Chapter 9
Natural Lagrangian Strain Measures of the Non-Linear Cosserat Continuum

Wojciech Pietraszkiewicz and Victor A. Eremeyev

Abstract Definitions of the Lagrangian stretch and wryness tensors in the non-linear Cosserat continuum are discussed applying three different methods. The resulting unique strain measures have several distinguishing features and are called the natural ones. They are expressed through the translation vector and either the rotation tensor or various finite rotation vector fields. The relation of the natural strain measures to those proposed in the representative literature is reviewed.

9.1 Introduction

The stretch and wryness tensors of the non-linear Cosserat continuum were originally defined by Cosserats [2] through components of some fields in the common Cartesian frame. Today their approach is hardly readable. During the last 50 years, the strain measures have been redefined by different authors in various forms using, for example, (a) components in two different curvilinear coordinate systems associated with the undeformed (reference) of deformed (actual) placements of the body, (b) components in the convective coordinate system, (c) Lagrangian or Eulerian descriptions, (d) different representations of the rotation group $SO(3)$ in terms of various finite rotation vectors, Euler angles, quaternions etc., (e) formally different tensor operations and sign conventions, as well as (f) requiring or not the strain measures to vanish in the undeformed placement of the body. Even the gradient and divergence operators as well as the Cauchy theorem influencing definitions of work-

W. Pietraszkiewicz (✉)
Institute of Fluid-Flow Machinery of the Polish Academy of Sciences, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland
e-mail: pietrasz@imp.gda.pl

V.A. Eremeyev
South Scientific Center of RASci & South Federal University, Milchakova str. 8a, 344090 Rostov on Don, Russia
e-mail: eremeyev.victor@gmail.com

G.A. Maugin, A.V. Metrikine (eds.), *Mechanics of Generalized Continua*, Advances in Mechanics and Mathematics 21,
DOI 10.1007/978-1-4419-5695-8_9,

conjugate pairs of the stress and strain measures are not defined in the same way in the literature. The strain measures introduced by Kafadar and Eringen [4] are among the most referred to in the literature, but even their derivation is not fully complete.

In this paper, we discuss three different methods of introducing the strain measures into the non-linear Cosserat continuum: (a) by a direct geometric approach, (b) defining the strain measures as the fields work-conjugate to the respective internal stress and couple-stress tensor fields, and (c) applying the principle of invariance under superposed rigid-body deformations to the strain energy density of the polar-elastic body. Each of the three ways allows one to associate different geometric and/or physical interpretations to the corresponding strain measures. In the discussion, we use the coordinate-free vector and tensor notation. Orientations of material particles in the reference and deformed placements, respectively, as well as their changes during deformation are described in the most general way by the proper orthogonal tensors. Our primary strain measures, called the natural ones, are of the relative type, for they are required to vanish in the reference placement.

In the reference (undeformed) placement $\kappa(\mathscr{B}) = B_\kappa \subset \mathscr{E}$, the material particle $X \in \mathscr{B}$ is given by its position vector $\mathbf{x} \in E$ relative to a point $o \in \mathscr{E}$ of the 3D physical space $\mathscr{E}$ and by three orthonormal directors $\mathbf{h}_a \in E$, $a, b = 1, 2, 3$, fixing orientation of X in the 3D vector space E.

In the actual (deformed) placement $\gamma(\mathscr{B}) = B_\gamma = \chi(B_\kappa) \subset \mathscr{E}$, the position of X becomes defined by the vector $\mathbf{y} \in E$, taken here for simplicity relative to the same point $o \in \mathscr{E}$, and by three orthonormal directors $\mathbf{d}_a \in E$. As a result, the finite displacement of the Cosserat continuum can be described by two following smooth mappings:

$$\mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}), \qquad \mathbf{d}_a = \mathbf{Q}(\mathbf{x})\mathbf{h}_a, \tag{9.1}$$

where $\mathbf{u} \in E$ is the translation vector and $\mathbf{Q} = \mathbf{d}_a \otimes \mathbf{h}_a \in \mathrm{SO}(3)$ is the proper orthogonal microrotation tensor. Two independent fields $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ describe translational and rotational degrees of freedom of the Cosserat continuum, respectively.

