Chapter 2

Linear Systems

2.1 Introduction

In this chapter we will be concerned with linear systems of the form

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ &\cdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t), \end{aligned}$$

where we assume that the functions a_{ij}, $1 \leq i,j \leq n$, $b_i, 1 \leq i \leq n$, are continuous real-valued functions on an interval I. We say that the collection of n functions $x_1,\ x_2, \cdots,\ x_n$ is a solution on I of this linear system provided each of these n functions is continuously differentiable on I and

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\ &\cdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t), \end{aligned}$$

for $t \in I$.

This system can be written as an equivalent vector equation

$$x' = A(t)x + b(t), \tag{2.1}$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x' := \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix},$$

and

$$A(t) := \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad b(t) := \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix},$$

W.G. Kelley and A.C. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Universitext 278, DOI 10.1007/978-1-4419-5783-2_2,

for $t \in I$. Note that the matrix functions A and b are continuous on I (a matrix function is continuous on I if and only if (iff) all of its entries are continuous on I). We say that an $n \times 1$ vector function x is a solution of (2.1) on I provided x is a continuously differentiable vector function on I (iff each component of x is continuously differentiable on I) and

$$x'(t) = A(t)x(t) + b(t),$$

for all $t \in I$.

Example 2.1 It is easy to see that the pair of functions x_1, x_2 defined by $x_1(t) = 2 + \sin t$, $x_2(t) = -t + \cos t$, for $t \in \mathbb{R}$ is a solution on $\mathbb{R}$ of the linear system

$$\begin{aligned} x_1' &= x_2 + t, \\ x_2' &= -x_1 + 1 \end{aligned}$$

and the vector function

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is a solution on $\mathbb{R}$ of the vector equation

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

$\triangle$

The study of equation (2.1) includes the nth-order scalar differential equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = r(t) \tag{2.2}$$

as a special case. To see this let y be a solution of (2.2) on I, that is, assume y has n continuous derivatives on I and

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = r(t), \quad t \in I.$$

Then let

$$x_i(t) := y^{(i-1)}(t),$$

for $t \in I$, $1 \le i \le n$. Then the $n \times 1$ vector function x with components x_i satisfies equation (2.1) on I if

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-1}(t) \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(t) \end{bmatrix}$$

for $t \in I$. The matrix function A is called the *companion matrix* of the differential equation (2.2). Conversely, it can be shown that if x is a solution of the vector equation (2.1) on I, where A and b are as before, then it follows that the scalar function y defined by $y(t) := x_1(t)$, for $t \in I$ is a solution of (2.2) on I. We next give an interesting example that leads to a four-dimensional linear system.

Example 2.2 (Coupled Vibrations) Consider the system of two masses m_1 and m_2 in Figure 1 connected to each other by a spring with spring constant k_2 and to the walls by springs with spring constants k_1 and k_3 respectively. Let $u(t)$ be the displacement of m_1 from its equilibrium position at time t and $v(t)$ be the displacement of m_2 from its equilibrium at time t. (We are taking the positive direction to be to the right.) Let c be the coefficient of friction for the surface on which the masses slide. An application of Newton's second law yields

$$\begin{aligned} m_1u'' &= -cu' - (k_1 + k_2)u + k_2v, \\ m_2v'' &= -cv' - (k_2 + k_3)v + k_2u. \end{aligned}$$

Here we have a system of two second-order equations, and we define $x_1 := u$, $x_2 := u'$, $x_3 := v$, and $x_4 := v'$, obtaining the first-order system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

△

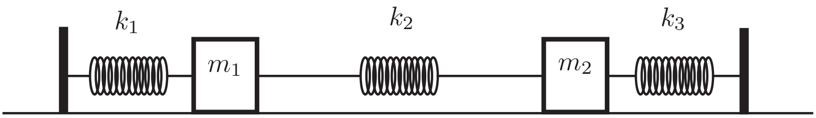

FIGURE 1. Coupled masses.

The following theorem is a special case of Theorem 8.65 (see also Corollary 8.18) .

Theorem 2.3 *Assume that the $n \times n$ matrix function A and the $n \times 1$ vector function b are continuous on an interval I. Then the IVP*

$$x' = A(t)x + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I$ and x_0 is a given constant $n \times 1$ vector, has a unique solution that exists on the whole interval I.

Note that it follows from Theorem 2.3 that in Example 2.2 if at any time t_0 the position and velocity of the two masses are known, then that uniquely determines the position and velocity of the masses at all other times.

We end this section by explaining why we call the differential equation (2.1) a *linear* vector differential equation.

Definition 2.4 A family of functions $\mathbb{A}$ defined on an interval I is said to be a *vector space* or *linear space* provided whenever $x, y \in \mathbb{A}$ it follows that for any constants $\alpha, \beta \in \mathbb{R}$

$$\alpha x + \beta y \in \mathbb{A}.$$

By the function $\alpha x + \beta y$ we mean the function defined by

$$(\alpha x + \beta y)(t) := \alpha x(t) + \beta y(t),$$

for $t \in I$. If $\mathbb{A}$ and $\mathbb{B}$ are vector spaces of functions defined on an interval I, then $L : \mathbb{A} \to \mathbb{B}$ is called a *linear operator* provided

$$L[\alpha x + \beta y] = \alpha L[x] + \beta L[y],$$

for all $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{A}$.

We now give an example of an important linear operator.

Example 2.5 Let $\mathbb{A}$ be the set of all $n \times 1$ continuously differentiable vector functions on an interval I and let $\mathbb{B}$ be the set of all $n \times 1$ continuous vector functions on an interval I and note that $\mathbb{A}$ and $\mathbb{B}$ are linear spaces. Define $L : \mathbb{A} \to \mathbb{B}$ by

$$Lx(t) = x'(t) - A(t)x(t),$$

for $t \in I$, where A is a given $n \times n$ continuous matrix function on I. To show that L is a linear operator, let $\alpha, \beta \in \mathbb{R}$, let $x, y \in \mathbb{A}$, and consider

$$\begin{aligned} L[\alpha x + \beta y](t) &= (\alpha x + \beta y)'(t) - A(t)(\alpha x + \beta y)(t) \\ &= \alpha x'(t) + \beta y'(t) - \alpha A(t)x(t) - \beta A(t)y(t) \\ &= \alpha [x'(t) - A(t)x(t)] + \beta [y'(t) - A(t)y(t)] \\ &= \alpha Lx(t) + \beta Ly(t) \\ &= (\alpha Lx + \beta Ly)(t), \end{aligned}$$

for $t \in I$. Hence

$$L[\alpha x + \beta y] = \alpha Lx + \beta Ly$$

and so $L : \mathbb{A} \to \mathbb{B}$ is a linear operator. △

Since the differential equation (2.1) can be written in the form

$$Lx = b,$$

where L is the linear operator defined in Example 2.5, we call (2.1) a *linear* vector differential equation. If b is not the trivial vector function, then the equation $Lx = b$ is called a *nonhomogeneous* linear vector differential equation and $Lx = 0$ is called the corresponding *homogeneous* linear vector differential equation.

2.2 The Vector Equation $x' = A(t)x$

To solve the nonhomogeneous linear vector differential equation

$$x' = A(t)x + b(t)$$

we will see later that we first need to solve the corresponding homogeneous linear vector differential equation

$$x' = A(t)x. \tag{2.3}$$

Hence we will first study the homogeneous vector differential equation (2.3). Note that if the vector functions $\phi_1, \phi_2, \cdots, \phi_k$ are solutions of $x' = A(t)x$ (equivalently, of $Lx = 0$, where L is as in Example 2.5) on I, then

$$\begin{aligned} &L[c_1\phi_1 + c_2\phi_2 + \cdots + c_k\phi_k] \\ &= c_1L[\phi_1] + c_2L[\phi_2] + \cdots + c_nL[\phi_k] \\ &= 0. \end{aligned}$$

This proves that any linear combination of solutions of (2.3) on I is a solution of (2.3) on I. Consequently, the set of all such solutions is a vector space. To solve (2.3), we will see that we want to find n *linearly independent* solutions on I (see Definition 2.9).

Definition 2.6 We say that the constant $n \times 1$ vectors $\psi_1, \psi_2, \cdots, \psi_k$ are *linearly dependent* provided there are constants $c_1, c_2, \cdots, c_k$, not all zero, such that

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_k\psi_k = 0,$$

where 0 denotes the $n \times 1$ zero vector. Otherwise we say that these k constant vectors are *linearly independent*.

Note that the constant $n \times 1$ vectors $\psi_1, \ \psi_2, \cdots, \psi_k$ are linearly independent provided that the only constants $c_1, c_2, \cdots, c_k$ that satisfy the equation

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_k\psi_k = 0,$$

are $c_1 = c_2 = \cdots = c_k = 0$.

Theorem 2.7 *Assume we have exactly n constant $n \times 1$ vectors*

$$\psi_1, \psi_2, \cdots, \psi_n$$

and C is the column matrix $C = [\psi_1\psi_2 \cdots \psi_n]$. Then $\psi_1, \psi_2, \cdots, \psi_n$ are linearly dependent iff $\det C = 0$.

Proof Let $\psi_1, \psi_2, \cdots, \psi_n$ and C be as in the statement of this theorem. Then

$$\det C = 0$$

if and only if there is a nontrivial vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

such that

$$C \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

if and only if

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n = 0,$$

where $c_1, c_2, \cdots, c_n$ are not all zero, if and only if

$$\psi_1, \psi_2, \cdots, \psi_n \quad \text{are linearly dependent.}$$

□

Example 2.8 Since

$$\det \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -3 & -1 & -3 \end{bmatrix} = 0,$$

the vectors

$$\psi_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$$

are linearly dependent by Theorem 2.7. △

Definition 2.9 Assume the $n \times 1$ vector functions $\phi_1, \phi_2, \cdots, \phi_k$ are defined on an interval I. We say that these k vector functions are *linearly dependent on* I provided there are constants $c_1, c_2, \cdots, c_k$, not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_k\phi_k(t) = 0,$$

for all $t \in I$. Otherwise we say that these k vector functions are *linearly independent on* I.

Note that the $n \times 1$ vector functions $\phi_1, \phi_2, \cdots, \phi_k$ are linearly independent on an interval I provided that the only constants $c_1, c_2, \cdots, c_k$ that satisfy the equation

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_k\phi_k(t) = 0,$$

for all $t \in I$, are $c_1 = c_2 = \cdots = c_k = 0$.

Any three 2×1 constant vectors are linearly dependent, but in the following example we see that we can have three linearly independent 2×1 vector functions on an interval I.

Example 2.10 Show that the three vector functions ϕ_1, ϕ_2, ϕ_3 defined by

$$\phi_1(t) = \begin{bmatrix} t \\ t \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \quad \phi_3(t) = \begin{bmatrix} t^3 \\ t \end{bmatrix}$$

are linearly independent on any nondegenerate interval I (a nondegenerate interval is any interval containing at least two points).

To see this, assume c_1, c_2, c_3 are constants such that

$$c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t) = 0,$$

for all $t \in I$. Then

$$c_1 \begin{bmatrix} t \\ t \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ t \end{bmatrix} + c_3 \begin{bmatrix} t^3 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for all $t \in I$. This implies that

$$c_1 t + c_2 t^2 + c_3 t^3 = 0, \tag{2.4}$$

for all $t \in I$. Taking three derivatives of both sides of equation (2.4), we have

$$6c_3 = 0.$$

Hence $c_3 = 0$. Letting $c_3 = 0$ in equation (2.4) and taking two derivatives of both sides of the resulting equation

$$c_1 t + c_2 t^2 = 0,$$

we get that

$$2c_2 = 0$$

and so $c_2 = 0$. It then follows that $c_1 = 0$. Hence the three vector functions ϕ_1, ϕ_2, ϕ_3 are linearly independent on I. △

In the next theorem when we say (2.5) gives us a *general solution* of (2.3) we mean all functions in this form are solutions of (2.3) and all solutions of (2.3) can be written in this form.

Theorem 2.11 *The linear vector differential equation* (2.3) *has n linearly independent solutions on I, and if $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions on I, then*

$$x = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n, \tag{2.5}$$

for $t \in I$, where $c_1, c_2, \cdots, c_n$ are constants, is a general solution of (2.3).

Proof Let $\psi_1, \psi_2, \cdots, \psi_n$ be n linearly independent constant $n \times 1$ vectors and let $t_0 \in I$. Then let ϕ_i be the solution of the IVP

$$x' = A(t)x, \quad x(t_0) = \psi_i,$$

for $1 \le i \le n$. Assume $c_1, c_2, \cdots, c_n$ are constants such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t) = 0,$$

for all $t \in I$. Letting $t = t_0$ we have

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0) = 0$$

or, equivalently,

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n = 0.$$

Since $\psi_1, \psi_2, \cdots, \psi_n$ are n linearly independent constant vectors, we have that

$$c_1 = c_2 = \cdots = c_n = 0.$$

It follows that the vector functions $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions on I. Hence we have proved the existence of n linearly independent solutions.

Next assume the vector functions $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions of (2.3) on I. Since linear combinations of solutions of (2.3) are solutions of (2.3), any vector function x of the form

$$x = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

is a solution of (2.3). It remains to show that every solution of (2.3) is of this form. Let $t_0 \in I$ and let

$$\xi_i := \phi_i(t_0),$$

$1 \leq i \leq n$. Assume $b_1, b_2, \cdots, b_n$ are constants such that

$$b_1\xi_1 + b_2\xi_2 + \cdots + b_n\xi_n = 0.$$

Then let

$$v(t) := b_1\phi_1(t) + b_2\phi_2(t) + \cdots + b_n\phi_n(t),$$

for $t \in I$. Then v is a solution of (2.3) on I with $v(t_0) = 0$. It follows from the uniqueness theorem (Theorem 2.3) that v is the trivial solution and hence

$$b_1\phi_1(t) + b_2\phi_2(t) + \cdots + b_n\phi_n(t) = 0,$$

for $t \in I$. But $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I, so

$$b_1 = b_2 = \cdots = b_n = 0.$$

But this implies that the constant vectors $\xi_1 := \phi_1(t_0)$, $\xi_2 := \phi_2(t_0)$, $\cdots$, $\xi_n := \phi_n(t_0)$ are linearly independent.

Let z be an arbitrary but fixed solution of (2.3). Let $t_0 \in I$, since $z(t_0)$ is an $n \times 1$ constant vector and $\phi_1(t_0)$, $\phi_2(t_0)$, $\cdots$, $\phi_n(t_0)$ are linearly independent $n \times 1$ constant vectors, there are constants $a_1, a_2, \cdots, a_n$ such that

$$a_1\phi_1(t_0) + a_2\phi_2(t_0) + \cdots + a_n\phi_n(t_0) = z(t_0).$$

By the uniqueness theorem (Theorem 2.3) we have that

$$z(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots + a_n\phi_n(t), \quad \text{for} \quad t \in I.$$

Hence

$$x = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

is a general solution of (2.3). $\square$

First we will see how to solve the vector differential equation

$$x' = Ax,$$

where A is a constant $n \times n$ matrix. We recall the definitions of eigenvalues and eigenvectors for a square matrix A.

