Chapter 10
Practical Applications of Simple Cosserat Methods

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Abstract Motivated by the need to construct models of slender elastic media that are versatile enough to accommodated non-linear phenomena under dynamical evolution, an overview is presented of recent practical applications of simple Cosserat theory. This theory offers a methodology for modeling non-linear continua that is physically accurate and amenable to controlled numerical approximation. By contrast to linear models, where non-linearities are sacrificed to produce a tractable theory, large deformations are within the range of validity of simple Cosserat models. The geometry of slender and shell-like bodies is exploited to produce a theory that contains as few degrees of freedom as is physically reasonable. In certain regimes it is possible to include fluid-structure interactions in Cosserat rod theory in order to model, for example, drill-string dynamics, undersea riser dynamics and cable-stayed bridges in light wind-rain conditions. The formalism also lends itself to computationally efficient, effective models of microscopic carbon nanotubes and macroscopic gravitational antennae.

10.1 Overview

The pioneering efforts of Euler and Bernoulli in the seventeenth and eighteenth centuries extended Newton's laws for discrete mass points to any continuous deformable body and initiated the recognition of the independent Angular Momentum Principle. With the introduction of the concept of the stress-tensor by Cauchy in the nineteenth century the essential framework of classical (non-relativistic) elastodynamics was complete. However, the implementation of this framework for the solu-

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tion of many practical engineering problems had to wait for effective mathematical tools. Naturally, the first use of the theory involved linearizations of the equations about stationary configurations commensurate with the linear response properties of particular materials. This led to the development of effective practical tools for solving linear partial differential equations on suitably shaped domains (e.g., Fourier and Laplace techniques) and their efficacy obscured for some time the need to address the fundamental non-linear nature of the basic equations or their boundary conditions. Indeed, until new mathematical tools became available, the theory was largely restricted to all pervading linear approximations. Although these had indisputable value it was finally appreciated that such approximations had limited domains of applicability and that the full theory was much richer in its scope.

If one takes the attitude that the methodology in constructing an effective model of a physical system should follow three distinct steps: formulation, analysis and interpretation then the analysis should consist solely in the application of precise mathematical processes exempt from any further ad-hoc physical simplifications [1]. The predominance of linearization methods for elastic problems involving strings, rods, beams, shells, etc. gave rise to somewhat haphazard modeling assumptions in which mathematical assumptions became unnecessarily involved with physical assumptions about the system under consideration. A typical example that pervades the majority of elementary texts even today is the derivation of the equation of motion of small transverse vibrations of an elastic string. Such derivations often assume that each material point is confined to a plane through its equilibrium position perpendicular to the line joining the ends of the string. Very few strings can execute purely transverse motion (without longitudinal deformation) and most derivations ignore the role played by the inextensibility of the string by assuming that the tension in the string is essentially constant. Such dubious assumptions are unnecessary since the wave equation follows most naturally from a systematic perturbation scheme applied to the exact (non-linear) equations of motion [1].

In recent years, an efficient approach to modeling the non-linear behavior of certain special classes of continuous mechanical structures has been developed. Based on the early work of the Cosserat brothers and others, the mathematical theory of Cosserat media is now well established. Antman [1] has formulated a complete theory of the mechanics of *slender* structures (rods) and *thin* structures (shells) in a recent book. The essential idea is to exploit the geometry of the structure to reduce the dimension of its configuration space. The genesis of the Cosserat approach lies in the application of the basic laws of Continuum Mechanics to a slender material continuum and the disengagement of these physical principles from the constitutive relations that model the material properties of particular structures. *Then the analysis of the model is based on mathematical methods in which approximations may be controlled.* Today one can often assess the validity of a particular approximation scheme in the absence of exact analytical solutions using computational techniques. The methods also permit an exploration of non-differentiable solutions (shock phenomena) that lie outside the remit of approaches based on the discretization of classical *differential* equations.

By contrast, one cannot a priori assume that approximations based on small displacements *about particular configurations* will necessarily provide accurate dynamical predictions over extended periods of time and one may cite ample evidence where linearized equations will completely miss dynamical behavior that arises from the underlying equations of motion, boundary conditions or material properties. The full (non-linear) equations of motion may admit distinct classes of solutions that are parametrically connected in the space of all solutions [14, 13, 2]. By contrast, stable solutions of the linearized equations are restricted to domains about particular configurations. Unstable linearized solutions signal that the approximation eventually breaks down and recourse to the full equations of motion becomes mandatory in order to understand how the system develops in time from prescribed initial and boundary conditions.