9.2 Strain Measures by Geometric Approach

Differentials of the independent kinematic fields are

$$\begin{aligned}
&\mathrm{d}\mathbf{x} = (\mathrm{Grad}\,\mathbf{x})\,\mathrm{d}\mathbf{x}, && \mathrm{d}\mathbf{y} = (\mathrm{grad}\,\mathbf{y})\,\mathrm{d}\mathbf{y} = (\mathrm{Grad}\,\mathbf{y})\,\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathrm{d}\mathbf{x},\\
&\mathrm{d}\mathbf{h}_a = \mathbf{B}\,\mathrm{d}\mathbf{x} \times \mathbf{h}_a, && \mathrm{d}\mathbf{d}_a = \mathbf{C}\,\mathrm{d}\mathbf{y} \times \mathbf{d}_a,\\
&\mathbf{B} = \frac{1}{2}\mathbf{h}_a \times \mathrm{Grad}\,\mathbf{h}_a, && \mathbf{C} = \frac{1}{2}\mathbf{d}_a \times \mathrm{grad}\,\mathbf{d}_a,
\end{aligned} \tag{9.2}$$

where Grad and grad are the gradient operators in B_κ and B_γ, and $\mathbf{B}$ and $\mathbf{C}$ are the microstructure curvature tensors in the undeformed (reference) and deformed (actual) placements of the Cosserat continuum, respectively. In particular, for any

nth order tensor field $\mathbf{Z}(\mathbf{x})$ we define its gradient in B_κ to satisfy $[\operatorname{Grad}\mathbf{Z}(\mathbf{x})]\mathbf{a} = (\mathrm{d}/\mathrm{d}t)\mathbf{Z}(\mathbf{x}+t\mathbf{a})|_{t=0}$ for any $t \in \mathbb{R}, \mathbf{a} \in E$.

The relative changes of lengths and orientations of the Cosserat continuum during deformation are governed by differences of differentials (9.2) brought by the tensor $\mathbf{Q}$ to two comparable orientations

$$\begin{gathered}
\mathrm{d}\mathbf{y} - \mathbf{Q}\,\mathrm{d}\mathbf{x} = \mathbf{X}\,\mathrm{d}\mathbf{x} = \mathbf{G}\,\mathrm{d}\mathbf{y}, \qquad \mathbf{C}\,\mathrm{d}\mathbf{y} - \mathbf{Q}\mathbf{B}\,\mathrm{d}\mathbf{x} = \mathbf{\Phi}\,\mathrm{d}\mathbf{x} = \mathbf{\Delta}\,\mathrm{d}\mathbf{y}, \\
\mathbf{Q}^T\,\mathrm{d}\mathbf{y} - \mathrm{d}\mathbf{x} = \mathbf{E}\,\mathrm{d}\mathbf{x} = \mathbf{Y}\,\mathrm{d}\mathbf{y}, \qquad \mathbf{Q}^T\mathbf{C}\,\mathrm{d}\mathbf{y} - \mathbf{B}\,\mathrm{d}\mathbf{x} = \mathbf{\Gamma}\,\mathrm{d}\mathbf{x} = \mathbf{\Psi}\,\mathrm{d}\mathbf{y}, \\
\mathbf{E} = \mathbf{Q}^T\mathbf{F} - \mathbf{I}, \qquad \mathbf{\Gamma} = \mathbf{Q}^T\mathbf{C}\mathbf{F} - \mathbf{B} = -\frac{1}{2}\boldsymbol{\varepsilon} : \left(\mathbf{Q}^T \operatorname{Grad}\mathbf{Q}\right),
\end{gathered} \tag{9.3}$$

