpha(t) \leq \beta(t)$ on $[-1, 1]$. Assume that $A$ and $B$ are constants satisfying,

$$\alpha(-1) = 1 \leq A \leq 2 = \beta(-1), \quad \text{and} \quad \alpha(1) = 1 \leq B \leq 2 = \beta(1).$$

Note that solutions of (7.24) are straight lines so the solution of the BVP

$$x'' = 0, \quad x(-1) = A, \quad x(1) = B$$

satisfies

$$\alpha(t) \leq x(t) \leq \beta(t),$$

for $t \in [-1, 1]$. (Compare this example to Theorems 7.20 and 7.34.) △

We begin with the case that the differential equation does not contain the first derivative.

**Theorem 7.20** *Assume $f$ is continuous on $[a, b] \times \mathbb{R}$ and $\alpha$, $\beta$ are lower and upper solutions of $x'' = f(t, x)$, respectively, with $\alpha(t) \leq \beta(t)$ on $[a, b]$. If $A$ and $B$ are constants such that $\alpha(a) \leq A \leq \beta(a)$ and $\alpha(b) \leq B \leq \beta(b)$, then the BVP*

$$x'' = f(t, x),$$
$$x(a) = A, \quad x(b) = B$$

*has a solution $x$ satisfying*

$$\alpha(t) \leq x(t) \leq \beta(t)$$

*on $[a, b]$.*

***Proof*** First we define on $[a, b] \times \mathbb{R}$ the modification $F$ of $f$ as follows:

$$F(t, x) := \begin{cases} f(t, \beta(t)) + \frac{x - \beta(t)}{1 + |x|}, & \text{if } x \geq \beta(t) \\ f(t, x), & \text{if } \alpha(t) \leq x \leq \beta(t) \\ f(t, \alpha(t)) + \frac{x - \alpha(t)}{1 + |x|}, & \text{if } x \leq \alpha(t). \end{cases}$$

Note that $F$ is continuous and bounded on $[a, b] \times \mathbb{R}$, and $F(t, x) = f(t, x)$, if $t \in [a, b]$, $\alpha(t) \leq x \leq \beta(t)$. Then by Theorem 7.17 the BVP

$$x'' = F(t, x),$$
$$x(a) = A, \quad x(b) = B,$$

has a solution $x$ on $[a,b]$. We claim that $x(t) \leq \beta(t)$ on $[a,b]$. Assume not; then $w(t) := x(t) - \beta(t) > 0$ at some points in $[a,b]$. Since $w(a) \leq 0$ and $w(b) \leq 0$, we get that $w(t)$ has a positive maximum at some point $t_0 \in (a,b)$. Hence

$$w(t_0) > 0, \quad w'(t_0) = 0, \quad w''(t_0) \leq 0.$$

This implies that

$$x(t_0) > \beta(t_0), \quad x'(t_0) = \beta'(t_0), \quad x''(t_0) \leq \beta''(t_0).$$

But

$$\begin{aligned} x''(t_0) - \beta''(t_0) &\geq F(t_0, x(t_0)) - f(t_0, \beta(t_0)) \\ &= f(t_0, \beta(t_0)) + \frac{x(t_0) - \beta(t_0)}{1 + |x(t_0)|} - f(t_0, \beta(t_0)) \\ &= \frac{x(t_0) - \beta(t_0)}{1 + |x(t_0)|} \\ &> 0, \end{aligned}$$

which is a contradiction. Hence $x(t) \leq \beta(t)$ on $[a,b]$. It is left as an exercise (see Exercise 7.18) to show that $\alpha(t) \leq x(t)$ on $[a,b]$. Since

$$\alpha(t) \leq x(t) \leq \beta(t),$$

for $t \in [a,b]$, it follows that $x$ is a solution of $x'' = f(t,x)$ on $[a,b]$. □

**Example 7.21** By Theorem 7.17 the BVP

$$x'' = -\cos x,$$

$$x(0) = 0 = x(1)$$

has a solution.

We now use Theorem 7.20 to find bounds on a solution of this BVP. It is easy to check that $\alpha(t) := 0$ and $\beta(t) := \frac{t(1-t)}{2}$, for $t \in [0,1]$ are lower and upper solutions, respectively, of $x'' = f(t,x,x')$ on $[0,1]$ satisfying the conditions of Theorem 7.20. Hence there is a solution $x$ of this BVP satisfying

$$0 \leq x(t) \leq \frac{t(1-t)}{2},$$

for $t \in [0,1]$. △

**Example 7.22** A BVP that comes up in the study of the viability of patches of plankton (microscopic plant and animal organisms) is the BVP

$$x'' = -rx\left(1 - \frac{1}{K}x\right),$$

$$x(0) = 0, \quad x(p) = 0,$$

where $p$ is the width of the patch of plankton, $x(t)$ is the density of the plankton $t$ units from one end of the patch of plankton, and $r > 0$, $K > 0$ are constants. It is easy to see that

$$\beta(t) = K$$

is an upper solution on $[0, p]$. We look for a lower solution of this equation of the form

$$\alpha(t) = a \sin\left(\frac{\pi t}{p}\right),$$

for some constant $a$. Consider

$$\begin{aligned}
&\alpha''(t) + r\alpha(t)\left[1 - \frac{1}{K}\alpha(t)\right] \\
&= -a\frac{\pi^2}{p^2}\sin\left(\frac{\pi t}{p}\right) + ra\sin\left(\frac{\pi t}{p}\right)\left[1 - \frac{a}{K}\sin\left(\frac{\pi t}{p}\right)\right] \\
&= a\sin\left(\frac{\pi t}{p}\right)\left\{-\frac{\pi^2}{p^2} + r\left[1 - \frac{a}{K}\sin\left(\frac{\pi t}{p}\right)\right]\right\}.
\end{aligned}$$