Definition 2.12 Let A be a given $n \times n$ constant matrix and let x be a column unknown n-vector. For any number λ the vector equation

$$Ax = \lambda x \tag{2.6}$$

has the solution $x = 0$ called the *trivial solution* of the vector equation. If λ_0 is a number such that the vector equation (2.6) with λ replaced by λ_0 has a nontrivial solution x_0, then λ_0 is called an *eigenvalue* of A and x_0 is called a corresponding *eigenvector*. We say λ_0, x_0 is an *eigenpair* of A.

Assume λ is an eigenvalue of A, then equation (2.6) has a nontrivial solution. Therefore,

$$(A - \lambda I)\, x = 0$$

has a nontrivial solution. From linear algebra we get that the *characteristic equation*

$$\det(A - \lambda I) = 0$$

is satisfied. If λ_0 is an eigenvalue, then to find a corresponding eigenvector we want to find a nonzero vector x so that

$$Ax = \lambda_0 x$$

or, equivalently,

$$(A - \lambda_0 I)\, x = 0.$$

Example 2.13 Find eigenpairs for

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

The characteristic equation of A is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0.$$

Simplifying, we have

$$\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0.$$

Hence the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -1$. To find an eigenvector corresponding to $\lambda_1 = -2$, we solve

$$(A - \lambda_1 I)x = (A + 2I)x = 0$$

or

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that

$$-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is an eigenpair for A. Similarly, we get that

$$-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenpair for A. $\triangle$

Theorem 2.14 *If λ_0, x_0 is an eigenpair for the constant $n \times n$ matrix A, then*

$$x(t) = e^{\lambda_0 t} x_0, \quad t \in \mathbb{R},$$

defines a solution x of

$$x' = Ax \tag{2.7}$$

on $\mathbb{R}$.

Proof Let

$$x(t) = e^{\lambda_0 t} x_0,$$

then

$$\begin{aligned} x'(t) &= \lambda_0 e^{\lambda_0 t} x_0 \\ &= e^{\lambda_0 t} \lambda_0 x_0 \\ &= e^{\lambda_0 t} A x_0 \\ &= A e^{\lambda_0 t} x_0 \\ &= Ax(t), \end{aligned}$$

for $t \in \mathbb{R}$. $\square$

Example 2.15 Solve the differential equation

$$x' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x.$$

From Example 2.13 we get that the eigenpairs for

$$A := \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

are

$$-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad -1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence by Theorem 2.14 the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \phi_2(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

are solutions on $\mathbb{R}$. Since the vector functions ϕ_1, ϕ_2 are linearly independent (show this) on $\mathbb{R}$, a general solution x is given by

$$x(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$t \in \mathbb{R}$. $\triangle$

Theorem 2.16 *If $x = u + iv$ is a complex vector-valued solution of* (2.3), *where u, v are real vector-valued functions, then u, v are real vector-valued solutions of* (2.3).

Proof Assume x is as in the statement of the theorem. Then

$$x'(t) = u'(t) + iv'(t) = A(t)\left[u(t) + iv(t)\right], \quad \text{for} \quad t \in I,$$

or

$$u'(t) + iv'(t) = A(t)u(t) + iA(t)v(t), \quad \text{for} \quad t \in I.$$

Equating real and imaginary parts, we have the desired results

$$u'(t) = A(t)u(t), \quad v'(t) = A(t)v(t), \quad \text{for} \quad t \in I. \quad \square$$

Example 2.17 Solve the differential equation

$$x' = \begin{bmatrix} 3 & 1 \\ -13 & -3 \end{bmatrix} x. \tag{2.8}$$

The characteristic equation of the coefficient matrix is

$$\lambda^2 + 4 = 0$$

and the eigenvalues are

$$\lambda_1 = 2i, \quad \lambda_2 = -2i.$$

To find an eigenvector corresponding to $\lambda_1 = 2i$, we solve

$$(A - 2iI)x = 0$$

or

$$\begin{bmatrix} 3-2i & 1 \\ -13 & -3-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence we want

$$(3 - 2i)x_1 + x_2 = 0.$$

It follows that

$$2i, \begin{bmatrix} 1 \\ -3+2i \end{bmatrix}$$

is an eigenpair for A. Hence by Theorem 2.14 the vector function ϕ defined by

$$\begin{aligned} \phi(t) &= e^{2it} \begin{bmatrix} 1 \\ -3+2i \end{bmatrix} \\ &= [\cos(2t) + i\sin(2t)] \begin{bmatrix} 1 \\ -3+2i \end{bmatrix} \\ &= \begin{bmatrix} \cos(2t) \\ -3\cos(2t) - 2\sin(2t) \end{bmatrix} + i \begin{bmatrix} \sin(2t) \\ 2\cos(2t) - 3\sin(2t) \end{bmatrix} \end{aligned}$$

is a solution. Using Theorem 2.16, we get that the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = \begin{bmatrix} \cos(2t) \\ -3\cos(2t) - 2\sin(2t) \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) - 3\sin(2t) \end{bmatrix}$$

are real vector-valued solutions of (2.8). Since ϕ_1, ϕ_2 are linearly independent (show this) on $\mathbb{R}$, we have by Theorem 2.11 that a general solution x of (2.8) is given by

$$x(t) = c_1 \begin{bmatrix} \cos(2t) \\ -3\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ 2\cos(2t) - 3\sin(2t) \end{bmatrix},$$

for $t \in \mathbb{R}$. △

Example 2.18 Let's solve the system in Example 2.2 involving two masses attached to springs for the special case that all the parameters are equal to one.

In this case we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & -1 \end{bmatrix}.$$

By expanding $\det(A - \lambda I)$ along the first row, we get the characteristic equation

$$\begin{aligned} 0 = \det(A - \lambda I) &= \lambda(\lambda+1)(\lambda^2+\lambda+2) + 2(\lambda^2+\lambda+2) - 1 \\ &= (\lambda^2+\lambda+2)^2 - 1 \\ &= (\lambda^2+\lambda+1)(\lambda^2+\lambda+3). \end{aligned}$$

Hence the eigenvalues of A are

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \; -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i.$$

As in the Example 2.17, the eigenpairs are computed to be

$$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \begin{bmatrix} 1 \\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \\ 1 \\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{bmatrix} \text{ and } -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i, \begin{bmatrix} 1 \\ -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i \\ -1 \\ \frac{1}{2} \mp \frac{\sqrt{11}}{2}i \end{bmatrix}$$

(use your calculator to check this). Complex vector solutions corresponding to the first two eigenpairs are

$$e^{(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)t} \begin{bmatrix} 1 \\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \\ 1 \\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{bmatrix},$$

and complex vector solutions for the remaining eigenpairs are obtained in a similar way. Finally, by multiplying out the complex solutions and taking real and imaginary parts, we obtain four real solutions. The first two are

$$e^{-\frac{t}{2}}\begin{bmatrix} \cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t \\ \cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t \end{bmatrix}, \quad e^{-\frac{t}{2}}\begin{bmatrix} \sin\frac{\sqrt{3}}{2}t \\ \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t \\ \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{bmatrix},$$

and the second two are

$$e^{-\frac{t}{2}}\begin{bmatrix} \cos(at) \\ -\frac{1}{2}\cos(at) - a\sin(at) \\ -\cos(at) \\ \frac{1}{2}\cos(at) + a\sin(at) \end{bmatrix}, \quad e^{-\frac{t}{2}}\begin{bmatrix} \sin(at) \\ a\cos(at) - \frac{1}{2}\sin(at) \\ -\sin(at) \\ -a\cos(at) + \frac{1}{2}\sin(at) \end{bmatrix},$$

where $a = \frac{\sqrt{11}}{2}$. We will show later (Example 2.26) that these four solutions are linearly independent on $\mathbb{R}$, so a general linear combination of them gives a general solution of the mass-spring problem. Note that as $t \to \infty$, the masses must experience exponentially decreasing oscillations about their equilibrium positions. △

If the matrix A has n linearly independent eigenvectors, then Theorem 2.14 can be used to generate a general solution of $x' = Ax$ (see Exercise 2.15). The following example shows that an $n \times n$ constant matrix may have fewer than n linearly independent eigenvectors.

Example 2.19 Consider the vector differential equation $x' = Ax$, where

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

The characteristic equation for A is

$$\lambda^2 - 4\lambda + 4 = 0,$$

so $\lambda_1 = \lambda_2 = 2$ are the eigenvalues. Corresponding to the eigenvalue 2 there is only one linearly independent eigenvector, and so we cannot use Theorem 2.14 to solve this differential equation. Later (see Example 2.36) we will use Putzer's algorithm (Theorem 2.35) to solve this differential equation. △

We now get some results for the linear vector differential equation (2.3). We define the matrix differential equation

$$X' = A(t)X, \tag{2.9}$$

where

$$X := \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & & \cdots & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

and

$$X' := \begin{bmatrix} x'_{11} & x'_{12} & \cdots & x'_{1n} \\ x'_{21} & x'_{22} & \cdots & x'_{2n} \\ \cdots & & \cdots & \\ x'_{n1} & x'_{n2} & \cdots & x'_{nn} \end{bmatrix}$$

are $n \times n$ matrix variables and A is a given $n \times n$ continuous matrix function on an interval I, to be the matrix differential equation corresponding to the vector differential equation (2.3). We say that a matrix function Φ is a solution of (2.9) on I provided Φ is a continuously differentiable $n \times n$ matrix function on I and

$$\Phi'(t) = A(t)\Phi(t),$$

for $t \in I$. The following theorem gives a relationship between the vector differential equation (2.3) and the matrix differential equation (2.9).

Theorem 2.20 *Assume A is a continuous $n \times n$ matrix function on an interval I and assume that Φ defined by*

$$\Phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_n(t)], \quad t \in I,$$

is the $n \times n$ matrix function with columns $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$. Then Φ is a solution of the matrix differential equation (2.9) *on I iff each column ϕ_i is a solution of the vector differential equation* (2.3) *on I for $1 \leq i \leq n$. Furthermore, if Φ is a solution of the matrix differential equation* (2.9)*, then*

$$x(t) = \Phi(t)c$$

is a solution of the vector differential equation (2.3) *for any constant $n \times 1$ vector c.*

Proof Assume $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of (2.3) on I and define the $n \times n$ matrix function Φ by

$$\Phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_n(t)], \quad t \in I.$$

Then Φ is a continuously differentiable matrix function on I and

$$\begin{aligned} \Phi'(t) &= [\phi'_1(t), \phi'_2(t), \cdots, \phi'_n(t)] \\ &= [A(t)\phi_1(t), A(t)\phi_2(t), \cdots, A(t)\phi_n(t)] \\ &= A(t)[\phi_1(t), \phi_2(t), \cdots, \phi_n(t)] \\ &= A(t)\Phi(t), \end{aligned}$$

for $t \in I$. Hence Φ is a solution of the matrix differential equation (2.9) on I. We leave it to the reader to show if Φ is a solution of the matrix differential equation (2.9) on I, then its columns are solutions of the vector differential equation (2.3) on I.

Next assume that the $n \times n$ matrix function Φ is a solution of the matrix differential equation (2.9) on I and let

$$x(t) := \Phi(t)c, \quad \text{for} \quad t \in I,$$

where c is a constant $n \times 1$ vector. Then

$$\begin{aligned} x'(t) &= \Phi'(t)c \\ &= A(t)\Phi(t)c \\ &= A(t)x(t), \end{aligned}$$

for $t \in I$. This proves the last statement in this theorem. □

Theorem 2.21 (Existence-Uniqueness Theorem) *Assume A is a continuous matrix function on an interval I. Then the IVP*

$$X' = A(t)X, \quad X(t_0) = X_0,$$

where $t_0 \in I$ and X_0 is an $n \times n$ constant matrix, has a unique solution X that is a solution on the whole interval I.

Proof This theorem follows from Theorem 2.3 and the fact that X is a solution of the matrix equation (2.9) iff each of its columns is a solution of the vector equation (2.3). □

We will use the following definition in the next theorem.

Definition 2.22 Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & & \cdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

Then we define the *trace* of $A(t)$ by

$$\operatorname{tr}[A(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t).$$

Theorem 2.23 (Liouville's Theorem) *Assume $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of the vector differential equation* (2.3) *on I and Φ is the matrix function with columns ϕ_1, ϕ_2, $\cdots$, ϕ_n. Then if $t_0 \in I$,*

$$\det \Phi(t) = e^{\int_{t_0}^t \operatorname{tr}[A(s)]\, ds} \det \Phi(t_0),$$

for $t \in I$.

Proof We will just prove this theorem in the case when $n = 2$. Assume ϕ_1, ϕ_2 are $n = 2$ solutions of the vector equation (2.1) on I and $\Phi(t)$ is the matrix with columns

$$\phi_1(t) = \begin{bmatrix} \phi_{11}(t) \\ \phi_{21}(t) \end{bmatrix}, \ \phi_2(t) = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \end{bmatrix}.$$

Let

$$u(t) = \det \Phi(t) = \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix},$$

for $t \in I$. Taking derivatives we get

$$\begin{aligned}
u'(t) &= \begin{vmatrix} \phi'_{11}(t) & \phi'_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} + \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi'_{21}(t) & \phi'_{22}(t) \end{vmatrix} \\
&= \begin{vmatrix} a_{11}(t)\phi_{11}(t) + a_{12}(t)\phi_{21}(t) & a_{11}(t)\phi_{12}(t) + a_{12}(t)\phi_{22}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} \\
&+ \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ a_{21}(t)\phi_{11}(t) + a_{22}(t)\phi_{21}(t) & a_{21}(t)\phi_{12}(t) + a_{22}(t)\phi_{22}(t) \end{vmatrix} \\
&= \begin{vmatrix} a_{11}(t)\phi_{11}(t) & a_{11}(t)\phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} + \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ a_{22}(t)\phi_{21}(t) & a_{22}(t)\phi_{22}(t) \end{vmatrix} \\
&= a_{11}(t) \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} + a_{22}(t) \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} \\
&= [a_{11}(t) + a_{22}(t)] \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{vmatrix} \\
&= \operatorname{tr}[A(t)] \det \Phi(t) \\
&= \operatorname{tr}[A(t)] u(t).
\end{aligned}$$

Solving the differential equation $u' = \operatorname{tr}[A(t)]u$, we get

$$u(t) = u(t_0) e^{\int_{t_0}^t \operatorname{tr}[A(s)]\, ds}, \quad \text{for} \quad t \in I,$$

or, equivalently,

$$\det \Phi(t) = e^{\int_{t_0}^t \operatorname{tr}[A(s)]\, ds} \det \Phi(t_0), \quad \text{for} \quad t \in I.$$

□

Corollary 2.24 *Assume $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of the vector equation (2.3) on I and Φ is the matrix function with columns $\phi_1, \phi_2, \cdots, \phi_n$. Then either*
(a) $\det \Phi(t) = 0$, *for all $t \in I$,*
or
(b) $\det \Phi(t) \neq 0$, *for all $t \in I$.*
Case (a) holds iff the solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly dependent on I, while case (b) holds iff the solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I.

