Chapter 6
Linear Cosserat Elasticity, Conformal Curvature and Bounded Stiffness

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Abstract We describe a principle of bounded stiffness and show that bounded stiffness in torsion and bending implies a reduction of the curvature energy in linear isotropic Cosserat models leading to the so called conformal curvature case $\frac{1}{2}\mu L_c^2 \|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\overline{A}\|^2$ where $\overline{A} \in \mathfrak{so}(3)$ is the Cosserat microrotation. Imposing bounded stiffness greatly facilitates the Cosserat parameter identification and allows a well-posed, stable determination of the one remaining length scale parameter L_c and the Cosserat couple modulus μ_c.

6.1 Introduction

Non-classical size-effects are becoming increasingly important for materials at the micro- and nanoscale regime. There are many possibilities in order to include size-effects on the continuum scale. One such prominent model is the Cosserat model. In its simplest isotropic linear version, the Cosserat model introduces six material

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parameters. However, parameter identification for Cosserat solids remains a difficult and challenging issue.

We motivate how to a priori reduce the number of curvature parameters in the linear, isotropic, centro-symmetric Cosserat model by requiring what we identify with *bounded stiffness*. First, we recall the Cosserat model and we motivate bounded stiffness in general. Imposing bounded stiffness reduces the curvature energy to the conformally invariant case, which is the weakest possible requirement for well-posedness of the linear isotropic Cosserat model [1].

It is well-known that a Cosserat solid displays size-effects. These size effects refer to a non-classical dependence of rigidity of an object upon one or more of its dimensions. In classical linear elasticity for a circular cylinder with radius a, the rigidity in tension is proportional to a^2, and the rigidity in torsion and bending is proportional to a^4. For the Cosserat solid, the ratio of rigidity to its classical value is increased: thinner samples of the same material respond stiffer. For certain parameter ranges of the Cosserat solid, this effect may be dramatic. For example, the rigidity in torsion could become proportional to a^2 so that the normalized torsional rigidity (normalized against the classical value) has a singularity proportional to $1/a^2$. However, Lakes [3] already notes: "... infinite stiffening effects are unphysical."

Our principle of bounded stiffness requires simply that the stiffness increase for thinner and thinner samples (normalized against the classical stiffness) should be bounded independently of the wire radius a, i.e., a singularity free response. In bending and torsion, we can directly read off the corresponding requirement. It leads in a straightforward way to what we term the conformal curvature case $\frac{1}{2}\mu L_c^2\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\overline{A}\|^2$. In separate contributions [10, 2, 7, 9, 1, 8], we have investigated, in more detail, this novel conformal curvature case from alternative perspectives.

6.2 The Linear Elastic Cosserat Model

For the *displacement* $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the *skew-symmetric infinitesimal microrotation* $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$, we consider the *two-field* minimization problem

$$I(u,\overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla\operatorname{axl}\overline{A}) - \langle f, u\rangle\,\mathrm{d}V \\ - \int_{\partial\Omega}\langle f_s, u\rangle - \langle M_s, u\rangle\,\mathrm{d}S \mapsto \min \quad \text{w.r.t.} \quad (u,\overline{A}), \tag{6.1}$$

under the following constitutive requirements and boundary conditions:

$$\overline{\varepsilon} = \nabla u - \overline{A}, \quad \textit{first Cosserat stretch tensor}$$

$$u_{|\Gamma} = u_{\mathrm{d}}, \quad \textit{essential displacement boundary conditions}$$

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2, \quad \textit{strain energy}$$

$$\phi := \operatorname{axl} \overline{A} \in \mathbb{R}^3, \qquad \overline{\mathfrak{k}} = \nabla \phi, \tag{6.2}$$
$$\| \operatorname{curl} \phi \|^2_{\mathbb{R}^3} = 4 \| \operatorname{axl} \operatorname{skew} \nabla \phi \|^2_{\mathbb{R}^3} = 2 \| \operatorname{skew} \nabla \phi \|^2_{\mathbb{M}^{3\times 3}},$$
$$W_{\text{curv}}(\nabla \phi) = \frac{\gamma + \beta}{2} \| \operatorname{dev} \operatorname{sym} \nabla \phi \|^2 + \frac{\gamma - \beta}{2} \| \operatorname{skew} \nabla \phi \|^2 + \frac{3\alpha + (\beta + \gamma)}{6} \operatorname{tr}[\nabla \phi]^2.$$

