Chapter 1

First-Order Differential Equations

1.1 Basic Results

In the scientific investigation of any phenomenon, mathematical models are used to give quantitative descriptions and to derive numerical conclusions. These models can take many forms, and one of the most basic and useful is that of a differential equation, that is, an equation involving the rate of change of a quantity. For example, the rate of decrease of the mass of a radioactive substance, such as uranium, is known to be proportional to the present mass. If $m(t)$ represents the mass at time t, then we have that m satisfies the differential equation

$$m' = -km,$$

where k is a positive constant. This is an *ordinary differential equation* since it involves only the derivative of mass with respect to a single independent variable. Also, the equation is said to be of *first-order* because the highest order derivative appearing in the equation is first-order. An example of a second-order differential equation is given by Newton's second law of motion

$$mx'' = f(t, x, x'),$$

where m is the (constant) mass of an object moving along the x-axis and located at position $x(t)$ at time t, and $f(t, x(t), x'(t))$ is the force acting on the object at time t.

In this chapter, we will consider only first-order differential equations that can be written in the form

$$x' = f(t, x), \tag{1.1}$$

where $f : (a, b) \times (c, d) \to \mathbb{R}$ is continuous, $-\infty \leq a < b \leq \infty$, and $-\infty \leq c < d \leq \infty$.

Definition 1.1 We say that a function x is a *solution* of (1.1) on an interval $I \subset (a, b)$ provided $c < x(t) < d$ for $t \in I$, x is a continuously differentiable function on I, and

$$x'(t) = f(t, x(t)),$$

for $t \in I$.

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Definition 1.2 Let $(t_0, x_0) \in (a, b) \times (c, d)$ and assume f is continuous on $(a, b) \times (c, d)$. We say that the function x is a solution of the initial value problem (IVP)

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.2}$$

on an interval $I \subset (a, b)$ provided $t_0 \in I$, x is a solution of (1.1) on I, and

$$x(t_0) = x_0.$$

The point t_0 is called the initial point for the IVP (1.2) and the number x_0 is called the initial value for the IVP (1.2).

Note, for example, that if $(a, b) = (c, d) = (-\infty, \infty)$, then the function m defined by $m(t) = 400e^{-kt}$, $t \in (-\infty, \infty)$ is a solution of the IVP

$$m' = -km, \quad m(0) = 400$$

on the interval $I = (-\infty, \infty)$.

Solving an IVP can be visualized (see Figure 1) as finding a solution of the differential equation whose graph passes through the given point (t_0, x_0).

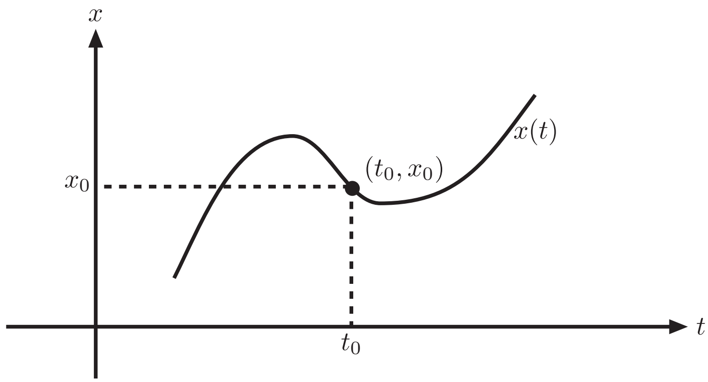

FIGURE 1. Graph of solution of IVP.

We state without proof the following important existence-uniqueness theorem for solutions of IVPs. Statements and proofs of some existence and uniqueness theorems will be given in Chapter 8.

Theorem 1.3 *Assume* $f : (a, b) \times (c, d) \to \mathbb{R}$ *is continuous, where* $-\infty \leq a < b \leq \infty$ *and* $-\infty \leq c < d \leq \infty$. *Let* $(t_0, x_0) \in (a, b) \times (c, d)$, *then the IVP* (1.2) *has a solution* x *with a maximal interval of existence* $(\alpha, \omega) \subset (a, b)$, *where* $\alpha < t_0 < \omega$. *If* $a < \alpha$, *then*

$$\lim_{t \to \alpha+} x(t) = c, \quad \textit{or} \quad \lim_{t \to \alpha+} x(t) = d$$

and if $\omega < b$, *then*

$$\lim_{t \to \omega-} x(t) = c, \quad \textit{or} \quad \lim_{t \to \omega-} x(t) = d.$$

If, in addition, the partial derivative of f with respect to x, f_x, is continuous on $(a, b) \times (c, d)$, then the preceding IVP has a unique solution.

We now give a couple of examples related to Theorem 1.3. The first example shows that if the hypothesis that the partial derivative f_x is continuous on $(a, b) \times (c, d)$ is not satisfied, then we might not have uniqueness of solutions of IVPs.

Example 1.4 (Nonuniqueness of Solutions to IVPs) If we drop an object from a bridge of height h at time $t = 0$ (assuming constant acceleration of gravity and negligible air resistance), then the height of the object after t units of time is $x(t) = -\frac{1}{2}gt^2 + h$. The velocity at time t is $x'(t) = -gt$, so by eliminating t, we are led to the IVP

$$x' = f(t, x) := -\sqrt{2g|h - x|}, \quad x(0) = h. \tag{1.3}$$

Note that this initial value problem has the constant solution $x(t) = h$, which corresponds to holding the object at bridge level without dropping it! We can find other solutions by separation of variables. If $h > x$, then

$$\int \frac{x'(t)\, dt}{\sqrt{2g(h - x(t))}} = -\int dt.$$

Computing the indefinite integrals and simplifying, we arrive at

$$x(t) = -\frac{g}{2}(t - C)^2 + h,$$

where C is an arbitrary constant. We can patch these solutions together with the constant solution to obtain for each $C > 0$

$$x(t) := \begin{cases} h, & \text{for} \quad t \leq C \\ h - \frac{g}{2}(t - C)^2, & \text{for} \quad t > C. \end{cases}$$

Thus for each $C > 0$ we have a solution of the IVP (1.3) that corresponds to releasing the object at time C. Note that the function f defined by $f(t, x) = -\sqrt{2g|h - x|}$ is continuous on $(-\infty, \infty) \times (-\infty, \infty)$ so by Theorem 1.3 the IVP (1.3) has a solution, but f_x does not exist when $x = h$ so we cannot use Theorem 1.3 to get that the IVP (1.3) has a unique solution. △

To see how bad nonuniqueness of solutions of initial value problems can be, we remark that in Hartman [**18**], pages 18–23, an example is given of a scalar equation $x' = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is continuous, where for every IVP (1.2) there is more than one solution on $[t_0, t_0 + \epsilon]$ and $[t_0 - \epsilon, t_0]$ for arbitrary $\epsilon > 0$.

