Chapter 5
Existence Theory of Nonlinear Dissipative Dynamics

Abstract In this chapter we present several applications of general theory to nonlinear dynamics governed by partial differential equations of dissipative type illustrating the ideas and general existence theory developed in the previous section. Most of significant dynamics described by partial differential equations can be written in the abstract form (4.1) with appropriate quasi-m-accretive operator A and Banach space X. The boundary value conditions are incorporated in the domain of A. The whole strategy is to find the appropriate operator A and to prove that it is quasi-m-accretive. The main emphasis here is on parabolic-like boundary value problems and the nonlinear hyperbolic equations although the area of problems covered by general theory is much larger.

5.1 Semilinear Parabolic Equations

The classical linear heat (or diffusion) equation perturbed by a nonlinear potential $\beta = \beta(y)$, where y is the state of system, is the simplest form of semilinear parabolic equation arising in applications and is treated below. The nonlinear potential β might describe exogeneous driving forces intervening over diffusion process or might induce unilateral state constraints.

The principal motivation for choosing multivalued functions β in examples below is to treat problems with a free (or moving) boundary as well as problems with discontinuous monotone nonlinearities. In the latter case, filling the jumps $[\beta(r_0 - 0), \beta(r_0 + 0)]$ of function β, we get a maximal monotone multivalued graph $\beta \subset \mathbf{R} \times \mathbf{R}$ for which the general existence theory applies.

To be more specific, assume that β is a maximal monotone graph such that $0 \in D(\beta)$, and Ω is an open and bounded subset of $\mathbf{R}^N$ with a sufficiently smooth boundary $\partial\Omega$ (for instance, of class C^2). Consider the parabolic boundary value problem

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$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \beta(y) \ni f & \text{in } \Omega \times (0,T) = Q, \\ y(x,0) = y_0(x) & \forall x \in \Omega, \\ y = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \end{cases} \tag{5.1}$$

where $y_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$ are given.

We may represent problem (5.1) as a nonlinear differential equation in the space $H = L^2(\Omega)$:

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{5.2}$$

where $A : L^2(\Omega) \to L^2(\Omega)$ is the operator defined by

$$\begin{aligned} Ay &= \{z \in L^2(\Omega);\ z = -\Delta y + w,\ w(x) \in \beta(y(x)),\ \text{a.e. } x \in \Omega\}, \\ D(A) &= \{y \in H_0^1(\Omega) \cap H^2(\Omega);\ \exists w \in L^2(\Omega),\ w(x) \in \beta(y(x)),\ \text{a.e. } x \in \Omega\}. \end{aligned} \tag{5.3}$$

Here, $(d/dt)y$ is the strong derivative of $y : [0,T] \to L^2(\Omega)$ and

$$\Delta y = \sum_{i=1}^{N} (\partial^2 y / \partial x_i^2)$$

is considered in the sense of distributions on Ω.

As a matter of fact, it is readily seen that if y is absolutely continuous from $[a,b]$ to $L^1(\Omega)$, then $dy/dt = \partial y/\partial t$ in $\mathscr{D}'((a,b);\ L^1(\Omega))$, and so a strong solution to equation (5.2) satisfies this equation in the sense of distributions in $(0,T) \times \Omega$. For this reason, whenever there is no any danger of confusion we write $\partial y/\partial t$ instead of dy/dt.

Recall (see Proposition 2.8) that A is maximal monotone (i.e., m-accretive) in $L^2(\Omega) \times L^2(\Omega)$ and $A = \partial\varphi$, where

$$\varphi(y) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\nabla y|^2 dx + \int_\Omega g(y) dx, & \text{if } y \in H_0^1(\Omega),\ g(y) \in L^1(\Omega), \\ +\infty, & \text{otherwise}, \end{cases}$$

and $\partial g = \beta$. Moreover, we have

$$\|y\|_{H^2(\Omega)} + \|y\|_{H_0^1(\Omega)} \le C(\|A^0 y\|_{L^2(\Omega)} + 1), \qquad \forall y \in D(A). \tag{5.4}$$

Writing equation (5.1) in the form (5.2), we view its solution y as a function of t from $[0,T]$ to $L^2(\Omega)$. The boundary conditions that appear in (5.1) are implicitly incorporated into problem (5.2) through the condition $y(t) \in D(A)$, $\forall t \in [0,T]$.

The function $y : \Omega \times [0,T] \to \mathbf{R}$ is called a *strong solution* to problem (5.1) if $y : [0,T] \to L^2(\Omega)$ is continuous on $[0,T]$, absolutely continuous on $(0,T)$, and satisfies

$$\begin{cases} \dfrac{d}{dt}y(x,t) - \Delta y(x,t) + \beta(y(x,t)) \ni f(x,t), & \text{a.e. } t \in (0,T),\ x \in \Omega, \\ y(x,0) = y_0(x), & \text{a.e. } x \in \Omega, \\ y(x,t) = 0, & \text{a.e. } x \in \partial\Omega,\ t \in (0,T). \end{cases} \tag{5.5}$$

Proposition 5.1. *Let* $y_0 \in L^2(\Omega)$ *and* $f \in L^2(0,T;L^2(\Omega)) = L^2(Q)$ *be such that* $y_0(x) \in \overline{D(\beta)}$, *a.e.* $x \in \Omega$. *Then, problem* (5.1) *has a unique strong solution*

$$y \in C([0,T];L^2(\Omega)) \cap W^{1,1}((0,T];L^2(\Omega))$$

that satisfies

$$t^{1/2} y \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)),\ t^{1/2}\frac{dy}{dt} \in L^2(0,T;L^2(\Omega)). \tag{5.6}$$

If, in addition, $f \in W^{1,1}([0,T];L^2(\Omega))$, *then* $y(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ *for every* $t \in (0,T]$ *and*

$$t\,\frac{dy}{dt} \in L^\infty(0,T;L^2(\Omega)). \tag{5.7}$$

If $y_0 \in H_0^1(\Omega),\ g(y_0) \in L^1(\Omega)$, *and* $f \in L^2(0,T;L^2(\Omega))$, *then*

$$\frac{dy}{dt} \in L^2(0,T;L^2(\Omega)), \qquad y \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{5.8}$$

Finally, if $y_0 \in D(A)$ *and* $f \in W^{1,1}([0,T];L^2(\Omega))$, *then*

$$\frac{dy}{dt} \in L^\infty(0,T;L^2(\Omega)), \qquad y \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \tag{5.9}$$

and

$$\frac{d^+}{dt}y(t) + (-\Delta y(t) + \beta(y(t)) - f(t))^0 = 0, \qquad \forall t \in [0,T]. \tag{5.10}$$

Proof. This is a direct consequence of Theorems 4.11 and 4.12, because, as seen in Proposition 2.8, we have

$$\overline{D(A)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(\beta)}, \quad \text{a.e. } x \in \Omega\}.$$

In particular, it follows that for $y_0 \in H_0^1(\Omega)$, $g(y_0) \in L^1(\Omega)$, and $f \in L^2(\Omega \times (0,T))$, the solution y to problem (5.1) belongs to the space

$$H^{2,1}(Q) = \left\{ y \in L^2(0,T;H^2(\Omega)),\ \frac{\partial y}{\partial t} \in L^2(Q) \right\}, \qquad Q = \Omega \times (0,T).$$

Problem (5.1) can be studied in the L^p setting, $1 \le p < \infty$ as well, if one defines the operator $A : L^p(\Omega) \to L^p(\Omega)$ as

$$Ay = \{z \in L^p(\Omega);\ z = -\Delta y + w,\ w(x) \in \beta(y)),\ \text{a.e. } x \in \Omega\}, \tag{5.11}$$

$$D(A) = \{y \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega);\ w \in L^p(\Omega) \text{ such that } w(x) \in \beta(y(x)),\ \text{a.e. } x \in \Omega\} \quad \text{if } p > 1, \tag{5.12}$$

$$D(A) = \{y \in W_0^{1,1}(\Omega);\ \Delta y \in L^1(\Omega),\ \exists w \in L^1(\Omega) \text{ such that } w(x) \in \beta(y(x)),\ \text{a.e. } x \in \Omega\} \quad \text{if } p = 1. \tag{5.13}$$

As seen earlier (Theorem 3.2), the operator A is m-accretive in $L^p(\Omega) \times L^p(\Omega)$ and so, also in this case, the general existence theory is applicable. □

Proposition 5.2. *Let $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];L^p(\Omega))$, $1 < p < \infty$. Then, problem* (5.1) *has a unique strong solution*

$$y \in C([0,T];L^p(\Omega)),$$

that satisfies

$$\frac{d}{dt} y \in L^\infty(0,T;L^p(\Omega)),\ y \in L^\infty(0,T;W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \tag{5.14}$$

$$\frac{d^+}{dt} y(t) + (-\Delta y(t) + \beta(y(t)) - f(t))^0 = 0, \qquad \forall t \in [0,T]. \tag{5.15}$$

If $y_0 \in \overline{D(A)}$ and $f \in L^1(0,T;L^p(\Omega))$, then (5.1) *has a unique mild solution*

$$y \in C([0,T];L^p(\Omega)).$$

Proof. Proposition 5.2 follows by Theorem 4.6 (recall that $X = L^p(\Omega)$ is uniformly convex for $1 < p < \infty$). □

Next, by Theorem 4.1 we have the following.

Proposition 5.3. *Assume $p = 1$. Then, for each $y_0 \in \overline{D(A)}$ and $f \in L^1(0,T;L^1(\Omega))$, problem* (5.1) *has a unique mild solution $y \in C([0,T];L^1(\Omega))$; that is,*

$$y(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t),$$

where y_ε is the solution to the finite difference scheme

$$\begin{aligned}
&y_\varepsilon^{i+1} = y_\varepsilon^i + \varepsilon \Delta y_\varepsilon^{i+1} - \varepsilon \beta(y_\varepsilon^{i+1}) + \int_{i\varepsilon}^{(i+1)\varepsilon} f(t)dt && \text{in } \Omega,\ i = 0,1,\dots,m,\\
& && m = \left[\tfrac{T}{\varepsilon}\right] + 1,\\
&y_\varepsilon^{i+1} \in H_0^1(\Omega) &&\\
&y_\varepsilon(t) = y_\varepsilon^i && \text{for } t \in (i\varepsilon,(i+1)\varepsilon).
\end{aligned}$$

Because the space $X = L^1(\Omega)$ is not reflexive, the mild solution to the Cauchy problem (5.2) in $L^1(\Omega)$ is only continuous as a function of t, even if y_0 and f are regular. However, also in this case we have a regularity property of mild solutions; that is, a smoothing effect on initial data, which resembles the case $p = 2$.

Proposition 5.4. *Let $\beta : \mathbf{R} \to \mathbf{R}$ be a maximal monotone graph, $0 \in \overline{D(\beta)}$, and $\beta = \partial g$. Let $f \in L^2(0,T;L^\infty(\Omega))$ and $y_0 \in L^1(\Omega)$ be such that $y_0(x) \in \overline{D(\beta)}$, a.e. $x \in \Omega$. Then, the mild solution $y \in C([0,T];L^1(\Omega))$ to problem* (5.1) *satisfies*

$$\|y(t)\|_{L^\infty(\Omega)} \le C\left(t^{-(N/2)}\|y_0\|_{L^1(\Omega)} + \int_0^t \|f(s)\|_{L^\infty(\Omega)}ds\right), \tag{5.16}$$

$$\begin{aligned}&\int_0^T\int_\Omega (t^{(N+4)/2}y_t^2 + t^{(N+2)/2}|\nabla y|^2)dx\,dt + T^{(N+4)/2}\int_\Omega |\nabla y(x,T)|^2dx\\ &\le C\left(\left(\|y_0\|_{L^1(\Omega)}^{4/(N+2)} + \int_0^T\int_\Omega |f|dx\,dt\right)^{(N+2)/2} + T^{(N+4)/2}\int_0^T\int_\Omega f^2dx\,dt\right).\end{aligned} \tag{5.17}$$

Proof. Without loss of generality, we may assume that $0 \in \beta(0)$. Also, let us assume first that $y_0 \in H_0^1(\Omega)\cap H^2(\Omega)$. Then, as seen in Proposition 5.1, problem (5.1) has a unique strong solution such that $t^{1/2}y_t \in L^2(Q)$, $t^{1/2}y \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) - \Delta y(x,t) + \beta(y(x,t)) \ni f(x,t), & \text{a.e. } (x,t) \in Q,\\ y(x,0) = y_0(x), & x \in \Omega,\\ y = 0, & \text{on } \Sigma. \end{cases} \tag{5.18}$$

Consider the linear problem

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = \|f(t)\|_{L^\infty(\Omega)} & \text{in } Q,\\ z(x,0) = |y_0(x)|, & x \in \Omega,\\ z = 0, & \text{on } \Sigma. \end{cases} \tag{5.19}$$

Subtracting these two equations and multiplying the resulting equation by $(y-z)^+$, and integrating on Ω we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(y-z)^+\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla(y-z)^+|^2dx \le 0, && \text{a.e. } t \in (0,T),\\ &(y-z)^+(0) \le 0 && \text{in } \Omega,\end{aligned}$$

because $z \ge 0$ and β is monotonically increasing. Hence, $y(x,t) \le z(x,t)$, a.e. in Q and so $|y(x,t)| \le z(x,t)$, a.e. $(x,t) \in Q$. On the other hand, the solution z to problem (5.19) can be represented as

$$z(x,t) = S(t)(|y_0|)(x) + \int_0^t S(t-s)(\|f(s)\|_{L^\infty(\Omega)})ds, \quad \text{a.e. } (x,t) \in Q,$$

where $S(t)$ is the semigroup generated on $L^1(\Omega)$ by $-\Delta$ with Dirichlet homogeneous conditions on $\partial\Omega$. We know, by the regularity theory of $S(t)$ (see also Theorem 5.4 below), that

$$\|S(t)u_0\|_{L^\infty(\Omega)} \le Ct^{-(N/2)}\|u_0\|_{L^1(\Omega)}, \qquad \forall u_0 \in L^1(\Omega),\ t>0.$$

Hence,

$$|y(x,t)| \le Ct^{-(N/2)}\|y_0\|_{L^1(\Omega)} + \int_0^t \|f(s)\|_{L^\infty(\Omega)}ds, \qquad (t,x)\in Q. \tag{5.20}$$

Now, for an arbitrary $y_0 \in L^1(\Omega)$ such that $y_0 \in \overline{D(\beta)}$, a.e. in Ω, we choose a sequence $\{y_0^n\} \subset H_0^1(\Omega)\cap H^2(\Omega)$, $y_0^n \in \overline{D(\beta)}$, a.e. in Q, such that $y_0^n \to y_0$ in $L^1(\Omega)$ as $n\to\infty$. (We may take, for instance, $y_0^n = S(n^{-1})(1+n^{-1}\beta)^{-1}y_0$.) If y_n is the corresponding solution to problem (5.1), then we know that $y_n \to y$ strongly in $C([0,T];L^1(\Omega))$, where y is the solution with the initial value y_0. By (5.20), it follows that y satisfies estimate (5.16).

Because $y(t)\in L^\infty(\Omega)\subset L^2(\Omega)$ for all $t>0$, it follows by Proposition 5.1 that $y \in W^{1,2}([\delta,T];L^2(\Omega))\cap L^2(\delta,T;H_0^1(\Omega)\cap H^2(\Omega))$ for all $0<\delta<T$ and it satisfies equation (5.18), a.e. in $Q=\Omega\times(0,T)$. (Arguing as before, we may assume that $y_0\in H_0^1(\Omega)\cap H^2(\Omega)$ and so $y_t, y \in L^2(0,T;L^2(\Omega))$.) To get the desired estimate (5.17), we multiply equation (5.18) by $y_t t^{k+2}$ and integrate on Q to get

$$\int_0^T\int_\Omega t^{k+2}y_t^2\,dx\,dt + \frac{1}{2}\int_0^T\int_\Omega t^{k+2}|\nabla y|_t^2\,dx\,dt + \int_0^T\int_\Omega t^{k+2}\frac{\partial}{\partial t}g(y)dx\,dt$$
$$= \int_0^T\int_\Omega t^{k+2}y_t f\,dx\,dt,$$

where $y_t = \partial y/\partial t$ and $\partial g = \beta$. This yields

$$\int_Q t^{k+2}y_t^2dx\,dt + \frac{T^{k+2}}{2}\int_\Omega |\nabla y(x,T)|^2dx + T^{k+2}\int_\Omega g(y(x,T))dx$$
$$\le \frac{k+2}{2}\int_Q t^{k+1}|\nabla y|^2dx\,dt + (k+2)\int_Q t^{k+1}g(y)dx\,dt$$
$$+\frac{1}{2}\int_0^T t^{k+2}y_t^2dx\,dt + \frac{1}{2}\int_Q t^{k+2}f^2dx\,dt.$$

Hence,

$$\int_Q t^{k+2}y_t^2dx\,dt + T^{k+2}\int_\Omega |\nabla y(x,T)|^2dx$$
$$\le (k+2)\int_Q t^{k+1}|\nabla y|^2dx\,dt + 2(k+2)\int_Q t^{k+1}\beta(y)dx + T^{k+2}\int_Q f^2dx\,dt.$$

(If β is multivalued, then $\beta(y)$ is of course the section of $\beta(y)$ arising in (5.18).)

Finally, writing $\beta(y)y$ as $(f+\Delta y - y_t)y$ and using Green's formula, we get

$$\begin{aligned}
&\int_Q t^{k+2} y_t^2 dx\,dt + T^{k+2} \int_\Omega |\nabla y(x,T)|^2 dx + \int_Q t^{k+1} |\nabla y|^2 dx\,dt \\
&\le (k+2)(k+1) \int_Q y^2 t^k dx\,dt \\
&\quad + T^{k+2} \int_Q f^2 dx\,dt + 2(k+2) \int_Q t^{k+1} |f|\,|y| dx\,dt \\
&\le C\left(\int_Q t^k y^2 dx\,dt + T^{k+2} \int_Q f^2 dx\,dt\right).
\end{aligned} \tag{5.21}$$

Next, we have, by the Hölder inequality

$$\int_\Omega y^2 dx \le \|y\|_{L^p(\Omega)}^{(N-2/N+2)} \|y\|_{L^1(\Omega)}^{4/(N+2)}$$

for $p = 2N(N-2)^{-1}$. Then, by the Sobolev embedding theorem,

$$\int_\Omega |y(x,t)|^2 dx \le \left(\int_\Omega |\nabla y(x,t)|^2 dx\right)^{N/(N+2)} \left(\int_\Omega |y(x,t)| dx\right)^{4/(N+2)}. \tag{5.22}$$

On the other hand, multiplying equation (5.18) by sign y and integrating on $\Omega \times (0,t)$, we get

$$\|y(t)\|_{L^1(\Omega)} \le \|y_0\|_{L^1(\Omega)} + \int_0^t \int_\Omega |f(x,s)| dx\,ds, \qquad t \ge 0,$$

because, as seen earlier (Section 3.2),

$$\int_\Omega \Delta y \,\mathrm{sign}\, y\, dx \le 0.$$

Then, by estimates (5.21) and (5.22), we get

$$\begin{aligned}
&\int_Q t^{k+2} y_t^2 dx\,dt + T^{k+2} \int_\Omega |\nabla y(x,T)|^2 dx + \int_Q t^{k+1} |\nabla y(x,t)|^2 dx\,dt \\
&\le C\Bigg(\left(\|y_0\|_{L^1(\Omega)}^{4/(N+2)} + \int_0^T \int_\Omega |f(x,t)| dx\,dt\right) \\
&\quad \times \int_0^t t^k \|\nabla y(t)\|_{L^2(\Omega)}^{2N/(N+2)} dt + T^{k+2} \int_Q f^2 dx\,dt\Bigg).
\end{aligned}$$

On the other hand, we have, for $k = N/2$,

$$\int_0^T t^k |\nabla y(t)|^{2N/(N+2)} dt \le \left(\int_0^T t^{k+1} |\nabla y(t)|^2 dt\right)^{N/(N+2)} T^{2/(N+2)}.$$

Substituting in the latter inequality, we get after some calculation involving the Hölder inequality

$$
\begin{aligned}
\int_Q t^{(N+4)/2} y_t^2 \,dx\,dt &+ \int_Q t^{(N+2)/2} |\nabla y(x,t)|^2 \,dx\,dt \\
&+ T^{(N+4)/2} \int_\Omega |\nabla y(x,T)|^2 \,dx \\
&\le C_1 \left(\|y_0\|_{L^1(\Omega)}^{4/(N+2)} + \int_Q |f(x,t)| \,dx\,dt \right)^{(N+2)/2} \\
&+ C_2 T^{(N+4)/2} \int_Q f^2(x,t)\,dx\,dt,
\end{aligned}
\tag{5.23}
$$

as claimed. □

In particular, it follows by Proposition 5.4 that the semigroup $S(t)$ generated by A (defined by (5.11) and (5.13) on $L^1(\Omega)$ has a smoothing effect on initial data; that is, for all $t > 0$ it maps $L^1(\Omega)$ into $D(A)$ and is differentiable on $(0, \infty)$.

In the special case where

$$
\beta(r) = \begin{cases} 0 & \text{if } r > 0, \\ \mathbf{R}^- & \text{if } r = 0, \end{cases}
$$

problem (5.1) reduces to the parabolic variational inequality (the obstacle problem)

$$
\begin{cases}
\dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } \{(x,t);\ y(x,t) > 0\}, \\
y \ge 0,\ \dfrac{\partial y}{\partial t} - \Delta y \ge f & \text{in } Q, \\
y(x,0) = y_0(x) & \text{in } \Omega,\ \ y = 0 \quad \text{on } \partial\Omega \times (0,T) = \Sigma.
\end{cases}
\tag{5.24}
$$

This is a problem with free (moving) boundary that is discussed in detail in the next section.

We also point out that Proposition 5.1 remains true for equations of the form

$$
\begin{cases}
\dfrac{\partial y}{\partial t} - \Delta y + \beta(x,y) \ni f & \text{in } Q, \\
y(x,0) = y_0(x) & \text{in } \Omega, \\
y = 0 & \text{on } \Sigma,
\end{cases}
$$

where $\beta : \Omega \times \mathbf{R} \to 2^{\mathbf{R}}$ is of the form $\beta(x,y) = \partial_y g(x,y)$ and $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a normal convex integrand on $\Omega \times \mathbf{R}$ sufficiently regular in X and with appropriate polynomial growth with respect to y. The details are left to the reader.

Now, we consider the equation

$$
\begin{cases}
\dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } \Omega \times (0,T) = Q, \\
\dfrac{\partial}{\partial \nu} y + \beta(y) \ni 0 & \text{on } \Sigma, \\
y(x,0) = y_0(x) & \text{in } \Omega,
\end{cases}
\tag{5.25}
$$

where $\beta \subset \mathbf{R} \times \mathbf{R}$ is a maximal monotone graph, $0 \in D(\beta)$, $y_0 \in L^2(\Omega)$, and $f \in L^2(Q)$. As seen earlier (Proposition 2.9), we may write (5.25) as

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) = f(t) & \text{in } (0,T), \\ y(0) = y_0, \end{cases}$$

where $Ay = -\Delta y$, $\forall y \in D(A) = \{y \in H^2(\Omega);\ 0 \in \partial y/\partial \nu + \beta(y), \text{ a.e. on } \partial\Omega\}$.

More precisely, $A = \partial\varphi$, where $\varphi : L^2(\Omega) \to \overline{\mathbf{R}}$ is defined by

$$\varphi(y) = \frac{1}{2}\int_\Omega |\nabla y|^2 dx + \int_{\partial\Omega} j(y) d\sigma, \qquad \forall y \in L^2(\Omega),$$

and $\partial j = \beta$.

Then, applying Theorems 4.11 and 4.12, we get the following.

Proposition 5.5. *Let $y_0 \in \overline{D(A)}$ and $f \in L^2(Q)$. Then, problem* (5.25) *has a unique strong solution $y \in C([0,T];L^2(\Omega))$ such that*

$$t^{1/2}\,\frac{dy}{dt} \in L^2(0,T;L^2(\Omega)),$$
$$t^{1/2} y \in L^2(0,T;H^2(\Omega)).$$

If $y_0 \in H^1(\Omega)$ and $j(y_0) \in L^1(\Omega)$, then

$$\frac{dy}{dt} \in L^2(0,T;L^2(\Omega)),$$
$$y \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)).$$

Finally, if $y_0 \in D(A)$ and $f, \partial f/\partial t \in L^2(\Omega)$, then

$$\frac{dy}{dt} \in L^\infty(0,T;L^2(\Omega)),$$
$$y \in L^\infty(0,T;H^2(\Omega))$$

and

$$\frac{d^+}{dt}\, y(t) - \Delta y(t) = f(t), \qquad \forall t \in [0,T].$$

It should be mentioned that one uses here the estimate (see (2.65))

$$\|u\|_{H^2(\Omega)} \le C(\|u - \Delta u\|_{L^2(\Omega)} + 1), \qquad \forall u \in D(A).$$

An important special case is

$$\beta(r) = \begin{cases} 0 & \text{if } r > 0, \\ (-\infty, 0] & \text{if } r = 0. \end{cases}$$

Then, problem (5.25) reads as

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } Q, \\ y\,\dfrac{\partial y}{\partial \nu} = 0,\ y \geq 0,\ \dfrac{\partial y}{\partial \nu} \geq 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{5.26}$$

A problem of this type arises in the control of a heat field. More generally, the thermostat control process is modeled by equation (5.26), where

$$\beta(r) = \begin{cases} a_1(r-\theta_1) & \text{if } -\infty < r < \theta_1, \\ 0 & \text{if } \theta_1 \leq r \leq \theta_2, \\ a_2(r-\theta_2) & \text{if } \theta_2 < r < \infty, \end{cases}$$

$a_i \geq 0,\ \theta_1 \in \mathbf{R},\ i = 1,2$. In the limit case, we obtain (5.26).

The black body radiation heat emission on $\partial\Omega$ is described by equation (5.26), where β is given by (the Stefan–Boltzman law)

$$\beta(r) = \begin{cases} \alpha(r^4 - y_1^4) & \text{for } r \geq 0, \\ -\alpha y_1^4 & \text{for } r < 0, \end{cases}$$

and, in the case of natural convection heat transfer,

$$\beta(r) = \begin{cases} ar^{5/4} & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases}$$

Note, also, that the Michaelis–Menten dynamic model of enzyme diffusion reaction is described by equation (5.1) (or (5.25)), where

$$\beta(r) = \begin{cases} \dfrac{r}{\lambda(r+k)} & \text{for } r > 0, \\ (-\infty, 0] & \text{for } r = 0, \\ \emptyset & \text{for } r < 0, \end{cases}$$

where λ, k are positive constants.