Proof The first statement of this theorem follows immediately from Liouville's formula in Theorem 2.23. The proof of the statements concerning linear independence and linear dependence is left as an exercise (see Exercise 2.24). □

It follows from Corollary 2.24 that if $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of the vector equation (2.3) on I and Φ is the matrix function with columns $\phi_1, \phi_2, \cdots, \phi_n$ and $t_0 \in I$, then $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I iff

$$\det \Phi(t_0) \neq 0, \quad \text{for any } t_0 \in I.$$

We show how to use this in the next example.

Example 2.25 Show that the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = \begin{bmatrix} \cos(2t) \\ -3\cos(2t) - 2\sin(2t) \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) - 3\sin(2t) \end{bmatrix},$$

for $t \in \mathbb{R}$ are linearly independent on $\mathbb{R}$.

In Example 2.17 we saw that ϕ_1, ϕ_2 are solutions of the vector differential equation (2.8) on $\mathbb{R}$. Let Φ be the matrix function with columns ϕ_1 and ϕ_2, respectively. Then

$$\det \Phi(0) = \begin{vmatrix} 1 & 0 \\ -3 & 2 \end{vmatrix} = 2 \neq 0.$$

Hence ϕ_1, ϕ_2 are linearly independent on R. △

Example 2.26 Let's show that the four real solutions computed in Example 2.18 involving the oscillations of two masses are linearly independent on $\mathbb{R}$.

If we evaluate each solution at $t = 0$, then we obtain the following determinant:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{11}}{2} \\ 1 & 0 & -1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{11}}{2} \end{vmatrix},$$

which has value $\sqrt{33}$, so linear independence is established. △

In the preceding examples, it was essential that the vector functions were solutions of a vector equation of the form (2.3). In the following example, two linearly independent vector functions on an interval I are shown to constitute a matrix with zero determinant at a point $t_0 \in I$.

Example 2.27 Show that the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} t \cdot |t| \\ 1 \end{bmatrix},$$

for $t \in \mathbb{R}$ are linearly independent on $\mathbb{R}$.

Assume c_1, c_2 are constants such that

$$c_1\phi_1(t) + c_2\phi_2(t) = 0,$$

for $t \in \mathbb{R}$. Then

$$c_1 \begin{bmatrix} t^2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \cdot |t| \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for $t \in \mathbb{R}$. This implies that

$$c_1 t^2 + c_2 t \cdot |t| = 0,$$

for $t \in \mathbb{R}$. Letting $t = 1$ and $t = -1$, we get the two equations

$$c_1 + c_2 = 0$$

$$c_1 - c_2 = 0,$$

respectively. This implies that $c_1 = c_2 = 0$, which gives us that ϕ_1, ϕ_2 are linearly independent on $\mathbb{R}$. Notice that if $\Phi(t) = [\phi_1(t), \phi_2(t)]$, then

$$\det \Phi(0) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0.$$

It must follow that

$$\phi_1(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} t \cdot |t| \\ 1 \end{bmatrix}$$

are not solutions of a two-dimensional vector differential equation of the form (2.3) on the interval $I = \mathbb{R}$. △

Definition 2.28 An $n \times n$ matrix function Φ is said to be a *fundamental matrix* for the vector differential equation (2.3) provided Φ is a solution of the matrix equation (2.9) on I and $\det \Phi(t) \neq 0$ on I.

Theorem 2.29 *An $n \times n$ matrix function Φ is a fundamental matrix for the vector differential equation* (2.3) *iff the columns of Φ are n linearly independent solutions of* (2.3) *on I. If Φ is a fundamental matrix for the vector differential equation* (2.3), *then a general solution x of* (2.3) *is given by*

$$x(t) = \Phi(t)c, \quad t \in I,$$

where c is an arbritrary $n \times 1$ constant vector. There are infinitely many fundamental matrices for the differential equation (2.3).

Proof Assume Φ is an $n \times n$ matrix function whose columns are linearly independent solutions of (2.3) on I. Since the columns of Φ are solutions of (2.3), we have by Theorem 2.20 that Φ is a solution of the matrix equation (2.9). Since the columns of Φ are linearly independent solutions of (2.9), we have by Corollary 2.24 that $\det \Phi(t) \neq 0$ on I. Hence Φ is a fundamental matrix for the vector equation (2.3). We leave it to the reader to show that if Φ is a fundamental matrix for the vector differential equation (2.3), then the columns of Φ are linearly independent solutions of (2.3) on I. There are infinitely many fundamental matrices for (2.3) since for any nonsingular $n \times n$ matrix X_0 the solution of the IVP

$$X' = A(t)X, \quad X(t_0) = X_0,$$

is a fundamental matrix for (2.3) (a nonsingular matrix is a matrix whose determinant is different than zero).

Next assume that Φ is a fundamental matrix for (2.3). Then by Theorem 2.20

$$x(t) = \Phi(t)c,$$

for any $n \times 1$ constant vector c, is a solution of (2.3). Now let z be an arbitrary but fixed solution of (2.3). Let $t_0 \in I$ and define

$$c_0 = \Phi^{-1}(t_0)z(t_0).$$

Then z and Φc_0 are solutions of (2.3) with the same vector value at t_0. Hence by the uniqueness of solutions to IVPs (Theorem 2.3),

$$z(t) = \Phi(t)c_0.$$

Therefore,

$$x(t) = \Phi(t)c, \quad \text{for } t \in I,$$

where c is an arbitrary $n \times 1$ constant vector defines a general solution of (2.3). □

Example 2.30 Find a fundamental matrix Φ for

$$x' = \begin{bmatrix} -2 & 3 \\ 2 & 3 \end{bmatrix} x. \tag{2.10}$$

Verify that Φ is a fundamental matrix and then write down a general solution of this vector differential equation in terms of this fundamental matrix.

The characteristic equation is

$$\lambda^2 - \lambda - 12 = 0$$

and so the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 4$. Corresponding eigenvectors are

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence the vector functions ϕ_1, ϕ_2 defined by

$$\phi_1(t) = e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \phi_2(t) = e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

for $t \in \mathbb{R}$ are solutions of (2.10). It follows from Theorem 2.20 that the matrix function Φ defined by

$$\Phi(t) = [\phi_1(t), \phi_2(t)] = \begin{bmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{bmatrix},$$

for $t \in \mathbb{R}$ is a matrix solution of the matrix equation corresponding to (2.9). Since

$$\det \Phi(t) = \begin{vmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{vmatrix} = 7e^t \neq 0,$$

for all $t \in \mathbb{R}$, Φ is a fundamental matrix of (2.10) on $\mathbb{R}$. It follows from Theorem 2.29 that a general solution x of (2.10) is given by

$$x(t) = \Phi(t)c = \begin{bmatrix} 3e^{-3t} & e^{4t} \\ -e^{-3t} & 2e^{4t} \end{bmatrix} c,$$

for $t \in \mathbb{R}$, where c is an arbritrary 2×1 constant vector. △

Theorem 2.31 *If Φ is a fundamental matrix for* (2.3)*, then $\Psi = \Phi C$ where C is an arbitrary $n \times n$ nonsingular constant matrix is a general fundamental matrix of* (2.3).

Proof Assume Φ is a fundamental matrix for (2.3) and set

$$\Psi = \Phi C,$$

where C is an $n \times n$ constant matrix. Then Ψ is continuously differentiable on I and

$$\begin{aligned}\Psi'(t) &= \Phi'(t)C \\ &= A(t)\Phi(t)C \\ &= A(t)\Psi(t).\end{aligned}$$

Hence $\Psi = \Phi C$ is a solution of the matrix equation (2.9). Now assume that C is also nonsingular. Since

$$\begin{aligned}\det[\Psi(t)] &= \det[\Phi(t)C] \\ &= \det[\Phi(t)]\det[C] \\ &\neq 0\end{aligned}$$

for $t \in I$, $\Psi = \Phi C$ is a fundamental matrix of (2.9). It remains to show any fundamental matrix is of the correct form. Assume Ψ is an arbitrary but fixed fundamental matrix of (2.3). Let $t_0 \in I$ and let

$$C_0 := \Phi^{-1}(t_0)\Psi(t_0).$$

Then C_0 is a nonsingular constant matrix and

$$\Psi(t_0) = \Phi(t_0)C_0$$

and so by the uniqueness theorem (Theorem 2.21)

$$\Psi(t) = \Phi(t)C_0, \quad \text{for } t \in I.$$

$\square$

2.3 The Matrix Exponential Function

In this section, we show how to compute a fundamental matrix for the linear system with constant coefficients

$$x' = Ax.$$

Specifically, we will compute the special fundamental matrix whose initial value is the identity matrix. This matrix function turns out to be an extension of the familiar exponential function from calculus. Here is the definition:

Definition 2.32 Let A be an $n \times n$ constant matrix. Then we define the matrix exponential function by e^{At} is the solution of the IVP

$$X' = AX, \quad X(0) = I,$$

where I is the $n \times n$ identity matrix.

Before we give a formula for e^{At} (see Theorem 2.35) we recall without proof the following very important result from linear algebra. In Exercise 2.29 the reader is asked to prove Theorem 2.33 for 2×2 matrices.

Theorem 2.33 (Cayley-Hamilton Theorem) *Every $n \times n$ constant matrix satisfies its characteristic equation.*

Example 2.34 Verify the Cayley-Hamilton Theorem (Theorem 2.33) directly for the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

The characteristic equation for A is

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = 0.$$

Now

$$\begin{aligned} A^2 - 3A - 10I &= \begin{bmatrix} 16 & 9 \\ 12 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which is what we mean by A satisfies its characteristic equation. △

Theorem 2.35 (Putzer Algorithm for Finding e^{At}) *Let $\lambda_1,\ \lambda_2,\ \cdots, \lambda_n$ be the (not necessarily distinct) eigenvalues of the matrix A. Then*

$$e^{At} = \sum_{k=0}^{n-1} p_{k+1}(t) M_k,$$

where $M_0 := I$,

$$M_k := \prod_{i=1}^{k} (A - \lambda_i I),$$

for $1 \leq k \leq n$ and the vector function p defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdots \\ p_n(t) \end{bmatrix},$$

for $t \in \mathbb{R}$, is the solution of the IVP

$$p' = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof Let

$$\Phi(t) := \sum_{k=0}^{n-1} p_{k+1}(t) M_k, \quad \text{for } t \in \mathbb{R},$$

where p_k, $1 \leq k \leq n$ and M_k, $0 \leq k \leq n$ are as in the statement of this theorem. Then by the uniqueness theorem (Theorem 2.21) it suffices to show that Φ satisfies the IVP

$$X' = AX, \quad X(0) = I.$$

First note that

$$\begin{aligned}
\Phi(0) &= \sum_{k=0}^{n-1} p_{k+1}(0) M_k \\
&= p_1(0) I \\
&= I.
\end{aligned}$$

Hence Φ satisfies the correct initial condition. Note that since the vector function p defined by

$$p(t) := \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdots \\ p_n(t) \end{bmatrix}$$

is the solution of the IVP

$$p' = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

we get that

$$\begin{aligned}
p_1'(t) &= \lambda_1 p_1(t), \\
p_i'(t) &= p_{i-1}(t) + \lambda_i p_i(t),
\end{aligned}$$

for $t \in \mathbb{R}$, $2 \leq i \leq n$.

Now consider, for $t \in \mathbb{R}$,

$$\begin{aligned}
& \Phi'(t) - A\Phi(t) \\
= \; & \sum_{k=0}^{n-1} p'_{k+1}(t)M_k - A\sum_{k=0}^{n-1} p_{k+1}(t)M_k \\
= \; & \lambda_1 p_1(t)M_0 + \sum_{k=1}^{n-1} \left[\lambda_{k+1}p_{k+1}(t) + p_k(t)\right] M_k - \sum_{k=0}^{n-1} p_{k+1}(t)AM_k \\
= \; & \lambda_1 p_1(t)M_0 + \sum_{k=1}^{n-1} \left[\lambda_{k+1}p_{k+1}(t) + p_k(t)\right] M_k \\
& - \sum_{k=0}^{n-1} p_{k+1}(t)\left[M_{k+1} + \lambda_{k+1}IM_k\right] \\
= \; & \sum_{k=1}^{n-1} p_k(t)M_k - \sum_{k=0}^{n-1} p_{k+1}(t)M_{k+1} \\
= \; & -p_n(t)M_n \\
= \; & 0,
\end{aligned}$$

since $M_n = 0$, by the Cayley-Hamilton theorem (Theorem 2.33). □

Example 2.36 Use the Putzer algorithm (Theorem 2.35) to find e^{At} when

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

The characteristic equation for A is

$$\lambda^2 - 4\lambda + 4 = 0,$$

so $\lambda_1 = \lambda_2 = 2$ are the eigenvalues. By the Putzer algorithm (Theorem 2.35),

$$e^{At} = \sum_{k=0}^{1} p_{k+1}(t)M_k = p_1(t)M_0 + p_2(t)M_1.$$

Now

$$M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$M_1 = A - \lambda_1 I = A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Now the vector function p given by

$$p(t) := \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix},$$

for $t \in \mathbb{R}$ is the solution of the IVP

$$p' = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence the first component p_1 of p solves the IVP

$$p_1' = 2p_1, \quad p_1(0) = 1$$

and so we get that

$$p_1(t) = e^{2t}.$$

Next the second component p_2 of p is a solution of the IVP

$$p_2' = e^{2t} + 2p_2, \quad p_2(0) = 0.$$

It follows that

$$p_2(t) = te^{2t}.$$

Hence

$$\begin{aligned} e^{At} &= p_1(t)M_0 + p_2(t)M_1 \\ &= e^{2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= e^{2t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}. \end{aligned}$$

It follows from Defintion 2.28 and Theorem 2.29 that a general solution of the vector differential equation

$$x' = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} x \tag{2.11}$$

is given by

$$\begin{aligned} x(t) &= e^{At}c \\ &= c_1 e^{2t}\begin{bmatrix} 1-t \\ -t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1+t \end{bmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$. Note that in Example 2.19 we pointed out that we can not solve the vector differential equation (2.11) using Theorem 2.14. △

Example 2.37 (Complex Eigenvalues) Use the Putzer algorithm (Theorem 2.35) to find e^{At} when

$$A := \begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix}.$$

The characteristic equation for A is

$$\lambda^2 + 4 = 0,$$

so $\lambda_1 = 2i, \lambda_2 = -2i$ are the eigenvalues. By the Putzer algorithm (Theorem 2.35),

$$e^{At} = \sum_{k=0}^{1} p_{k+1}(t)M_k = p_1(t)M_0 + p_2(t)M_1.$$

Now

$$M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$M_1 = A - \lambda_1 I = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix}.$$

