Chapter 4
The Cauchy Problem in Banach Spaces

Abstract This chapter is devoted to the Cauchy problem associated with nonlinear quasi-accretive operators in Banach spaces. The main result is concerned with the convergence of the finite difference scheme associated with the Cauchy problem in general Banach spaces and in particular to the celebrated Crandall–Liggett exponential formula for autonomous equations, from which practically all existence results for the nonlinear accretive Cauchy problem follow in a more or less straightforward way.

4.1 The Basic Existence Results

Mild Solutions

Let X be a real Banach space with the norm $\|\cdot\|$ and dual X^* and let $A \subset X \times X$ be a quasi-accretive set of $X \times X$, or in other terminology, $A : D(A) \subset X \to X$ is an operator (eventually multivalued) such that $A + \omega I$ is accretive for some $\omega \in \mathbf{R}$. We refer to Section 3.1 for definitions and basic properties of quasi-accretive (or ω-accretive) operators.

Consider the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{4.1}$$

where $y_0 \in X$ and $f \in L^1(0,T;X)$.

Definition 4.1. A strong solution to (4.1) is a function $y \in W^{1,1}((0,T];X) \cap C([0,T];X)$ such that

$$f(t) - \frac{dy}{dt}(t) \in Ay(t), \quad \text{a.e. } t \in (0,T),\ y(0) = y_0.$$

Here, $W^{1,1}((0,T];X) = \{y \in L^1(0,T;X);\ y' \in L^1(\delta,T;X), \forall \delta \in (0,T)\}$.

V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4419-5542-5_4,

It is readily seen that any strong solution to (4.1) is unique and is a continuous function of f and y_0. More precisely, we have:

Proposition 4.1. *Let A be ω-accretive, $f_i \in L^1(0,T;X)$, $y_0^i \in \overline{D(A)}$, $i=1,2$, and let $y_i \in W^{1,1}((0,T];X)$, $i=1,2$, be corresponding strong solutions to problem* (4.1). *Then,*

$$\begin{aligned}\|y_1(t)-y_2(t)\| &\le e^{\omega t}\|y_0^1-y_0^2\|+\int_0^t e^{\omega(t-s)}[y_1(s)-y_2(s),f_1(s)-f_2(s)]_s ds\\ &\le e^{\omega t}\|y_0^1-y_0^2\|+\int_0^t e^{\omega(t-s)}\|f_1(s)-f_2(s)\|ds, \qquad \forall t\in[0,T].\end{aligned} \tag{4.2}$$

Here (see Proposition 3.7)

$$[x,y]_s = \inf_{\lambda>0}\lambda^{-1}(\|x+\lambda y\|-\|x\|) = \max\{(y,x^*);\ x^*\in\Phi(x)\} \tag{4.3}$$

and $\|x\|\Phi(x)=J(x)$ is the duality mapping of X; that is, $\Phi(x)=\partial\|x\|$.

The main ingredient of the proof is the following chain differentiation rule lemma.

Lemma 4.1. *Let $y=y(t)$ be an X-valued function on $[0,T]$. Assume that $y(t)$ and $\|y(t)\|$ are differentiable at $t=s$. Then,*

$$\|y(s)\|\,\frac{d}{ds}\,\|y(s)\| = \left(\frac{dy}{ds}(s),w\right), \qquad \forall w\in J(y(s)). \tag{4.4}$$

Here, $J: X\to X^*$ is the duality mapping of X.

Proof. Let $\varepsilon>0$. We have

$$(y(s+\varepsilon)-y(s),w) \le (\|y(s+\varepsilon)\|-\|y(s)\|)\|w\|, \qquad \forall w\in J(y(s)),$$

and this yields

$$\left(\frac{dy}{ds}(s),w\right) \le \frac{d}{ds}\,\|y(s)\|\,\|y(s)\|.$$

Similarly, from the inequality

$$(y(s-\varepsilon)-y(s),w) \le (\|y(s-\varepsilon)\|-\|y(s)\|)\|w\|,$$

we get

$$\left(\frac{d}{ds}y(s),w\right) \ge \frac{d}{ds}\,\|y(s)\|\,\|y(s)\|,$$

as claimed.

In particular, it follows by (4.4) that

$$\frac{d}{ds}\,\|y(s)\| = \left[y(s),\frac{dy}{ds}(s)\right]_s. \qquad \square \tag{4.5}$$

Proof of Proposition 4.1. We have

$$\frac{d}{ds}(y_1(s)-y_2(s))+Ay_1(s)-Ay_2(s)\ni f_1(s)-f_2(s), \quad \text{a.e. } s\in(0,T). \tag{4.6}$$

On the other hand, because A is ω-accretive, we have (see (3.16))

$$[y_1(s)-y_2(s),Ay_1(s)-Ay_2(s)]_s \geq -\omega\|y_1(s)-y_2(s)\|$$

and so, by (4.5) and (4.6), we see that

$$\frac{d}{ds}\|y_1(s)-y_2(s)\| \leq [y_1(s)-y_2(s),f_1(s)-f_2(s)]_s+\omega\|y_1(s)-y_2(s)\|, \quad \text{a.e. } s\in(0,T).$$

Then, integrating on $[0,t]$, we get (4.2), as claimed.

Proposition 4.1 shows that, as far as the uniqueness and continuous dependence of solution of data are concerned, the class of quasi-accretive operators A offers a suitable framework for the Cauchy problem. For this reason, such a nonlinear system is also called *quasi-accretive*. However, for the existence we must extend the notion of the solution for the Cauchy problem (4.1) from differentiable to continuous functions.

Definition 4.2. Let $f\in L^1(0,T;X)$ and $\varepsilon>0$ be given. An ε-*discretization* on $[0,T]$ of the equation $y'+Ay\ni f$ consists of a partition $0=t_0\leq t_1\leq t_2\leq\cdots\leq t_N$ of the interval $[0,t_N]$ and a finite sequence $\{f_i\}_{i=1}^N\subset X$ such that

$$t_i-t_{i-1}<\varepsilon \quad \text{for } i=1,\dots,N,\ T-\varepsilon<t_N\leq T, \tag{4.7}$$

$$\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\|f(s)-f_i\|ds<\varepsilon. \tag{4.8}$$

We denote by $D_A^\varepsilon(0=t_0,t_1,\dots,t_N;f_1,\dots,f_N)$ this ε-discretization.

A $D_A^\varepsilon(0=t_0,t_1,\dots,t_N;f_1,\dots,f_N)$ *solution* to (4.1) is a piecewise constant function $z:[0,t_N]\to X$ whose values z_i on $(t_{i-1},t_i]$ satisfy the finite difference equation

$$\frac{z_i-z_{i-1}}{t_i-t_{i-1}}+Az_i\ni f_i, \qquad i=1,\dots,N. \tag{4.9}$$

Such a function $z=\{z_i\}_{i=1}^N$ is called an ε-*approximate solution* to the Cauchy problem (4.1) if it further satisfies

$$\|z(0)-y_0\|\leq\varepsilon. \tag{4.10}$$

Definition 4.3. A *mild solution* of the Cauchy problem (4.1) is a function $y\in C([0,T];X)$ with the property that for each $\varepsilon>0$ there is an ε-approximate solution z of $y' + Ay \ni f$ on $[0,T]$ such that $\|y(t) - z(t)\| \le \varepsilon$ for all $t \in [0,T]$ and $y(0) = x$.

Let us note that every strong solution $y \in C([0,T];X) \cap W^{1,1}((0,T];X)$ to (4.1) is a mild solution. Indeed, let $0 = t_0 \le t_1 \le \cdots \le t_N$ be an ε-discretization of $[0,T]$ such that

$$\left\| \frac{d}{dt} y(t) - \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}} \right\| \le \varepsilon, \qquad t_i - t_{i-1} \le \delta,\ i = 1,2,\dots,N,$$

and

$$\int_{t_{i-1}}^{t_i} \|f(t) - f(t_i)\| dt \le \varepsilon(t_i - t_{i-1}).$$

Then, the step function $z : [0,T] \to X$ defined by $z = y(t_i)$ on $(t_{i-1}, t_i]$ is a solution to the ε-discretization D_A^ε $(0 = t_0, t_1, \dots, t_n; f_1, \dots, f_n)$, and, if we choose the discretization $\{t_j\}$ so that $\|y(t) - y(s)\| \le \varepsilon$ for $t,s \in (t_{i-1}, t_i)$, we have by (4.2) that $\|y(t) - z(t)\| \le \varepsilon$ for all $t \in [0,T]$, as claimed.

Theorem 4.1 below is the main result of this section.

Theorem 4.1. *Let A be ω-accretive, $y_0 \in \overline{D(A)}$, and $f \in L^1(0,T;X)$. For each $\varepsilon > 0$, let problem* (4.1) *have an ε-approximate solution. Then, the Cauchy problem* (4.1) *has a unique mild solution y. Moreover, there is a continuous function $\delta = \delta(\varepsilon)$ such that $\delta(0) = 0$ and if z is an ε-approximate solution of* (4.1)*, then*

$$\|y(t) - z(t)\| \le \delta(\varepsilon) \quad \textit{for} \ \ t \in [0, T - \varepsilon]. \tag{4.11}$$

Let $f, g \in L^1(0,T;X)$ and $y, \bar{y}$ be mild solutions to (4.1) *corresponding to f and g, respectively. Then,*

$$\begin{aligned} \|y(t) - \bar{y}(t)\| &\le e^{\omega(t-s)} \|y(s) - \bar{y}(s)\| \\ &+ \int_s^t e^{\omega(t-\tau)} [y(\tau) - \bar{y}(\tau), f(\tau) - g(\tau)]_s d\tau \\ &\textit{for} \ \ 0 \le s < t \le T. \end{aligned} \tag{4.12}$$

This important result, which represents the core of the existence theory of evolution processes governed by accretive operators is proved below in several steps. It is interesting that, as Theorem 4.1 amounts to saying, the existence of a unique mild solution for (4.1) is the consequence of two assumptions on A: ω-accretivity and existence of an ε-approximate solution. The latter is implied by the quasi-m-accretivity or a weaker condition of this type. Indeed, we have

Theorem 4.2. *Let C be a closed convex cone of X and let A be ω-accretive in $X \times X$ such that*

$$D(A) \subset C \subset \bigcap_{0<\lambda<\lambda_0} R(I + \lambda A) \quad \textit{for some}\ \lambda > 0. \tag{4.13}$$

Let $y_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$ be such that $f(t) \in C$, a.e. $t \in (0,T)$. Then, problem (4.1) *has a unique mild solution y. If y and $\bar{y}$ are two mild solutions to* (4.1) *corresponding to f and g, respectively, then*

$$\begin{aligned}\|y(t)-\bar{y}(t)\| &\le e^{\omega(t-s)}\|y(s)-\bar{y}(s)\| \\ &+\int_s^t e^{\omega(t-\tau)}[y(\tau)-\bar{y}(\tau),f(\tau)-g(\tau)]_s d\tau \quad \text{for } 0\le s<t\le T.\end{aligned} \tag{4.14}$$

Proof. Let $f \in L^1(0,T;X)$ and let f_i be the nodal approximation of f; that is,

$$f_i = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(s)ds, \qquad i = 1,2,...,N,$$

where $\{t_i\}_{i=1}^N$, $t_0 = 0$, is a partition of the interval $[0,t_N]$ such that $t_i - t_{i-1} < \varepsilon$, $t-\varepsilon < t_N < T$. By assumption (4.13), it follows that, for ε small enough, the function $z = z_i$ on $(t_{i-1},t_i]$, $z_0 = y_0$, is well defined by (4.9) and it is an ε-approximate solution to (4.1). (It is readily seen by assumption (4.2) and the ω-accretivity of A that equation (4.9) has a unique solution $\{z_i\}_{i=0}^N$.) Thus, Theorem 4.1 is applicable and so problem (4.1) has a unique solution satisfying (4.14). □

In particular, by Theorem 4.2 we obtain the following.

Corollary 4.1. *Let A be quasi-m-accretive. Then, for each $y_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$ there is a unique mild solution y to* (4.1).

In the sequel, we frequently refer to the map $(y_0, f) \to y$ from $\overline{D(A)} \times L^1(0,T;X)$ to $C([0,T];X)$ as the *nonlinear evolution associated with* A. It should be noted that, in particular, the range condition (4.13) holds if $C = X$ and A is ω-m-accretive in $X \times X$.

In the particular case when $f \equiv 0$, if A is ω-accretive and

$$R(I+\lambda A) \supset \overline{D(A)} \quad \text{for all small } \lambda > 0, \tag{4.15}$$

then we have, by Theorem 4.1:

Theorem 4.3 (Crandall and Liggett [24]). *Let A be ω-accretive, satisfying the range condition* (4.15) *and $y_0 \in \overline{D(A)}$. Then, the Cauchy problem*

$$\begin{aligned}&\frac{dy}{dt} + Ay \ni 0, \qquad t > 0, \\ &y(0) = y_0,\end{aligned} \tag{4.16}$$

has a unique mild solution y. Moreover,

$$y(t) = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} y_0 \tag{4.17}$$

uniformly in t on compact intervals.

Indeed, in this case, if $t_0 = 0$, $t_i = i\varepsilon$, $i = 1,...,N$, then the solution z_ε to the ε-discretization $D_A^\varepsilon(0 = t_0, t_1, ..., t_N)$ is given by the iterative scheme

$$z_\varepsilon(t) = (I + \varepsilon A)^{-i} y_0 \quad \text{for } t \in ((i-1)\varepsilon, i\varepsilon].$$

Hence, by (4.11), we have

$$\|y(t) - (I + \varepsilon A)^{-i} y_0\| \le \delta(\varepsilon) \quad \text{for } (i-1)\varepsilon < t \le i\varepsilon,$$

which implies the exponential formula (4.17) with uniform convergence on compact intervals. We note that, in particular, the range conditions (4.13) and (4.15) are automatically satisfied if A is quasi-m-accretive; that is, if $\omega I + A$ is m-accretive for some real ω. The solution y to (4.16) given by exponential formula (4.17) is also denoted by $e^{-At} y_0$.

Corollary 4.2. *Let A be quasi-m-accretive and* $y_0 \in \overline{D(A)}$. *Then the Cauchy problem* (4.16) *has a unique mild solution y given by the exponential formula* (4.17).

We now apply Theorem 4.2 to the mild solutions $y = y(t)$ and $\bar{y} = x$ to the equations

$$y' + Ay \ni f \quad \text{in } (0,T),$$

and

$$y' + Ay \ni v \quad \text{in } (0,T),\ v \in Ax,$$

respectively. We have, by (4.14),

$$\|y(t) - x\| \le e^{\omega(t-s)} \|y(s) - x\| + \int_s^t [y(\tau) - x, f(\tau) - v]_s e^{\omega(t-\tau)} d\tau, \qquad \forall\, 0 \le s < t \le T,\ [x,v] \in A. \tag{4.18}$$

Such a function $y \in C([0,T];X)$ is called an *integral solution* to equation (4.1).

We may conclude, therefore, that under the assumptions of Theorem 4.2 the Cauchy problem (4.1) has an integral solution, which coincides with the mild solution of this problem. On the other hand, it turns out that the integral solution is unique (see Bénilan and Brezis [11]) and under the assumptions of Theorem 4.2 (in particular, if A is ω-m-accretive) these two notions coincide.

It should be mentioned that in finite-dimensional spaces, Theorem 4.1 reduces to the classical Peano convergence scheme for solutions to the Cauchy problem which is valid for any continuous operator A. However, in infinite dimensions there are classical counterexamples which show that continuity alone is not enough for the existence of solutions. On the other hand, in most of significant infinite-dimensional examples the operator A is not continuous. This is the case with nonlinear boundary value problems of parabolic or hyperbolic type where the domain $D(A)$ of operator A is a proper subset of X and so A is unbounded. More is said about this in Chapter 5.

If X is the Euclidean space $\mathbf{R}^N$ and $A = \psi : \mathbf{R}^N \to \mathbf{R}^N$ is a measurable and monotone function; that is,

$$(\psi(x) - \psi(y), x - y)_N \ge 0, \qquad \forall x, y \in \mathbf{R}^N,$$

where $(\cdot,\cdot)_N$ is the scalar product of $\mathbf{R}^N$, then the Cauchy problem

$$\begin{aligned} &\frac{dy}{dt}(t) + \psi(y(t)) = 0, \qquad t \geq 0, \\ &y(0) = y_0 \end{aligned} \tag{4.19}$$

is not, generally, well posed.

This can be seen from the following elementary example

$$\frac{dy}{dt}(t) + \mathrm{sgn}_0 y(t) = 0, \qquad t \geq 0,\ y(0) = y_0,$$

where $\mathrm{sgn}_0 y = y/|y|$. However, if we replace ψ by the Filipov mapping

$$\widetilde{\psi}(x) = \bigcap_{\delta>0} \bigcap_{m(E)=0} \overline{\mathrm{conv}\ \psi(B_\delta(x) \setminus E)}, \qquad \forall x \in \mathbf{R}^N,$$

which, as seen in Proposition 2.5, is m-accretive in $\mathbf{R}^N \times \mathbf{R}^N$, then the corresponding Cauchy problem; that is,

$$\begin{aligned} &\frac{dy}{dt}(t) + \widetilde{\psi}(y(t)) \ni 0, \qquad t \geq 0, \\ &y(0) = y_0, \end{aligned}$$

has by Theorem 4.1 a unique solution y. This is the so-called Filipov solution to (4.19) which exists locally even for nonmonotone functions ψ.

Let us now come back to the proof of Theorem 4.1.

Let z be a solution to an ε-discretization $D_A^\varepsilon(0 = t_1, t_1, \ldots, t_N; f_1, \ldots, f_N)$ and let w be a solution to $D_A^\varepsilon(0 = s_0, s_1, \ldots, s_M; g_1, \ldots, g_M)$ with the nodal values z_i and w_j, respectively. We set $a_{ij} = \|z_i - w_j\|$, $\delta_i = (t_i - t_{i-1})$, $\gamma_j = (s_j - s_{j-1})$.

We begin with the following estimate for the solutions to finite difference scheme (4.7)–(4.9).

Lemma 4.2. *For all* $1 \leq i \leq N,\ 1 \leq j \leq M$, *we have*

$$\begin{aligned} a_{ij} \leq &\left(1 - \omega\, \frac{\delta_i \gamma_j}{\delta_i + \gamma_j}\right)^{-1} \left(\frac{\gamma_j}{\delta_i + \gamma_j}\, a_{i-1,j} + \frac{\delta_i}{\delta_i + \gamma_j}\, a_{i,j-1} \right. \\ &\left. + \frac{\delta_i \gamma_j}{\delta_i + \gamma_j}\, [z_i - w_j, f_i - g_j]_s \right). \end{aligned} \tag{4.20}$$

Moreover, for all $[x, v] \in A$ *we have*

$$a_{i,0} \leq \alpha_{i,1} \|z_0 - x\| + \|w_0 - x\| + \sum_{k=1}^{i} \alpha_{i,k} \delta_k (\|f_k\| + \|v\|), \qquad 0 \leq i \leq N, \tag{4.21}$$

and

$$a_{0,j} \leq \beta_{j,1} \|w_0 - x\| + \|z_0 - x\| + \sum_{k=1}^{j} \beta_{j,k} \gamma_k (\|g_k + \|v\|), \qquad 0 \leq j \leq M, \tag{4.22}$$

where

$$\alpha_{i,k} = \prod_{m=k}^{i} (1-\omega\delta_m)^{-1}, \qquad \beta_{j,k} = \prod_{m=k}^{j} (1-\omega\gamma_m)^{-1}. \tag{4.23}$$

Proof. We have

$$f_i + \delta_i^{-1}(z_{i-1} - z_i) \in Az_i, \qquad g_j + \gamma_j^{-1}(w_{j-1} - w_j) \in Aw_j, \tag{4.24}$$

and, because A is ω-accretive, this yields (see (3.16))

$$[z_i - w_j, f_i + \delta_i^{-1}(z_{i-1} - z_i) - g_j - \gamma_j^{-1}(w_{j-1} - w_j)]_s \geq -\omega\|z_i - w_j\|.$$

Hence,

$$\begin{aligned} -\omega\|z_i - w_j\| &\leq [z_i - w_j, f_i - g_j]_s + \delta_i^{-1}[z_i - w_j, z_{i-1} - z_i]_s \\ &+ \gamma_j^{-1}[z_i - w_j, w_j - w_{j-1}]_s \\ &\leq [z_i - w_j, f_i - g_j]_s - \delta_i^{-1}(\|z_i - w_j\| - \|z_{i-1} - w_j\|) \\ &- \gamma_j^{-1}(\|z_i - w_j\| - \|z_i - w_{j-1}\|), \end{aligned}$$

and rearrranging we obtain (4.20).