We note that more general boundary value problems of the form

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \gamma(y) \ni f & \text{in } Q, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu} + \beta(y) \ni 0 & \text{on } \Sigma, \end{cases}$$

where β and γ are maximal monotone graphs in $\mathbf{R}\times\mathbf{R}$ such that $0\in D(\beta)$, $0\in D(\gamma)$ can be written in the form (5.2) where $A=\partial\varphi$ and $\varphi: L^2(\Omega)\to\overline{\mathbf{R}}$ is defined by

$$\varphi(y)=\begin{cases}\frac{1}{2}\int_\Omega |\nabla y|^2dx+\int_\Omega g(y)dx+\int_{\partial\Omega} j(y)d\sigma & \text{if } y\in H^1(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

and $\partial g=\gamma$, $\partial j=\beta$.

We may conclude, therefore, that for $f\in L^2(\Omega)$ and $y_0\in H^1(\Omega)$ such that $g(y_0)\in L^1(\Omega)$, $j(y_0)\in L^1(\partial\Omega)$ the preceding problem has a unique solution $y\in W^{1,2}([0,T];L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$.

On the other hand, semilinear parabolic problems of the form (5.1) or (5.25) arise very often as feedback systems associated with the linear heat equation. For instance, the feedback relay control

$$u=-\rho\ \text{sign}\ y, \tag{5.27}$$

where

$$\text{sign}\, r=\begin{cases}\frac{r}{|r|} & \text{if } r\neq 0,\\ [-1,1] & \text{if } r=0,\end{cases}$$

applied to the controlled heat equation

$$\begin{cases}\frac{\partial y}{\partial t}-\Delta y=u & \text{in } \Omega\times\mathbf{R}^+,\\ y=0 & \text{on } \partial\Omega\times\mathbf{R}^+,\\ y(x,0)=y_0(x) & \text{in } \Omega\end{cases} \tag{5.28}$$

transforms it into a nonlinear equation of the form (5.1); that is,

$$\begin{cases}\frac{\partial y}{\partial t}-\Delta y+\rho\ \text{sign}\, y\ni 0 & \text{in } \Omega\times\mathbf{R}^+,\\ y=0 & \text{on } \partial\Omega\times\mathbf{R}^+,\\ y(x,0)=y_0(x) & \text{in } \Omega.\end{cases} \tag{5.29}$$

This is the closed-loop system associated with the feedback law (5.27) and, according to Proposition 5.4, for every $y_0\in L^1(\Omega)$, it has a unique strong solution $y\in C(\mathbf{R}^+;L^2(\Omega))$ satisfying

$$y(t)\in L^\infty(\Omega),\qquad \forall t>0,$$
$$t^{(N+4)/4}y_t\in L^2_{\text{loc}}(\mathbf{R}^+;L^2(\Omega)),\qquad t^{(N+2)/4}y\in L^2_{\text{loc}}(\mathbf{R}^+;H^1(\Omega)).$$

(Of course, if $y_0\in L^2(\Omega)$, then y has sharper properties provided by Proposition 5.1.)

Let us observe that the feedback control (5.27) belongs to the constraint set $\{u \in L^\infty(\Omega \times \mathbf{R}^+); \|u\|_{L^\infty(\Omega\times\mathbf{R}^+)} \le \rho\}$ and steers the initial state y_0 into the origin in a finite time T. Here is the argument. We assume first that $y_0 \in L^\infty(\Omega)$ and consider the function $w(x,t) = \|y_0\|_{L^\infty(\Omega)} - \rho t$. On the domain $\Omega \times (0, \rho^{-1}\|y_0\|_{L^\infty(\Omega)}) = Q_0$, we have

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + \rho \text{ sign } w \ni 0 & \text{in } Q_0, \\ w(0) = \|y_0\|_{L^\infty(\Omega)} & \text{in } \Omega, \\ w \ge 0 & \text{on } \partial\Omega \times (0, \rho^{-1}\|y_0\|_{L^\infty(\Omega)}). \end{cases} \tag{5.30}$$

Then, subtracting equations (5.29) and (5.30) and multiplying by $(y-w)^+$ (or, simply, applying the maximum principle), we get

$$(y-w)^+ \le 0 \quad \text{in } Q_0.$$

Hence, $y \le w$ in Q_0. Similarly, it follows that $y \ge -w$ in Q_0 and, therefore,

$$|y(x,t)| \le \|y_0\|_{L^\infty(\Omega)} - \rho t, \qquad \forall (x,t) \in Q_0.$$

Hence, $y(t) \equiv 0$ for all $t \ge T = \rho^{-1}\|y_0\|_{L^\infty(\Omega)}$. Now, if $y_0 \in L^1(\Omega)$, then inserting in system (5.28) the feedback control

$$u(t) = \begin{cases} 0 & \text{for } 0 \le t \le \varepsilon, \\ -\rho \text{ sign } y(t) & \text{for } t > \varepsilon, \end{cases}$$

we get a trajectory $y(t)$ that steers y_0 into the origin in the time

$$T(y_0) < \varepsilon + \rho^{-1}\|y(\varepsilon)\|_{L^\infty(\Omega)} \le \varepsilon + C(\rho\varepsilon^{N/2})^{-1}\|y_0\|_{L^1(\Omega)},$$

where C is independent of ε and y_0 (see estimate (5.16)). If we choose $\varepsilon > 0$ that minimizes the right-hand side of the latter inequality, then we get

$$T(y_0) \le \left(\frac{CN}{2\rho}\|y_0\|_{L^1(\Omega)}\right)^{2/(N+2)} + \left(\frac{N}{2}\right)^{-(N/(N+2))} \left(\frac{C}{\rho}\|y_0\|_{L^1(\Omega)}\right)^{2/(N+2)}.$$

We have, therefore, proved the following null controllability result for system (5.28).

Proposition 5.6. *For any $y_0 \in L^1(\Omega)$ and $\rho > 0$ there is $u \in L^\infty(\Omega \times \mathbf{R}^+)$, $\|u\|_{L^\infty(\Omega\times\mathbf{R}^+)} < \rho$, that steers y_0 into the origin in a finite time $T(y_0)$.*

Remark 5.1. Consider the nonlinear parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + |y|^{p-1}y = 0, & \text{in } \Omega \times \mathbf{R}^+, \\ y(x,0) = y_0(x), & x \in \Omega, \\ y = 0, & \text{on } \partial\Omega \times \mathbf{R}^+, \end{cases} \tag{5.31}$$

where $0 < p < (N+1)/N$ and $y_0 \in L^1(\Omega)$. By Proposition 5.4, we know that the solution y satisfies the estimates

$$\|y(t)\|_{L^\infty(\Omega)} \le Ct^{-(N/2)}|y_0\|_{L^1(\Omega)},$$
$$\|y(t)\|_{L^1(\Omega)} \le C\|y_0\|_{L^1(\Omega)},$$

for all $t > 0$.

Now, if y_0 is a bounded Radon measure on Ω; that is, $y_0 \in M(\Omega) = (C_0(\overline{\Omega}))^*$ ($C_0(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$), there is a sequence $\{y_0^j\} \subset C_0(\Omega)$ such that $\|y_0^j\|_{L^1(\Omega)} \le C$ and $y_0^j \to y_0$ weak-star in $M(\Omega)$. Then, if y^j is the corresponding solution to equation (5.31) it follows from the previous estimates that (see Brezis and Friedman [17])

$$\begin{aligned} y^j &\to y \quad &&\text{in } L^q(Q), \qquad 1 < q < \frac{N+2}{N}, \\ |y^j|^{p-1}y^j &\to |y|^{p-1}y \quad &&\text{in } L^1(Q). \end{aligned}$$

This implies that y is a generalized (mild) solution to equation (5.31).

If $p > (N+2)/N$, there is no solution to (5.31).

Remark 5.2. Consider the semilinear parabolic equation (5.1), where β is a continuous monotonically increasing function, $f \in L^p(Q)$, $p > 1$, and $y_0 \in W_0^{p,2-(2/p)}(\Omega)$, $g(y_0) \in L^1(\Omega)$, $g(r) = \int_0^r |\beta(s)|^{p-2}\beta(s)ds$. Then, the solution y to problem (5.1) belongs to $W_p^{2,1}(Q)$ and

$$\|y\|^p_{W_p^{2,1}(Q)} \le C\left(\|f\|^p_{L^p(\Omega)} + \|y_0\|^p_{W_0^{p,2-(2/p)}(\Omega)} + \int_\Omega g(y_0)dx\right).$$

Here, $W_p^{2,1}(Q)$ is the space

$$\left\{y \in L^p(Q);\ \frac{\partial^{r+s}}{\partial t^r \partial x^s} y \in L^p(Q),\, 2r+s \le 2\right\}.$$

For $p = 2$, $W_2^{2,1}(Q) = H^{2,1}(Q)$.

Indeed, if we multiply equation (5.1) by $|\beta(y)|^{p-2}\beta(y)$ we get the estimate (as seen earlier in Proposition 5.1, for f and y_0 smooth enough this problem has a unique solution $y \in W^{1,\infty}([0,T];L^p(\Omega))$, $y \in L^\infty(0,T;W^{2,p}(\Omega))$)

$$\begin{aligned} &\int_\Omega g(y(x,t))dx + \int_0^t \int_\Omega |\beta(y(x,s))|^p dx ds \\ &\le \int_0^t \int_\Omega |\beta(y(x,s))|^{p-1}|f(x,s)|dxds + \int_\Omega g(y_0(x))dx \\ &\le \left(\int_0^t \int_\Omega |\beta(y(x,s))|^p dxds\right)^{1/q} \left(\int_0^t \int_\Omega |f(x,s)|^p dxds\right)^{1/p}, \end{aligned}$$

where $1/p+1/q=1$. In particular, this implies that

$$\|\beta(y)\|_{L^p(Q)} \le C(\|f\|_{L^p(Q)} + \|g(y_0)\|_{L^1(\Omega)})$$

and by the L^p estimates for linear parabolic equations (see, e.g., Ladyzenskaya, Solonnikov, and Ural'ceva [31] and Friedman [27]) we find the estimate (5.34), which clearly extends to all $f \in L^p(Q)$ and $y_0 \in W_0^{p,2-(2/p)}(\Omega)$, $g(y_0) \in L^1(\Omega)$.

Nonlinear Parabolic Equations of Divergence Type

Several physical diffusion processes are described by the continuity equation

$$\frac{\partial y}{\partial t} + \mathrm{div}_x\, \mathbf{q} = f,$$

where the flux $\mathbf{q}$ of the diffusive material is a nonlinear function β of local density gradient ∇y. Such an equation models nonlinear interaction phenomena in material science and in particular in mathematical models of crystal growth as well as in image processing (see Section 2.4). This class of problems can be written as

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) - \mathrm{div}_x\, \beta(\nabla(y(x,t))) \ni f(x,t), & x \in \Omega,\ t \in (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{5.32}$$

where $\beta : \mathbf{R}^N \to \mathbf{R}^N$ is a maximal monotone graph satisfying conditions (2.138) and (2.139) (or, in particular, conditions (2.134) and (2.135) of Theorem 2.15).

In the space $X = L^2(\Omega)$ consider the operator A defined by (2.155) and thus represent (5.32) as a Cauchy problem in X; that is,

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) \ni f(t), \ \ t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{5.33}$$

In Section 2.4, we studied in detail the stationary version of (5.37) (i.e., $Ay = f$) and we have proven (Theorem 2.18) that A is maximal monotone (m-accretive) and so, by Theorem 4.6, we obtain the following existence result.

Proposition 5.7. *Let* $f \in W^{1,1}([0,T];L^2(\Omega))$, $y_0 \in W_0^{1,p}(\Omega)$ *be such that* $\mathrm{div}\, \eta_0 \in L^2(\Omega)$ *for some* $\eta_0 \in (L^q(\Omega))^N$, $\eta_0 \in \beta(\nabla y_0)$, *a.e. in* Ω. *Then, there is a unique strong solution* y *to* (5.32) (*equivalently to* (5.33)) *such that*

$$y \in L^\infty(0,T;W_0^{1,p}(\Omega)) \cap W^{1,\infty}([0,T];L^2(\Omega))$$

$$\frac{d^+}{dt} y(t) - \mathrm{div}_x\, \eta(t) = f(t), \qquad \forall t \in [0,T],$$

where $\eta \in L^\infty(0,T;L^2(\Omega))$, $\eta(t,x) \in \beta(\nabla y(x,t))$, *a.e.* $(x,t) \in \Omega \times (0,T) = Q$. *Moreover, if* $\beta = \partial j$, *then the strong solution* y *exists for all* $y_0 \in L^2(\Omega)$ *and* $f \in L^2(Q)$.

The last part of Proposition 5.7 follows by Theorem 4.11, because, as seen earlier in Theorem 2.18, in this latter case $A = \partial\varphi$.

Now, if we refer to Theorem 2.19 and Remark 2.4 we may infer that Proposition 5.7 remains true under conditions $\beta = \partial j$ and (2.161) and (2.162). We have, therefore, the following.

Proposition 5.8. *Let* β *satisfy conditions* (2.161) and (2.162). *Then, for each* $y_0 \in L^2(\Omega)$ *and* $f \in L^2(0,T;L^2(\Omega))$ *there is a unique strong solution to* (5.32) *or to the equation with Neumann boundary conditions* $\beta(\nabla y(x)) \cdot \nu(x) = 0$ *in the following weak sense,*

$$\frac{d}{dt}\int_\Omega y(x,t)v(x)dx + \int_\Omega \eta(x,t)\cdot\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx, \qquad \forall v \in C^1(\overline{\Omega}),$$
$$\eta(x,t) \in \beta(\nabla y(x,t)), \quad a.e.\ (x,t) \in \Omega \times (0,T),$$
$$y(x,0) = y_0(x).$$

Now, if we refer to the singular diffusion boundary value problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \mathrm{div}_x\,(\mathrm{sign}\ (\nabla y)) \ni f & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), \end{cases}$$

it has for each $y_0 \in BV^0(\Omega)$ a unique strong solution

$$y \in W^{1,2}([0,T];L^2(\Omega)) \cap C([0,T];L^2(\Omega))$$

with $\|Dy(t)\| \in W^{1,\infty}([0,T])$ (similarly for the case of Neumann boundary conditions).

Indeed, as seen earlier, it can be written as a first-order equation of subgradient type in $L^2(\Omega)$,

$$\begin{cases} \dfrac{dy}{dt}(t) + \partial\varphi(y(t)) \ni f(t), & t \in (0,T), \\ y(0) = y_0, \end{cases}$$

where φ is given by (2.182). Then, the existence follows by Theorem 4.11.

By (2.149) and the Trotter–Kato theorem (see Theorem 4.14), we know that the solution y is the limit in $C([0,T];L^2(\Omega))$ of solution y_ε to the problem

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \varepsilon\Delta y_\varepsilon - \mathrm{div}_x\beta_\varepsilon(\nabla y_\varepsilon) = f & \text{in } \Omega \times (0,T) \\ y_\varepsilon = 0 \quad \text{on } \partial\Omega; \qquad y_\varepsilon(x,0) = y_0(x), \end{cases}$$

where β_ε is the Yosida approximation of $\beta = \text{sign}$.

As noticed earlier, this equation is relevant in image restoration techniques and crystal-faceted growth theory. In particular, for $f(t) \equiv f_e \in L^2(\Omega)$ it follows by Theorem 4.13 that

$$\lim_{t\to\infty} y(t) = y_e \quad \text{strongly in } L^2(\Omega),$$

where y_e is an equilibrium solution; that is, $\partial\varphi(y_e) \ni f_e$.

In image processing, the solution $y = y(\cdot,t)$ might be seen as a family of restored images with the scale parameter t. The parabolic equation (5.32) itself acts as a filter that processes the original corrupted version $f = f(x)$.

Semilinear Parabolic Equation in $\mathbf{R}^N$

We consider here equation (5.1) in $\Omega = \mathbf{R}^N$; that is,

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \beta(y) \ni f & \text{in } (0,T)\times\mathbf{R}^N, \\ y(0,x) = y_0(x) & x \in \mathbf{R}^N, \\ y(t,\cdot) \in L^1(\mathbf{R}^N) & \forall t \in (0,T). \end{cases} \tag{5.34}$$

With respect to the case of bounded domain Ω previously studied, this problem presents some peculiarities and the more convenient functional space to study it is $L^1(\mathbf{R}^N)$.

We write (5.34) as a differential equation in $X = L^1(\mathbf{R}^N)$ of the form

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) \ni f(t), & t \in (0,T), \\ y(0) = y_0, \end{cases}$$

where $A : D(A) \subset L^1(\mathbf{R}^N) \to \mathbf{R}^N$ is defined by

$$\begin{aligned} Ay &= \{z \in L^1(\mathbf{R}^N);\ z = -\Delta y + w,\ w \in \beta(y),\ \text{a.e. in } \mathbf{R}^N\}, \\ D(A) &= \{y \in L^1(\mathbf{R}^N);\ \Delta y \in L^1(\mathbf{R}^N),\ \exists w \in L^1(\mathbf{R}^N), \\ &\qquad \text{such that } w(x) \in \beta(y(x)),\ \text{a.e. } x \in \mathbf{R}^N\}. \end{aligned}$$

By Theorem 3.3 we know that, if $N = 1,2,3$, then A is m-accretive in $L^1(\mathbf{R}^N) \times L^1(\mathbf{R}^N)$.

Then, by Theorem 4.1, which neatly applies to this situation, we get the following existence result.

Proposition 5.9. *Let $y_0 \in L^1(\mathbf{R}^N)$ and $f \in L^1(0,T;\mathbf{R}^N)$ be such that $\Delta y_0 \in L^1(\mathbf{R}^N)$ and $\exists w \in L^1(\mathbf{R}^N)$, $w(x) \in \beta(y_0(x))$, a.e. $x \in \mathbf{R}^N$. Then, problem* (5.34) *has a unique mild solution $y \in C([0,T];L^1(\mathbf{R}^N))$. In other words,*

$$y(t) = \lim_{\varepsilon\to 0} y_\varepsilon(t) \quad \textit{strongly in } L^1(\mathbf{R}^n) \textit{ for each } t \in [0,T], \tag{5.35}$$

where y_ε *is the solution to the finite difference scheme*

$$\begin{aligned} &y_\varepsilon(t) = y_\varepsilon^i \quad \textit{for } t \in (i\varepsilon,(i+1)\varepsilon), \qquad i = 0,1,\ldots,M,\\ &y_\varepsilon^{i+1} - y_\varepsilon^i - \varepsilon\Delta y_\varepsilon^{i+1} + \varepsilon\beta(y_\varepsilon^{i+1}) \ni \int_{i\varepsilon}^{(i+1)\varepsilon} f(t)dt \quad \textit{in } \mathbf{R}^n,\\ &y_\varepsilon^i \in L^1(\mathbf{R}^N), \qquad i = 0,1,\ldots,M = \left[\tfrac{T}{\varepsilon}\right]. \end{aligned} \tag{5.36}$$

5.2 Parabolic Variational Inequalities

An important class of multivalued nonlinear parabolic-like boundary value problem is the so-called parabolic variational inequalities which we briefly present below in an abstract setting.

Here and throughout in the sequel, V and H are real Hilbert spaces such that V is dense in H and $V \subset H \subset V'$ algebraically and topologically. We denote by $|\cdot|$ and $\|\cdot\|$ the norms of H and V, respectively, and by $(\cdot,\cdot)$ the scalar product in H and the pairing between V and its dual V'. The norm of V' is denoted $\|\cdot\|_*$. The space H is identified with its own dual.

We are given a linear continuous and symmetric operator A from V to V' satisfying the coercivity condition

$$(Ay,y) + \alpha|y|^2 \ge \omega\|y\|^2, \qquad \forall y \in V, \tag{5.37}$$

for some $\omega > 0$ and $\alpha \in \mathbf{R}$. We are also given a lower semicontinuous convex function $\varphi : V \to \overline{\mathbf{R}} = (-\infty,+\infty]$, $\varphi \not\equiv +\infty$.

For $y_0 \in V$ and $f \in L^2(0,T;V')$, consider the following problem.

Find $y \in L^2(0,T;V) \cap C([0,T];H) \cap W^{1,2}([0,T];V')$ *such that*

$$\begin{cases} (y'(t)+Ay(t),y(t)-z) + \varphi(y(t)) - \varphi(z) \le (f(t),y(t)-z),\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,T),\ \forall z \in V,\\ y(0) = y_0. \end{cases} \tag{5.38}$$

Here, $y' = dy/dt$ is the strong derivative of the function $y : [0,T] \to V'$. In terms of the subgradient mapping $\partial\varphi : V \to V'$, problem (5.38) can be written as

$$\begin{cases} y'(t) + Ay(t) + \partial\varphi(y(t)) \ni f(t), \quad \text{a.e. } t \in (0,T),\\ y(0) = y_0. \end{cases} \tag{5.39}$$

This is an abstract variational inequality of parabolic type. In applications to partial differential equations, V is a Sobolev subspace of $H = L^2(\Omega)$ (Ω is an open subset of $\mathbf{R}^N$), A is an elliptic operator on Ω, and the unknown function $y : \Omega \times [0,T] \to \mathbf{R}$ is viewed as a function of t from $[0,T]$ to $L^2(\Omega)$.

As seen earlier in Section 4.1, in the special case where $\varphi = I_K$ is the indicator function of a closed convex subset K of V; that is,

$$\varphi(y) = 0 \quad \text{if } y \in K, \qquad \varphi(y) = +\infty \quad \text{if } y \notin K, \tag{5.40}$$

the variational inequality (5.38) reduces to the reflection problem

$$\begin{cases} y(t) \in K, & \forall t \in [0,T], \\ (y'(t) + Ay(t), y(t) - z) \le (f(t), y(t) - z), & \text{a.e. } t \in (0,T), \forall z \in K, \\ y(0) = y_0. \end{cases} \tag{5.41}$$

Regarding the existence for problem (5.38), we have the following.

Theorem 5.1. *Let $f \in W^{1,2}([0,T];V')$ and $y_0 \in V$ be such that*

$$\{Ay_0 + \partial\varphi(y_0) - f(0)\} \cap H \ne \emptyset. \tag{5.42}$$

Then, problem (5.38) *has a unique solution $y \in W^{1,2}([0,T];V) \cap W^{1,\infty}([0,T];H)$ and the map $(y_0, f) \to y$ is Lipschitz from $H \times L^2(0,T;V')$ to $C([0,T];H) \cap L^2(0,T;V)$. If $f \in W^{1,2}([0,T];V')$ and $\varphi(y_0) < \infty$, then problem* (5.38) *has a unique solution $y \in W^{1,2}([0,T];H) \cap C_w([0,T];V)$. If $f \in L^2(0,T;H)$ and $\varphi(y_0) < \infty$, then problem* (5.38) *has a unique solution $y \in W^{1,2}([0,T];H) \cap C_w([0,T];V)$, that satisfies*

$$y'(t) = (f(t) - Ay(t) - \partial\varphi(y(t)))^0, \quad \textit{a.e. } t \in (0,T).$$

Here $C_w([0,T];V)$ is the space of weakly continuous functions from $(0,T)$ to V; that is, from $(0,T)$ to V endowed with the weak topology.

Proof. Consider the operator $L : D(A) \subset H \to H$,

$$\begin{aligned} Ly &= \{Ay + \partial\varphi(y)\} \cap H, \qquad \forall y \in D(L), \\ D(L) &= \{y \in V;\ \{Ay + \partial\varphi(y)\} \cap H \ne \emptyset\}. \end{aligned}$$

Note that $\alpha I + L$ is maximal monotone in $H \times H$ (I is the identity operator in H). Indeed, by hypothesis (5.37), the operator $\alpha I + A$ is continuous and positive definite from V to V'. Because $\partial\varphi : V \to V'$ is maximal monotone we infer by Theorem 2.6 (or by Corollary 2.6) that $\alpha I + L$ is maximal monotone from V to V' and, consequently, in $H \times H$.

Then, by Theorem 4.6, for every $y_0 \in D(L)$ and $g \in W^{1,1}([0,T];H)$ the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt}(t) + Ly(t) \ni g(t), & \text{a.e. in } (0,T), \\ y(0) = y_0, \end{cases}$$

has a unique strong solution $y \in W^{1,\infty}([0,T];H)$. Let us observe that $\partial\varphi_\alpha = \alpha I + L$, where $\varphi_\alpha : H \to \overline{\mathbf{R}}$ is given by

$$\varphi_\alpha(y) = \frac{1}{2}(Ay + \alpha y, y) + \varphi(y), \qquad \forall y \in H. \tag{5.43}$$

Indeed, φ_α is convex and lower semicontinuous in H because

$$\lim_{\|y\|\to\infty} \frac{\varphi_\alpha(y)}{\|y\|} = \infty$$

and φ_α is lower semicontinuous on V.

On the other hand, it is readily seen that $\alpha I + L \subset \partial\varphi_\alpha$, and because $\alpha I + L$ is maximal monotone, we infer that $\alpha I + L = \partial\varphi_\alpha$, as claimed. In particular, this implies that $\overline{D(L)} = \overline{D(\varphi_\alpha)} = \overline{D(\varphi)}$ (in the topology of H).

Now, let $y_0 \in V$ and $f \in W^{1,2}([0,T];V')$, satisfying condition (5.42).