The next example shows even if the hypotheses of Theorem 1.3 hold the solution of the IVP might only exist on a proper subinterval of (a, b).

Example 1.5 Let k be any nonzero constant. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(t, x) = kx^2$ is continuous and has a continuous partial derivative with respect to x. By Theorem 1.3, the IVP

$$x' = kx^2, \quad x(0) = 1$$

has a unique solution with a maximal interval of existence (α, ω). Using separation of variables, as in the preceding example, we find

$$x(t) = \frac{1}{C - kt}.$$

When we apply the initial condition $x(0) = 1$, we have $C = 1$, so that the solution of the IVP is

$$x(t) = \frac{1}{1 - kt},$$

with maximal interval of existence $(-\infty, 1/k)$ if $k > 0$ and $(1/k, \infty)$ if $k < 0$. In either case, $x(t)$ goes to infinity as t approaches $1/k$ from the appropriate direction.

Observe the implications of this calculation in case $x(t)$ is the density of some population at time t. If $k > 0$, then the density of the population is growing, and we conclude that growth cannot be sustained at a rate proportional to the square of density because the density would have to become infinite in finite time! On the other hand, if $k < 0$, the density is declining, and it is theoretically possible for the decrease to occur at a rate proportional to the square of the density, since $x(t)$ is defined for all $t > 0$ in this case. Note that $\lim_{t\to\infty} x(t) = 0$ if $k < 0$. △

1.2 First-Order Linear Equations

An important special case of a first-order differential equation is the first-order linear differential equation given by

$$x' = p(t)x + q(t), \tag{1.4}$$

where we assume that $p : (a, b) \to \mathbb{R}$ and $q : (a, b) \to \mathbb{R}$ are continuous functions, where $-\infty \le a < b \le \infty$. In Chapter 2, we will study systems of linear equations involving multiple unknown functions. The next theorem shows that a single linear equation can always be solved in terms of integrals.

Theorem 1.6 (Variation of Constants Formula) *If $p : (a, b) \to \mathbb{R}$ and $q : (a, b) \to \mathbb{R}$ are continuous functions, where $-\infty \le a < b \le \infty$, then the unique solution x of the IVP*

$$x' = p(t)x + q(t), \quad x(t_0) = x_0, \tag{1.5}$$

where $t_0 \in (a, b)$, $x_0 \in \mathbb{R}$, is given by

$$x(t) = e^{\int_{t_0}^t p(\tau)\, d\tau} x_0 + e^{\int_{t_0}^t p(\tau)\, d\tau} \int_{t_0}^t e^{-\int_{t_0}^s p(\tau)\, d\tau} q(s)\, ds,$$

$t \in (a, b)$.

Proof Here the function f defined by $f(t, x) = p(t)x + q(t)$ is continuous on $(a, b) \times (-\infty, \infty)$ and $f_x(t, x) = p(t)$ is continuous on $(a, b) \times (-\infty, \infty)$. Hence by Theorem 1.3 the IVP (1.5) has a unique solution with a maximal interval of existence $(\alpha, \omega) \subset (a, b)$ [the existence and uniqueness of the solution of the IVP (1.5) and the fact that this solution exists on the whole interval (a, b) follows from Theorem 8.65]. Let

$$x(t) := e^{\int_{t_0}^{t} p(\tau)\, d\tau} x_0 + e^{\int_{t_0}^{t} p(\tau)\, d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} p(\tau)\, d\tau} q(s)\, ds$$

for $t \in (a, b)$. We now show that x is the solution of the IVP (1.5) on the whole interval (a, b). First note that $x(t_0) = x_0$ as desired. Also,

$$\begin{aligned} x'(t) &= p(t) e^{\int_{t_0}^{t} p(\tau)\, d\tau} x_0 + p(t) e^{\int_{t_0}^{t} p(\tau)\, d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} p(\tau) d\tau} q(s)\, ds + q(t) \\ &= p(t) \left[e^{\int_{t_0}^{t} p(\tau)\, d\tau} x_0 + e^{\int_{t_0}^{t} p(\tau)\, d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} p(\tau) d\tau} q(s)\, ds \right] + q(t) \\ &= p(t) x(t) + q(t) \end{aligned}$$

for $t \in (a, b)$. □

In Theorem 2.40, we generalize Theorem 1.6 to the vector case. We now give an application of Theorem 1.6.

Example 1.7 (Newton's Law of Cooling) Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose that the object has an initial temperature of 40 degrees. If the temperature of the surrounding medium is $70 + 20e^{-2t}$ degrees after t minutes and the constant of proportionality is $k = -2$, then the initial value problem for the temperature $x(t)$ of the object at time t is

$$x' = -2(x - 70 - 20e^{-2t}), \quad x(0) = 40.$$

By the variation of constants formula, the temperature of the object after t minutes is

$$\begin{aligned} x(t) &= 40 e^{\int_0^t -2 d\tau} + e^{\int_0^t -2 d\tau} \int_0^t e^{\int_0^s 2 d\tau} (140 + 40e^{-2s})\, ds \\ &= 40e^{-2t} + e^{-2t} \int_0^t \left(140e^{2s} + 40\right)\, ds \\ &= 40e^{-2t} + e^{-2t}[70(e^{2t} - 1) + 40t] \\ &= 10(4t - 3)e^{-2t} + 70. \end{aligned}$$

Sketch the graph of x. Does the temperature of the object exceed 70 degrees at any time t? △

1.3 Autonomous Equations

If, in equation (1.1), f depends only on x, we get the autonomous differential equation

$$x' = f(x). \tag{1.6}$$

We always assume $f : \mathbb{R} \to \mathbb{R}$ is continuous and usually we assume its derivative is also continuous. The fundamental property of autonomous differential equations is that translating any solution of the autonomous differential equation along the t-axis produces another solution.