where $\mathbf{E}$, $\mathbf{\Gamma}$ are the relative Lagrangian stretch and wryness tensors, $\mathbf{G}$, $\mathbf{\Delta}$ are the relative Eulerian strain measures, while $\mathbf{X}$, $\mathbf{\Phi}$, $\mathbf{Y}$, $\mathbf{\Psi}$ are the relative two-point deformation measures. In (9.3), $\mathbf{I}$ is the identity (metric) tensor of $E \otimes E$, the $3\mathrm{rd}$-order skew tensor $\boldsymbol{\varepsilon} = -\mathbf{I} \times \mathbf{I}$ is the Ricci tensor of $E \otimes E \otimes E$, and the double dot-product $:$ of two $3\mathrm{rd}$-order tensors $\mathbf{A}$, $\mathbf{P}$ represented in the base $\mathbf{h}_a$ is defined as $\mathbf{A} : \mathbf{P} = A_{\mathrm{amn}} P_{\mathrm{mnb}} \mathbf{h}_a \otimes \mathbf{h}_b$.

Let us note some interesting features of the relative Lagrangian strain measures:

1. They are given in the common coordinate-free notation; their various component representations can easily be generated, if necessary.
2. Definitions of the measures are valid for finite translations and rotations as well as for unrestricted stretches and changes of microstructure orientation of the Cosserat body.
3. The measures are expressed in terms of the rotation tensor $\mathbf{Q}$; for any specific parametrization of the rotation group $\mathrm{SO}(3)$ by various finite rotation vectors, Euler angles, quaternions, etc. appropriate expressions for the measures can easily be found, if necessary.
4. The measures vanish in the rigid-body deformation $\mathbf{y} = \mathbf{O}\mathbf{x} + \mathbf{a}$, $\mathbf{d}_a = \mathbf{O}\mathbf{h}_a$ with a constant vector $\mathbf{a}$ and a constant proper orthogonal tensor $\mathbf{O}$ defined for the whole body.
5. In the absence of deformation from the reference placement, that is, when $\mathbf{F} = \mathbf{Q} = \mathbf{I}$, the measures vanish identically.
6. The measures are not symmetric, in general: $\mathbf{E}^T \neq \mathbf{E}$, $\mathbf{\Gamma}^T \neq \mathbf{\Gamma}$.

In our purely geometric approach, there is no need for discussing whether these measures might be defined as transposed ones or with opposite signs. The derivation process itself is concise, elegant, direct, and seems to be the most complete one in the literature.

9.3 Principle of Virtual Work and Work-Conjugate Strain Measures

Already Reissner [7] noted that the internal structure of two local equilibrium equations of the Cosserat elastic body requires specific two strain measures expressed in terms of independent translation and rotation vectors as the only field variables. We develop this idea here in the general case of the non-linear Cosserat continuum using the coordinate-free approach.

If the local coordinate-free form of the equilibrium conditions of the Cosserat continuum derived in Appendix of [5] are multiplied by two arbitrary smooth vector fields $\mathbf{v}$, $\boldsymbol{\omega} \in E$, then we generate the integral identity

$$\int_{B_\kappa} \big\{(\mathrm{Div}\,\mathbf{T} + \mathbf{f}) \cdot \mathbf{v} + \big[\mathrm{Div}\,\mathbf{M} - \mathrm{ax}\big(\mathbf{F}\mathbf{T} - \mathbf{T}^T\mathbf{F}^T\big) + \mathbf{m}\big] \cdot \boldsymbol{\omega}\big\}\,\mathrm{d}v \\ - \int_{\partial B_{\kappa f}} \big\{(\mathbf{n}\mathbf{T} - \mathbf{t}^*) \cdot \mathbf{v} + (\mathbf{n}\mathbf{M} - \mathbf{m}^*) \cdot \boldsymbol{\omega}\big\}\,\mathrm{d}a = 0. \tag{9.4}$$