Now the vector function p given by

$$p(t) := \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

for $t \in \mathbb{R}$ must be a solution of the IVP

$$p' = \begin{bmatrix} 2i & 0 \\ 1 & -2i \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence p_1 is the solution of the IVP

$$p_1' = (2i)p_1, \quad p_1(0) = 1,$$

and we get that

$$p_1(t) = e^{2it}.$$

Next p_2 is the solution of the IVP

$$p_2' = e^{2it} - (2i)p_2, \quad p_2(0) = 0.$$

It follows that

$$\begin{aligned} p_2(t) &= \frac{1}{4i}e^{2it} - \frac{1}{4i}e^{-2it} \\ &= \frac{1}{2}\sin(2t), \end{aligned}$$

for $t \in \mathbb{R}$. Hence

$$\begin{aligned} e^{At} &= p_1(t)M_0 + p_2(t)M_1 \\ &= e^{2it}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\sin(2t)\begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \\ &= [\cos(2t) + i\sin(2t)]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\sin(2t)\begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \\ &= \begin{bmatrix} \cos(2t) + \frac{1}{2}\sin(2t) & -\frac{1}{2}\sin(2t) \\ \frac{5}{2}\sin(2t) & \cos(2t) - \frac{1}{2}\sin(2t) \end{bmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$. △

Example 2.38 Use the Putzer algorithm (Theorem 2.35) to help you solve the vector differential equation

$$x' = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} x. \tag{2.12}$$

Let A be the coefficient matrix in (2.12). The characteristic equation for A is

$$(\lambda - 2)^2 (\lambda - 3) = 0,$$

so $\lambda_1 = \lambda_2 = 2, \lambda_3 = 3$ are the eigenvalues of A. By the Putzer algorithm (Theorem 2.35),

$$e^{At} = \sum_{k=0}^{2} p_{k+1}(t)M_k = p_1(t)M_0 + p_2(t)M_1 + p_3(t)M_2.$$

Now

$$M_0 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_1 = (A - \lambda_1 I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} M_2 &= (A - \lambda_2 I)(A - \lambda_1 I) \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Now the vector function p given by

$$p(t) := \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix},$$

for $t \in \mathbb{R}$ solves the IVP

$$p' = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since p_1 is the solution of the IVP

$$p_1' = 2p_1, \quad p_1(0) = 1,$$

we get that

$$p_1(t) = e^{2t}.$$

Next p_2 is the solution of the IVP

$$p_2' = e^{2t} + 2p_2, \quad p_2(0) = 0.$$

It follows that

$$p_2(t) = te^{2t}.$$

Finally, p_3 is the solution of the IVP

$$p_3' = te^{2t} + 3p_3, \quad p_3(0) = 0.$$

Solving this IVP, we obtain

$$p_3(t) = -te^{2t} - e^{2t} + e^{3t}.$$

Hence

$$\begin{aligned} e^{At} &= p_1(t)M_0 + p_2(t)M_1 + p_3(t)M_2 \\ &= e^{2t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &+ \left(-te^{2t} - e^{2t} + e^{3t}\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t} - e^{2t} & 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

Hence a general solution x of (2.12) is given by

$$\begin{aligned} x(t) &= e^{At}c \\ &= \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t} - e^{2t} & 0 & e^{3t} \end{bmatrix} c \\ &= c_1\begin{bmatrix} e^{2t} \\ te^{2t} \\ e^{3t} - e^{2t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$. △

In the following theorem we give some properties of the matrix exponential.

Theorem 2.39 *Assume A and B are $n \times n$ constant matrices. Then*

(i) $\frac{d}{dt}e^{At} = Ae^{At}$, *for $t \in \mathbb{R}$,*

(ii) $\det\left[e^{At}\right] \neq 0$, *for $t \in \mathbb{R}$ and e^{At} is a fundamental matrix for* (2.7),

(iii) $e^{At}e^{As} = e^{A(t+s)}$, *for $t, s \in \mathbb{R}$,*

(iv) $\{e^{At}\}^{-1} = e^{-At}$, *for $t \in \mathbb{R}$ and, in particular,*

$$\{e^{A}\}^{-1} = e^{-A},$$

(v) *if $AB = BA$, then $e^{At}B = Be^{At}$, for $t \in \mathbb{R}$ and, in particular,*

$$e^{A}B = Be^{A},$$

(vi) *if $AB = BA$, then $e^{At}e^{Bt} = e^{(A+B)t}$, for $t \in \mathbb{R}$ and, in particular,*

$$e^{A}e^{B} = e^{(A+B)},$$

(vii) $e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots$, *for $t \in \mathbb{R}$,*

(viii) *if P is a nonsingular matrix, then $e^{PBP^{-1}} = Pe^{B}P^{-1}$.*

Proof The result (i) follows immediately from the definition of e^{At}.

Since e^{At} is the identity matrix at $t = 0$ and $\det(I) = 1 \neq 0$, we get from Corollary 2.24 that $\det(e^{At}) \neq 0$, for all $t \in \mathbb{R}$ and so (ii) holds.

We now prove that (iii) holds. Fix $s \in \mathbb{R}$, let t be a real variable and let

$$\Phi(t) := e^{At}e^{As} - e^{A(t+s)}.$$

Then

$$\begin{aligned}\Phi'(t) &= Ae^{At}e^{As} - Ae^{A(t+s)}\\ &= A\left[e^{At}e^{As} - e^{A(t+s)}\right]\\ &= A\Phi(t),\end{aligned}$$

for $t \in \mathbb{R}$. So Φ is a solution of the matrix equation $X' = AX$. Also, $\Phi(0) = e^{As} - e^{As} = 0$, so by the uniqueness theorem, Theorem 2.21, $\Phi(t) = 0$ for $t \in \mathbb{R}$. Hence

$$e^{At}e^{As} = e^{A(t+s)},$$

for $t \in \mathbb{R}$. Since $s \in \mathbb{R}$ is arbitrary, (iii) holds.

To show that (iv) holds, we get by using (iii)

$$\begin{aligned}e^{At}e^{-At} &= e^{At}e^{A(-t)}\\ &= e^{A(t+(-t))}\\ &= I,\end{aligned}$$

for $t \in \mathbb{R}$. This implies that

$$\{e^{At}\}^{-1} = e^{-At},$$

for $t \in \mathbb{R}$. Letting $t = 1$, we get that

$$\{e^{A}\}^{-1} = e^{-A},$$

and so (iv) holds.

The proof of (v) is similar to the proof of (iii) and is left as an exercise (see Exercise 2.39).

To prove (vi), assume $AB = BA$ and let

$$\Phi(t) := e^{At}e^{Bt} - e^{(A+B)t}.$$

Then using the product rule and using (iv),

$$\begin{aligned}\Phi'(t) &= Ae^{At}e^{Bt} + e^{At}Be^{Bt} - (A+B)e^{(A+B)t}\\ &= Ae^{At}e^{Bt} + Be^{At}e^{Bt} - (A+B)e^{(A+B)t}\\ &= (A+B)\left[e^{At}e^{Bt} - e^{(A+B)t}\right]\\ &= (A+B)\Phi(t),\end{aligned}$$

for $t \in \mathbb{R}$. Also, $\Phi(0) = I - I = 0$, so by the uniqueness theorem, Theorem 2.21, $\Phi(t) = 0$ for $t \in \mathbb{R}$. Hence

$$e^{(A+B)t} = e^{At}e^{Bt},$$

for $t \in \mathbb{R}$. Letting $t = 1$, we get that

$$e^{(A+B)} = e^A e^B$$

and hence (vi) holds.

We now prove (vii). It can be shown that the infinite series of matrices

$$I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots$$

converges for $t \in \mathbb{R}$ and that this infinite series of matrices can be differentiated term by term. Let

$$\Phi(t) := I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots,$$

for $t \in \mathbb{R}$. Then

$$\begin{aligned} \Phi'(t) &= A + A^2\frac{t}{1!} + A^3\frac{t^2}{2!} + \cdots \\ &= A\left[I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots\right] \\ &= A\Phi(t), \end{aligned}$$

for $t \in \mathbb{R}$. Since $\Phi(0) = I$, we have by the uniqueness theorem, Theorem 2.21, $\Phi(t) = e^{At}$, for $t \in \mathbb{R}$. Hence

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots,$$

for $t \in \mathbb{R}$ and so (vii) holds.

Finally, (viii) follows from (vii). □

Theorem 2.40 (Variation of Constants Formula) *Assume that A is an $n \times n$ continuous matrix function on an interval I, b is a continuous $n \times 1$ vector function on I, and Φ is a fundamental matrix for* (2.3)*. Then the solution of the IVP*

$$x' = A(t)x + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I$ and $x_0 \in \mathbb{R}^n$, is given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds,$$

for $t \in I$.

Proof The uniqueness of the solution of the given IVP follows from Theorem 2.3. Let Φ be a fundamental matrix for (2.3) and set

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds,$$

for $t \in I$. Then

$$\begin{aligned} x'(t) &= \Phi'(t)\Phi^{-1}(t_0)x_0 + \Phi'(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds + \Phi(t)\Phi^{-1}(t)b(t) \\ &= A(t)\Phi(t)\Phi^{-1}(t_0)x_0 + A(t)\Phi(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds + b(t) \\ &= A(t)\left[\Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds\right] + b(t) \\ &= A(t)x(t) + b(t), \end{aligned}$$

for $t \in I$. Also,

$$\begin{aligned} x(t_0) &= \Phi(t_0)\Phi^{-1}(t_0)x_0 \\ &= x_0. \end{aligned}$$

□

Corollary 2.41 *Assume A is an $n \times n$ constant matrix and b is a continuous $n \times 1$ vector function on an interval I. Then the solution x of the IVP*

$$x' = Ax + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I, x_0 \in \mathbb{R}^n$ is given by (the reader should compare this to the variation of constants formula in Theorem 1.6)

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)}b(s)\,ds,$$

for $t \in I$.

Proof Letting $\Phi(t) = e^{At}$ in the general variation of constants formula in Theorem 2.40, we get, using the fact that $\{e^{At}\}^{-1} = e^{-At}$,

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)b(s)\,ds \\ &= e^{At}e^{-At_0}x_0 + e^{At}\int_{t_0}^t e^{-As}b(s)\,ds \\ &= e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)}b(s)\,ds, \end{aligned}$$

for $t \in I$. □

In the next theorem we see under a strong assumption (2.13) we can find a fundamental matrix for the nonautonomous case $x' = A(t)x$.

Theorem 2.42 *Assume $A(t)$ is a continuous $n \times n$ matrix function on an interval I. If*

$$A(t)A(s) = A(s)A(t) \tag{2.13}$$

for all $t, s \in I$, then

$$\Phi(t) := e^{\int_{t_0}^t A(s)\, ds}$$

(defined by it's power series (2.14)*) is a fundamental matrix for $x' = A(t)x$ on I.*

Proof Let

$$\Phi(t) := e^{\int_{t_0}^t A(s)\, ds} := \sum_{k=0}^{\infty} \frac{1}{k!} \left[\int_{t_0}^t A(s) ds \right]^k. \tag{2.14}$$

We leave it to the reader to show that the infinite series of matrices in (2.14) converges uniformly on each closed subinterval of I. Differentiating term by term and using the fact that (2.13) implies that $A(t) \int_{t_0}^t A(s)\, ds = \int_{t_0}^t A(s)\, ds\, A(t)$ we get

$$\begin{aligned} \Phi'(t) &= A(t) \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left[\int_{t_0}^t A(s) ds \right]^{k-1} \\ &= A(t) \sum_{k=0}^{\infty} \frac{1}{k!} \left[\int_{t_0}^t A(s) ds \right]^k \\ &= A(t)\Phi(t). \end{aligned}$$

Since

$$\det \Phi(t_0) = \det I = 1 \neq 0,$$

we have that $\Phi(t)$ is a fundamental matrix for the vector differential equation $x' = A(t)x$. □

Note that condition (2.13) holds when either $A(t)$ is a diagonal matrix or $A(t) \equiv A$, a constant matrix.

Example 2.43 Use the variation of constants formula to solve the IVP

$$x' = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix},$$

we have from Example 2.36 that

$$e^{At} = e^{2t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}.$$

From the variation of constants formula given in Corollary 2.41,

$$\begin{aligned}
x(t) &= e^{2t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&+ \int_0^t e^{2(t-s)}\begin{bmatrix} 1-t+s & t-s \\ -t+s & 1+t-s \end{bmatrix}\begin{bmatrix} e^{2s} \\ 2e^{2s} \end{bmatrix} ds \\
&= e^{2t}\begin{bmatrix} 2-2t+t \\ -2t+1+t \end{bmatrix} + e^{2t}\int_0^t \begin{bmatrix} 1-t+s+2t-2s \\ -t+s+2+2t-2s \end{bmatrix} ds \\
&= e^{2t}\begin{bmatrix} 2-t \\ 1-t \end{bmatrix} + e^{2t}\int_0^t \begin{bmatrix} 1+t-s \\ 2+t-s \end{bmatrix} ds \\
&= e^{2t}\begin{bmatrix} 2-t \\ 1-t \end{bmatrix} + e^{2t}\begin{bmatrix} t+\frac{t^2}{2} \\ 2t+\frac{t^2}{2} \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} 2+\frac{t^2}{2} \\ 1+t+\frac{t^2}{2} \end{bmatrix},
\end{aligned}$$

for $t \in \mathbb{R}$. △

Example 2.44 Use the variation of constants formula to solve the IVP

$$x' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2te^t \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Let

$$A := \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We will find e^{At} by an alternate method. Note that

$$e^{At} = e^{(B+C)t},$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $BC = CB$,

$$e^{At} = e^{(B+C)t} = e^{Bt}e^{Ct}.$$

It follows from Exercise 2.41 that

$$e^{Bt} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}.$$

Also,

$$\begin{aligned} e^{Ct} &= I + C\frac{t}{1!} + C^2\frac{t^2}{2!} + C^3\frac{t^3}{3!} + \cdots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2!} \\ &= \begin{bmatrix} 1 & 2t & 2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} e^{At} &= e^{Bt}e^{Ct} \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 2t & 2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 2te^t & 2t^2e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}. \end{aligned}$$

From the variation of constants formula given in Corollary 2.41

$$\begin{aligned} x(t) &= \begin{bmatrix} e^t & 2te^t & 2t^2e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &\quad + \int_0^t \begin{bmatrix} e^{t-s} & 2(t-s)e^{t-s} & 2(t-s)^2e^{t-s} \\ 0 & e^{t-s} & 2(t-s)e^{t-s} \\ 0 & 0 & e^{t-s} \end{bmatrix} \begin{bmatrix} 2se^s \\ 0 \\ 0 \end{bmatrix} ds \\ &= \begin{bmatrix} 2e^t + 2te^t \\ e^t \\ 0 \end{bmatrix} + e^t \int_0^t \begin{bmatrix} 2s \\ 0 \\ 0 \end{bmatrix} ds \\ &= \begin{bmatrix} 2e^t + 2te^t \\ e^t \\ 0 \end{bmatrix} + \begin{bmatrix} t^2e^t \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2e^t + 2te^t + t^2e^t \\ e^t \\ 0 \end{bmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$. △

Example 2.45 Use Theorem 2.40 and Theorem 2.42 to help you solve the IVP

$$x' = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{2}{t} \end{bmatrix} x + \begin{bmatrix} t^2 \\ t \end{bmatrix}, \quad x(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

on the interval $I = (0, \infty)$. By Theorem 2.42,

$$\begin{aligned} \Phi(t) = e^{\int_1^t A(s)\,ds} &= e^{\int_1^t \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{2}{s} \end{bmatrix} ds} \\ &= e^{\begin{bmatrix} \ln t & 0 \\ 0 & 2\ln t \end{bmatrix}} \\ &= \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} \end{aligned}$$

is a fundamental matrix for

$$x' = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{2}{t} \end{bmatrix} x.$$

Using the variation of constants formula in Theorem 2.40 we get the solution $x(t)$ of the given IVP is given by

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(1)x_0 + \Phi(t)\int_1^t \Phi^{-1}(s)b(s)ds \\ &= \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} I \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} \int_1^t \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix} \begin{bmatrix} s^2 \\ s \end{bmatrix} ds \\ &= \begin{bmatrix} t \\ -2t^2 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} \int_1^t \begin{bmatrix} s \\ \frac{1}{s} \end{bmatrix} ds \\ &= \begin{bmatrix} t \\ -2t^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t^3 - \frac{1}{2}t \\ t^2 \ln t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}t^3 + \frac{1}{2}t \\ -2t^2 + t^2 \ln t \end{bmatrix}. \end{aligned}$$

$\triangle$

Definition 2.46 Let $\mathbb{R}^n$ denote the set of all $n \times 1$ constant vectors. Then a *norm* on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ having the following properties:

(i) $\|x\| \geq 0$, for all $x \in \mathbb{R}^n$,
(ii) $\|x\| = 0$ iff $x = 0$,
(iii) $\|cx\| = |c| \cdot \|x\|$ for all $c \in \mathbb{R}$, $x \in \mathbb{R}^n$,
(iv) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.