Hence if

$$r > \frac{\pi^2}{p^2},$$

and $0 < a < K$ is sufficiently small, then $\alpha(t) = a\sin\left(\frac{\pi t}{p}\right)$ is a lower solution of our given differential equation satisfying

$$\alpha(t) \le \beta(t).$$

The hypotheses of Theorem 7.20 are satisfied; hence we conclude that if the width $p$ of the patch of plankton satisfies

$$p > \frac{\pi}{\sqrt{r}},$$

then our given BVP has a solution $x$ satisfying

$$a\sin\left(\frac{\pi t}{p}\right) \le x(t) \le K,$$

for $t \in [0, p]$. This implies that if the width of the patch of plankton satisfies

$$p > \frac{\pi}{\sqrt{r}},$$

then the patch is viable. △

**Theorem 7.23** (Uniqueness Theorem) *Assume that $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and for each fixed $(t, x') \in [a, b] \times \mathbb{R}$, $f(t, x, x')$ is strictly increasing with respect to $x$. Then the BVP*

$$x'' = f(t, x, x'),$$

$$x(a) = A, \quad x(b) = B$$

*has at most one solution.*

***Proof*** Assume $x$ and $y$ are distinct solutions of the given BVP. Without loss of generality, we can assume that there are points $t \in (a, b)$ such that $x(t) > y(t)$. Let

$$w(t) := x(t) - y(t),$$

for $t \in [a, b]$. Since $w(a) = w(b) = 0$ and $w(t) > 0$ at some points in $(a, b)$, $w$ has a positive maximum at some point $d \in (a, b)$. Hence

$$w(d) > 0, \quad w'(d) = 0, \quad w''(d) \leq 0.$$

But

$$\begin{aligned} w''(d) &= x''(d) - y''(d) \\ &= f(d, x(d), x'(d)) - f(d, y(d), y'(d)) \\ &= f(d, x(d), x'(d)) - f(d, y(d), x'(d)) \\ &> 0, \end{aligned}$$

where we have used that for each fixed $(t, x') \in [a, b] \times \mathbb{R}$, $f(t, x, x')$ is strictly increasing with respect to $x$. This is a contradiction and the proof is complete. □

In the next example we show that in Theorem 7.23 you cannot replace *strictly increasing* in the statement of Theorem 7.23 by *nondecreasing*.

**Example 7.24** Consider the differential equation

$$x'' = f(t, x, x') := |x'|^p,$$

where $0 < p < 1$.

Note that $f : [-a, a] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and for each fixed $(t, x') \in [-a, a] \times \mathbb{R}$, $f(t, x, x')$ is nondecreasing with respect to $x$. From Exercise 7.22 there is a solution of this differential equation of the form $x(t) = \kappa |t|^\alpha$, for some $\kappa > 0$ and some $\alpha > 2$. Since all constant functions are also solutions, we get that there are BVPs that do not have at most one solution. △

The function $f(t, x, x') := |x'|^p$ in Example 7.24 does not satisfy a Lipschitz condition with respect to $x'$ on $[-a, a] \times \mathbb{R}^2$. The next theorem, which we state without proof, shows that we do get uniqueness if we assume $f$ satisfies a Lipschitz condition with respect to $x'$.

**Theorem 7.25** (Uniqueness Theorem) *Assume that $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and for each fixed $(t, x') \in [a, b] \times \mathbb{R}$, $f(t, x, x')$ is nondecreasing with respect to $x$. Further assume that $f$ satisfies a Lipschitz condition with respect to $x'$ on each compact subset of $[a, b] \times \mathbb{R}^2$. Then the BVP*

$$x'' = f(t, x, x'),$$

$$x(a) = A, \quad x(b) = B$$

*has at most one solution.*

**Theorem 7.26** *Assume that $f(t,x)$ is continuous on $[a,b]\times\mathbb{R}$ and for each fixed $t\in[a,b]$, $f(t,x)$ is nondecreasing in $x$. Then the BVP*

$$x'' = f(t,x),$$

$$x(a) = A, \quad x(b) = B$$

*has a unique solution.*

***Proof*** Let $L$ be the line segment joining $(a,A)$ and $(b,B)$, and let $M$ and $m$ be, respectively, the maximum and minimum values of $f$ on $L$. Since $f$ is nondecreasing in $x$, if $(t,x)$ is above $L$, then $f(t,x)\ge m$, and if $(t,x)$ is below $L$, then $f(t,x)\le M$. Define

$$\beta(t) = \frac{m}{2}t^2 + p, \quad \alpha(t) = \frac{M}{2}t^2 + q,$$

where $p$ is chosen to be large enough that $\beta$ is above $L$ on $[a,b]$ and $q$ is small enough that $\alpha$ is below $L$ on $[a,b]$. Then $\beta(t) > \alpha(t)$, for $a\le t\le b$, $\beta(a) > A > \alpha(a)$, $\beta(b) > B > \alpha(b)$, and

$$\beta''(t) = m \le f(t,\beta(t)), \quad \alpha''(t) = M \ge f(t,\alpha(t)),$$

for $a\le t\le b$. We can apply Theorem 7.20 to obtain a solution $x$ of the BVP so that $\alpha(t)\le x(t)\le\beta(t)$, for $a\le t\le b$. Uniqueness follows from Theorem 7.25. □

**Example 7.27** Consider the BVP

$$x'' = c(t)x + d(t)x^3 + e(t),$$

$$x(a) = A, \quad x(b) = B,$$

where $c(t)$, $d(t)$, and $e(t)$ are continuous on $[a,b]$. If further $c(t)\ge 0$ and $d(t)\ge 0$, then by Theorem 7.26 this BVP has a unique solution. △