An example of this situation [15] arises in modeling the dynamics of on-shore drill-strings. Traditional models can be constructed based on Euler–Bernoulli linearized beam theory for planar vibrations coupled to axial vibrations of a heavy drill-string confined to the vertical bore cavity of an active drilling assembly. The drill-string is driven in a rotary fashion from the top and reacts to non-linear friction at the attached drill-bit. Thus the boundary conditions are generally non-linear functions of the drill-string configuration and their linearization limits the temporal evolution to a superposition of small amplitude stable eigenmodes. Without such a linearization, the drill-string can execute a complex motion in space eventually making contact with the bore-cavity along its length. A similar phenomenon occurs if one executes the evolution as a perturbation of the drill-string about a distinct *whirling motion* [10] in which the entire string vibrates in a plane rotating about a vertical axis in the bore cavity. The existence of these different dynamical configurations and the attendant clash with the bore-cavity renders a linearized approach ineffective and recourse to a Cosserat model is both simpler and more reliable. This not only accommodates naturally the coupling between lateral, axial and torsional motions but enables the inherently non-linear interaction of the drill-string with the bore cavity to be modeled.

In the context of vertical steel marine risers attached to floating platforms, similar boundary interactions can induce parametric excitations that feed energy among the various elastic modes of vibration of the riser (axial, lateral and torsional motion about its length) in unsuspected ways and the use of a Cosserat description eliminates the need for ad-hoc assumptions about the significance of the various energy transfer channels. For flexible risers that are not vertically constrained, the need for a non-perturbative approach becomes even more important since one often deals with forces and torques in which the riser executes larger curved motions in space as well as extended dynamical interactions along its length where contact with the sea-bed is in evidence. Problems of riser collisions, either with other risers externally or with internal elements, are further situations where linearized methods become suspect. In both cases, internal dampings of the structure (viscoelastic or possibly of memory type (hysteretic)) provide further sources of non-linearity that are accessible to a time domain evolution in the Cosserat description.

Recent applications of Cosserat methods have included the non-linear analysis of complex vibration states of drill-strings, marine risers, MEMS structures, rigid gears, gravitational antennae and carbon nanotubes. It has been shown that the inherent non-linearities in the simple Cosserat theory of rods [1] can be exploited for the design of space-based slender elastic gravitational wave antennae [17, 16]. Such structures are more sensitive to perturbation by gravitational waves and respond to broader frequency ranges than linearized elasticity models might suggest.

A Cosserat model of a drill-string is a flexible Cosserat rod connected at one end to a rotary drive and connected at the other end to a heavy drill bit subject to non-linear frictional forces. A Cosserat model of a marine riser is a flexible Cosserat rod vertically immersed in sea water and connected to the sea bed. A Cosserat model of a cable in a cable stayed bridge is a high tension Cosserat rod immersed in air. In the models above, the fluid-structure interaction in which the external forces and torques on the structure arise from its contact with a fluid play a role. One must model such forces and torques in terms of the structure configuration variables and properties of the external stimuli, and the dynamics of such stimuli are coupled to the motion of the structure by appropriate boundary conditions. In certain regimes such interactions can be rendered tractable to analysis, particularly in the presence of laminar potential flow with possible vorticity.

Long stay cables are important structural components of cable-stayed bridges. Due to their large flexibility and small structural damping, they are prone to vibration induced by motion of their supports and/or aerodynamic forces such as wind and rain loadings. Under the simultaneous occurrence of light-to-moderate wind and rain, large amplitude vibrations of stay cables have been observed in a number of cable-stayed bridges worldwide. The mechanism leading to rain–wind induced vibration in stay cables has recently become of concern to bridge engineers and scientists in various countries and, clearly, this phenomenon involves a complex interplay between the cables, rain rivulets and air. A fully detailed mathematical analysis would require the equations of multi-phase fluid dynamics, a model for accretion and fluid–solid adhesion and the continuum mechanics of an elastic structure. However, such an approach is extremely complicated, unwieldy and involves considerable computing power. Fortunately, in low ambient wind speeds, experiments with an artificial mobile rivulet on a fixed cylinder subject to aerodynamic loading do indicate an approach to steady rivulet oscillation. A fully dynamical Cosserat model of a cable section and a mobile rain rivulet was developed in which the complex fluid–structure interaction was approximated in two distinct ways. The first employed an approximation that permitted the use of data extrapolated from wind-tunnel measurements [8]. The second approach [7] modeled the aerodynamic interaction in terms of a sub-critical vortex description [5] which took account of the effects due to the boundary layers between the cable, rivulet and air. This was based on the observation that the instantaneous Reynolds number associated with the relative air-cable velocity is in the sub-critical range and hence dynamical vortex shedding should play a role. It is likely that the fluid vorticity generated by a cable will interact with other cables downstream and contribute significantly to their ensuing motion. One advantage of the second modeling approach is that it can be naturally generalized to accommodate more than one cable. A method for constructing flows containing point vortices in the vicinity of a pair of cable sections was recently proposed in [6], and a fluid model employing vortex sheets was developed in [4].