Here $\mathbf{f}$ and $\mathbf{m}$ are the volume force and couple vectors applied at any point $y = \chi(x)$ of the deformed body, but measured per unit volume of B_κ, $\mathbf{t}^*(\mathbf{x})$ and $\mathbf{m}^*(\mathbf{x})$ the external force and couple vector fields prescribed on the part $\partial B_{\gamma f}$, but measured per unit area of $\partial B_{\kappa f}$, $\mathbf{t}_{(n)} = \mathbf{n}\mathbf{T}_{(n)}$ and $\mathbf{m}_{(n)} = \mathbf{n}\mathbf{M}$ the surface traction and couple vector fields applied at any point of $\partial P_\gamma \in \partial B_\gamma$, but measured per unit area of $\partial P_\kappa \in \partial B_\kappa$, expressible as linear functions of the nominal type stress $\mathbf{T}$ and couple-stress $\mathbf{M}$ tensors, respectively, $\mathbf{n}$ the unit vector externally normal to ∂P_κ, $\mathrm{ax}\,\mathbf{A}$ the axial vector of the skew tensor $\mathbf{A}$, and the divergence operator Div is defined to satisfy $[\mathrm{Div}\,\mathbf{Z}(\mathbf{x})]\mathbf{a} = \mathrm{Div}[\mathbf{Z}(\mathbf{x})\mathbf{a}]$ for any $\mathbf{a} \in E$.

The vector field $\mathbf{v}$ can be interpreted as the kinematically admissible virtual translation $\mathbf{v} \equiv \delta\mathbf{y}$ and the vector field $\boldsymbol{\omega}$ as the kinematically admissible virtual rotation $\boldsymbol{\omega} \equiv \mathrm{ax}(\delta\mathbf{Q}\mathbf{Q}^T)$ in B_γ such that $\mathbf{v} = \boldsymbol{\omega} = \mathbf{0}$ on $\partial B_{\kappa d} = \partial B_\kappa \backslash \partial B_{\kappa f}$, where δ is the symbol of virtual change (variation). Then using the divergence theorem Identity (9.4) can be transformed into the principle of virtual work of the non-linear Cosserat continuum

$$\int_{B_\kappa} \big[\mathbf{T}^T : (\mathrm{Grad}\,\mathbf{v} - \boldsymbol{\Omega}\mathbf{F}) + \mathbf{M}^T : \mathrm{Grad}\,\boldsymbol{\omega}\big]\,\mathrm{d}v \\ = \int_{B_\kappa} (\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \boldsymbol{\omega})\,\mathrm{d}v + \int_{\partial B_{\kappa f}} (\mathbf{t}^* \cdot \mathbf{v} + \mathbf{m}^* \cdot \boldsymbol{\omega})\,\mathrm{d}a. \tag{9.5}$$

But we can show that $\delta\mathbf{E} = \mathbf{Q}^T(\mathrm{Grad}\,\mathbf{v} - \boldsymbol{\Omega}\mathbf{F})$ and $\delta\boldsymbol{\Gamma} = \mathbf{Q}^T\,\mathrm{Grad}\,\boldsymbol{\omega}$, where $\boldsymbol{\Omega} = \boldsymbol{\omega} \times \mathbf{I}$, and the internal virtual work density under the first volume integral of (9.5) can now be given by the expressions

$$\sigma = \mathbf{T}^T : (\mathbf{Q}\delta\mathbf{E}) + \mathbf{M}^T : (\mathbf{Q}\delta\boldsymbol{\Gamma}) = \mathbf{S} : \delta\mathbf{E} + \mathbf{K} : \delta\boldsymbol{\Gamma}, \tag{9.6}$$

where $\mathbf{S} = \mathbf{Q}^T\mathbf{T}^T$, $\mathbf{K} = \mathbf{Q}^T\mathbf{M}^T$ are the stress and couple-stress tensors whose natural components are referred entirely to the reference (undeformed) placement. The stress measures $\mathbf{S}$, $\mathbf{K}$ require the relative Lagrangian strain measures $\mathbf{E}$, $\boldsymbol{\Gamma}$ as their work-conjugate counterparts.