To get estimates (4.21), (4.22), we note that, inasmuch as A is ω-accretive, we have (see (3.3))

$$\|z_i - x\| \leq (1-\delta_i\omega)^{-1}\|z_i - x + \delta_i(f_i + \delta_i^{-1}(z_{i-1} - z_i) - v)\|,$$

respectively,

$$\|w_j - x\| \leq (1-\gamma_j\omega)^{-1}\|w_j - x + \gamma_j(g_j + \gamma_j^{-1}(w_{j-1} - w_j) - v)\|,$$

for all $[x,v] \in A$. Hence,

$$\begin{aligned} \|z_i - x\| &\leq (1-\delta_i\omega)^{-1}\|z_{i-1} - x\| + (1-\delta_i\omega)^{-1}\delta_i(\|f_i\| + \|v\|) \\ \|w_j - x\| &\leq (1-\gamma_j\omega)^{-1}\|w_{j-1} - x\| + (1-\gamma_j\omega)^{-1}\gamma_j(\|g_i\| + \|v\|) \end{aligned}$$

and (4.21), (4.22) follow by a simple calculation. □

In order to get, by (4.20), explicit estimates for a_{ij}, we invoke a technique frequently used in stability analysis of finite difference numerical schemes.

Namely, consider the functions ψ and φ on $[0,T]$ that satisfy the linear first order hyperbolic equation

$$\frac{\partial\psi}{\partial t}(t,s) + \frac{\partial\psi}{\partial s}(t,s) - \omega\psi(t,s) = \varphi(t,s) \quad \text{for } 0 \leq t \leq T,\ 0 \leq s \leq T, \tag{4.25}$$

and the boundary conditions

$$\psi(t,s) = b(t-s) \quad \text{for } t=0 \text{ or } s=0, \tag{4.26}$$

where $b \in C([-T,T])$ and φ is defined later on.

There is a close relationship between equation (4.25) and inequality (4.20). Indeed, let us define the grid

$$D = \{(t_i, s_j);\ 0 = t_0 \le t_1 \le \cdots \le t_N < T,\ 0 = s_0 \le s_1 \le \cdots \le s_M < T\}$$

and approximate (4.25) by the difference equations

$$\begin{aligned} &\frac{\psi_{i,j} - \psi_{i-1,j}}{\delta_i} + \frac{\psi_{i,j} - \psi_{i,j-1}}{\gamma_j} - \omega \psi_{ij} = \varphi_{i,j} \\ &\text{for } i = 1,\dots,N,\ j = 1,\dots,M, \end{aligned} \tag{4.27}$$

where $\delta_i = t_i - t_{i-1}$, $\gamma_j = s_j - s_{j-1}$, and $\varphi_{i,j}$ is a piecewise constant approximation of φ defined below. After some rearrangement we obtain

$$\begin{aligned} \psi_{i,j} = \left(1 - \omega \frac{\delta_i \gamma_j}{\delta_i + \gamma_j}\right)^{-1} &\left(\tfrac{\gamma_j}{\delta_i + \gamma_j} \psi_{i-1,j} + \tfrac{\delta_i}{\delta_i + \gamma_j} \psi_{i,j-1} + \tfrac{\delta_i \gamma_j}{\delta_i + \gamma_j} \varphi_{i,j} \right), \\ & i = 1,\dots,N,\ j = 1,\dots,M. \end{aligned} \tag{4.28}$$

In the following we take

$$\varphi(t,s) = \|f(t) - g(s)\|, \qquad \varphi_{i,j} = \|f_i - g_j\|, \qquad i = 1,\dots,N,\ j = 1,\dots,M,$$

where f_i and g_j are the nodal approximations of $f, g \in L^1(0,T;X)$, respectively.

Integrating equations (4.25) and (4.26), via the characteristics method, we get

$$\begin{aligned} &\psi(t,s) = G(b,\varphi)(t,s) \\ &= \begin{cases} e^{\omega s} b(t-s) + \displaystyle\int_0^s e^{\omega(s-\tau)} \varphi(t-s+\tau, \tau)\, d\tau & \text{if } 0 \le s < t \le T, \\ e^{\omega t} b(t-s) + \displaystyle\int_0^t e^{\omega(t-\tau)} \varphi(\tau, s-t+\tau)\, d\tau & \text{if } 0 \le t < s \le T. \end{cases} \end{aligned} \tag{4.29}$$

We set $\Omega = (0,T) \times (0,T)$, and for every measurable function $\varphi : [0,T] \times [0,T] \to \mathbf{R}$ we set

$$\begin{aligned} \|\varphi\|_\Omega = \inf\{\|f\|_{L^1(0,T)} + \|g\|_{L^1(0,T)};\ |\varphi(t,s)| \le |f(t)| + |g(s)|,& \\ \text{a.e. } (t,s) \in \Omega\}.& \end{aligned} \tag{4.30}$$

Let $\Omega(\Delta) = [0,t_N] \times [0,s_M]$ and $B : [-s_M, t_N] \to \mathbf{R}$, $\phi : \Omega(\Delta) \to \mathbf{R}$ be piecewise constant functions; that is, here are $b_{i,j}, \phi_{i,j} \in \mathbf{R}$ such that $b(0) = B(0)$ and

$$\begin{aligned} B(r+s) &= b_{ij} \quad \text{for } t_{i-1} < r \le r_i,\ -s_j \le s < -s_{j-1}, \\ \phi(t,s) &= \phi_{i,j} \quad \text{for } (t,s) \in (t_{i-1}, t_i] \times (s_{j-1}, s_j]. \end{aligned}$$

Observe, by (4.29), via the Banach fixed point theorem, that if the mesh $m(\Delta) = \max\{(\delta_i, \gamma_j);\ i,j\}$ of Δ is sufficiently small, then the system (4.28) with the boundary value conditions

$$\psi_{i,j} = b_{i,j} \quad \text{for } i=0 \text{ or } j=0, \tag{4.31}$$

has a unique solution $\{\psi_{ij}\}$, $i=1,\dots,N,\ j=1,\dots,M$.

Denote by $\Psi = H_\Delta(B,\phi)$ the piecewise constant function on Ω defined by

$$\Psi = \psi_{i,j} \quad \text{on } (t_{i-1},t_i] \times (s_{j-1},s_j]; \tag{4.32}$$

that is, the solution to (4.28), (4.31).

Lemma 4.3 below provides the convergence of the finite difference scheme (4.27), (4.31) as $m(\Delta) \to 0$.

Lemma 4.3. *Let $b \in C([-T,T])$ and $\varphi \in L^1(\Omega)$ be given. Then,*

$$\|G(b,\varphi) - H_\Delta(B,\phi)\|_{L^\infty(\Omega(\Delta))} \to 0 \tag{4.33}$$

as

$$m(\Delta) + \|b - B\|_{L^\infty(-s_M,t_N)} + \|\varphi - \phi\|_{\Omega(\Delta)} \to 0.$$

Proof. In order to avoid a tedious calculus, we prove (4.33) in the accretive case only (i.e., $\omega = 0$).

Let us prove first the estimate

$$\|H_\Delta(B,\phi)\|_{L^\infty(\Omega(\Delta))} \le \|B\|_{L^\infty(-s_M,t_N)} + \|\phi\|_{\Omega(\Delta)}. \tag{4.34}$$

Indeed, we have $H_\Delta(B,\phi) = H_\Delta(B,0) + H_\Delta(0,\phi)$, and by (4.30), (4.32) we see that the values of $H_\Delta(B,0)$ are convex combinations of the values of B.

Hence,

$$\|H_\Delta(B,0)\|_{L^\infty(\Omega(\Delta))} \le \|B\|_{L^\infty(-s_M,t_N)}.$$

It remains to show that

$$\|H_\Delta(0,\phi)\|_{L^\infty(\Omega(\Delta))} \le \|\phi\|_{\Omega(\Delta)}.$$

By the definition (4.30) of the $\|\cdot\|_{\Omega(\Delta)}$-norm, we have

$$\|\phi\|_{\Omega(\Delta)} = \inf\left\{\sum_{i=1}^N \delta_i\alpha_i + \sum_{j=1}^M \gamma_j\beta_j;\ \alpha_i + \beta_j \ge |\phi_{i,j}|,\ \alpha_i,\beta_j \ge 0\right\}.$$

Now, let $g_{i,j} = \alpha_i + \beta_j \ge |\phi_{i,j}|$ and set

$$d_{i,j} = \sum_{k=1}^i \alpha_k\delta_k + \sum_{k=1}^j \beta_k\gamma_k.$$

It is readily seen that $\psi_{i,j} = d_{i,j}$ satisfy the system (4.28) where $\phi_{i,j} = g_{i,j}$. Hence, $d = H_\Delta(\widetilde{B}, g)$ provided $d_{i,j} = \widetilde{b}_{i,j}$ for $i = 0$ or $j = 0$, where $d = \{d_{i,j}\}$, $\widetilde{B} = \{\widetilde{b}_{i,j}\}$ and $g = \{g_{i,j}\}$. Inasmuch as $g_{i,j} \geq |\phi_{i,j}|$, we have

$$d = H_\Delta(\widetilde{B}, g) \geq H_\Delta(0, \phi) \geq |H_\Delta(0, \phi)$$

if $b_{i,j} \geq 0$. Hence,

$$\|H_\Delta(0,\phi)\|_{L^\infty(\Omega(\Delta))} \leq \|d\|_{L^\infty(\Omega(\Delta))} \leq \|\phi\|_{\Omega(\Delta)},$$

as claimed.

Now, let $\widetilde{\psi} = G(\widetilde{b}, \widetilde{\varphi})$ and assume first that $\widetilde{\psi}_{tt}, \widetilde{\psi}_{ss} \in L^\infty(\Omega)$. Then, by (4.25) we see that $\widetilde{\psi}_{i,j} = \widetilde{\psi}(t_i, s_j)$ satisfy the system

$$\frac{\widetilde{\psi}_{i,j} - \widetilde{\psi}_{i-1,j}}{\delta_i} + \frac{\widetilde{\psi}_{i,j} - \widetilde{\psi}_{i,j-1}}{\gamma_j} = \widetilde{\varphi}_{i,j} + e_{i,j}, \qquad \widetilde{\psi}_{i,0} = \widetilde{b}(t_i), \qquad \widetilde{\psi}_{0,j} = \widetilde{b}(-s_j),$$
$$i = 0, 1, ..., N, \; j = 0, 1, ..., M,$$

where $e = \{e_{ij}\}$ satisfies the estimate

$$|e_{ij}| \leq \gamma_j \|\widetilde{\psi}_{ss}\|_{L^\infty(\Omega)} + \delta_i \|\widetilde{\psi}_{tt}\|_{L^\infty(\Omega)}, \qquad \forall i, j.$$

Then, by (4.34), this yields

$$\begin{aligned} &\|G(\widetilde{b}, \widetilde{\varphi}) - H_\Delta(B, \phi)\|_{L^\infty(\Omega(\Delta))} \\ &\leq \|B - \widetilde{b}\|_{L^\infty(-s_M, t_N)} + \|\widetilde{\varphi} - \phi\|_{\Omega(\Delta)} + \|e\|_{\Omega(\Delta)} \\ &\leq \|B - \widetilde{b}\|_{L^\infty(-s_M, t_N)} + \|\widetilde{\varphi} - \phi\|_{\Omega(\Delta)} \\ &\quad + Cm(\Omega)(\|\widetilde{\psi}_{tt}\|_{L^\infty(\Omega)} + \|\widetilde{\psi}_{ss}\|_{L^\infty(\Omega)}). \end{aligned} \tag{4.35}$$

Now, let $\varphi \in L^1(\Omega)$, $b \in C([-T, T])$, and $\widetilde{b} \in C^2([-T, T])$, $\widetilde{\varphi} \in C^2(\widetilde{\Omega})$. Then, $\widetilde{\psi} = G(\widetilde{b}, \widetilde{\varphi})$ is smooth, and by (4.35) we have

$$\begin{aligned} &\|G(b, \varphi) - H_\Delta(B, \phi)\|_{L^\infty(\Omega(\Delta))} \\ &\leq \|G(b, \varphi) - G(\widetilde{b}, \widetilde{\varphi})\|_{L^\infty(\Omega(\Delta))} + \|G(\widetilde{b}, \widetilde{\varphi}) - H_\Delta(B, \phi)\|_{L^\infty(\Omega(\Delta))} \\ &\leq 2\|b - \widetilde{b}\|_{L^\infty(-s_M, t_N)} + C\|\varphi - \widetilde{\varphi}\|_{\Omega(\Delta)} + \|B - b\|_{L^\infty(-s_M, t_N)} \\ &\quad + \|\widetilde{\varphi} - \phi\|_{\Omega(\Delta)} + Cm(\Delta)(\|\widetilde{\psi}_{tt}\|_{L^\infty(\Omega)} + \|\widetilde{\psi}_{ss}\|_{L^\infty(\Omega)}). \end{aligned} \tag{4.36}$$

Given $\eta > 0$, we may choose $\widetilde{b}$ and $\widetilde{\varphi}$ such that $\|b - \widetilde{b}\|_{L^\infty(-s_M, t_N)}$, $\|\varphi - \widetilde{\varphi}\|_{\Omega(\Delta)} \leq \eta$. Then (4.36) implies (4.33), as desired. □

Proof of Theorem 4.1 (Continued). We apply Lemma 4.3, where $\varphi(t,s) = \|f(t) - g(s)\|$, $\phi = \{\phi_{i,j}\}$, $\phi_{i,j} = \|f_i - g_j\|$, $1 \leq j \leq M$, $1 \leq i \leq N$, f_i and g_j are the nodal values of f and g, respectively, and

$$B(t) = b_{i,0} \quad \text{for } t_{i-1} < t \le t_i,\ i = 1,...,N,$$
$$B(s) = b_{0,j} \quad \text{for } -s_j < s \le -s_{j-1},\ j = 1,...,M.$$

Here, $b_{i,0}$ is the right-hand side of (4.21) and $b_{0,j}$ is the right-hand side of (4.22). It is easily seen that, for $\varepsilon \to 0$,

$$B(t) \to b(t) = e^{\omega t}\|z_0 - x\| + \|w_0 - x\| + \int_0^t e^{\omega(t-\tau)}(\|f(\tau)\| + \|v\|)d\tau, \quad \forall t \in [0,T],$$

and

$$B(s) \to b(-s) = e^{\omega s}\|w_0 - x\| + \|z_0 - x\| + \int_0^s e^{\omega(s-\tau)}(\|g(\tau)\| + \|v\|)d\tau, \quad \forall s \in [-T,0].$$

By (4.8), we have

$$\|\varphi - \phi\|_{\Omega(\Delta)} \le 2\varepsilon$$

and, by Lemma 4.2,

$$a_{i,j} = \|z_i - w_j\| \le H_\Delta(B,\phi)_{i,j}, \qquad \forall i, j.$$

Then, by Lemma 4.3, we see that, for every $\eta > 0$, we have

$$\|z(t) - w(s)\| \le G(b,\varphi)(t,s) + \eta, \qquad \forall s,t \in [0,T], \tag{4.37}$$

as soon as $0 < \varepsilon < \nu(\eta)$.

If $f \equiv g$ and $z_0 = w_0$, then $G(b,\varphi)(t,t) = e^{\omega t}b(0) = 2e^{\omega t}\|z_0 - x\|$ and so, by (4.37),

$$\|z(t) - w(t)\| \le \eta + 2e^{\omega t}\|z - x\|, \qquad \forall x \in D(A),\ t \in [0,T],$$

for all $0 < \varepsilon \le \nu(\eta)$. Because $\|z_0 - s_0\| \le \varepsilon$, $y_0 \in \overline{D(A)}$, and x is arbitrary in $D(A)$, it follows that the sequence z_ε of ε-approximate solutions satisfies the Cauchy criterion and so $y(t) = \lim_{\varepsilon\to 0} z_\varepsilon(t)$ exists uniformly on $[0,T]$. Now, we take the limit as $\varepsilon \to 0$ in (4.36) with $s = t + h$, $g \equiv f$, and $z_0 = w_0 = y_0$. We get

$$\|y(t+h) - y(t)\| \le G(b,\varphi)(t+h,t) = e^{\omega t}(e^{\omega h} + 1)\|y_0 - x\|$$
$$+ \int_0^h e^{\omega(h-\tau)}(\|f(\tau)\| + \|v\|)d\tau + \int_0^t e^{\omega(t-\tau)}\|f(\tau+h) - f(\tau)\|d\tau, \qquad \forall [x,v] \in A,$$

and therefore y is continuous on $[0,T]$. □

Now, by (4.37) we have, for $f \equiv g,\ t = s$,

$$\|z(t) - y(t)\| \le \delta(\varepsilon), \qquad \forall t \in [0,T],$$

where z is any ε-approximate solution and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, we take $t = s$ in (4.37) and let ε tend to zero. Then, by (4.29), we get the inequality

$$\|y(t) - \bar{y}(t)\| \le e^{\omega t}\|y(0) - \bar{y}(0)\| + \int_0^t e^{\omega(t-\tau)}\|f(\tau) - g(\tau)\|d\tau.$$

To obtain (4.12), we apply inequality (4.37), where

$$\varphi(t,s) = [y(t) - \bar{y}(t), f(t) - g(s)]_s \quad \text{and} \quad t = s.$$

Then, by (4.29), we see that

$$G(h,\varphi)(t,t) = e^{\omega t}\|y(0) - \bar{y}(0)\| + \int_0^t e^{\omega(t-s)}[y(s) - \bar{y}(s), f(s) - g(s)]_s ds,$$

and so (4.12) follows for $s = 0$ and, consequently, for all $s \in (0,t)$.

Thus, the proof of Theorem 4.1 is complete.

The convergence theorem can be made more precise for the autonomous equation (4.16); that is, for $f \equiv 0$.

Corollary 4.3. *Let A be ω-accretive and satisfy condition* (4.15)*, and let $y_0 \in \overline{D(A)}$. Let y be the mild solution to problem* (4.16) *and let y_ε be an ε-approximate solution to* (4.16) *with $y_\varepsilon(0) = y_0$. Then,*

$$\|y_\varepsilon(t) - y(t)\| \le C_T(\|y_0 - x\| + |Ax|(\varepsilon + t^{1/2}\varepsilon^{1/2})), \qquad \forall t \in [0,T], \tag{4.38}$$

for all $x \in D(A)$. In particular, we have

$$\left\| y(t) - \left(I + \frac{t}{n}A\right)^{-n} y_0 \right\| \le C_T(\|y_0 - x\| + tn^{1/2}|Ax|) \tag{4.38$'$}$$

for all $t \in [0,T]$ and $x \in D(A)$. Here, C_T is a positive constant independent of x and y_0 and $|Ax| = \inf\{\|z\|;\ z \in Ax\}$.

Proof. The mappings $y_0 \to y$ and $y_0 \to y_\varepsilon$ are Lipschitz continuous with Lipschitz constant $e^{\omega T}$, thus it suffices to prove estimate (4.38) for $y_0 \in D(A)$.

By estimate (4.36), we have, for all $T > 0$,

$$\begin{aligned}
&\|G(b,0) - H_\Delta(B,0)\|_{L^\infty(\Omega(\Delta))} \\
&\le \|b - \widetilde{b}\|_{L^\infty(-T,T)} + \|B - \widetilde{b}\|_{L^\infty(-T,T)} + C\varepsilon(\|\widetilde{\psi}_{tt}\|_{L^\infty(\Omega)} + \|\widetilde{\psi}_{ss}\|_{L(\Omega)}),
\end{aligned}$$

where $\overline{\psi} = G(\widetilde{b},0)$, $\widetilde{b}$ is a sufficiently smooth function on $[-T,T]$, $\Omega = (0,T) \times (0,T)$, and C is independent of ε, b, and B. We apply this inequality for B and b as in the proof of Theorem 4.1; that is,

$$b(t) = \omega^{-1}(e^{\omega|t|} - 1)|Ax|, \qquad \forall t \in [-T,T].$$

Then, we have

$$b'(t) = e^{\omega|t|}|Ax|\,\mathrm{sign}\,t,$$

and we approximate the signum function $\mathrm{sign}\,t$ by

$$\theta(t) = \begin{cases} \dfrac{t}{\lambda} & \text{for } |t| \le \lambda, \\ \dfrac{t}{|t|} & \text{for } |t| > \lambda, \end{cases}$$

and so, we construct a smooth approximation $\widetilde{b}$ of b such that

$$\widetilde{b}(0) = 0, \widetilde{b}'(t) = e^{\omega|t|}Ax\theta(t),$$

and

$$\widetilde{b}''(t) = \omega\theta(t)|Ax|e^{\omega|t|} + \theta'(t)|Ax|e^{\omega|t|}.$$

Hence,

$$\sup\{|\widetilde{b}''(s)|;\ 0 \le s \le t\} \le e^{\omega|t|}|Ax|(\omega + \lambda^{-1})$$

and, therefore,

$$\begin{aligned}&\|b - \widetilde{b}\|_{L^\infty(-t,t)} + C\varepsilon(\|\widetilde{\psi}_{tt}\|_{L^\infty((0,t)\times(0,t))} + \|\widetilde{\psi}_{ss}\|_{L^\infty((0,t)\times(0,t))})\\ &\le Ct\varepsilon|Ax|(1+\lambda^{-1}) + C\lambda|Ax|, \qquad \forall t \in [0,T],\end{aligned}$$

where C depends on T only.