Here, f are given volume forces while u_{d} are Dirichlet boundary conditions for the displacement at $\Gamma \subset \partial \Omega$ where $\Omega \subset \mathbb{R}^3$ denotes a bounded Lipschitz domain. Surface tractions, volume couples and surface couples can be included in the standard way. The strain energy W_{mp} and the curvature energy W_{curv} are the most general isotropic quadratic forms in the *infinitesimal non-symmetric first Cosserat strain tensor* $\overline{\varepsilon} = \nabla u - \overline{A}$ and the *micropolar curvature tensor* $\overline{\mathfrak{k}} = \nabla \operatorname{axl} \overline{A} = \nabla \phi$ (curvature-twist tensor). The parameters μ, λ [MPa] are the classical Lamé moduli and α, β, γ are further micropolar curvature moduli with dimension $[\text{Pa} \cdot \text{m}^2] = [\text{N}]$ of a force. The additional parameter $\mu_c \geq 0$ [MPa] in the strain energy is the *Cosserat couple modulus*. For $\mu_c = 0$ the two fields of displacement u and microrotations $\overline{A} \in \mathfrak{so}(3)$ decouple, and one is left formally with classical linear elasticity for the displacement u. The strong form Cosserat balance equations are given by

$$\begin{aligned}
&\operatorname{Div} \sigma = f, \quad \textit{balance of linear momentum} \\
&-\operatorname{Div} m = 4\mu_c \cdot \operatorname{axl} \operatorname{skew} \overline{\varepsilon}, \quad \textit{balance of angular momentum} \\
&\sigma = 2\mu \cdot \operatorname{sym} \overline{\varepsilon} + 2\mu_c \cdot \operatorname{skew} \overline{\varepsilon} + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot \mathbb{1} \\
&\quad = (\mu + \mu_c) \cdot \overline{\varepsilon} + (\mu - \mu_c) \cdot \overline{\varepsilon}^T + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot \mathbb{1} \\
&\quad = 2\mu \cdot \operatorname{dev} \operatorname{sym} \overline{\varepsilon} + 2\mu_c \cdot \operatorname{skew} \overline{\varepsilon} + K \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot \mathbb{1}, \\
&m = (\gamma + \beta) \operatorname{dev} \operatorname{sym} \nabla \phi + (\gamma - \beta) \operatorname{skew} \nabla \phi + \frac{3\alpha + (\gamma + \beta)}{3} \operatorname{tr}[\nabla \phi] \mathbb{1}, \\
&\phi = \operatorname{axl} \overline{A}, \qquad u_{|\Gamma} = u_{\text{d}}.
\end{aligned} \tag{6.3}$$

For simplicity we do not make an explicit statement for the boundary conditions which are satisfied by the microrotations. Note only that if the microrotations $\overline{A} \in \mathfrak{so}(3)$ remain free at the boundary, we would have $m.\mathbf{n}_{|\partial \Omega} = 0$. This Cosserat model can be considered with basically three different sets of moduli for the curvature energy which in each step relaxes the curvature energy. The situations are characterized by possible estimates for the curvature energy:

1. $W_{\text{curv}}(\overline{\mathfrak{k}}) \geq c^+ \| \overline{\mathfrak{k}} \|^2$,
2. $W_{\text{curv}}(\overline{\mathfrak{k}}) \geq c^+ \| \operatorname{sym} \overline{\mathfrak{k}} \|^2$,
3. $W_{\text{curv}}(\overline{\mathfrak{k}}) \geq c^+ \| \operatorname{dev} \operatorname{sym} \overline{\mathfrak{k}} \|^2$.

The different estimates give rise to the introduction of representative cases:

1. (*Pointwise positive case*) $\frac{1}{2}\mu L_c^2\|\nabla\phi\|^2$. This corresponds to $\alpha = \beta = 0$, $\gamma = \mu L_c^2$.
2. (*Symmetric case*) $\frac{1}{2}\mu L_c^2\|\operatorname{sym}\nabla\phi\|^2$. This corresponds to $\alpha = 0, \beta = \gamma$ and $\gamma = \frac{1}{2}\mu L_c^2$.
3. (*Conformal case*) $\frac{1}{2}\mu L_c^2\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 = \frac{1}{2}\mu L_c^2(\|\operatorname{sym}\nabla\phi\|^2 - \frac{1}{3}\operatorname{tr}[\nabla\phi]^2)$. This corresponds to $\beta = \gamma$ and $\gamma = \frac{1}{2}\mu L_c^2$ and $\alpha = -\frac{1}{3}\mu L_c^2$. In terms of the polar ratio $\Psi = (\beta+\gamma)/(\alpha+\beta+\gamma)$, it corresponds to the limit value $\Psi = 3/2$.