Example 2.47 Three important examples of norms on $\mathbb{R}^n$ are

(i) the *Euclidean norm* (l_2 norm) defined by

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

(ii) the *maximum norm* (l_∞ norm) defined by

$$\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\},$$

(iii) the *traffic norm* (l_1 norm) defined by

$$\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|.$$

We leave it to the reader to check that these examples are actually norms. △

A sequence $\{x^k\}_{n=1}^{\infty}$ in $\mathbb{R}^n$ is said to converge with respect to a norm $\|\cdot\|$ on $\mathbb{R}^n$ if there is a $x_0 \in \mathbb{R}^n$ such that

$$\lim_{k\to\infty} \|x^k - x_0\| = 0.$$

It can be shown that a sequence in $\mathbb{R}^n$ converges with respect to one norm on $\mathbb{R}^n$ iff it converges with respect to any norm on $\mathbb{R}^n$ (think about this for the three norms just listed). Unless otherwise stated we will let $\|\cdot\|$ represent any norm on $\mathbb{R}^n$.

Definition 2.48 Let $\phi(t, x_0) = e^{At}x_0$ denote the solution of the IVP

$$x' = Ax, \quad x(0) = x_0.$$

(i) We say that the trivial solution of (2.7) is *stable on* $[0,\infty)$ provided given any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|y_0\| < \delta$, then

$$\|\phi(t, y_0)\| < \epsilon,$$

for $t \geq 0$.

(ii) We say that the trivial solution of (2.7) is *unstable on* $[0,\infty)$ provided it is not stable on $[0,\infty)$.

(iii) We say that the trivial solution of (2.7) is *globally asymptotically stable on* $[0,\infty)$ provided it is stable on $[0,\infty)$ and for any $y_0 \in \mathbb{R}^n$,

$$\lim_{t\to\infty} \phi(t, y_0) = 0.$$

The next theorem shows how the eigenvalues of A determine the stability of the trivial solution of $x' = Ax$.

Theorem 2.49 (Stability Theorem) *Assume A is an $n\times n$ constant matrix.*

(i) *If A has an eigenvalue with positive real part, then the trivial solution is unstable on* $[0,\infty)$.

(ii) *If all the eigenvalues of A with zero real parts are simple (multiplicity one) and all other eigenvalues of A have negative real parts, then the trivial solution is stable on* $[0,\infty)$.

(iii) *If all the eigenvalues of A have negative real parts, then the trivial solution of $x' = Ax$ is globally asymptotically stable on* $[0,\infty)$.

Proof We will prove only part (iii) of this theorem. The proof of part (i) is Exercise 2.46. A proof of part (ii) can be based on the Putzer algorithm (Theorem 2.35) and is similar to the proof of part (iii) of this theorem.

We now prove part (iii) of this theorem. By part (ii) the trivial solution is stable on $[0,\infty)$, so it remains to show that every solution approaches the zero vector as $t \to \infty$. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A, and choose

$\delta > 0$ so that $\Re(\lambda_k) \le -\delta < 0$ for $k = 1, 2, \cdots, n$. Let $x_0 \in \mathbb{R}^n$, then by Putzer's algorithm (Theorem 2.35)

$$\phi(t, x_0) = e^{At}x_0 = \sum_{k=0}^{n-1} p_{k+1}(t)M_k x_0.$$

Since p_1 solves the IVP

$$p_1' = \lambda_1 p_1, \quad p_1(0) = 1,$$

we get

$$|p_1(t)| = |e^{\lambda_1 t}| \le e^{-\delta t},$$

for all $t \ge 0$. Next, p_2 satisfies

$$p_2' = \lambda_2 p_2 + p_1, \quad p_2(0) = 0,$$

so by the variation of constants formula (Theorem 1.6 or Corollary 2.41),

$$p_2(t) = \int_0^t e^{\lambda_2(t-s)}e^{\lambda_1 s}\, ds.$$

Since $|e^{\lambda_1 s}| \le e^{-\delta s}$ and $|e^{\lambda_2(t-s)}| \le e^{-\delta(t-s)}$ for $t \ge s$,

$$|p_2(t)| \le \int_0^t e^{-\delta(t-s)}e^{-\delta s}\, ds = te^{-\delta t}.$$

We can continue in this way (by induction) to show

$$|p_k(t)| \le \frac{t^{k-1}}{(k-1)!}e^{-\delta t},$$

for $k = 1, 2, \cdots, n$. It follows that each $p_k(t) \to 0$, as $t \to \infty$, and consequently that

$$\phi(t, x_0) = \sum_{k=0}^{n-1} p_{k+1}(t)M_k x_0 \to 0,$$

as $t \to \infty$. □

Example 2.50 In the example involving the vibration of two coupled masses (see Example 2.18), we showed that the eigenvalues were $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $\lambda = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$. Since all four eigenvalues have negative real parts, the origin is globally asymptotically stable. △

Example 2.51 Determine the stability of the trivial solution of

$$x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x,$$

on $[0, \infty)$.

The characteristic equation is

$$\lambda^2 + 1 = 0,$$

and hence the eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$. Since both eigenvalues have zero real parts and both eigenvalues are simple, the trivial solution is stable on $[0, \infty)$. △

Example 2.52 Determine the stability of the trivial solution of

$$x' = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x,$$

on $[0, \infty)$.

The characteristic equation is

$$(\lambda^2 + 4\lambda + 5)(\lambda - 1) = 0$$

and hence the eigenvalues are $\lambda_1 = -2 + i$, $\lambda_2 = -2 - i$, and $\lambda_3 = 1$. Since the eigenvalue λ_3 has a positive real part, the trivial solution is unstable on $[0, \infty)$. △

2.4 Induced Matrix Norm

Definition 2.53 Assume that $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Let M_n denote the set of all $n \times n$ real matrices. We define the matrix norm on M_n induced by the vector norm by

$$\|A\| := \sup_{\|x\|=1} \|Ax\|.$$

Note that we use the same notation for the vector norm and the corresponding matrix norm, since from context it should be clear which norm we mean. To see that $\|A\|$ is well defined, assume there is a sequence of points $\{x_k\}$ in $\mathbb{R}^n$ with $\|x_k\| = 1$ such that

$$\lim_{k \to \infty} \|Ax_k\| = \infty.$$

Since $\|x_k\| = 1$, $k = 1, 2, 3, \cdots$, there is a convergent subsequence $\{x_{k_j}\}$. Let

$$x_0 := \lim_{j \to \infty} x_{k_j}.$$

But then

$$\lim_{j \to \infty} \|Ax_{k_j}\| = \|Ax_0\|,$$

which gives us a contradiction. Using a similar argument (see Exercise 2.51), it is easy to prove that

$$\|A\| := \max_{\|x\|=1} \|Ax\|.$$

This induced matrix norm is a norm on M_n (see Exercise 2.50).

Theorem 2.54 *The matrix norm induced by the vector norm $\|\cdot\|$ is given by*

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

In particular,

$$\|Ax\| \leq \|A\| \cdot \|x\|,$$

for all $x \in \mathbb{R}^n$.

Proof This result follows from the following statement. If $x \neq 0$, then

$$\frac{\|Ax\|}{\|x\|} = \|A\left(\frac{x}{\|x\|}\right)\| = \|Ay\|,$$

where $y = \frac{x}{\|x\|}$ is a unit vector. □

Theorem 2.55 *The matrix norm induced by the traffic norm (l_1 norm) is given by*

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

Proof Let $\|\cdot\|_1$ be the traffic norm on $\mathbb{R}^n$, let $A \in M_n$, $x \in \mathbb{R}^n$, and consider

$$\begin{aligned}
\|Ax\|_1 &= \| \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \cdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{pmatrix} \|_1 \\
&= |\sum_{j=1}^{n} a_{1j}x_j| + \cdots + |\sum_{j=1}^{n} a_{nj}x_j| \\
&\le \sum_{j=1}^{n} |a_{1j}||x_j| + \cdots + \sum_{j=1}^{n} |a_{nj}||x_j| \\
&= \sum_{i=1}^{n} |a_{i1}||x_1| + \cdots + \sum_{i=1}^{n} |a_{in}||x_n| \\
&\le \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \sum_{j=1}^{n} |x_j| \\
&= \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \|x\|_1.
\end{aligned}$$

It follows that

$$\|A\|_1 \le \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

We next prove the reverse inequality. To see this, pick j_0, $1 \le j_0 \le n$, so that

$$\sum_{i=1}^{n} |a_{ij_0}| = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

Let e_{j_0} be the unit vector in $\mathbb{R}^n$ in the j_0 direction. Then

$$\begin{aligned}
\|A\|_1 &= \max_{\|x\|=1} \|Ax\|_1 \\
&\geq \|Ae_{j_0}\|_1 \\
&= \| \begin{pmatrix} a_{1j_0} \\ \cdots \\ a_{nj_0} \end{pmatrix} \|_1 \\
&= \sum_{i=1}^{n} |a_{ij_0}| \\
&= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.
\end{aligned}$$

It follows that

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

□

The proof of the next theorem is Exercise 2.53.

Theorem 2.56 *The matrix norm induced by the maximum vector norm is given by*

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

One could also prove the following result.

Theorem 2.57 *The matrix norm induced by the Euclidean vector norm is given by*

$$\|A\|_2 = \sqrt{\lambda_0},$$

where λ_0 is the largest eigenvaue of A^TA.

Note that if I is the identity matrix, then

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1.$$

This fact will be used frequently throughout the remainder of this section.

We now define the Lozinski measure of a matrix A and we will see that this measure can sometimes be used in determining the global asymptotic stability of the vector equation $x' = Ax$.

Definition 2.58 Assume $\|\cdot\|$ is a matrix norm on M_n induced by a vector norm $\|\cdot\|$ on $\mathbb{R}^n$. Then we define the Lozinski measure $\mu : M_n \to \mathbb{R}$ by

$$\mu(A) = \lim_{h\to 0+} \frac{\|I + hA\| - 1}{h}.$$

The limit in the preceding definition exists by Exercise 2.49.

The Lozinski measure is not a norm, but satisfies the properties given in the following theorem.

Theorem 2.59 (Properties of μ) *The function μ satisfies the following properties:*

(i) $\mu(\alpha A) = \alpha\mu(A)$, *for* $A \in M_n$, $\alpha \geq 0$,
(ii) $|\mu(A)| \leq \|A\|$ *for* $A \in M_n$,
(iii) $\mu(A+B) \leq \mu(A) + \mu(B)$ *for* $A, B \in M_n$,
(iv) $|\mu(A) - \mu(B)| \leq \|A - B\|$ *for* $A, B \in M_n$,
(v) $Re(\lambda) \leq \mu(A)$ *for all eigenvalues* λ *of* A.

Proof Since

$$\mu(0A) = \mu(0) = \lim_{h\to 0+} \frac{\|I + h0\| - 1}{h} = 0 = \|0\|,$$

part (i) is true for $\alpha = 0$. Now assume that $\alpha > 0$ and consider

$$\begin{aligned}
\mu(\alpha A) &= \lim_{h\to 0+} \frac{\|I + h\alpha A\| - 1}{h} \\
&= \lim_{k\to 0+} \frac{\|I + kA\| - 1}{\frac{k}{\alpha}} \\
&= \alpha \lim_{k\to 0+} \frac{\|I + kA\| - 1}{k} \\
&= \alpha\mu(A)
\end{aligned}$$

and hence (i) is true. Part (ii) follows from the inequalities

$$|(\|I + hA\| - 1)| \leq \|I\| + h\|A\| - 1 = h\|A\|$$

and the definition of $\mu(A)$. To see that (iii) is true, consider

$$\begin{aligned}
\mu(A+B) &= \lim_{h\to 0+} \frac{\|I + h(A+B)\| - 1}{h} \\
&= \lim_{k\to 0+} \frac{\|I + \frac{k}{2}(A+B)\| - 1}{\frac{k}{2}} \\
&= \lim_{k\to 0+} \frac{\|2I + k(A+B)\| - 2}{k} \\
&\leq \lim_{k\to 0+} \frac{\|I + kA\| + \|I + kB\| - 2}{k} \\
&= \lim_{k\to 0+} \frac{\|I + kA\| - 1}{k} + \lim_{k\to 0+} \frac{\|I + kB\| - 1}{k} \\
&= \mu(A) + \mu(B).
\end{aligned}$$

To prove (iv), note that

$$\begin{aligned}
\mu(A) &= \mu(A - B + B) \\
&\leq \mu(A - B) + \mu(B).
\end{aligned}$$

