Chapter 4
Cracks in Cosserat Continuum—Macroscopic Modeling

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Abstract Modeling of particulate and layered materials (e.g., concrete and rocks, rock masses) by Cosserat continua involves characteristic internal lengths which can be commensurate with the microstructural size (the particle size or layer thickness). When fracture propagation in such materials is considered, the criterion of their growth is traditionally based on the parameters of the crack-tip stress singularities referring to the distances to the crack tip smaller than the characteristic lengths and hence smaller than the microstructural size. This contradicts the very notion of continuum modeling which refers to the scales higher than the microstructural size. We propose a resolution of this contradiction by considering an intermediate asymptotics corresponding to the distances from the crack tip larger than the microstructural sizes (the internal Cosserat lengths) but yet smaller than the crack length. The approach is demonstrated using examples of shear crack in particulate and bending crack in layer materials.

4.1 Introduction

In a variety of materials with microstructure, there are two important types whose adequate continuum modeling requires the consideration of rotational degrees of freedom—the particulate and layered materials with week bonding. Such materials represent a wide class of natural and structural materials including rocks and rock masses, soils, concrete and composites. As the continuum modeling is based on a homogenization procedure which introduces the averaging scale (e.g., the volume element size), the distances smaller than the averaging scale cannot be addressed

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G.A. Maugin, A.V. Metrikine (eds.), *Mechanics of Generalized Continua*, Advances in Mechanics and Mathematics 21,
DOI 10.1007/978-1-4419-5695-8_4, © Springer Science+Business Media, LLC 2010

in such a modeling even if all distances are equally accessible in the continuum. When the higher order continua such as Cosserat continuum are employed, they bring their own characteristic lengths. Then the interplay between the characteristic lengths of the continuum and the scale of averaging becomes important and in some cases can control the results of the modeling. This paper investigates crack propagation in the Cosserat continua whose characteristic lengths commensurate with the microstructural length of the material (the particle size or the layer thickness).

4.2 The Length Scales in Continuum Description

Continuum modeling is based on the introduction of the volume element of size H satisfying the following double inequality under the hypothesis of separation of scales:

$$l_m \ll H \ll L, \tag{4.1}$$

where l_m is the characteristic size of the material microstructure (e.g., the grain or defect size, the distance between the microstructural elements, etc.), L is the characteristic size of the area under consideration, the wave length, etc. The first inequality in (4.1) is required for the volume element to be representative. The second inequality stipulates that the characteristic length l_g of redistribution of the fields involved in the description be considerably greater than H since the fields in the continuum description are approximated by uniform within H. In other words, the variations of the field over distance H should be negligible. This can be quantified as follows.

Consider, for the sake of simplicity, a scalar function $f(x)$ continuously differentiable in a vicinity of 0, which signifies the origin of a suitable coordinate frame. If the variations of f are negligible over H then

$$\left|f(H) - f(0)\right| \ll \left|f(0)\right|.$$

Keeping only the linear term in the Taylor expansion, this condition gives

$$l_g = \left|\frac{f(0)}{f'(0)}\right| \gg H. \tag{4.2}$$

If the function is regular at the origin then $l_g \sim L$ and (4.2) are equivalent to the right-hand inequality (4.1). If, however, the field is singular at the origin then we shall write (4.2) at a point $x \neq 0$ and then allow x tend to zero. In particular, when the singularity is a power law, $|f(x)| \sim x^a$, as in the cases of corner singularities, Condition (4.2) is replaced by $x/a \gg H$. This corresponds to the asymptotics

$$x/H \to \infty \tag{4.3}$$

which is the outer asymptotics of the singularity. Thus in dealing with singularities, one first has to obtain the asymptotics (4.3) by fixing x and tending H (and subsequently l_m) to zero and then tending L to infinity. Upon rescaling this leads to the inner asymptotics $x \to 0$.