**Example 7.28** The following BVP serves as a simple model in the theory of combustion (see Williams [**54**]):

$$\epsilon x'' = x^2 - t^2,$$

$$x(-1) = x(1) = 1.$$

In the differential equation, $\epsilon$ is a positive parameter related to the speed of the reaction, $x$ is a positive variable associated with mass, and $t$ measures directed distance from the flame. Note that the right-hand side of the differential equation is increasing in $x$ only for $x>0$, so that Theorem 7.26 cannot be immediately applied. However, if we observe that $\alpha\equiv 0$ is a lower solution, then the proof of Theorem 7.26 yields a positive upper solution, and we can conclude that the BVP has a unique positive solution. The uniqueness of this positive solution follows by the proof of Theorem 7.23. △

**Definition 7.29** Assume $\alpha$ and $\beta$ are continuous functions on $[a,b]$ with $\alpha(t) \le \beta(t)$ on $[a,b]$, and assume $c > 0$ is a given constant; then we say that $F(t,x,x')$ is the modification of $f(t,x,x')$ associated with the triple $\alpha(t)$, $\beta(t)$, $c$ provided

$$F(t,x,x') := \begin{cases} g(t,\beta(t),x') + \frac{x-\beta(t)}{1+|x|}, & \text{if } x \ge \beta(t), \\ g(t,x,x'), & \text{if } \alpha(t) \le x \le \beta(t), \\ g(t,\alpha(t),x') + \frac{x-\alpha(t)}{1+|x|}, & \text{if } x \le \alpha(t), \end{cases}$$

where

$$g(t,x,x') := \begin{cases} f(t,x,c), & \text{if } x' \ge c, \\ f(t,x,x'), & \text{if } |x'| \le c, \\ f(t,x,-c), & \text{if } x' \le -c. \end{cases}$$

We leave it to the reader to show that $g$, $F$ are continuous on $[a,b]\times\mathbb{R}^2$, $F$ is bounded on $[a,b] \times \mathbb{R}^2$, and

$$F(t,x,x') = f(t,x,x'),$$

if $t \in [a,b]$, $\alpha(t) \le x \le \beta(t)$, and $|x'| \le c$.

**Theorem 7.30** *Assume $f$ is continuous on $[a,b] \times \mathbb{R}^2$ and $\alpha$, $\beta$ are lower and upper solutions of $x'' = f(t,x,x')$, respectively, with $\alpha(t) \le \beta(t)$ on $[a,b]$. Further assume that solutions of IVPs for $x'' = f(t,x,x')$ are unique. If there is a $t_0 \in [a,b]$ such that*

$$\alpha(t_0) = \beta(t_0), \quad \alpha'(t_0) = \beta'(t_0),$$

*then $\alpha(t) \equiv \beta(t)$ on $[a,b]$.*

***Proof*** Assume $\alpha$, $\beta$, and $t_0$ are as in the statement of this theorem and it is not true that $\alpha(t) \equiv \beta(t)$ on $[a,b]$. We will only consider the case where there are points $t_1, t_2$ such that $a \le t_0 \le t_1 < t_2 \le b$,

$$\alpha(t_1) = \beta(t_1), \quad \alpha'(t_1) = \beta'(t_1),$$

and

$$\alpha(t) < \beta(t), \quad \text{on} \quad (t_1, t_2].$$

Pick $c > 0$ so that

$$|\alpha'(t)| < c, \quad |\beta'(t)| < c,$$

for $t \in [t_1, t_2]$. Let $F$ be the modification of $f$ with respect to the triple $\alpha$, $\beta$, $c$ for the interval $[t_1, t_2]$. By Theorem 7.17, the BVP

$$x'' = F(t,x,x'), \quad x(t_1) = \alpha(t_1), \quad x(t_2) = x_2,$$

where $\alpha(t_2) < x_2 < \beta(t_2)$ has a solution $x$. We claim that $x(t) \le \beta(t)$ on $[t_1, t_2]$. Assume not; then there is a $d \in (t_1, t_2)$ such that $w(t) := x(t) - \beta(t)$ has a positive maximum on $[t_1, t_2]$ at $d$. It follows that

$$w(d) > 0, \quad w'(d) = 0, \quad w''(d) \le 0,$$

and so

$$x(d) > \beta(d), \quad x'(d) = \beta'(d), \quad x''(d) \leq \beta''(d).$$

But

$$\begin{aligned} w''(d) &= x''(d) - \beta''(d) \\ &\geq F(d, x(d), x'(d)) - f(d, \beta(d), \beta'(d)) \\ &= f(d, \beta(d), \beta'(d)) + \frac{x(d) - \beta(d)}{1 + |\beta(d)|} - f(d, \beta(d), \beta'(d)) \\ &= \frac{x(d) - \beta(d)}{1 + |\beta(d)|} \\ &> 0, \end{aligned}$$

which is a contradiction. Hence $x(t) \leq \beta(t)$ on $[t_1, t_2]$. Similarly, $\alpha(t) \leq x(t)$ on $[t_1, t_2]$. Thus

$$\alpha(t) \leq x(t) \leq \beta(t),$$

on $[t_1, t_2]$. Pick $t_3 \in [t_1, t_2)$ so that

$$x(t_3) = \alpha(t_3), \quad x'(t_3) = \alpha'(t_3),$$

and

$$\alpha(t) < x(t),$$

on $(t_3, t_2]$. Since

$$|x'(t_3)| = |\alpha'(t_3)| < c,$$

we can pick $t_4 \in (t_3, t_2]$ so that

$$|x'(t)| < c,$$

on $[t_3, t_4]$. Then $x$ is a solution of $x'' = f(t, x, x')$ on $[t_3, t_4]$ and hence $x$ is an upper solution of $x'' = f(t, x, x')$ on $[t_3, t_4]$. Let $F_1$ be the modification of $f$ with respect to the triple $\alpha, x, c$ for the interval $[t_3, t_4]$. Let $x_4 \in (\alpha(t_4), x(t_4))$, then by Theorem 7.17 the BVP