9.4 Invariance of the Polar-Elastic Strain Energy Density

In the polar-elastic body, the constitutive relations are defined through the strain energy density W_κ per unit volume of the undeformed placement κ. In general, the density W_κ can be assumed in the following form:

$$W_\kappa = W_\kappa(\mathbf{y}, \mathbf{F}, \mathbf{Q}, \operatorname{Grad} \mathbf{Q}; \mathbf{x}, \mathbf{B}). \tag{9.7}$$

But W_κ in (9.7) should satisfy the principle of invariance under the superposed rigid-body deformations. After appropriate transformations we can show that this requires the form of W_κ to be reduced to

$$W_\kappa = W_\kappa(\mathbf{E} + \mathbf{I}, \mathbf{I} \times \mathbf{\Gamma}; \mathbf{x}, \mathbf{B}) = \overline{W}_\kappa(\mathbf{E}, \mathbf{\Gamma}; \mathbf{x}, \mathbf{B}). \tag{9.8}$$

This again confirms that the relative Lagrangian strain measures $\mathbf{E}$, $\mathbf{\Gamma}$ are required to be the independent fields in the polar-elastic strain energy density in order it to be invariant under the superposed rigid-body deformation.

The geometric approach, the structure of equilibrium conditions and invariance of the polar-elastic strain energy density all require the tensors $\mathbf{E}$, $\mathbf{\Gamma}$ as the most appropriate Lagrangian strain measures for the non-linear Cosserat continuum. We call the measures the *natural* stretch and wryness tensors, respectively.

9.5 The Vectorial Parameterization

While the three components of $\mathbf{u}$ in (9.3) are all independent, the nine components of $\mathbf{Q}$ in (9.3) are subject to six constraints following from the orthogonality conditions $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$, so that only three rotational parameters of $\mathbf{Q}$ are independent. In many applications, it is more convenient to use the strain measures expressed in terms of six displacement parameters all of which are independent.

In the literature, many techniques how to parameterize the rotation group $\mathrm{SO}(3)$ were developed, which can roughly be classified as vectorial and non-vectorial ones. Various finite rotation vectors as well as the Cayley–Gibbs and exponential map parameters are examples of the vectorial parametrization, for they all have three independent scalar parameters as Cartesian components of a generalized vector in the 3D vector space E. The non-vectorial parameterizations are expressed either in terms of three scalar parameters that cannot be treated as vector components, such as Euler-type angles, for example, or through more scalar parameters subject to additional constraints, such as unit quaternions, Cayley–Klein parameters, or direction cosines. Each of these expressions may appear to be more convenient than others when solving specific problems of the non-linear Cosserat continuum.

The microrotation tensor $\mathbf{Q}$ represents the isometric and orientation-preserving transformation of the 3D vector space E into itself. By the Euler theorem, such a transformation can be expressed in terms of the angle of rotation ϕ about the axis of rotation described by the eigenvector $\mathbf{e}$ corresponding to the real eigenvalue $+1$ of $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{e} = +\mathbf{e}$, $\cos\phi = \frac{1}{2}(\operatorname{tr}\mathbf{Q} - \mathbf{1})$, $\sin\phi\mathbf{e} = \frac{1}{2}\operatorname{ax}(\mathbf{Q} - \mathbf{Q}^T)$, where $\operatorname{tr}\mathbf{A}$ is the trace of the second-order tensor $\mathbf{A}$. In terms of $\mathbf{e}$ and ϕ, the microrotation tensor $\mathbf{Q}$ can be expressed by the Gibbs [3] formula

$$\mathbf{Q} = \cos\phi\,\mathbf{I} + (1 - \cos\phi)\mathbf{e}\otimes\mathbf{e} + \sin\phi\,\mathbf{e}\times\mathbf{I}. \tag{9.9}$$

In the vectorial parametrization of $\mathbf{Q}$, one introduces a scalar function $p(\phi)$ generating three components of the finite rotation vector $\mathbf{p}\in E$ defined as $\mathbf{p} = p(\phi)\mathbf{e}$, see, for example, [1]. The generating function $p(\phi)$ has to be an odd function of ϕ with the limit behavior $\lim_{\phi\to 0}(p(\phi)/\phi) = \kappa$, where κ is a positive real normalization factor (usually 1 or $1/2$), and $p(0) = 0$. Then the tensor $\mathbf{Q}$ can be represented as

$$\mathbf{Q} = \cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\mathbf{p}\otimes\mathbf{p} + \frac{\sin\phi}{p}\mathbf{p}\times\mathbf{I}. \tag{9.10}$$