All three cases are mathematically well-posed [1, 5]. The pointwise positive Case 1 is usually considered in the literature. Case 2 leads to a symmetric couple-stress tensor m, and a new motivation for Case 3 is the subject of this contribution. In a plane strain problem, all three cases coincide and only one curvature parameter matters, thus not permitting to discern any relation between the three curvature parameters.

Case 3 is called the conformal curvature case since the curvature energy is invariant under superposed infinitesimal conformal mappings, i.e., mappings $\phi_C : \mathbb{R}^3 \mapsto \mathbb{R}^3$ that satisfy $\operatorname{dev}\operatorname{sym}\nabla\phi_C = 0$. Such mappings infinitesimally preserve shapes and angles [7]. In that case, the couple stress tensor m is symmetric and trace-free. In Case 2 and Case 3, the constitutive couple stress/curvature tensor relation cannot be inverted, but the system of equations is nevertheless well-posed.

6.3 The Idea of Bounded Stiffness

Let us turn our attention to the practical aspects of the problem of determining material parameters. We investigate the question for which parameter values (μ_c, α, β, γ) the linear elastic Cosserat model can be considered to be a consistent description for a continuous solid showing size-effects. We assume the continuous solid to be available in any small size we can think of, the possibility of which is certainly included in the very definition of a continuous solid. Note that this assumption *excludes*, e.g., man made grid-structures but includes, e.g., polycrystalline material. We are investigating the situation when one or several dimensions of the specimen get small. Denoting by a such a dimension, the limit $a \to 0$ is purely formal in the sense that we are only interested in the leading order behavior for small, but not arbitrarily small a. Understanding this limiting process $a \to 0$ opens us, indirectly, the possibility to bound the stiffness of the material at smallest reasonable specimen size away from unrealistic orders of magnitude. Our conclusions are based on simple three-dimensional boundary value problems for which analytical solutions are available.

By examining the bending and torsion analytical formula and calculating the stiffness increase at small wire radius a, we are forced to conclude that the conformal curvature case $\frac{1}{2}\mu L_c^2\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ is the only one possible [8]. Any other combination of parameters will lead to unphysical stiffening effects. The requirement of bounded stiffness is also very natural if we compare with atomistic simulations; in such a case, all response curves will show bounded energy *and* bounded stiffness.

One should keep in mind, however, that the internal length scale L_c remains a phenomenological parameter in the Cosserat model, the value of which is not necessarily determined by a given microstructural length scale, although this is often tacitly understood. On the contrary, L_c could also be chosen large in which case the ratio L_c/a may be very large, although wires with given radius a can still be experimentally investigated.

The remainder of this paragraph is a free adaption of a statement given by Metrikine in [4, p. 740] to our situation: No material at no scale is perfectly rigid. Therefore, their stiffness is bounded at all scales. This is one of the most fundamental principles in modern physics, and any general model, which is supposed to be applicable at the complete length scale, must satisfy it. All researchers agree with that statement. Many models, however, are designed to work only at a specific length scale. Homogeneous continuum elasticity models, for example, are all applicable only at a relatively large length scale (i.e., the length scale of the corresponding mechanical processes is much larger than the characteristic length of the material microstructure). Should such models comply with bounded stiffness? There is no consensus among researchers as to how to answer this question in the case that unbounded stiffness is associated with those length scales at which the model in question is not applicable according to its initial assumptions.

The authors of this paper advocate the following answer to the above question. Imagine two models of a material which, with the same accuracy, describe static behavior of the material at a desired length scale. Imagine further that one model has bounded stiffness whereas the other has not, but its singular stiffness is associated with the length scales outside the considered length scale range. In this case, the bounded stiffness model should be preferred.

It should not only be done because it complies with bounded stiffness. More importantly, models with bounded stiffness profit from an insensitive parameter identification. This is important for experimental identification of material properties.

Whether or not the Cosserat model with conformal curvature shows bounded stiffness in all possible boundary value problems is not yet clear. We surmise that the micromorphic model [6] is a good candidate for that purpose.