Hence

$$\mu(A) - \mu(B) \le \|A - B\|.$$

Interchanging A and B, we have that

$$\mu(B) - \mu(A) \le \|B - A\| = \|A - B\|.$$

Altogether we get the desired result

$$|\mu(A) - \mu(B)| \le \|A - B\|.$$

Finally, to prove (v), let λ_0 be an eigenvalue of A. Let x_0 be a corresponding eigenvector with $\|x_0\| = 1$. Consider

$$\begin{aligned}
\lim_{h\to 0+} \frac{\|(I + hA)x_0\| - 1}{h} &= \lim_{h\to 0+} \frac{\|(1 + h\lambda_0)x_0\| - 1}{h} \\
&= \lim_{h\to 0+} \frac{|1 + h\lambda_0| \cdot \|x_0\| - 1}{h} \\
&= \lim_{h\to 0+} \frac{|1 + h\lambda_0| - 1}{h} \\
&= \lim_{h\to 0+} \left(\frac{|1 + h\lambda_0| - 1}{h} \cdot \frac{|1 + h\lambda_0| + 1}{|1 + h\lambda_0| + 1} \right) \\
&= \lim_{h\to 0+} \frac{|1 + h\lambda_0|^2 - 1}{h[|1 + h\lambda_0| + 1]} \\
&= \lim_{h\to 0+} \frac{(1 + h\lambda_0)(1 + h\bar{\lambda}_0) - 1}{h[|1 + h\lambda_0| + 1]} \\
&= \lim_{h\to 0+} \frac{(\lambda_0 + \bar{\lambda}_0)h + h^2\lambda_0\bar{\lambda}_0}{h[|1 + h\lambda_0| + 1]} \\
&= \frac{\lambda_0 + \bar{\lambda}_0}{2} \\
&= Re(\lambda_0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{h\to 0+} \frac{\|(I + hA)x_0\| - 1}{h} &\le \lim_{h\to 0+} \frac{\|(I + hA)\| \cdot \|x_0\| - 1}{h} \\
&= \lim_{h\to 0+} \frac{\|I + hA\| - 1}{h} \\
&= \mu(A).
\end{aligned}$$

So we have that (v) holds. □

Corollary 2.60 *If $\mu(A) < 0$, then the trivial solution of the vector equation $x' = Ax$ is globally asymptotically stable.*

Proof This follows from part (v) of Theorem 2.59 and Theorem 2.49. □

Theorem 2.61 *The following hold:*

(i) *if* μ_1 *corresponds to the traffic vector norm, then, for* $A \in M_n$

$$\mu_1(A) = \max_{1 \le j \le n} \left\{ a_{jj} + \sum_{i=1, i \ne j}^{n} |a_{ij}| \right\};$$

(ii) *if* μ_∞ *corresponds to the maximum vector norm, then*

$$\mu_\infty(A) = \max_{1 \le i \le n} \left\{ a_{ii} + \sum_{j=1, j \ne i}^{n} |a_{ij}| \right\};$$

(iii) *if* μ_2 *corresponds to the Euclidean norm, then*

$$\mu_2(A) = \max\{\lambda : \lambda \text{ is an eigenvalue of } \tfrac{1}{2}(A + A^T)\}.$$

Proof We will prove part (i) here. The proof of part (ii) is Exercise 2.59 and the proof of part (iii) is nontrivial, but is left to the reader. Using Theorem 2.55

$$\begin{aligned}
\mu_1(A) &= \lim_{h \to 0+} \frac{\|I + hA\|_1 - 1}{h} \\
&= \lim_{h \to 0+} \max_{1 \le j \le n} \frac{\sum_{i=1, i \ne j}^{n} |ha_{ij}| + |1 + ha_{jj}| - 1}{h} \\
&= \max_{1 \le j \le n} \lim_{h \to 0+} \frac{h \sum_{i=1, i \ne j}^{n} |a_{ij}| + ha_{jj}}{h} \\
&= \max_{1 \le j \le n} \left\{ a_{jj} + \sum_{i=1, i \ne j}^{n} |a_{ij}| \right\}.
\end{aligned}$$

□

Example 2.62 The trivial solution of the vector equation

$$x' = \begin{pmatrix} -3.3 & .3 & 3 & -.4 \\ 1 & -2 & 1 & 1.3 \\ -1.2 & .4 & -5 & .2 \\ -1 & .8 & .5 & -2 \end{pmatrix} x \tag{2.15}$$

is globally asymptotically stable because

$$\mu_1(A) = -.1 < 0,$$

where A is the coefficient matrix in (2.15). Note that $\mu_\infty(A) = 1.3 > 0$. △

2.5 Floquet Theory

Differential equations involving periodic functions play an important role in many applications. Let's consider the linear system

$$x' = A(t)x,$$

where the $n \times n$ matrix function A is a continuous, periodic function with smallest positive period ω. Such systems are called Floquet systems and the study of Floquet systems is called Floquet theory. A natural question is whether a Floquet system has a periodic solution with period ω. Although this is not neccessarily the case, it is possible to characterize all the solutions of such systems and to give conditions under which a periodic solution does exist. Fortunately, the periodic system turns out to be closely related to a linear system with constant coefficients, so the properties of these systems obtained in earlier sections can be applied. In particular, we can easily answer questions about the stability of periodic systems.
First we need some preliminary results about matrices.

Theorem 2.63 (Jordan Canonical Form) *If A is an $n \times n$ constant matrix, then there is a nonsingular $n \times n$ constant matrix P so that $A = PJP^{-1}$, where J is a block diagonal matrix of the form*

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix},$$

where either J_i is the 1×1 matrix $J_i = [\lambda_i]$ or

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_i & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix},$$

$1 \leq i \leq k$, and the λ_i's are the eigenvalues of A.

Proof We will only prove this theorem for 2×2 matrices A. For a proof of the general result, see Horn and Johnson [**24**]. There are two cases:

Case 1: A has two linearly independent eigenvectors x^1 and x^2.

In this case we have eigenpairs λ_1, x^1 and λ_2, x^2 of A, where λ_1 and λ_2 might be the same. Let P be the matrix with columns x^1 and x^2, that is,

$$P = [x^1 \; x^2].$$

Then

$$\begin{aligned} AP &= A[x^1 \; x^2] \\ &= [Ax^1 \; Ax^2] \\ &= [\lambda_1 x^1 \; \lambda_2 x^2] \\ &= PJ, \end{aligned}$$

where

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Hence

$$A = PJP^{-1},$$

where P is of the correct form.

Case 2: Assume A has only one linearly independent eigenvector x^1.

Let v be a vector that is independent of x^1. By the Cayley-Hamilton Theorem (Theorem 2.33),

$$(A - \lambda_1 I)(A - \lambda_1 I)v = 0,$$

so

$$(A - \lambda_1 I)v = cx^1,$$

for some $c \neq 0$. Define $x^2 = v/c$, so that

$$(A - \lambda_1 I)x^2 = x^1.$$

Set

$$P = [x^1 \; x^2].$$

Then

$$\begin{aligned} AP &= A[x^1 \; x^2] \\ &= [Ax^1 \; Ax^2] \\ &= [\lambda_1 x^1 \; \lambda_1 x^2 + x^1] \\ &= PJ, \end{aligned}$$

where

$$J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

Hence

$$A = PJP^{-1},$$

where P is of the correct form.

□

Theorem 2.64 (Log of a Matrix) *If C is an $n \times n$ nonsingular matrix, then there is a matrix B such that*

$$e^B = C.$$

Proof We will just prove this theorem for 2×2 matrices. Let μ_1, μ_2 be the eigenvalues of C. Since C is nonsingular $\mu_1, \mu_2 \neq 0$. First we prove the result for two special cases.

Case 1. Assume

$$C = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

In this case we seek a diagonal matrix

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix},$$

so that $e^B = C$. That is, we want to choose b_1 and b_2 so that

$$e^B = \begin{bmatrix} e^{b_1} & 0 \\ 0 & e^{b_2} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}.$$

Hence we can just take

$$B = \begin{bmatrix} \ln \mu_1 & 0 \\ 0 & \ln \mu_2 \end{bmatrix}.$$

Case 2. Assume

$$C = \begin{bmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{bmatrix}.$$

In this case we seek a matrix B of the form

$$B = \begin{bmatrix} a_1 & a_2 \\ 0 & a_1 \end{bmatrix},$$

so that $e^B = C$. That is, we want to choose a_1 and a_2 so that

$$e^B = \begin{bmatrix} e^{a_1} & a_2 e^{a_1} \\ 0 & e^{a_1} \end{bmatrix} = \begin{bmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{bmatrix}.$$

Hence we can just take

$$B = \begin{bmatrix} \ln \mu_1 & \frac{1}{\mu_1} \\ 0 & \ln \mu_1 \end{bmatrix}.$$

Case 3. C is an arbitrary 2×2 nonsingular constant matrix. By the Jordan canonical form theorem (Theorem 2.63) there is a nonsingular matrix P such that $C = PJP^{-1}$, where

$$J = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{bmatrix}.$$

By the previous two cases there is a matrix B_1 so that

$$e^{B_1} = J.$$

Let

$$B := PB_1P^{-1};$$

then, using part (viii) in Theorem 2.39,

$$e^B = e^{PB_1P^{-1}} = Pe^{B_1}P^{-1} = C.$$

□

Example 2.65 Find a log of the matrix

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

The characteristic equation for C is

$$\lambda^2 - 6\lambda + 5 = 0$$

and so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 5$. The Jordan canonical form (see Theorem 2.63) of C is

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Eigenpairs of C are

$$1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad 5, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

From the proof of Theorem 2.64, if we let

$$P := \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix},$$

then

$$PB_1P^{-1}$$

is a log of C provided B_1 is a log of J. Note that by the proof of Theorem 2.64,

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & \ln 5 \end{bmatrix}$$

is a log of J. Hence a log of C is given by

$$\begin{aligned} B &= PB_1P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \ln 5 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}\ln 5 & \frac{1}{4}\ln 5 \\ \frac{3}{4}\ln 5 & \frac{3}{4}\ln 5 \end{bmatrix}. \end{aligned}$$

$\triangle$

Before we state and prove Floquet's theorem (Theorem 2.67) we give a motivating example.

Example 2.66 Consider the scalar differential equation

$$x' = (\sin^2 t)\, x.$$

A general solution of this differential equation is

$$\phi(t) = ce^{\frac{1}{2}t - \frac{1}{4}\sin(2t)}.$$

Note that even though the coefficient function in our differential equation is periodic with minimum period π, the only period π solution of our differential equation is the trivial solution. But note that all nontrivial solutions are of the form

$$\phi(t) = p(t)e^{bt},$$

where $p(t) = ce^{-\frac{1}{4}\sin(2t)} \neq 0$, for all $t \in \mathbb{R}$, is a continuously differentiable function on $\mathbb{R}$ that is periodic with period π (which is the minimum positive period of the coefficient function in our differential equation) and $b = \frac{1}{2}$ is a constant (1×1 matrix). △

Floquet's theorem shows that any fundamental matrix for the Floquet system $x' = A(t)x$ can be written in a form like shown in Example 2.66.

Theorem 2.67 (Floquet's Theorem) *If Φ is a fundamental matrix for the Floquet system $x' = A(t)x$, where the matrix function A is continuous on $\mathbb{R}$ and has minimum positive period ω, then the matrix function Ψ defined by $\Psi(t) := \Phi(t + \omega)$, $t \in \mathbb{R}$ is also a fundamental matrix. Furthermore there is a nonsingular, continuously differentiable $n \times n$ matrix function P which is periodic with period ω and an $n \times n$ constant matrix B (possibly complex) so that*

$$\Phi(t) = P(t)e^{Bt},$$

for all $t \in \mathbb{R}$.

Proof Assume Φ is a fundamental matrix for the Floquet system $x' = A(t)x$. Define the matrix function Ψ by

$$\Psi(t) = \Phi(t + \omega),$$

for $t \in \mathbb{R}$. Then

$$\begin{aligned}\Psi'(t) &= \Phi'(t + \omega)\\ &= A(t + \omega)\Phi(t + \omega)\\ &= A(t)\Psi(t).\end{aligned}$$

Since $\det \Psi(t) \neq 0$ for all $t \in \mathbb{R}$, Ψ is a fundamental matrix of $x' = A(t)x$. Hence the first statement of the theorem holds.

Since Φ and Ψ are fundamental matrices for $x' = A(t)x$, Theorem 2.31 implies that there is a nonsingular constant matrix C so that

$$\Phi(t + \omega) = \Phi(t)C, \quad \text{for } t \in \mathbb{R}.$$

By Theorem 2.64 there is a matrix B such that

$$e^{B\omega} = C.$$

Then define the matrix function P by

$$P(t) = \Phi(t)e^{-Bt},$$

for $t \in \mathbb{R}$. Obviously, P is a continuously differentiable, nonsingular matrix function on $\mathbb{R}$. To see that P is periodic with period ω consider

$$\begin{aligned}P(t + \omega) &= \Phi(t + \omega)e^{-Bt-B\omega}\\ &= \Phi(t)Ce^{-B\omega}e^{-Bt}\\ &= \Phi(t)e^{-Bt}\\ &= P(t).\end{aligned}$$

Finally note that

$$\Phi(t) = P(t)e^{Bt},$$

for all $t \in \mathbb{R}$. $\square$

Definition 2.68 Let Φ be a fundamental matrix for the Floquet system $x' = A(t)x$. Then the eigenvalues μ of

$$C := \Phi^{-1}(0)\Phi(\omega)$$

are called the *Floquet multipliers* of the Floquet system $x' = A(t)x$.

Fundamental matrices for $x' = A(t)x$ are not unique, so we wonder if Floquet multipliers are well defined in Definition 2.68. To see that Floquet multipliers are well defined, let Φ and Ψ be fundamental matrices for the Floquet system $x' = A(t)x$ and let

$$C := \Phi^{-1}(0)\Phi(\omega)$$

and let

$$D := \Psi^{-1}(0)\Psi(\omega).$$

We want to show that C and D have the same eigenvalues. Since Φ and Ψ are fundamental matrices of $x' = A(t)x$, Theorem 2.31 yields a nonsingular constant matrix M such that

$$\Psi(t) = \Phi(t)M$$

for all $t \in \mathbb{R}$. It follows that

$$\begin{aligned} D &= \Psi^{-1}(0)\Psi(\omega) \\ &= M^{-1}\Phi^{-1}(0)\Phi(\omega)M \\ &= M^{-1}CM. \end{aligned}$$

Therefore, C and D are similar matrices (see Exercise 2.14) and hence have the same eigenvalues. Hence Floquet multipliers are well defined.