Similarly, we have

$$\|B - \widetilde{b}\|_{L^\infty(-t,t)} \le C(\varepsilon + \lambda)|Ax|.$$

Finally,

$$\|G(b,0) - H_\Delta(B,0)\|_{L^\infty(\Omega_t(\Delta))} \le C(\varepsilon + \lambda + t\varepsilon\lambda^{-1})|Ax|,$$

where $\Omega_t = (0,t) \times (0,t)$. This implies that (see the proof of Theorem 4.1)

$$\|y_\varepsilon(t) - y(t)\| \le G(b,0)(t,t) + C|Ax|(\varepsilon + \lambda + t\varepsilon\lambda^{-1})$$

for all $t \in [0,T]$ and all $\lambda > 0$. For $\lambda = (t\varepsilon)^{1/2}$, this yields

$$\|y_\varepsilon(t) - y(t)\| \le C|Ax|(\varepsilon + t^{1/2}\varepsilon^{1/2}), \qquad \forall t \in [0,T],$$

which completes the proof. □

Regularity of Mild Solutions

A question of great interest is that of circumstances under which the mild solutions are strong solutions. One may construct simple examples which show that in a general Banach space this might be false. However, if the space is reflexive, then under natural assumptions on A, f, and y_ε the answer is positive.

Theorem 4.4. *Let X be reflexive and let A be closed and ω-accretive, and let A satisfy assumption* (4.13). *Let $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];X)$ be such that $f(t) \in C$, $\forall t \in [0,T]$. Then, problem* (4.1) *has a unique mild strong solution y which is strong solution and $y \in W^{1,\infty}([0,T];X)$. Moreover, y satisfies the estimate*

$$\left\| \frac{dy}{dt}(t) \right\| \le e^{\omega t}|f(0)-Ay_0| + \int_0^t e^{\omega(t-s)} \left\| \frac{df}{ds}(s) \right\| ds, \quad \text{a.e. } t \in (0,T), \tag{4.39}$$

where $|f(0)-Ay_0| = \inf\{\|w\|;\ w \in f(0)-Ay_0\}$.

In particular, we have the following theorem.

Theorem 4.5. *Let X be a reflexive Banach space and let A be an ω-m-accretive operator. Then, for each $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];X)$, problem* (4.1) *has a unique strong solution $y \in W^{1,\infty}([0,T];X)$ that satisfies estimate* (4.39).

Proof of Theorem 4.4. Let y be the mild solution to problem (4.1) provided by Theorem 4.2. We apply estimate (4.14), where $y(t) := y(t+h)$ and $g(t) := f(t+h)$. We get

$$\begin{aligned}\|y(t+h)-y(t)\| &\le \|y(h)-y(0)\|e^{\omega t} + \int_0^t \|f(s+h)-f(s)\|e^{\omega(t-s)}ds \\ &\le Ch + \|y(h)-y(0)\|e^{\omega t},\end{aligned}$$

because $f \in W^{1,1}([0,T];X)$ (see Theorem 1.18 and Remark 1.2). Now, applying the same estimate (4.14) to y and y_0, we get

$$\|y(h)-y_0\| \le \int_0^h \|f(s)-\xi\|e^{\omega(h-s)}ds \le \int_0^h |Ay_0 - f(s)|ds, \quad \forall \xi \in Ay_0,\ h \in [0,T].$$

We may conclude, therefore, that the mild solution y is Lipschitz on $[0,T]$. Then, by Theorem 1.17, it is, a.e., differentiable and belongs to $W^{1,\infty}([0,T];X)$. Moreover, we have

$$\left\| \frac{dy}{dt}(t) \right\| = \lim_{h\to 0} \frac{\|y(t+h)-y(t)\|}{h} \le e^{\omega t}|Ay_0 - f(0)| + \int_0^t \left\| \frac{df}{ds}(s) \right\| e^{\omega(t-s)}ds, \quad \text{a.e. } t \in (0,T).$$

Now, let $t \in [0,T]$ be such that

$$\frac{dy}{dt}(t) = \lim_{h\to 0} \frac{1}{h}(y(t+h)-y(t))$$

exists. By inequality (4.18), we have

$$\|y(t+h)-x\| \leq e^{\omega h}\|y(t)-x\| + \int_t^{t+h} e^{\omega(t+h-s)}[y(\tau)-x, f(\tau)-w]_s d\tau,$$
$$\forall [x,w] \in A.$$

Noting that

$$[v-x, u-v]_s \leq \|u-x\| - \|v-x\|, \qquad \forall u,v,x \in X,$$

we get

$$[y(t)-x, y(t+h)-y(t)]_s$$
$$\leq (e^{\omega h}-1)\|y(t)-x\| + \int_t^{t+h} e^{\omega(t+h-\tau)}[y(\tau)-x, f(\tau)-w]_s d\tau.$$

Because the bracket $[u,v]_s$ is upper semicontinuous in (u,v), and positively homogeneous and continuous in v (see Proposition 3.7), this yields

$$\left[y(t)-x, \frac{dy}{dt}(t)\right]_s - \omega\|y(t)-x\| \leq [y(t)-x, f(t)-w]_s, \qquad \forall [x,w] \in A.$$

Taking into account part (v) of Proposition 3.7, this implies that there is $\xi \in J(y(t)-x)$ such that (J is the duality mapping)

$$\left(\frac{dy}{dt}(t) - \omega(y(t)-x) - f(t) - w, \xi\right) \leq 0. \tag{4.40}$$

Inasmuch as the function y is differentiable in t, we have

$$y(t-h) = y(t) - h\frac{d}{dt}y(t) + hg(h), \tag{4.41}$$

where $g(h) \to 0$ for $h \to 0$. On the other hand, by condition (4.13), for every h sufficiently small and positive, there are $[x_h, w_h] \in A$ such that

$$y(t-h) + hf(t) = x_h + hw_h.$$

Substituting successively in (4.30) and in (4.41) we get

$$(1-\omega h)\|y(t)-x_h\| \leq h\|g(h)\|, \qquad \forall h \in (0, \lambda_0).$$

Hence, $x_h \to y(t)$ and $w_h \to f(t) - dy(t)/dt$ as $h \to 0$. Because A is closed, we conclude that

$$\frac{dy}{dt}(t) + Ay(t) \ni f(t),$$

as claimed.

Remark 4.1. In particular, Theorems 4.1–4.5 remain true for equations of the form

$$\begin{cases} \frac{dy}{dt}(t) + Ay(t) + Fy(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{4.42}$$

where A is m-accretive in $X \times X$ and $F : X \to X$ is Lipschitzian. Indeed, in this case, as easily seen, the operator $A + F$ is quasi-m-accretive; that is, $A + F + \omega I$ is m-accretive for $\omega = \|F\|_{\text{Lip}}$.

More can be said about the regularity of a strong solution to problem (4.1) if the space X is uniformly convex.

Theorem 4.6. *Let A be ω-m-accretive, $f \in W^{1,1}([0,T];X)$, $y_0 \in D(A)$ and let X be uniformly convex along with the dual X^*. Then, the strong solution to problem* (4.1) *is everywhere differentiable from the right, $(d^+/dt)y$ is right continuous, and*

$$\frac{d^+}{dt} y(t) + (Ay(t) - f(t))^0 = 0, \qquad \forall t \in [0,T), \tag{4.43}$$

$$\left\| \frac{d^+}{dt} y(t) \right\| \le e^{\omega t} \|(Ay_0 - f(0))^0\| + \int_0^t e^{\omega(t-s)} \left\| \frac{df}{ds}(s) \right\| ds, \qquad \forall t \in [0,T). \tag{4.44}$$

Here, $(Ay - f)^0$ is the element of minimum norm in the set $Ay - f$.

Proof. Because X and X^* are uniformly convex, Ay is a closed convex subset of X for every $x \in D(A)$ (see Section 3.1) and so, $(Ay(t) - f(t))^0$ is well defined.

Let $y \in W^{1,\infty}([0,T];X)$ be the strong solution to (4.1). We have

$$\frac{d}{dh}(y(t+h) - y(t)) + Ay(t+h) \ni f(t+h), \quad \text{a.e. } h > 0,\ t \in (0,T),$$

and because A is ω-accretive, this yields

$$\left(\frac{d}{dh}(y(t+h) - y(t)), \xi \right) \le \omega \|y(t+h) - y(t)\|^2 + (f(t+h) - \eta(t), \xi), \qquad \forall \eta(t) \in Ay(t),$$

where $\xi = J(y(t+h) - y(t))$.

Then, by Lemma 4.1, we get

$$\|y(t+h) - y(t)\| \le \int_0^h e^{\omega(h-s)} \|\eta(t) - f(t+s)\| ds, \tag{4.45}$$

which yields

$$\left\| \frac{dy}{dt}(t) \right\| \le \|f(t) - \eta(t)\|, \qquad \forall \eta(t) \in Ay(t), \quad \text{a.e. } t \in (0,T).$$

In other words,

$$\left\| \frac{dy}{dt}(t) \right\| \leq \|(Ay(t) - f(t))^0\|, \quad \text{a.e. } t \in (0,T),$$

and because $dy(t)/dt + Ay(t) \ni f(t)$, a.e. $t \in (0,T)$, we conclude that

$$\frac{dy}{dt}(t) + (Ay(t) - f(t))^0 = 0, \quad \text{a.e. } t \in (0,T). \tag{4.46}$$

Observe also that, for all h, y satisfies the equation

$$\frac{d}{dt}(y(t+h) - y(t)) + Ay(t+h) - Ay(t) \ni f(t+h) - f(t), \quad \text{a.e. in } (0,T).$$

Multiplying this equation by $J(y(t+h) - y(t))$ and using the ω-accretivity of A, we see by Lemma 4.1 that

$$\frac{d}{dt}\|y(t+h) - y(t)\| \leq \omega\|y(t+h) - y(t)\| + \|f(t+h) - f(t)\|,$$
$$\text{a.e. } t, t+h \in (0,T),$$

and therefore

$$\begin{aligned} &\|y(t+h) - y(t)\| \\ &\leq e^{\omega(t-s)}\|y(s+h) - y(s)\| + \int_s^t e^{\omega(t-\tau)}\|f(\tau+h) - f(\tau)\| d\tau. \end{aligned} \tag{4.47}$$

Finally,

$$\begin{aligned} \left\| \frac{dy}{dt}(t) \right\| &\leq e^{\omega(t-s)} \left\| \frac{dy}{ds}(s) \right\| + \int_s^t e^{\omega(t-\tau)} \left\| \frac{df}{d\tau}(\tau) \right\| d\tau, \\ &\qquad \text{a.e. } 0 < s < t < T. \end{aligned} \tag{4.48}$$

Similarly, multiplying the equation

$$\frac{d}{dt}(y(t) - y_0) + Ay(t) \ni f(t), \quad \text{a.e. } t \in (0,T),$$

by $J(y(t) - y_0)$ and, integrating on $(0,t)$, we get the estimate

$$\|y(t) - y_0\| \leq \int_0^t e^{\omega(t-s)}\|(Ay_0 - f(s))^0\| ds, \qquad \forall t \in [0,T], \tag{4.49}$$

and, substituting in (4.47) with $s = 0$, we get

$$\begin{aligned} \left\| \frac{d}{dt} y(t) \right\| &\leq e^{\omega t}\|(Ay_0 - f(0))^0\| + \int_0^t e^{\omega(t-s)} \left\| \frac{df}{ds}(s) \right\| ds, \\ &\qquad \text{a.e. } t \in (0,T). \end{aligned} \tag{4.50}$$

Because A is demiclosed (see Proposition 3.4) and X is reflexive, it follows by (4.46) and (4.50) that $y(t) \in D(A)$, $\forall t \in [0,T]$, and

$$\|(Ay(t) - f(t))^0\| \le C, \qquad \forall t \in [0,T]. \tag{4.51}$$

Let us show now that (4.46) extends to all $t \in [0,T]$. For t arbitrary but fixed in $[0,T]$, consider $h_n \to 0$ such that $h_n > 0$ for all n and

$$\frac{y(t+h_n) - y(t)}{h_n} \rightharpoonup \xi \quad \text{in } X \text{ as } n \to 0.$$

By (4.46) and the previous estimates, we see that

$$\|\xi\| \le \|(Ay(t) - f(t))^0\|, \qquad \forall t \in [0,T], \tag{4.52}$$

and $\xi \in f(t) - Ay(t)$ because A is demiclosed. Indeed, we have

$$f(t) - \xi = w - \lim_{n\to\infty} \frac{1}{h_n} \int_t^{t+h_n} \eta(s)ds,$$

where $\eta \in L^\infty(0,T;X)$ and $\eta(t) \in Ay(t)$, $\forall t \in [0,T]$.

We set $\eta_n(s) = \eta(t + sh_n)$ and $y_n(s) = y(t + sh_n)$. If we denote again by A the realization of A in $L^2(0,T;X) \times L^2(0,T;X)$, we have $y_n \to y(t)$ in $L^2(0,T;X)$, $\eta_n \to f(t) - \xi$ weakly in $L^2(0,T;X)$.

Because A is demiclosed in $L^2(0,T;X) \times L^2(0,T;X)$ we have that $f(t) - \xi \in Ay(t)$, as claimed. Then, by (4.52) we conclude that $\xi = (Ay(t) - f(t))^0$ and, therefore,

$$\frac{d^+}{dt} y(t) = \lim_{h\downarrow 0} \frac{y(t+h) - y(t)}{h} = -(Ay(t) - f(t))^0, \qquad \forall t \in [0,T).$$

Next, we see by (4.47) that

$$\left\| \frac{d^+}{dt} y(t) \right\| \le e^{\omega(t-s)} \left\| \tfrac{d^+}{dt} y(s) \right\| + \textstyle\int_s^t e^{\omega(t-\tau)} \left\| \tfrac{df}{d\tau}(\tau) \right\| d\tau, \qquad 0 \le s \le t \le T. \tag{4.53}$$

Let $t_n \to t$ be such that $t_n > t$ for all n. Then, on a subsequence, again denoted by t_n,

$$\frac{d^+ y(t_n)}{dt} = -(Ay(t_n) - f(t_n))^0 \rightharpoonup \xi,$$

where $-\xi \in Ay(t) - f(t)$ (because A is demiclosed). On the other hand, it follows by (4.53) that

$$\|\xi\| \le \limsup_{n\to\infty} \|(Ay(t_n) - f(t_n))^0\| \le \|(Ay(t(-f(t))^0\|.$$

Hence, $\xi = -(Ay(t) - f(t))^0$ and $(d^+/dt)y(t_n) \to \xi$ strongly in X (because X is uniformly convex). We have, therefore, proved that $(d^+/dt)y(t)$ is right continuous on $[0,T)$, thereby completing the proof. □

In particular, it follows by Theorem 4.6 that, if A is quasi-m-accretive, $y_0 \in D(A)$, and X, X^* are uniformly convex, then the solution y to the autonomous problem (4.16) is everywhere differentiable from the right and

$$\frac{d^+}{dt}y(t) + A^0 y(t) = 0, \qquad \forall t \geq 0, \tag{4.54}$$

where A^0 is the minimal section of A. Moreover, the function $t \to A^0 y(t)$ is continuous from the right on $\mathbf{R}^+$.

It turns out that this result remains true under weaker conditions on A. Namely, one has the following.

Theorem 4.7. *Let A be ω-accretive, closed, and satisfy the condition*

$$\overline{\operatorname{conv} D(A)} \subset \bigcap_{0<\lambda<\lambda_0} R(I+\lambda A) \quad \textit{for some } \lambda_0 > 0. \tag{4.55}$$

Let X and X^* be uniformly convex. Then, for every $x \in D(A)$ the set Ax has a unique element of minimum norm A^0x, and for every $y_0 \in D(A)$ the Cauchy problem (4.16) *has a unique strong solution $y \in W^{1,\infty}([0,\infty);X)$, which is everywhere differentiable from the right and*

$$\frac{d^+}{dt}y(t) + A^0 y(t) = 0, \qquad \forall t \geq 0. \tag{4.56}$$

Moreover, the function $t \to A^0 y(t)$ is continuous from the right and

$$\left\| \frac{d^+}{dt}y(t) \right\| \leq e^{\omega t} \|A^0 y_0\|, \qquad \forall t \geq 0. \tag{4.57}$$

The result extends to nonhomogeneous equation (4.1) with $f \in W^{1,\infty}([0,T];X)$.

Proof. We assume first that A is demiclosed in $X \times X$.

Define the set $B \subset X \times Y$ by

$$Bx = \overline{\operatorname{conv} Ax}, \qquad x \in D(B) = D(A).$$

It is readily seen that B is ω-accretive. Moreover, by (4.55) it follows that

$$D(A) \subset \bigcap_{0<\lambda<\lambda_0} R(I+\lambda B).$$

Let $x \in D(A)$. Then, $x_\lambda = (I+\lambda A)^{-1}x$ and $y_\lambda = A_\lambda x$ are well defined for $0 < \lambda < \lambda_0$. Moreover, $\|A_\lambda x\| \leq |Ax| = \inf\{\|w\|;\ w \in Ax\}$ and $x_\lambda \to x$ for $\lambda \to 0$ (see Proposition 3.2). Let $\lambda_n \to 0$ be such that $A_{\lambda_n} x \rightharpoonup y$. Because $A_{\lambda_n} x \in Ax_{\lambda_n}$ and A is demiclosed, it follows that $y \in Ax$. On the other hand, we have

$$\|A_\lambda x\| = \|B_\lambda x\| \le |Bx| = \|B^0 x\|.$$

(B^0x exists and is unique because the set Bx is convex, closed, and X is uniformly convex.) This implies that $y = B^0x \in Ax$. Hence, Ax has a unique element of minimum norm A^0x. Then we may apply Theorem 4.6 to deduce that the strong solution y to problem (4.16) (which exists and is unique by Theorem 4.5) satisfies (4.56) and (4.57). (In the proof of Theorem 4.6, the quasi-m-accretivity has been used only to assure the existence of a strong solution, the demiclosedness of A, and the existence of A^0.)

To complete the proof, we turn now to the case where A is only closed. Let $\widetilde{A}$ be the closure of A in $X \times X_w$; that is, the smallest demiclosed extension of A. Clearly, $D(A) \subset D(\widetilde{A}) \subset \overline{D(A)}$ and $\widetilde{A}$ satisfies condition (4.55). Moreover, because the duality mapping J is continuous, it is easily seen that $\widetilde{A}$ is ω-accretive. Then, applying the first part of the proof, we conclude that problem

$$\frac{d^+u}{dt} + \widetilde{A}^0 u = 0 \qquad \text{in } [0,\infty),$$
$$u(0) = y_0,$$

has a unique solution u satisfying all the conditions of the theorem. To conclude the proof, it suffices to show that $D(\widetilde{A}) = D(A)$ and $\widetilde{A}^0 = A^0$.