Let $\{y_0^n\} \subset D(L)$ and $\{f_n\} \subset W^{1,2}([0,T];H)$ be such that

$$\begin{aligned} y_0^n &\to y_0 && \text{strongly in } H, \text{ weakly in } V,\\ f_n &\to f && \text{strongly in } L^2(0,T;V'),\\ \frac{d}{dt} f_n &\to \frac{df}{dt} && \text{strongly in } L^2(0,T;V'). \end{aligned}$$

Let $y_n \in W^{1,\infty}([0,T];H)$ be the corresponding solution to the Cauchy problem

$$\begin{cases} \dfrac{dy_n}{dt}(t) + Ly_n(t) \ni f_n(t), & \text{a.e. in } (0,T),\\ y_n(0) = y_0^n. \end{cases} \tag{5.44}$$

If we multiply (5.44) by $y_n - y_0$ and use condition (5.37), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|y_n(t) - y_0|^2 + \omega\|y_n(t) - y_0\|^2\\ &\le \alpha|y_n(t) - y_0|^2 + (f_n(t) - \xi, y_n(t) - y_0), \quad \text{a.e. } t \in (0,T), \end{aligned} \tag{5.45}$$

where $\xi \in Ay_0 + \partial\varphi(y_0) \subset V'$. After some calculation involving Gronwall's lemma, this yields

$$|y_n(t) - y_0|^2 + \int_0^t \|y_n(s) - y_0\|^2 ds \le C, \qquad \forall n \in \mathbf{N},\ t \in [0,T]. \tag{5.46}$$

Now, we use the monotonicity of $\partial\varphi$ along with condition (5.37) to get by (5.44) that

$$\frac{1}{2}\frac{d}{dt}|y_n(t)-y_m(t)|^2+\omega\|y_n(t)-y_m(t)\|^2$$
$$\leq \alpha|y_n(t)-y_m(t)|^2+\|f_n(t)-f_m(t)\|_*\|y_n(t)-y_m(t)\|, \quad \text{a.e. } t\in(0,T).$$

Integrating on $(0,t)$, and using Gronwall's lemma, we obtain the inequality

$$|y_n(t)-y_m(t)|^2+\int_0^T\|y_n(t)-y_m(t)\|^2dt$$
$$\leq C\left(|y_0^n-y_0^m|^2+\int_0^t\|f_n(t)-f_m(t)\|^2dt\right).$$

Thus, there is $y\in C([0,T];H)\cap L^2(0,T;V)$ such that

$$y_n\to y \quad \text{in } C([0,T];H)\cap L^2(0,T;V). \tag{5.47}$$

Now, again using equation (5.44), we get

$$\frac{1}{2}\frac{d}{dt}|y_n(t+h)-y_n(t)|^2+\omega\|y_n(t+h)-y_n(t)\|^2$$
$$\leq \alpha|y_n(t+h)-y_n(t)|^2+\|f_n(t+h)-f_n(t)\|_*\|y_n(t+h)-y_n(t)\|,$$

for all $t,h\in(0,T)$ such that $t+h\in(0,T)$. This yields

$$|y_n(t+h)-y_n|^2+\int_0^{T-h}\|y_n(t+h)-y_n(t)\|^2dt$$
$$\leq C\left(|y_n(h)-y_0^n|^2+\int_0^{T-h}\|f_n(t+h)-f_n(t)\|_*^2dt\right)$$

and, letting n tend to $+\infty$,

$$|y(t+h)-y(t)|^2+\int_0^{T-h}\|y(t+h)-y(t)\|^2dt$$
$$\leq C\left(|y(h)-y_0|^2+\int_0^{T-h}\|f(t+h)-f(t)\|_*^2dt\right), \tag{5.48}$$
$$\forall t\in[0,T-h].$$

Next, by (5.45) we see that, if $\xi\in Ay_0+\partial\varphi(y_0)$ is such that $f(0)-\xi\in H$, then we have

$$\frac{1}{2}\frac{d}{dt}|y_n(t)-y_0|^2+\omega\|y_n(t)-y_0\|^2$$
$$\leq \alpha|y_n(t)-y_0|^2+\|f_n(t)-f_n(0)\|_*\|y_n(t)-y_0^n\|+|f_n(0)-\xi|\,|y_n(t)-y_0^n|.$$

Integrating and letting $n\to\infty$, we get by the Gronwall inequality

$$|y(t) - y_0| \leq C\left(\int_0^t \|f(s) - f(0)\|_* ds + |f(0) - \xi|t\right), \qquad \forall t \in [0,T].$$

This yields, eventually with a new positive constant C,

$$|y(t) - y_0| \leq Ct, \qquad \forall t \in [0,T].$$

Along with (5.48), the latter inequality implies that y is H-valued, absolutely continuous on $[0,T]$, and

$$|y'(t)|^2 + \int_0^t \|y'(t)\|^2 dt \leq C\left(|y_0|^2 + \int_0^T \|f'(t)\|_*^2 dt + 1\right), \quad \text{a.e. } t \in (0,T),$$

where $y' = dy/dt,\ f' = df/dt$. Hence, $y \in W^{1,\infty}([0,T];H) \cap W^{1,2}([0,T];V)$.

Let us show now that y satisfies equation (5.38) (equivalently, (5.39)). By (5.44), we have

$$\frac{1}{2}\,\frac{d}{dt}\,|y_n(t) - z|^2 \leq (f_n(t) - \alpha y_n(t) - \eta, y_n(t) - z), \quad \text{a.e. } t \in (0,T),$$

where $z \in D(L)$ and $\eta \in Lz$. This yields

$$\frac{1}{2}\,(|y_n(t+\varepsilon) - z|^2 - |y_n(t) - z|^2 \leq \int_t^{t+\varepsilon} (f_n(s) + \alpha y_n(s) - \eta, y_n(s) - z))ds$$

and, letting $n \to \infty$,

$$\frac{1}{2}\,(|y(t+\varepsilon) - z|^2 - |y(t) - z|^2) \leq \int_t^{t+\varepsilon} (f(s) + \alpha y(s) - \eta, y(s) - z)\,ds.$$

Finally, this yields

$$(y(t+\varepsilon) - y(t), y(t) - z) \leq \int_t^{t+\varepsilon} (f(s) + \alpha y(s) - \eta, y(s) - z) ds.$$

Because y is, a.e., H-differentiable on $(0,T)$, we get

$$(y'(t) - \alpha y(t) + \eta - f(t), y(t) - z) \leq 0, \quad \text{a.e. } t \in (0,T),$$

for all $[z,\eta] \in L$. Now, because L is maximal monotone in $H \times H$, we conclude that

$$f(t) \in y'(t) + Ly(t), \quad \text{a.e. } t \in (0,T),$$

as desired.

Now, if $(y_0^i, f_i),\ i = 1,2$, satisfy condition (5.42) and the y_i are the corresponding solutions to equation (5.39), by assumption (5.37) it follows that

$$|y_1(t)-y_2(t)|^2+\int_0^T \|y_1(t)-y_2(t)\|^2dt$$
$$\leq C\left(|y_0^1-y_0^2|^2+\int_0^T \|f_1(t)-f_2(t)\|_*^2dt\right), \qquad \forall t\in[0,T].$$

Now, assume that $f\in W^{1,2}([0,T];V')$ and $y_0\in D(\varphi)$. Then, as seen earlier, we may rewrite equation (5.39) as

$$\begin{cases} y'(t)+\partial\varphi_\alpha(y(t))-\alpha y(t)\ni f(t), & \text{a.e. } t\in(0,T),\\ y(0)=y_0, \end{cases} \tag{5.49}$$

where $\varphi_\alpha : H\to\overline{\mathbf{R}}$ is defined by (5.43). For $f=f_n$ and $y_0=y_0^n$, $y=y_n$, we have the estimate

$$|y_n'(t)|^2+\frac{d}{dt}\,\varphi_\alpha(y_n(t))-\frac{\alpha}{2}\,\frac{d}{dt}\,|y_n(t)|^2\leq (f_n(t),y_n'(t)), \quad \text{a.e. } t\in(0,T).$$

This yields

$$\int_0^T |y_n'(t)|^2dt+\varphi_\alpha(y_n(t))\leq (f_n(0),y_n^0)+\int_0^T \|f_n'(t)\|_*\|y_n(t)\|dt-\frac{\alpha}{2}\,|y_n^0|^2.$$

Finally,

$$\int_0^T |y_n'(t)|^2dt+\|y_n(t)\|^2\leq C(\|f_n\|_{W^{1,2}([0,T];V')}+|y_n^0|^2)\leq C.$$

Then, arguing as before, we see that the function y given by (5.47) belongs to $W^{1,2}([0,T];H)\cap L^\infty(0,T;V)$ and is a solution to equation (5.38).

Because $y\in C([0,T];H)\cap L^\infty(0,T;V)$, it is readily seen that y is weakly continuous from $[0,T]$ to V.

If $f\in L^\infty(0,T;H)$ and $y_0\in D(\varphi_\alpha)$, we may apply Theorem 5.1 to equation (5.49) to arrive at the same result. □

Theorem 5.2. *Let $y_0\in K$ and $f\in W^{1,2}([0,T];V')$ be given such that*

$$(f(0)-Ay_0-\xi_0,y_0-v)\geq 0, \qquad \forall v\in K, \tag{5.50}$$

for some $\xi_0\in H$.

Then, (5.41) *has a unique solution $y\in W^{1,\infty}([0,T];H)\cap W^{1,2}([0,T];V)$.*

If $y_0\in K$ and $f\in W^{1,2}([0,T];V')$, then system (5.41) *has a unique solution $y\in W^{1,2}([0,T];H)\cap C_w([0,T];V)$. If $f\in L^2(0,T;H)$ and $y_0\in K$, then* (5.41) *has a unique solution $y\in W^{1,2}([0,T];H)\cap C_w([0,T];V)$. Assume in addition that*

$$(Ay,y)\geq\omega\|y\|^2, \qquad \forall y\in V, \tag{5.51}$$

for some $\omega>0$, and that there is $h\in H$ such that

$$(I+\varepsilon A_H)^{-1}(y+\varepsilon h)\in K, \qquad \forall\varepsilon>0,\ \forall y\in K. \tag{5.52}$$

Then, $Ay\in L^2(0,T;H)$.

Proof. The first part of the theorem is an immediate consequence of Theorem 5.1. Now, assume that $f\in L^2(0,T;H)$, $y_0\in K$, and conditions (5.51) and (5.52) hold. Let $y\in W^{1,2}([0,T];H)\cap C_w([0,T];V)$ be the solution to (5.41). If in (5.41) we take $z=(I+\varepsilon A_H)^{-1}(y+\varepsilon h)$ (we recall that $A_H y = Ay\cap H$), we get

$$\begin{aligned}&(y'(t)+A(t),A_\varepsilon(t)-(I+\varepsilon A_H)^{-1}h)\\&\le (f(t),A_\varepsilon y(t)-(I+\varepsilon A_H)^{-1}h),\quad \text{a.e. } t\in(0,T),\end{aligned}$$

where $A_\varepsilon = A(I+\varepsilon A_H)^{-1}=\varepsilon^{-1}(I-(I+\varepsilon A_H)^{-1})$. Because, by monotonicity of A,

$$(Ay,A_\varepsilon y)\ge |A_\varepsilon y|^2,\qquad \forall y\in D(A_H)=\{y;\ Ay\in H\}$$

and

$$\frac{1}{2}\frac{d}{dt}(A_\varepsilon y(t),y(t))=(y'(t),A_\varepsilon(t)),\quad \text{a.e. } t\in(0,T),$$

we get

$$\begin{aligned}&(A_\varepsilon y(t),y(t))+\int_0^t |A_\varepsilon y(s)|^2ds\\&\le (A_\varepsilon y_0,y_0)+2\int_0^t (A_\varepsilon y(s)-(I+\varepsilon A_H)^{-1}f(s),h)ds\\&+\int_0^t |f(s)|^2ds+2(y(t)-y_0,(I+\varepsilon A_H)^{-1}h),\quad \text{a.e. } t\in(0,T).\end{aligned}$$

Hence,

$$\int_0^T |A_\varepsilon y(t)|^2dt+(A_\varepsilon(t),y(t))\le C,\qquad \forall\varepsilon>0,\ t\in[0,T],$$

and, by Proposition 2.3, we conclude that $Ay\in L^2(0,T;H)$, as claimed. □

Now, we prove a variant of Theorem 5.1 in the case where $\varphi: V\to\overline{\mathbf{R}}$ is lower semicontinuous on H. (It is easily seen that this happens, for instance, if $\varphi(u)/\|u\|\to+\infty$ as $\|u\|\to\infty$.

Proposition 5.10. *Let* $A:V\to V'$ *be a linear, continuous, symmetric operator satisfying condition* (5.37) *and let* $\varphi:H\to\overline{\mathbf{R}}$ *be a lower semicontinuous convex function. Furthermore, assume that there is C independent of* ε *such that either*

$$(Ay,\nabla\varphi_\varepsilon(y))\ge -C(1+|\nabla\varphi_\varepsilon(y)|)(1+|y|),\qquad \forall y\in D(A_H), \tag{5.53}$$

or

$$\varphi((I+\varepsilon A_H)^{-1}(y+\varepsilon h))\le\varphi(y)+C,\qquad \forall\varepsilon>0,\ \forall y\in H, \tag{5.54}$$

for some $h\in H$*, where* $A_\alpha=\alpha I+A_H$.

Then, for every $y_0 \in \overline{D(\varphi) \cap V}$ *and every* $f \in L^2(0,T;H)$, *problem* (5.41) *has a unique solution* $y \in W^{1,2}((0,T];H) \cap C([0,T];H)$ *such that* $t^{1/2}y' \in L^2(0,T;H)$, $t^{1/2}Ay \in L^2(0,T;H)$. *If* $y_0 \in D(\varphi) \cap V$, *then* $y \in W^{1,2}([0,T];H) \cap C([0,T];V)$. *Finally, if* $y_0 \in D(A_H) \cap D(\partial\varphi)$ *and* $f \in W^{1,1}([0,T];H)$, *then* $y \in W^{1,\infty}([0,T];H)$.

Here, φ_ε is the regularization of φ.

Proof. As seen previously, the operator

$$A_\alpha y = \alpha y + Ay, \qquad \forall y \in D(A_\alpha) = D(A_H),$$

is maximal monotone in $H \times H$. Then, by Theorem 2.6 (if condition (5.53) holds) and, respectively, Theorem 2.1 (under assumption (5.54)), $A_\alpha + \partial\varphi$ is maximal monotone in $H \times H$ and

$$|A_\alpha y| \le C(|(A_\alpha + \partial\varphi)^0(y)| + |y| + 1), \qquad \forall y \in D(A_H) \cap D(\partial\varphi).$$

Moreover, $A_\alpha + \partial\varphi = \partial\varphi^\alpha$, where (see (5.43))

$$\varphi^\alpha(y) = \frac{1}{2}\,(Ay,y) + \varphi(y) + \frac{\alpha}{2}\,|y|^2, \qquad \forall y \in V,$$

and writing equation (5.39) as

$$y' + \partial\varphi^\alpha(y) - \alpha y \ni f, \qquad \text{a.e. in } (0,T),$$
$$y(0) = y_0,$$

it follows by Theorem 4.1 that there is a strong solution y to equation (5.43) satisfying the conditions of the theorem. Note, for instance, that if $y_0 \in D(\varphi) \cap V$, then $y \in W^{1,2}([0,T];H)$ and $\varphi^\alpha(y) \in W^{1,1}([0,T])$. Because y is continuous from $[0,T]$ to H and bounded in V, this implies that y is weakly continuous from $[0,T]$ to V. Now, because $t \to \varphi^\alpha(y(t))$ is continuous and $\varphi : H \to \overline{\mathbf{R}}$ is lower semicontinuous, we have

$$\lim_{t_n \to t}(Ay(t_n), y(t_n)) \le (Ay(t), y(t)), \qquad \forall t \in [0,T],$$

and this implies that $y \in C([0,T];V)$, as claimed. □

Corollary 5.1. *Let* $A : V \to V'$ *be a linear, continuous, and symmetric operator satisfying condition* (5.37) *and let* K *be a closed convex subset of* H *with*

$$(I + \varepsilon A_\alpha)^{-1}(y + \varepsilon h) \in K, \qquad \forall \varepsilon > 0,\ \forall y \in K, \tag{5.55}$$

for some $h \in H$. *Then, for every* $y_0 \in K$ *and* $f \in L^2(0,T;H)$, *the variational inequality* (5.41) *has a unique solution*

$$y \in W^{1,2}([0,T];H) \cap C([0,T];V) \cap L^2(0,T;D(A_H)).$$

Moreover, one has

$$\frac{dy}{dt}(t) + (A_H y(t) - f(t) - N_K(y(t)))^0 = 0, \quad a.e.\ t \in (0,T),$$

where $N_K(y) \subset L^2(\Omega)$ *is the normal cone at* K *in* y*.*

The parabolic variational inequalities represent a rigorous and efficient way to treat dynamic diffusion problems with a free or moving boundary. As an example, consider the *obstacle parabolic problem*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } \{(x,t) \in Q;\ y(x,t) > \psi(x)\}, \\ \dfrac{\partial y}{\partial t} - \Delta y \geq f & \text{in } Q = \Omega \times (0,T), \\ y(x,t) \geq \psi(x) & \forall (x,t) \in Q, \\ \alpha_1 y + \alpha_2 \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y_0(x) & x \in \Omega, \end{cases} \tag{5.56}$$

where Ω is an open bounded subset of $\mathbf{R}^N$ with smooth boundary (of class $C^{1,1}$, for instance), $\psi \in H^2(\Omega)$, and $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > 0$.

This is a problem of the form (5.41), where

$$H = L^2(\Omega), \qquad V = H^1(\Omega),$$

and $A \in L(V,V')$ is defined by

$$(Ay,z) = \int_\Omega \nabla y \cdot \nabla z\, dx + \frac{\alpha_1}{\alpha_2} \int_{\partial\Omega} yz\, d\sigma, \qquad \forall y,z \in H^1(\Omega), \tag{5.57}$$

if $\alpha_2 \neq 0$, or

$$(Ay,z) = \int_\Omega \nabla y \cdot \nabla z\, dx, \qquad \forall y,z \in H_0^1(\Omega), \tag{5.58}$$

if $\alpha_2 = 0$. (In this case, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$.)

The set $K \subset V$ is given by

$$K = \{y \in H^1(\Omega);\ y(x) \geq \psi(x), \quad \text{a.e. } x \in \Omega\}, \tag{5.59}$$

and condition (5.55) is satisfied if

$$\alpha_1 \psi + \alpha_2 \frac{\partial \psi}{\partial \nu} \leq 0, \quad \text{a.e. on } \partial\Omega. \tag{5.60}$$

Note also that $A_H : D(A_H) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$A_H y = -\Delta y, \quad \text{a.e. in } \Omega, \qquad \forall y \in D(A_H),$$
$$D(A_H) = \left\{ y \in H^2(\Omega);\ \alpha_1 y + \alpha_2 \frac{\partial y}{\partial \nu} = 0, \quad \text{a.e. on } \partial\Omega \right\},$$

and

$$\|y\|_{H^2(\Omega)} \le C(\|A_H y\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}), \qquad \forall y \in D(A_H),$$

Then, we may apply Corollary 5.1 to get the following.

Corollary 5.2. *Let* $f \in L^2(Q)$, $y_0 \in H^1(\Omega)$ $(y_0 \in H_0^1(\Omega)$ *if* $\alpha_2 = 0)$ *be such that* $y_0 \ge \psi$, *a.e. in* Ω. *Assume also that* $\psi \in H^1(\Omega)$ *satisfies condition* (5.60). *Then, problem* (5.56) *has a unique solution*

$$y \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C([0,T];H_0^1(\Omega)).$$

Noting that

$$N_K(y) = \{v \in L^2(\Omega);\ v(x) \in \beta(y(x) - \psi(x)), \quad \text{a.e. } x \in \Omega\},$$

where $\beta : \mathbf{R} \to 2^{\mathbf{R}}$ is given by

$$\beta(r) = \begin{cases} 0 & r > 0, \\ \mathbf{R}^- & r = 0, \\ \emptyset & r < 0, \end{cases}$$

it follows by Corollary 5.1 that the solution y satisfies the equation

$$\frac{d}{dt} y(t) + (-\Delta y(t) + \beta(y(t) - \psi) - f(t))^0 = 0, \quad \text{a.e. } t \in (0,T).$$

Hence, the solution y to problem (5.56) given by Corollary 5.2 satisfies the system

$$\begin{cases} \dfrac{\partial}{\partial t} y(x,t) - \Delta y(x,t) = f(x,t), & \text{a.e. in } \{(x,t) \in Q;\ y(x,t) > \psi(x)\}, \\ \dfrac{\partial}{\partial t} y(x,t) = \max\{f(x,t) + \Delta\psi(x), 0\}, & \text{a.e. in } \{(x,t);\ y(x,t) = \psi(x)\}, \end{cases} \tag{5.61}$$

because $y(\cdot,t) \in H^2(\Omega)$ and so $\Delta y(x,t) = \Delta\psi(x)$, a.e. in $\{y(x,t) = \psi(x)\}$.

It follows, also, that the solution y to the obstacle problem (5.56) is given by

$$y(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t) \quad \text{in } C([0,T];L^2(\Omega)),$$

where y_ε is the solution to the penalized problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y - \dfrac{1}{\varepsilon}(y - \psi)^- = f & \text{in } Q, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ \alpha_1 y + \alpha_1 \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma. \end{cases} \tag{5.62}$$

Now, let us consider the obstacle problem (5.56) with nonhomogeneous boundary conditions; that is,

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } \{(x,t) \in Q;\ y(x,t) > \psi(x)\}, \\ \dfrac{\partial y}{\partial t} - \Delta y \geq f,\ y \geq 0 & \text{in } Q, \\ \alpha y + \dfrac{\partial y}{\partial \nu} = g & \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\ y = 0 & \text{on } \Sigma_2 = \Gamma_2 \times (0,T), \\ y(x,0) = y_0(x) & \text{on } \Omega, \end{cases} \tag{5.63}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \cap \Gamma_2 = \emptyset$, and $g \in L^2(\Sigma_1)$.

If we take

$$V = \{y \in H^1(\Omega); \qquad y = 0 \quad \text{on } \Gamma_2\},$$

define $A : V \to V'$ by

$$(Ay,z) = \int_\Omega \nabla y \cdot \nabla z\, dx + \alpha \int_{\Gamma_1} yz\, dx, \qquad \forall y,z \in V,$$

and $f_0 : [0,T] \to V'$ by

$$(f_0(t),z) = \int_{\Gamma_1} g(x,t)z(x)dx, \qquad \forall z \in V,$$

we may write problem (5.63) as

$$\begin{aligned} \left(\frac{dy}{dt}(t) + Ay(t), y(t) - z\right) &\leq (F(t), y(t) - z), \qquad \forall z \in K,\ \text{a.e. } t \in (0,T), \\ y(0) &= y_0, \end{aligned} \tag{5.64}$$

where $F = f + f_0 \in L^2(0,T;V')$ and K is defined by (5.59).

Equivalently,

$$\begin{aligned} \int_\Omega \frac{\partial y}{\partial t}(x,t)(y(x,t) - z(x))dx &+ \int_\Omega \nabla y(x,t) \cdot \nabla(y(x,t) - z(x))dx \\ &+ \alpha \int_{\Gamma_1} f(x,t)(y(x,t) - z(x))dx \\ &\leq \int_\Omega f(x,t)(y(x,t) - z(x))dx \\ &+ \int_{\Gamma_1} g(x,t)(y(x,t) - z(x))dx, \\ &\qquad \forall z \in K,\ t \in [0,T]. \end{aligned} \tag{5.65}$$

Applying Theorem 5.2, we get the following.

Corollary 5.3. *Let* $f \in W^{1,2}([0,T];L^2(\Omega))$*,* $g \in W^{1,2}([0,T];L^2(\Gamma_1))$*, and* $y_0 \in K$*. Then, problem* (5.65) *has a unique solution*

$$y \in W^{1,2}([0,T];V) \cap C_w([0,T];V).$$

If, in addition,

$$\begin{cases} \dfrac{\partial y_0}{\partial \nu} + \alpha y_0 = g(x,0), & \text{a.e. on } \{x \in \Gamma_1;\ y_0(x) > \psi(x)\}, \\ \dfrac{\partial \psi}{\partial \nu} + \alpha \psi \le g(x,0), & \text{a.e. on } \{x \in \Gamma_1;\ y_0(x) = \psi(x)\}, \end{cases} \tag{5.66}$$

then $y \in W^{1,2}([0,T];V) \cap W^{1,\infty}([0,T];L^2(\Omega))$.

(We note that condition (5.66) implies (5.50).)

It is readily seen that the solution y to (5.65) satisfies (5.63) in a certain generalized sense. Indeed, assuming that the set $E = \{(x,t);\ y(x,t) > \psi(x)\}$ is open and taking $z = y(x,t) \pm \rho\varphi$ in (5.65), where $\varphi \in C_0^\infty(E)$ and ρ is sufficiently small, we see that

$$\frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } \mathscr{D}'(E). \tag{5.67}$$

It is also obvious that

$$\frac{\partial y}{\partial t} - \Delta y \ge f \quad \text{in } \mathscr{D}'(Q). \tag{5.68}$$

Regarding the boundary conditions, by (5.65), (5.67), and (5.68), it follows that

$$\frac{\partial y}{\partial \nu} + \alpha y = g \quad \text{in } \mathscr{D}'(E \cap \Sigma_1),$$

respectively,

$$\frac{\partial y}{\partial \nu} + \alpha y \ge g \quad \text{in } \mathscr{D}'(\Sigma_1).$$

In other words,

$$\begin{cases} \dfrac{\partial y}{\partial \nu} + \alpha y = g & \text{on } \{(x,t) \in \Sigma_1;\ y(x,t) > \psi(x)\}, \\ \dfrac{\partial \psi}{\partial \nu} + \alpha \psi \ge g & \text{on } \{(x,t) \in \Sigma_1;\ y(x,t) = \psi(x)\}. \end{cases}$$

Hence, if g satisfies the compatibility condition

$$\frac{\partial \psi}{\partial \nu} + \alpha \psi \le g \quad \text{on } \Sigma_1,$$

then the solution y to problem (5.65) satisfies the required boundary conditions on Σ_1.

Also in this case, the solution y given by Corollary 5.3 can be obtained as the limit as $\varepsilon \to 0$ of the solution y_ε to the equation

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \beta_\varepsilon(y_\varepsilon - \psi) = f & \text{in } \Omega \times (0,T), \\ y_\varepsilon(x,0) = y_0(x) & \text{in } \Omega, \\ \dfrac{\partial y_\varepsilon}{\partial \nu} + \alpha y_\varepsilon = g \quad \text{on } \Sigma_1, & y_\varepsilon = 0 \quad \text{on } \Sigma_2, \end{cases} \tag{5.69}$$

where

$$\beta_\varepsilon(r) = -\left(\frac{1}{\varepsilon}\right) r^-, \qquad \forall r \in \mathbf{R}.$$

If $Q^+ = \{(x,t) \in Q;\ y(x,t) > \psi(x)\}$, we may view y as the solution to the free boundary problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in } Q^+, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ \alpha_1 y + \alpha_2 \dfrac{\partial y}{\partial \nu} = 0 \quad \text{on } \Sigma, & y = \psi,\ \dfrac{\partial y}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} \quad \text{on } \partial Q^+(t), \end{cases} \tag{5.70}$$

where $\partial Q^+(t)$ is the boundary of the set $Q^+(t) = \{x \in \Omega;\ y(x,t) > \psi(x)\}$. We call $\partial Q^+(t)$ the *moving boundary* and ∂Q^+ the *free boundary* of problem (5.70).