Theorem 1.8 *If x is a solution of the autonomous differential equation* (1.6) *on an interval (a, b), where $-\infty \leq a < b \leq \infty$, then for any constant c, the function y defined by $y(t) := x(t-c)$, for $t \in (a+c, b+c)$ is a solution of* (1.6) *on $(a + c, b + c)$.*

Proof Assume x is a solution of the autonomous differential equation (1.6) on (a, b); then x is continuously differentiable on (a, b) and

$$x'(t) = f(x(t)),$$

for $t \in (a, b)$. Replacing t by $t - c$ in this last equation, we get that

$$x'(t - c) = f(x(t - c)),$$

for $t \in (a + c, b + c)$. By the chain rule of differentiation we get that

$$\frac{d}{dt}[x(t - c)] = f(x(t - c)),$$

for $t \in (a + c, b + c)$. Hence if $y(t) := x(t - c)$ for $t \in (a + c, b + c)$, then y is continuously differentiable on $(a + c, b + c)$ and we get the desired result that

$$y'(t) = f(y(t)),$$

for $t \in (a + c, b + c)$. □

Definition 1.9 If $f(x_0) = 0$ we say that x_0 is an *equilibrium point* for the differential equation (1.6). If, in addition, there is a $\delta > 0$ such that $f(x) \neq 0$ for $|x - x_0| < \delta$, $x \neq x_0$, then we say x_0 is an *isolated equilibrium point.*

Note that if x_0 is an equilibrium point for the differential equation (1.6), then the constant function $x(t) = x_0$ for $t \in \mathbb{R}$ is a solution of (1.6) on $\mathbb{R}$.

Example 1.10 (Newton's Law of Cooling) Consider again Newton's law of cooling as in Example 1.7, where in this case the temperature of the surrounding medium is a constant 70 degrees. Then we have that the temperature $x(t)$ of the object at time t satisfies the differential equation

$$x' = -2(x - 70).$$

Note that $x = 70$ is the only equilibrium point. All solutions can be written in the form

$$x(t) = De^{-2t} + 70,$$

where D is an arbitrary constant. If we translate a solution by a constant amount c along the t-axis, then

$$x(t - c) = De^{-2(t-c)} + 70 = De^{2c}e^{-2t} + 70$$

is also a solution, as predicted by Theorem 1.8. Notice that if the temperature of the object is initially greater than 70 degrees, then the temperature will decrease and approach the equilibrium temperature 70 degrees as t goes to infinity. Temperatures starting below 70 degrees will increase toward the limiting value of 70 degrees. A simple graphical representation of this behavior is a "phase line diagram," (see Figure 2) showing the equilibrium point and the direction of motion of the other solutions.

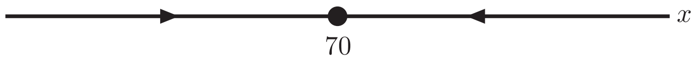

Figure 2. Phase line diagram of $x' = -2(x - 70)$.

△

Definition 1.11 Let ϕ be a solution of (1.6) with maximal interval of existence (α, ω). Then the set

$$\{\phi(t) : t \in (\alpha, \omega)\}$$

is called an *orbit* for the differential equation (1.6).

Note that the orbits for

$$x' = -2(x - 70)$$

are the sets

$$(-\infty, 70), \quad \{70\}, \quad (70, \infty).$$

A convenient way of thinking about phase line diagrams is to consider $x(t)$ to be the position of a point mass moving along the x-axis and $x'(t) = f(x(t))$ to be its velocity. The phase line diagram then gives the direction of motion (as determined by the sign of the velocity). An orbit is just the set of all locations of a continuous motion.

Theorem 1.12 *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then two orbits of* (1.6) *are either disjoint sets or are the same set.*

Proof Let ϕ_1 and ϕ_2 be solutions of (1.6). We will show that if there are points t_1, t_2 such that

$$\phi_1(t_1) = \phi_2(t_2),$$

then the orbits corresponding to ϕ_1 and ϕ_2 are the same. Let

$$x(t) := \phi_1(t - t_2 + t_1);$$

then by Theorem 1.8 we have that x is a solution of (1.6). Since

$$x(t_2) = \phi_1(t_1) = \phi_2(t_2),$$

we have by the uniqueness theorem (Theorem 1.3) that x and ϕ_2 are the same solutions. Hence $\phi_1(t - t_2 + t_1)$ and $\phi_2(t)$ correspond to the same solution. It follows that the orbits corresponding to ϕ_1 and ϕ_2 are the same. □

Example 1.13 (Logistic Growth) The logistic law of population growth (Verhulst [**52**], 1838) is

$$N' = rN\left(1 - \frac{N}{K}\right), \tag{1.7}$$

where N is the number of individuals in the population, $r(1 - N/K)$ is the *per capita growth rate* that declines with increasing population, and $K > 0$ is the *carrying capacity* of the environment. With $r > 0$, we get the phase line diagram in Figure 3. What are the orbits of the differential equation in this case?

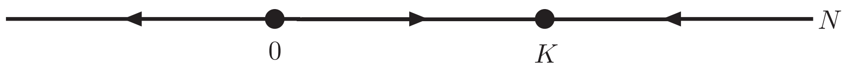

Figure 3. Phase line diagram of $N' = rN(1 - N/K)$.

We can use the phase line diagram to sketch solutions of the logistic equation. In order to make the graphs more accurate, let's first calculate the second derivative of N by differentiating both sides of the differential equation.

$$\begin{aligned} N'' &= rN'\left(1 - \frac{N}{K}\right) - r\frac{NN'}{K} \\ &= rN'\left(1 - \frac{2N}{K}\right) \\ &= r^2N\left(1 - \frac{N}{K}\right)\left(1 - \frac{2N}{K}\right). \end{aligned}$$

It follows that $N''(t) > 0$ if either $N(t) > K$ or $0 < N(t) < K/2$ and $N''(t) < 0$ if either $N(t) < 0$ or $K/2 < N(t) < K$. With all this in mind, we get the graph of some of the solutions of the logistic (Verhulst) equation in Figure 4.

△

Phase line diagrams are a simple geometric device for analyzing the behavior of solutions of autonomous equations. In later chapters we will study higher dimensional analogues of these diagrams, and it will be useful to have a number of basic geometric concepts for describing solution behavior. The following definitions contain some of these concepts for the one-dimensional case.