Let $x \in D(\widetilde{A})$. Then, for each λ, there is $[x_\lambda, y_\lambda] \in A \subset \widetilde{A}$ such that

$$x = x_\lambda - \lambda y_\lambda \qquad \text{for } 0 < \lambda < \lambda_0.$$

We have $x_\lambda = (I + \lambda A)^{-1}x$ and $y_\lambda = A_\lambda x = \widetilde{A}_\lambda x$. Because $x \in D(\widetilde{A})$, we have that $x_\lambda \xrightarrow{\lambda \to 0} x$ and $\|y_\lambda\| \le |\widetilde{A}x| = \|\widetilde{A}^0 x\|$. As $\widetilde{A}$ is demiclosed and X is uniformly convex, this implies, by a standard device, that $y_\lambda \to \widetilde{A}^0 x$ as $\lambda \to 0$. Finally, because A is closed, this yields $\widetilde{A}^0 x \in Ax$ and $x \in D(A)$. Hence, $D(\widetilde{A}) = D(A)$ and $\widetilde{A}^0 x = A^0 x$, $\forall x \in D(A)$. The proof of Theorem 4.7 is complete. □

Remark 4.2. If the space X^* is uniformly convex, A is quasi-m-accretive, $f \in W^{1,1}([0,T];X)$, and $y_0 \in D(A)$, then the strong solution $y \in W^{1,\infty}([0,T];X)$ to problem (4.1) (see Theorem 4.4) can be obtained as

$$y(t) = \lim_{\lambda \to 0} y_\lambda(t) \quad \text{in } X, \text{ uniformly on } [0,T], \tag{4.58}$$

where $y_\lambda \in C^1([0,T];X)$ are the solutions to the Yosida approximating equation

$$\begin{cases} \dfrac{dy_\lambda}{dt}(t) + A_\lambda y_\lambda(t) = f(t), & t \in [0,T], \\ y_\lambda(0) = y_0, \end{cases} \tag{4.59}$$

where $A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1})$ for $0 < \lambda < \lambda_0$. Here is the argument that also provides a simple proof of Theorem 4.4 in this special case. By Lemma 4.2, we have

$$\frac{1}{2}\,\frac{d}{dt}\,\|y_\lambda(t)-y_\mu(t)\|^2+(A_\lambda y_\lambda(t)-A_\mu y_\mu(t),J(y_\lambda(t)-y_\mu(t)))=0,$$
$$\text{a.e. } t\in(0,T),\quad \text{for all } \lambda,\mu\in(0,\lambda_0).$$

Inasmuch as A is ω-accretive and $A_\lambda y\in A(I+\lambda A)^{-1}y$, we get that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\|y_\lambda(t)-y_\mu(t)\|^2+(A_\lambda y_\lambda(t)-A_\mu y_\mu(t),J(y_\lambda(t)-y_\mu(t))\\&-J((I+\lambda A)^{-1}y_\lambda(t)-(1+\mu A)^{-1}y_\mu(t)))\\&\le\omega\|(1+\lambda A)^{-1}y_\lambda(t)-(1+\mu A)^{-1}y_\mu(t)\|^2,\quad \text{a.e. } t\in(0,T).\end{aligned}\tag{4.60}$$

On the other hand, multiplying the equation

$$\frac{d^2y_\lambda}{dt^2}+\frac{d}{dt}A_\lambda y_\lambda(t)=\frac{df}{dt},\quad \text{a.e. } t\in(0,T),$$

by $J(dy_\lambda/dt)$, it yields

$$\frac{1}{2}\,\frac{d}{dt}\,\left\|\frac{dy_\lambda}{dt}(t)\right\|^2\le\left\|\frac{df}{dt}(t)\right\|\,\left\|\frac{dy_\lambda}{dt}(t)\right\|+\omega\left\|\frac{dy_\lambda}{dt}(t)\right\|,\quad \text{a.e. } t\in(0,T),$$

because A_λ is ω-accretive. This implies that

$$\begin{aligned}\left\|\frac{dy_\lambda}{dt}(t)\right\|&\le e^{\omega t}\left\|\frac{dy_\lambda}{dt}(0)\right\|+\int_0^t e^{\omega(t-s)}\left\|\frac{df}{ds}(s)\right\|ds\\&\le e^{\omega t}|Ay_0-f(0)|+\int_0^t e^{\omega(t-s)}\left\|\frac{df}{ds}(s)\right\|ds.\end{aligned}\tag{4.61}$$

Hence, $\|A_\lambda y_\lambda(t)\|\le C$, $\forall\lambda\in(0,\lambda_0)$, and $\|y_\lambda(t)-(1+\lambda A)^{-1}y_\lambda(t)\|\le C\lambda$. Because J is uniformly continuous on bounded sets, it follows by (4.60) that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\|y_\lambda(t)-y_\mu(t)\|^2\le\omega\|(I+\lambda A)^{-1}y_\lambda(t)-(I+\mu A)^{-1}y_\mu(t)\|^2\\&+(\|A_\lambda y_\lambda(t)\|+\|A_\mu y_\mu(t)\|)\|J(y_\lambda(t)-y_\mu(t))-J((I+\lambda A)^{-1}y_\lambda(t)-(I+\mu A)^{-1}y_\mu(t))\|\\&\le\omega\|y_\lambda(t)-y_\mu(t)\|^2+C(\lambda+\mu)\\&+\|J(y_\lambda(t)-y_\mu(t))-J((I+\lambda A)^{-1}y_\lambda(t)-(1+\mu A)^{-1}y_\mu(t))\|,\end{aligned}$$

because $\|(I+\lambda A)^{-1}y_\lambda-y_\lambda\|=\lambda\|A_\lambda y_\lambda\|\le C\lambda$. Then, taking into account that J is uniformly continuous and that, by (4.59) and (4.61), $\{\|A_\lambda y_\lambda\|\}$ is bounded, the latter implies, via Gronwall's lemma, that $\{y_\lambda\}$ is a Cauchy sequence in the space $C([0,T];X)$ and $y(t)=\lim_{\lambda\to0}y_\lambda(t)$ exists in X uniformly on $[0,T]$. Let $[x,w]$ be arbitrary in A and let $x_\lambda=x+\lambda w$. Multiplying equation (4.59) by $J(y_\lambda(t)-x_\lambda)$ and integrating on $[s,t]$, we get

$$\frac{1}{2}\|y_\lambda(t)-x_\lambda\|^2$$
$$\leq \frac{1}{2}\|y_\lambda(s)-x_\lambda\|^2 e^{\omega(t-s)} + \int_s^t e^{\omega(t-\tau)}(f(\tau)-w, J(y_\lambda(\tau)-x_\lambda))d\tau,$$

and, letting $\lambda \to 0$,

$$\frac{1}{2}\|y(t)-x\|^2$$
$$\leq \frac{1}{2}\|y(s)-x\|^2 e^{\omega(t-s)} + \int_s^t e^{\omega(t-\tau)}(f(\tau)-w, J(y_\lambda(\tau)-x))d\tau,$$

because J is continuous. This yields

$$\left(\frac{y(t)-y(s)}{t-s}, J(y(s)-x)\right) \leq \frac{1}{2}\|y(s)-x\|^2(e^{\omega(t-s)}-1)(t-s)^{-1} + \frac{1}{t-s}\int_s^t e^{\omega(t-\tau)}(f(\tau)-w, J(y_\lambda(\tau)-x))d\tau, \tag{4.62}$$

because, as seen earlier,

$$\frac{1}{2}\|y(t)-x\|^2 - \frac{1}{2}\|y(s)-x\|^2 \geq (y(t)-x, J(y(s)-x)).$$

By (4.61), we see that y is absolutely continuous on $[0,T]$ and $dy/dt \in L^\infty(0,T;X)$. Hence, y is, a.e., differentiable on $(0,T)$. If $s = t_0$ is a point where y is differentiable, by (4.62) we see that

$$\left(f(t_0) - \frac{dy}{dt}(t_0) - w + \omega(y(t_0)-x), J(y(t_0)-x)\right) \geq 0, \qquad \forall [x,w] \in A.$$

Because $A + \omega I$ is m-accretive, this implies that

$$f(t_0) - \frac{dy}{dt}(t_0) \in Ay(t_0).$$

Hence, y is the strong solution to problem (4.1).

Local Lipschitzian Perturbations

Consider the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) + Fy(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{4.63}$$

where A is quasi-m-accretive in $X \times X$ and $F : X \to X$ is locally Lipschitz; that is,

$$\|Fu - Fv\| \le L_R \|u - v\|, \qquad \forall u, v \in B_R,\ \forall R > 0, \tag{4.64}$$

where $B_R = \{u \in X;\ \|u\| \le R\}$.

We have the following.

Theorem 4.8. *Let X be a reflexive Banach space and let A be a quasi-m-accretive operator in X. Let $f \in W^{1,1}([0,T];X)$ and let $F : X \to X$ be locally Lipschitz. Then, for each $y_0 \in D(A)$ there is $T(y_0) \in (0,T)$ and a function $y \in W^{1,\infty}([0,T(y_0)];X)$ such that*

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) + Fy(t) \ni f(t), & \text{a.e. } t \in (0, T(y_0)), \\ y(0) = y_0. \end{cases} \tag{4.65}$$

Assume further that

$$(Fy, w) \ge -\gamma_1 \|y\|^2 + \gamma_2, \qquad \forall [y, w] \in J. \tag{4.66}$$

Then, the solution y to (4.65) *is global; that is, it exists on all of* $[0,T]$.

Proof. We truncate F on X as follows

$$F_R(y) = \begin{cases} F(y) & \text{if } \|y\| \le R \\ F\left(\dfrac{Ry}{\|y\|}\right) & \text{if } \|y\| > R \end{cases} \tag{4.67}$$

and notice that F_R is Lipschitz on X:

$$\|F_R(x) - F_R(y)\| \le L_R^1 \|x - y\|, \qquad \forall x, y \in X, \tag{4.68}$$

for some $L_R^1 > 0$. The latter is obvious if $\|x\|, \|y\| \le R$ or if $\|x\|, \|y\| > R$. If $\|x\| \le R$ and $\|y\| > R$, we have

$$\begin{aligned} &\|F_R(x) - F_R(y)\| = \left\| F(x) - F\left(\frac{Ry}{\|y\|}\right) \right\| \le L_R \left\| x - \frac{Ry}{\|y\|} \right\| \\ &\le L_R R^{-1} \|x\|y\| - Ry\| \le L_R R^{-1} \|R(x-y) + x(\|y\| - R)\| \le 2L_R \|x - y\|. \end{aligned} \tag{4.69}$$

Then, (4.69) implies that F_R is Lipschitz continuous and so $A + F_R$ is quasi-*m*-accretive. Hence for each $R > 0$ there is a unique strong solution y_R to equation

$$\begin{cases} \dfrac{dy_R}{dt}(t) + Ay_R(t) + F_R(y_R(t)) \ni f(t), & \text{a.e. } t \in (0, T), \\ y_R(0) = y_0. \end{cases} \tag{4.70}$$

Multiplying (4.70) by $w \in J(y_R)$ and using the quasi-accretivity of A, we get (without any loss of generality we assume that $0 \in A0$)

$$\frac{d}{dt} \|y_R(t)\| \le L_R^1 \|y_R(t)\| + \|f(t)\|, \quad \text{a.e. } t \in (0, T)$$

and therefore

$$\|y_R(t) \leq e^{L_R^1 t}\|y_0\| + \int_0^t e^{L_R^1(t-s)}\|f(s)\|ds \leq e^{L_R^1 t}\|y_0\| + \frac{M}{L_R^1}(e^{L_R^1 t} - 1), \qquad \forall t \in (0,T).$$

This yields

$$\|y_R(t)\| \leq R$$

for $0 \leq t \leq T_R$ and $R > 0$ sufficiently large if $T_R > 0$ is suitably chosen.

Hence on $[0,T_R]$, $\|y_R(t)\| \leq R$ and so equation (4.70) reduces on this interval to (4.63). This means that (4.63) has a unique solution y on $[0,T_R]$.

If we assume (4.66), then by (4.70) we see that

$$\frac{1}{2}\frac{d}{dt}\|y_R(t)\|^2 \leq \gamma_1\|y_R(t)\|^2 + \gamma_2, \quad \text{a.e. } t \in (0,T).$$

Hence

$$\|y_R(t)\|^2 \leq e^{2\gamma_1 t}\|y_0\|^2 + \frac{\gamma_2}{\gamma_1}(e^{2\gamma_1 T} - 1) \leq R \quad \text{for } t \in [0,T]$$

if R is sufficiently large. Hence, for such R, y_R is the solution to (4.65) on all of $[0,T]$. □

The Cauchy Problem Associated with Demicontinuous Monotone Operators

We are given a Hilbert space H and a reflexive Banach space V such that $V \subset H$ continuously and densely. Denote by V' the dual space. Then, identifying H with its own dual, we may write

$$V \subset H \subset V'$$

algebraically and topologically.

The norms of V and H are denoted $\|\cdot\|$ and $|\cdot|$, respectively. We denote by (v_1, v_2) the pairing between $v_1 \in V'$ and $v_2 \in V$; if $v_1, v_2 \in H$, this is the ordinary inner product in H. Finally, we denote by $\|\cdot\|_*$ the norm of V' (which is the dual norm). In addition to these spaces, we are given a single-valued, monotone operator $A : V \to V'$. We assume that A is demicontinuous and coercive from V to V'.

We begin with the following simple application of Theorem 4.6.

Theorem 4.9. *Let $f \in W^{1,1}([0,T];H)$ and $y_0 \in V$ be such that $Ay_0 \in H$. Then, there exists one and only one function $y : [0,T] \to V$ that satisfies*

$$y \in W^{1,\infty}([0,T];H), \qquad Ay \in L^\infty(0,T;H), \tag{4.71}$$

$$\begin{cases} \dfrac{dy}{dt}(t) + Ay(t) = f(t), \quad \text{a.e. } t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{4.72}$$

Moreover, y is everywhere differentiable from the right (in H) and

$$\frac{d^+}{dt}y(t)+Ay(t)=f(t), \qquad \forall t\in[0,T).$$

Proof. Define the operator $A_H : H \to H$,

$$A_H u = Au, \qquad \forall u \in D(A_H)=\{u\in V;\ Au\in H\}. \tag{4.73}$$

By hypothesis, the operator $u \to u+Au$ is monotone, demicontinuous, and coercive from V to V'. Hence, it is surjective (see, e.g., Corollary 2.1) and so, A_H is m-accretive (maximal monotone) in $H\times H$. Then, we may apply Theorem 4.6 to conclude the proof. □

Now, we use Theorem 4.9 to derive a classical existence result due to Lions [40].

Theorem 4.10. *Let $A : V \to V'$ be a demicontinuous monotone operator that satisfies the conditions*

$$(Au,u)\ge \omega\|u\|^p+C_1, \qquad \forall u\in V, \tag{4.74}$$

$$\|Au\|_*\le C_2(1+\|u\|^{p-1}), \quad \forall u\in V, \tag{4.75}$$

where $\omega>0$ and $p>1$. Given $y_0\in H$ and $f\in L^q(0,T;V')$, $1/p+1/q=1$, there exists a unique absolutely continuous function $y:[0,T]\to V'$ that satisfies

$$y\in C([0,T];H)\cap L^p(0,T;V)\cap W^{1,q}([0,T];V'), \tag{4.76}$$

$$\frac{dy}{dt}(t)+Ay(t)=f(t), \quad a.e.\ t\in(0,T), \qquad y(0)=y_0, \tag{4.77}$$

where d/dt is considered in the strong topology of V'.

Proof. Assume that $y_0\in D(A_H)$ and $f\in W^{1,1}([0,T];H)$. By Theorem 4.9, there is $y\in W^{1,\infty}([0,T];H)$ with $Ay\in L^\infty(0,T;H)$ satisfying (4.77). Then, by assumption (4.74), multiplying equation by $y(t)$ (scalarly in H), we have

$$\frac{1}{2}\frac{d}{dt}|y(t)|^2+\omega\|y(t)\|^p\le\|f(t)\|_*\|y(t)\|, \quad \text{a.e. } t\in(0,T)$$

(see Theorem 1.18) and, therefore,

$$|y(t)|^2+\int_0^t\|y(s)\|^p ds\le C\left(|y_0|^2+\int_0^t\|f(s)\|_*^q ds\right), \qquad \forall t\in[0,T]. \tag{4.78}$$

Then, by (4.75), we get

$$\int_0^T\left\|\frac{dy}{dt}(t)\right\|_*^q dt\le C\left(|y_0|^2+\int_0^T\|f(t)\|_*^q dt\right). \tag{4.79}$$

(We denote by C several positive constants independent of y_0 and f.) Let us show now that $D(A_H)$ is a dense subset of H. Indeed, if x is any element of H, we set $x=$

$(I+\varepsilon A_H)^{-1}x$ (I is the unity operator in H). Multiplying the equation $x_\varepsilon + \varepsilon A x_\varepsilon = x$ by x_ε, it follows by (4.74) and (4.75) that

$$|x_\varepsilon|^2 + \omega\varepsilon\|x_\varepsilon\|^p \le |x_\varepsilon|\,|x| + C\varepsilon, \qquad \forall \varepsilon > 0,$$

and

$$\|x_\varepsilon - x\|_* \le \varepsilon\|Ax\|_* \le C\varepsilon(\|x_\varepsilon\|^{p-1}+1), \qquad \forall \varepsilon > 0.$$

Hence, $\{x_\varepsilon\}$ is bounded in H and $x_\varepsilon \to x$ in V' as $\varepsilon \to 0$. Therefore, $x_\varepsilon \rightharpoonup x$ in H as $\varepsilon \to 0$, which implies that $D(A_H)$ is dense in H.

Now, let $y_0 \in H$ and $f \in L^q(0,T;V')$. Then, there are the sequences $\{y_0^n\} \subset D(A_H)$, $\{f_n\} \subset W^{1,1}([0,T];H)$ such that

$$y_0^n \to y_0 \quad \text{in } H, \qquad f_n \to f \quad \text{in } L^q(0,T;V'),$$

as $n \to \infty$. Let $y_n \in W^{1,\infty}([0,T];H)$ be the solution to problem (4.77), where $y_0 = y_0^n$ and $f = f_n$. Because A is monotone, we have

$$\frac{1}{2}\,\frac{d}{dt}\,|y_n(t) - y_m(t)|^2 \le (f_n(t) - f_m(t), y_n(t - y_m(t)), \quad \text{a.e. } t \in (0,T).$$

Integrating from 0 to t, we get

$$\begin{aligned} &|y_n(t) - y_m(t)|^2 \\ &\le |y_n^0 - y_m^0|^2 + 2\left(\int_0^t \|f_n(s) - f_m(s)\|_*^q ds\right)^{1/q}\left(\int_0^t \|y_m(s) - y_m(s)\|^p ds\right)^{1/p}. \end{aligned} \tag{4.80}$$

On the other hand, it follows by estimates (4.78) and (4.79) that $\{y_n\}$ is bounded in $L^p(0,T;V)$ and $\{dy_n/dt\}$ is bounded in $L^q(0,T;V')$. Then, it follows by (4.80) that $y(t) = \lim_{n\to\infty} y_n(t)$ exists in H uniformly in t on $[0,T]$. Moreover, extracting a further subsequence if necessary, we have

$$\begin{aligned} y_n &\to y && \text{weakly in } L^p(0,T;V), \\ \frac{y_n}{dt} &\to \frac{dy}{dt} && \text{weakly in } L^q(0,T;V'), \end{aligned}$$

where dy/dt is considered in the sense of V'-valued distributions on$(0,T)$. In particular, we have proved that $y \in C([0,T];H) \cap L^p(0,T;V) \cap W^{1,q}([0,T];V')$. It remains to prove that y satisfies, a.e., on $(0,T)$ equation (4.77).

Let $x \in V$ be arbitrary but fixed. Multiplying the equation

$$\frac{dy_n}{dt} + Ay_n = f_n, \quad \text{a.e. } t \in (0,T)$$

by $y_n - x$ and integrating on (s,t), we get

$$\frac{1}{2}(|y_n(t) - x|^2 - |y_n(s) - x|^2) \le \int_s^t (f_n(\tau) - Ax, y_n(\tau) - x)d\tau.$$

Letting $n \to \infty$, it yields

$$\frac{1}{2}(|y(t)-x|^2 - |y(s)-x|^2) \le \int_s^t (f(\tau) - Ax, y(\tau) - x)d\tau.$$

Hence,

$$\left(\frac{y(t)-y(s)}{t-s}, y(s)-x\right) \le \frac{1}{t-s}\int_s^t (f(\tau)-Ax, y(\tau)-x)d\tau. \tag{4.81}$$

We know that y is, a.e., differentiable from $(0,T)$ into V' and

$$f(t_0) = \lim_{h\downarrow 0} \frac{1}{h}\int_{t_0}^{t_0+h} f(s)ds, \quad \text{a.e. } t_0 \in (0,T).$$

Let t_0 be such a point where y is differentiable. By (4.81), it follows that

$$\left(\frac{dy}{dt}(t_0) - f(t_0) + Ax, y(t_0) - x\right) \le 0,$$

and because x is arbitrary in V and A is maximal monotone in $V \times V'$, this implies that

$$\frac{dy}{dt}(t_0) + Ay(t_0) = f(t_0),$$

as claimed. □

It should be noted that compared with Theorem 4.6 and the previous results on the Cauchy problem (4.1), Theorem 4.10 provides a strong solution (in the V'-sense) under quite weak conditions on initial data and the nonhomogeneous term f. However, this class of problems is confined to those that have a variational formulation in a dual pairing (V,V').

As we show later on in Section 4.3, Theorem 4.10 remains true for time-dependent operators $A(t) : V \to V'$ satisfying assumptions (4.74) and (4.75).

Continuous Semigroups of Contractions

Definition 4.4. Let C be a closed subset of a Banach space X. A *continuous semigroup of contractions on* C is a family of mappings $\{S(t);\ t \ge 0\}$ that maps C into itself with the properties:

(i) $S(t+s)x = S(t)S(s)x,\ \forall x \in C,\ t,s \ge 0.$

(ii) $S(0)x = x,\ \forall x \in C.$

(iii) For every $x \in C$, the function $t \to S(t)x$ is continuous on $[0,\infty)$.