In problem (5.70), the noncoincidence set Q^+ as well as the free boundary ∂Q^+ is not known a priori and represents unknowns of the problem. In problem (5.41) or (5.65), the free boundary does not appear explicitly, but in this formulation the problem is nonlinear and multivalued.

Perhaps the best-known example of a parabolic free boundary problem is the classical Stefan problem, which we briefly describe in what follows and which has provided one of the principal motivations of the theory of parabolic variational inequalities.

The Stefan Problem

This problem describes the conduction of heat in a medium involving a phase charge. To be more specific, consider a unit volume of ice Ω at temperature $\theta < 0$. If a uniform heat source of intensity F is applied, then the temperature increases at rate E/C_1 until it reaches the melting point $\theta = 0$. Then, the temperature remains at zero until ρ units of heat have been supplied to transform the ice into water (ρ is the latent heat). After all the ice has melted the temperature begins to increase at the rate h/C_2 (C_1 and C_2 are specific heats of ice and water, respectively). During the process, the variation of the internal energy $e(t)$ is therefore given by

$$e(t) = C(\theta(t)) + \rho H(\theta(t)),$$

where

$$C(\theta) = \begin{cases} C_1\theta & \text{for } \theta \leq 0, \\ C_2\theta & \text{for } \theta > 0, \end{cases}$$

and H is the Heaviside graph

$$H(\theta) = \begin{cases} 1 & \theta > 0, \\ [0,1] & \theta = 0, \\ 0 & \theta < 0. \end{cases}$$

In other words, we have

$$e = \gamma(\theta) = \begin{cases} C_1\theta & \text{if } \theta < 0, \\ [0,\rho] & \text{if } \theta = 0, \\ C_2\theta + \rho & \text{if } \theta > 0. \end{cases} \tag{5.71}$$

The function γ is called the *enthalpy* of the system.

Now, let $Q = \Omega \times (0,\infty)$ and denote by Q_-, Q_+, Q_0 the regions of Q, where $\theta < 0$, $\theta > 0$, and $\theta = 0$, respectively. We set $S_+ = \partial Q_+$, $S_- = \partial Q_-$, and $S = S_+ \cup S_-$.

If $\theta = \theta(x,t)$ is the temperature distribution in Q and $q = q(x,t)$ the heat flux, then, according to the Fourier law,

$$q(x,t) = -k\nabla\theta(x,t), \tag{5.72}$$

where k is the thermal conductivity. Consider the function

$$K(\theta) = \begin{cases} k_1\theta & \text{if } \theta < 0, \\ k_2\theta & \text{if } \theta > 0, \end{cases}$$

where k_1, k_2 are the thermal conductivity of the ice and water, respectively.

If f is the external heat source, then the conservation law yields

$$\frac{d}{dt}\int_{\Omega^*} e(x,t)dx = -\int_{\partial\Omega^*} (q(x,t),\nu)d\sigma + \int_{\Omega^*} F(x,t)dx$$

for any subdomain $\Omega^* \times (t_1,t_2) \subset Q$ (ν is the normal to $\partial\Omega^*$) if e and q are smooth. Equivalently,

$$\begin{aligned} &\int_{\Omega^*} e_t(x,t)dx + \int_{S\cap\Omega^*} [|e(t)|]V(t)dt \\ &= -\int_{\Omega^*} \operatorname{div} q(x,t)dx + \int_{\partial\Omega^*\cap S} [|(q(t),\nu)|]d\sigma + \int_{\Omega^*} F(x,t)dx, \end{aligned}$$

where $V(t) = -N_t\|N_t\|$ is the true velocity of the interface S ($N = (N_1,N_2)$ is the unit normal to S) and $[|\cdot|]$ is the jump along S.

The previous inequality yields

$$\begin{aligned} &\frac{\partial}{\partial t} e(x,t) + \operatorname{div} q(x,t) = F(x,t) \quad &&\text{in } Q \setminus S, \\ &[|e(t)|] N_x + [|(q(t), N_t|] = 0 &&\text{on } S. \end{aligned} \tag{5.73}$$

Taking into account equations (5.71)–(5.73), we get the system

$$\begin{cases} C_1 \dfrac{\partial \theta}{\partial t} - k_1 \Delta \theta = f & \text{in } Q_-, \\ C_2 \dfrac{\partial \theta}{\partial t} - k_2 \Delta \theta = f & \text{in } Q_+, \end{cases} \tag{5.74}$$

$$\begin{cases} (k_2 \nabla \theta^+ - k_1 \nabla \theta^-) \cdot N_x = \rho N_t & \text{on } S, \\ \theta^+ = \theta^- = 0 & \text{on } S. \end{cases} \tag{5.75}$$

If we represent the interface S by the equation $t = \sigma(x)$, then (5.75) reads

$$\begin{cases} (k_1 \nabla \theta^+ - k_2 \nabla \theta^-) \cdot \nabla \sigma = -\rho & \text{in } S, \\ \theta^+ = \theta^- = 0. \end{cases} \tag{5.76}$$

The usual boundary and initial value conditions can be associated with equations (5.74) and (5.76), for instance,

$$\theta = 0 \qquad \text{in } \partial\Omega \times (0,T), \tag{5.77}$$

$$\theta(x,0) = \theta_0(x) \qquad \text{in } \Omega, \tag{5.78}$$

or Neumann boundary conditions on $\partial\Omega$.

This is the classical two-phase Stefan problem. Here, we first study with the methods of variational inequalities a simplified model described by the one-phase Stefan problem

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - \Delta \theta = 0 & \text{in } Q_+ = \{(x,t) \in Q;\ \sigma(x) < t < T\}, \\ \theta = 0 & \text{in } Q_- = \{(x,t) \in Q;\ 0 < t < \sigma(x)\}, \\ \nabla_x(x,t) \cdot \nabla \sigma(x) = -\rho & \text{on } S = \{(x,t);\ t = \sigma(x)\}, \\ \theta = 0 & \text{in } S \cup Q_-, \\ \theta \geq 0 & \text{in } Q_+. \end{cases} \tag{5.79}$$

These equations model the melting of a body of ice $\Omega \subset \mathbf{R}^3$ maintained at $\theta^0 C$. Therefore, assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are disjoint and Γ_1 is in contact with a heating medium with temperature θ_1; $t = \sigma(x)$ is the equation of the interface (moving boundary) S_t, which separates the liquid phase (water) and solid (ice). Thus, to equations (5.79) we must add the boundary conditions

$$\begin{cases} \dfrac{\partial \theta}{\partial \nu} + \alpha(\theta - \theta_1) = 0 & \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\ \theta = 0 & \text{on } \Sigma_2 = \Gamma_2 \times (0,T) \end{cases} \tag{5.80}$$

and the initial value conditions

$$\theta(x,0) = \theta_0(x) > 0,\ \forall x \in \Omega_0, \qquad \theta(x,0) = 0,\ \forall x \in \Omega \setminus \Omega_0. \tag{5.81}$$

There is a simple device due to G. Duvaut [21] that permits us to reduce problem (5.79)–(5.81) to a parabolic variational inequality. To this end, consider the function

$$y(x,t) = \begin{cases} \displaystyle\int_{\sigma(x)}^t \theta(x,s)ds & \text{if } x \in \Omega \setminus \Omega_0,\ t > \sigma(x), \\ \displaystyle\int_0^t \theta(x,s)ds & \text{if } x \in \Omega_0,\ t \in [0,T], \\ 0 & \text{if } (x,t) \in Q_-, \end{cases} \tag{5.82}$$

and let

$$f_0(x,t) = \begin{cases} -\rho & \text{if } x \in \Omega \setminus \Omega_0,\ 0 < t < T, \\ \theta_0(x) & \text{if } x \in \Omega_0,\ 0 < t < T. \end{cases} \tag{5.83}$$

Lemma 5.1. *Let $\theta \in H^1(Q)$ and $\sigma \in H^1(\Omega)$. Then,*

$$\frac{\partial y}{\partial t} - \Delta y = f_0 \chi \quad in \ \mathscr{D}'(Q), \tag{5.84}$$

where χ is the characteristic function of Q_+.

Proof. By (5.82), we have

$$\frac{\partial y}{\partial t}(\varphi) = \int_{Q_+} \theta(x,t)\varphi(x,t)dxdt, \qquad \forall \varphi \in C_0^\infty(Q).$$

On the other hand, we have

$$\begin{aligned} (y_x, \varphi) &= -y(\varphi_x) \\ &= -\int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)}^T \varphi_x(x,t)dt \int_{\sigma(x)}^t \theta(x,s)ds \\ &\quad -\int_{\Omega_0} dx \int_0^T \varphi_x(x,t)dt \int_0^t \theta(x,s)ds \\ &= -\int_{\Omega\setminus\Omega_0} dx \operatorname{div}\left(\int_{\sigma(x)}^T \varphi(x,t)dt \int_{\sigma(x)}^t \theta(x,s)ds\right) \\ &= \int_{\Omega\setminus\Omega_0} dx \left(\int_{\sigma(x)}^T \varphi(x,t)dt \int_{\sigma(x)}^t \theta_x(x,s)ds\right) \\ &\quad -\int_{\Omega_0} dx \operatorname{div}\left(\int_0^T \varphi(x,t)dt \int_0^t \theta(x,s)ds\right) \end{aligned}$$

$$= \int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)}^{T} \varphi(x,t)dt \int_{\sigma(x)}^{t} \theta_x(x,s)s$$
$$+ \int_{\Omega_0} dx \int_0^T \varphi(x,t)dt \int_{\sigma(x)}^{t} \theta_x(x,s)ds.$$

(Here, $y_x = \nabla_x y,\ \varphi_x = \nabla_x \varphi$.) This yields

$$\Delta y(\varphi) = -y_x(\varphi_x) = -\int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)}^{T} \varphi_x(x,t)dt \cdot \int_{\sigma(x)}^{T} \theta_x(x,s)ds$$
$$-\int_{\Omega_0} dx \int_0^T \varphi_x(x,t)dt \cdot \int_0^t \theta_x(x,s)ds$$

and, by the divergence formula, we get

$$\Delta y(\varphi) = \int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)}^{T} dt \left(\int_{\sigma(x)}^{t} \Delta\theta(x,s)ds\, \varphi(x,t)\right)$$
$$+ \int_{\Omega_0} ds \int_0^T dt \left(\int_0^t \Delta\theta(x,s)ds\, \varphi(x,t)\right), \qquad \forall \varphi \in C_0^\infty(Q),$$

because $\nabla_x\theta(x,\sigma(x)) \cdot \nabla\sigma(x) = -\rho$, $\forall x \in \Omega \setminus \Omega_0$. Then, by equations (5.79), we see that

$$\left(\frac{\partial y}{\partial t} - \Delta y\right)(\varphi) = -\int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)} dt \left(\int_{\sigma(x)}^{t} \theta_t(x,s)ds - \theta(s,t)\right)\varphi(x,t)$$
$$-\int_{\Omega_0} dx \int_0^T dt \left(\int_0^t \theta_t(x,s)ds - \theta(x,t)\right)\varphi(x,t)$$
$$-\rho \int_{\Omega\setminus\Omega_0} dx \int_{\sigma(x)}^{T} \varphi(x,t)dt$$
$$= \int_{Q_+} f_0(x,t)\varphi(x,t)dxdt,$$

as claimed. □

By Lemma 5.1 we see that the function y satisfies the obstacle problem

$$\begin{cases} y \ge 0,\ \dfrac{\partial y}{\partial t} - \Delta y \ge f_0 & \text{in } Q, \\ \dfrac{\partial y}{\partial t} - \Delta y = f_0 & \text{in } \{(x,t) \in Q;\ y(x,t) > 0\}, \\ y = 0 & \text{in } \{(x,t) \in Q;\ \sigma(x) > t\}, \end{cases} \tag{5.85}$$

and the boundary value conditions

$$\frac{\partial}{\partial \nu}\frac{\partial y}{\partial t} = -\alpha\left(\frac{\partial y}{\partial t} - \theta_1\right) \quad \text{on } \Sigma_1, \qquad \frac{\partial y}{\partial t} = 0 \quad \text{on } \Sigma_2, \tag{5.86}$$

(see (5.80) and (5.82)). Then, by Corollary 5.2, we have the following.

Corollary 5.4. *Let* $\theta_1 \in L^2(\Sigma_1)$ *be given. Then, problem* (5.85) and (5.86) *has a unique* (*generalized*) *solution* $y \in W^{1,\infty}([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];H^1(\Omega))$.

Keeping in mind that $S_t = \partial\{(x,t);\ y(x,t) = 0\}$, we can derive from Corollary 5.4 an existence result for the one-phase Stefan problem (5.79)–(5.81).

Other mathematical models for physical problems involving a free boundary such as the oxygen diffusion in an absorbing tissue (Elliott and Ockendon [23]) or electrochemical machining processes lead by similar devices to parabolic variational inequalities of the same type. It should be mentioned also that dynamics of elasto-plastic materials as well as the phase transition in systems composed of different metals are better described by parabolic variational inequalities, eventually combined with linear hyperbolic equations. This is the case for instance with Fremond's model of thermomechanical dynamics of shape memory delay. The phase transition often manifests a hysteretic behavior due to irreversible changes in process dynamics and the study of hypothesis models is another source of variational inequalities although the hysteresis operator, in general, is not monotone in the sense described above. However, some standard hysteresis equations (stop and play, for instance) are expressed in terms of variational inequalities. (We refer to Visintin book's [42] for a treatment of these problems.)

5.3 The Porous Media Diffusion Equation

The nonlinear diffusion equation models the dynamic of density in a substance undergoing diffusion described by Fick's first law (or Darcy's law). It also models phase transition dynamics (the Stefan problem) or other physical processes that are of diffusion type (heat propagation, filtration, or dynamics of biological groups). Such an equation can be schematically written as

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta\beta(y) \ni f & \text{in } \Omega \times (0,T) = Q, \\ \beta(y) = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{5.87}$$

where Ω is a bounded and open subset of $\mathbf{R}^N$ with smooth boundary, and $\beta : \mathbf{R} \to 2^{\mathbf{R}}$ is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ such that $0 \in D(\beta)$.

The steady-state equation associated with (5.87) is just the stationary porous media equation studied in Sections 2.2 and 3.2.

The function $y \in C([0,T];L^1(\Omega))$ is called a generalized solution to problem (5.87) if

$$\int_Q (y\varphi_t + \beta(y)\Delta\varphi)\,dx\,dt + \int_Q f\varphi\,dx\,dt + \int_\Omega y_0\varphi(x,0)\,dx = 0 \tag{5.88}$$

for all $\varphi \in C^{2,1}(\overline{Q})$ such that $\varphi(x,T) = 0$ in Ω and $\varphi = 0$ on Σ.

Let us first briefly describe some specific diffusive-like problems that lead to equations of this type.

1. *The flow of gases in porous media.* Let y be the density of a gas that flows through a porous medium that occupies a domain $\Omega \subset \mathbf{R}^3$ and let $\bar{v}$ be the pore velocity. If p denotes the pressure, we have $p = p_0 y^\alpha$ for $\alpha \geq 1$. Then, the conservation law equation

$$k_1 \frac{\partial y}{\partial t} + \operatorname{div}(y\bar{v}) = 0$$

combined with Darcy's law

$$\gamma\bar{v} = -k_2 \nabla p$$

(k_1 is the porosity of the medium, k_2 the permeability, and γ the viscosity) yields the *porous medium equation*

$$\frac{\partial y}{\partial t} - \delta\Delta y^{\alpha+1} = 0 \quad \text{in } Q, \tag{5.89}$$

where

$$\delta = k_2 p_0 (k_1(\alpha+1)\gamma)^{-1}.$$

Equation (5.89) is also relevant in the study of other mathematical models, such as population dynamics. The case where $-1 < \alpha < 0$ is that of fast diffusion processes arising in physics of plasma. In particular, the case

$$\beta(x) = \begin{cases} \log x & \text{for } x > 0 \\ -\infty & \text{for } x \leq 0 \end{cases}$$

emerges from the central limit approximation to Carleman's model of Boltzman equations. Nonlinear diffusion equations of the form (5.87) perturbed by a term of transport; that is,

$$\frac{\partial y}{\partial t} - \Delta\beta(y) + \operatorname{div} K(y) \ni f$$

with appropriate boundary conditions arise in the dynamics of underground water flows and are known in the literature as the Richards equation. The special case

$$\beta(y) = \begin{cases} \beta_0(y) & \text{for } y < y_s, \\ [\beta_0(y_s), +\infty) & \text{for } y = y_s, \\ \emptyset & \text{for } y > y_s, \end{cases}$$

where $\beta_0 : \mathbf{R} \to \mathbf{R}$ is a continuous and monotonically increasing function, models the dynamics of saturated–unsaturated underground water flows. The treatment of such an equation with methods of nonlinear accretive differential equations is given in Marinoschi [34, 35].

2. *Two-phase Stefan problem.* We come back to the two-phase Stefan problem (5.74), (5.75), (5.77), (5.78); that is

$$\begin{cases} C_1\theta_t - k_1\Delta\theta = f & \text{in } Q_-\{(x,t);\ \theta(x,t) < 0\} \\ C_2\theta_t - k_2\Delta\theta = f & \text{in } Q_+ = \{(x,t);\ \theta(x,t) > 0\}, \\ (k_1\nabla\theta^+ - k_2\nabla\theta^-)\cdot\nabla\sigma(x) = -\rho & \text{on } S, \end{cases} \tag{5.90}$$

where $t = \sigma(x)$ is the equation of the interface S.

We may write system (5.90) as

$$\frac{\partial}{\partial t}\gamma(\theta) - \Delta K(\theta) \ni f \quad \text{in } Q, \tag{5.91}$$

where $\gamma : \mathbf{R} \to 2^{\mathbf{R}}$ is given by (5.71). Indeed, for every test function $\varphi \in C_0^\infty(Q)$ we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t}\gamma(\theta) - \Delta K(\theta)\right)(\varphi) \\ &= -\int_Q (\gamma(\theta)\varphi_t + K(\theta)\Delta\varphi)\,dx\,dt \\ &= C_1\int_{Q_-}\theta_t\varphi\,dx\,dt + C_2\int_{Q_+}\theta_t\,dx\,dt - k_1\int_{Q_-}\varphi\Delta\theta\,dx\,dt \\ &\quad -k_2\int_{Q_+}\varphi\Delta\theta\,dx\,dt + \int_S\left(k_2\,\frac{\partial\theta^+}{\partial\nu} - k_1\,\frac{\partial\theta^-}{\partial\nu}\right)\varphi\,ds - \rho\int_{Q_+}\varphi_t\,dx\,dt \\ &= \int_{Q_-}(C_1\theta_t - k_1\Delta\theta)\varphi\,dx\,dt + \int_{Q_+}(C_2\theta_t - k_2\Delta\theta)\varphi\,dx\,dt \\ &\quad + \int_S((k_2\nabla\theta^+ - k_1\nabla\theta^-)\cdot\nabla\sigma + \rho)\,dx = 0. \end{aligned} \tag{5.92}$$

If we denote by β the function $\gamma^{-1}K$; that is,

$$\beta(r) = \begin{cases} k_1C_1^{-1}r & \text{for } r < 0, \\ 0 & \text{for } 0 \le r < \rho, \\ k_2C_2^{-1}(r-\rho) & \text{for } r \ge \rho, \end{cases} \tag{5.93}$$

we may write (5.91) in the form (5.87).

Problem (5.87) can be treated as a nonlinear accretive Cauchy problem in two functional spaces: $H^{-1}(\Omega)$ and $L^1(\Omega)$.

3. *The Hilbert space approach.* In the space $H^{-1}(\Omega)$, consider the operator

$$\begin{aligned} A = \{\ [y,w] \in (H^{-1}(\Omega)\cap L^1(\Omega)) \times H^{-1}(\Omega);\ w = -\Delta v, \\ v \in H_0^1(\Omega),\ v(x) \in \beta(y(x)), \quad \text{a.e. } x \in \Omega\}. \end{aligned}$$

We assume that

$$\beta^{-1} \text{ is everywhere defined and bounded on the bounded subsets of } \mathbf{R}. \tag{5.94}$$

Then, by Proposition 2.10, A is maximal monotone in $H^{-1}(\Omega) \times H^{-1}(\Omega)$. More precisely, $A = \partial\varphi$, where $\varphi : H^{-1}(\Omega) \to \overline{\mathbf{R}}$ is defined by

$$\varphi(y) = \begin{cases} \int_\Omega j(y(x))dx & \text{if } y \in L^1(\Omega) \cap H^{-1}(\Omega),\ j(y) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\partial j = \beta$.

Then, we may write problem (5.87) as

$$\begin{aligned} &\frac{dy}{dt} + Ay \ni f \qquad \text{in } (0,T), \\ &y(0) = y_0, \end{aligned} \tag{5.95}$$

and so, by Theorem 4.11, we obtain the following existence result.

Theorem 5.3. *Let β be a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ satisfying condition (5.94). Let $f \in L^1(0,T;H^{-1}(\Omega))$ and let $y_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $y_0(x) \in \overline{D(\beta)}$, a.e. $x \in \Omega$. Then, there is a unique pair of functions $y \in C([0,T];H^{-1}(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega))$ and $v : Q \to \mathbf{R}$, such that $v(t) \in H_0^1(\Omega)$, $\forall t \in [0,T]$ satisfying*

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta v = f, & \text{a.e. in } Q = \Omega \times (0,T), \\ v(x,t) \in \beta(y(x,t)), & \text{a.e. } (x,t) \in Q, \\ y(x,0) = y_0(x), & \text{a.e. in } \Omega. \end{cases} \tag{5.96}$$

$$t^{1/2}\,\frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega)), \qquad t^{1/2} v \in L^2(0,T;H_0^1(\Omega)). \tag{5.97}$$

Moreover, if $j(y_0) \in L^1(\Omega)$, then

$$\frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega)), \qquad v \in L^2(0,T;H_0^1(\Omega)). \tag{5.98}$$

If $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];H^{-1}(\Omega))$, then

$$\frac{\partial y}{\partial t} \in L^\infty(0,T;H^{-1}(\Omega)), \qquad v \in L^\infty(0,T;H_0^1(\Omega)). \tag{5.99}$$

We note that the derivative $\partial y/\partial t$ in (5.96) is the strong derivative dy/dt of the function $t \to y(\cdot,t)$ from $[0,T]$ into $H^{-1}(\Omega)$, and it coincides with the derivative $\partial y/\partial t$ in the sense of distributions on Q. It is readily seen that the solution y (see Theorem 5.3) is a generalized solution to (5.87) in the sense of definition (5.88).

4. *The L^1-approach.* In the space $X = L^1(\Omega)$, consider the operator

$$\begin{aligned} A = \{[y,w] \in L^1(\Omega) \times L^1(\Omega);\ w = -\Delta v, \\ v \in W_0^{1,1}(\Omega),\ v(x) \in \beta(y(x)), \quad \text{a.e. } x \in \Omega\}. \end{aligned} \tag{5.100}$$

We have seen earlier (Theorem 3.5) that A is m-accretive in $L^1(\Omega) \times L^1(\Omega)$. Then, applying the general existence Theorem 4.2, we obtain the following.

Proposition 5.11. *Let β be a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ such that $0 \in \overline{\beta(0)}$. Then, for every $f \in L^1(0,T;L^1(\Omega))$ and every $y_0 \in L^1(\Omega)$, such that $y_0(x) \in \overline{D(\beta)}$, a.e. $x \in \Omega$, the Cauchy problem*

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) \ni f(t) \quad \text{in } (0,T), \\ y(0) = y_0, \end{cases} \tag{5.101}$$

has a unique mild solution $y \in C([0,T];L^1(\Omega))$.

We note that $\overline{D(A)} = \{y_0 \in L^1(\Omega);\ y_0(x) \in \overline{D(\beta)}$, a.e. $x \in \Omega\}$.

Indeed, $(1+\varepsilon\beta)^{-1}y_0 \to y_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, if $y_0 \in \overline{D(\beta)}$, a.e. $x \in \Omega$, and $(I+\varepsilon A)^{-1}y_0 \to y_0$ if $j(y_0) \in L^1(\Omega)$.

Proposition 5.11 amounts to saying that

$$y(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t) \quad \text{in } L^1(\Omega), \text{ uniformly on } [0,T],$$

where y_ε is the solution to the difference equations

$$\begin{cases} \dfrac{1}{\varepsilon}(y_\varepsilon(t) - y_\varepsilon(t-\varepsilon)) - \Delta v_\varepsilon(t) = f_\varepsilon(t) & \text{in } \Omega \times (0,T), \\ v_\varepsilon(x,t) \in \beta(y_\varepsilon(x,t)), & \text{a.e. in } \Omega \times (0,T), \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ y_\varepsilon(t) = y_0 & \text{for } t \le \varepsilon,\ x \in \Omega. \end{cases} \tag{5.102}$$

The function $t \to v_\varepsilon(t) \in W_0^{1,1}(\Omega)$ is piecewise constant on $[0,T]$ and $f_\varepsilon(t) = f_i$, $\forall t \in [i\varepsilon,(i+1)\varepsilon]$ is a piecewise constant approximation of $f : [0,T] \to L^1(\Omega)$.

By (5.102), it is readily seen that y is a generalized solution to problem (5.87). In particular, it follows by Proposition 5.11 that the operator A defined by (5.100) generates a semigroup of nonlinear contractions $S(t) : \overline{D(A)} \to \overline{D(A)}$. This semigroup is not differentiable in $L^1(\Omega)$, but in some special situations it has regularity properties comparable with those of the semigroup generated by the Laplace operator on $L^2(\Omega)$ under Dirichlet boundary conditions. In fact, we have the following smoothing effect of nonlinear semigroup $S(t)$ with respect to the initial data.

Theorem 5.4. *Let $\beta \in C^1(\mathbf{R} \setminus \{0\}) \cap C(\mathbf{R})$ be a monotone function satisfying the conditions*

$$\beta(0) = 0, \qquad \beta'(r) \ge C|r|^{\alpha-1}, \qquad \forall r \ne 0, \tag{5.103}$$

where $\alpha > 0$ if $N \le 2$ and $\alpha > (N-2)/N$ if $N \ge 3$. Then, $S(t)(L^1(\Omega)) \subset L^\infty(\Omega)$ for every $t > 0$,

$$\|S(t)y_0\|_{L^\infty(\Omega)} \le Ct^{-(N/(N\alpha+2-N))}\|y_0\|_{L^1(\Omega)}^{2/(2+N(\alpha-1))}, \qquad \forall t>0, \tag{5.104}$$

and $S(t)(L^p(\Omega)) \subset L^p(\Omega)$ *for all* $t>0$ *and* $1 \le p < \infty$.