$$x'' = F_1(t, x, x'), \quad x(t_3) = \alpha(t_3), \quad x(t_4) = x_4,$$

has a solution $y$ on $[t_3, t_4]$. By a similar argument we can show that

$$\alpha(t) \leq y(t) \leq x(t),$$

on $[t_3, t_4]$. Now we can pick $t_5 \in (t_3, t_4]$ so that $x$ and $y$ differ at some points in $(t_3, t_5]$ and

$$|y'(t)| < c,$$

for $t \in [t_3, t_5]$. Then $y$ is a solution of $x'' = f(t, x, x')$ on $[t_3, t_5]$. But now we have that $x$, $y$ are distinct solutions of the same IVP (same initial conditions at $t_3$), which contradicts the uniqueness of solutions of IVPs. $\square$

**Corollary 7.31** *Assume that the linear differential equation*

$$x'' + p(t)x' + q(t)x = 0 \tag{7.25}$$

*has a positive upper solution $\beta$ on $[a, b]$. Then* (7.25) *is disconjugate on $[a, b]$.*

***Proof*** Assume that (7.25) has a positive upper solution $\beta$ on $[a,b]$ but that (7.25) is not disconjugate on $[a,b]$. It follows that there is a solution $x$ of (7.25) and points $a \le t_1 < t_2 \le b$ such that $x(t_1) = x(t_2) = 0$ and $x(t) > 0$ on $(t_1, t_2)$. But then there is a constant $\lambda > 0$ so that if $\alpha(t) := \lambda x(t)$, then

$$\alpha(t) \le \beta(t), \quad t \in [t_1, t_2],$$

and $\alpha(c) = \beta(c)$ for some $c \in (a,b)$. It follows that $\alpha'(c) = \beta'(c)$. Since $\alpha$ is a solution, it is also a lower solution. Applying Theorem 7.30, we get that $\alpha(t) \equiv \beta(t)$ on $[t_1, t_2]$, which is a contradiction. □

## 7.5 Nagumo Condition

In this section we would like to extend Theorem 7.20 to the case where the nonlinear term is of the form $f(t,x,x')$; that is, also depends on $x'$. In Exercise 7.24 you are asked to show that $x(t) := 4 - \sqrt{4-t}$ is a solution of the differential equation $x'' = 2(x')^3$ on $[0,4)$. Note that this solution is bounded on $[0,4)$, but its derivative is unbounded on $[0,4)$. This is partly due to the fact that the nonlinear term $f(t,x,x') = 2(x')^3$ grows too fast with respect to the $x'$ variable. We now define a Nagumo condition, which is a growth condition on $f(t,x,x')$ with respect to $x'$. In Theorem 7.33 we will see that this Nagumo condition will give us an a priori bound on the derivative of solutions of $x'' = f(t,x,x')$ that satisfy

$$\alpha(t) \le x(t) \le \beta(t),$$

for $t \in [a,b]$. We then will use this result to prove Theorem 7.34, which extends Theorem 7.20 to the case where $f$ also depends on $x'$ but satisfies a Nagumo condition. We end this section by giving several applications of Theorem 7.34.

**Definition 7.32** We say that $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a *Nagumo condition* with respect to the pair $\alpha(t)$, $\beta(t)$ on $[a,b]$ provided $\alpha, \beta : [a,b] \to \mathbb{R}$ are continuous, $\alpha(t) \le \beta(t)$ on $[a,b]$, and there is a function $h : [0,\infty) \to (0,\infty)$ such that

$$|f(t,x,x')| \le h(|x'|),$$

for all $t \in [a,b]$, $\alpha(t) \le x \le \beta(t)$, $x' \in \mathbb{R}$ with

$$\int_\lambda^\infty \frac{s\, ds}{h(s)} > \max_{a \le t \le b} \beta(t) - \min_{a \le t \le b} \alpha(t),$$

where

$$\lambda := \max\left\{ \frac{|\beta(b) - \alpha(a)|}{b-a}, \frac{|\alpha(b) - \beta(a)|}{b-a} \right\}.$$

**Theorem 7.33** *Assume that $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a Nagumo condition with respect to $\alpha(t), \beta(t)$ on $[a,b]$, where $\alpha(t) \le \beta(t)$ on $[a,b]$. Assume $x$ is a solution of $x'' = f(t,x,x')$ satisfying*

$$\alpha(t) \le x(t) \le \beta(t), \quad \text{on} \quad [a,b];$$

*then there exists a constant $N > 0$, independent of $x$ such that $|x'(t)| \leq N$ on $[a, b]$.*

***Proof*** Let $\lambda$ be as in Definition 7.32. Choose $N > \lambda$ so that

$$\int_\lambda^N \frac{s\,ds}{h(s)} > \max_{t\in[a,b]} \beta(t) - \min_{t\in[a,b]} \alpha(t).$$

Let $x$ be a solution of $x'' = f(t, x, x')$ satisfying

$$\alpha(t) \leq x(t) \leq \beta(t),$$

for $t \in [a, b]$. Then, using the mean value theorem, it can be shown that there is a $t_0 \in (a, b)$ such that

$$|x'(t_0)| = \left|\frac{x(b) - x(a)}{b - a}\right| \leq \lambda.$$

We claim that $|x'(t)| \leq N$ on $[a, b]$. Assume not; then $|x'(t)| > N$ at some points in $[a, b]$. We will only consider the case where there is a $t_1 \in [a, t_0)$ such that $x'(t_1) < -N$. (Another case is considered in Exercise 7.27.) Choose $t_1 < t_2 < t_3 \leq t_0$ so that

$$x'(t_2) = -N, \quad x'(t_3) = -\lambda,$$

and

$$-N < x'(t) < -\lambda, \quad \text{on} \quad (t_2, t_3).$$

On $[t_2, t_3]$,

$$\begin{aligned} |x''(t)| &= |f(t, x(t), x'(t))| \\ &\leq h(|x'(t)|) \\ &= h(-x'(t)). \end{aligned}$$