Definition 1.14 Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then we let $\phi(\cdot, x_0)$ denote the solution of the IVP

$$x' = f(x), \quad x(0) = x_0.$$

Definition 1.15 We say that an equilibrium point x_0 of the differential equation (1.6) is *stable* provided given any $\epsilon > 0$ there is a $\delta > 0$ such that

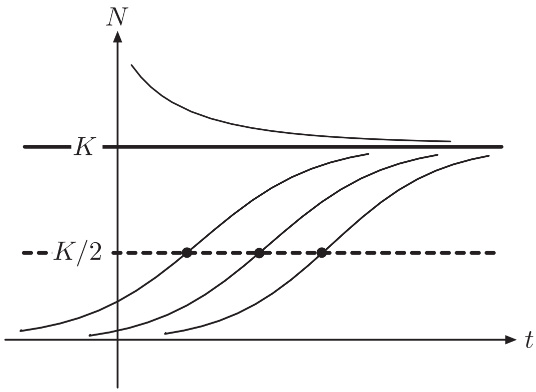

Figure 4. Graph of some solutions of $N' = rN(1 - N/K)$.

whenever $|x_1 - x_0| < \delta$ it follows that the solution $\phi(\cdot, x_1)$ exists on $[0, \infty)$ and

$$|\phi(t, x_1) - x_0| < \epsilon,$$

for $t \geq 0$. If, in addition, there is a $\delta_0 > 0$ such that $|x_1 - x_0| < \delta_0$ implies that

$$\lim_{t \to \infty} \phi(t, x_1) = x_0,$$

then we say that the equilibrium point x_0 is *asymptotically stable*. If an equilibrium point is not stable, then we say that it is *unstable*.

For the differential equation $N' = rN(1 - N/K)$ the equilibrium point $N_1 = 0$ is unstable and the equilibrium point $N_2 = K$ is asymptotically stable (see Figures 3 and 4).

Definition 1.16 We say that F is a *potential energy function* for the differential equation (1.6) provided

$$f(x) = -F'(x).$$

Theorem 1.17 *If F is a potential energy function for* (1.6)*, then $F(x(t))$ is strictly decreasing along any nonconstant solution x. Also, x_0 is an equilibrium point of* (1.6) *iff $F'(x_0) = 0$. If x_0 is an isolated equilibrium point of* (1.6) *such that F has a local minimum at x_0, then x_0 is asymptotically stable.*

Proof Assume F is a potential energy function for (1.6), assume x is a nonconstant solution of (1.6), and consider

$$\begin{aligned} \frac{d}{dt}F(x(t)) &= F'(x(t))x'(t) \\ &= -f^2(x(t)) \\ &< 0. \end{aligned}$$

Hence the potential energy function F is strictly decreasing along nonconstant solutions. Since $f(x_0) = 0$ iff $F'(x_0) = 0$, x_0 is an equilibrium point of (1.6) iff $F'(x_0) = 0$.

Let x_0 be an isolated equilibrium point of (1.6) such that F has a local minimum at x_0, and choose an interval $(x_0 - \delta, x_0 + \delta)$ such that $F'(x) > 0$ on $(x_0, x_0 + \delta)$ and $F'(x) < 0$ on $(x_0 - \delta, x_0)$. Suppose $x_1 \in (x_0, x_0 + \delta)$. Then $F(\phi(t, x_1))$ is strictly decreasing, so $\phi(t, x_1)$ is decreasing, remains in the interval $(x_0, x_0 + \delta)$, and converges to some limit $l \geq x_0$. We will show that $l = x_0$ by assuming $l > x_0$ and obtaining a contradiction. If $l > x_0$, then there is a positive constant C so that $F'(\phi(t, x_1)) \geq C$ for $t \geq 0$ (Why is the right maximal interval of existence for the solution $\phi(t, x_1)$ the interval $[0, \infty)$?). But

$$\phi(t, x_1) - x_1 = \int_0^t \left(-F'(\phi(s, x_1))\right)\, ds \leq -Ct,$$

for $t \geq 0$, which implies that $\phi(t, x_1) \to -\infty$ as $t \to \infty$, a contradiction. We conclude that $\phi(t, x_1) \to x_0$ as $t \to \infty$. Since the case $x_1 \in (x_0 - \delta, x_0)$ is similar, we have that x_0 is asymptotically stable. □

Example 1.18 By finding a potential energy function for

$$x' = -2(x - 70),$$

draw the phase line diagram for this differential equation.

Here a potential energy function is given by

$$\begin{aligned} F(x) &= -\int_0^x f(u)\, du \\ &= -\int_0^x -2(u - 70)\, du \\ &= x^2 - 140x. \end{aligned}$$

In Figure 5 we graph $y = F(x)$ and using Theorem 1.17 we get the phase line diagram below the graph of the potential energy function. Notice that $x = 70$ is an isolated minimum for the potential energy function.

△

1.4 Generalized Logistic Equation

We first do some calculations to derive what we will call the generalized logistic equation. Assume p and q are continuous functions on an interval I and let $x(t)$ be a solution of the first order linear differential equation

$$x' = -p(t)x + q(t) \tag{1.8}$$

with $x(t) \neq 0$ on I. Then set

$$y(t) = \frac{1}{x(t)}, \quad t \in I.$$

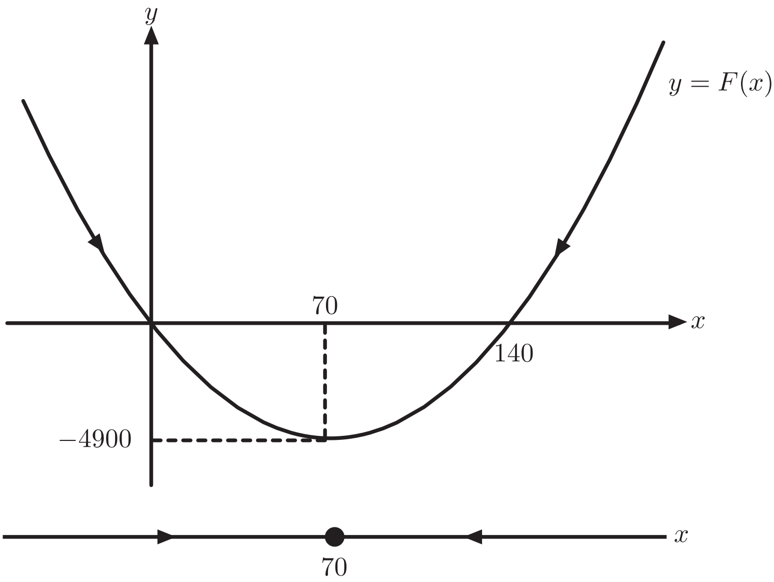

Figure 5. Potential energy function and phase line diagram for $x' = -2(x-70)$.