Recent years have seen a surge of interest in carbon nanotubes with the aim to engineer nano-scale devices. Calculations of the electronic and mechanical properties of carbon nanotubes often rely on *ab-initio* techniques employing density-functional theory. However, numerical implementations of such schemes are intensive and can require considerable computing power. Cosserat methods afford a natural alternative for formulating more efficient models of nanotubes. This approach was used to successfully describe the equilibria of a nanotube under an external load [11] and investigate its radial breathing modes [3].

The following is a brief summary of the Cosserat models used to investigate the above physical systems.

10.2 Simple Cosserat Methods

The motion of a Cosserat rod may be represented in terms of the motion in space of the line of centroids of its cross-sections and the material deformation about that line. The configuration of such a rod is modeled mathematically by a space-curve with structure: i.e., as a principal SO(3) bundle over a moving space-curve. This structure defines the relative orientation of neighboring cross-sections along the rod. Specifying a unit vector $\mathbf{d}_1$ (which may be identified with the normal to the cross-section) at each point along the rod centroid enables the state of flexure to be related to the angle between this vector and the tangent to the space-curve. Specifying a second unit vector $\mathbf{d}_2$ orthogonal to the first vector (thereby placing it in the plane of the cross-section) permits an encoding of the state of bending and twist along its length. Thus a field of two mutually orthogonal unit vectors along the rod provides three continuous dynamical degrees of freedom that, together with the continuous three degrees of freedom describing a space-curve relative to some arbitrary origin in space, define a simple Cosserat rod model.

The general mathematical theory of non-linear Newtonian elasticity is well established. The dynamics of the Cosserat rod theory follows as a well defined limit of a three-dimensional continuum and is conventionally formulated in a Lagrangian picture in which material elements are labeled by s. In the following, objects associated with an un-deformed *reference* configuration are superscripted with 0.

The dynamical evolution of a (transversely homogeneous) simple Cosserat rod with reference length L^0, reference mass density, $s \in [0, L^0] \mapsto \rho^0(s)$, and cross-sectional area, $s \in [0, L^0] \mapsto A(s)$, is governed by Euler's dynamical laws:

$$\rho^0 A\ddot{\mathbf{r}} = \mathbf{n}' + \mathbf{f}, \qquad \partial_t(\rho^0 \mathscr{I}(\mathbf{w})) = \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \mathbf{l} \tag{10.1}$$

involving implicitly the triad of orthonormal vector fields $\{\mathbf{d}_1(s,t), \mathbf{d}_2(s,t), \mathbf{d}_3(s,t)\}$ (directors) over the space-curve: $s \in [0, L^0] \mapsto \mathbf{r}(s,t)$ at time t where $\mathbf{r}' = \partial_s \mathbf{r}$, $\dot{\mathbf{r}} = \partial_t \mathbf{r}$, etc. The external force and torque densities (including couplings to external gravitational, friction, electromagnetic and fluid forces) are denoted $\mathbf{f}$ and $\mathbf{l}$, and $s \in [0, L^0] \mapsto \rho^0 \mathscr{I}$ is a moment of inertia tensor. In these field equations, the *contact force fields* $\mathbf{n}$ and *contact torque fields* $\mathbf{m}$ are related to the "*strain*" *fields* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ by constitutive relations. The strains are themselves defined in terms of the configuration variables $\mathbf{r}$ and $\mathbf{d}_k$ for $k = 1, 2, 3$ by the relations: $\mathbf{r}' = \mathbf{v}$,