It follows that

$$-\frac{x'(t)x''(t)}{h(-x'(t))} \leq -x'(t),$$

for $t \in [t_2, t_3]$. Integrating both sides from $t_2$ to $t_3$, we get

$$\int_{t_2}^{t_3} \frac{-x'(t)x''(t)}{h(-x'(t))} dt \leq x(t_2) - x(t_3).$$

This implies that

$$\int_{-x'(t_2)}^{-x'(t_3)} \frac{-s}{h(s)} ds = \int_N^\lambda \frac{-s}{h(s)} ds \leq x(t_2) - x(t_3).$$

It follows that

$$\int_\lambda^N \frac{s}{h(s)} ds \leq x(t_2) - x(t_3) \leq \max_{t\in[a,b]} \beta(t) - \min_{t\in[a,b]} \alpha(t),$$

which is a contradiction. □

**Theorem 7.34** *Assume that* $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and* $\alpha, \beta$ *are lower and upper solutions, respectively, of* $x'' = f(t,x,x')$ *on* $[a,b]$ *with* $\alpha(t) \le \beta(t)$ *on* $[a,b]$. *Further assume that* $f$ *satisfies a Nagumo condition with respect to* $\alpha(t), \beta(t)$ *on* $[a,b]$. *Assume* $A, B$ *are constants satisfying*

$$\alpha(a) \le A \le \beta(a), \quad \alpha(b) \le B \le \beta(b);$$

*then the BVP*

$$x'' = f(t,x,x'), \quad x(a) = A, \quad x(b) = B \tag{7.26}$$

*has a solution satisfying*

$$\alpha(t) \le x(t) \le \beta(t),$$

*for* $t \in [a,b]$.

***Proof*** Pick $N_1 > 0$ sufficiently large so that $|\alpha'(t)| < N_1$ and $|\beta'(t)| < N_1$ on $[a,b]$. Let $\lambda$ be as in Definition 7.32 and pick $N_2 > \lambda$ sufficiently large so that

$$\int_\lambda^{N_2} \frac{sds}{h(s)} > \max_{t\in[a,b]} \beta(t) - \min_{t\in[a,b]} \alpha(t).$$

Then let $N := \max\{N_1, N_2\}$ and let $F(t,x,x')$ be the modification of $f(t,x,x')$ associated with the triple $\alpha(t)$, $\beta(t)$, $N$. Since $F$ is continuous and bounded on $[a,b] \times \mathbb{R}^2$, we have by Theorem 7.17 that the BVP

$$x'' = F(t,x,x'),$$

$$x(a) = A, \quad x(b) = B$$

has a solution $x$ on $[a,b]$. We claim that $x(t) \le \beta(t)$ on $[a,b]$. Assume not; then $w(t) := x(t) - \beta(t) > 0$ at some points in $[a,b]$. Since $w(a) \le 0$ and $w(b) \le 0$, we get that $w(t)$ has a positive maximum at some point $\xi \in (a,b)$. Hence

$$w(\xi) > 0, \;\; w'(\xi) = 0, \;\; w''(\xi) \le 0.$$

This implies that

$$x(\xi) > \beta(\xi), \;\; x'(\xi) = \beta'(\xi), \;\; x''(\xi) \le \beta''(\xi).$$

Note that $|x'(\xi)| = |\beta'(\xi)| < N$. But

$$\begin{aligned} x''(\xi) - \beta''(\xi) &\ge F(\xi, x(\xi), x'(\xi)) - f(\xi, \beta(\xi), \beta'(\xi)) \\ &= g(\xi, \beta(\xi), \beta'(\xi)) + \frac{x(\xi) - \beta(\xi)}{1 + |x(\xi)|} - f(\xi, \beta(\xi), \beta'(\xi)) \\ &= \frac{x(\xi) - \beta(\xi)}{1 + |x(\xi)|} \\ &> 0, \end{aligned}$$

which is a contradiction. Hence $x(t) \le \beta(t)$ on $[a,b]$. It is left as an exercise to show that $\alpha(t) \le x(t)$ on $[a,b]$.

Now if $|x'(t)| \le N$ on $[a,b]$, then $x$ would be our desired solution of the BVP (7.26). As in the proof of Theorem 7.33, we have by the mean value theorem that there is a $t_0 \in (a,b)$ such that

$$|x'(t_0)| = \left|\frac{x(b)-x(a)}{b-a}\right| \le \lambda < N,$$

where $\lambda$ is as in Definition 7.32. We claim $|x'(t)| \le N$ on $[a,b]$. Assume not, then there is a maximal interval $[t_1,t_2]$ containing $t_0$ in its interior such that $|x'(t)| \le N$ on $[t_1,t_2]$, where either $t_1 > a$ and $|x'(t_2)| = N$ or $t_2 < b$ and $|x'(t_2)| = N$. We then proceed as in the proof of Theorem 7.33 to get a contradiction. □