It follows that

$$\begin{aligned} y'(t) &= -\frac{x'(t)}{x^2(t)} \\ &= [p(t)x(t) - q(t)]\, y^2(t) \\ &= [p(t) - q(t)y(t)]\, y(t), \quad t \in I. \end{aligned}$$

We call the differential equation

$$y' = [p(t) - q(t)y]\, y \tag{1.9}$$

the generalized logistic equation. Above we proved that if $x(t)$ is a nonzero solution of the linear equation (1.8) on I, then $y(t) = \frac{1}{x(t)}$ is a nonzero solution of the generalized logistic equation (1.9) on I. Conversely, if $y(t)$ is a nonzero solution of the generalized logistic equation (1.9) on I, then (Exercise 1.28) $x(t) = \frac{1}{y(t)}$ is a nonzero solution of the linear equation (1.8) on I.

We now state the following theorem.

Theorem 1.19 *If* $y_0 \neq 0$ *and*

$$\frac{1}{y_0} + \int_{t_0}^{t} q(s) e^{\int_{t_0}^{s} p(\tau)\, d\tau}\, ds \neq 0, \quad t \in I,$$

then the solution of the IVP

$$y' = [p(t) - q(t)y]\, y, \quad y(t_0) = y_0, \quad t_0 \in I \tag{1.10}$$

is given by

$$y(t) = \frac{e^{\int_{t_0}^t p(\tau)\, d\tau}}{\frac{1}{y_0} + \int_{t_0}^t q(s) e^{\int_{t_0}^s p(\tau)\, d\tau}\, ds}. \tag{1.11}$$

Proof Note that by Theorem 1.3 every IVP (1.10) has a unique solution. Assume $y_0 \neq 0$ and let $x_0 = \frac{1}{y_0}$. By the variation of constants formula in Theorem 1.6 the solution of the IVP

$$x' = -p(t)x + q(t), \quad x(t_0) = x_0$$

is given by

$$\begin{aligned} x(t) &= e^{-\int_{t_0}^t p(\tau)\, d\tau} x_0 + e^{-\int_{t_0}^t p(\tau)\, d\tau} \int_{t_0}^t e^{\int_{t_0}^s p(\tau)\, d\tau} q(s)\, ds \\ &= e^{-\int_{t_0}^t p(\tau)\, d\tau} \frac{1}{y_0} + e^{-\int_{t_0}^t p(\tau)\, d\tau} \int_{t_0}^t e^{\int_{t_0}^s p(\tau)\, d\tau} q(s)\, ds \end{aligned}$$

which is nonzero on I by assumption. It follows that the solution of the IVP (1.10) is given by

$$y(t) = \frac{1}{e^{-\int_{t_0}^t p(\tau)\, d\tau} \frac{1}{y_0} + e^{-\int_{t_0}^t p(\tau)\, d\tau} \int_{t_0}^t q(s) e^{\int_{t_0}^s p(\tau)\, d\tau}\, ds}.$$

Multiplying the numerator and denominator by $e^{\int_{t_0}^t p(\tau)\, d\tau}$ we get the desired result (1.11). □

In applications (e.g., population dynamics) one usually has that

$$p(t) = q(t)K$$

where $K > 0$ is a constant. In this case the generalized logistic equation becomes

$$y' = p(t)\left[1 - \frac{y}{K}\right] y. \tag{1.12}$$

The constant solutions $y(t) = 0$ and $y(t) = K$ are called equilibrium solutions of (1.12). The constant K is called the carrying capacity (saturation level).

We now state the following corollary of Theorem 1.19.

Corollary 1.20 *If* $y_0 \neq 0$ *and*

$$\frac{1}{y_0} - \frac{1}{K} + \frac{1}{K} e^{\int_{t_0}^t p(s)\, ds} \neq 0, \quad t \in I,$$

then the solution of the IVP

$$y' = p(t)\left[1 - \frac{y}{K}\right] y, \quad y(t_0) = y_0 \tag{1.13}$$

is given by

$$y(t) = \frac{e^{\int_{t_0}^t p(s)\, ds}}{\frac{1}{y_0} - \frac{1}{K} + \frac{1}{K} e^{\int_{t_0}^t p(s)\, ds}}. \tag{1.14}$$

Proof This follows from Theorem 1.19, where we use the fact that

$$\begin{aligned}\int_{t_0}^t e^{\int_{t_0}^s p(\tau)\, d\tau} q(s)\, ds &= \frac{1}{K} \int_{t_0}^t e^{\int_{t_0}^s p(\tau)\, d\tau} p(s)\, ds \\ &= \frac{1}{K} \left[e^{\int_{t_0}^t p(s)\, ds} - 1 \right].\end{aligned}$$

□

The following theorem gives conditions under which the solutions of the generalized logistic equation (1.12) with nonnegative initial conditions behave very similar to the corresponding solutions of the autonomous logistic equation (1.7).

Theorem 1.21 *Assume* $p : [t_0, \infty) \to [0, \infty)$ *is continuous and* $\int_{t_0}^\infty p(t)\, dt = \infty$*. Let* $y(t)$ *be the solution of the IVP* (1.13) *with* $y_0 > 0$*, then* $y(t)$ *exists on* $[t_0, \infty)$*. Also if* $0 < y_0 < K$*, then* $y(t)$ *is nondecreasing with* $\lim_{t\to\infty} y(t) = K$*. If* $y_0 > N$*, then* $y(t)$ *is nonincreasing with* $\lim_{t\to\infty} y(t) = K$*.*

Proof Let $y(t)$ be the solution of the IVP (1.13) with $y_0 > 0$. Then from (1.14)

$$y(t) = \frac{e^{\int_{t_0}^t p(s)\, ds}}{\frac{1}{y_0} - \frac{1}{K} + \frac{1}{K} e^{\int_{t_0}^t p(s)\, ds}}. \tag{1.15}$$

By the uniqueness of solutions of IVP's the solution $y(t)$ is bounded below by K and hence $y(t)$ remains positive to the right of t_0. But since

$$y'(t) = p(t) \left[1 - \frac{y(t)}{K} \right] y(t) \leq 0,$$

$y(t)$ is decreasing. It follows from Theorem 1.3 that $y(t)$ exists on $[t_0, \infty)$ and from (1.15) we get that $\lim_{t\to\infty} y(t) = K$.

Next assume that $0 < y_0 < K$. Then by the uniqueness of solutions of IVP's we get that

$$0 < y(t) < K,$$

to the right of t_0. It follows that $y(t)$ is a solution on $[t_0, \infty)$ and by (1.15) we get that $\lim_{t\to\infty} y(t) = K$. Also $0 < y(t) < K$, implies that

$$y'(t) = p(t) \left[1 - \frac{y(t)}{K} \right] y(t) \geq 0,$$

so $y(t)$ is nondecreasing on $[t_0, \infty)$. □

1.5 Bifurcation

Any unspecified constant in a differential equation is called a *parameter*. One of the techniques that is used to study differential equations is to let a parameter vary and to observe the resulting changes in the behavior of the solutions. Any large scale change is called a *bifurcation* and the value of the parameter for which the change occurs is called a *bifurcation point*. We end this chapter with some simple examples of bifurcations.