Proof. First, we establish the estimates

$$\begin{aligned}\|(I+\lambda A)^{-1}f\|_p^p + C\lambda\left(\int_\Omega |(I+\lambda A)^{-1}f|^{((p+\alpha-1)N)/(N-2)}dx\right)^{(N-2)/N}&\\ \le \|f\|_p^p, \qquad \forall f \in L^p(\Omega),\ \lambda>0,&\end{aligned} \tag{5.105}$$

for $N>2$, and

$$\begin{aligned}|(I+\lambda A)^{-1}f\|_p + C\lambda\left(\int_\Omega |(I+\lambda A)^{-1}f|^{(p+1-\alpha)q}dx\right)^{1/q} \le \int_\Omega |f|^p dx,&\\ \forall q>1,&\end{aligned} \tag{5.106}$$

if $N=2$. Here $\|\cdot\|_p$ is the L^p norm in Ω, C is independent of $p\ge 1$, and A is the operator defined by (5.100).

We set $u=(I+\lambda A)^{-1}f$; that is,

$$\begin{cases} u-\lambda\Delta\beta(u)=f & \text{in } \Omega,\\ \beta(u)=0 & \text{on } \partial\Omega. \end{cases} \tag{5.107}$$

We recall that $\beta(u)\in W_0^{1,q}(\Omega)$, where $1<q<N/(N-2)$ (see Corollary 3.1).

Multiplying equation (5.107) by $|u|^{p-1}\operatorname{sign} u$ and integrating on Ω, we get

$$\int_\Omega |u|^p dx + \lambda p(p-1)\int_\Omega \beta'(u)|u|^{p-2}|\nabla u|^2 dx \le \int_\Omega |f|^p dx.$$

Now, using the identity

$$|u|^{p+\alpha-3}|\nabla u|^2 = \frac{4}{(p+\alpha-1)^2}\left|\nabla |u|^{(p+\alpha-1)/2}\right|^2, \quad \text{a.e. in } \Omega$$

and condition (5.103), we get

$$\int_\Omega |u|^p dx + \frac{4\lambda p(p-1)}{(p+\alpha-1)^2}\int_\Omega \left|\nabla|u|^{(p+\alpha-1)/2}\right|^2 dx \le C\int_\Omega |f|^p dx. \tag{5.108}$$

On the other hand, by the Sobolev embedding theorem

$$\int_\Omega \left|\nabla|u|^{(p+\alpha-1)/2}\right|^2 dx \le C\left(\int_\Omega |u|^{(p+\alpha-1)N/(N-2)}dx\right)^{(N-2)/N} \quad \text{if } N>2,$$

and

$$\int_\Omega \left|\nabla |u|^{(p+\alpha-1)/2}\right|^2 dx \le C\left(\int_\Omega |u|^{(p+\alpha-1)/q} dx\right)^{1/q}, \qquad \forall q>1,$$

for $N=2$. Then, substituting these inequalities into (5.108), we get (5.105) and (5.106), respectively.

We set $J_\lambda = (I+\lambda A)^{-1}$ and

$$\varphi(u) = \|u\|_p^p, \qquad \psi(u) = C\|u\|_{(p+\alpha-1)N/(N-2)}^{p+\alpha-1}.$$

Then, inequality (5.105) can be written as

$$\varphi(J_\lambda f) + \lambda\,\psi(J_\lambda f) \le \varphi(f), \qquad \forall f \in L^p(\Omega).$$

This yields

$$\varphi(J_\lambda^k f) + \lambda\,\psi(J_\lambda^k f) = \varphi(J_\lambda^{k-1}), \qquad \forall k.$$

Summing these equations from $k=1$ to $k=n$, and taking $\lambda = t/n$, yields

$$\varphi(J_{t/n}^n f) + \sum_{k=1}^n \frac{1}{n}\,\psi(J_{t/n}^k f) = \varphi(f).$$

Recalling that, by Theorem 4.3, $J_{t/n}^n f \to S(t)$ for $n\to\infty$, the latter equation implies that

$$\varphi(S(t)f) + \int_0^t \psi(S(\tau)f)d\tau = \varphi(f), \qquad \forall t \ge 0. \tag{5.109}$$

In particular, it follows that the function $t \to \varphi(S(t)f)$ is decreasing and so is $t \to \psi(S(t)f)$. Then, by (5.109), we see that $\varphi(S(t)f) + t\psi(S(t)f) \le \varphi(f)$, $\forall t>0$; that is,

$$\|S(t)f\|_p^p + Ct\|S(t)f\|_{(p+\alpha-1)N/(N-2)}^{p+\alpha-1} \le \|f\|_p^p, \qquad \forall t>0, \tag{5.110}$$

where C is independent of p and f.

Let p_n be inductively defined by

$$p_{n+1} = (p_n+\alpha-1)\,\frac{N}{N-2}.$$

Then, by (5.110), we see that

$$\|S(t_{n+1})f\|_{p_{n+1}}^{(N/(N-2))p_{n+1}} \le \frac{\|S(t_n)f\|_{p_n}^{p_n}}{C(t_{n+1}-t_n)},$$

where $t_0=0$ and $t_{n+1}>t_n$. Choosing $t_{n+1}-t_n = t/(2^{n+1})$, we get after some calculation that

$$\limsup_{n\to\infty} \|S(t)f\|_{p_{n+1}}^{((N-2)/N)np_{n+1}} \le C\|f\|_{p_0}\left(\frac{2}{t}\right)^{\mu}, \qquad \forall t>0,$$

where $\mu = N/2$, because p_n is given by

$$p_n = \left(\frac{N}{N-2}\right)^n p_0 + \frac{N_\alpha}{2(N-2)}\left(\left(\frac{N}{N-2}\right)^n - 1\right)$$

(here, we have used the fact that $\alpha > (N-2)/N$), we get the final estimate

$$\|S(t)f\|_\infty \le C\|f\|_{p_0}^{2p_0/(2p_0+N(\alpha-1))} t^{-(N/(2p_0+N(\alpha-1)))}, \qquad \forall p_0 \ge 1,$$

as claimed.

The case $N = 2$ follows similarly. Moreover, by inequality (5.105) and the exponential formula defining $S(t)$, it follows that

$$\|S(t)f\|_p \le \|f\|_p, \qquad \forall p \in L^p(\Omega), \qquad t \ge 0.$$

This completes the proof of Theorem 5.4. □

The Porous Media Equation in $\mathbf{R}^N$

Consider now equation (5.87) in $\Omega = \mathbf{R}^N$, for $N = 1,2,3$:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta\beta(y) \ni f & \text{in } \mathbf{R}^N \times (0,T), \\ y(0,x) = y_0(x), & x \in \mathbf{R}^N, \\ \beta(y(t)), y(t) \in L^1(\mathbf{R}^n), & \forall t \in [0,T]. \end{cases} \tag{5.111}$$

where $\partial/\partial t$ and Δ are taken in the sense of distributions on $(0,T)\times\mathbf{R}^N$ (see (5.88)). We may rewrite equation (5.111) in the form (5.83) on the space $X = L^1(\mathbf{R}^N)$, where

$$Ay = \{-\Delta w;\ w(x) \in \beta(y(x)), \text{ a.e. } x \in \Omega,\ w, \Delta w \in L^1(\mathbf{R}^N)\},\ \forall y \in D(A),$$
$$D(A) = \{y \in L^1(\mathbf{R}^N);\ \exists w \in L^1(\mathbf{R}^N),\ \Delta w \in L^1(\mathbf{R}^N),\ w(x) \in \beta(y(x)), \text{ a.e. } x \in \mathbf{R}^N\},$$

where Δw is taken in the sense of distributions. Here β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ such that $0 \in \beta(0)$ and $0 \in \operatorname{int} D(\beta)$ if $N = 1,2$. Then, as shown earlier in Theorem 3.7, A is m-accretive in $L^1(\mathbf{R}^N)\times\mathbf{R}^N$ and so, by Theorem 4.1, we obtain the following.

Proposition 5.12. *Assume that $f \in L^1(0,T;L^1(\mathbf{R}^N))$ and $y_0 \in L^1(\mathbf{R}^N)$ is such that $\exists w \in L^1(\mathbf{R}^N)$, $\Delta w \in L^1(\mathbf{R}^N)$, $w(x) \in \beta(y_0(x))$, a.e. $x \in \mathbf{R}^N$. Then, problem* (5.111) *has a unique mild solution $y \in C([0,T];L^1(\mathbf{R}^N))$.*

Remark 5.3. The continuity of solutions to (5.111) with respect to φ is studied in the work of Bénilan and Crandall [9]. In this context, we mention also the work of Brezis and Crandall [16] and Alikakos and Rostamian [1].

Localization of Solutions to Porous Media Equations

A nice feature of solutions to the porous media equation are finite time extinction for the fast diffusion equation (i.e., $\beta(y) = y^\alpha$, $0 < \alpha < 1$), and propagation with finite velocity for the low diffusion equation (i.e., $1 < \alpha < \infty$). We refer the reader to the work of Pazy [36] and to the recent book of Antontsev, Diaz, and Shmarev [2] for detailed treatment of this phenomena. (See also the Vasquez monograph [40] for a detailed study of the localization of solutions to a porous media equation.) Here, we briefly discuss the extinction in finite time.

Proposition 5.13. *Let $y \in C([0,\infty); L^1(\Omega) \cap H^{-1}(\Omega))$ be the solution to equation*

$$\frac{\partial y}{\partial t} - \mu\Delta(|y|^\alpha \operatorname{sign} y) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{5.112}$$

where $y_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$, $\mu > 0$, $0 < \alpha < 1$ if $N = 1,2$ and $1/5 \le \alpha < 1$ if $N = 3$. Then,

$$y(x,t) = 0 \quad \text{for } t \ge T(y_0),$$

where

$$T(y_0) = \frac{|y_0|_{-1}^{1-\alpha}}{\mu\gamma^{1+\alpha}}\,.$$

If $\alpha = 0$ and $N = 1$, then $y(x,t) = 0$ for $t \ge (|y_0|_{-1})/\mu\gamma$.

Proof. Assume first that $N > 1$. As seen earlier, the equation has a unique smooth solution $y \in W^{1,2}([0,T]; H^{-1}(\Omega))$ for each $T > 0$. Multiplying scalarly in $H^{-1}(\Omega)$ equation (5.112) by y and integrating on $(0,T)$, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,|y(t)|_{-1}^2 + \mu\int_\Omega |y(s,x)|^{\alpha+1} dx = 0, \qquad \forall t \ge 0.$$

Now, by the Sobolev embedding theorem (see Theorem 1.4), we have

$$\gamma|y(s)|_{-1} \le |y(s)|_{L^{\alpha+1}(\Omega)} \quad \text{for all } \alpha > 0 \text{ if } N = 1,2 \text{ and for } \alpha \ge \frac{N-2}{N+2} \text{ if } N \ge 3.$$

(Here, $|\cdot|_{-1}$ is the $H^{-1}(\Omega)$ norm.) This yields

$$\frac{d}{dt}\,|y(t)|_{-1}^2 + 2\mu\gamma^{\alpha+1}|y(t)|_{-1}^{\alpha+1} \le 0, \qquad \forall t \ge 0,$$

and therefore

$$\frac{d}{dt}\,|y(t)|_{-1}^{1-\alpha} + \mu\gamma^{1+\alpha} \le 0, \quad \text{a.e. } t > 0.$$

Hence,

$$|y(t)|_{-1} = 0 \quad \text{for } t \ge \frac{|y_0|_{-1}^{1-\alpha}}{\mu\gamma^{1+\alpha}}.$$

If $N = 1$, then, multiplying scalarly in $H^{-1}(\Omega)$ equation (5.112) by $y(t)$, we get

$$\frac{1}{2}\frac{d}{dt}|y(t)|_{-1}^2 + \mu|y(t)|_{L^1(\Omega)} \le 0, \quad \text{a.e. } t > 0.$$

This yields (we have $|y|_{L^1(\Omega)} \ge \gamma|y_0|_{-1}$):

$$|y(t)|_{-1} + \mu\gamma t \le |y_0|_{-1}, \qquad \forall t \ge 0$$

and, therefore,

$$|y(t)|_{-1} = 0 \quad \text{for } t \ge \frac{|y_0|_{-1}}{\mu\gamma}. \qquad \square$$

Remark 5.4. The extinction in finite time is a significant nonlinear behavior of solutions to fast diffusion porous media equations and this implies that the diffusion process reaches its critical state (which is zero in this case) in finite time. The case $\alpha = 0$ models an important class of diffusion processes with self-organized criticality, the so-called Bak's sand-pile model.

5.4 The Phase Field System

Consider the parabolic system

$$\begin{cases} \dfrac{\partial}{\partial t}\theta(t,x) + \ell\dfrac{\partial\varphi}{\partial t}(t,x) - k\Delta\theta(t,x) = f_1(t,x), & \text{in } Q = \Omega\times(0,T), \\ \dfrac{\partial}{\partial t}\varphi(t,x) - \alpha\Delta\varphi(t,x) - \kappa(\varphi(t,x) - \varphi^3(t,x)) & \\ \qquad\qquad\qquad\qquad + \delta\theta(t,x) = f_2(t,x), & \text{in } Q, \\ \theta(0,x) = \theta_0(x), \qquad \varphi(0,x) = \varphi_0(x), & x\in\Omega, \\ \theta = 0, \qquad \varphi = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{5.113}$$

where $\ell, k, \alpha, \kappa, \delta$ are positive constants. This system, called in the literature the *phase-field system*, was introduced as a model of a phase transition process in physics and, in particular, the melting and solidification phenomena. (See Caginalp [18].) In this latter case, $\theta = \theta(t,x)$ is the temperature, whereas φ is the phase-field transition function. The two-phase Stefan problem presented above can be viewed as a particular limit case of this model. In fact, it can be obtained from the two-phase Stefan model of phase transition by the following heuristic argument.

As seen earlier, the two-phase Stefan problem (5.74) and (5.75) can be rewritten as

$$\frac{\partial}{\partial t}\gamma(\theta) - \Delta K(\theta) = f \quad \text{in } \mathscr{D}'(\Omega\times(0,T)),$$

where γ is the multivalued graph (5.71); that is, $\gamma = C + \rho H$. Equivalently,

$$\frac{\partial}{\partial t}\varphi(\theta)\theta - \Delta K(\theta) = f \quad \text{in } \mathscr{D}'(\Omega\times(0,T)), \tag{5.114}$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ is given by the graph

$$\varphi(\theta) = \begin{cases} C_1 & \text{if } \theta < 0, \\ C_2 + \dfrac{\rho}{\theta} & \text{if } \theta > 0. \end{cases} \tag{5.115}$$

The idea behind Caginalp's model of phase transition is to replace the multivalued graph φ by a function $\varphi = \varphi(t,x)$, called the *phase function* and equation (5.114) by

$$\varphi \frac{\partial \theta}{\partial t} + \theta \frac{\partial \varphi}{\partial t} - \Delta K(\theta) = f. \tag{5.116}$$

The phase function φ should be interpreted as a measure of phase transition and more precisely as the proportion related to the first phase and the second one. For instance, in the case of liquid–solid transition, one has, formally, $\varphi \geq 1$ in the liquid zone $\{(t,x);\ u(t,x) > 0\}$ and $\varphi < 0$ in the solid zone $\{(t,x);\ u(t,x) < 0\}$. In general, however, φ remains in an interval $[\varphi_*, \varphi^*]$ which is determined by the specific physical model. This is the reason why φ is taken as the solution to a parabolic equation of the Ginzburg–Landau type

$$\frac{\partial \varphi}{\partial t} - \alpha \Delta \varphi - \kappa(\varphi - \varphi^3) + \delta \theta = f_2, \tag{5.117}$$

which is the basic mathematical model of phase transition. Equations (5.116) and (5.117) lead, after further simplifications, to system (5.113).

As regards the existence in problem (5.113), we have the following.

Theorem 5.5. *Assume that* $\varphi_0, \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\Omega \subset \mathbf{R}^N$, $N = 1,2,3$, *and that* $f_1, f_2 \in W^{1,2}([0,T]; L^2(\Omega))$. *Then, there is a unique solution* (θ, φ) *to system* (5.113) *satisfying*

$$(\theta, \varphi) \in (W^{1,\infty}([0,T]; L^2(\Omega)))^2 \cap (L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)))^2. \tag{5.118}$$

Proof. We set $y = \theta + \ell\varphi$ and reduce system (5.113) to

$$\begin{cases} \dfrac{\partial}{\partial t} y - k\Delta y + k\ell\Delta\varphi = f_1 & \text{in } Q, \\ \dfrac{\partial}{\partial t} \varphi - \alpha\Delta\varphi - \kappa(\varphi - \varphi^3) + \delta(y - \ell\varphi) = f_2 & \text{in } Q, \\ y(0) = y_0 = \theta_0 + \ell\varphi_0, \qquad \varphi(0) = \varphi_0 \text{ in } \Omega, \qquad y = \varphi = 0 & \text{on } \Sigma. \end{cases} \tag{5.119}$$

In the space $X = L^2(\Omega) \times L^2(\Omega)$ consider the operator $A : X \to X$,

$$A \begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} -k\Delta y + k\ell\Delta\varphi \\ -\alpha\Delta\varphi - \kappa(\varphi - \varphi^3) + \delta(y - \ell\varphi) \end{pmatrix}$$

with the domain $D(A) = \{(y,\varphi) \in (H^2(\Omega)\cap H_0^1(\Omega))^2;\ \varphi \in L^6(\Omega)\}$. Then, system (5.119) can be written as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} y \\ \varphi \end{pmatrix} + A\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, & t \in (0,T), \\ \begin{pmatrix} y \\ \varphi \end{pmatrix}(0) = \begin{pmatrix} y_0 \\ \varphi_0 \end{pmatrix}. \end{cases} \tag{5.120}$$

In order to apply Theorem 4.4 to (5.120), we check that A is quasi-m-accretive in X. To this aim we endow the space $X = L^2(\Omega)\times L^2(\Omega)$ with an equivalent Hilbertian norm provided by the scalar product

$$\left\langle \begin{pmatrix} y \\ \varphi \end{pmatrix}, \begin{pmatrix} \widetilde{y} \\ \widetilde{\varphi} \end{pmatrix} \right\rangle = a(y,\widetilde{y})_{L^2(\Omega)} + (\varphi,\widetilde{\varphi})_{L^2(\Omega)},$$

where $a = \alpha/k\ell^2$. Then, as easily seen, we have

$$\left\langle A\begin{pmatrix} y \\ \varphi \end{pmatrix} - A\begin{pmatrix} y^* \\ \varphi^* \end{pmatrix}, \begin{pmatrix} y \\ \varphi \end{pmatrix} - \begin{pmatrix} y^* \\ \varphi^* \end{pmatrix} \right\rangle$$
$$\geq \eta(\|\nabla(y-y^*)\|^2_{L^2(\Omega)} + \|\nabla(\varphi-\varphi^*)\|^2_{L^2(\Omega)}) - \omega(\|y-y^*\|^2_{L^2(\Omega)} + \|\varphi-\varphi^*\|^2_{L^2(\Omega)}),$$

for some $\omega,\eta > 0$. Clearly, this implies that A is quasi-accretive; that is, $A+\omega I$ is accretive.

Now, consider for $g_1, g_2 \in L^2(\Omega)$ the equation

$$\lambda\begin{pmatrix} y \\ \varphi \end{pmatrix} + A\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}; \tag{5.121}$$

that is,

$$\begin{cases} \lambda y - k\Delta y + k\ell\Delta\varphi = g_1 & \text{in } \Omega, \\ \lambda\varphi - \alpha\Delta\varphi - \kappa(\varphi-\varphi^3) + \delta(y-\ell\varphi) = g_2, & \\ y = \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.122}$$

System (5.122) can be equivalently rewritten as

$$\begin{pmatrix} \lambda y \\ (\lambda-\kappa-\ell\delta)\varphi + \delta y \end{pmatrix} + A_0\begin{pmatrix} y \\ \varphi \end{pmatrix} + F\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{5.123}$$

where $F, A_0 : L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)\times L^2(\Omega)$ are given by

$$A_0 \begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} -k\Delta y + k\ell\Delta\varphi \\ -\alpha\Delta\varphi \end{pmatrix}$$

$$D(A_0) = (H^2(\Omega) \times H_0^1(\Omega))^2$$

and

$$F \begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \kappa\varphi^3 \end{pmatrix}$$

$$D(F) = L^2(\Omega) \times L^6(\Omega).$$

By the Lax–Milgram lemma (Lemma 1.3), it is easily seen that A_0 is m-accretive and coercive in $X = L^2(\Omega) \times L^2(\Omega)$. On the other hand, F is quasi-m-accretive and

$$\left\langle A_0 \begin{pmatrix} y \\ \varphi \end{pmatrix}, F \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\rangle \geq 0, \qquad \forall \begin{pmatrix} y \\ \varphi \end{pmatrix} \in D(A_0).$$

Hence, by Proposition 3.8, $A_0 + F$ is quasi-m-accretive and this implies that (5.123) has a solution for λ sufficiently large. □

Remark 5.5. The liquid and solid regions in the case of a melting solidification problem are those that remain invariant by the flow $t \to (\theta(t), \varphi(t))$. This is one way of determining in specific physical models the range interval $[\varphi_*, \varphi^*]$ of phase-field function φ. A more general nonlinear phase-field model is proposed and studied by Bonetti, Colli, Fabrizio, and Gilardi [12] in connection with a phase transition model proposed by Fremond [26]. More precisely, under our notation this system is of the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial t}(G(\varphi)) - \lambda\Delta \log u = f, \\ \mu\,\dfrac{\partial \varphi}{\partial t} - \nu\Delta\varphi + F'(\varphi) + uG'(\varphi) = 0, \end{cases}$$

and the above functional treatment applies as well to this general problem.

5.5 The Equation of Conservation Laws

We consider here the Cauchy problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(y) = 0 & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ y(x,0) = y_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{5.124}$$

where $a = (a_1, ..., a_N)$ is a continuous map from $\mathbf{R}$ to $\mathbf{R}^N$ satisfying the condition

$$\limsup_{|r|\to 0} \frac{\|a(r)\|}{|r|} < \infty,$$

and $y_0 \in L^1(\mathbf{R}^N)$.

This equation can be treated as a nonlinear Cauchy problem in the space $X = L^1(\mathbf{R}^N)$. In fact, we have seen earlier (Theorem 3.8) that the first-order differential operator $y \to \sum_{i=1}^N (\partial/\partial x_i)\, a_i(y)$ admits an m-accretive extension $A \subset L^1(\mathbf{R}^N) \times L^1(\mathbf{R}^N)$ defined as the closure in $L^1(\mathbf{R}^N) \times L^1(\mathbf{R}^N)$ of the operator A_0 given by Definition 3.2.

Then, by Theorem, 4.3, the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt} + Ay \ni 0 \quad \text{in } (0,+\infty), \\ y(0) = y_0, \end{cases}$$

has for every $y_0 \in \overline{D(A)}$ a unique mild solution $y(t) = S(t)y_0$ given by the exponential formula (4.17) or, equivalently,

$$y(t) = \lim_{\varepsilon\to 0} y_\varepsilon(t) \quad \text{uniformly on compact intervals,}$$

where y_ε is the solution to difference equation

$$\begin{aligned} \varepsilon^{-1}(y_\varepsilon(t) - y_\varepsilon(t-\varepsilon)) + Ay_\varepsilon(t) &= 0 \qquad \text{for } t > \varepsilon, \\ y_\varepsilon(t) &= y_0 \qquad \text{for } t < 0. \end{aligned} \tag{5.125}$$

We call such a function $y(t) = S(t)y_0$ a *semigroup solution* or *mild solution* to the Cauchy problem (5.124).

We see in Theorem 5.6 below that this solution is in fact an entropy solution to the equation of conservation laws.

Theorem 5.6. *Let $y = S(t)y_0$ be the semigroup solution to problem* (5.124). *Then,*

(i) $S(t)L^p(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ *for all* $1 \le p < \infty$ *and*

$$\|S(t)y_0\|_{L^p(\mathbf{R}^N)} \le \|y_0\|_{L^p(\mathbf{R}^N)}, \qquad \forall y_0 \in \overline{D(A)} \cap L^p(\mathbf{R}^N). \tag{5.126}$$

(ii) *If* $y_0 \in \overline{D(A)} \cap L^\infty(\mathbf{R}^N)$, *then*

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^N} \big(|y(x,t)-k|\varphi_t(x,t) \\ &+ \operatorname{sign}_0(y(x,t)-k)(a(y(x,t)) - a(k)) \cdot \varphi_x(x,t)\big)\, dx\, dt \ge 0 \end{aligned} \tag{5.127}$$

for every $\varphi \in C_0^\infty(\mathbf{R}^N \times (0,T))$ *such that* $\varphi \ge 0$, *and all* $k \in \mathbf{R}^N$ *and* $T > 0$.

Here $\varphi_t = \partial\varphi/\partial t$ and $\varphi_x = \nabla_x \varphi$.

Inequality (5.127) is Kruzkhov's [30] definition of entropy solution to the Cauchy problem (5.124) and its exact significance is discussed below.

Proof of Theorem 5.6. Because, as seen in the proof of Theorem 3.8, $(I+\lambda A)^{-1}$ maps $L^p(\mathbf{R}^N)$ into itself and

$$\|(I+\lambda A)^{-1}u\|_{L^p(\mathbf{R}^N)} \le \|u\|_{L^p(\mathbf{R}^N)}, \qquad \forall \lambda > 0,\ u \in L^p(\mathbf{R}^N) \text{ for } 1 \le p \le \infty,$$

we deduce (i) by the exponential formula (4.17).