Taking the gradient of (9.10) and substituting it into (9.3), after appropriate transformations the natural Lagrangian stretch $\mathbf{E}$ and wryness $\boldsymbol{\Gamma}$ tensors can be represented in terms of the finite rotation vector $\mathbf{p}$ by the general relations

$$\mathbf{E} = \left[\cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\mathbf{p}\otimes\mathbf{p} - \frac{\sin\phi}{p}\mathbf{p}\times\mathbf{I}\right](\mathbf{I} + \operatorname{Grad}\mathbf{u}) - \mathbf{I}, \tag{9.11}$$

$$\boldsymbol{\Gamma} = \left[\frac{\sin\phi}{p}\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p}\otimes\mathbf{p} - \frac{1-\cos\phi}{p^2}\mathbf{p}\times\mathbf{I}\right]\operatorname{Grad}\mathbf{p}. \tag{9.12}$$

Among definitions of $\mathbf{p}$ used in the literature, let us mention the finite rotation vectors defined as

$$\boldsymbol{\theta} = 2\tan\frac{\phi}{2}\mathbf{e}, \qquad \boldsymbol{\phi} = \phi\mathbf{e}, \qquad \boldsymbol{\varpi} = \sin\phi\mathbf{e}, \qquad \boldsymbol{\rho} = \tan\frac{\phi}{2}\mathbf{e}, \tag{9.13}$$

$$\boldsymbol{\sigma} = 2\sin\frac{\phi}{2}\mathbf{e}, \qquad \boldsymbol{\mu} = 4\tan\frac{\phi}{4}\mathbf{e}, \qquad \boldsymbol{\beta} = 4\sin\frac{\phi}{4}\mathbf{e}, \tag{9.14}$$

where the generating functions are $\theta = 2\tan(\phi/2)$, ϕ, $\varpi = \sin\phi$, $\rho = \tan(\phi/2)$, $\sigma = 2\sin(\phi/2)$, $\mu = 4\tan(\phi/4)$, and $\beta = 4\sin(\phi/4)$, respectively.

The explicit formulae for $\mathbf{E}$ and $\boldsymbol{\Gamma}$ expressed in terms of the corresponding finite rotation vectors (9.13) and (9.14) are summarized in Tables 9.1 and 9.2, see [6].

When the values of $\mathbf{u}$ and ϕ as well as their spatial gradients are infinitesimal $\|\mathbf{u}\| \ll 1$, $\|\operatorname{Grad}\mathbf{u}\| \ll 1$, $|\phi| \ll 1$, $\|\operatorname{Grad}\phi\| \ll 1$, we also have $\sin\phi\approx\phi$, $\cos\phi\approx 1$, and $p(\phi)\approx\kappa\phi$. Then it follows that $\mathbf{p}\approx\kappa\boldsymbol{\vartheta}$, $\mathbf{Q}\approx\mathbf{I}+\boldsymbol{\vartheta}\times\mathbf{I}$, where $\boldsymbol{\vartheta} = \phi\mathbf{e}$ is now the infinitesimal rotation vector. Then from (9.11) and (9.12) we obtain $\mathbf{E}\approx\boldsymbol{e}\equiv\operatorname{Grad}\mathbf{u} - \boldsymbol{\vartheta}\times\mathbf{I}$, $\boldsymbol{\Gamma}\approx\boldsymbol{\gamma}\equiv\operatorname{Grad}\boldsymbol{\vartheta}$. The infinitesimal strain measures $\boldsymbol{e}$, $\boldsymbol{\gamma}$ or their transpose were used in many papers and books on linear Cosserat continuum.