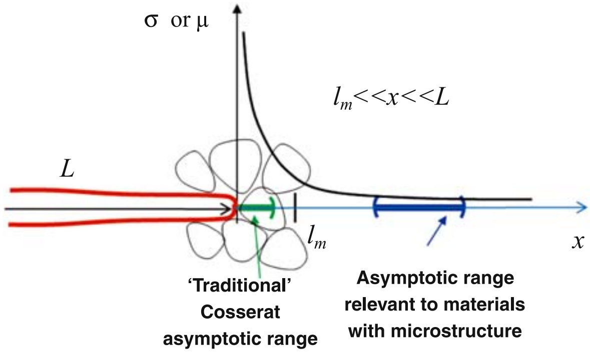

Fig. 4.1 In particulate material, the traditional asymptotic range falls within the microstructural elements. The proposed asymptotics works on the boundary between the microstructural scale and the scale of the crack

Standard continua do not inherit a microstructural length, hence this asymptotics becomes redundant. For non-standard continua possessing characteristic lengths, the asymptotics (4.3) can be put to use to achieve considerable simplification. The practical way of doing this is to neglect the terms of the type of $Hf'(x)$ as compared to $f(x)$. By continuing the expansion in the Taylor series, one can obtain that in this asymptotics the terms of the type of $H^2 f''(x), \ldots$ should also be neglected.

In essence, a continuum constructed to represent a heterogeneous material cannot address the distances shorter than the volume element size. This fact manifests itself in the vicinity of a stress singularity: The stress distribution computed in the continuum may misrepresent the stress state of the original homogeneous material in the vicinity of the crack tip. Instead, the inner region of the outer asymptotics should be used. Therefore, instead of the traditional modeling based just on the $x \to 0$ singularity, one needs to obtain the outer asymptotics first and then use it to determine the singularity, see Fig. 4.1.

Of course, if the continuum is a classical one without a characteristic length then the complete solution and the outer asymptotics coincide, so the traditional approach does not change. The difference only comes for continua with internal crack lengths, such as Cosserat continua.

The characteristic lengths, l, are related to the microstructural length of the material through the values of the material constants. In principle, the characteristic lengths can be considerably higher than the microstructural length such that if the latter is beyond the resolution of the continuum the formers are not. We, however, distinguish one important case when $l \sim H$ such that the Cosserat characteristic lengths are also beyond the resolution of the continuum. We call it the *small scale Cosserat continuum*. The following sections will provide two examples of small scale continua and consider the simplest cases of crack propagation in them.

4.3 Small Scale Isotropic Cosserat Continuum as a Model of Particulate Material

We start with an isotropic Cosserat continuum model of particulate materials (such as rock, concrete, ceramics). A model was proposed in [1] whereby the particulate material was represented as an assembly of spherical particles connected at random points. Each connection was considered as a combination of 6 elastic springs: 1 tensile and 2 shear springs, as well as 1 torsional and 2 bending springs. The stiffnesses of the springs were estimated by considering the springs as elastic cylinders of diameter commensurate with the grain size loaded either by uniform stress (for the normal and shear stiffnesses) or linearly distributed stress with vanishing average (for the torsional and bending stiffnesses). After applying homogenization by differential expansions (e.g., [7]) an isotropic Cosserat continuum was obtained with the characteristic lengths of the order of the grain size. Thus $l \sim H$, i.e., the Cosserat characteristic lengths are also beyond the resolution of the continuum, and hence the asymptotics of $l \to 0$ discussed in the previous section applies.

Consider, for the sake of simplicity, a two dimensional Cosserat continuum in the plane (x_1, x_2). The Lamé equations read (e.g., [5, 4])

$$\begin{cases} (\lambda+2\mu)[\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_1 \partial x_2}]+(\mu+\alpha)[\frac{\partial^2 u_1}{\partial x_2^2}-\frac{\partial^2 u_2}{\partial x_1 \partial x_2}]+2\alpha\frac{\partial \varphi_3}{\partial x_2}=0, \\ (\lambda+2\mu)[\frac{\partial^2 u_1}{\partial x_1 \partial x_2}+\frac{\partial^2 u_2}{\partial x_2^2}]+(\mu+\alpha)[\frac{\partial^2 u_2}{\partial x_1^2}-\frac{\partial^2 u_1}{\partial x_1 \partial x_2}]-2\alpha\frac{\partial \varphi_3}{\partial x_1}=0, \\ l_2^2[\frac{\partial^2 \varphi_3}{\partial x_1^2}+\frac{\partial^2 \varphi_3}{\partial x_2^2}]-\varphi_3+\frac{1}{2}[\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}]=0. \end{cases} \tag{4.4}$$

The only characteristic length present here is the length l_2 of the following group of lengths which characterize the isotropic Cosserat continuum:

$$l_1=\sqrt{B/(4\mu)}; \qquad l_2=\sqrt{B/(4\alpha)}; \qquad l=\sqrt{l_1^2+l_2^2}, \tag{4.5}$$

where λ and μ are the Lamé constants (μ is the shear modulus), α is the Cosserat shear modulus, B is the bending stiffness.