6.4 Infinitesimal Conformal Mappings

The maps $u_C : \mathbb{R}^3 \mapsto \mathbb{R}^3$ that satisfy $\operatorname{dev}\operatorname{sym}\nabla u_C = 0$ are called infinitesimal conformal mappings. They form a ten-dimensional vector space and are given in closed form by

$$u_C(x) = \frac{1}{2}\big(2\langle \operatorname{axl}(\widehat{W}), x\rangle x - \operatorname{axl}(\widehat{W})\|x\|^2\big) + [\widehat{p}\mathbb{1} + \widehat{A}].x + \widehat{b}, \tag{6.4}$$

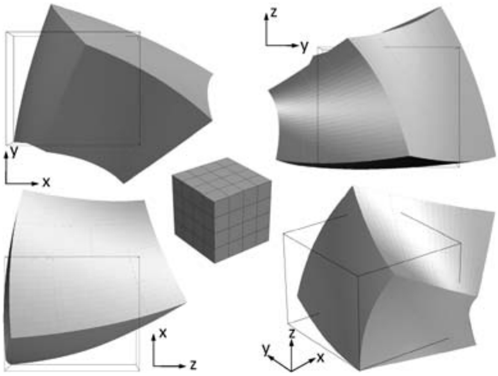

Fig. 6.1 Deformed and undeformed shape of the unit cube under infinitesimal conformal transformation u_C as boundary condition using the linear elastic Cosserat model and Case 3 assumption with $\widehat{W}_{12} = \widehat{W}_{13} = \widehat{W}_{23} = 3$, $\widehat{p} = -4$, $\widehat{A}_{12} = \widehat{A}_{13} = \widehat{A}_{23} = 4$ and $\widehat{b} = 0$, DOFs $= 4300$, quadratic elements

where $\widehat{W}, \widehat{A} \in \mathfrak{so}(3)$, $\widehat{b} \in \mathbb{R}^3$ and $\widehat{p} \in \mathbb{R}$ are arbitrary constants. The conformal mappings may be used to construct a universal solution of the Cosserat model if the conformal curvature expression (Case 3) is considered. The solution is invariant of the Cosserat coupling constant μ_c and the internal length scale L_c.[1] To understand this claim, let us consider the boundary value problem of linear Cosserat elasticity in strong form with conformal curvature:

$$\begin{aligned} &\operatorname{Div}\sigma(\nabla u, \overline{A}) = f, \qquad -\operatorname{Div} m = 4\mu_c \cdot \operatorname{axl}(\operatorname{skew}\nabla u - \overline{A}),\\ &\sigma = 2\mu \cdot \operatorname{dev}\operatorname{sym}\nabla u + 2\mu_c \cdot \operatorname{skew}(\nabla u - \overline{A}) + K \cdot \operatorname{tr}[\nabla u] \cdot \mathbb{1}, \\ &m = \mu L_c^2 \cdot \operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}(\overline{A}), \qquad u_{|\partial\Omega} = u_C, \end{aligned} \tag{6.5}$$

where K is the bulk modulus. Inserting for u an *infinitesimal conformal map* u_C (Fig. 6.1) which is defined by (6.4) and choosing $\overline{A}(x) = \operatorname{anti}(\frac{1}{2}\operatorname{curl} u(x))$ simplifies the equations to

$$\begin{aligned} &\operatorname{Div}\sigma(\nabla u, \overline{A}) = \widehat{f}, \qquad -\operatorname{Div} m = 0,\\ &\sigma = 2\mu \cdot \operatorname{dev}\operatorname{sym}\nabla u + K \cdot \operatorname{tr}[\nabla u] \cdot \mathbb{1}, \\ &m = \mu L_c^2 \operatorname{dev}\operatorname{sym}\nabla\left[\frac{1}{2}\operatorname{curl} u(x)\right]. \end{aligned} \tag{6.6}$$

$\operatorname{Div}\sigma(\nabla u_C) = 3K\operatorname{axl}(\widehat{W})$, for $\widehat{p}$ is constant and $\operatorname{Div}[\langle\widehat{k}, x\rangle\mathbb{1}] = \widehat{k}$. Since the boundary value problem $\operatorname{Div}\sigma(\nabla u) = 3K\operatorname{axl}(\widehat{W})$, $u_{|_\Gamma}(x) = u_C(x)$ for a given constant $\widehat{W} \in \mathfrak{so}(3)$ admits a unique solution, this solution is already given by $u(x) = u_C(x)$. We have therefore obtained an inhomogeneous, three-dimensional analytical solution for the boundary value problem of linear Cosserat elasticity with constant body forces $\widehat{f} = 3K\operatorname{axl}(\widehat{W})$. To gain further understanding of the conformal Cosserat model (Case 3) we subject a regular and rectangular network of beams