Table 9.1 The natural Lagrangian stretch tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\mathbf{E}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$[1+\frac{\theta^2}{4}]^{-1}[(1-\frac{\theta^2}{4})\mathbf{I}+\frac{1}{2}\boldsymbol{\theta}\otimes\boldsymbol{\theta}-\boldsymbol{\theta}\times\mathbf{I}](\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\phi} \equiv \phi\mathbf{e}$	$(-2\pi, 2\pi)$	$[\cos\phi\,\mathbf{I}+\frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\otimes\boldsymbol{\phi}-\frac{\sin\phi}{\phi}\boldsymbol{\phi}\times\mathbf{I}](\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$[\cos\phi\,\mathbf{I}+\frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\otimes\boldsymbol{\varpi}-\boldsymbol{\varpi}\times\mathbf{I}](\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\rho^2}[(1-\rho^2)\mathbf{I}+2\boldsymbol{\rho}\otimes\boldsymbol{\rho}-2\boldsymbol{\rho}\times\mathbf{I}](\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$[(1-\frac{1}{2}\sigma^2)\mathbf{I}+\frac{1}{2}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}-\cos\frac{\phi}{2}\boldsymbol{\sigma}\times\mathbf{I}](\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$(1+\frac{\mu^2}{16})^{-2}\{[1-\frac{\mu^2}{16}(\frac{3}{8}-\frac{\mu^2}{16})]\mathbf{I}+\frac{1}{2}\boldsymbol{\mu}\otimes\boldsymbol{\mu}$ $-(1-\frac{\mu^2}{16})\boldsymbol{\mu}\otimes\mathbf{I}\}(\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$\{[1-\frac{\beta^2}{2}(1-\frac{\beta^2}{16})]\mathbf{I}+\frac{1}{2}(1-\frac{\beta^2}{8})\boldsymbol{\beta}\otimes\boldsymbol{\beta}$ $-\sqrt{1-\frac{\beta^2}{16}}(1-\frac{\beta^2}{8})\boldsymbol{\beta}\times\mathbf{I}\}(\mathbf{I}+\operatorname{Grad}\mathbf{u})-\mathbf{I}$

Table 9.2 The natural Lagrangian wryness tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\boldsymbol{\Gamma}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$[1+\frac{\theta^2}{4}]^{-1}(\mathbf{I}-\frac{1}{2}\boldsymbol{\theta}\times\mathbf{I})\operatorname{Grad}\boldsymbol{\theta}$
$\boldsymbol{\phi} \equiv \phi\mathbf{e}$	$(-2\pi, 2\pi)$	$[\frac{\sin\phi}{\phi}\mathbf{I}+\frac{\phi-\sin\phi}{\phi^3}\boldsymbol{\phi}\otimes\boldsymbol{\phi}-\frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\times\mathbf{I}]\operatorname{Grad}\boldsymbol{\phi}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$[\mathbf{I}+\frac{1}{\varpi^2}(\frac{1}{\cos\phi}-1)\boldsymbol{\varpi}\otimes\boldsymbol{\varpi}-\frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\times\mathbf{I}]\operatorname{Grad}\boldsymbol{\varpi}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$\frac{2}{1+\rho^2}(\mathbf{I}-\boldsymbol{\rho}\times\mathbf{I})\operatorname{Grad}\boldsymbol{\rho}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\mathbf{e}$	$(-\pi, \pi)$	$[\cos\frac{\phi}{2}\mathbf{I}-\frac{1}{4\cos(\phi/2)}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}-\frac{1}{2}\boldsymbol{\sigma}\times\mathbf{I}]\operatorname{Grad}\boldsymbol{\sigma}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$[1+\frac{\mu^2}{16}]^{-2}[(1-\frac{\mu^2}{16})\mathbf{I}+\frac{1}{8}\boldsymbol{\mu}\otimes\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{\mu}\times\mathbf{I}]\operatorname{Grad}\boldsymbol{\mu}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\mathbf{e}$	$(-2\pi, 2\pi)$	$[\sqrt{1-\frac{\beta^2}{16}}(1-\frac{\beta^2}{8})\mathbf{I}+\frac{1-(1-\beta^2/8)(1-\beta^2/16)}{\beta^2\sqrt{1-\beta^2/16}}\boldsymbol{\beta}\otimes\boldsymbol{\beta}$ $-\frac{1}{2}(1-\frac{\beta^2}{16})\boldsymbol{\beta}\times\mathbf{I}]\operatorname{Grad}\boldsymbol{\beta}$

9.6 Review of Some Other Lagrangian Non-Linear Strain Measures

In Table 9.3, we present a review of various definitions of the Lagrangian strain measures proposed in 14 representative papers in the field. In those works, different notation, sign conventions, notions of gradient and divergence operators, coordinate systems, description of rotations, etc. are applied. To compare them with our natural strain measures $(9.3)_3$, we bring them into the common coordinate-free form using the microrotation tensor $\mathbf{Q}$, see [5].