To prove inequality (5.126), consider the solution y to equation (5.125), where $y_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $A_0 = A$. (Recall that $L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \subset R(I+\lambda A)^{-1}$ for all $\lambda > 0$.) Then, $\|y_\varepsilon(t)\|_{L^p(\mathbf{R}^N)} \le \|y_0\|_{L^p(\mathbf{R}^N)}$ for $p = 1,\infty$ and so, by Definition 3.2 and by (5.125), we have

$$\begin{aligned}&\int_{\mathbf{R}^N} (\mathrm{sign}_0(y_\varepsilon(x,t)-k)(a(y_\varepsilon(x,t)) - a(k))) \cdot \varphi_x(x,t)\\ &+ \varepsilon(y_\varepsilon(x,t-\varepsilon) - y_\varepsilon(x,t))\,\mathrm{sign}_0(y_\varepsilon(x,t)-k)\varphi(x,t))dx \ge 0,\\ &\qquad \forall k \in \mathbf{R},\ \varphi \in C_0^\infty(\mathbf{R}^N \times (0,T)),\ \varphi \ge 0,\ t \in (0,T).\end{aligned} \tag{5.128}$$

On the other hand, we have

$$\begin{aligned}&(y_\varepsilon(x,t-\varepsilon) - y_\varepsilon(x,t))\mathrm{sign}_0(y_\varepsilon(x,t)-k)\\ &= (y_\varepsilon(x,t-\varepsilon)-k)\mathrm{sign}_0(y_\varepsilon(x,t)-k) - (y_\varepsilon(x,t)-k)\mathrm{sign}_0(y_\varepsilon(x,t)-k)\\ &\le z_\varepsilon(x,t-\varepsilon) - z_\varepsilon(x,t),\end{aligned}$$

where $z_\varepsilon(x,t) = |y_\varepsilon(x,t)-k|$.

Substituting the latter into (5.128) and integrating on $\mathbf{R}^N \times [0,T]$, we get

$$\begin{aligned}&\int_0^T \int_{\mathbf{R}^N} (\mathrm{sign}_0(y_\varepsilon(x,t)-k)(a(y_\varepsilon(x,t)) - a(k)) \cdot \varphi_x(x,t)\\ &+ \varepsilon^{-1}(z_\varepsilon(x,t-\varepsilon) - z_\varepsilon(x,t))\varphi(x,t))dxdt \ge 0.\end{aligned}$$

This yields

$$\begin{aligned}&\int_0^T \int_{\mathbf{R}^N} (\mathrm{sgn}_0(y_\varepsilon(x,t)-k)(a(y_e(x,t)) - a(k)) \cdot \varphi_x(x,t))dxdt\\ &-\varepsilon^{-1}\int_0^\varepsilon \int_{\mathbf{R}^N} |y_\varepsilon(x,t)-k|\varphi(x,t)dxdt + \varepsilon^{-1}\int_0^T \int_{\mathbf{R}^N} z_\varepsilon(x,t)\varphi(x,t)dxdt\\ &+\varepsilon^{-1}\int_{T-\varepsilon}^T \int_{\mathbf{R}^N} z_\varepsilon(x,t)(\varphi(x,t+\varepsilon) - \varphi(x,t))dxdt \ge 0.\end{aligned}$$

Now, letting ε tend to zero, we get (5.127) because $y_\varepsilon(t) \to y(t)$ uniformly on $[0,T]$ in $L^1(\mathbf{R}^N)$ and $\varepsilon^{-1}(z_\varepsilon(x,t-\varepsilon) - z_\varepsilon(x,t)) \to |y(x,t)-k|$. This completes the proof of Theorem 5.5.

As mentioned earlier, equation (5.124) is known in the literature as the *equation of conservation laws* and has a large spectrum of applications in mechanics and was extensively studied in recent years. A function $\eta : \mathbf{R} \to \mathbf{R}$ is called an *entropy* of system (5.124) if there is a function $q : \mathbf{R} \to \mathbf{R}^n$ (the *entropy flux* associated with entropy η) such that $\nabla^2 q \geq 0$ and

$$\nabla q_j(y) = \nabla \eta(y) \cdot \nabla a_j(y), \qquad \forall y \in \mathbf{R}^N,\ j = 1, ..., N.$$

(Such a pair (η, q) is called an *entropy pair*.)

The bounded measurable function $y : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ is called an *entropy solution* to (5.124) if, for all convex entropy pairs (η, q),

$$\frac{\partial}{\partial t}\,\eta(y(t,x)) + \mathrm{div}_x\, q(y(t,x)) \leq 0 \quad \text{in } \mathscr{D}'(\mathbf{R}^N \times (0,T));$$

that is,

$$\int_0^T \int_{\mathbf{R}^N} (\eta(y(t,x))\varphi_t(t,x) + q(y(t,x)) \cdot \varphi_x(t,x))dtdx \geq 0$$

for all $\varphi \in C_0^\infty((0,T) \times \mathbf{R}^N)$, $\varphi \geq 0$.

If take $\eta(y) \equiv |y-k|$ and $q(y) \equiv \mathrm{sign}_0(y-k)(a(y)-a(k))$, we see that y satisfies equation (5.127). The existence and uniqueness of the entropy solution were proven by S. Kruzkhov [30]. (See also Bénilan and Kruzkhov [11] for some recent results.) Recalling that the resolvent $(I + \lambda A)^{-1}$ of the operator A can be approximated by the family of approximating equation (3.74), one might deduce via the Trotter–Kato Theorem 4.14 that the entropy solution y can also be obtained as the limit for $\varepsilon \to 0$ to solutions y_ε to the parabolic nonlinear equation

$$\frac{\partial y}{\partial t} - \varepsilon \Delta y + (a(y))_x = 0,$$

in $\mathbf{R}^N$ which is related to Hopf's viscosity solution approach to nonlinear conservation laws equations.

5.6 Semilinear Wave Equations

The linear wave equation perturbed by a nonlinear term in speed can be conveniently written as a first order differential equation in an appropriate Hilbert space defined below and treated so by the general existence theory developed in Chapter 4.

We are given two real Hilbert spaces V and H such that $V \subset H \subset V'$ and the inclusion mapping of V into H is continuous and densely defined. We have denoted by V' the dual of V and H is identified with its own dual. As usual, we denote by $\|\cdot\|$ and $|\cdot|$ the norms of V and H, respectively, and by $(\cdot,\cdot)$ the duality pairing between V and V' and the scalar product of H.

We consider the second-order Cauchy problem

$$\frac{d^2y}{dt^2} + Ay + B\left(\frac{dy}{dt}\right) \ni f, \qquad y(0) = y_0, \qquad \frac{dy}{dt}(0) = y_1, \tag{5.129}$$

where A is a linear continuous and symmetric operator from V to V' and $B \subset V \times V'$ is maximal monotone operator. We assume further that

$$(Ay, y) + \alpha|y|^2 \geq \omega\|y\|^2, \qquad \forall y \in V, \tag{5.130}$$

where $\omega > 0$ and $\alpha \in \mathbf{R}$.

One principal motivation and model for equation (5.129) is the nonlinear hyperbolic boundary value problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + \beta\left(\dfrac{\partial y}{\partial t}\right) \ni f(x,t) & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), \qquad \dfrac{dy}{dt}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{5.131}$$

where β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ and Ω is a bounded open subset of $\mathbf{R}^N$ with a smooth boundary.

As regards problem (5.129), we have the following existence result.

Theorem 5.7. *Let $f \in W^{1,1}([0,T];H)$ and $y_0 \in V$, $y_1 \in D(B)$ be given such that*

$$\{Ay_0 + By_1\} \cap H \neq \emptyset. \tag{5.132}$$

Then, there is a unique function $y \in W^{1,\infty}([0,T];V) \cap W^{2,\infty}([0,T];H)$ that satisfies

$$\begin{cases} \dfrac{d^+}{dt}\left(\dfrac{dy}{dt}\right)(t) + Ay(t) + B\left(\dfrac{d^+}{dt}\, y(t)\right) \ni f(t), \qquad \forall t \in [0,T], \\ y(0) = y_0, \qquad \dfrac{dy}{dt}(0) = y_1, \end{cases} \tag{5.133}$$

where $d^+/dt(dy/dt)$ is considered in the topology of H and $(d^+/dt)y$ in V.

Proof. Let $X = V \times H$ be the Hilbert space with the scalar product

$$\langle U_1, U_2 \rangle = (Au_1, u_2) + \alpha(u_1, u_2) + (v_1, v_2),$$

where $U_1 = [u_1, v_1]$, $U_2 = [u_2, v_2]$.

In the space X, define the operator $\mathscr{A} : D(\mathscr{A}) \subset X \to X$ by

$$\begin{cases} D(\mathscr{A}) = \{[u,v] \in V \times H; \{Au + Bv\} \cap H \neq \emptyset\}, \\ \mathscr{A}[u,v] = [-v; \{Au + Bv\} \cap H] + \sigma[u,v], \;\; [u,v] \in D(\mathscr{A}), \end{cases} \tag{5.134}$$

where

$$\sigma = \sup\left\{ \frac{\alpha(u,v)}{((Au,u)+\alpha|u|^2+|v|^2)};\ u \in V,\ v \in H \right\}.$$

We may write equation (5.129) as a first-order differential system

$$\begin{cases} \dfrac{dy}{dt} - z = 0 \quad \text{in } (0,T), \\ \\ \dfrac{dy}{dt} + Ay + Bz \ni f. \end{cases}$$

Equivalently,

$$\begin{cases} \dfrac{dt}{dt} U(t) + \mathscr{A}U(t) - \sigma U(t) \ni F(t), \qquad t \in (0,T), \\ \\ U(0) = U_0, \end{cases} \tag{5.135}$$

where

$$U(t) = [y(t), z(t)], \qquad F(t) = [0, f(t)], \qquad U_0 = [y_0, y_1].$$

It is easily seen that $\mathscr{A}$ is monotone in $X \times X$. Let us show that it is maximal monotone; that is, $R(I + \mathscr{A}) = V \times H$, where I is the unity operator in $V \times H$. To this end, let $[g,h] \in V \times H$ be arbitrarily given. Then, the equation $U + \mathscr{A}U \ni [g,h]$ can be written as

$$\begin{cases} y - z + \sigma y = g, \\ \\ z + Ay + Bz + \sigma z \ni h. \end{cases}$$

Substituting $y = (1+\sigma)^{-1}(z+g)$ in the second equation, we obtain

$$(1+\sigma)z + (1+\sigma)^{-1}Az + Bz \ni h - (1+\sigma)^{-1}Ag.$$

Under our assumptions, the operator $z \xrightarrow{\Gamma} (1+\sigma)z + (1-\sigma)^{-1}Az$ is continuous, positive, and coercive from V to V'. Then, $R(\Gamma + B) = V'$ (see Corollary 2.6, and so the previous equation has a solution $z \in D(B)$ and a fortiori $[g,h] \in R(I + \mathscr{A})$.

Then, the conclusions of Theorem 5.7 follow by Theorem 4.6 because there is a unique solution $U \in W^{1,\infty}([0,T]; V \times H)$ to problem (5.135) satisfying

$$\frac{d^+}{dt} U(t) + \mathscr{A}U(t) - \sigma U(t) \ni F(t), \qquad \forall t \in [0,T) :$$

$$\begin{cases} \dfrac{d^+}{dt} y(t) = z(t), & \forall t \in [0,T), \\ \dfrac{d^+}{dt} z(t) + Ay(t) + B(z(t)) \ni f(t), & \forall t \in [0,T), \end{cases}$$

where $(d^+/dt)y$ is in the topology of V whereas $(d^+/dt)z$ is in the topology of H. □

The operator B that arises in equation (5.129) might be multivalued. Moreover, if $B = \partial\varphi$, where $\varphi : V \to \overline{\mathbf{R}}$ is a lower semicontinuous convex function, problem (5.129) reduces to a variational inequality of hyperbolic type.

In order to apply Theorem 5.7 to the hyperbolic problem (5.131), we take $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$, $A = -\Delta$, and $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $B = \partial\varphi$, where $\varphi : H_0^1(\Omega) \to \overline{\mathbf{R}}$ is the function

$$\varphi(y) = \int_\Omega j(y(x))dx, \qquad \forall y \in H_0^1(\Omega),\ \beta = \partial j. \tag{5.136}$$

The operator B is an extension of the operator $(B_0 y)(x) = \{w \in L^2(\Omega);\ w(x) \in \beta(y(x))$, a.e. $x \in \Omega\}$, from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. It should be said that, in general, the operator B does not coincide with B_0. The simplest example is $j(r) = 0$ if $0 \le r \le 1$, $j(r) = +\infty$ otherwise. In this case, $\partial\varphi = \partial I_K$, where $K = \{y \in H_0^1(\Omega); 0 \le y(x) \le 1$, a.e., $x \in \Omega\}$. Then $\mu \in \partial\varphi(y)$ satisfies $\mu(y-z) \ge 0$, $\forall z \in K$ and, therefore, $\mu(\varphi) = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Hence, μ is a measure with support on $\partial\Omega$. More generally (see Brezis [13]), if φ is defined by (5.136), then $\mu \in \partial\varphi(y) \in H^{-1}(\Omega)$, and then μ is a bounded measure on Ω and $\mu = \mu_a dx + \mu_s$ where the absolutely continuous part $\mu_a \in L^1(\Omega)$ has the property that $\mu_a(x) \in \beta(y(x))$, a.e. $x \in \Omega$. However, if $D(\beta) = \mathbf{R}$, then, by Lemma 2.2, if $\mu \in H^{-1}(\Omega) \cap L^1(\Omega)$ is such that $\mu(x) \in \beta(y(x))$, a.e. $x \in \Omega$, then $\mu \in By$.

Then, by Theorem 5.7, we get the following.

Corollary 5.5. *Let β be a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ and let $B = \partial\varphi$, where φ is defined by* (5.136). *Let $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $y_1 \in H_0^1(\Omega)$, and $f \in L^2(Q)$ be such that $\partial f/\partial t \in L^2(Q)$ and*

$$\mu_0(x) \in \beta(y_1(x)), \quad \textit{a.e. } x \in \Omega \quad \textit{for some } \mu_0 \in L^2(\Omega). \tag{5.137}$$

Then, there is a unique function $y \in C([0,T]; H_0^1(\Omega))$ such that

$$\frac{\partial y}{\partial t} \in C([0,T];L^2(\Omega)) \cap C([0,T];H_0^1(\Omega)), \qquad \frac{\partial^2 y}{dt^2} \in L^\infty(0,T;L^2(\Omega)) \tag{5.138}$$

$$\begin{cases} \dfrac{d^+}{dt}\dfrac{\partial y}{\partial t}(t) - \Delta y(t) + B\left(\dfrac{\partial}{\partial t}y(t)\right) \ni f(t), & \forall t \in [0,T), \\ y(x,0) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(x,0) = y_1(x), & \textit{in } \Omega, \\ y = 0, & \textit{on } \partial\Omega \times (0,T). \end{cases} \tag{5.139}$$

Assume further that $D(\beta) = \mathbf{R}$. Then, $\Delta y(t) \in L^1(\Omega)$ for all $t \in [0,T)$ and

$$\frac{d^+}{dt}\frac{dy}{dt}(x,t) - \Delta y(x,t) + \mu(x,t) = f(x,t), \qquad x \in \Omega,\ t \in [0,T), \tag{5.140}$$

where $\mu(x,t) \in \beta((\partial y/\partial t)(x,t))$, a.e. $x \in \Omega$.

(We note that condition (5.139) implies (5.132).)

Problems of the form (5.131) arise in wave propagation and description of the dynamics of an elastic solid. For instance, if $\beta(r) = r|r|$, this equation models the behavior of an elastic membrane with the resistance proportional to the velocity.

If $j(r) = |r|$, then $\beta(r) = \text{sign}\, r$ and so equation (5.139) is of multivalued type.

As another example, consider the unilateral hyperbolic problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \Delta y + f & \text{in } \left\{(x,t) \in Q;\ \dfrac{\partial y}{\partial t}(x,t) > \psi(x)\right\}, \\ \dfrac{\partial^2 y}{\partial t^2} \geq \Delta y + f,\ \dfrac{\partial}{\partial t} y \geq \psi & \text{in } Q, \\ y = 0 & \text{on } \partial\Omega \times [0,T), \\ y(x,0) = y_0(x),\ \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{5.141}$$

where $\psi \in H^2(\Omega)$ is such that $\psi \leq 0$, a.e. on $\partial\Omega$. This is a reflection-type problem for the linear wave equation with constraints on velocity that exhibits a free boundary type behavior with moving boundary.

Clearly, we may write this variational inequality in the form (5.129), where $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $A = -\Delta$, and $B \subset H_0^1(\Omega) \times H^{-1}(\Omega)$ is defined by

$$Bu = \{w \in H^{-1}(\Omega);\ (w, u - v) \geq 0,\ \forall v \in K\}$$

for all $u \in D(B) = K = \{u \in H_0^1(\Omega);\ u \geq \psi,\ \text{a.e. in } \Omega\}$.

By Theorem 5.7, we have therefore the following existence result for problem (5.141).

Corollary 5.6. *Let $f, f_t \in L^2(Q)$ and $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $y_1 \in H_0^1(\Omega)$ be such that $y_1(x) \geq \psi(x)$, a.e. $x \in \Omega$. Then, there is a unique function $y \in W^{1,\infty}([0,T]; H_0^1(\Omega))$ with $\partial y/\partial t \in W^{1,\infty}([0,T]; L^2(\Omega))$ satisfying*

$$\begin{cases} \displaystyle\int_\Omega \left(\frac{d^+}{dt}\frac{\partial y}{\partial t}(x,t)\left(\frac{\partial y}{\partial t}(x,t) - u(x)\right) + \nabla y(x,t) \cdot \nabla\left(\frac{\partial y}{\partial t}(x,t) - u(x)\right)\right) dx \\ \qquad \leq \displaystyle\int_\Omega f(x,t)\left(\frac{\partial y}{\partial t}(x,t) - u(x)\right) dx, \qquad \forall u \in K,\ \forall t \in [0,T), \\ y(x,0) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(x,0) = y_1(x), \qquad \forall x \in \Omega. \end{cases} \tag{5.142}$$

Problem (5.142) is a variational (or weak) formulation of the free boundary problem (5.141).

The Klein–Gordon Equation

We consider now the hyperbolic boundary value problem

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y + g(y) = f & \text{in } \Omega \times (0,T) = Q, \\ y(x,0) = y_0(x), \qquad \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \end{cases} \tag{5.143}$$

where Ω is a bounded and open subset of $\mathbf{R}^N$, with a sufficiently smooth boundary (of class C^2, for instance), and $g \in W^{1,\infty}(\mathbf{R})$ satisfies the following conditions.

(i) $|g'(r)| \le L(1+|r|^p)$, a.e. $r \in \mathbf{R}$, where $0 \le p \le 2/(N-2)$ if $N > 2$, and p is any positive number if $1 \le N \le 2$;
(ii) $rg(r) \ge 0$, $\forall r \in \mathbf{R}$.

In the special case where $g(y) = \mu |y|^\rho y$, assumptions (i) and (ii) are satisfied for $0 < \rho \le 2/(N-2)$ if $N > 2$, and for $\rho \ge 0$ if $N \le 2$. For $\rho = 2$, this is the classical Klein–Gordon equation, arising in the quantum field theory (see Reed and Simon [37]).

In the sequel, we denote by ψ the primitive of g, which vanishes at 0: $\psi(r) = \int_0^r g(t)dt$, $\forall r \in \mathbf{R}$.

Theorem 5.8. *Let* $f, (\partial f/\partial t) \in L^2(Q)$ *and* $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $y_1 \in H_0^1(\Omega)$ *be such that* $\psi(y_0) \in L^1(\Omega)$. *Then, under assumptions* (i) and (ii) *there is a unique function y that satisfies*

$$\begin{cases} y \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T];H_0^1(\Omega)), \\ \frac{\partial y}{\partial t} \in C([0,T];H_0^1(\Omega)), \qquad \frac{\partial^2 y}{\partial t^2} \in L^\infty(0,T;L^2(\Omega)), \\ \psi(y) \in L^\infty(0,T;L^1(\Omega)), \end{cases} \tag{5.144}$$

and

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y + g(y) = f, & \text{a.e. in } Q, \\ y(x,0) = y_0(x), \qquad \frac{\partial y}{\partial t}(x,0) = y_1(x), & \text{a.e. } x \in \Omega. \end{cases} \tag{5.145}$$

Proof. As in the previous case, we write equation (5.143) as a first-order differential equation in $X = H_0^1(\Omega) \times L^2(\Omega)$; that is,

$$\frac{\partial y}{\partial t} - z = 0, \qquad \frac{dz}{dt} - \Delta y + g(y) = f \quad \text{in } [0,T]. \tag{5.146}$$

Equivalently,

$$\begin{cases} \frac{d}{dt}U(t) + A_0 U(t) + GU(t) = F(t), & t \in [0,T], \\ U(0) = [y_0, y_1], \end{cases} \tag{5.147}$$

where $U(t) = [y(t), z(t)]$, $G(U) = [0, g(y)]$, $A_0 U = [-z, -\Delta y]$, and $F(t) = [0, f(t)]$.

The space $X = H_0^1(\Omega) \times L^2(\Omega)$ is endowed with the usual norm:

$$\|U\|_X^2 = \|y\|_{H_0^1(\Omega)}^2 + \|z\|_{L^2(\Omega)}^2, \qquad U = [y, z].$$

It should be said that although the operator $A_0 + G$ is not quasi-m-accretive in the space X, the Cauchy problem (5.147) can be treated with the previous method.

We note first that the operator G is locally Lipschitz on X. Indeed, we have

$$\|G(y_1, z_1) - G(y_2, z_2)\|_X = \|g(y_1) - g(y_2)\|_{L^2(\Omega)}.$$

On the other hand, we have

$$\begin{aligned}|g(y_1) - g(y_2)| &\le \left|\int_0^1 g'(\lambda y_1 + (1-\lambda)y_2)d\lambda(y_1 - y_2)\right| \\ &\le L|y_1 - y_2|\int_0^1 (1 + |\lambda(y_1 - y_2) + y_2|^p)d\lambda \\ &\le C|y_1 - y_2|(\max(|y_1|^p, |y_2|^p) + 1), \qquad \forall y_1, y_2 \in \mathbf{R}.\end{aligned}$$

Hence, for any $z \in L^2(\Omega)$ and $y_i \in H_0^1(\Omega)$, $i = 1, 2$, we have

$$\begin{aligned}&\int_\Omega z(x)(g(y_1(x)) - g(y_2(x)))dx \\ &\le C\int_\Omega |z(x)|\,|y_1(x) - y_2(x)|(\max(|y_1(x)|^p, |y_2(x)|^p) + 1)dx\end{aligned}$$

and, therefore, by the Hölder inequality,

$$\begin{aligned}\int_\Omega z(g(y_1) - g(y_2))dx \le C\|z\|_{L^2(\Omega)}\|y_1 - y_2\|_{L^\beta(\Omega)} \max(\|y_1\|_{L^{2p}(\Omega)}^p, \|y_2\|_{L^{2p}(\Omega)}^p) \\ + C\|z\|_{L^2(\Omega)}\|y_1 - y_2\|_{L^2(\Omega)},\end{aligned}$$

where

$$\frac{1}{\beta} + \frac{1}{\delta} + \frac{1}{2} = 1.$$

Now, we take in the latter inequality $\delta = N$ and $\beta = 2N/(N-2)$. We get

$$\begin{aligned}&\|g(y_1) - g(y_2)\|_2 \\ &\le C\|y_1 - y_2\|_{2N/(N-2)} \max(\|y_1\|_{Np}^p, \|y_2\|_{Np}^p) + C\|y_1 - y_2\|_2, \qquad \forall y_1, y_2 \in H_0^1(\Omega).\end{aligned}$$

Then, by the Sobolev embedding theorem and assumption (i), we have

$$\begin{aligned}\|y_i\|_{Np} &\le C_i\|y_i\|_{H_0^1(\Omega)}, \qquad i = 1, 2, \\ \|y_1 - y_2\|_{2N/(N-2)} &\le C_0\|y_1 - y_2\|_{H_0^1(\Omega)}.\end{aligned}$$

(We have denoted by $\|\cdot\|_p$ the L^p norm.) This yields

$$\|g(y_1)-g(y_2)\|_2 \leq C\|y_1-y_2\|_{H_0^1(\Omega)}(\max(\|y_1\|^p_{H_0^1(\Omega)}, \|y_2\|^p_{H_0^1(\Omega)})+1)$$

and, therefore,

$$\begin{aligned}&\|G(y_1,z_1)-G(y_2,z_2)\|_X\\ &\leq C\|y_1-y_2\|_{H_0^1(\Omega)}(1+\max(\|y_1\|^p_{H_0^1(\Omega)}, \|y_2\|^p_{H_0^1(\Omega)})),\\ &\qquad\qquad\qquad\qquad \forall y_1,y_2 \in H_0^1(\Omega),\end{aligned} \tag{5.148}$$

as claimed. □

To prove the existence of a local solution, we use the truncation method presented in Section 4.1 (see Theorem 4.8).

Let $r>0$ be arbitrary but fixed. Define the operator $\widetilde{G}: X \to X$,

$$\widetilde{G}(y,z) = \begin{cases} G(y,z) & \text{if } \|y\|_{H_0^1(\Omega)} \leq r,\\ G\left(r\dfrac{y}{\|y\|_{H_0^1(\Omega)}}, z\right) & \text{if } \|y\|_{H_0^1(\Omega)} > r.\end{cases}$$

By (5.148), we see that the operator $\widetilde{G}$ is Lipschitz on X. Hence, A_0+G is ω-m-accretive on X and, by Theorem 4.6, we conclude that the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U(t)+A_0U(t)+\widetilde{G}U(t)=F(t), & \text{a.e. } t\in(0,T),\\ U(0)=[y_0,y_1], \end{cases} \tag{5.149}$$

has a unique solution $U \in W^{1,\infty}([0,T];X)$. This implies that there is a unique $y \in W^{1,\infty}([0,T];H_0^1(\Omega))$ with $dy/dt \in W^{1,\infty}([0,T];L^2(\Omega))$ such that

$$\begin{cases} \dfrac{d^2y}{dt^2}(t)-\Delta y(t)+\widetilde{g}(y(t))=f(t), & \text{a.e. } t\in(0,T),\\ y(0)=y_0,\ \dfrac{dy}{dt}(0)=y_1 & \text{in } \Omega, \end{cases} \tag{5.150}$$

where $\widetilde{g}: H_0^1(\Omega) \to L^2(\Omega)$ is defined by

$$\widetilde{g}(y) = \begin{cases} g(y) & \text{if } \|y\|_{H_0^1(\Omega)} \leq r,\\ g\left(r\dfrac{y}{\|y\|_{H_0^1(\Omega)}}\right) & \text{if } \|y\|_{H_0^1(\Omega)} > r.\end{cases}$$

Choose r sufficiently large such that $\|y_0\|_{H_0^1(\Omega)} < r$. Then, there is an interval $[0,T_r]$ such that $\|y(t)\|_{H_0^1(\Omega)} \leq r$ for $t \in [0,T_r]$ and $\|y(t)\|_{H_0^1(\Omega)} > r$ for $t > T_r$. We have therefore

$$\frac{\partial^2 y}{\partial t^2} - \Delta y + g(y) = f \quad \text{in } \Omega\times(0,T_r),$$

and multiplying this by y_t and integrating on $\Omega \times (0,t)$, we get the energy equality

$$\|y_t(t)\|_2^2 + \|y(t)\|^2_{H_0^1(\Omega)} + 2\int_\Omega \psi(y(x,t))dx$$
$$= \|y_1\|_2^2 + \|y_0\|^2_{H_0^1(\Omega)} + 2\int_\Omega \psi(y_0(x))dx + 2\int_0^t \int_\Omega f y_s \, dx\, ds.$$

Because $\psi(y) \geq 0$ and $\psi(y_0) \in L^1(\Omega)$, by Gronwall's lemma we see that

$$\|y_t(t)\|_2 \leq (\|y_1\|_2^2 + \|y_0\|^2_{H_0^1(\Omega)} + 2\|\psi(y_0)\|_{L^1(\Omega)})^{1/2} + \int_0^{T_r} \|f(s)\|_2 ds$$

and, therefore,

$$\|y_t(t)\|_2^2 + \|y(t)\|^2_{H_0^1(\Omega)} + 2\int_\Omega \psi(y(x,t))dx$$
$$\leq \|y_1\|_2^2 + \|y_0\|^2_{H_0^1(\Omega)} + 2\int_\Omega \psi(y_0)dx + \left(\int_0^t \|f(s)\|_2^2 ds\right)^{1/2}$$
$$\times \left((\|y_1\|_2^2 + \|y_0\|^2_{H_0^1(\Omega)} + 2\|\psi(y_0)\|_{L^1(\Omega)})^{1/2} + \int_0^{T_r} \|f(s)\|_2 ds \right).$$

The latter estimate shows that, given $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $T > 0$, and $f \in L^2(Q_T)$, there is a sufficiently large r such that $\|y(t)\|_{H_0^1(\Omega)} \leq r$ for $t \in [0,T]$. We may infer, therefore, that for r large enough the function y found as the solution to (5.150) is, in fact, a solution to equation (5.145) satisfying all the conditions of Theorem 5.8.