Example 2.69 Find the Floquet multipliers for the scalar differential equation

$$x' = (\sin^2 t)\, x.$$

In Example 2.66 we saw that a nontrivial solution of this differential equation is

$$\phi(t) = e^{\frac{1}{2}t - \frac{1}{4}\sin(2t)}.$$

Hence

$$c := \phi^{-1}(0)\phi(\pi) = e^{\frac{\pi}{2}}$$

and so $\mu = e^{\frac{\pi}{2}}$ is the Floquet multiplier for this differential equation. $\triangle$

Example 2.70 Find the Floquet multipliers for the Floquet system

$$x' = \begin{bmatrix} 1 & 1 \\ 0 & \frac{(\cos t + \sin t)}{(2 + \sin t - \cos t)} \end{bmatrix} x.$$

Solving this equation first for x_2 and then x_1, we get that

$$x_2(t) = \beta(2 + \sin t - \cos t),$$

$$x_1(t) = \alpha e^t - \beta(2 + \sin t),$$

for $t \in \mathbb{R}$. It follows that a fundamental matrix for our Floquet system is

$$\Phi(t) = \begin{bmatrix} -2 - \sin t & e^t \\ 2 + \sin t - \cos t & 0 \end{bmatrix}.$$

Since

$$C = \Phi^{-1}(0)\Phi(2\pi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi} \end{bmatrix},$$

the Floquet multipliers are $\mu_1 = 1$ and $\mu_2 = e^{2\pi}$. △

Theorem 2.71 *Let $\Phi(t) = P(t)e^{Bt}$ be as in Floquet's theorem (Theorem 2.67). Then x is a solution of the Floquet system $x' = A(t)x$ iff the vector function y defined by $y(t) = P^{-1}(t)x(t)$, $t \in \mathbb{R}$ is a solution of $y' = By$.*

Proof Assume x is a solution of the Floquet system $x' = A(t)x$. Then

$$x(t) = \Phi(t)x_0,$$

for some $n \times 1$ constant vector x_0. Let $y(t) = P^{-1}(t)x(t)$. Then

$$\begin{aligned} y(t) &= P^{-1}(t)\Phi(t)x_0 \\ &= P^{-1}(t)P(t)e^{Bt}x_0 \\ &= e^{Bt}x_0, \end{aligned}$$

which is a solution of

$$y' = By.$$

Conversely, assume y is a solution of $y' = By$ and set

$$x(t) = P(t)y(t).$$

Since y is a solution of $y' = By$, there is an $n \times 1$ constant vector y_0 such that

$$y(t) = e^{Bt}y_0.$$

It follows that

$$\begin{aligned} x(t) &= P(t)y(t) \\ &= P(t)e^{Bt}y_0 \\ &= \Phi(t)y_0, \end{aligned}$$

which is a solution of the Floquet system $x' = A(t)x$. □

Theorem 2.72 *Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Floquet multipliers of the Floquet system $x' = A(t)x$. Then the trivial solution is*

(i) globally asymptotically stable on $[0, \infty)$ iff $|\mu_i| < 1$, $1 \leq i \leq n$;

(ii) stable on $[0, \infty)$ provided $|\mu_i| \leq 1$, $1 \leq i \leq n$ and whenever $|\mu_i| = 1$, μ_i is a simple eigenvalue;

(iii) unstable on $[0,\infty)$ provided there is an i_0, $1 \leq i_0 \leq n$, such that $|\mu_{i_0}| > 1$.

Proof We will just prove this theorem for the two-dimensional case. Let $\Phi(t) = P(t)e^{Bt}$ and C be as in Floquet's theorem. Recall that in the proof of Floquet's theorem, B was picked so that

$$e^{B\omega} = C.$$

By the Jordan canonical form theorem (Theorem 2.63) there are matrices M and J so that

$$B = MJM^{-1},$$

where either

$$J = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \quad \text{or} \quad J = \begin{bmatrix} \rho_1 & 1 \\ 0 & \rho_1 \end{bmatrix},$$

where ρ_1, ρ_2 are the eigenvalues of B. It follows that

$$\begin{aligned} C &= e^{B\omega} \\ &= e^{MJM^{-1}\omega} \\ &= Me^{J\omega}M^{-1} \\ &= MKM^{-1}, \end{aligned}$$

where either

$$K = \begin{bmatrix} e^{\rho_1\omega} & 0 \\ 0 & e^{\rho_2\omega} \end{bmatrix}, \quad \text{or} \quad K = \begin{bmatrix} e^{\rho_1\omega} & \omega e^{\rho_1\omega} \\ 0 & e^{\rho_1\omega} \end{bmatrix}.$$

Since the eigenvalues of K are the same (see Exercise 2.14) as the eigenvalues of C, we get that the Floquet multipliers are

$$\mu_i = e^{\rho_i\omega},$$

$i = 1, 2$, where it is possible that $\rho_1 = \rho_2$. Since

$$|\mu_i| = e^{Re(\rho_i)\omega},$$

we have that

$$\begin{aligned} |\mu_i| < 1 \quad &\text{iff} \quad Re(\rho_i) < 0 \\ |\mu_i| = 1 \quad &\text{iff} \quad Re(\rho_i) = 0 \\ |\mu_i| > 1 \quad &\text{iff} \quad Re(\rho_i) > 0. \end{aligned}$$

By Theorem 2.71 the equation

$$x(t) = P(t)y(t)$$

gives a one-to-one correspondence between solutions of the Floquet system $x' = A(t)x$ and $y' = By$. Note that there is a constant $Q_1 > 0$ so that

$$\|x(t)\| = \|P(t)y(t)\| \leq \|P(t)\|\|y(t)\| \leq Q_1\|y(t)\|,$$

for $t \in \mathbb{R}$ and since $y(t) = P^{-1}(t)x(t)$ there is a constant $Q_2 > 0$ such that

$$\|y(t)\| = \|P^{-1}(t)x(t)\| \leq \|P^{-1}(t)\|\|x(t)\| \leq Q_2\|x(t)\|,$$

for $t \in \mathbb{R}$. The conclusions of this theorem then follow from Theorem 2.49. $\square$

Theorem 2.73 *The number μ_0 is a Floquet multiplier of the Floquet system $x' = A(t)x$ iff there is a nontrivial solution x such that*

$$x(t+\omega) = \mu_0 x(t),$$

for all $t \in \mathbb{R}$. Consequently, the Floquet system has a nontrivial periodic solution of period ω if and only if $\mu_0 = 1$ is a Floquet multiplier.

Proof First assume μ_0 is a Floquet multiplier of the Floquet system $x' = A(t)x$. Then μ_0 is an eigenvalue of

$$C := \Phi^{-1}(0)\Phi(\omega),$$

where Φ is a fundamental matrix of $x' = A(t)x$. Let x_0 be an eigenvector corresponding to μ_0 and define the vector function x by

$$x(t) = \Phi(t)x_0, \quad t \in \mathbb{R}.$$

Then x is a nontrivial solution of $x' = A(t)x$ and

$$\begin{aligned} x(t+\omega) &= \Phi(t+\omega)x_0 \\ &= \Phi(t)Cx_0 \\ &= \Phi(t)\mu_0 x_0 \\ &= \mu_0 x(t), \end{aligned}$$

for all $t \in \mathbb{R}$.

Conversely, assume there is a nontrivial solution x such that

$$x(t+\omega) = \mu_0 x(t),$$

for all $t \in \mathbb{R}$. Let Ψ be a fundamental matrix of our Floquet system, then

$$x(t) = \Psi(t)y_0,$$

for all $t \in \mathbb{R}$ for some nontrivial vector y_0. By Floquet's theorem the matrix function $\Psi(\cdot + \omega)$ is also a fundamental matrix. Hence $x(t+\omega) = \mu_0 x(t)$, so $\Psi(t+\omega)y_0 = \mu_0 \Psi(t)y_0$ and therefore

$$\Psi(t)Dy_0 = \Psi(t)\mu_0 y_0,$$

where $D := \Psi^{-1}(0)\Psi(\omega)$. It follows that

$$Dy_0 = \mu_0 y_0,$$

and so μ_0 is an eigenvalue of D and hence is a Floquet multiplier of our Floquet system. $\square$

Theorem 2.74 *Assume $\mu_1, \mu_2, \cdots, \mu_n$ are the Floquet multipliers of the Floquet system $x' = A(t)x$. Then*

$$\mu_1 \mu_2 \cdots \mu_n = e^{\int_0^\omega \operatorname{tr}[A(t)]\, dt}.$$

Proof Let Φ be the solution of the matrix IVP

$$X' = A(t)X, \quad X(0) = I.$$

Then Φ is a fundamental matrix for $x' = A(t)x$ and

$$C := \Phi^{-1}(0)\Phi(\omega) = \Phi(\omega).$$

Then we get that

$$\begin{aligned} \mu_1\mu_2\cdots\mu_n &= \det C \\ &= \det\Phi(\omega) \\ &= e^{\int_0^\omega \operatorname{tr}[A(t)]\,dt}\det\Phi(0) \\ &= e^{\int_0^\omega \operatorname{tr}[A(t)]\,dt}, \end{aligned}$$

where we have used Liouville's theorem (Theorem 2.23), which is the desired result.

□

Example 2.75 (Hill's Equation) Consider the scalar differential equation (Hill's equation)

$$y'' + q(t)y = 0,$$

where we assume that q is a continuous periodic function on $\mathbb{R}$ with minimum positive period ω. G. W. Hill [**21**] considered equations of this form when he studied planetary motion. There are many applications of Hill's equation in mechanics, astronomy, and electrical engineering. For a more thorough study of Hill's equation than is given here, see [**35**]. Writing Hill's equation as a system in the standard way, we get the Floquet system

$$x' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} x.$$

By the Floquet multipliers of Hill's equation we mean the Floquet multipliers of the preceding Floquet system. It follows from Theorem 2.74 that the Floquet multipliers of Hill's equation satisfy

$$\mu_1\mu_2 = 1.$$

△

Example 2.76 (Mathieu's Equation) A special case of Hill's equation is Mathieu's equation,

$$y'' + (\alpha + \beta\cos t)y = 0,$$

where α and β are real parameters. We will assume that $\beta \neq 0$. Note that the Floquet multipliers of Mathieu's equation depend on α and β. From Theorem 2.74 the Floquet multipliers of Mathieu's equation satisfy

$$\mu_1(\alpha,\beta)\mu_2(\alpha,\beta) = 1.$$

Let

$$\gamma(\alpha,\beta) := \mu_1(\alpha,\beta) + \mu_2(\alpha,\beta).$$

Then the Floquet multipliers of Mathieu's equation satisfy the quadratic equation

$$\mu^2 - \gamma(\alpha, \beta)\mu + 1 = 0.$$

In particular,

$$\mu_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}.$$

There are five cases to consider:

Case 1: $\gamma > 2$.

In this case the Floquet multipliers satisfy

$$0 < \mu_2 < 1 < \mu_1.$$

It then follows from Theorem 2.72 that the trivial solution of Mathieu's equation is unstable on $[0, \infty)$ in this case. Using Exercise 2.70, we can show in this case that there is a general solution of Mathieu's equation of the form

$$y(t) = c_1 e^{\sigma t} p_1(t) + c_2 e^{-\sigma t} p_2(t),$$

where $\sigma > 0$ and p_i, $i = 1, 2$, are continuously differentiable functions on $\mathbb{R}$ which are periodic with period 2π.

Case 2: $\gamma = 2$.

In this case

$$\mu_1 = \mu_2 = 1.$$

It follows, using Exercise 2.70, that there is a nontrivial solution of period 2π. It has been proved in [**35**] that there is a second linearly independent solution that is unbounded. In particular, the trivial solution is unstable on $[0, \infty)$.

Case 3: $-2 < \gamma < 2$.

In this case the Floquet multipliers are not real and $\mu_2 = \overline{\mu}_1$ In this case, using Exercise 2.70, there is a general solution of the form

$$y(t) = c_1 e^{i\sigma t} p_1(t) + c_2 e^{-i\sigma t} p_2(t),$$

where $\sigma > 0$ and the p_i, $i = 1, 2$ are continuously differentiable functions on $\mathbb{R}$ that are periodic with period 2π. In this case it then follows from Theorem 2.72 that the trivial solution is stable on $[0, \infty)$.

Case 4: $\gamma = -2$.

In this case

$$\mu_1 = \mu_2 = -1.$$

It follows from Exercise 2.67 that there is a nontrivial solution that is periodic with period 4π. It has been shown in [**35**] that there is a second linearly independent solution that is unbounded. In particular, the trivial solution is unstable on $[0, \infty)$.

Case 5: $\gamma < -2$.

In this case the Floquet multipliers satisfy

$$\mu_2 < -1 < \mu_1 < 0.$$

In this case it follows from Theorem 2.72 that the trivial solution is unstable on $[0, \infty)$. △

A very interesting fact concerning Mathieu's equation is that if $\beta > 0$ is fixed, then there are infinitely many intervals of α values, where the trivial solution of Mathieu's equation is alternately stable and unstable on $[0, \infty)$ (see [**35**]).

2.6 Exercises

2.1 Show that the characteristic equation for the constant coefficient scalar differential equation $y'' + ay' + by = 0$ is the same as the characteristic equation for the companion matrix of this differential equation.

2.2 Let $\mathbb{A}$ be the set of all continuous scalar functions on an interval I and define $M : \mathbb{A} \to \mathbb{A}$ by

$$Mx(t) = \int_a^t x(s)\, ds,$$

for $t \in I$, where a is a fixed point in I. Prove that M is a linear operator.

2.3 Determine in each case if the constant vectors are linearly dependent or linearly independent. Prove your answer.

(i)

$$\psi_1 = \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

(ii)

$$\psi_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(iii)

$$\psi_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 1 \\ 13 \\ 2 \\ -1 \end{bmatrix}$$

2.4 Determine if the scalar functions ϕ_1, ϕ_2 defined by $\phi_1(t) = \ln(t)$, $\phi_2(t) = \ln(t^2)$ for $t \in (0, \infty)$ are linearly dependent or linearly independent on $(0, \infty)$. Prove your answer.

2.5 Determine if the given functions are linearly dependent or linearly independent on the given interval I. Prove your answers.

(i) $x_1(t) = 4 \sin t,\ x_2(t) = 7 \sin(-t), \quad I = \mathbb{R}$
(ii) $x_1(t) = 2 \sin(2t),\ x_2(t) = -3 \cos(2t),\ x_3(t) = 8, \quad I = \mathbb{R}$
(iii) $x_1(t) = e^{3t} t,\ x_2(t) = e^{-2t}, \quad I = \mathbb{R}$
(iv) $x_1(t) = e^{3t} t,\ x_2(t) = e^{3t+4}, \quad I = \mathbb{R}$

(v) $x_1(t) = \sin t,\ x_2(t) = \cos t,\ x_3(t) = \sin(t + \frac{\pi}{4}), \quad I = \mathbb{R}$

2.6 Prove that if $x_1, x_2, \cdots, x_k$ are k functions defined on I and if one of them is identically zero on I, then $x_1, x_2, \cdots, x_k$ are linearly dependent on I.

2.7 Prove that two functions x, y defined on I are linearly dependent on I iff one of them is a constant times the other.

2.8 Determine if the scalar functions ϕ_1, ϕ_2, ϕ_3 defined by $\phi_1(t) = 3$, $\phi_2(t) = 3\sin^2 t$, $\phi_3(t) = 4\cos^2 t$ for $t \in \mathbb{R}$ are linearly dependent or linearly independent on $\mathbb{R}$. Prove your answer.

2.9 Determine if the scalar functions ϕ_1, ϕ_2 defined by $\phi_1(t) = t^2 + 1$, $\phi_2(t) = 2t^2 + 3t - 7$ for $t \in \mathbb{R}$ are linearly independent or linearly dependent on $\mathbb{R}$. Verify your answer.

2.10 Determine if the scalar functions $\phi_1, \phi_2, \phi_3, \phi_4$ defined by $\phi_1(t) = \sin^2 t$, $\phi_2(t) = \cos^2 t$, $\phi_3(t) = \tan^2 t$, $\phi_4(t) = \sec^2 t$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ are linearly dependent or linearly independent on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Prove your answer.