(iv) $\|S(t)x - S(t)y\| \le \|x-y\|,\ \forall t \ge 0,\ x,y \in C.$

More generally, if instead of (iv) we have

(v) $\|S(t)x - S(t)y\| \le e^{\omega t}\|x-y\|,\ \forall t \ge 0,\ x,y \in C,$

we say that $S(t)$ is a continuous ω-quasi-contractive semigroup on C.

The operator $A_0 : D(A_0) \subset C \to X$, defined by

$$A_0 x = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \qquad x \in D(A_0), \tag{4.82}$$

where $D(A_0)$ is the set of all $x \in C$ for which the limit (4.82) exists, is called the *infinitesimal generator* of the semigroup $S(t)$.

As in the case of strongly continuous semigroups of linear continuous operators, there is a close relationship between the continuous semigroups of contractions and accretive operators. Indeed, it is easily seen that $-A_0$ is accretive in $X \times X$. More generally, if $S(t)$ is quasi-contractive, then $-A_0$ is ω-accretive. Keeping in mind the theory of C_0-semigroups of contractions, one might suspect that there is a one-to-one correspondence between the class of continuous semigroups of contractions and that of m-accretive operators.

As seen in Theorem 4.3, if X is a Banach space and A is an ω-accretive mapping satisfying the range condition (4.15) (in particular, if A is ω-m-accretive), then, for every $y_0 \in \overline{D(A)}$, the Cauchy problem (4.16) has a unique mild solution $y(t) = S_A(t)y_0 = e^{-At}y_0$ given by the exponential formula (4.17); that is,

$$S_A(t)y_0 = \lim_{n \to \infty} \left(I + \frac{t}{n}A\right)^{-n} y_0. \tag{4.83}$$

(For this reason, $S_A(t)$ is, sometimes, denoted by e^{-At}.) We have the following.

Proposition 4.2. *$S_A(t)$ is a continuous ω-quasi-contractive semigroup on $C = \overline{D(A)}$.*

Proof. It is obvious that conditions (ii)–(iv) are satisfied as a consequence of Theorem 4.3. To prove (i), we note that, for a fixed $s > 0$, $y_1(t) = S_A(t+s)x$ and $y_2(t) = S_A(t)S_A(s)x$ are both mild solutions to the problem

$$\begin{cases} \dfrac{dy}{dt} + Ay = 0, & t \geq 0, \\ y(0) = S_A(s)x, \end{cases}$$

and so, by uniqueness of the solution we have $y_1 \equiv y_2$.

Let us assume now that X, X^* are uniformly convex Banach spaces and that A is an ω-accretive set that is closed and satisfies condition (4.55):

$$\overline{\operatorname{conv} D(A)} \subset \bigcap_{0<\lambda<\lambda_0} R(I + \lambda A) \quad \text{for some } \lambda_0 > 0. \tag{4.84}$$

Then, by Theorem 4.7, for every $x \in D(A)$, $S_A(t)x$ is differentiable from the right on $[0, +\infty)$ and

$$-A^0 x = \lim_{t \downarrow 0} \frac{S_A(t)x - x}{t}, \qquad \forall x \in D(A).$$

Hence, $-A^0 \subset A_0$, where A_0 is the infinitesimal generator of $S_A(t)$. □

As a matter of fact, we may prove in this case the following partial extension of Hille–Philips theorem in continuous semigroups of contractions. (See A. Pazy [45].)

Proposition 4.3. *Let X and X^* be uniformly convex and let A be an ω-accretive and closed set of $X \times X$ satisfying condition* (4.84). *Then, there is a continuous ω-quasi-contractive semigroup $S(t)$ on $\overline{D(A)}$, whose generator A_0 coincides with $-A^0$.*

Proof. For simplicity, we assume that $\omega = 0$. We have already seen that A^0 (the minimal section of A) is single-valued, everywhere defined on $D(A)$, and $-A_0 x = A^0 x$, $\forall x \in D(A)$. Here, A_0 is the infinitesimal generator of the semigroup $S_A(t)$ defined on $\overline{D(A)}$ by the exponential formula (4.17). We prove that $D(A_0) = D(A)$. Let $x \in D(A_0)$. Then

$$\limsup_{h \downarrow 0} \frac{\|S_A(t+h)x - S_A(t)x\|}{h} < \infty, \qquad \forall t \geq 0,$$

and, by the semigroup property (i), it follows that $t \to S_A(t)x$ is Lipschitz continuous on every compact interval $[0,T]$. Hence, $t \to S_A(t)x$ is a.e. differentiable on $(0,\infty)$ and

$$\frac{d}{dt}\, S_A(t)x = A_0 S_A(t)x, \quad \text{a.e. } t > 0.$$

Now, because $y(t) = S_A(t)x$ is a mild solution to (4.16), that is, a.e. differentiable and $(d/dt)y(0) = A_0 x$, it follows by Theorem 4.5 that $S_A(t)x$ is a strong solution to (4.16):

$$\frac{d}{dt}\, S_A(t)x + A^0 S_A(t)x = 0, \quad \text{a.e. } t > 0.$$

Now,

$$-A_0 x = \lim_{h \downarrow 0} \frac{1}{h} \int_0^h A^0 S_A(t)x\, dt,$$

and this implies as in the proof of Theorem 4.6 that $x \in D(A)$ and $-A_0 x \in Ax$ (as seen in the proof of Theorem 4.7, we may assume that A is demiclosed). This completes the proof. □

If X is a Hilbert space, it has been proven by Y. Komura [38] that every continuous semigroup of contractions $S(t)$ on a closed convex set $C \subset X$ is generated by an m-accretive set A; that is, there is an m-accretive set $A \subset X \times X$ such that $-A^0$ is an infinitesimal generator of $S(t)$. Moreover, the domain of the infinitesimal generator of a semigroup of contractions on a closed convex subset $C \subset X$ is dense in C. These remarkable results resemble the classical properties of semigroups of linear contractions in Banach spaces.

Remark 4.3. There is a simple way due to Dafermos and Slemrod [27] to transform the nonhomogeneous Cauchy problem (4.1) into a homogeneous problem. Let us assume that $f \in L^1(0,\infty;X)$ and denote by Y the product space $Y = X \times L^1(0,\infty;X)$ endowed with the norm

$$\|\{x,f\}\|_Y = \|x\| + \int_0^\infty \|f(t)\| dt, \ \ (x,f) \in Y.$$

Let $\mathscr{A} : Y \to Y$ be the (multivalued) operator

$$\mathscr{A}(x,f) = \{Ax - f(0), -f'\}, \qquad (x,f) \in D(\mathscr{A}),$$
$$D(\mathscr{A}) = D(A) \times W^{1,1}([0,\infty);X),$$

where $f' = df/dt$.

It is readily seen that if y is a solution to problem (4.1), then $Y(t) = \{y(t), f_t(s)\}$, where $f_t(s) = f(t+s)$ is the solution to the homogeneous Cauchy problem

$$\frac{d}{dt}Y(t) + \mathscr{A}Y(t) \ni 0, \qquad t \geq 0,$$
$$Y(0) = \{y_0, f\}.$$

On the other hand, if A is ω-m-accretive in $X \times X$, so is $\mathscr{A}$ in $Y \times Y$.

This result is, in particular, useful because it can lead (see Theorem 4.3) to an exponential representation formula for solutions to the nonautonomous equation (4.1) but we omit the details.

Remark 4.4. If A is m-accretive, $f \equiv 0$, and y_e is a stationary (equilibrium) solution to (4.1) (i.e., $0 \in Ay_e$), then we see by estimate (4.14) that the solution $y = y(t)$ to (4.1) is bounded on $[0,\infty)$. More precisely, we have

$$\|y(t) - y_e\| \leq \|y(0) - y_e\|, \qquad \forall t \geq 0.$$

Moreover, if A is strongly accretive (i.e., $A - \gamma I$ is accretive for some $\gamma > 0$), then

$$\|y(t) - y_e\| \leq e^{-\gamma t}\|y(0) - y_0\|, \qquad \forall t \geq 0,$$

which amounts to saying that the trajectory $\{y(t),\ t \geq 0\}$ approaches as $t \to \infty$ the equilibrium solution y_e of the system. This means that the dynamic system associated with (4.1) is *dissipative* and, in this sense, sometimes we refer to equations of the form (4.1) as *dissipative systems*.

Nonlinear Evolution Associated with Subgradient Operators

Here, we study problem (4.1) in the case where A is the subdifferential $\partial\varphi$ of a lower semicontinuous convex function φ from a Hilbert space H to $\overline{\mathbf{R}} = (-\infty, +\infty]$. In other words, consider the problem

$$\begin{cases} \dfrac{dy}{dt}(t) + \partial\varphi(y(t)) \ni f(t), & \text{in } (0,T), \\ y(0) = y_0, \end{cases} \tag{4.85}$$

in a real Hilbert space H with the scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. It turns out that the nonlinear evolution generated by $A = \partial\varphi$ on $\overline{D(A)}$ has regularity properties that in the linear case are characteristic of analytic semigroups.

If $\varphi : H \to \overline{\mathbf{R}}$ is a lower semicontinuous, convex function, then its subdifferential $A = \partial\varphi$ is maximal monotone (equivalently, m-accretive) in $H \times H$ and $\overline{D(A)} = \overline{D(\varphi)}$ (see Theorem 2.8 and Proposition 2.3). Then, by Theorem 4.2, for every $y_0 \in \overline{D(A)}$ and $f \in L^1(0,T;H)$ the Cauchy problem (4.85) has a unique mild solution $y \in C([0,T];H)$, which is a strong solution if $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];H)$ (Theorem 4.4).

Theorem 4.11 below amounts to saying that y remains a strong solution to (4.85) on every interval $[\delta,T]$ even if $y_0 \notin D(A)$ and f is not absolutely continuous. In other words, the evolution generated by $\partial\varphi$ has a smoothing effect on initial data and on the right-hand side f of (4.85). (Everywhere in the following, H is identified with its own dual.)

Theorem 4.11. *Let $f \in L^2(0,T;H)$ and $y_0 \in \overline{D(A)}$. Then the mild solution y to problem* (4.1) *belongs to $W^{1,2}([\delta,T];H)$ for every $0 < \delta < T$, and*

$$y(t) \in D(A), \qquad a.e.\ t \in (0,T), \tag{4.86}$$

$$t^{1/2}\frac{dy}{dt} \in L^2(0,T;H) \qquad \varphi(u) \in L^1(0,T), \tag{4.87}$$

$$\frac{dy}{dt}(t) + \partial\varphi(y(t)) \ni f(t), \qquad a.e.\ t \in (0,T). \tag{4.88}$$

Moreover, if $y_0 \in D(\varphi)$, then

$$\frac{dy}{dt} \in L^2(0,T;H), \qquad \varphi(y) \in W^{1,1}([0,T]). \tag{4.89}$$

The main ingredient of the proof is the following chain rule differentiation lemma.

Lemma 4.4. *Let $u \in W^{1,2}([0,T];H)$ and $g \in L^2(0,T;H)$ be such that $g(t) \in \partial\varphi(u(t))$, a.e., $t \in (0,T)$. Then, the function $t \to \varphi(u(t))$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\varphi(u(t)) = \left(g(t), \frac{du}{dt}(t)\right), \quad a.e.\ t \in (0,T). \tag{4.90}$$

Proof. Let φ_λ be the regularization of φ; that is,

$$\varphi_\lambda(u) = \inf\left\{\frac{|u-v|^2}{2\lambda} + \varphi(v);\ v \in H\right\}, \qquad u \in H,\ \lambda > 0.$$

We recall (see Theorem 2.9) that φ_λ is Fréchet differentiable on H and

$$\nabla\varphi_\lambda = (\partial\varphi)_\lambda = \lambda^{-1}(I - (I + \lambda\partial\varphi)^{-1}), \qquad \lambda > 0.$$

Obviously, the function $t \to \varphi_\lambda(u(t))$ is absolutely continuous (in fact, it belongs to $W^{1,2}([0,T];H)$) and

$$\frac{d}{dt}\varphi_\lambda(u(t)) = \left((\partial\varphi)_\lambda(u(t)), \frac{du}{dt}(t)\right), \quad \text{a.e. } t \in (0,T).$$

Hence,

$$\varphi_\lambda(u(t)) - \varphi_\lambda(u(s)) = \int_s^t \left((\partial\varphi)_\lambda(u(\tau)), \frac{du}{dt}(\tau)\right) d\tau, \qquad \forall s < t,$$

and, letting λ tend to zero, we obtain that

$$\varphi(u(t)) - \varphi(u(s)) = \int_s^t \left((\partial\varphi)^0(u(\tau)), \frac{du}{d\tau}(\tau)\right) d\tau, \qquad 0 \le s < t.$$

By the Lebesgue dominated convergence theorem, the function $t \to (\partial\varphi)^0(u(t))$ is in $L^2(0,T;H)$ and so $t \to \varphi(u(t))$ is absolutely continuous on $[0,T]$. ($(\partial\varphi)^0 = A^0$ is the minimal section of A.) Let t_0 be such that $\varphi(u(t))$ is differentiable at $t = t_0$. We have

$$\varphi(u(t_0)) \le \varphi(v) + (g(t_0), u(t_0) - v), \qquad \forall v \in H.$$

This yields, for $v = u(t_0 - \varepsilon)$,

$$\frac{d}{dt}\varphi(u(t_0)) \le \left(g(t_0), \frac{du}{dt}(t_0)\right).$$

Now, by taking $v = u(t_0 + \varepsilon)$ we get the opposite inequality, and so (4.90) follows. □

Proof of Theorem 4.11. Let x_0 be an element of $D(\partial\varphi)$ and $y_0 \in \partial\varphi(x_0)$. If we replace the function φ by $\widetilde{\varphi}(y) = \varphi(y) - \varphi(x_0) - (y_0, u - x_0)$, equation (4.85) reads

$$\frac{dy}{dt}(t) + \partial\widetilde{\varphi}(y(t)) \ni f(t) - y_0.$$

Hence, without any loss of generality, we may assume that

$$\min\{\varphi(u);\ u \in H\} = \varphi(x_0) = 0.$$

Let us assume first that $y_0 \in D(\partial\varphi)$ and $f \in W^{1,2}([0,T];H)$; that is, $df/dt \in L^2(0,T;H)$. Then, by Theorem 4.2, the Cauchy problem in (4.85) has a unique strong solution $y \in W^{1,\infty}([0,T];H)$. The idea of the proof is to obtain a priori estimates in $W^{1,2}([\delta,T];H)$ for y, and after this to pass to the limit together with the initial values and forcing term f.

To this end, we multiply equation (4.85) by $t(dy/dt)$. By Lemma 4.4, we have

$$t\left|\frac{dy}{dt}(t)\right|^2 + t\frac{d}{dt}\varphi(y(t)) = t\left(f(t), \frac{dy}{dt}(t)\right), \quad \text{a.e. } t \in (0,T).$$

Hence,

$$\int_0^T t\left|\frac{dy}{dt}(t)\right|^2 dt + T\varphi(y(T)) = \int_0^T t\left(f(t),\frac{dy}{dt}(t)\right)dt + \int_0^T \varphi(y(t))dt$$

and, therefore,

$$\int_0^T t\left|\frac{dy}{dt}(t)\right|^2 dt \le \int_0^T t|f(t)|^2 dt + 2\int_0^T \varphi(y(t))dt \tag{4.91}$$

because $\varphi \ge 0$ in H.

Next, we use the obvious inequality

$$\varphi(y(t)) \le (w(t), y(t)-x_0), \qquad \forall w(t) \in \partial\varphi(y(t))$$

to get

$$\varphi(y(t)) \le \left(f(t) - \frac{dy}{dt}(t), y(t)-x_0\right), \quad \text{a.e. } t \in (0,T),$$

which yields

$$\int_0^T \varphi(y(t))dt \le \frac{1}{2}|y(0)-x_0|^2 + \int_0^T |f(t)|\,|y(t)-x_0|dt.$$

Now, multiplying equation (4.85) by $y(t)-x_0$ and integrating on $[0,t]$, yields

$$|y(t)-x_0| \le |y(0)-x_0| + \int_0^t |f(s)|ds, \qquad \forall t \in [0,T].$$

Hence,

$$2\int_0^T \varphi(y(t))dt \le \left(|y(0)-x_0| + \int_0^T |f(t)|dt\right)^2. \tag{4.92}$$

Now, combining estimates (4.91) and (4.92), we get

$$\int_0^T t\left|\frac{dy}{dt}(t)\right|^2 dt \le \int_0^T t|f(t)|^2 dt + 2\left(|y_0-x_0| + \int_0^T |f(t)|dt\right)^2. \tag{4.93}$$

Multiplying equation (4.85) by dy/dt, we get

$$\left|\frac{dy}{dt}(t)\right|^2 + \frac{d}{dt}\varphi(y(t)) = \left(f(t),\frac{dy}{dt}(t)\right), \quad \text{a.e. } t \in (0,T).$$

Hence,

$$\frac{1}{2}\int_0^t \left|\frac{dy}{dt}(s)\right|^2 ds + \varphi(y(t)) \le \frac{1}{2}\int_0^t |f(s)|^2 ds + \varphi(y_0). \tag{4.94}$$

Now, let us assume that $y_0 \in \overline{D(\partial\varphi)}$ and $f \in L^2(0,T;H)$. Then, there exist subsequences $\{y_0^n\} \subset D(\partial\varphi)$ and $\{f_n\} \subset W^{1,2}([0,T];H)$ such that $y_0^n \to y_0$ in H and $f_n \to f$ in $L^2(0,T;H)$ as $n \to \infty$. Denote by $y_n \in W^{1,\infty}([0,T];H)$ the corresponding solutions to (4.86). Because $\partial\varphi$ is monotone, we have (see Proposition 4.1)

$$|y_n(t) - y_m(t)| \le |y_0^n - y_0^m| + \int_0^t |f_n(s) - f_m(s)|ds.$$

Hence, $y_n \to y$ in $C([0,T];H)$. On the other hand, this clearly implies that

$$\frac{dy_n}{dt} \to \frac{dy}{dt} \quad \text{in } \mathscr{D}'(0,T;H),$$

(i.e., in the sense of vectorial H-valued distributions on $(0,t)$), and, by estimate (4.93), it follows that $t^{1/2}(dy/dt) \in L^2(0,T;H)$. Hence, y is absolutely continuous on every interval $[\delta,T]$ and $y \in W^{1,2}([\delta,T];H)$ for all $0 < \delta < T$.

Moreover, by estimate (4.92), written for $y = y_n$, we deduce by virtue of Fatou's lemma that $\varphi(y) \in L^1(0,T)$ and

$$\int_0^T \varphi(y(t))dt \le \liminf_{n\to\infty} \int_0^T \varphi(y_n(t))dt \le \left(|y_0 - x| + \int_0^T |f(t)|dt\right)^2.$$

We may infer, therefore, that y satisfies estimates (4.92) and (4.93). Moreover, y satisfies equation (4.85). Indeed, we have

$$\frac{1}{2}|y_n(t) - x|^2 \le \frac{1}{2}|y_n(s) - x|^2 + \int_s^t (f_n(\tau) - w, y_n(\tau) - x)d\tau$$

for all $0 \le x < t \le T$ and all $[x,w] \in \partial\varphi$. This yields for all $0 \le s < t \le T$ and all $[x,w] \in \partial\varphi$,

$$\frac{1}{2}(|y(t) - x|^2 - |y(s) - x|^2) \le \int_s^t (f(\tau) - w, y(\tau) - x)d\tau$$

and, therefore,

$$\left(\frac{y(t) - y(s)}{t - s}, y(s) - x\right) \le \frac{1}{t - s}\int_s^t (f(\tau) - w, y(\tau) - x)d\tau.$$

Letting $s \to t$, we get, a.e. $t \in (0,T)$,

$$\left(\frac{dy}{dt}(t), y(t) - x\right) \le (f(t) - w, y(t) - x)$$

for all $[x,w] \in A$, and because $A = \partial\varphi$ is maximal monotone, this implies that $y(t) \in D(A)$ and $(d/dt)y(t) \in f(t) - Ay(t)$, a.e. $t \in (0,T)$, as desired.