$\mathbf{d}'_k = \mathbf{u} \times \mathbf{d}_k$, $\dot{\mathbf{d}}_k = \mathbf{w} \times \mathbf{d}_k$. The latter ensures that the triad remains orthonormal under evolution. The last equation identifies $\mathbf{w} = \frac{1}{2}\sum_{k=1}^{3} \mathbf{d}_k \times \dot{\mathbf{d}}_k$ with the local angular velocity vector field of the director triad. To close the above equations of motion constitutive relations appropriate to the rod must be specified. These relate the contact forces and torques to the strains $\mathbf{u}$, $\mathbf{v}$, their time rates of change for viscoelastic materials and their possible dependence on the history of the evolution of the structure. In general, the model accommodates continua whose characteristics (density, cross-sectional area, rotary inertia) vary with s and the thermodynamic state of the material [9]. For a system of several rods coupled to localized regions with mass or rotary inertia, one matches rod degrees of freedom according to junction conditions describing the coupling [12]. If the rod characteristics change discontinuously conditions on the contact forces and torques on either side of the discontinuity must be satisfied.

Mechanically deforming a carbon nanotube will lead to changes in the relative positions of the nanotube's carbon atoms. Bending and twisting the nanotube may lead to significant cross-section deformation outside the remit of the simple Cosserat rod theory described above. Motivated by the efficiency and wide domain of applicability of simple Cosserat rod theory, an extended shell-like model (a *Cosserat tube*) was developed [11, 3] to describe such cross-section deformation.

It is widely accepted that a carbon nanotube may be effectively modeled as an *elastic* continuum over spatial scales that greatly exceed interatomic distances. The mechanical properties of a simple Cosserat tube may be motivated using the *Kirchoff* [1] constitutive relations for a simple Cosserat rod. The balance laws (10.1) for such a rod may be recovered from an action principle [3], and a concise description of a Cosserat tube may be developed by demanding that the tube's action reduces to the rod's action when the tube's cross-section is constrained to remain in its undeformed state. The position $\mathbf{R}$ of a material point (s, σ, ζ) in the wall of a Cosserat

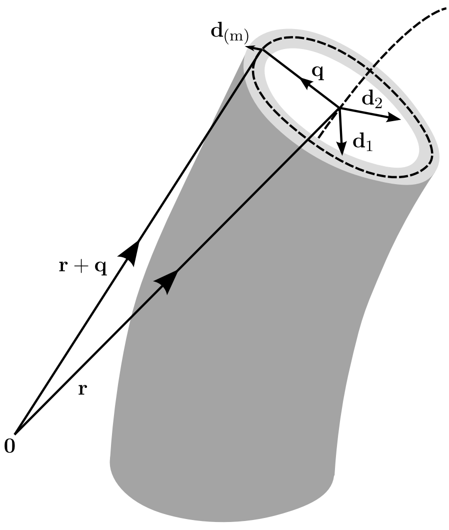

Fig. 10.1 The directors $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_{(\mathrm{m})}$ and the position vectors $\mathbf{r}$, $\mathbf{r}+\mathbf{q}$ of the (*dashed*) space-curves representing the tube's "bulk" and the cross-section are shown. The vector fields $\mathbf{d}_{(\mathrm{m})}$ and $\mathbf{q}$ lie in the plane spanned by $\mathbf{d}_1$, $\mathbf{d}_2$

tube can be written

$$\mathbf{R}(s,\sigma,\zeta,t) = \mathbf{r}(s,t) + \mathbf{q}(s,\sigma,t) + \zeta\,\mathbf{d}_{(\mathrm{m})}(s,\sigma,t) \tag{10.2}$$

with

$$\mathbf{q}(s,\sigma,t) = q_1(s,\sigma,t)\,\mathbf{d}_1(s,t) + q_2(s,\sigma,t)\,\mathbf{d}_2(s,t)$$

where the unit vector field $\mathbf{d}_{(\mathrm{m})}$ is an additional "microstructure" director and the functions q_1, q_2, $\mathbf{d}_{(\mathrm{m})}$ encode the shape of the cross-section (see Fig. 10.1). The elastic potential $\mathscr{V}$ of the tube,