[1] Here, even a strong variation in shear modulus $\mu(x)$ would be allowed (as well as a strong variation in the couple modulus $\mu_c(x)$ and internal length scale $L_c(x)$. Only the bulk modulus K must be constant.

to an infinitesimal conformal displacement (6.4). In our comparison, we use $\widehat{b} = 0$, $\widehat{A} = 0$ and some generic values for $\widehat{p}$ and $\widehat{W}$. The area of the squared structure is 1 and the beams are characterized by a quadratic cross-section of dimension 0.05. Thus, the area of the cross-section amounts to 0.0025 and the moment of inertia to 5.208×10^{-7} against bending. The Young's modulus is set to $E = 1$. We use the Bernoulli beam theory of second order. Thus, displacements and rotations are limited to a reasonable amount. First, only nodes on the boundary are conformally displaced, all other organize themselves by minimizing strain and curvature energy of the beams. In Fig. 6.2, one can see the initial rectangular beam structure, the boundary conditions and the displacement vectors bringing all nodes of the boundary into the conformally corresponding points. The nodes within the structure meet the condition, which is fulfilled for all beams balance of momentum and balance of angular momentum.

The right picture in Fig. 6.2 indicates that curvature appears nearly everywhere in all beams. The maximum value of this curvature is about 6.24. Now, all nodes of the structure are conformally displaced. Thus, only the curvature energy of the beams can be minimized. In Fig. 6.3, one can see the initial rectangular beam structure, the boundary conditions and the displacement vectors bringing all nodes into their conformally corresponding points. The beams preserve the balance of angular momentum. The right picture in Fig. 6.3 indicates that curvature appears also nearly everywhere in all beams. The maximum value of this curvature is about 7.02. While the infinitesimal conformal mapping does not give rise to a curvature contribution in the Cosserat model with conformal curvature (Case 3), we clearly see that the beam-network response is always with curvature. This allows us to already con-

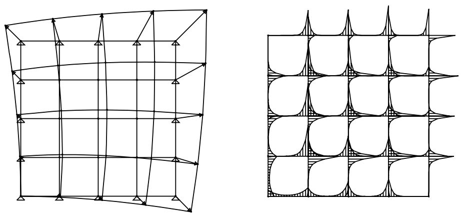

Fig. 6.2 (*Left*) Initial structure with boundary conditions, conformal displacement of boundary nodes and deformed mesh. (*Right*) Trend of curves (plotted on undeformed mesh) indicates the curvature of beams

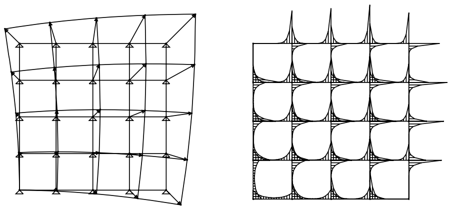

Fig. 6.3 (*Left*) Initial structure with boundary conditions, conformal displacement of all nodes and deformed mesh. (*Right*) Trend of curves (plotted on undeformed mesh) indicates the curvature of beams

clude that *the conformal Cosserat model cannot be identified with a homogenized beam model*. We rather expect a homogenized beam model to give rise to a uniform positive definite curvature expression as embodied in Case 1 of our classification.

Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing two-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \mathrm{tr}[XY^T]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following, we omit the indices $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}[X] = \langle X, \mathbb{1}\rangle$. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ so that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\mathrm{dev}\, X = X\frac{1}{3}\mathrm{tr}[X]\mathbb{1} \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\mathrm{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3) \mid X^T X = \mathbb{1}, \det X = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \mid X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\mathrm{axl}\,\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. Note that $(\mathrm{axl}\,\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$ so that

$$\mathrm{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \qquad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk} \cdot \mathrm{axl}\,\overline{A}_k, \tag{6.7}$$
$$\|\overline{A}\|^2_{\mathbb{M}^{3\times3}} = 2\|\,\mathrm{axl}\,\overline{A}\|^2_{\mathbb{R}^3}, \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times3}} = 2\langle \mathrm{axl}\,\overline{A}, \mathrm{axl}\,\overline{B}\rangle_{\mathbb{R}^3},$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \mathrm{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \tag{6.8}$$
$$\mathrm{axl}\big(\mathrm{skew}(a \otimes b)\big) = -\frac{1}{2} a \times b,$$

and $2\,\mathrm{skew}(b \otimes a) = \mathrm{anti}(a \times b) = \mathrm{anti}(\mathrm{anti}(a).b)$. Moreover, $\mathrm{curl}\, u = 2\,\mathrm{axl}(\mathrm{skew}\,\nabla u)$.

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