As l_2 is the only length parameter explicit in (4.4), the asymptotics of small volume elements requires that $l_2 \to 0$, which reduces the third equation of (4.4) to

$$\varphi_3=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right), \tag{4.6}$$

implying that the asymptotics of small characteristic size formally leads to the *Cosserat continuum with constrained rotations*. We can use the corresponding solutions to obtain the required asymptotics.

4.4 Shear Crack in Small Scale Isotropic Cosserat Continuum: A Mechanism of In-Plane Growth

Traditional treatment of cracks in Cosserat continuum [4, 2] is based on extracting of asymptotics $r \ll l$, where r is the distance to the crack tip. Subsequently, the Cosserat continuum with constrained rotations had to be employed to avoid insurmountable technical difficulties. Here we use the above intermediate asymptotics and obtain the Cosserat continuum with constrained rotations as a consequence of the asymptotics rather than a palliative.

Consider a shear crack of length $2L$ under uniform loading. The boundary conditions for this problem are

$$\sigma_{21} = \tau_0, \qquad \sigma_{22} = 0, \qquad \mu_{23} = 0 \quad \text{for } -L \le x_1 \le L,\ x_2 = 0, \tag{4.7}$$

while the stresses and displacements are continuous outside $[-L, L]$.

We represent the crack as a continuous distribution of dislocations (displacement discontinuities) and disclinations (rotation discontinuities in the case of Cosserat continuum) of unknown density in the otherwise continuous material and then equating the stress produced by all dislocations at the points of the crack contour to the boundary conditions. However, due to symmetry of boundary conditions (4.7), only dislocations with Burgers vector parallel to x_1 need to be taken into account [1].

A solution for this dislocation in the Cosserat continuum with constrained rotations (as required by the small scale asymptotics) can be found in [2], from where the main terms of asymptotics $r \gg l$ are:

$$\begin{aligned}
\frac{\sigma_{11}}{\mu b} &= \frac{1}{r}\left[-\frac{3\sin\theta + \sin 3\theta}{4\pi(1-\nu)} + O\left(\frac{l^2}{r^2}\right)\right],\\
\frac{\sigma_{22}}{\mu b} &= \frac{1}{r}\left[\frac{\sin 3\theta - \sin\theta}{4\pi(1-\nu)} + O\left(\frac{l^2}{r^2}\right)\right],\\
\frac{\sigma_{12}}{\mu b} &= \frac{1}{r}\left[\frac{\cos\theta + \cos 3\theta}{4\pi(1-\nu)} + O\left(\frac{l^2}{r^2}\right)\right],\\
\frac{\sigma_{21}}{\mu b} &= \frac{1}{r}\left[\frac{\cos\theta + \cos 3\theta}{4\pi(1-\nu)} + O\left(\frac{l^2}{r^2}\right)\right],\\
\mu_{13} &= 2\frac{\mu b l^2}{\pi r^2}\cos 2\theta + O\left(\frac{l^2}{r^2}\right),\\
\mu_{23} &= 2\frac{\mu b l^2}{\pi r^2}\sin 2\theta + O\left(\frac{l^2}{r^2}\right).
\end{aligned}\tag{4.8}$$

Here b is the value of the Burgers vector, $r^2 = x_1^2 + x_2^2$, θ is the polar angle.