**Example 7.35** Consider the BVP

$$x'' = x^2 + (x')^4, \quad x(0) = A, \quad x(1) = B. \tag{7.27}$$

Suppose that $A$, $B$ are positive, and $c$ is a number bigger than both $A$ and $B$. Then $\alpha(t) := 0$, $\beta(t) := c$ are lower and upper solutions, respectively, of the differential equation in (7.27) on $[0,1]$ with $\alpha(t) \le \beta(t)$ on $[0,1]$. Note that the $\lambda$ in Definition 7.32 is given by $\lambda = c$. Define the function $h$ by $h(s) := c^2 + s^4$ and note that

$$\begin{aligned}\int_\lambda^\infty \frac{s}{h(s)} ds &= \int_c^\infty \frac{s}{c^2+s^4} ds \\ &= \frac{\pi}{4c} - \frac{1}{2c}\arctan c.\end{aligned}$$

The inequality

$$\int_\lambda^\infty \frac{s}{h(s)} ds > c$$

leads to the inequality

$$\frac{\pi}{2} - \arctan c > 2c^2.$$

Let $c_0$ be the positive solution of

$$\frac{\pi}{2} - \arctan c = 2c^2.$$

Using our calculator, $c_0 \approx .69424$. It follows that if $A$ and $B$ are strictly between 0 and $c_0$, the BVP has a solution $x$ satisfying

$$0 < x(t) < c_0, \quad t \in [0,1]$$

(why can we have strict inequalities here?).

△

As a final example, we show how upper and lower solutions can be used to verify approximations of solutions of singular perturbation problems of the type studied in Chapter 4.

**Example 7.36** Let $\epsilon$ be a small positive parameter and consider the BVP

$$\epsilon x'' + 2x' + x^2 = 0,$$
$$x(0) = 3, \quad x(1) = 1.$$

If we substitute a perturbation series

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots,$$

we obtain

$$2x_0' + x_0^2 = 0,$$

which has the family of solutions

$$x_0(t) = \frac{2}{t+C}.$$

As in Chapter 4, we observe that it is consistent to have a solution with boundary layer at $t = 0$ since for such a solution, the large positive second derivative term $\epsilon x''$ will balance the negative first derivative term $2x'$ in the differential equation. Consequently, we impose the condition $x_0(1) = 1$, which yields

$$x_0(t) = \frac{2}{t+1}.$$

We will leave it as an exercise (Exercise 7.32) to show that the boundary layer correction at $t = 0$ is $e^{-2t/\epsilon}$. Consequently, a formal approximation of a solution to the BVP is (for small $\epsilon$)

$$A(t, \epsilon) = \frac{2}{t+1} + e^{-\frac{2t}{\epsilon}}.$$

As a first step in verifying the formal approximation, we define a trial upper solution of the form

$$\beta(t, \epsilon) = \frac{2}{t+1} + e^{(-2+a\epsilon)t/\epsilon} + \epsilon f(t, \epsilon),$$

where $a$ is a positive constant and $f$ is a nonnegative continuous function to be determined. Note that

$$\beta(t, \epsilon) - A(t, \epsilon) = e^{-2t/\epsilon}\left(e^{at} - 1\right) + \epsilon f \geq 0,$$

for $t \geq 0$ and $\epsilon > 0$, and the maximum of $e^{-2t/\epsilon}(e^{at} - 1)$ occurs when $e^{at} = 2/(2 - a\epsilon)$, so for this value of $t$

$$e^{-\frac{2t}{\epsilon}}\left(e^{at} - 1\right) = e^{-\frac{2t}{\epsilon}}\left(\frac{a\epsilon}{2 - a\epsilon}\right) = \mathcal{O}(\epsilon),$$

as $\epsilon \to 0$. It follows that

$$\beta(t, \epsilon) - A(t, \epsilon) = \mathcal{O}(\epsilon) \quad (\epsilon \to 0)$$

uniformly for $0 \leq t \leq 1$.

To check if $\beta$ is an upper solution for the BVP, we compute

$$\begin{aligned}\epsilon\beta'' &+ 2\beta' + \beta^2 \qquad (7.28)\\ &= \epsilon\left[\frac{4}{(t+1)^3} + \frac{(a\epsilon-2)^2}{\epsilon^2}e^{(a\epsilon-2)t/\epsilon} + \epsilon f''\right]\\ &\quad +2\left[-\frac{2}{(t+1)^2} + \frac{a\epsilon-2}{\epsilon}e^{(a\epsilon-2)t/\epsilon} + \epsilon f'\right]\\ &\quad + \left[\frac{2}{t+1} + e^{(a\epsilon-2)t/\epsilon} + \epsilon f\right]^2.\end{aligned}$$

After some simplification, (7.28) reduces to

$$e^{(a\epsilon-2)t/\epsilon}\left[a(a\epsilon-2) + \frac{4}{t+1} + e^{(a\epsilon-2)t/\epsilon} + 2\epsilon f\right] + \frac{4\epsilon}{(t+1)^3}$$
$$+\epsilon\left[\epsilon f'' + 2f' + \frac{4}{t+1}f\right] + \epsilon^2 f^2.$$

Choose $f$ to be a nonnegative solution of

$$\epsilon f'' + 2f' + 4f = -5,$$

specifically,

$$f = \frac{5}{4}\left[e^{-\lambda(1-t)} - 1\right] \quad (0 \le t \le 1),$$

where $\lambda$ solves $\epsilon\lambda^2 + 2\lambda + 4 = 0$, namely,

$$\lambda = \frac{\sqrt{1-4\epsilon}-1}{\epsilon} = -2 + \mathcal{O}(\epsilon) \quad (\epsilon \to 0).$$

Returning to our calculation, we have that (7.28) is less than or equal to

$$e^{(a\epsilon-2)t/\epsilon}\left[a(a\epsilon-2) + \frac{4}{t+1} + e^{(a\epsilon-2)t/\epsilon} + 2\epsilon f\right]$$
$$+\epsilon\left[\frac{4}{(t+1)^3} + \epsilon f^2 - 5\right].$$

Choose $a = 6$ and $\epsilon$ small enough that $\epsilon \le 1/6$, $2\epsilon f \le 1$, and $\epsilon f^2 \le 1$. Then

$$\epsilon\beta'' + 2\beta' + \beta^2 \le 0, \quad (0 \le t \le 1),$$

so $\beta$ is an upper solution.