2.11 Find four two dimensional vector functions that are linearly independent on $\mathbb{R}$ and prove that they are linearly independent on $\mathbb{R}$.

2.12 Find two scalar functions that are linearly independent on $\mathbb{R}$, but linearly dependent on $(0, \infty)$. Prove your answer.

2.13 Find the inverse of each of the following matrices:

(i) $A = \begin{bmatrix} 2 & -6 \\ -1 & 4 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

2.14 Two $n \times n$ matrices A, B are said to be *similar* if there is a nonsingular $n \times n$ matrix M such that $A = M^{-1}BM$. Prove that similar matrices have the same eigenvalues.

2.15 Show that if an $n \times n$ matrix has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (not necessarily distinct) with linearly independent eigenvectors

$$x_1, x_2 \cdots, x_n,$$

respectively, then a general solution of $x' = Ax$ is $x(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$.

2.16 Show that if a square matrix has all zeros either above or below the main diagonal, then the numbers down the diagonal are the eigenvalues of the matrix.

2.17 If λ_0 is an eigenvalue of A, find an eigenvalue of

(i) A^T (the transpose of A)

(ii) A^n, where n is a positive integer
(iii) A^{-1}, provided $\det(A) \neq 0$

Be sure to verify your answers.

2.18 Show that any $n+1$ solutions of (2.3) on (c,d) are linearly dependent on (c,d).

2.19 Show that the characteristic equation for any 2×2 constant matrix A is

$$\lambda^2 - \mathrm{tr}[A]\lambda + \det(A) = 0,$$

where $\mathrm{tr}[A]$ is defined in Definition 2.22. Use this result to find the characteristic equation for each of the following matrices:

(i) $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

2.20 Using Theorem 2.14, solve the following differential equations:

(i) $x' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} x$

2.21 Solve the IVP

$$x' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x.$$

2.22 Work each of the following:

(i) Find eigenpairs for the matrix $A = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix} x$

(ii) Use your answer in (i) to find two linearly independent solutions (prove that they are linearly independent) of $x' + Ax$ on $\mathbb{R}$.

(iii) Use your answer in (ii) to find a fundamental matrix for $x' = Ax$.

(iv) Use your answer in (iii) and Theorem 2.31 to find e^{At}.

2.23 (Complex Eigenvalues) Using Theorem 2.14, solve the following differential equations:

(i) $x' = \begin{bmatrix} 0 & 9 \\ -9 & 0 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x$

2.24 Prove the last statement in Corollary 2.24.

2.25 Using Theorem 2.14, solve the following differential equations:

(i) $x' = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x$

2.26 Use Theorem 2.14 to find a fundamental matrix for

$$x' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} x.$$

2.27 Show that the matrix function Φ defined by

$$\Phi(t) := \begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix},$$

for $t \in (0, \infty)$ is a fundamental matrix for the vector differential equation

$$x' = \begin{bmatrix} 0 & 1 \\ -\frac{6}{t^2} & \frac{4}{t} \end{bmatrix} x.$$

2.28 Show that the matrix function Φ defined by

$$\Phi(t) := \begin{bmatrix} e^{-t} & 2 \\ 1 & e^t \end{bmatrix},$$

for $t \in \mathbb{R}$ is a fundamental matrix for the vector differential equation

$$x' = \begin{bmatrix} 1 & -2e^{-t} \\ e^t & -1 \end{bmatrix} x.$$

Find the solution satisfying the initial condition

$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

2.29 Verify the Cayley-Hamilton Theorem (Theorem 2.33) directly for the general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

2.30 Verify the Cayley-Hamilton Theorem (Theorem 2.33) directly for the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

2.31 Use the Putzer algorithm (Theorem 2.35) to find e^{At} for each of the following:

$$\text{(i)}\ \ A = \begin{bmatrix} 2 & 1 \\ -9 & -4 \end{bmatrix}$$

$$\text{(ii)}\ \ A = \begin{bmatrix} 10 & 4 \\ -9 & -2 \end{bmatrix}$$

$$\text{(iii)}\ \ A = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{bmatrix}$$

$$\text{(iv)}\ \ A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

2.32 (Multiple Eigenvalue) Use the Putzer algorithm (Theorem 2.35) to find e^{At} given that

$$A = \begin{bmatrix} -5 & 2 \\ -2 & -1 \end{bmatrix}.$$

2.33 (Complex Eigenvalues) Use the Putzer algorithm (Theorem 2.35) to find e^{At} for each of the following:

$$\text{(i)}\ \ A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$$

$$\text{(ii)}\ \ A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

$$\text{(iii)}\ \ A = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$$

2.34 Use the Putzer algorithm (Theorem 2.35) to help you solve each of the following differential equations:

$$\text{(i)}\ \ x' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x$$

$$\text{(ii)}\ \ x' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} x$$

$$\text{(iii)}\ \ x' = \begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix} x$$

$$\text{(iv)}\ \ x' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x$$

2.35 Solve each of the following differential equations:

(i) $x' = \begin{bmatrix} -4 & 3 \\ -2 & 3 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{bmatrix} x$

2.36 Work each of the following:

(i) Solve the differential equations for the vibrating system in Example 2.2 in case there is no friction: $m_1 = m_2 = k_1 = k_2 = k_3 = 1$ and $c = 0$.
(ii) Decide whether the trivial solution is stable in this case, and discuss the implications of your answer for the vibrating system.
(iii) Find the solution that satisfies the initial conditions $u(0) = 1$, $u'(0) = v(0) = v'(0) = 0$. Also, sketch a graph of the solution.

2.37 Work each of the following:

(i) Solve the differential equations for the vibrating system in Example 2.2 for the parameter values $m_1 = m_2 = k_1 = k_2 = k_3 = 1$, $c = 2$. (*Note*: The eigenvalues are not distinct in this case.)
(ii) Show that the trivial solution is globally asymptotically stable.

2.38 Find 2×2 matrices A and B such that

$$e^A e^B \neq e^{A+B}.$$

2.39 Show that if A and B are $n \times n$ constant matrices and $AB = BA$, then

$$e^{At} B = B e^{At},$$

for $t \in \mathbb{R}$. Also, show that $e^A B = B e^A$.

2.40 Use the Putzer algorithm (Theorem 2.35) to find

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t}.$$

Repeat the same problem using Theorem 2.39 part (*vii*). Use your answer and Theorem 2.39 part (*iii*) to prove the addition formulas for the trigonometric sine and cosine functions:

$$\begin{aligned} \sin(t+s) &= \sin(t)\cos(s) + \sin(s)\cos(t) \\ \cos(t+s) &= \cos(t)\cos(s) - \sin(t)\sin(s). \end{aligned}$$

2.41 Show that

$$e^{\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \ddots & 0 \\ 0 & 0 & \lambda_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & e^{\lambda_n} \end{bmatrix}.$$

2.42 Use the variation of constants formula to solve each of the following IVPs:

$$\text{(i)}\ x' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{(ii)}\ x' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ e^{3t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{(iii)}\ x' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ te^{2t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{(iv)}\ x' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2.43 Use the variation of constants formula to solve each of the following IVP's

$$\text{(i)}\ x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{(ii)}\ x' = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2.44 Use Theorem 2.40 and Theorem 2.42 to help you solve the IVP

$$x' = \begin{bmatrix} 2t & 0 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

2.45 Let $\epsilon > 0$. Graph $\{x \in \mathbb{R}^2 : \|x\| < \epsilon\}$ when

(i) $\|\cdot\|_2$ is the Euclidean norm (l_2 norm
(ii) $\|\cdot\|_\infty$ is the maximum norm (l_∞ norm)
(iii) $\|\cdot\|_1$ is the traffic norm (l_1 norm)

2.46 Prove: If A has an eigenvalue with positive real part, then there is a solution x of $x' = Ax$ so that $\|x(t)\| \to \infty$ as $t \to \infty$.

2.47 Determine the stability of the trivial solution for each of the following:

$$\text{(i)}\ x' = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} x$$

$$\text{(ii)}\ x' = \begin{bmatrix} 5 & 8 \\ -7 & -10 \end{bmatrix} x$$

(iii) $x' = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} x$

2.48 Determine the stability of the trivial solution for each of the following:

(i) $x' = \begin{bmatrix} -1 & -6 \\ 0 & 4 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} 0 & -9 \\ 4 & 0 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x$

2.49 Show that the limit in Definition 2.58 exists by showing, for $0 < \theta < 1, h > 0$, that $\|I + \theta hA\| \leq \theta\|I + hA\| + (1 - \theta)$, which implies that

$$\frac{\|I + \theta hA\| - 1}{\theta h} \leq \frac{\|I + hA\| - 1}{h}$$

and by showing that $\frac{\|I+hA\|-1}{h}$ is bounded below by $-\|A\|$.

2.50 Show that a matrix norm induced by a vector norm on $\mathbb{R}^n$ is a norm on M_n.

2.51 Show that the matrix norm on M_n induced by the vector norm $\|\cdot\|$ is given by

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

2.52 Show that if $A, B \in M_n$, then $\|AB\| \leq \|A\| \cdot \|B\|$.

2.53 Prove Theorem 2.56.

2.54 Find the matrix norm of the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

corresponding to the maximum norm $\|\cdot\|_\infty$, the traffic norm $\|\cdot\|_1$, and the Euclidean norm $\|\cdot\|_2$, respectively. Also, find $\mu_\infty(A)$, $\mu_1(A)$, $\mu_2(A)$.

2.55 Find the matrix norm of the matrix

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -3 & 2 \\ 0 & -1 & -5 \end{bmatrix}$$

corresponding the maximum norm $\|\cdot\|_\infty$ and the traffic norm $\|\cdot\|_1$, respectively. Also, find $\mu_\infty(A)$ and $\mu_1(A)$.

2.56 Determine if the trivial solution of the vector equation

$$x' = \begin{pmatrix} -2.2 & .3 & 1 & -.4 \\ 1.5 & -3 & -1 & .3 \\ -1.2 & .8 & -3 & .7 \\ -.1 & .3 & .3 & -1 \end{pmatrix} x$$

is globally asymptotically stable or not.

2.57 Determine if the trivial solution of the vector equation

$$x' = \begin{pmatrix} -2 & 1.7 & .1 & .1 \\ 1.8 & -4 & 1 & 1 \\ 0 & 1 & -2 & .5 \\ .4 & 1 & .5 & -2 \end{pmatrix} x$$

is globally asymptotically stable or not.

2.58 Work each of the following:

(i) Find the matrix norm of

$$A = \begin{pmatrix} -6 & 2 & -3 \\ 2.5 & -7 & 4 \\ -2 & 1 & -8 \end{pmatrix}$$

(ii) Find the matrix norm of A corresponding to the traffic norm $\|\cdot\|_1$.

(iii) Find the Lozinski measure $\mu_1(A)$.

(iv) What can you say about the stability of the trivial solution of $x' = Ax$?

2.59 Prove part (ii) of Theorem 2.61.

2.60 Find a log of each of the following matrices:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

2.61 Find the Floquet multipliers for each of the following scalar equations:

(i) $x' = (2\sin(3t))\, x$

(ii) $x' = (\cos^2 t)\, x$

(iii) $x' = (-1 + \sin(4t))\, x$

2.62 Assume the scalar function a is continuous on $\mathbb{R}$ and $a(t+\omega) = a(t)$ for $t \in \mathbb{R}$, where $\omega > 0$. Prove directly by solving the scalar differential equation $x' = a(t)x$ that every nontrivial solution is of the form $x(t) = p(t)e^{rt}$ for $t \in \mathbb{R}$, where $p(t) \neq 0$ for $t \in \mathbb{R}$ is a continuously differentiable function on $\mathbb{R}$ that is periodic with period ω and r is the average value of $a(t)$ on $[0, \omega]$, [i.e. $r = \frac{1}{\omega}\int_0^\omega a(t)\ dt$]. Show that $\mu = e^{r\omega}$ is the Floquet multiplier for $x' = a(t)x$. In particular, show that all solutions of $x' = a(t)x$ are periodic with period ω iff $\int_0^\omega a(t)\ dt = 0$.

2.63 Find the Floquet multipliers for each of the following Floquet systems:

(i) $x' = \begin{bmatrix} 3 & 0 \\ 0 & \sin^2 t \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} -1+\cos t & 0 \\ \cos t & -1 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} -1 & 0 \\ \sin\ t & -1 \end{bmatrix} x$

(iv) $x' = \begin{bmatrix} -3+2\sin t & 0 \\ 0 & -1 \end{bmatrix} x$

2.64 Determine the stability of the trivial solution for each of the differential equations in Exercise 2.61 by looking at the Floquet multipliers that you found in Exercise 2.61.

2.65 Find the Floquet multipliers for each of the following Floquet systems and by looking at the Floquet multipliers determine the stability of the trivial solution on $[0,\infty)$:

(i) $x' = \begin{bmatrix} -3+2\sin t & 0 \\ 0 & -1 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} \cos(2\pi t)+1 & 0 \\ \cos(2\pi t) & 1 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} -2 & 0 \\ \sin(2t) & -2 \end{bmatrix} x$

2.66 Show that

$$\Phi(t) = \begin{bmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{bmatrix}$$

is a fundamental matrix for the Floquet system

$$x' = \begin{bmatrix} -\sin(2t) & \cos(2t)-1 \\ \cos(2t)+1 & \sin(2t) \end{bmatrix} x.$$

Find the Floquet multipliers for this Floquet system. What do the Floquet multipliers tell you about the stability of the trivial solution?

2.67 Show that if $\mu = -1$ is a Floquet multiplier for the Floquet system $x' = A(t)x$, then there is a nontrivial periodic solution with period 2ω.

2.68 Without finding the Floquet multipliers, find the product of the Floquet multipliers of the Floquet system

$$x' = \begin{bmatrix} 2 & \sin^2 t \\ \cos^2 t & \sin t \end{bmatrix} x.$$

2.69 Show that

$$x(t) = \begin{bmatrix} -e^{\frac{t}{2}} \cos t \\ e^{\frac{t}{2}} \sin t \end{bmatrix}$$

is a solution of the Floquet system

$$x' = \begin{bmatrix} -1+(\frac{3}{2})\cos^2 t & 1-(\frac{3}{2})\cos t \sin t \\ -1-(\frac{3}{2})\sin t\ \cos t & -1+(\frac{3}{2})\sin^2 t \end{bmatrix} x.$$

Using just the preceding solution, find a Floquet multiplier for the proceding system. Without solving the system, find the other Floquet multiplier. What can you say about the stability of the trivial solution? Show that for all t the coefficient matrix in the proceding Floquet system has eigenvalues with negative real parts. This example is due to Markus and Yamabe [**36**].

2.70 (Floquet Exponents) Show that if μ_0 is a Floquet multiplier for the Floquet system $x' = A(t)x$, then there is a number ρ_0 (called a Floquet exponent) such that there is a nontrivial solution $x_0(t)$ of the Floquet system $x' = A(t)x$ of the form

$$x_0(t) = e^{\rho_0 t} p_0(t),$$

where p_0 is a continuously differentiable function on $\mathbb{R}$ that is periodic with period ω.