The uniqueness of y satisfying (5.144) and (5.145) is the consequence of the fact that such a function is the solution (along with $z = \partial y/\partial t$) to the ω-accretive differential equation (5.149).

By the previous proof, it follows that, if one merely assumes that

$$y_0 \in H_0^1(\Omega), \qquad y_1 \in L^2(\Omega), \qquad \psi(y_0) \in L^1(\Omega),$$

then there is a unique function $y \in C([0,T]; H_0^1(\Omega))$, $\partial y/\partial t \in C([0,T]; L^2(\Omega))$, that satisfies equation (5.143) in a mild sense. However, if $\psi(y_0) \notin L^1(\Omega)$ or, if one drops assumption (ii), then the solution to (5.143) exists locally in time, only; that is, in a neighborhood of the origin.

Under appropriate assumptions on g and β, the above existence results extend to equations of the form

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y + \beta\left(\dfrac{\partial y}{\partial t}\right) + g(y) = f & \text{in } Q, \\ y(x,0) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

(See Haraux [28].) In Barbu, Lasiecka and Rammaha [5], the local and global existence of generalized solutions is studied in the case of more general equations of the form

$$\frac{\partial^2 y}{\partial t^2} - \Delta y + |y|^k \beta\left(\frac{\partial y}{\partial t}\right) = |y|^{p-1} y \quad \text{in } \Omega \times (0,T),$$

where $\beta(r) \le C_0 r^m$, $\int_0^r \beta(s)ds \ge C r^{m+1}$, $0 \le k < N/(N+2)$, $1 < p < \infty$.

It turns out that, if $1 < p \le k+m$, then there is a global solution but every solution is only local and blows up if p is greater than $m+k$. For other recent results in this context we refer also to the work of Serrin, Todorova, and Vitillaro [38].

5.7 Navier–Stokes Equations

The classical Navier–Stokes equations

$$\begin{cases} y_t(x,t) - \nu_0 \Delta y(x,t) + (y \cdot \nabla) y(x,t) = f(x,t) + \nabla p(x,t), \\ \hfill x \in \Omega,\ t \in (0,T) \\ (\nabla \cdot y)(x,t) = 0, \hfill \forall (x,t) \in \Omega \times (0,T) \\ y = 0 \hfill \text{on } \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), \hfill x \in \Omega \end{cases} \tag{5.151}$$

describe the non-slip motion of a viscous, incompressible, Newtonian fluid in an open domain $\Omega \subset \mathbf{R}^N$, $N = 2,3$. Here $y = (y_1, y_2, \dots, y_N)$ is the velocity field, p is the pressure, f is the density of an external force, and $\nu_0 > 0$ is the viscosity of the fluid.

We have used the following standard notation

$$\begin{cases} \nabla \cdot y = \operatorname{div} y = \sum_{i=1}^N D_i y_i, \qquad D_i = \dfrac{\partial}{\partial x_i}, \qquad i = 1,\dots,N \\ (y \cdot \nabla) y = \sum_{i=1}^N y_i D_i y_j, \qquad j = 1,\dots,N. \end{cases}$$

By a classical device due to J. Leray, the boundary value problem (5.151) can be written as an infinite-dimensional Cauchy problem in an appropriate function space on Ω. To this end we introduce the following spaces

$$H = \{y \in (L^2(\Omega))^N;\ \nabla \cdot y = 0,\ y \cdot \nu = 0 \text{ on } \partial\Omega\} \tag{5.152}$$

$$V = \{y \in (H_0^1(\Omega))^N;\ \nabla \cdot y = 0\}. \tag{5.153}$$

Here ν is the outward normal to $\partial\Omega$.

The space H is a closed subspace of $(L^2(\Omega))^N$ and it is a Hilbert space with the scalar product

$$(y,z) = \int_\Omega y \cdot z\,dx \tag{5.154}$$

and the corresponding norm $|y| = \left(\int_\Omega |y|^2 dx\right)^{1/2}$. (We denote by the same symbol $|\cdot|$ the norm in $\mathbf{R}^N$, $(L^2(\Omega))^N$, and H, respectively.) The norm of the space V is denoted by $\|\cdot\|$:

$$\|y\| = \left(\int_\Omega |\nabla y(x)|^2 dx\right)^{1/2}. \tag{5.155}$$

We denote by $P : (L^2(\Omega))^N \to H$ the orthogonal projection of $(L^2(\Omega))^N$ onto H (the Leray projector) and set

$$a(y,z) = \int_\Omega \nabla y \cdot \nabla z\,dx, \qquad \forall y,z \in V. \tag{5.156}$$

$$A = -P\Delta,\ D(A) = (H^2(\Omega))^N \cap V. \tag{5.157}$$

Equivalently,

$$(Ay,z) = a(y,z), \qquad \forall y,z \in V. \tag{5.157$'$}$$

The *Stokes operator* A is self-adjoint in H, $A \in L(V,V')$ (V' is the dual of V with the norm denoted by $\|\cdot\|_{V'}$) and

$$(Ay,y) = \|y\|^2, \qquad \forall y \in V. \tag{5.158}$$

Finally, consider the trilinear functional

$$b(y,z,w) = \int_\Omega \sum_{i,j=1}^N y_i D_i z_j w_j\,dx, \qquad \forall y,z,w \in V \tag{5.159}$$

and we denote by $B : V \to V'$ the nonlinear operator defined by

$$By = P(y \cdot \nabla)y \tag{5.160}$$

or, equivalently,

$$(By,w) = b(y,y,w), \qquad \forall w \in V. \tag{5.160$'$}$$

Let $f \in L^2(0,T;V')$ and $y_0 \in H$. The function $y : [0,T] \to H$ is said to be a *weak solution* to equation (5.151) if

$$y \in L^2(0,T;V') \cap C_w([0,T];H) \cap W^{1,1}([0,T];V') \tag{5.161}$$

$$\begin{cases} \dfrac{d}{dt}(y(t),\psi) + \nu_0 a(y(t),\psi) + b(y(t),y(t),\psi) = (f(t),\psi), \quad \text{a.e. } t \in (0,T), \\ y(0) = y_0, \qquad \forall \psi \in V. \end{cases} \tag{5.162}$$

(Here $(\cdot,\cdot)$ is, as usual, the pairing between V, V' and the scalar product of H.)

Equation (5.162) can be equivalently written as

$$\begin{cases} \frac{dy}{dt}(t) + \nu_0 Ay(t) + By(t) = f(t), & \text{a.e. } t \in (0,T) \\ y(0) = y_0 \end{cases} \tag{5.163}$$

where dy/dt is the strong derivative of function $y : [0,T] \to V'$.

The function y is said to be the *strong solution* to (5.151) if $y \in W^{1,1}([0,T];H) \cap L^2(0,T;D(A))$ and (5.163) holds with $dy/dt \in L^1(0,T;H)$ the strong derivative of function $y : [0,T] \to H$.

There is a standard approach to existence theory for the Navier–Stokes equation (5.163) based on the Galerkin approximation scheme (see, e.g., Temam [39]). The method we use here relies on the general results on the nonlinear Cauchy problem of monotone type developed before and, although it leads to a comparable result, it provides a new insight into existence theory of this problem.

It should be said that equation (5.163) is not of monotone type in H, but it can be treated, however, into this framework by an argument described below.

Before proceeding with the existence for problem (1.1), we pause briefly to present some fundamental properties of the trilinear functional b defining the inertial operator B (see Constantin and Foias [19], Temam [39]).

Proposition 5.14. *Let* $1 \le N \le 3$. *Then*

$$b(y,z,w) = -b(y,w,z), \ \forall y,z,w \in V \tag{5.164}$$

$$|b(y,z,w)| \le C\|y\|_{m_1}\|z\|_{m_2+1}\|w\|_{m_3}, \ \forall u \in V_{m_1},\ v \in V_{m_2},\ w \in V_{m_3} \tag{5.165}$$

where $m_i \ge 0,\ i = 1,2,3$ *and*

$$\begin{aligned} m_1 + m_2 + m_3 &\ge \frac{N}{2} \quad \text{if } m_i \ne \frac{N}{2}, \quad \forall i = 1,2,3, \\ m_1 + m_2 + m_3 &> \frac{N}{2} \quad \text{if } m_i = \frac{N}{2}, \quad \text{for some } i = 1,2,3. \end{aligned} \tag{5.166}$$

Here $V_{m_i} = V \cap (H_0^{m_i}(\Omega))^N$.

Proof. It suffices to prove (5.165) for $y,z,w \in \{y \in (C_0^\infty(\Omega))^N;\ \nabla \cdot y = 0\}$. We have

$$\begin{aligned} b(y,z,w) &= \int_\Omega y_i D_i z_j w_j dx = \int_\Omega (y_i D_i(z_j w_j) - y_i D_i w_j z_j) dx \\ &= -\int_\Omega y_i D_i w_j z_j dx = -b(y,z,w) \end{aligned}$$

because $\nabla \cdot y = 0$. By Hölder's inequality we have

$$|b(y,z,w)| \le |y_i|_{q_1} |D_i z_j|_{q_2} |w_j|_{q_3}, \qquad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \le 1. \tag{5.167}$$

(Here $|\cdot|_q$ is the norm of $L^q(\Omega)$.) On the other hand, by the Sobolev embedding theorem we have (see Theorem 1.5)

$$H^{m_i}(\Omega) \subset L^{q_i}(\Omega) \quad \text{for } \frac{1}{q_i} = \frac{1}{2} - \frac{m_i}{N}$$

if $m_i < N/2$. Then, (5.167) yields

$$|b(y,z,w)| \le C\|y\|_{m_1}\|z\|_{m_2+1}\|w\|_{m_3}$$

if $m_i < N/2$, $i = 1,2,3$.

If one m_i is larger than $N/2$ the previous inequality still remains true because, in this case,

$$H^{m_i}(\Omega) \subset L^\infty(\Omega).$$

If $m_i = N/2$ then

$$H^{m_i}(\Omega) \subset \bigcap_{q>2} L^q(\Omega)$$

and so (5.167) holds for $1/q_2 + 1/q_3 < 1$ and $q_1 = \varepsilon$ where

$$\frac{1}{\varepsilon} = 1 - \frac{1}{q_2} - \frac{1}{q_3}\cdot$$

Then (5.165) follows for $m_1 + m_2 + m_3 > N/2$ as claimed.

We have also the interpolation inequality

$$\|u\|_m \le c\|u\|_\ell^{1-\alpha}\|u\|_{\ell+1}^\alpha, \quad \text{for } \alpha = m - \ell \in [0,1]. \tag{5.168}$$

In particular, it follows by Proposition 5.14 that B is continuous from V to V'. Indeed, we have

$$(By - Bz, w) = b(y, y-z, w) + b(y-z, z, w), \qquad \forall w \in V$$

and this yields (notice that $\|\cdot\| = \|\cdot\|_1$ and $|Ay| = |y|_2$)

$$|(By - Bz, w)| \le C(\|y\|\|y-z\|\|w\| + \|y-z\|\|z\|\|w\|).$$

Hence

$$\|By - Bz\|_{V'} \le C\|y-z\|(\|y\| + \|z\|), \qquad \forall y,z \in V. \tag{5.169}$$

We would like to treat (5.163) as a nonlinear Cauchy problem in the space H. However, because the operator $\nu_0 A + B$ is not quasi-m-accretive in H, we first consider a quasi-m-accretive approximation of the form taken in the proof of Theorem 4.8.

For each $M > 0$ define the operator $B_M : V \to V'$ (see (4.67))

$$B_M y = \begin{cases} By & \text{if } \|y\| \le M, \\ \dfrac{M^2}{\|y\|^2} By & \text{if } \|y\| > M, \end{cases}$$

and consider the operator $\Gamma_M : D(\Gamma_M) \subset H \to H$

$$\Gamma_M = \nu_0 A + B_M, \qquad D(\Gamma_M) = D(A). \tag{5.170}$$

Let us show that Γ_M is well defined. Indeed, we have

$$|\Gamma_M y| \le \nu_0 |Ay| + |B_M y|, \qquad \forall y \in D(A).$$

On the other hand, by (5.165) for $m_1 = 1,\ m_2 = 1/2,\ m_3 = 0$, we have for $\|y\| \le M$

$$|(B_M y, w)| = |b(y,y,w)| \le C\|y\|^{3/2}|Ay|^{1/2}|w|$$

because $\|y\|_{3/2} \le \|y\|^{1/2}|Ay|^{1/2}$. Hence

$$|B_M y| \le C|Ay|^{1/2}\|y\|^{3/2}, \qquad \forall y \in D(A).$$

Similarly, we get for $\|y\| > M$

$$|B_M y| \le \frac{CM^2}{\|y\|^2}\,|Ay|^{1/2}\|y\|^{3/2} \le C|Ay|^{1/2}\|y\|^{3/2}.$$

This yields

$$|\Gamma_M y| \le \nu_0|Ay| + C|Ay|^{1/2}\|y\|^{3/2}, \qquad \forall y \in D(A) \tag{5.171}$$

as claimed. □

Lemma 5.2. *There is α_M such that $\Gamma_M + \alpha_M I$ is m-accretive in $H \times H$.*

Proof. We show first that for each $\nu > 0$

$$((\Gamma_M + \lambda)y - (\Gamma_M + \lambda)z, y - z) \ge \frac{\nu}{2}\,\|y-z\|^2, \qquad \forall y,z \in D(A),\ \text{ for } \lambda \ge C_M^\nu.$$

To this end we prove that

$$|(B_M y - B_M z, y - z)| \le \frac{\nu}{2}\,\|y-z\|^2 + C_M|y-z|^2. \tag{5.172}$$

We treat only the case $N = 3$ because $N = 2$ follows in a similar way.

Let $\|y\|, \|z\| \le M$. Then we have

$$\begin{aligned}(B_M y - B_M z, y - z) &= (By - Bz, y - z) = b(y,y,y-z) - b(z,z,y-z)\\ &= b(y-z,y,y-z) + b(z,y-z,y-z) = b(y-z,y,y-z).\end{aligned}$$

Hence, by Proposition 5.14, for $m_1 = 1,\ m_2 = 0,\ m_3 = 1/2$ we have

$$\begin{aligned}|(B_M y - B_M z, y-z)| &= |b(y-z,y,y-z)| \le C\|y-z\|\|y\|\|y-z\|_{1/2}\\ &\le C\|y-z\|^{3/2}\|y\||y-z|^{1/2}\\ &\le C_M\|y-z\|^{3/2}|y-z|^{1/2}\\ &\le \frac{\nu}{2}\|y-z\|^2 + C_M|y-z|^2\end{aligned}$$

as desired.

Now consider the case where $\|y\| > M$, $\|z\| > M$. We have

$$\begin{aligned}&(B_M y - B_M z, y-z)\\ &= \frac{M^2}{\|y\|^2}(b(y,y,y-z) - b(z,z,y-z)) + \left(\frac{M^2}{\|y\|^2} - \frac{M^2}{\|z\|^2}\right) b(z,z,y-z)\\ &= \frac{M^2}{\|y\|^2} b(y-z,y,y-z) + M^2\left(\frac{\|z\|^2 - \|y\|^2}{\|y\|^2\|z\|^2}\right) b(z,z,y-z).\end{aligned}$$

This yields

$$\begin{aligned}|(B_M y - B_M z, y-z)| \le{}& \frac{CM^2}{\|y\|}\|y-z\|^{3/2}|y-z|^{1/2}\\ &+ \frac{CM^2}{\|y\|^2\|z\|^2}\left|\,\|z\|^2 - \|y\|^2\right|\|z\|\,\|y-z\|_{1/2}\\ \le{}& \frac{\nu}{2}\|y-z\|^2 + C_M^1|y-z|^2.\end{aligned}$$

Assume now that $\|y\| > M,\ \|z\| \le M$. We have

$$\begin{aligned}&|(B_M y - B_M z, y-z)| = \left|\frac{M^2}{\|y\|^2} b(y,y,y-z) - b(z,z,y-z)\right|\\ &\le \left|\frac{M^2}{\|y\|^2} - 1\right| |b(z,z,y-z)| + \frac{M^2}{\|y\|^2}|b(y,y,y-z) - b(z,z,y-z)|\\ &\le C\,\frac{\|y\|^2 - M^2}{\|y\|^2}\|z\|^2\|y-z\|^{1/2}|y-z|^{1/2} + \frac{M^2}{\|y\|^2}|b(y-z,y,y-z)|\\ &\le C_M^1\|y-z\|^{3/2}|y-z|^{1/2}\end{aligned}$$

which again implies (5.172), as claimed.

We note also that by (5.169) it follows that

$$\|B_M y - B_M z\|_{V'} \le C\|y-z\|(\|y\| + \|z\|), \qquad \forall y,z \in V, \tag{5.173}$$

where C is independent of M.

Let us now proceed with the proof of α_M-m-accretivity of Γ_M. Consider the operator

$$F_M u = \nu_0 Au + B_M u + \alpha_M u, \ \forall u \in D(F_M)$$
$$D(F_M) = \{u \in V;\ \nu_0 Au + B_M u \in H\}. \tag{5.174}$$

By (5.172) we see that for $\alpha_M \geq C_M$ the operator $u \to \nu_0 Au + B_M u + \alpha_M u$ is monotone, coercive, and continuous from V to V'. Hence its restriction to H; that is, F_M is maximal monotone (m-accretive) in $H \times H$. To complete the proof it suffices to show that $D(F_M) = D(A)$ for α_M large enough. (Clearly $D(A) \subset D(F_M)$.)

Note first that by (5.165) we have

$$|(B_M y, w)| \leq C|b(y,y,w)| \leq C\|y\| \|y\|_{3/2} |w|, \qquad \forall w \in H,$$

and this yields by interpolation (see (5.168))

$$|B_M(y)| \leq C\|y\|^{3/2} |Ay|^{1/2} \leq C_M |Ay|^{1/2}.$$

Hence

$$|Ay| \leq \frac{1}{\nu_0}(|\Gamma_M y| + |B_M y|) \leq \frac{1}{\nu_0}(|\Gamma_M y| + C_M |Ay|^{1/2}), \qquad \forall y \in D(A);$$

that is,

$$|Ay| \leq C_M(|\Gamma_M y| + 1), \qquad \forall y \in D(A). \tag{5.175}$$

Now we consider the operators

$$F_M^1 = \nu_0(1-\varepsilon)A, \qquad D(F_M^1) = D(A)$$
$$F_M^2 = \varepsilon\nu_0 A + B_M + \alpha_M I, \qquad D(F_M^2) = \{u \in V;\ \varepsilon\nu_0 Au + B_M u \in H\},$$

where α_M is large enough so that F_M^2 is m-accretive in $H \times H$. (We have seen above that such an α_M exists.)

We have

$$\begin{aligned}\left|F_M^2(y)\right| &\leq \varepsilon\nu_0|Ay| + |B_M y| + \alpha_M|y| \\ &\leq \varepsilon\nu_0|Ay| + C_M|Ay|^{1/2} + \alpha_M|y| \leq \varepsilon(1+\delta)|Ay| + \alpha_M|y| + C_M^1 \\ &\leq \frac{\varepsilon(1+\delta)}{\nu_0(1-\varepsilon)}\left|F_M^1(y)\right| + \alpha_M|y| + C_M^1,\ \forall y \in D(A) = D(F_M^1).\end{aligned}$$

Thus for ε small enough it follows by Proposition 3.9 that $F_M^1 + F_M^2$ with the domain $D(A)$ is m-accretive in $H \times H$. Because $F_M = F_M^1 + F_M^2$ on $D(A) \subset D(F_M)$ we infer that $D(F_M) = D(A)$ as claimed. □

For each $M > 0$ consider the equation

$$\begin{cases} \dfrac{dy}{dt}(t) + \nu_0 Ay(t) + B_M y(t) = f(t), & t \in (0,T) \\ y(0) = y_0. \end{cases} \tag{5.176}$$

Proposition 5.15. *Let $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];H)$ be given. Then there is a unique solution $y_M \in W^{1,\infty}([0,T];H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$ to equation* (5.176)*. Moreover, $(d^+/dt)y_M(t)$ exists for all $t \in [0,T)$ and*

$$\frac{d^+}{dt} y_M(t) + \nu_0 A y_M(t) + B_M y_M(t) = f(t), \qquad \forall t \in [0,T). \tag{5.177}$$

Proof. This follows by Theorem 4.4. Because $\Gamma_M y_M = \nu_0 A y_M + B_M y_M \in L^\infty(0,T;H)$, by (5.175) we infer that $A y_M \in L^\infty(0,T;H)$. As $dy_M/dt \in L^\infty(0,T;H)$, we conclude also that $y_M \in C([0,T];V) \cap L^\infty(0,T;D(A))$, as claimed. □

Now we are ready to formulate the main existence result for the strong solutions to Navier–Stokes equation (5.151) ((5.151)′).

Theorem 5.9. *Let $N = 2,3$ and $f \in W^{1,1}([0,T];H)$, $y_0 \in D(A)$ where $0 < T < \infty$. Then there is a unique function $y \in W^{1,\infty}([0,T^*);H) \cap L^\infty(0,T^*;D(A)) \cap C([0,T^*];V)$ such that*

$$\begin{cases} \dfrac{dy(t)}{dt} + \nu_0 A y(t) + B y(t) = f(t), & \text{a.e. } t \in (0,T^*), \\ y(0) = y_0, \end{cases} \tag{5.178}$$

for some $T^* = T^*(\|y_0\|) \le T$. If $N = 2$ then $T^* = T$. Moreover, $y(t)$ is right differentiable and

$$\frac{d^+}{dt} y(t) + \nu_0 A y(t) + B y(t) = f(t), \qquad \forall t \in [0,T^*). \tag{5.179}$$

Proof. The idea of the proof is to show that for M sufficiently large the flow $y_M(t)$, defined by Proposition 5.15, is independent of M on each interval $[0,T]$ if $N = 2$ or on $[0,T(y_0)]$ if $N = 3$. Let y_M be the solution to (5.176); that is,

$$\begin{cases} \dfrac{dy_M}{dt}(t) + \nu_0 A y_M(t) + B_M y_M(t) = f(t), & \text{a.e. } t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{5.180}$$

If we multiply (5.180) by y_M and integrate on $(0,t)$, we get

$$|y_M(t)|^2 + \nu_0 \int_0^t \|y_M(s)\|^2 ds \le C\left(|y_0|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \qquad \forall M.$$

Next, we multiply (5.180) (scalarly in H) by $A y_M(t)$. We get

$$\frac{1}{2}\frac{d}{dt}\|y_M(t)\|^2 + \nu_0 |A y_M(t)|^2 \le |(B_M y_M(t), A y_M(t))| + |f(t)||A y_M|, \qquad \text{a.e. } t \in (0,T).$$

This yields

$$
\begin{aligned}
&\|y_M(t)\|^2+\nu_0\int_0^t |Ay_M(s)|^2 ds \\
&\le C\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(t)|^2 dt+\int_0^t |(B_M y_M, Ay_M)| ds\right).
\end{aligned}
\tag{5.181}
$$

On the other hand, for $N=3$, by (5.165) we have (the case $N=2$ is treated separately below)

$$
\begin{aligned}
|(B_M y_M, Ay_M)| &< |b(y_M, y_M, Ay_M)| \\
&\le C\|y_M\|\,\|y_M\|_{3/2}|Ay_M| \\
&\le C\|y_M\|^{3/2}|Ay_M|^{3/2}, \quad \text{a.e. } t\in(0,T).
\end{aligned}
$$

(Everywhere in the following C is independent of M,ν_0.) Then, by (5.181) we have

$$
\begin{aligned}
&\|y_M(t)\|^2+\nu_0\int_0^t |Ay_M(s)|^2 ds \\
&\le C\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(t)|^2 dt+\int_0^t |Ay_M(s)|^{3/2}\|y_M(s)\|^{3/2} ds\right) \\
&\le C\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(t)|^2 dt+\frac{1}{\nu_0}\int_0^t \|y_M(s)\|^6 ds\right)+\frac{\nu}{2}\int_0^t |Ay_M(s)|^2 ds, \\
&\qquad\qquad \forall t\in[0,T].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\|y_M(t)\|^2+\frac{\nu_0}{2}\int_0^t |Ay_M(s)|^2 ds \\
&\le C_0\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(s)|^2 ds+\frac{1}{\nu_0}\int_0^t \|y_M(s)\|^6 ds\right).
\end{aligned}
\tag{5.182}
$$

Next, we consider the integral inequality

$$
\|y_M(t)\|^2 \le C_0\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(s)|^2 ds+\frac{1}{\nu_0}\int_0^t \|y_M(s)\|^6 ds\right). \tag{5.183}
$$

We have

$$
\|y_M(t)\|^2 \le \varphi(t), \qquad \forall t\in(0,T),
$$

where

$$
\begin{aligned}
\varphi' &\le \frac{C_0}{\nu_0}\,\varphi^3, \qquad \forall t\in(0,T) \\
\varphi(0) &= C_0\left(\|y_0\|^2+\frac{1}{\nu_0}\int_0^T |f(s)|^2 ds\right).
\end{aligned}
$$

This yields

$$
\varphi(t)\le\left(\frac{\nu_0\varphi^3(0)}{\nu_0-3t\varphi^3(0)}\right)^{1/3}, \qquad \forall t\in\left(0,\frac{\nu_0}{3\varphi^3(0)}\right).
$$

Hence

$$\|y_M(t)\|^2 \le \left(\frac{\nu_0\varphi^3(0)}{\nu_0 - 3t\varphi^3(0)}\right)^{1/3}, \qquad \forall t \in (0,T^*), \tag{5.184}$$

where

$$T^* = \frac{\nu_0}{3C_0^3\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(s)|^2 ds\right)^3}\cdot$$

Then, by (5.182) we get

$$\|y_M(t)\|^2 + \frac{\nu_0}{2}\int_0^t |Ay_M(s)|^2 ds \le C_1(\delta)\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \quad 0 < t < T^* - \delta. \tag{5.185}$$

For $N = 2$, we have (see (5.165))

$$\begin{aligned} |(B_M y_M, Ay_M)| &\le C|y_M|^{1/2}\|y_M\||Ay_M|^{3/2} \\ &\le \frac{\nu_0}{2}|Ay_M|^2 + \frac{C}{\nu_0}\|y_M\|^4. \end{aligned}$$

This yields

$$\begin{aligned} &\|y_M(t)\|^2 + \frac{\nu_0}{2}\int_0^t |Ay_M(s)|^2 ds \\ &\le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt + \frac{1}{\nu_0}\int_0^t \|y_M(s)\|^4 ds\right). \end{aligned}$$

Then, by (5.182) and the Gronwall lemma, we obtain

$$\|y_M(t)\|^2 + \frac{\nu_0}{2}\int_0^t |Ay_M(s)|^2 ds \le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \quad \forall t \in (0,T). \tag{5.186}$$

By (5.184), (5.186) we infer that for M large enough, $\|y_M(t)\| \le M$ on $(0,T^*)$ if $N = 3$ or on the whole of $(0,T)$ if $N = 2$.