$$\begin{aligned}\mathscr{V} = \int_0^{L^0} \mathrm{d}s \int_{\mathscr{C}^0(s)} \mathrm{d}A^0 \\ \times \Big[\frac{1}{2}G\big(\partial_s\mathbf{R}\cdot\mathbf{d}_1 - \partial_s\mathbf{R}^0\cdot\mathbf{d}_1^0\big)^2 \\ + \frac{1}{2}G\big(\partial_s\mathbf{R}\cdot\mathbf{d}_2 - \partial_s\mathbf{R}^0\cdot\mathbf{d}_2^0\big)^2 \\ + \frac{1}{2}E\big(\partial_s\mathbf{R}\cdot\mathbf{d}_3 - \partial_s\mathbf{R}^0\cdot\mathbf{d}_3^0\big)^2 \\ + \frac{1}{2}G\big(\partial_\sigma\mathbf{R}\cdot\mathbf{d}_{(\mathrm{m})} - \partial_\sigma\mathbf{R}^0\cdot\mathbf{d}_{(\mathrm{m})}^0\big)^2 \\ + \frac{1}{2}E\big(\partial_\sigma\mathbf{R}\cdot\mathbf{d}_3\times\mathbf{d}_{(\mathrm{m})} - \partial_\sigma\mathbf{R}^0\cdot\mathbf{d}_3^0\times\mathbf{d}_{(\mathrm{m})}^0\big)^2\Big]\end{aligned} \tag{10.3}$$

vanishes in the reference (un-deformed) configuration, where G is the shear modulus, E is Young's modulus, ρ^0 is the mass density of the reference configuration and $\mathrm{d}A^0$ is the area element of the cross-section $\mathscr{C}^0(s)$ in the reference configuration. The variable s is the arc parameter of a space-curve running along the tube ($s \in [0, L^0]$) in the reference configuration and σ is the arc parameter of a closed space-curve within a cross-section in the reference configuration (see the dashed space-curves in Fig. 10.1). The action S_{tube} for the Cosserat tube taken to be

$$S_{\mathrm{tube}}[\mathbf{r},\mathbf{d}_1,\mathbf{d}_2,q_1,q_2,\mathbf{d}_{(\mathrm{m})}] = \int \mathrm{d}t\Big[\int_0^{L^0}\mathrm{d}s\int_{\mathscr{C}^0(s)}\mathrm{d}A^0\,\frac{1}{2}\rho^0\partial_t\mathbf{R}\cdot\partial_t\mathbf{R} - \mathscr{V}\Big] \tag{10.4}$$

and equations of motion for $\mathbf{r}$, $\mathbf{d}_k$, q_1, q_2, $\mathbf{d}_{(\mathrm{m})}$ are obtained by varying S_{tube} subject to appropriate boundary conditions.

Under a constrained variation, S_{tube} reduced to an action S_{rod} for a simple Cosserat rod with Kirchoff constitutive properties. This follows by writing

$$S_{\mathrm{rod}}[\mathbf{r},\mathbf{d}_1,\mathbf{d}_2] = S_{\mathrm{tube}}[\mathbf{r},\mathbf{d}_1,\mathbf{d}_2,\check{q}_1,\check{q}_2,\check{\mathbf{d}}_{(\mathrm{m})}] \tag{10.5}$$

and enforcing the constraints

$$\begin{aligned} \mathbf{d}_{(\mathrm{m})} &= \check{\mathbf{d}}_{(\mathrm{m})} \equiv \cos(\theta)\,\mathbf{d}_1 + \sin(\theta)\,\mathbf{d}_2, \\ q_1 &= \check{q}_1 \equiv R^0 \cos(\theta), \\ q_2 &= \check{q}_2 \equiv R^0 \sin(\theta) \end{aligned} \tag{10.6}$$

prior to variation with respect to $\mathbf{r}$, $\mathbf{d}_1$, $\mathbf{d}_2$. The variable $\theta = \sigma/R^0$ is the angle around the rod, where the constant R^0 is the radius of the circular space-curve representing the rod's cross-section (the circular dashed curve in Fig. 10.1). The constraints (10.6) remove the final two terms from the integrand in the potential (10.3). Euler's dynamical laws (10.1) for a simple Cosserat rod with Kirchoff constitutive relations are then recovered as the Euler–Lagrange equations derived from the action S_{rod}.

10.3 Conclusion

Cosserat methods afford dynamical models of non-linear continua that are physically accurate and yet computationally amenable. By contrast to linear models, where non-linearities are sacrificed to produce a tractable theory, large deformations are within the range of validity of simple Cosserat models. The geometry of slender and shell-like bodies can be exploited to develop efficient tools for analyzing the dynamics of a broad range of important physical systems.

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