The main asymptotic terms for stress (4.8) coincide with the solution for a conventional dislocation in the classical isotropic elastic continuum [3]. Furthermore, moment stress μ_{23} created by the dislocation on the crack line $(\theta = 0, \pm\pi)$ is equal to zero. This means that the dislocation does not contribute to the moment stress on the crack line. Therefore, the boundary conditions (4.7) can be satisfied by the appropriate distribution of the dislocations without disclinations. (There is no coupling between the dislocations and disclination for shear cracks.) Subsequently, the problem reduces to the conventional one for a shear crack whose solution for a uniform load is known. Furthermore, since we are only interested in the stress concentration at the crack tip, we can obtain a general result using the expression for the displacement discontinuity at the crack tip. We use the expression for the distribution of the relevant displacement component u_1 in a vicinity of a Mode II crack tip [8] as well as its discontinuity across the crack line and its derivative, which is the dislocation density, ρ_1:

$$\begin{aligned} u_1 &= \frac{K_{\mathrm{II}}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[2(1-\nu)+\cos^2\frac{\theta}{2}\right], \\ \rho_1(r) &= \frac{2K_{\mathrm{II}}(1-\nu)}{\mu\sqrt{2\pi r}}. \end{aligned} \tag{4.9}$$

Here (r,θ) is the polar coordinate frame with the origin placed at the crack tip, K_{II} is the conventional Mode II stress intensity factor, ν is the Poisson's ratio.

Using the expression for the dislocation-generated moment stresses (4.8) one obtains the moment stress at the crack tip on the line of crack continuation ($\theta = 0$) whose leading asymptotic term (as $r \to 0$) reads

$$\mu_{13} = \frac{K_{\mathrm{II}}(1-\nu)}{\sqrt{2\pi}r^{3/2}}l^2, \qquad \mu_{23} = 0. \tag{4.10}$$

This is a stronger singularity than the classical one. However, it does not contribute to the energy release rate because of the absence of reciprocal rotation discontinuity in Mode II. In the spirit of the philosophy proposed, the way to treat it is to compare the values of $\mu_{13}(l)$, which are finite. In particular, the criterion of crack propagation can be formulated in terms of the critical value of the moment stress needed to break the bonds and make the particles rotate. It is important that the maximum moment stress is acting at the continuation of the crack, which forms a mechanism of often observed in-plane shear crack propagation.

4.5 Anisotropic Cosserat Continuum for Layered Material with Sliding Layers

In 2D layered materials, the role of rotation is played by the bending angle (the derivative of layer deflection). For the corresponding Cosserat continuum, in the case of freely sliding layers the displacement and rotation fields, in the absence of body forces and moments, are governed by the following set of equations [6]:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}=0,\qquad \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}=0,\qquad \frac{\partial\mu_{31}}{\partial x_1}+\sigma_{12}-\sigma_{21}=0,$$
$$\sigma_{11}=A_{11}\gamma_{11}+A_{12}\gamma_{22},\qquad \sigma_{22}=A_{12}\gamma_{11}+A_{22}\gamma_{22},\tag{4.11}$$
$$\sigma_{21}=0,\qquad \sigma_{12}=G\gamma_{12},\qquad \mu_{13}=B\kappa_{13},$$

where

$$\gamma_{11}=\frac{\partial u_1}{\partial x_1},\qquad \gamma_{21}=\frac{\partial u_1}{\partial x_2}+\varphi_3,\qquad \gamma_{22}=\frac{\partial u_2}{\partial x_2},$$
$$\gamma_{12}=\frac{\partial u_2}{\partial x_1}-\varphi_3,\qquad \kappa_{13}=\frac{\partial\varphi_3}{\partial x_1},\tag{4.12}$$

E, ν are the layer's Young's modulus, and Poisson's ratio, $G=E/(2(1+\nu))$, b is the layer thickness,

$$A_{11}=A_{22}=\frac{(1-\nu)E}{(1+\nu)(1-2\nu)},$$
$$A_{12}=\frac{\nu E}{(1+\nu)(1-2\nu)},\qquad B=\frac{Eb^2}{12(1-\nu^2)}.\tag{4.13}$$

4.6 Bending Crack

We consider now a special type of the crack—a bending crack which is oriented normal to the layering and is represented as a continuous distribution of disclinations, which are discontinuities in rotations, see Fig. 4.2. The leading terms of the outer asymptotics of stress and moment stress are [6]:

$$\sigma_{22}(0,x_2)=-\frac{Eb^{1/2}}{4(1-\nu^2)\sqrt{2\pi}3^{1/4}}\frac{1}{\sqrt{|x_2|}},$$
$$\mu_{13}(0,x_2)=-\frac{Eb^{3/2}3^{1/4}}{24(1-\nu^2)\sqrt{2\pi}}\frac{\mathrm{sgn}(x_2)}{\sqrt{|x_2|}}.\tag{4.14}$$