Example 1.22 We consider the differential equation

$$x' = \lambda(x - 1),$$

where λ is a parameter. In Figure 6 the phase line diagrams for this differential equation when $\lambda < 0$ and $\lambda > 0$ are drawn. There is a drastic change in the phase line diagrams as λ passes through zero (the equilibrium point $x = 1$ loses its stability as λ increases through zero). Because of this we say bifurcation occurs when $\lambda = 0$.

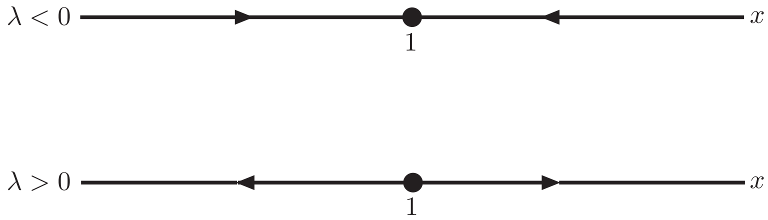

Figure 6. Phase line diagrams for $x' = \lambda(x - 1)$, $\lambda < 0$, $\lambda > 0$.

△

Example 1.23 (Saddle-Node Bifurcation) Now consider the equation

$$x' = \lambda + x^2.$$

If $\lambda < 0$, then there is a pair of equilibrium points, one stable and one unstable. When $\lambda = 0$, the equilibrium points collide, and for $\lambda > 0$, there are no equilibrium points (see Figure 7). In this case, the bifurcation that occurs at $\lambda = 0$ is usually called a *saddle-node* bifurcation.

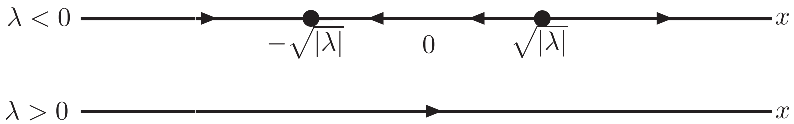

Figure 7. Phase line diagrams for $x' = \lambda + x^2$, $\lambda < 0$, $\lambda > 0$.

△

Example 1.24 (Transcritical Bifurcation) Consider

$$x' = \lambda x - x^2.$$

For $\lambda < 0$, there is an unstable equilibrium point at $x = \lambda$ and a stable equilibrium point at $x = 0$. At $\lambda = 0$, the two equilibrium points coincide. For $\lambda > 0$, the equilibrium at $x = 0$ is unstable, while the one at $x = \lambda$ is stable, so the equilibrium points have switched stability! See Figure 8. This type of bifurcation is known as a *transcritical* bifurcation.

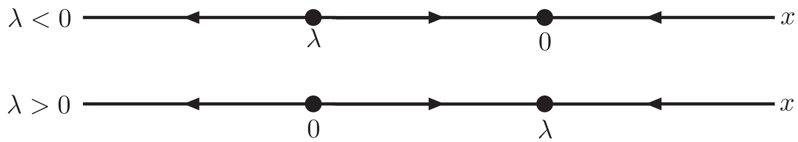

FIGURE 8. Phase line diagrams for $x' = \lambda x - x^2$, $\lambda < 0$, $\lambda > 0$.

△

Example 1.25 (Pitchfork Bifurcation) We will draw the *bifurcation diagram* (see Figure 9) for the differential equation

$$x' = f(\lambda, x) := (x - 1)(\lambda - 1 - (x - 1)^2). \tag{1.16}$$

Note that the equations

$$x = 1, \quad \lambda = 1 + (x - 1)^2$$

give you the equilibrium points of the differential equation (1.16). We graph these two equilibrium curves in the λx-plane (see Figure 9). Note for each $\lambda \leq 1$ the differential equation (1.16) has exactly one equilibrium point and for each $\lambda > 1$ the differential equation (1.16) has exactly three equilibrium points. When part of an equilibrium curve is dashed it means the corresponding equilibrium points are unstable, and when part of an equilibrium curve is solid it means the corresponding equilibrium points are stable. To determine this stability of the equilibrium points note that at points on the pitchfork we have $f(\lambda, t) = 0$, at points above the pitchfork $f(\lambda, t) < 0$, at points below the pitchfork $f(\lambda, t) > 0$, at points between the top two forks of the pitchfork $f(\lambda, t) > 0$, and at points between the lower two forks of the pitchfork $f(\lambda, t) < 0$. Note that the equilibrium point $x = 1$ is asymptotically stable for $\lambda < 1$ and becomes unstable for $\lambda > 1$. We say we have *pitchfork bifurcation* at $\lambda = 1$.

△

In the final example in this section we give an example of hysteresis.

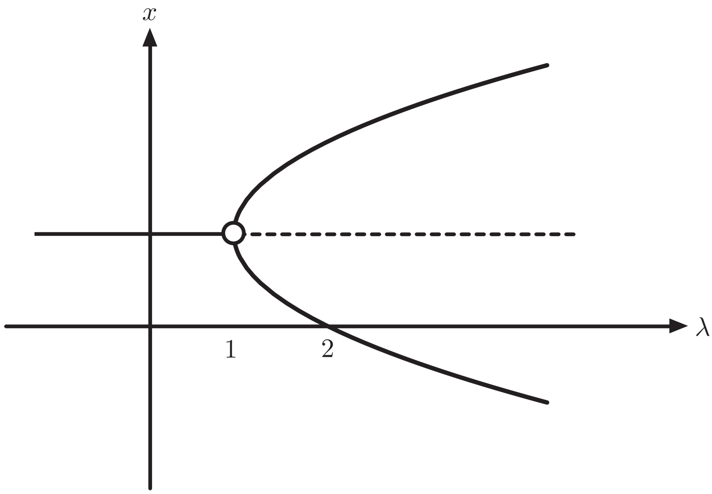

Figure 9. Bifurcation diagram for (1.16).