Assume now that $y_0 \in D(\varphi)$. We choose in this case $y_0^n = (I + n^{-1}\partial\varphi)^{-1}y_0 \in D(\partial\varphi)$ and note that $y_0^n \to y_0$ as $n \to \infty$, and

$$\varphi(y_0^n) \le \varphi(y_0) + (\partial\varphi_n(y_0), (I + n^{-1}\partial\varphi)^{-1}y_0 - y_0) \le \varphi(y_0), \qquad \forall n \in \mathbf{N}^*.$$

Then, by estimate (4.94), we have

$$\frac{1}{2}\int_0^t \left|\frac{dy_n}{ds}(s)\right|^2 ds + \varphi(y_n(t)) \le \frac{1}{2}\int_0^t \left|\frac{df_n}{ds}(s)\right|^2 ds + \varphi(y_0)$$

and, letting $n \to \infty$, we find the estimate

$$\frac{1}{2}\int_0^t \left|\frac{dy}{dt}(s)\right|^2 ds + \varphi(y(t)) \le \frac{1}{2}\int_0^t \left|\frac{df}{ds}(s)\right|^2 ds + \varphi(y_0), \qquad t \in [0,T], \quad (4.95)$$

because $\{dy_n/dt\}$ is weakly convergent to dy/dt in $L^2(0,T;H)$ and φ is lower semicontinuous in H. This completes the proof of Theorem 4.11.

In the sequel, we denote by $W^{1,p}((0,T];H)$, $1 \le p \le \infty$, the space of all $y \in L^p(0,T;H)$ such that $dy/dt \in L^p(\delta,T;H)$ for every $\delta \in (0,T)$.

Theorem 4.12. *Assume that* $y_0 \in \overline{D(A)}$ *and* $f \in W^{1,1}([0,T];H)$. *Then, the solution* y *to problem* (4.85) *satisfies*

$$t\,\frac{dy}{dt} \in L^\infty(0,\infty;H),\ y(t) \in D(A), \qquad \forall t \in (0,T], \quad (4.96)$$

$$\frac{d^+}{dt}\,y(t) + (Ay(t) - f(t))^0 = 0, \qquad \forall t \in (0,T]. \quad (4.97)$$

Proof. By equation (4.85), we have

$$\frac{d}{dt}\,|y(t+h) - y(t)| \le |f(t+h) - f(t)|, \quad \text{a.e. } t, t+h \in (0,T).$$

Hence,

$$\left|\frac{dy}{dt}(t)\right| \le \left|\frac{dy}{ds}(s)\right| + \int_s^t \left|\frac{df}{dt}(\tau)\right| d\tau, \quad \text{a.e. } 0 < s < t < T. \quad (4.98)$$

This yields

$$\frac{1}{2}\,s\left|\frac{dy}{dt}(t)\right|^2 \le s\left|\frac{dy}{ds}(s)\right|^2 + s\left(\int_s^t \left|\frac{df}{d\tau}(\tau)\right| d\tau\right)^2, \quad \text{a.e. } 0 < s < t < T.$$

Then, integrating from 0 to t and using estimate (4.93), we get

$$t\left|\frac{dy}{dt}(t)\right| \le \left(\int_0^t s|f(s)|^2 ds + 2\left(|y(0) - x_0| + \int_0^t |f(s)|ds\right)^2 + \frac{t^2}{2}\left(\int_0^t \left|\frac{df}{d\tau}(\tau)\right| d\tau\right)^2\right)^{1/2}, \quad \text{a.e. } t \in (0,T). \quad (4.99)$$

In particular, it follows by (4.99) that

$$\limsup_{\substack{h\to 0\\ h>0}} \left|\frac{y(t+h) - y(t)}{h}\right| < \infty, \qquad \forall t \in [0,T].$$

Hence, the weak closure E of

$$\left\{\frac{(y(t+h)-y(t))}{h}\right\} \qquad \text{for } h \to 0$$

is nonempty for every $t \in [0,T)$. Let η be an element of E. We have proved earlier the inequality

$$\left(\frac{y(t+h)-y(t)}{h}, y(t)-x\right) \le \frac{1}{h}\int_t^{t+h} (f(\tau)-w, y(\tau)-x)d\tau$$

for all $[x,w] \in \partial\varphi$ and $t, t+h \in (0,T)$. This yields

$$(\eta, y(t)-x) \le (f(t)-w, y(t)-x), \qquad \forall t \in (0,T),$$

and, because $[x,w]$ is arbitrary in $\partial\varphi$, we conclude, by maximal monotonicity of A, that $y(t) \in D(A)$ and $f(t)-\eta \in Ay(t)$. Hence, $y(t) \in D(A)$ for every $t \in (0,T)$. Then, by Theorem 4.6, it follows that

$$\frac{d^+}{dt} y(t) + (Ay(t)-f(t))^0 = 0, \qquad \forall t \in (0,T), \tag{4.100}$$

because, for every $\varepsilon > 0$ sufficiently small, $y(\varepsilon) \in D(A)$ and so (4.100) holds for all $t > \varepsilon$. □

In particular, it follows by Theorem 4.12 that the semigroup $S(t) = e^{-At}$ generated by $A = \partial\varphi$ on $\overline{D(A)}$ maps $\overline{D(A)}$ into $D(A)$ for all $t > 0$ and

$$t\left|\frac{d^+}{dt} S(t)y_0\right| \le C, \qquad \forall t > 0.$$

More precisely, we have the following.

Corollary 4.4. *Let $S(t) = e^{-At}$ be the continuous semigroup of contractions generated by $A = \partial\varphi$ on $\overline{D(A)}$. Then, $S(t)\overline{D(A)} \subset D(A)$ for all $t > 0$, and*

$$\left|\frac{d^+}{dt} S(t)y_0\right| = |A^0 S(t)y_0| \le |A^0 x| + \frac{1}{t}|x-y_0|, \qquad \forall t > 0, \tag{4.101}$$

for all $y_0 \in \overline{D(A)}$ and $x \in D(A)$.

Proof. Multiplying equation (4.85) (where $f \equiv 0$) by $t(dy/dt)$ and integrating on $(0,t)$, we get

$$\int_0^t s\left|\frac{dy}{ds}(s)\right|^2 ds + t\varphi(y(t)) \le \int_0^T \varphi(y(s))ds, \qquad \forall t > 0.$$

Next, we multiply the same equation by $y(t)-x$ and integrate on $(0,t)$. We get

$$\frac{1}{2}|y(t)-x|^2+\int_0^t \varphi(y(s))ds \le \frac{1}{2}|y(0)-x|^2+t\varphi(x).$$

Combining these two inequalities, we obtain

$$\begin{aligned}\int_0^t s\left|\frac{dy}{ds}(s)\right|^2 &\le \frac{1}{2}(|y(0)-x|^2-|y(t)-x|^2+t(\varphi(x)-\varphi(y(t))\\ &\le \frac{1}{2}(|y(0)-x|^2-|y(t)-x|^2+t(A^0x,x-y(t))\\ &\le \frac{1}{2}|y(0)-x|^2+\frac{t^2|A^0x|^2}{2}, \qquad \forall t>0.\end{aligned}$$

Because, by formula (4.98) the function $t \to |(d/dt)y(t)|$ (and consequently $t \to |(d^+/dt)y(t)|$) is monotonically decreasing, this implies (4.101). □

Remark 4.5. Theorems 4.11 and 4.12 clearly remain true for equations of the form

$$\begin{cases}\dfrac{dy}{dt}(t)+\partial\varphi(y(t))-\omega y(t)\ni f(t), \ \text{a.e. in } (0,T),\\ y(0)=y_0,\end{cases}$$

where $\omega \in \mathbf{R}$ and also for Lipschitzian perturbations of $\partial\varphi$. The proof is exactly the same and so it is omitted.

A nice feature of nonlinear semigroups generated by subdifferential operators in Hilbert space is their longtime behavior. Namely, one has the following result due to Bruck [18].

Theorem 4.13. *Let $A=\partial\varphi$, where $\varphi: H \to (-\infty,+\infty]$ is a convex l.s.c. function such that $(\partial\varphi)^{-1}(0) \neq \emptyset$. Then, for each $y_0 \in \overline{D(A)}$ there is $\xi \in (\partial\varphi)^{-1}(0)$ such that*

$$\xi = w\text{-}\lim_{t\to\infty} e^{-At}y_0. \tag{4.102}$$

Proof. If we multiply the equation

$$\frac{d}{dt}y(t)+Ay(t)\ni 0, \quad \text{a.e. } t>0,$$

by $y(t)-y_0$, where $x \in (\partial\varphi)^{-1}(0)$, we obtain that

$$\frac{1}{2}\frac{d}{dt}|y(t)-x|^2 \le 0, \quad \text{a.e. } t>0,$$

because $A=\partial\varphi$ and, therefore, $(Ay(t),y(t)-x)\ge 0$, $\forall t \ge 0$. This implies that $\{y(t)\}_{t\ge 0}$ is bounded and we denote by K the so-called weak ω-limit set associated with the trajectory $\{y(t)\}_{t\ge 0}$; that is,

$$K=\left\{w\text{-}\lim_{t_n\to\infty} y(t_n)\right\}.$$

Let us notice that $K \subset (\partial\varphi)^{-1}(0)$. Indeed, if $y(t_n) \rightharpoonup \xi$, for some $\{t_n\} \to \infty$, then we see by (4.101) that

$$\lim_{n\to\infty} \frac{dy}{dt}(t_n) = 0$$

and because A is demiclosed, this implies that $0 \in A\xi$ (i.e., $\xi \in A^{-1}(0) = (\partial\varphi)^{-1}(0)$). On the other hand, $t \to |y(t) - x|^2$ is decreasing for each $x \in (\partial\varphi)^{-1}(0)$ and, in particular, for each $x \in K$.

Let ξ_1, ξ_2 be two arbitrary elements of K given by

$$\xi_1 = w\text{-}\lim_{n'\to\infty} y(t_{n'}), \qquad \xi_2 = w\text{-}\lim_{n''\to\infty} y(t_{n''}),$$

where $t_{n'} \to \infty$ and $t_{n''} \to \infty$ as $n' \to \infty$ and $n'' \to \infty$, respectively.

Because $\lim_{t\to\infty} |y(t) - x|^2$ exists for each $x \in K \subset (\partial\varphi)^{-1}(0)$, we have

$$\begin{aligned}
\lim_{n'\to\infty} |y(t_{n'}) - \xi_1|^2 &= \lim_{n''\to\infty} |y(t_{n''}) - \xi_1|^2, \\
\lim_{n''\to\infty} |y(t_{n''}) - \xi_2|^2 &= \lim_{n'\to\infty} |y(t_{n'}) - \xi_2|^2.
\end{aligned}$$

The latter implies by an elementary calculation that $|\xi_1 - \xi_2|^2 = 0$. Hence, K consists of a single point and this completes the proof of (4.102). □

Remark 4.6. In particular, it follows by Theorem 4.13 that, for each $y_0 \in \overline{D(A)}$, the solution $y(t) = e^{-At}y_0$, $A = \partial\varphi$ is weakly convergent to an equilibrium point $\xi \in \arg\min_{u\in H} \varphi(u)$ of system (4.14). There is a discrete version which asserts that the sequence $\{y_n\}$ defined by

$$y_{n+1} = y_n - h\partial\varphi(y_{n+1}), \qquad n = 0, 1, \ldots,\ h > 0,$$

is weakly convergent in H to an element $\xi \in (\partial\varphi)^{-1}(0)$; that is, to a minimum point for φ on H. The proof is completely similar. This discrete version of Theorem 4.13, known in convex optimization as the steepest descent algorithm is at the origin of a large category of gradient type algorithms.

Remark 4.7. If, under assumptions of Theorem 4.13, the trajectory $\{y(t)\}_{t\geq 0}$ is relatively compact in H (this happens for instance if each level set $\{x;\ \varphi(x) \leq \lambda\}$ is compact), then (4.102) is strengthening to

$$y(t) = e^{-At}y_0 \to \xi \quad \text{strongly in } H \text{ as } t \to \infty.$$

The longtime behavior of trajectories $\{y(t);\ t > 0\}$ to nonlinear equation (4.1) and their convergence for $t \to \infty$ to an equilibrium solution $\xi \in A^{-1}(0)$ is an important problem largely studied in the literature by different methods including dynamic topology (the Lasalle principle) or by accretivity arguments of the type presented above. Without entering into details we refer to the works of Dafermos and Slemrod [27], Haraux [31] and also to the book of Moroşanu [42].

The Reflection Problem on Closed Convex Sets

Let A be a self-adjoint positive operator in Hilbert space H and let K be a closed convex subset of H. Then, the function $\varphi : H \to \overline{\mathbf{R}}$ defined by

$$\varphi(u) = \begin{cases} \frac{1}{2}(Au,u) + I_K(u), & \forall u \in K \cap D(A^{1/2}), \\ +\infty, & \text{otherwise} \end{cases}$$

(I_K indicator function of K) is convex and l.s.c. Moreover, if there is $h \in H$ such that

$$(I + \lambda A)^{-1}(x + \lambda h) \in K, \qquad \forall \lambda > 0,\ x \in K,$$

then $A + \partial I_K$ is maximal monotone (see Theorem 2.11) and so $\partial\varphi = A + \partial I_K$ with $D(\partial\varphi) = D(A) \cap K$.

For this special form of φ, equation (4.85) reduces to the variational inequality

$$\begin{cases} \left(\frac{dy}{dt}(t) + Ay(t) - f(t), y(t) - z\right) \le 0, & \forall z \in K,\ t \in (0,T), \\ y(0) = y_0, \qquad y(t) \in K, & \forall t \in [0,T], \end{cases} \tag{4.103}$$

which is similar to that considered in Section 2.3.

A more general situation is discussed in Section 5.2 below. Here, we confine ourselves to noting that the solution $y \in W^{1,2}([0,T];H)$ to (4.103), which exists and is unique for $y_0 \in K$ and $f \in L^2(0,T;H)$, satisfies the system

$$\begin{cases} \frac{dy}{dt}(t) + Ay(t) = f(t) & \text{if } y(t) \in \overset{\circ}{K}, \\ \frac{dy}{dt}(t) + Ay(t) = -\eta_K(t) + f(t) & \text{if } y(t) \in \partial K, \end{cases}$$

where $\eta_K(t) \in N_K(y(t))$, the normal cone to K on the boundary ∂K. (Here, $\overset{\circ}{K}$ is the interior of K if nonempty.) For instance, if $K = \{u \in H;\ |u| \le \rho\}$, then we have

$$\begin{cases} \frac{dy}{dt}(t) + Ay(t) = f(t) & \text{on } \{t;\ |y(t)| < \rho\}, \\ \frac{dy}{dt}(t) + Ay(t) = -\lambda y(t) + f(t) & \text{on } \{t;\ |y(t)| = \rho\}, \end{cases}$$

for some $\lambda \ge 0$. The parameter λ must be viewed as a Lagrange multiplier that arises from constraint $y(t) \in K$, $\forall t \ge 0$.

For this reason, problem (4.103) is also called the *reflection problem on K* associated with linear equation $dy/dt + Ay = 0$ and under this interpretation it is relevant not only in the dynamic theory of free boundary problems, but also in the theory of stochastic processes with optimal stopping time arising in the theory of financial markets (see, e.g., Barbu and Marinelli [8]).

The Brezis–Ekeland Variational Principle

It turns out that the Cauchy problem (4.85) can be equivalently represented as a minimization problem in the space $L^2(0,T;H)$ or $W^{1,2}([0,T];H)$ which is quite surprising because, in general, the Cauchy problem is not of variational type.

In fact, if $\varphi : H \to \overline{\mathbf{R}}$ is convex, l.s.c., and φ^* is its conjugate function we have by Proposition 1.5 that

$$\varphi(y) + \varphi^*(p) \geq (y,p), \qquad \forall y, p \in H,$$

with equality if and only if $p \in \partial\varphi(y)$. Then, we may equivalently write (4.85) as

$$\begin{aligned}\frac{dy}{dt}(t) + z(t) = f(t), \qquad &\varphi(y(t)) + \varphi^*(z(t)) = (y(t), z(t)), \quad \text{a.e. } t \in (0,T),\\ y(0) = y_0.&\end{aligned}$$

Hence, if $y \in W^{1,2}([0,T];H)$ is the solution to (4.85), where $y_0 \in D(\varphi)$ (see Theorem 4.11), then we have

$$\varphi(y(t)) + \varphi^*\left(f(t) - \frac{dy}{dt}(t)\right) = \left(y(t), f(t) - \frac{dy}{dt}(t)\right), \quad \text{a.e. } t \in (0,T),$$

and the latter is equivalent to (4.85). This yields

$$\int_0^T \left(\varphi(y(t)) + \varphi^*\left(f(t) - \frac{dy}{dt}(t)\right) - (y(t), f(t))\right) dt + \frac{1}{2}|y(T)|^2 - \frac{1}{2}|y_0|^2 = 0$$

and we have also that

$$\begin{aligned} y = \arg\min\Bigg\{ &\int_0^T \left[\varphi(\theta(t)) + \varphi^*\left(f(t) - \frac{d\theta}{dt}(t)\right) - (\theta(t), f(t))\right] dt \\ &+ \frac{1}{2}|\theta(T)|^2 - \frac{1}{2}|y_0|^2; \quad \theta \in W^{1,2}([0,T];H),\ \theta(0) = y_0\Bigg\}. \end{aligned} \tag{4.104}$$

This means that the Cauchy problem (4.85) is equivalent to the minimization problem (4.104). This is the *Brezis–Ekeland principle* and it reveals an interesting connection between the subpotential Cauchy problem and convex optimization, which found many interesting applications in the theory of variational inequalities (see, e.g., Stefanelli [51], and Visintin [53]).

However, the function $\Phi : W^{1,2}([0,T];H) \to \overline{\mathbf{R}}$, defined by the right-hand side of (4.104), is convex and lower semicontinuous but, in general, not coercive (this happens if $D(\varphi) = H$ only) and so, one cannot derive Theorem 4.11 directly from the existence of a minimizer y in problem (4.104).

4.2 Approximation and Structural Stability of Nonlinear Evolutions

The Trotter–Kato Theorem for Nonlinear Evolutions

One might expect the solution to Cauchy problem (4.1) to be continuous with respect to the operator A, that is, with respect to small structural variations of the problem. We show below that this indeed happens in a certain precise sense and for a certain notion of convergence defined in the space of quasi-m-accretive operators.

Consider in a general Banach space X a sequence A_n of subsets of $X \times X$. The subset of $X \times X$, $\liminf A_n$ is defined as the set of all $[x,y] \in X \times X$ such that there are sequences x_n, y_n, $y_n \in A_n x_n$, $x_n \to x$ and $y_n \to y$ as $n \to \infty$. If A_n are quasi-m-accretive, there is a simple resolvent characterization of $\liminf A_n$. (See Attouch [1, 2].)

Proposition 4.4. *Let $A_n + \omega I$ be m-accretive for $n = 1,2\dots$. Then $A \subset \liminf A_n$ if and only if*

$$\lim_{n\to\infty} (I+\lambda A_n)^{-1}x = (I+\lambda A)^{-1}x, \qquad \forall x \in X, \tag{4.105}$$

for $0 < \lambda < \omega^{-1}$.

Proof. Assume that (4.105) holds and let $[x,y] \in A$ be arbitrary but fixed. Then, we have

$$(I+\lambda A)^{-1}(x+\lambda y) = x, \qquad \forall \lambda \in (0, \omega^{-1})$$

and, by (4.105),

$$(I+\lambda A_n)^{-1}(x+\lambda y) \to (I+\lambda A)^{-1}(x+\lambda y) = x.$$

In other words, $x_n = (I+\lambda A_n)^{-1}(x+\lambda y) \to x$ as $n \to \infty$ and $x_n + \lambda y_n = x + \lambda y$, $y_n \in A x_n$. Hence, $y_n \to y$ as $n \to \infty$, and so $[x,y] \in \liminf A_n$.

Conversely, let us assume now that $A \subset \liminf A_n$. Let x be arbitrary in X and let $x_0 = (I+\lambda A)^{-1}x$; that is,

$$x_0 + \lambda y_0 = x, \quad \text{where } y_0 \in A x_0.$$

Then, there are $[x_n, y_n] \in A_n$ such that $x_n \to x_0$ and $y_n \to y_0$ as $n \to \infty$. We have

$$x_n + \lambda y_n = z_n \to x_0 + \lambda y_0 = x \quad \text{as } n \to \infty.$$

Hence,

$$(I+\lambda A_n)^{-1}x \to x_0 = (I+\lambda A)^{-1}y_0 \quad \text{for } 0 < \lambda < \omega^{-1},$$

as claimed. □

In the literature, such a convergence is called *convergence in the sense of graphs*.

Theorem 4.14 below is the nonlinear version of the Trotter–Kato theorem from the theory of C_0-semigroups and, roughly speaking, it amounts to saying that if A_n

is convergent to A in the sense of graphs, then the dynamic (evolution) generated by A_n is uniformly convergent to that generated by A (see Pazy [45]).

Theorem 4.14. *Let A_n be ω-m-accretive in $X \times X$, $f^n \in L^1(0,T;X)$ for $n = 1,2,\ldots$ and let y_n be mild solution to*

$$\frac{dy_n}{dt}(t) + A_n y_n(t) \ni f^n(t) \quad in \ [0,T], \quad y_n(0) = y_0^n. \tag{4.106}$$

Let $A \subset \liminf A_n$ and assume that

$$\lim_{n\to\infty} \left(\int_0^T \|f^n(t) - f(t)\| dt + \|y_0^n - y_0\| \right) = 0. \tag{4.107}$$

Then, $y_n(t) \to y(t)$ uniformly on $[0,T]$, where y is the mild solution to problem (4.106).