We leave it to the reader to check that $A$ itself is a lower solution. It follows from Theorem 7.34 that for positive values of $\epsilon$ that are small enough to satisfy the preceding inequalities, the BVP has a solution $x(t,\epsilon)$ such that

$$A(t,\epsilon) \le x(t,\epsilon) \le \beta(t,\epsilon), \quad (0 \le t \le 1).$$

Since $\beta(t,\epsilon) - A(t,\epsilon) = \mathcal{O}(\epsilon)$, we also have

$$0 \le x(t,\epsilon) - A(t,\epsilon) = \mathcal{O}(\epsilon),$$

as $\epsilon \to 0$, uniformly for $0 \le t \le 1$. △

For a general discussion of the analysis of singular perturbation problems by the method of upper and lower solutions, see Chang and Howes [**7**] and Kelley [**30**].

## 7.6 Exercises

**7.1** Show that if $\lim_{n\to\infty} x_n = x$, in a NLS $\mathbb{X}$, then the sequence $\{x_n\}$ is a Cauchy sequence.

**7.2** Show that if $\mathbb{X}$ is the set of all continuous functions on $[a,\infty)$ satisfying $\lim_{t\to\infty} x(t) = 0$, where $\|\cdot\|$ is defined by $\|x\| = \max\{|x(t)| : a \le t < \infty\}$, then $\mathbb{X}$ is a Banach space.

**7.3** Assume that $\mathbb{X}$ is the set of all continuous functions on $[a,b]$ and assume $\rho$ is a positive continuous function on $[a,b]$, where $\|\cdot\|$ is defined by $\|x\| = \max\left\{\frac{|x(t)|}{\rho(t)} : a \le t \le b\right\}$. Show that $\mathbb{X}$ is a Banach space.

**7.4** Show that if $X$ is the Banach space of continuous functions on $[0,1]$ with the max (sup) norm defined by $\|x\| := \max\{|x(t)| : t \in [0,1]\}$ and the operator $T : X \to X$ is defined by

$$Tx(t) = t + \int_0^t sx(s)ds, \quad t \in [0,1],$$

then $T$ is a contraction mapping on $X$.

**7.5** Show that if in the contraction mapping theorem (Theorem 7.5), $\alpha \ge 1$, then the theorem is false.

**7.6** Find a Banach space $\mathbb{X}$ and a mapping $T$ on $\mathbb{X}$ such that $\|Tx - Ty\| < \|x - y\|$ for all $x, y \in \mathbb{X}$, $x \ne y$, such that $T$ has no fixed point in $\mathbb{X}$.

**7.7** Show that if in the contraction mapping thoerem (Theorem 7.5) we replace "$T$ is a contraction mapping" by "there is an integer $m \ge 1$ such that $T^m$ is a contraction mapping," then the theorem holds with the appropriate change in the inequality (7.1).

**7.8** For what values of $n$ and $m$ can we apply Theorem 7.6 to conclude that

$$[t^n x']' + t^m x = \frac{1}{t^2 - 1}$$

has a solution that goes to zero as $t \to \infty$?

**7.9** Show that the hypotheses of Theorem 7.7 concerning the BVP

$$x'' = \cos x, \quad x(0) = 0, \quad x(1) = 0$$

hold. Assume that $x_0(t) \equiv 0$ and let $\{x_n\}$ be the corresponding sequence of Picard iterates as defined in Theorem 7.5. Find the first Picard iterate $x_1$ and use (7.1) to find out how good an approximation it is to the actual solution of the given BVP.

**7.10** Show that the hypotheses of Theorem 7.7 concerning the BVP

$$x'' = \frac{2}{1+t^2}x, \quad x(0) = 1, \quad x(1) = 2$$

hold. Assume that your initial guess is $x_0(t) = 1 + t^2$ and let $\{x_n\}$ be the corresponding sequence of Picard iterates as defined in Theorem 7.5. Find the first four Picard iterates. Explain your answers. Next find the first Picard iterate $x_1$ for the BVP

$$x'' = \frac{1}{1+t^2}x, \quad x(0) = 1, \quad x(1) = 2,$$

where again your initial guess is $x_0(t) = 1 + t^2$ and use (7.1) to find out how good an approximation it is to the actual solution of the given BVP.

**7.11** Show that the hypotheses of Theorem 7.7 concerning the BVP

$$x'' = -2\sin x, \quad x(0) = 0, \quad x(1) = 1$$

hold. Assume that $x_0(t) = t$ and let $\{x_n\}$ be the corresponding sequence of Picard iterates as defined in Theorem 7.5. Find the first Picard iterate $x_1$ and use (7.1) to find out how good an approximation it is to the actual solution of the given BVP.

**7.12** Use Theorem 7.9 to find $L$ so that if $b - a < L$, then the BVP

$$x'' = -\sqrt{3x^2+1}, \quad x(a) = A, \quad x(b) = B$$

has for each $A, B \in \mathbb{R}$ a unique solution.

**7.13** Use Theorem 7.9 to find $L$ so that if $b - a < L$, then the BVP

$$x'' = -\frac{9}{1+t^2}x, \quad x(a) = A, \quad x(b) = B,$$

for each $A, B \in \mathbb{R}$, has a unique solution.

**7.14** Show that if $u$ is a nontrivial soultion of (7.21) with $u(0) = 0$, then if $u'(t) \neq 0$ on $[0, \infty)$, then $\rho(K, L) := \infty$. Otherwise $\rho(K, L)$ is the first point to the right of 0 where $u'(t)$ is zero.