Example 1.26 (Hysteresis) The bifurcation diagram for the differential equation

$$x' = \lambda + x - x^3 \tag{1.17}$$

is given in Figure 10. Note that if we start with $\lambda < \lambda_1 := -\frac{2}{3\sqrt{3}}$ and slowly increase λ, then for all practical purposes solutions stay close to the smallest equilibrium point until when λ passes through the value $\lambda_2 := \frac{2}{3\sqrt{3}}$, where the solution quickly approaches the largest equilibrium point for $\lambda > \lambda_2$. On the other hand, if we start with $\lambda > \lambda_2$ and start decreasing λ, solutions stay close to the largest equilibrium point until λ decreases through the value λ_1, where all of a sudden the solution approaches the smallest equilibrium point. △

There are lots of interesting examples of hysteresis in nature. Murray [**37**], pages 4–8, discusses the possible existence of hysteresis in a population model for the spuce budworm. For an example of hysteresis concerning the temperature in a continuously stirred tank reactor see Logan [**33**], pages 430–434. Also for an interesting example concerning the buckling of a wire arc see Iooss and Joseph [**28**], pages 25–28. Hysteresis also occurs in the theory of elasticity.

1.6 Exercises

1.1 Find the maximal interval of existence for the solution of the IVP

$$x' = (\cos t)\, x^2, \quad x(0) = 2.$$

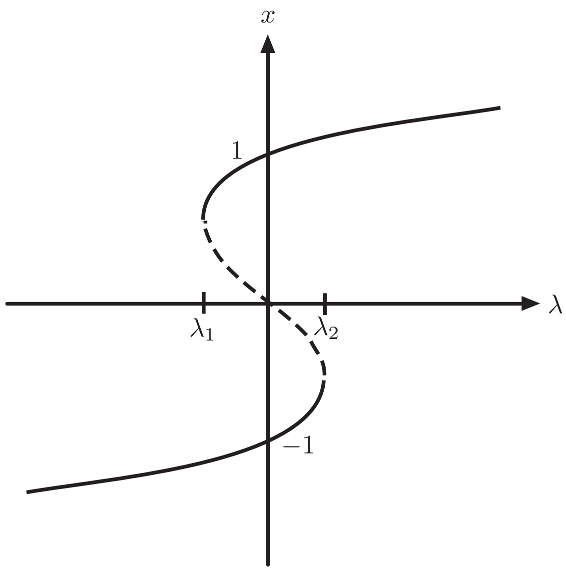

Figure 10. Hysteresis.

1.2 Find the maximal interval of existence for the solution of the IVP

$$x' = \frac{2tx^2}{1+t^2}, \quad x(0) = x_0$$

in terms of x_0.

1.3 Show that the IVP

$$x' = x^{\frac{1}{3}}, \quad x(0) = 0$$

has infinitely many solutions. Explain why Theorem 1.3 does not apply to give you uniqueness.

1.4 Assume that $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ is defined by $f(t,x) = x^{\frac{1}{3}}$, for $(t,x) \in \mathbb{R} \times (0,\infty)$. Show that for any $(t_0, x_0) \in \mathbb{R} \times (0, \infty)$ the IVP

$$x' = f(t,x), \quad x(t_0) = x_0$$

has a unique solution. Find the maximum interval of existence for the solution of this IVP when $(t_0, x_0) = (1,1)$.

1.5 Use the variation of constants formula in Theorem 1.6 to solve the following IVPs:

(i) $x' = 2x + e^{2t}, \quad x(0) = 3$
(ii) $x' = 3x + 2e^{3t}, \quad x(0) = 2$
(iii) $x' = \tan(t)x + \sec(t), \quad x(0) = -1$
(iv) $x' = \frac{2}{t}x + t, \quad x(1) = 2$

1.6 Use the variation of constants formula in Theorem 1.6 to solve the following IVPs:

(i) $x' = -2x + e^{at}, \quad x(0) = 2$
(ii) $x' = 4x + 3t, \quad x(0) = 1$
(iii) $tx' - 3x = 1 - 3\ln t, \quad x(0) = 1$

1.7 Draw the phase line diagram for $x' = -2x$. Show that there are infinitely many solutions that give you the orbits $(0, \infty)$ and $(-\infty, 0)$ respectively.

1.8 Draw the phase line diagrams for each of the following:

(i) $x' = -x + x^3$
(ii) $x' = x^4$
(iii) $x' = x^2 + 4x + 2$
(iv) $x' = x^3 - 3x^2 + 3x - 1$

1.9 Draw the phase line diagrams for each of the following:

(i) $x' = \cosh x$
(ii) $x' = \cosh x - 1$
(iii) $x' = (x - a)^2$
(iv) $x' = \sin x$
(v) $x' = \sin(2x)$
(vi) $x' = e^x$
(vii) $x' = \sinh^2(x - b)$
(viii) $x' = \cos x - 1$

1.10 Show that if x is a nonzero solution of the linear equation $x' = -rx + \frac{r}{K}$, where r and K are positive constants, then $N = \frac{1}{x}$ is a solution of the logistic equation (1.7). Use this and the variation of constants formula to solve the IVP $N' = rN\left(1 - \frac{N}{K}\right), \ N(0) = \frac{K}{2}$.

1.11 Solve the logistic equation (1.7) by using the method of separation of variables. Note in general we can not solve the more general logistic equation (1.9) by the method of separation of variables.

1.12 (Bernoulli's Equation) The differential equation $y' = p(t)y + q(t)y^\alpha$, $\alpha \neq 0, 1$ is called *Bernoulli's equation*. Show that if y is a nonzero solution of Bernoulli's equation, then $x = y^{1-\alpha}$ is a solution of the linear equation $x' = (1 - \alpha)p(t)x + (1 - \alpha)q(t)$.

1.13 Use Exercise 1.12 to solve the following Bernoulli equations:

(i) $x' = -\frac{1}{t}x + tx^2$
(ii) $x' = x + e^t x^2$
(iii) $x' = -\frac{1}{t}x + \frac{1}{tx^2}$

1.14 Assume that p and q are continuous functions on $\mathbb{R}$ which are periodic with period $T > 0$. Show that the linear differential equation (1.4) has a periodic solution x with positive period T iff the differential equation (1.4) has a solution x satisfying $x(0) = x(T)$.

1.15 Show that the equilibrium point $x_0 = 0$ for the differential equation $x' = 0$ is stable but not asymptotically stable.

1.16 Determine the stability (*stable, unstable, or asymptotically stable*) of the equilibrium points for each of the differential equations in Exercise 1.8.

1.17 A yam is put in a 200^oC oven at time $t = 0$. Let $T(t)$ be the temperature of the yam in degrees Celsius at time t minutes later. According to Newton's law of cooling, $T(t)$ satisfies the differential equation

$$T' = -k(T - 200),$$

where k is a positive constant. Draw the phase diagram for this differential equation and then draw a possible graph of various solutions with various initial temperatures at $t = 0$. Determine the stability of all equilibrium points for this differential equation.