Proof. Let $D^\varepsilon_{A^n}(0 = t_0, t_1, \ldots, t_N; f_1^n, \ldots, f_N^n)$ be an ε-discretization of problem (4.106) and let $D^\varepsilon_A(0 = t_0, t_1, \ldots, t_n; f_1, \ldots, f_N)$ be the corresponding ε-discretization for (4.1). We take $t_i = i\varepsilon$ for all i. Let $y_{\varepsilon,n}$ and y_ε be the corresponding ε-approximate solutions; that is,

$$y_{\varepsilon,n}(t) = y^i_{\varepsilon,n},\ y_\varepsilon(t) = y^i_\varepsilon \quad \text{for } t \in (t_{i-1}, t_i],$$

where $y^0_{\varepsilon,n} = y^n_0$, $y^0_\varepsilon = y_0$, and

$$y^i_{\varepsilon,n} + \varepsilon A_n y^i_{\varepsilon,n} \ni y^{i-1}_{\varepsilon,n} + \varepsilon f_i^n, \qquad i = 1,\ldots,N, \tag{4.108}$$

$$y^i_\varepsilon + \varepsilon A y^i_\varepsilon \ni y^{i-1}_\varepsilon + \varepsilon f_i, \qquad i = 1,\ldots,N. \tag{4.109}$$

By the definition of $\liminf A_n$, for every $\eta > 0$ there is $[\bar{y}^i_{\varepsilon,n}, w^i_{\varepsilon,n}] \in A_n$ such that

$$\|\bar{y}^i_{\varepsilon,n} - y^i_\varepsilon\| + \|w^i_{\varepsilon,n} - w^i_\varepsilon\| \le \eta \quad \text{for } n \ge \delta(\eta,\varepsilon). \tag{4.110}$$

Here, $w^i_\varepsilon = (1/\varepsilon)(y^{i-1}_\varepsilon + \varepsilon f_i - y^i_\varepsilon) \in A y^i_\varepsilon$. Then, using the ω-accretivity of A_n, by (4.108)–(4.110) it follows that

$$\|\bar{y}^i_{\varepsilon,n} - y^i_{\varepsilon,n}\| \le (1-\varepsilon\omega)^{-1} \|\bar{y}^{i-1}_{\varepsilon,n} - y^{i-1}_{\varepsilon,n}\| + \varepsilon(1-\varepsilon\omega)^{-1}\|f_i^n - f_i\| + C\varepsilon\eta, \qquad \forall i,$$

for $n \ge \delta(\eta,\varepsilon)$. This yields

$$\|\bar{y}^i_{\varepsilon,n} - y^i_{\varepsilon,n}\| \le C\eta + C\varepsilon \sum_{k=1}^{i} (1-\varepsilon\omega)^{-k} \|f_k^n - f_k\|, \qquad i = 1,\ldots,N.$$

Hence,

$$\|y^i_{\varepsilon,n} - y^i_\varepsilon\| \le C\eta + C\varepsilon \sum_{k=1}^{i} (1-\varepsilon\omega)^{-k} \|f_k^n - f_k\|, \qquad i = 1,\ldots,N,$$

for $n \ge \delta(\varepsilon,\eta)$.

We have shown, therefore, that, for $n \geq \delta(\varepsilon,\eta)$,

$$\|y_{\varepsilon,n}(t) - y_\varepsilon(t)\| \leq C\left(\eta + \int_0^T \|f^n(t) - f(t)\|dt\right), \qquad \forall t \in [0,T], \tag{4.111}$$

where C is independent of n and ε.

Now, we have

$$\|y_n(t) - y(t)\| \leq \|y_n(t) - y_{\varepsilon,n}(t)\| + \|y_{\varepsilon,n}(t) - y_\varepsilon(t)\| + \|y_\varepsilon(t) - y(t)\|, \quad \forall t \in [0,T). \tag{4.112}$$

Let η be arbitrary but fixed. Then, by Theorem 4.1, we have

$$\|y_\varepsilon(t) - y(t)\| \leq \eta, \qquad \forall t \in [0,T], \quad \text{if } 0 < \varepsilon < \varepsilon_0(\eta).$$

Also, by estimate (4.37) in the proof of Theorem 4.1, we have

$$\|y_{\varepsilon,n}(t) - y_n(t)\| \leq \eta, \qquad \forall t \in [0,T],$$

for all $0 < \varepsilon < \varepsilon_1(\eta)$, where $\varepsilon_1(\eta)$ does not depend on n. Thus, by (4.111) and (4.112), we have

$$\|y_n(t) - y(t)\| \leq C\left(\eta + \int_0^T \|f^n(t) - f(t)\|dt\right), \qquad \forall t \in [0,T]$$

for n sufficiently large and any $\eta > 0$. □

Corollary 4.5. *Let A be ω-m-accretive, $f \in L^1(0,T;X)$, and $y_0 \in \overline{D(A)}$. Let $y_\lambda \in C^1([0,T];X)$ be the solution to the approximating Cauchy problem*

$$\frac{dy}{dt}(t) + A_\lambda y(t) = f(t) \quad \text{in } [0,T], \qquad y(0) = y_0,\ 0 < \lambda < \frac{1}{\omega}, \tag{4.113}$$

where $A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1})$. Then, $\lim_{\lambda \to 0} y_\lambda(t) = y(t)$ uniformly in t on $[0,T]$, where y is the mild solution to problem (4.1).

Proof. It is easily seen that $A \subset \liminf_{\lambda \to 0} A_\lambda$. Indeed, for $\alpha \in (0,1/\omega)$ we set

$$x_\lambda = (I + \alpha A_\lambda)^{-1}x,\ \ u = (I + \alpha A)^{-1}x, \qquad \forall \lambda > 0.$$

After some calculation, we see that

$$x_\lambda + \alpha A\left(\left(1 + \frac{\lambda}{\alpha}\right)x_\lambda - \frac{\lambda}{\alpha}x\right) \ni x.$$

Subtracting this equation from $u + \alpha Au \ni x$ and using the ω-accretivity of A, we get

$$\|x_\lambda - u\|^2 \leq \alpha\omega\left\|\left(1 + \frac{\lambda}{\alpha}\right)x_\lambda - \frac{\lambda}{\alpha}x - u\right\|^2 + \frac{\lambda}{\alpha}(x_\lambda - u, x - x_\lambda).$$

Hence, $\lim_{\lambda\to 0} x_\lambda = u = (I+\alpha A)^{-1}x$ for $0<\alpha<1/\lambda$, and so we may apply Theorem 4.14. □

Remark 4.8. If X is a Hilbert space and $S_n(t)$ is the semigroup generated by A_n on X, then, according to a result due to H. Brezis, condition (4.105) is equivalent to the following one. For every $x\in\overline{D(A)}$, $\exists\{x_n\}\subset D(A_n)$ such that $x_n\to x$ and $S_n(t)x_n\to S(t)x$, $\forall t>0$, where $S(t)$ is the semigroup generated by A on $\overline{D(A)}$.

Theorem 4.14 is useful in proving the stability and convergence of a large class of approximation schemes for problem (4.1). For instance, if A is a nonlinear partial differential operator on a certain space of functions defined on a domain $\Omega\subset\mathbf{R}^m$, then very often the A_n arise as finite element approximations of A on a subspace X_n of X. Another important class of convergence results covered by this theorem is the homogenization problem (see, e.g., Attouch [2] and references given there).

Nonlinear Chernoff Theorem and Lie–Trotter Products

We prove here the nonlinear version of the famous Chernoff theorem (see Chernoff [21]), along with some implications for the convergence of the Lie–Trotter product formula for nonlinear semigroups of contractions.

Theorem 4.15. *Let X be a real Banach space, A be an accretive operator satisfying the range condition* (4.15)*, and let $C=\overline{D(A)}$ be convex. For each $t>0$, let $F(t):C\to C$ satisfy:*

(i) $\|F(t)x-F(t)u\|\le\|x-u\|,\qquad \forall x,y\in C\quad \textit{and}\quad t\in[0,T].$

(ii) $\displaystyle\lim_{t\downarrow 0}\left(I+\lambda\,\frac{I-F(t)}{t}\right)^{-1}x=(I+\lambda A)^{-1}x,\qquad \forall x\in C,\ \lambda>0.$

Then, for each $x\in C$ and $t>0$,

$$\lim_{n\to 0}\left(F\left(\frac{t}{n}\right)\right)^n x=S_A(t)x, \tag{4.114}$$

uniformly in t on compact intervals.

Here, $S_A(t)$ is the semigroup generated by A on $C=\overline{D(A)}$. (See (4.82).) It should be said that in the special case where $F(t)=(I+tA)^{-1}$, Theorem 4.15 reduces to the exponential formula (4.17) in Theorem 4.3.

The main ingredient of the proof is the following convergence result.

Proposition 4.5. *Let $C\subset X$ be nonempty, closed, and convex, let $F:C\to C$ be a nonexpansive operator, and let $h>0$. Then, the Cauchy problem*

$$\frac{du}{dt}+h^{-1}(I-F)u=0,\qquad u(0)=x\in C, \tag{4.115}$$

has a unique solution $u\in C^1([0,\infty);X)$, such that $u(t)\in C$, for all $t\ge 0$.

Moreover, the following estimate holds

$$\|F^n x - u(t)\| \leq \left(\left(n - \frac{t}{h}\right)^2 + n\right)^{1/2} \|x - Fx\|, \qquad \forall t \geq 0, \tag{4.116}$$

for all $n \in \mathbf{N}$. In particular, for $t = nh$ we have

$$\|F^n x - u(nh)\| \leq n^{1/2} \|x - Fx\|, \qquad n = 1, 2, \ldots, t \geq 0. \tag{4.117}$$

Proof. The initial value problem (4.115) can be written equivalently as

$$u(t) = e^{-(t/h)} x + \int_0^t e^{-((t-s)/h)} F u(s) ds, \qquad \forall t \geq 0,$$

and it has a unique solution $u(t) \in C$, $\forall t \geq 0$, by the Banach fixed point theorem. Making the substitution $t \to t/h$, we can reduce the problem to the case $h = 1$.

Multiplying equation (4.115) by $J(u(t) - x)$, where $J : X \to X^*$ is the duality mapping, we get

$$\frac{d}{dt} \|u(t) - x\| \leq \|Fx - x\|, \quad \text{a.e. } t > 0,$$

because $I - F$ is accretive. Hence,

$$\|u(t) - x\| \leq t \|Fx - x\|, \qquad \forall t \geq 0. \tag{4.118}$$

On the other hand, we have

$$u(t) - F^n x = e^{-t}(x - F^n x) + \int_0^t e^{s-t} (F u(s) - F^n x) ds$$

and

$$\|x - F^n x\| \leq \sum_{k=1}^n \|F^{k-1} x - F^k x\| \leq n \|x - Fx\|, \qquad \forall n.$$

Hence,

$$\|u(t) - F^n x\| \leq n e^{-t} \|x - Fx\| + \int_0^t e^{s-t} \|u(s) - F^{n-1} x\| ds.$$

We set $\varphi_n(t) = \|u(t) - F^n x\| \, \|x - Fx\|^{-1} e^t$. Then, we have

$$\varphi_n(t) \leq n + \int_0^t \varphi_{n-1}(x) ds, \qquad \forall t \geq 0, \ n = 1, 2, \ldots, \tag{4.119}$$

and, by (4.118), we see that

$$\varphi_0(t) \leq t e^t, \qquad \forall t \geq 0. \tag{4.120}$$

Solving iteratively (4.119) and (4.120), we get

$$\begin{aligned}\varphi_n(t) &\le \sum_{k=1}^{n} \frac{kt^{n-k}}{(n-k)!} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\varphi_0(s)ds \\ &= \sum_{k=1}^{n} \frac{kt^{n-k}}{(n-k)!} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\sum_{j=1}^{\infty}\frac{s^{j+1}}{j!}ds \\ &= \sum_{k=1}^{n} \frac{kt^{n-k}}{(n-k)!} + \sum_{j=0}^{\infty}\frac{1}{(n-1)!j!}\int_0^t (t-s)^{n-1}s^{j+1}ds.\end{aligned}$$

Because

$$\int_0^t (t-s)^{n-1}s^{j+1}ds = \frac{t^{n+j+1}(j+1)!(n-1)!}{(n+j+1)!},$$

we obtain that

$$\begin{aligned}\varphi_n(t) &\le \sum_{k=0}^{n}\frac{(n-k)t^k}{k!} + \sum_{j=0}^{\infty}\frac{(j+1)t^{n+j+1}}{(n+j+1)!} = \sum_{k=0}^{\infty}\frac{(n-k)t^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}|n-k| \le \left(\sum_{k=0}^{\infty}\frac{(n-k)^2t^k}{k!}\right)^{1/2} e^{t/2}.\end{aligned}$$

Hence,

$$\varphi_n(t) \le e^t((n-t)^{-1}+t)^{1/2}, \qquad \forall t \ge 0,$$

as claimed. □

Proof of Theorem 4.15. We set $A_h = h^{-1}(I - F(h))$ and denote by $S_h(t)$ the semigroup generated by A_h on $C = \overline{D(A)}$ (Theorem 4.3). We also use the standard notation

$$J_\lambda = (I+\lambda A)^{-1}, \qquad J_\lambda^h = (I+\lambda A_h)^{-1}.$$

Because $J_\lambda^h x \to J_\lambda x$, $\forall x \in C$, as $h \to 0$, it follows by Theorem 4.14 that, for every $x \in C$,

$$S_h(t)x \to S_A(t)x \quad \text{uniformly in } t \text{ on compact intervals.} \tag{4.121}$$

Next, by Proposition 4.5, we have that

$$\begin{aligned}\|S_h(nh)x - F^n(h)x\| &\le \|S_h(nh)J_\lambda^h x - F^n(h)J_\lambda^h x\| + 2\|x - J_\lambda^h x\| \\ &\le \|x - J_\lambda^h x\|(2+\lambda^{-1}hn^{1/2}).\end{aligned}$$

Now, we fix $x \in D(A)$ and $h = n^{-1}t$. Then, the previous inequality yields

$$\begin{aligned}&\left\|S_{t/n}(t)x - F^n\left(\frac{t}{n}\right)x\right\| \le (2+\lambda^{-1}tn^{-(1/2)})(\|x-J_\lambda x\| + \|J_\lambda^{t/n}x\|) \\ &\le (2+\lambda^{-1}tn^{-(1/2)})(\lambda|Ax| + \|J_\lambda^{t/n}x - J_\lambda x\|), \qquad \forall t>0,\ \lambda>0.\end{aligned}$$

Finally,

$$\left\|S_{t/n}(t)x - F^n\left(\frac{t}{n}\right)x\right\| \le 2\lambda|Ax| + tn^{-(1/2)}|Ax| \\ + (2+\lambda^{-1}tn^{-(1/2)})\|J_\lambda^{t/n}x - J_\lambda x\|, \\ \forall t > 0,\ \lambda > 0. \tag{4.122}$$

Now, fix $\lambda > 0$ such that $2\lambda|Ax| \le \varepsilon/3$. Then, by (ii), we have

$$(2+\lambda^{-1}tn^{-(1/2)})\|J_\lambda^{t/n}x - J_\lambda x\| \le \frac{\varepsilon}{3} \quad \text{for } n > N(\varepsilon),$$

and so, by (4.121) and (4.122), we conclude that, for $n \to \infty$,

$$F^n\left(\frac{t}{n}\right)x \to S_A(t)x \quad \text{uniformly in } t \text{ on every } [0,T]. \tag{4.123}$$

Now, because

$$\|S_A(t)x - S_A(t)y\| \le |x-y|, \qquad \forall t \ge 0,\ x,y \in C,$$

and

$$\left\|F^n\left(\frac{1}{n}\right)x - F^n\left(\frac{t}{n}\right)y\right\| \le \|x-y\|, \qquad \forall t \ge 0,\ x,y \in C,$$

(4.123) extends to all $x \in \overline{D(A)} = C$. The proof of Theorem 4.15 is complete.

Remark 4.9. The conclusion of Theorem 4.15 remains unchanged if A is ω-accretive, satisfies the range condition (4.15), and $F(t) : C \to C$ are Lipschitzian with Lipschitz constant $L(t) = 1 + \omega t + o(t)$ as $t \to 0$. The proof is essentially the same and relies on an appropriate estimate of the form (4.117) for Lipschitz mappings on C.

Given two m-accretive operators $A, B \subset X \times X$ such that $A+B$ is m-accretive, one might expect that

$$S_{A+B}(t)x = \lim_{n\to\infty}\left(S_A\left(\frac{t}{n}\right)S_B\left(\frac{t}{n}\right)\right)^n x, \qquad \forall t \ge 0, \tag{4.124}$$

for all $x \in \overline{D(A)\cap D(B)}$. This is the *Lie–Trotter product* formula and one knows that it is true for C_0-semigroups of contractions and in other situations (see Pazy [45], p. 92). It is readily seen that (4.124) is equivalent to the convergence of the fractional step method scheme for the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt} + Ay + By \ni 0 & \text{in } [0,T], \\ y(0) = y_0; \end{cases} \tag{4.125}$$

that is,

$$\begin{cases} \dfrac{dy}{dt} + Ay \ni 0 & \text{in } [i\varepsilon,(i+1)\varepsilon],\ i=0,1,\dots,N-1,\ T=N\varepsilon, \\ y^+(i\varepsilon) = z(\varepsilon), & i=0,1,\dots,N-1, \\ y^+(0) = y_0, \end{cases} \tag{4.126}$$

$$\begin{aligned} &\frac{dz}{dt} + Bz \ni 0 \quad \text{in } [0,\varepsilon], \\ &z(0) = y^-(i\varepsilon). \end{aligned} \tag{4.127}$$

In a general Banach space, the Lie–Trotter formula (4.124) is not convergent even for regular operators B unless $S_A(t)$ admits a graph infinitesimal generator A: for all $[x,y] \in A$ there is $x_h \to x$ as $h \to 0$ such that $h^{-1}(x_h - S_A(h)x) \to y$ (Bénilan and Ismail [12]). However, there are known several situations in which formula (4.124) is true and one is described in Theorem 4.16 below.

Theorem 4.16. *Let X and X^* be uniformly convex and let A,B be m-accretive single-valued operators on X such that $A+B$ is m-accretive and $S_A(t), S_B(t)$ map $\overline{D(A)\cap D(B)}$ into itself. Then,*

$$S_{A+B}(t)x = \lim_{n\to\infty} \left(S_A\left(\frac{t}{n}\right) S_B\left(\frac{t}{n}\right)\right)^n x, \qquad \forall x \in \overline{D(A)\cap D(B)}, \tag{4.128}$$

and the limit is uniform in t on compact intervals.

Proof. We verify the hypotheses of Theorem 4.15, where $F(t) = S_A(t)S_B(t)$ and $C = \overline{D(A)\cap D(B)}$. To prove (ii), it suffices to show that

$$\lim_{t\downarrow 0} \frac{x - F(t)x}{x} = Ax + Bx, \qquad \forall x \in D(A)\cap D(B). \tag{4.129}$$

Indeed, if

$$x_t = \left(I + \lambda\, \frac{I - F(t)}{t}\right)^{-1} x$$

and

$$x_0 = (I + \lambda(A+B))^{-1}x,$$

then we have

$$x_t + \frac{\lambda}{t}\,(x_t - F(t)x_t) = x \tag{4.130}$$

and, respectively,

$$x_0 + \lambda A x_0 + \lambda B x_0 = x. \tag{4.131}$$

Subtracting (4.130) from (4.131), we may write

$$x_t - x_0 + \frac{\lambda}{t}\,((I - F(t))x_t - (I + F(t)x_0)) + \lambda\left(Ax_0 + Bx_0 - \frac{x_0 - F(t)x_0}{t}\right) = 0.$$

Multiplying this by $J(x_t - x_0)$, where J is the duality mapping of X, and using (4.129) and the accretiveness of $I - F(t)$, it follows that

$$\lim_{t\downarrow 0} \|x_t - x_0\| \le \lambda \lim_{t\downarrow 0} \left\| Ax_0 + Bx_0 - \frac{x_0 - F(t)x_0}{t} \right\| = 0.$$

Hence, $\lim_{t\downarrow 0} x_t = x_0$, which implies (ii).