**7.15** Prove the remaining cases in Theorem 7.14.

**7.16** (Forced Pendulum Problem) Find a number $B$ so that if $0 < b < B$, then the BVP

$$\theta'' + \frac{g}{l}\sin\theta = F(t), \quad \theta(0) = A, \quad \theta'(b) = m,$$

where $g$ is the acceleration due to gravity, $l$ is the length of the pendulum, and $A$ and $m$ are constants has a unique solution.

**7.17** Use Theorem 7.17 and its proof to show that

$$x'' = \frac{x}{x^2+1}, \quad x(0) = 0, \quad x(1) = 2$$

has a solution $x$ and to find upper and lower bounds on $x'(t)$.

**7.18** Show, as claimed in the proof of Theorem 7.20, that $\alpha(t) \le x(t)$ on $[a, b]$.

**7.19** By adapting the method of Example 7.21, find bounds on a solution of the BVP

$$x'' = \sin x - 2, \quad x(0) = x(1) = 0.$$

**7.20** Show that the BVP

$$x'' = -x(2 - x),$$

$$x(0) = 0, \quad x(\pi) = 0$$

has a solution $x$ satisfying $\sin t \le x(t) \le 2$ on $[0, \pi]$.

**7.21** Show that the BVP

$$x'' = x + x^3,$$

$$x(a) = A, \quad x(b) = B$$

has a solution $x$ satisfying $|x(t)| \le C$ on $[a, b]$, where $C := \max\{|A|, |B|\}$.

**7.22** Consider the differential equation in Example 7.24

$$x'' = f(t, x, x') := |x'|^p,$$

where $0 < p < 1$. Show that there is a solution of this differential equation of the form $x(t) = \kappa |t|^\alpha$ for some $\kappa > 0$ and some $\alpha > 2$.

**7.23** What, if anything, can you say about uniqueness or existence of solutions of the given BVPs (give reasons for your answers)?

(i)

$$x'' = \cos(t^2 x^2 (x')^3),$$

$$x(a) = A, \quad x(b) = B$$

(ii)

$$x'' = t^2 + x^3 - (x')^2,$$

$$x(a) = A, \quad x(b) = B$$

(iii)

$$x'' = -e^{4t} + t^2 + \frac{2}{1 + (x')^2},$$

$$x(a) = A, \quad x(b) = B$$

**7.24** Show that $x(t) := 4 - \sqrt{4 - t}$ is a solution of the differential equation $x'' = 2(x')^3$ that is bounded on $[0, 4)$ but whose derivative is unbounded on $[0, 4)$.

**7.25** For what numbers $a < b$, $A$, $B$, does the BVP

$$x'' = \arctan x, \quad x(a) = A, \quad x(b) = B$$

have a unique solution?

**7.26** Show that the equation

$$c = \sqrt{2}\cosh\left(\frac{c}{4}\right)$$

has two solutions $c_1, c_2$ satisfying $0 < c_1 < 2 < c_2 < 14$. Use your calculator to approximate $c_1$ and $c_2$. Show that if $c$ is a solution of $c = \sqrt{2}\cosh\left(\frac{c}{4}\right)$, then

$$x(t) := \log\left(\frac{\cosh^2\left(\frac{c}{4}\right)}{\cosh^2\left[\frac{c}{2}(t-\frac{1}{2})\right]}\right)$$

is a solution of the BVP

$$x'' = -e^x, \quad x(0) = 0, \quad x(1) = 0.$$

**7.27** In the proof of Theorem 7.33, verify the case where there is a $t_1 \in (t_0, b]$ such that $x'(t_1) < -N$.

**7.28** Show that the BVP

$$x'' = -x + x^3 + (x')^2, \quad x(a) = A, \quad x(b) = B,$$

where $0 \leq A, B \leq 1$, has a solution $x$ satisfying $0 \leq x(t) \leq 1$ for $t \in [a, b]$.

**7.29** Assume that $\alpha, \beta \in C[a, b]$ with $\alpha(t) \leq \beta(t)$ on $[a, b]$. Show that if

$$f(t, x, x') := \frac{1}{1+t^2} + (\sin x)(x')^2,$$

then $f$ satisfies a Nagumo condition with respect to $\alpha$, $\beta$ on $[a, b]$.

**7.30** Show that if $0 \leq A, B \leq \frac{\pi}{5}$, then the BVP

$$x'' = x + (x')^4, \quad x(a) = A, \quad x(b) = B$$

has a solution for all sufficiently large $b$.

**7.31** Show that the BVP

$$x'' = \frac{2}{9}(\sin t)x + (x')^4, \quad x(0) = A, \quad x(2) = B,$$

where $-.5 \leq A \leq .5$, $-.5 \leq B \leq .5$, has a solution $x$ satisfying $-.5 \leq x(t) \leq .5$ for $t \in [0, 2]$.

**7.32** Verify that the boundary layer correction in Example 7.36 is $e^{-2t/\epsilon}$.

**7.33** In Example 7.36, show that the approximation $A$ is itself a lower solution.

**7.34** Consider the BVP

$$\epsilon^2 x'' = x + x^3, \quad x(0) = x(1) = 1,$$

where $\epsilon$ is a small positive parameter. Show that

$$\beta(t, \epsilon) = e^{-t/\epsilon} + e^{(t-1)/\epsilon}$$

is an upper solution and find a lower solution. Sketch the solution and describe any boundary layers.

**7.35** For the singular perturbation problem

$$\epsilon x'' + x' + x^2 = 0, \quad x(0) = 3, \quad x(1) = .5,$$

compute a formal approximation and verify it by the method of upper and lower solutions.