The results summarized in Table 9.1 show that the stretch and wryness tensors introduced in many papers do not agree with each other and with our Lagrangian

Table 9.3 Definitions of the stretch and wryness tensors

Paper	The stretch tensor	The wryness tensor
Kafadar and Eringen (1971)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Stojanovic (1972)	$\mathbf{F}^T\mathbf{F}$	$\mathbf{F}^T\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}\,\mathrm{Grad}\,\mathbf{Q}^T)$
Besdo (1974)	$\mathbf{Q}-\mathbf{I}$	$\mathbf{F}[\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{F}^{-1}\,\mathrm{Grad}\,\mathbf{F}) + \mathbf{B}]$ $-\mathbf{Q}(\boldsymbol{\Gamma}+\mathbf{B})$
Shkutin (1980)	$\mathbf{F}^T\mathbf{Q}-\mathbf{I}$	$-\frac{1}{2}[\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})]^T + \mathbf{B}^T$
Badur and Pietraszkiewicz (1986)	$\mathbf{Q}^T\mathbf{F}$	$\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Reissner (1987)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}[\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})]^T$
Zubov (1990)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}[\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})]^T$
Dłużewski (1993)	$\mathbf{Q}^T\mathbf{F}$	$\mathbf{Q}^T\,\mathrm{Grad}\,\boldsymbol{\phi}$
Merlini (1997)	$\mathbf{F}-\mathbf{Q}$,	$-\mathbf{Q}\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$,
	$\mathbf{Q}^T\mathbf{F}-\mathbf{I}$	$-\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Steinmann and Stein (1997)	$\mathbf{Q}^T\mathbf{F}$	$-\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Nikitin and Zubov (1998)	$\mathbf{Q}^T\mathbf{F}$	$-\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Grekova and Zhilin (2001)	$\mathbf{F}^T\mathbf{Q}$	$\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
Nistor (2002)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}[\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})]^T$
Ramezani and Naghdabadi (2007)	$\mathbf{F}^T\mathbf{Q}$	$\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$
The present paper and [5]	$\mathbf{Q}^T\mathbf{F}-\mathbf{I}$	$-\frac{1}{2}\boldsymbol{\varepsilon} : (\mathbf{Q}^T \,\mathrm{Grad}\,\mathbf{Q})$

strain measures defined in (9.3). Most definitions differ only by transpose of the measures, or by opposite signs, or the measures do not vanish in the absence of deformation. Such differences are not essential for the theory, although one should be aware of them. But we have also discovered a few strain measures which are incompatible with our Lagrangian stretch and wryness tensors. One should avoid such incompatible strain measures when analyzing problems of physical importance using the Cosserat continuum model.

References

1. Bauchau, O.A., Trainelli, L., The vectorial parameterization of rotation. Nonlinear Dyn. **32**, 71–92 (2003)
2. Cosserat, E., Cosserat, F., Théorie des corps déformables. Hermann et Fils, Paris (1909)
3. Gibbs, J.W., Vector Analysis. Yale University Press, New Haven (1901)
4. Kafadar, C.B., Eringen, A.C., Micropolar media—I. The classical theory. Int. J. Eng. Sci. **9**, 271–305 (1971)
5. Pietraszkiewicz, W., Eremeyev, V.A., On natural strain measures of the non-linear micropolar continuum. Int. J. Solids Struct. **46**(34), 774–787 (2009)
6. Pietraszkiewicz, W., Eremeyev, V.A., On vectorially parameterized natural strain measures the non-linear Cosserat continuum. Int. J. Solids Struct. **46**(11–12), 2477–2480 (2009)
7. Reissner, E., On kinematics and statics of finite-strain force and moment stress elasticity. Stud. Appl. Math. **52**, 93–101 (1973)