Hence $B_M y_M = By_M$ on $(0,T^*)$ (respectively on $(0,T)$) and so $y_M = y$ is a solution to (5.178). This completes the proof of existence.

Uniqueness. If y_1, y_2 are two solutions to (5.178), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|y_1(t) - y_2(t)|^2 + \nu_0\|y_1(t) - y_2(t)\|^2 \\ &\le |(B(y)(t) - By_2(t), y_1(t) - y_2(t))| \\ &= |b(y_1(t), y_1(t), y_1(t) - y_2(t)) - b(y_2(t), y_2(t), y_1(t) - y_2(t))| \\ &\le C\|y_1(t) - y_2(t)\|^2(\|y_1(t)\| + \|y_2(t)\|), \quad \text{a.e. } t \in (0,T^*). \end{aligned}$$

Hence, $y_1 \equiv y_2$.

It is useful to note that the solution y to (5.178) satisfies the estimates

$$|y(t)|^2 + \nu_0 \int_0^t \|y(s)\|^2 ds \le C\left(|y_0|^2 + \frac{1}{\nu_0}\int_0^T |f(s)|^2 ds\right) \tag{5.187}$$

and (for $N = 3$)

$$\begin{aligned} &\|y(t)\|^2 + \nu_0 \int_0^t |Ay(s)|^2 ds \\ &\le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^{T^*} |f(t)|^2 dt\right)\left(\int_0^t \frac{ds}{T^*-t} + 1\right), \qquad t \in (0,T^*), \end{aligned} \tag{5.188}$$

whereas, for $N = 2$,

$$\begin{aligned} \|y(t)\|^2 + \nu_0 \int_0^t |Ay(s)|^2 ds \le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \\ \forall t \in (0,T), \end{aligned} \tag{5.189}$$

where C is independent of y_0 and f.

If $N = 2$, we have a sharper estimate for y. Indeed, if we multiply (5.178) by tAy and integrate on $(0,t)$, we get after integration by parts

$$\begin{aligned} &\frac{t}{2}\|y(t)\|^2 + \nu_0 \int_0^t s|Ay(s)|^2 ds \\ &= -\int_0^t (sb(y(s),y(s),Ay(s)) - s(f(s),Ay(s)))ds + \frac{1}{2}\int_0^t \|y(s)\|^2 ds \\ &\le C\int_0^t s|Ay(s)|^{3/2}|y(s)|^{1/2}\|y(s)\|ds + \frac{\nu_0}{2}\int_0^t s|Ay(s)|^2 ds \\ &\quad + \frac{1}{2}\int_0^t s|f(s)|^2 ds + \frac{1}{2}\int_0^t \|y(s)\|^2 ds. \end{aligned}$$

Then, by (5.188), we get the estimate

$$\begin{aligned} t\|y(t)\|^2 + \nu_0 \int_0^t s|Ay(s)|^2 ds \le C\left(|y_0|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \\ \forall t \in (0,T). \end{aligned} \tag{5.190}$$

Estimates (5.186), (5.188), and (5.190) suggest that equation (5.151) could have a strong solution y under weaker assumptions on y_0 and f. We show below that this is indeed the case. □

Theorem 5.10. *Let $y_0 \in H$, $f \in L^2(0,T;H)$, $T > 0$, and $N = 2$. Then there is a unique solution*

$$y \in C(]0,T];V) \cap C_w([0,T];H) \cap L^2(0,T;V),$$
$$t^{1/2} y \in L^2(0,T;D(A)) \cap L^\infty(0,T;V),$$
$$t^{1/2}\,\frac{dy}{dt} \in L^2(0,T;H), \qquad \frac{dy}{dt} \in L^{2/(1+\varepsilon)}(0,T;V')$$

to equation (5.178)*; that is,*

$$\begin{cases} \dfrac{dy}{dt}(t) + \nu_0 Ay(t) + By(t) = f(t), & a.e.\ t \in (0,T) \\ y(0) = y_0. \end{cases} \tag{5.191}$$

If $y_0 \in V$*, then* $y \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$.

Proof. Let $\{y_0^j\} \subset D(A)$ and $\{f_j\} \subset W^{1,1}([0,T];H)$ be such that

$$y_0^j \to y_0 \quad \text{strongly in } H,$$
$$f_j \to f \quad \text{strongly in } L^2(0,T;H).$$

By (5.187), (5.190), we have

$$|y_j(t)|^2 + \int_0^T \|y_j(t)\|^2 dt + t\|y_j(t)\|^2 + \int_0^T t|Ay_j(t)|^2 dt \le C, \qquad t \in (0,T).$$

Then, by (5.165), we obtain that

$$\int_0^T \|By_j(t)\|_{V'}^{2/(1+\varepsilon)} dt + \int_0^T t|By_j(t)|^2 dt \le C, \qquad \forall \varepsilon > 0$$

because

$$|(By_j,\varphi)| = |b(y_j,y_j,\varphi)| \le C|y_j|^{1/2}\|y_j\||Ay_j|^{1/2}|\varphi|$$

and

$$|(By_j,\varphi)| \le C\|y_j\|_\varepsilon \|y_j\|\|\varphi\|.$$

This yields

$$|By_j| \le C|y_j|^{1/2}\|y_j\||Ay_j|^{1/2},$$
$$\|By_j\|_{V'} \le C\|y_j\|_\varepsilon \|y_j\| \le C\|y_j\|^{1+\varepsilon}|y_j|^{1-\varepsilon}.$$

Hence

$$\int_0^T \left(\left\| \frac{dy_j(t)}{dt} \right\|_{V'}^{2/(1+\varepsilon)} + t\left| \frac{dy_j(t)}{dt} \right|^2 \right) dt \le C.$$

Because the embeddings $D(A) \subset V \subset H \subset V'$ are compact, it follows by the Ascoli–Arzelà theorem that on a subsequence, again denoted y_j, we have

$$
\begin{aligned}
y_j(t) &\overset{j\to\infty}{\longrightarrow} y(t) && \text{in } C([0,T];V') \\
y_j &\longrightarrow y && \text{weak-star in } L^\infty(0,T;H), \\
& && \text{weakly in } L^2(0,T;V), \\
\sqrt{t}\,\frac{dy_j}{dt} &\longrightarrow \sqrt{t}\,\frac{dy}{dt} && \text{weakly in } L^2(0,T;H) \\
Ay_j &\longrightarrow Ay && \text{weakly in } L^2(0,T;V'), \\
\sqrt{t}\,y_j &\longrightarrow \sqrt{t}\,y && \text{weak-star in } L^\infty(0,T;V), \\
& && \text{weakly in } L^2(0,T;D(A)).
\end{aligned}
$$

Moreover, by the Aubin compactness theorem, we have

$$
\begin{aligned}
\sqrt{t}\,y_j(t) &\longrightarrow \sqrt{t}\,y(t) && \text{uniformly in } H \text{ on } [0,T] \\
\sqrt{t}\,y_j &\longrightarrow \sqrt{t}\,y && \text{strongly in } L^2(0,T;V).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
\big|(By_j(t)-By(t),\varphi)\big| &\le \big|b(y_j(t)-y(t),y_j(t),\varphi)\big| + \big|b(y(t),y_j(t)-y(t),\varphi)\big| \\
&\le C\big|y_j(t)-y(t)\big|^{1/2}\big\|y_j(t)-y(t)\big\|^{1/2}\big|Ay_j(t)\big|^{1/2}\big\|y_j(t)\big\|^{1/2}|\varphi| \\
&+ C\|y(t)\|^{1/2}\big\|y_j(t)-y(t)\big\|^{1/2}|y(t)|^{1/2}\big|A(y_j(t)-y(t))\big|^{1/2}|\varphi|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\big|By_j(t)-By(t)\big| \le C\big\|y_j(t)-y(t)\big\|^{1/2}\big(\big|Ay_j(t)\big|^{1/2}\big|y_j(t)-y(t)\big|^{1/2}\big|y_j(t)\big|^{1/2} \\
+ \|y(t)\|^{1/2}\big|A(y_j(t)-y(t))\big|^{1/2}\big|y_j(t)\big|^{1/2}\big).
\end{aligned}
$$

We have, therefore,

$$
\int_0^T t^2\big|By_j(t)-By(t)\big|^2 dt \to 0 \quad \text{as } j\to\infty.
$$

Letting $j\to\infty$, we conclude that y satisfies, a.e. on $(0,T)$, equation (5.191) and that

$$
\begin{aligned}
t\|y(t)\|^2 + |y(t)|^2 + \int_0^T (\|y(t)\|^2 + t|Ay(t)|^2)dt \le C, \\
\int_0^T \left(\left\| \frac{dy}{dt}(t) \right\|_{V'}^{2/(1+\varepsilon)} + t\left| \frac{dy}{dt}(t) \right|^2 \right) dt \le C,
\end{aligned}
$$

where d/dt is considered in the sense of distributions.

If $y_0 \in V$, then we have

$$
\big\|y_j(t)\big\|^2 + \nu_0 \int_0^T \big|Ay_j(t)\big|^2 dt \le C
$$

and this implies the last part of the theorem. This completes the proof. (The uniqueness follows as in the proof of Theorem 5.9.) □

Theorem 5.11. *Let $N = 3$, $y_0 \in V$, and $f \in L^2(0,T;H)$. Then there is*

$$T_0^* = T(\|y_0\|, \|f\|_{L^2(0,T;H)})$$

such that on $(0,T_0^*)$ equation (5.151) *has a unique solution*

$$y \in L^\infty(0,T_0^*;V) \cap L^2(0,T_0^*;D(A)) \cap C([0,T_0^*];H)$$
$$\frac{dy}{dt} \in L^2(0,T_0^*;H), \qquad By \in L^2(0,T_0^*;H).$$

Proof. Let $\{y_0^j\}$ and $\{f_j\}$ be as in the proof of Theorem 5.10 ($y_0^j \to y_0$ in V this time.) By the above estimates (see (5.188)), we have

$$\|y_j(t)\|^2 + \nu_0 \int_0^{T_0^*} |Ay_j(t)|^2 dt \le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(s)|^2 ds\right), \qquad \forall t \in [0,T_0^*),$$

where $T_0^* < T^* < T$.

We also have (see (5.165))

$$|By_j(t)| \le C\|y_j(t)\|^{3/2}|Ay_j(t)|^{1/2}|y_j(t)|^{1/2} \le C_1|Ay_j(t)|^{1/2}, \qquad \forall t \in (0,T_0^*).$$

Hence,

$$\int_0^{T_0^*} \left(|By_j(t)|^2 + \left|\frac{dy_j}{dt}(t)\right|^2\right) dt \le C.$$

Hence, on a subsequence

$$\begin{aligned}
y_j(t) \to y(t) \quad & \text{strongly in } H \text{ uniformly on } [0,T] \\
& \text{weak-star in } L^\infty(0,T;V) \\
\frac{dy_j}{dt} \to \frac{dy}{dt} \quad & \text{weakly in } L^2(0,T;H) \\
Ay_j \to Ay \quad & \text{weakly in } L^2(0,T;H) \\
By_j \to \eta \quad & \text{weakly in } L^2(0,T;H).
\end{aligned}$$

Moreover, by the Aubin compactness theorem we have $y_j \to y$ strongly in $L^2(0,T;V)$. Note also that, by (5.165), we have

$$|(By_j - By, \varphi)| \le C(\|y_j - y\|^{3/2}|A(y_j - y)|^{1/2} + \|y_j - y\|\,\|y\|_{3/2})|\varphi|.$$

Hence,

$$\int_0^T |By_j - By|dt \leq C\left(\int_0^T \|y_j - y\|^2 dt\right)^{1/2}\left(\left(\int_0^T \|y_j - y\||A(y_j - y)|dt\right)^{1/2}\right.$$
$$\left. + \int_0^T |Ay|^{1/2}\|y\|^{3/2}dt\right) \leq C\int_0^T \|y_j - y\|^2 dt \to 0 \quad \text{as } j \to 0$$

and, therefore,

$$By_j \to By \quad \text{strongly in } L^1(0,T;H),$$

which implies that $\eta = By$. Hence, y is a strong solution on $(0,T_0^*)$. The uniqueness is immediate. □

The main existence result for a weak solution to equation (5.151) ((5.151)′) is Leray's theorem below.

Theorem 5.12. *Let $y_0 \in H$, $f \in L^2(0,T;V')$. Then there is at least one weak solution y^* to equation* (5.151). *Moreover,*

$$\frac{dy^*}{dt} \in L^{4/3}(0,T;V') \qquad \textit{for } N = 3. \tag{5.192}$$

$$\frac{dy^*}{dt} \in L^{2/(1+\varepsilon)}(0,T;V') \quad \textit{for } N = 2. \tag{5.193}$$

If $N = 2$, there is a unique weak solution satisfying (5.193).

Proof. We return to approximating equation (5.176) and note the estimates

$$|y_M(t)|^2 + \int_0^T \|y_M(t)\|^2 dt \leq C\left(|y_0|^2 + \int_0^T |f(t)|_*^2 dt\right). \tag{5.194}$$

(For simplicity, we denote below by $|\cdot|_*$ the norm $\|\cdot\|_{V'}$ of V'.) We also have by (5.165)

$$|(B_M y_M(t), w)| \leq C\|y_M(t)\|_{1/2}\|y_M(t)\|\|w\| \leq C|y_M(t)|^{1/2}\|y_M(t)\|^{3/2}\|w\|.$$

Hence, $|B_M y_M|_* \leq C\|y_M\|^{3/2}|y_M|^{1/2}$ and, therefore,

$$\int_0^T |B_M y_M(t)|_*^{4/3} dt \leq C\left(|y_0|^2 + \int_0^T |f(t)|_*^2 dt\right) \tag{5.195}$$

$$\int_0^T \left|\frac{dy_M}{dt}(t)\right|_*^{4/3} dt \leq C\left(|y_0|^2 + \int_0^T |f(t)|_*^2 dt\right). \tag{5.196}$$

For $N = 2$ we have (see (5.165)) for $m_1 = \varepsilon$, $m_2 = 0$, $m_3 = 1$,

$$|B_M y_M(t)|_* \leq C|y_M(t)|^{1-\varepsilon}\|y_M(t)\|^{1+\varepsilon} \leq C_1\|y_M(t)\|^{1+\varepsilon}.$$

Hence,

$$\int_0^T \left(\left| \frac{dy_M}{dt} \right|_*^{2/(1+\varepsilon)} + |B_M y_M|_*^{2/(1+\varepsilon)} \right) dt \le C \quad \text{for } N = 2. \tag{5.197}$$

Assume now that $y_0 \in H$ and $f \in L^2(0,T;V')$.

Let $y_0^j \in D(A)$ and $\{f_j\} \subset W^{1,1}([0,T];H)$ be such that

$$y_0^j \to y_0 \ \text{ in } H, \qquad f_j \to f \ \text{ in } L^2(0,T;V').$$

Let y_j be the corresponding solution to equation $(5.151)'$. By estimates (5.195)–(5.197), we have for a constant C independent of M,

$$\int_0^T \left(\|y_j\|^2 + \left\| \frac{dy_j}{dt} \right\|_*^{4/3} + |B_M y_j|_*^{4/3} \right) dt + |y_j(t)|^2 \le C \tag{5.198}$$

if $N = 3$, and

$$\int_0^T \left(\|y_j(t)\|^2 + \left\| \frac{dy_j}{dt} \right\|_*^{2/(1+\varepsilon)} + |B_M y_j|_*^{2/(1+\varepsilon)} \right) dt + |y_j(t)|^2 \le C \tag{5.199}$$

if $N = 2$.

Hence, on a subsequence we have

$$\begin{aligned}
y_j &\to y_M && \text{weakly in } L^2(0,T;V)\\
Ay_j &\to Ay_M && \text{weakly in } L^2(0,T;V')\\
\frac{dy_j}{dt} &\to \frac{dy_M}{dt} && \text{weakly in } L^{4/3}(0,T;V') \text{ if } N=3\\
& && \text{weakly in } L^{2/(1+\varepsilon)}(0,T;V') \text{ if } N=2\\
B_M y_j &\to \eta_M && \text{weakly in } L^{4/3}(0,T;V') \text{ if } N=3\\
& && \text{weakly in } L^{2/(1+\varepsilon)}(0,T;V') \text{ if } N=2.
\end{aligned}$$

Moreover, recalling inequality (5.172) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|y_j(t) - y_k(t)|^2 + \frac{\nu_0}{2}\|y_j(t) - y_k(t)\|^2\\
&\le \alpha_M |y_j(t) - y_k(t)|^2 + |f_j(t) - f_k(t)| \|y_j(t) - y_k(t)\|_* .
\end{aligned}$$

By Gronwall's lemma we have

$$|y_j(t) - y_k(t)|^2 \le \left| y_0^j - y_0^k \right|^2 + C\int_0^T |f_j(t) - f_k(t)|_*^2 dt$$

and, therefore,

$$\int_0^T \left\|y_j(t)-y_k(t)\right\|^2 dt \le C\left(\left|y_0^j - y_0^k\right|^2 + \int_0^T \left|f_j(t)-f_k(t)\right|_*^2 dt\right).$$

Hence,

$$y_j \to y_M \quad \text{strongly in } L^2(0,T;V)\cap C([0,T];H).$$

Clearly, we have

$$\begin{cases} \dfrac{dy_M}{dt}(t) + \nu A y_M(t) + \eta_M(t) = f(t), & \text{a.e. } t\in(0,T) \\ y_M(0) = y_0. \end{cases}$$

On the other hand, by (5.165), where $m_1 = 1$, $m_2 = 0$, $m_3 = 1$, it follows that

$$\left|B_M y_j - B_M y_M\right|_* \le C\left\|y_j - y_j\right\|(\left\|y_j\right\| + \|y_M\|).$$

Hence,

$$B_M y_j \to B_M y_M = \eta_M \quad \text{strongly in } L^1(0,T;V').$$

We have shown therefore that for each $y_0 \in H$ and $f \in L^2(0,T;V')$ the equation

$$\begin{cases} \dfrac{dy_M}{dt}(t) + \nu A y_M(t) + B_M y_M(t) = f(t), & \text{a.e. } t\in(0,T) \\ y_M(0) = y_0 \end{cases} \tag{5.200}$$

has a solution $y_M \in L^2(0,T;V)\cap C([0,T];H)$ with $dy_M/dt \in L^{4/3}(0,T;V')$ if $N=3$, $dy_M/dt \in L^{2/(1+\varepsilon)}(0,T;V')$ if $N=2$. Moreover, y_M satisfies estimates (5.194)–(5.196).

Now, we let $M\to\infty$. Then on a subsequence, again denoted M, we have

$$\begin{aligned}
y_M &\to y^* && \text{weak-star in } L^\infty(0,T;H) \\
& && \text{weakly in } L^2(0,T;V) \\
\frac{dy_M}{dt} &\to \frac{dy^*}{dt} && \text{weakly in } L^{4/3}(0,T;V') \text{ if } N=3 \\
& && \text{weakly in } L^{2/(1+\varepsilon)}(0,T;V') \text{ if } N=2 \\
Ay_M &\to Ay^* && \text{weakly in } L^2(0,T;V') \\
B_M y_M &\to \eta && \text{weakly in } L^{4/3}(0,T;V') \text{ if } N=3 \\
& && \text{weakly in } L^{2/(1+\varepsilon)}(0,T;V') \text{ if } N=2.
\end{aligned}$$

We have

$$\begin{cases} \dfrac{dy^*}{dt}(t) + \nu_0 A y^*(t) + \eta(t) = f(t), & \text{a.e. in } (0,T) \\ y^*(0) = y_0. \end{cases} \tag{5.201}$$

To conclude the proof it remains to be shown that $\eta(t) = By^*(t)$, a.e. $t\in(0,T)$.

We note first that, by Aubin's compactness theorem, for $M\to\infty$,

$$y_M \to y^* \quad \text{strongly in } L^2(0,T;H).$$

We note also that by (5.194) we have $m\{t;\ \|y_M(t)\| > M\} \le C/M^2$.

Let $\varphi \in L^\infty(0,T;\mathscr{V})$. Then, we have

$$\begin{aligned}
&\int_0^T |(B_M y_M - By^*, \varphi)|dt \\
&\le \int_{E_M} |(By_M - By^*, \varphi)|dt + C\int_{E_M^c} \|\varphi\|(|y_M|^{1/2}\|y_M\|^{3/2} + |y^*|^{1/2}\|y^*\|^{3/2})dt,
\end{aligned}$$

where $E_M = \{t;\ \|y_M(t)\| > M\}$. Hence, by estimates (5.194) we have

$$\begin{aligned}
&\int_0^T |(B_M y_M - By^*, \varphi)|dt \\
&\le \int_0^T (|b(y_M - y^*, y_M, \varphi)| + |b(y^*, y_M - y^*, \varphi)|)dt + CM^{-2}\|\varphi\|_{L^\infty(0,T;V)}.
\end{aligned}$$

Recalling that $y_M \to y^*$ strongly in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$, we get

$$\lim_{M\to\infty} \int_0^T (B_M y_M - By^*, \varphi)dt = 0, \qquad \forall \varphi \in L^2(0,T;\mathscr{V}),$$

where $\mathscr{V} = \{\varphi \in C_0^\infty(\Omega);\ \operatorname{div}\varphi = 0\}$. Hence, $\eta = By^*$ and this concludes the proof.

If $N = 2$, the solution is unique. Indeed, for two such solutions y_1, y_2 we have

$$\frac{1}{2}\frac{d}{dt}|y_1 - y_2|^2 + \nu_0\|y_1 - y_2\|^2 + b(y_1 - y_2, y_1, y_1 - y_2) = 0, \quad \text{a.e. } t \in (0,T).$$

This yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|y_1 - y_2|^2 + \nu_0\|y_1 - y_2\|^2 &\le C\|y_1 - y_2\|_{1/2}\|y_1\|\|y_1 - y_2\|_{1/2} \\
&\le C|y_1 - y_2|\|y_1 - y_2\|\|y_1\|.
\end{aligned}$$

By Gronwall's lemma, we get $y_1 = y_2$. □

Remark 5.6. The existence results presented in this section are classic and can be found in a slightly different form in the monographs of Temam [39], Constantin and Foias [19]. However, the semigroup approach used here is new and it closely follows the work of Barbu and Sritharan [6].

Perhaps the main advantage of the semigroup approach is that one can apply the general theory developed in Chapter 4 to get existence, regularity, and approximation results for Navier–Stokes equations.

In fact, as shown earlier, the Navier–Stokes flow $t \to y(t)$ is the restriction to $[0,T]$ of the flow $t \to y_M(t)$ generated by an equation of quasi-m-accretive type.

Bibliographical Remarks

There is an extensive literature on semilinear parabolic equations, parabolic variational inequalities, and the Stefan problem (see Lions [33], Duvaut and Lions [22], Friedman [27]and Elliott and Ockendon [23] for significant results and complete references on this subject). Here, we were primarily interested in the existence results that arise as direct consequences of the general theory developed previously, and we tried to put in perspective those models of free boundary problems that can be formulated as nonlinear differential equations of accretive type. The L^1-space semigroup approach to the nonlinear diffusion equation was initiated by Bénilan [8] (see also Konishi [29]), and the $H^{-1}(\Omega)$ approach is due to Brezis [15]. The smoothing effect of the semigroup generated by the semilinear elliptic operator in $L^1(\Omega)$ (Proposition 5.5) is due to Evans [24, 25]. The analogous result for the nonlinear diffusion operator in $L^1(\Omega)$ (Theorem 5.4) was first established by Bénilan [8], and Véron [41], but the proof given here is essentially due to Pazy [36]. For other related contributions to the existence and regularity of solutions to the porous medium equation, we refer to Bénilan, Crandall, and Pierre [10], and Brezis and Crandall [16]. The semigroup approach to the conservation law equation (Theorem 5.6) is due to Crandall [20]. Theorem 5.7 along with other existence results for abstract hyperbolic equations has been established by Brezis [15] (see also Haraux's book [28] and Barbu [4]). The semigroup approach to Navier–Stokes equations was developed in the works of Barbu [3] and Barbu and Sritharan [6] (see also Barbu and Sritharan [7] and Lefter [32] for other results in this direction).

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