1.18 Given that the function F defined by $F(x) = x^3 + 3x^2 - x - 3$, for $x \in \mathbb{R}$ is a potential energy function for $x' = f(x)$, draw the phase line diagram for $x' = f(x)$.

1.19 Given that the function F defined by $F(x) = 4x^2 - x^4$, for $x \in \mathbb{R}$ is a potential energy function for $x' = f(x)$, draw the phase line diagram for $x' = f(x)$.

1.20 For each of the following differential equations find a potential energy function and use it to draw the phase line diagram:

(i) $x' = x^2$
(ii) $x' = 3x^2 - 10x + 6$
(iii) $x' = 8x - 4x^3$
(iv) $x' = \frac{1}{x^2+1}$

1.21 Find and graph a potential energy function for the equation

$$x' = -a(x - b), \quad a, b > 0$$

and use this to draw a phase line diagram for this equation.

1.22 Determine the stability (*stable, unstable, or asymptotically stable*) of the equilibrium points for each of the differential equations in Exercise 1.20.

1.23 Given that a certain population $x(t)$ at time t is known to satisfy the differential equation

$$x' = ax \ln\left(\frac{b}{x}\right),$$

when $x > 0$, where $a > 0$, $b > 0$ are constants, find the equilibrium population and determine its stability.

1.24 Use the phase line diagram and take into account the concavity of solutions to graph various solutions of each of the following differential equations

(i) $x' = 6 - 5x + x^2$

(ii) $x' = 4x^2 + 3x^3 - x^4$
(iii) $x' = 2 + x - x^2$

1.25 A tank initially contains 1000 gallons of a solution of water and 5 pounds of some solute. Suppose that a solution with the same solute of concentration .1 pounds per gallon is flowing into the tank at the rate of 2 gallons per minute. Assume that the tank is constantly stirred so that the concentration of solute in the tank at each time t is essentially constant throughout the tank.

(i) Suppose that the solution in the tank is being drawn off at the rate of 2 gallons per minute to maintain a constant volume of solution in the tank. Show that the number $x(t)$ of pounds of solute in the tank at time t satisfies the differential equation

$$x' = .2 - \frac{x}{500},$$

and compute $x(t)$;

(ii) Suppose now that the solution in the tank is being drawn off at the rate of 3 gallons per minute so that the tank is eventually drained. Show that the number $y(t)$ of pounds of solute in the tank at time t satisfies the equation

$$y' = .2 - \frac{3y}{1000 - t},$$

for $0 < t < 1000$, and compute $y(t)$.

1.26 (Terminal Velocity) Let m be the mass of a large object falling rapidly toward the earth with velocity $v(t)$ at time t. (We take downward velocity to be positive in this problem.) If we take the force of gravity to be constant, the standard equation of motion is

$$mv' = mg - kv^2,$$

where g is the acceleration due to gravity and $-kv^2$ is the upward force due to air resistance:

(i) Sketch the phase line diagram and determine which of the equilibrium points is asymptotically stable. Why is "terminal velocity" an appropriate name for this number?

(ii) Assume the initial velocity is $v(0) = v_0$, and solve the IVP. Show that the solution v approaches the asymptotically stable equilibrium as $t \to \infty$.

1.27 Assume that $x_0 \in (a, b)$, $f(x) > 0$ for $a < x < x_0$, and $f(x) < 0$ for $x_0 < x < b$. Show that x_0 is an asymptotically stable equilibrium point for $x' = f(x)$.

1.28 Show that if $y(t)$ is a nonzero solution of the generalized logistic equation (1.9) on I, then $x(t) = \frac{1}{y(t)}$ is a nonzero solution of the linear equation (1.8) on I.

1.29 If in Theorem 1.21 we replace $\int_{t_0}^{\infty} p(t)\, dt = \infty$ by $\int_{t_0}^{\infty} p(t)\, dt = L$, where $0 \leq L < \infty$, what can we say about solutions of the IVP (1.13) with $y_0 > 0$

1.30 (Harvesting) Work each of the following:

(i) Explain why the following differential equation could serve as a model for logistic population growth with harvesting if λ is a positive parameter:

$$N' = rN\left(1 - \frac{N}{K}\right) - \lambda N.$$

(ii) If $\lambda < r$, compute the equilibrium points, sketch the phase line diagram, and determine the stability of the equilibria.

(iii) Show that a bifurcation occurs at $\lambda = r$. What type of bifurcation is this?

1.31 Solve the generalized logistic equation (1.13) by the method of separation of variables.

1.32 (Gene Activation) The following equation occurs in the study of gene activation:

$$x' = \lambda - x + \frac{4x^2}{1 + x^2}.$$

Here $x(t)$ is the concentration of gene product at time t.

(i) Sketch the phase line diagram for $\lambda = 1$.

(ii) There is a small value of λ, say λ_0, where a bifurcation occurs. Estimate λ_0, and sketch the phase line diagram for some $\lambda \in (0, \lambda_0)$.

(iii) Draw the bifurcation diagram for this differential equation.

1.33 For each of the following differential equations find values of λ where bifurcation occurs. Draw phase line diagrams for values of λ close to the value of λ where bifurcation occurs.

(i) $x' = \lambda - 4 - x^2$

(ii) $x' = x^3(\lambda - x)$

(iii) $x' = x^3 - x + \lambda$

1.34 Assume that the population $x(t)$ of rats on a farm at time t (in weeks) satisfies the differential equation $x' = -.1x(x - 25)$. Now assume that we decide to kill rats at a constant rate of λ rats per week. What would be the new differential equation that x satisfies? For what value of λ in this new differential equation does bifurcation occur? If the number of rats we kill per week is larger than this bifurcation value what happens to the population of rats?

1.35 Give an example of a differential equation with a parameter λ which has pitchfork bifurcation at $\lambda = 0$, where $x_0 = 0$ is an unstable equilibrium point for $\lambda < 0$ and is a stable equilibrium point for $\lambda > 0$.

1.36 Draw the bifurcation diagram for each of the following

(i) $x' = \lambda - x^2$
(ii) $x' = (\lambda - x)(\lambda - x^3)$

1.37 In each of the following draw the bifurcation diagram and give the value(s) of λ where bifurcation occurs:

(i) $x' = (\lambda - x)(\lambda - x^2)$
(ii) $x' = \lambda - 12 + 3x - x^3$
(iii) $x' = \lambda - \frac{x^2}{1+x^2}$
(iv) $x' = x \sin x + \lambda$

In which of these does hysteresis occur?