The crack can be modeled as a distribution of disclinations sitting on the x_2 axis, as shown in Fig. 4.2, then the only stress component the disclinations produce there is μ_{13}. This leads to an integral equation whose solution for a semi-infinite crack loaded by a couple of concentrated moments, m, applied at a distance a from its tip is [6]:

$$\mu_{13}\sim\frac{\mathrm{M}_3}{(x_2)^{1/4}},\qquad \mathrm{M}_3=-\frac{m\sqrt{2}}{2\pi a^{3/4}}.\tag{4.15}$$

The stress singularity here is much weaker than in the Mode II crack (Sect. 4.4). Similarly to the case of a crack in particulate materials, the criterion of crack prop-

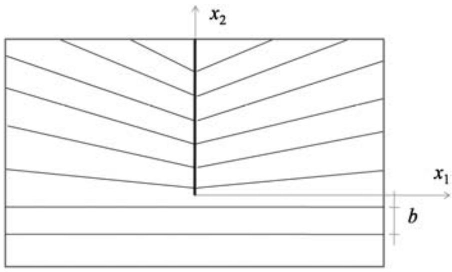

Fig. 4.2 Bending crack in layered material

agation can be formulated in terms of $\mu_{13}(b)$ compared with the bending moment per unit area required to break the layers.

4.7 Conclusions

Modeling particulate materials with the small-scale Cosserat continuum—a continuum with characteristic scales of the order of the microstructural size—justifies the asymptotics of small characteristic lengths. In 2D isotropic Cosserat continuum, this asymptotics formally produces a Cosserat continuum with constraint rotations, which is the first simplification of the proposed modeling. Furthermore, the stress singularities should be considered at the distances greater than the characteristic lengths, as opposed to the traditional approach.

The Mode II crack in a small scale isotropic Cosserat continuum produces moment stress with a strong ($3/2$ power) singularity controlled by the conventional Mode II stress intensity factor. The bending crack (discontinuity in rotations) in a small scale Cosserat model of layered material with sliding layers produces moment stress with a weak ($1/4$ power) singularity. In both cases, the singularity can only be interpreted up to the distances to the crack tip exceeding the microstructural length of the material. Subsequently, the moment stress values at these distances are to be used in the (force) criteria of crack propagation.

Acknowledgements We acknowledge financial support from the Australian Research Council via Discovery Grants DP0773839 (EP) and DP0559737 (AVD). We also wish to thank Greg Sevel for his help with the paper formatting.

References

1. Dyskin, A.V., Pasternak, E.: Rotational mechanism of in-plane shear crack growth in rocks under compression. In: Potvin, Y., Carter, J., Dyskin, A., Jeffrey, R. (eds.) 1st Southern Hemisphere International Rock Mechanics Symposium SHIRMS 2008, vol. 2, pp. 111–120. Australian Centre for Geomechanics, Crawley (2008)

2. Gourgiotis, P., Georgiadis, H.: Distributed dislocation approach for cracks in couple-stress elasticity: shear modes. Int. J. Fract. **147**, 83–102 (2007)
3. Landau, L., Lifshitz, E.: Theory of Elasticity. Braunschweig, Oxford (1959)
4. Morozov, N.: Mathematical Issues of the Crack Theory. Nauka, Moscow (1984). In Russian
5. Nowacki, W.: The linear theory of micropolar elasticity. In: Nowacki, W., Olszak, W. (eds.) Micropolar Elasticity, pp. 1–43. Springer, Berlin (1974)
6. Pasternak, E., Dyskin, A., Mhlhaus, H.B.: Cracks of higher modes in Cosserat continua. Int. J. Fract. **140**, 189–199 (2006)
7. Pasternak, E., Mhlhaus, H., Dyskin, A.: On the possibility of elastic strain localisation in a fault. Pure Appl. Geophys. **161**, 2309–2326 (2004)
8. Tada, H., Paris, P., Irwin, G.: The Stress Analysis of Cracks. Handbook, 3rd edn. ASME Press, New York (1985)