To prove (4.129), we write $t^{-1}(x - F(t)x)$ as

$$t^{-1}(x - F(t)x) = t^{-1}(x - S_A(t)x) + t^{-1}(S_A(t)x - S_A(t)S_B(t)x).$$

Because $t^{-1}(x - S_A(t)x) \to Ax$ as $t \to 0$ (Theorem 4.7), it remains to prove that

$$z_t = t^{-1}(S_A(t)x - S_A(t)S_B(t)x) \to Bx \quad \text{as } t \to 0. \tag{4.132}$$

Because $S_A(t)$ is nonexpansive, we have

$$\|z_t\| \le t^{-1}\|S_B(t)x - x\| \le \|Bx\|, \qquad \forall t > 0. \tag{4.133}$$

On the other hand, inasmuch as $I - S_A(t)$ is accretive, we have

$$\left(\frac{u - S_A(t)u}{t} + \frac{S_A(t)x - S_B(t)x}{t} - z_t, J(u - S_A(t)x) \right) > 0, \qquad \forall u \in C,\ t > 0. \tag{4.134}$$

Let $t_n \to 0$ be such that $z_{t_n} \rightharpoonup z$. Then, by (4.134), we have that

$$(Au + Bx - Ax - z, J(u - x)) \ge 0, \qquad \forall u \in D(A),$$

because $J : X \to X^*$ is continuous and

$$t^{-1}(x - S_B(t)x) \to Bx, \qquad t^{-1}(x - S_A(t)x) \to Ax.$$

Inasmuch as A is m-accretive, this implies that $Ax + z - Bx = Ax$ (i.e., $z = Bx$). On the other hand, by (4.133), recalling that X is uniformly convex, it follows that $z_{t_n} \to Bx$ (strongly). Then, (4.132) follows, and the proof of Theorem 4.16 is complete. □

Remark 4.10. Theorem 4.16, which is essentially due to Brezis and Pazy [16] was extended by Kobayashi [35] to multivalued operators A and B in a Hilbert space H. More precisely, if A, B and $A + B$ are maximal monotone and if there is a nonempty closed convex set $C \subset \overline{D(A) \cap D(B)}$ such that $(I + \lambda A)^{-1} C \subset C$ and $(I + \lambda B)^{-1} C \subset C$, $\forall \lambda > 0$, then

$$S_{A+B}(t)x = \lim_{n\to\infty} \left(S_A\left(\frac{t}{n}\right) S_B\left(\frac{t}{n}\right) \right)^n x, \qquad \forall x \in C,$$

uniformly in t on compact intervals. For some extensions to Banach spaces we refer to Reich [49].

4.3 Time-Dependent Cauchy Problems

This section is concerned with the evolution problem

$$\begin{cases} \frac{dy}{dt}(t) + A(t)y(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{4.135}$$

where $\{A(t)\}_{t\in[0,T]}$ is a family of quasi-m-accretive operators in $X \times X$.

The existence problem for (4.135) is a difficult one and not completely solved even for linear operators $A(t)$. In general, one cannot expect a positive and convenient answer to the existence problem for (4.135) if one takes into account that in most applications to partial differential equations the domain $D(A(t))$ might not be independent of time. However, we can identify a few classes of time-dependent problems for which the Cauchy problem (4.135) is well posed.

Nonlinear Demicontinuous Evolutions in Duality Pair of Spaces

Let V be a reflexive Banach space and H be a real Hilbert space identified with its own dual such that $V \subset H \subset V'$ algebraically and topologically. The existence result given below is the time-dependent analogue of Theorem 4.10.

Theorem 4.17. *Let $\{A(t);\ t \in [0,T]\}$ be a family of nonlinear, monotone, and demicontinuous operators from V to V' satisfying the assumptions:*

(i) *The function $t \to A(t)u(t)$ is measurable from $[0,T]$ to V' for every measurable function $u : [0,T] \to V$.*

(ii) $(A(t)u,u) \ge \omega\|u\|^p + C_1,\ \forall u \in V,\ t \in [0,T]$.

(iii) $\|A(t)u\|_{V'} \le C_1(1+\|u\|^{p-1}),\ \forall u \in V,\ t \in [0,T]$, *where $\omega > 0$, $p > 1$.*

Then, for every $y_0 \in H$ and $f \in L^q(0,T;V')$, $1/p+1/q=1$, there is a unique absolutely continuous function $y \in W^{1,q}([0,T];V')$ that satisfies

$$\begin{aligned} y \in C([0,T];H) &\cap L^p(0,T;V), \\ \frac{dy}{dt}(t) + A(t)y(t) &= f(t), \quad \text{a.e. } t \in (0,T), \\ y(0) &= y_0. \end{aligned} \tag{4.136}$$

Proof. For the sake of simplicity, we assume first that $p \ge 2$. Consider the spaces

$$\mathcal{V} = L^p(0,T;V), \qquad \mathcal{H} = L^2(0,T;H), \qquad \mathcal{V}' = L^q(0,T;V').$$

Clearly, we have

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$$

algebraically and topologically.

Let $y_0 \in H$ be arbitrary and fixed and let $B : \mathscr{V} \to \mathscr{V}'$ be the operator

$$Bu = \frac{du}{dt}, \qquad u \in D(B) = \left\{ u \in \mathscr{V};\ \frac{du}{dt} \in \mathscr{V}',\ u(0) = y_0 \right\},$$

where d/dt is considered in the sense of vectorial distributions on $(0,T)$. We note that $D(B) \subset W^{1,q}(0,T;V') \cap L^q(0,T;V) \subset C([0,T];H)$, so that $y(0) = y_0$ makes sense.

Let us check that B is maximal monotone in $\mathscr{V} \times \mathscr{V}'$. Because B is clearly monotone, by virtue of Theorem 2.3, it suffices to show that $R(B + \Phi_p) = \mathscr{V}'$, where

$$\Phi_p(u(t)) = F(u(t))\|u(t)\|^{p-2}, \qquad u \in \mathscr{V},$$

and $F : V \to V'$ is the duality mapping of V. Indeed, for every $f \in \mathscr{V}'$ the equation

$$Bu + \Phi_p(u) = f,$$

or, equivalently,

$$\frac{du}{dt} + F(u)\|u\|^{p-2} = f \quad \text{in } [0,T],\ u(0) = y_0,$$

has, by virtue of Theorem 4.10, a unique solution

$$u \in C([0,T];H) \cap L^p(0,T;V), \qquad \frac{du}{dt} \in L^q(0,T;V').$$

(Renorming the spaces V and V', we may assume that V and V' are strictly convex and F is demicontinuous and that so is the operator $u \to F(u)\|u\|^{p-2}$.) Hence, B is maximal monotone in $\mathscr{V} \times \mathscr{V}'$.

Define the operator $A_0 : \mathscr{V} \to \mathscr{V}'$ (the realization of A in pair $\mathscr{V}, \mathscr{V}'$) by

$$(A_0 u)(t) = A(t)u(t), \quad \text{a.e. } t \in (0,T).$$

Clearly, A_0 is monotone, demicontinuous, and coercive from $\mathscr{V}$ to $\mathscr{V}'$ because so is $A(t) : V \to V'$.

Then, by Corollaries 2.2 and 2.6, $A_0 + B$ is maximal monotone and surjective. Hence, $R(A_0 + B) = \mathscr{V}'$, which completes the proof.

The proof in the case $1 < p < 2$ is completely similar if we take $\mathscr{V} = L^p(0,T;V) \cap L^2(0,T;H)$ and replace $A(t)$ by $A(t) + \lambda I$ for some $\lambda > 0$. The details are left to the reader. $\square$

Remark 4.11. It should be said that Theorem 4.17 applies neatly to the parabolic boundary value problem

$$\begin{aligned}
\frac{\partial y}{\partial t}(x,t) - \sum_{|\alpha|\le m} D^\alpha(A_\alpha(t,x,y,D^\beta y)) &= f(x,t), \qquad && (x,t)\in\Omega\times(0,T)\\
y(x,0) &= y_0(x), && x\in\Omega\\
D^\beta y &= 0 && \text{on } \partial\Omega \text{ for } |\beta|<m,
\end{aligned}$$

where $A_\alpha : [0,T]\times\Omega\times\mathbf{R}^{mN}\to\mathbf{R}^{mN}$ are measurable in (t,x), continuous in other variables and satisfy for each $t\in[0,T]$ assumptions (i)–(iii) in Remark 2.6.

Then we apply Theorem 4.17 for $V = W_0^{m,p}(\Omega), V' = W^{-m,q}(\Omega)$ and $A(t): V\to V'$ defined by

$$(A(t)y,z) = \sum_{|\alpha|\le m}\int_\Omega A_\alpha(t,x,y(x),D^\beta y(x))\cdot D^\alpha y(x)dx, \qquad \forall y,z\in W_0^{m,p}(\Omega).$$

Hence, for $f\in L^q(0,T;W^{-m,q}(\Omega))$, $y_0\in L^2(\Omega)$, there is a unique solution

$$\begin{aligned}
&y\in L^p(0,T;W_0^{m,p}(\Omega))\cap C([0,T];L^2(\Omega))\\
&\frac{dy}{dt}\in L^q(0,T;W^{-m,q}(\Omega)).
\end{aligned}$$

Subpotential Time-Dependent Evolutions

Let $X=H$ be a real Hilbert space and $A(t)=\partial\varphi(t,y)$, $t\in[0,T]$, where $\varphi(t): H\to\overline{\mathbf{R}}=(-\infty,\infty]$ is a family of convex and lower semicontinuous functions satisfying the following conditions.

(k) For each measurable function $y:[0,T]\to H$, the function $t\to\varphi(t,y(t))$ is measurable on $(0,T)$.

(kk) $\varphi(t,y)\le\varphi(s,y)+\alpha|t-s|(\varphi(s,y)+|y|^2+1)$ for all $y\in H$ and $0\le s\le t\le T$.

Here α is a nonnegative constant.

We note that, in particular, assumption (kk) implies that $D\varphi(s,\cdot)\subset D\varphi(t,\cdot)$ for all $0\le s\le t\le T$. A standard example of such a family $\{\varphi(t,\cdot)\}_t$ is

$$\varphi(t,\cdot)=I_{K(t)}, \qquad t\in[0,T],$$

where $\{K(t)\}_t$ is an increasing family of closed convex subsets such that the function $t\to P_{K(t)}y(t)$ is measurable for each measurable function $y:[0,T]\to H$. Here, $P_{K(t)}=(I+\lambda\partial I_{K(t)})^{-1}$ is the projection operator on $K(t)$ and the last assumption implies of course (k) for $\varphi(t)=I_{K(t)}$.

Theorem 4.18. *Assume that $\varphi:[0,T]\times H\to\overline{\mathbf{R}}=(-\infty,\infty]$ satisfies hypotheses* (k), (kk). *Then, for each $y_0\in D(\varphi(0,\cdot))$ and $f\in L^2(0,T;H)$, there is a unique pair of functions $y\in W^{1,2}([0,T];H)$ and $\eta\in L^2(0,T;H)$ such that*

$$\begin{aligned} \eta(t) \in \partial\varphi(t,y(t)), &\quad a.e.\ t \in (0,T), \\ \frac{dy}{dt}(t) + \eta(t) = f(t), &\quad a.e.\ t \in (0,T), \\ y(0) = y_0. & \end{aligned} \tag{4.137}$$

This means that y is solution to (4.135), where $A(t) = \partial\varphi(t,\cdot)$.

Proof. It suffices to prove the existence in the sense of (4.137) for the equation

$$\begin{aligned} \frac{dy}{dt}(t) + \partial\varphi(t,y(t)) + \lambda_0 y(t) \ni f(t), &\quad \text{a.e. } t \in (0,T), \\ y(0) = y_0, & \end{aligned} \tag{4.138}$$

where $\lambda_0 > 0$ is arbitrary but fixed. Indeed, by the substitution $e^{\lambda_0 t} y \to y$, equation (4.138) reduces to

$$\frac{dy}{dt}(t) + e^{\lambda_0 t}\partial\varphi(t, e^{-\lambda_0 t} y(t)) \ni e^{\lambda_0 t} f(t), \qquad t \in [0,T);$$

that is,

$$\frac{dy}{dt} + \partial\widetilde{\varphi}(t,y) \ni e^{\lambda_0 t} f, \qquad t \in (0,T),$$

where $\widetilde{\varphi}(t,y) = e^{2\lambda_0 t}\varphi(t, e^{-\lambda_0 t} y)$ and $e^{\lambda_0 t}\partial\varphi(t, e^{-\lambda_0 t} y) = \partial\widetilde{\varphi}(t,y)$.

Clearly, $\widetilde{\varphi}$ satisfies assumptions (k), (kk).

Now, we may rewrite equation (4.138) in the space $\mathscr{H} = L^2(0,T;H)$ as

$$By + \mathscr{A}y + \lambda_0 y \ni f, \tag{4.139}$$

where

$$By = \frac{dy}{dt}, \quad D(B) = \{y \in W^{1,2}([0,T];H)\ y(0) = y_0\},$$

$$\mathscr{A}y = \{\eta \in L^2(0,T;H);\ \eta(t) \in \partial\varphi(t,y(t)), \quad \text{a.e. } t \in (0,T)\},$$

$$D(\mathscr{A}) = \{y \in L^2(0,T;H),\ \exists\eta \in L^2(0,T;H),\ \eta(t) \in \partial\varphi(t,y(t)), \quad \text{a.e. } t \in (0,T)\}.$$

Because, as easily seen, $\mathscr{A}$ is maximal monotone in $\mathscr{H} \times \mathscr{H}$ and $\mathscr{A} \subset \partial\varphi$, we infer that $\mathscr{A} = \partial\phi$, where $\phi : \mathscr{H} \to (-\infty,+\infty]$ is the convex function

$$\phi(y) = \int_0^T \varphi(t,y(t))dt. \tag{4.140}$$

By assumption (k), it follows via Fatou's lemma that ϕ is also lower semicontinuous and nonidentically $+\infty$ on $\mathscr{H}$. (The latter follows by (kk).)

To prove the existence for equation (4.138) (equivalently (4.139)), we apply Proposition 3.9. To this end it suffices to check the inequality

$$\phi((I+\lambda B)^{-1}y) \leq \phi(y) + C\lambda(\phi(y) + |y|^2_{\mathscr{H}} + 1), \qquad \forall y \in \mathscr{H}. \tag{4.141}$$

We notice that

$$(I+\lambda B)^{-1}y = e^{-(t/\lambda)}y_0 + \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}y(s)ds, \qquad \forall \lambda > 0,\ t \in (0,T),$$

and this yields (by convexity of $y \to \varphi(t,y)$ and by (kk))

$$\begin{aligned}
\phi((I+\lambda B)^{-1}y) &= \int_0^T \varphi\left(t, e^{-(t/\lambda)}y_0 + \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}y(s)ds\right)dt \\
&\leq \int_0^T \left(e^{-(t/\lambda)}\varphi(t,y_0) + \frac{1}{\lambda}\int_0^t e^{-(t-s)/\lambda}\varphi(t,y(s))ds\right)dt \\
&\leq C\lambda(1-e^{-(T/\lambda)})\varphi(0,y_0) + \alpha T(\varphi(0,y_0) + |y_0|^2 + 1)) \\
&\quad + \frac{1}{\lambda}\int_0^T dt \int_0^t e^{-(t-s)/\lambda}\varphi(s,y(s))ds \\
&\quad + \frac{\alpha}{\lambda}\int_0^T dt \int_0^t e^{-(t-s)/\lambda}(\varphi(s,y(s)) + 1 + |y(s)|^2)|t-s|ds \\
&\leq \frac{1}{\lambda}\int_0^T \varphi(s,y(s))ds \int_s^T e^{-(t-s)/\lambda}dt \\
&\quad + \frac{\alpha}{\lambda}\int_0^T (\varphi(s,y(s)) + |y(s)|^2)ds \int_s^T e^{-(t-s)/\lambda}|t-s|dt \\
&\quad + C\lambda(\varphi(0,y_0) + |y_0|^2 + 1 \\
&\leq \phi(y) + C\lambda(\varphi(0,y_0) + \phi(y) + |y|^2_{\mathscr{H}} + 1). \qquad \square
\end{aligned}$$

Time-Dependent *m*-Accretive Evolution

We consider here equation (4.135) under the following assumptions.

(j) $\{A(t)\}_{t\in[0,T]}$ is a family of *m*-accretive operators in X such that, for all $\lambda > 0$,

$$\begin{aligned}\|A_\lambda(t)y - A_\lambda(s)y\| &\leq C|t-s|(\|A_\lambda(t)y\| + \|y\| + 1), \\ &\forall y \in X,\ \forall s,t \in [0,T].\end{aligned} \tag{4.142}$$

Here, $A_\lambda(t)$ is the Yosida approximation of $y \to A(t,y)$. (See (3.1).)

Unlike the previous situations considered here, condition (4.142) has the unpleasant consequence that the domain of $A(t)$ is independent of t; that is, $D(A(t)) \equiv D(A(0))$, $\forall t \in [0,T]$. This assumption is, in particular, too restrictive if we want to treat partial differential equations with time-dependent boundary value conditions, but it is, however, satisfied in a few significant cases involving partial differential equations with smooth time-dependent nonlinearities.

Theorem 4.19. *Assume that X is a reflexive Banach space with uniformly convex dual X^*. If $\{A(t)\}$ satisfies assumption* (j)*, then, for each $f \in W^{1,1}([0,T];X)$ and $y_0 \in D \equiv D(A(t))$, there is a unique function $y \in W^{1,\infty}([0,T];X)$ such that*

$$\begin{cases} \dfrac{dy}{dt}(t) + A(t)y(t) \ni f(t), & a.e.\ t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{4.143}$$

Proof. We start, as usual, with the approximating equation

$$\begin{aligned} \frac{dy_\lambda}{dt} + A_\lambda(t)y_\lambda(t) &= f(t), \quad t \in (0,T), \\ y_\lambda(0) &= y_0, \end{aligned} \tag{4.144}$$

which has a unique solution $y_\lambda \in C^1([0,T];X)$. By (4.142) and (4.144) and the accretivity of $A_\lambda(t)$, we see that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|y_\lambda(t+h) - y_\lambda(t)\|^2 \\ &\le (A_\lambda(t+h)y_\lambda(t) - A_\lambda(t)y_\lambda(t), J(y_\lambda(t+h) - y_\lambda(t))) \\ &\le C|h|\,\|y_\lambda(t+h) - y_\lambda(t)\|(\|A_\lambda(t)y_\lambda(t)\| + \|y_\lambda(t)\| + 1), \qquad \forall t, t+h \in [0,T]. \end{aligned}$$

This yields

$$\begin{aligned} &\|y_\lambda(t+h) - y_\lambda(t)\| \\ &\le C\int_0^t (\|A_\lambda(s)y_\lambda(s)\| + \|y_\lambda(s)\| + 1)ds + \|y_\lambda(h) - y_0\|. \end{aligned} \tag{4.145}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|y_\lambda(h) - y_\lambda(0)\|^2 = &-(A_\lambda(t)y_\lambda(t), J(y_\lambda(t) - y_0)) \\ &+ (f(t), J(y_\lambda(t) - y_0)), \quad \text{a.e. } t \in (0,T), \end{aligned}$$

and therefore

$$\begin{aligned} \|y_\lambda(h) - y_0\| &\le \int_0^h \|A_\lambda(s)y_0\|ds + \|f\|_{L^\infty(0,T;H)}h \\ &\le h(\|A_\lambda(0)y_0\| + \|f\|_{L^\infty(0,T;H)}). \end{aligned}$$

Then, substituting into (4.144) and letting $h \to 0$, we obtain that

$$\begin{aligned} \left\|\frac{dy_\lambda}{dt}(t)\right\| \le C\Big(&\int_0^t (\|A_\lambda(s)y_\lambda(s)\| + \|y_\lambda(s)\| + 1)ds \\ &+ \|A^0(0)y_0\| + \|f\|_{L^\infty(0,T;H)}\Big), \qquad \forall \lambda > 0. \end{aligned} \tag{4.146}$$

On the other hand, by (4.144) we also have that

$$\|y_\lambda(t)\| \le C, \qquad \forall t \in [0,T],\ \lambda > 0.$$

By (4.144) and (4.146), we get via Gronwall's lemma that

$$\left\| \frac{dy_\lambda}{dt}(t) \right\| + \|A_\lambda(t) y_\lambda(t)\| \le C, \qquad \forall \lambda > 0,\ t \in [0,T]. \tag{4.147}$$

Then, by (4.147) we find as in the proof of Theorem 4.6 that the sequence $\{y_\lambda\}_\lambda$ is Cauchy in $C([0,T];X)$ and $y = \lim_{\lambda\to 0} y_\lambda$ is the solution to (4.143). The details are left to the reader. □

4.4 Time-Dependent Cauchy Problem Versus Stochastic Equations

The above methods apply as well to stochastic differential equations in Hilbert spaces with additive Gaussian noise because, as we show below, these equations can be reduced to time-dependent deterministic equations depending on a random parameter. Below we treat only two problems of this type and refer to standard monographs for complete treatment.

Consider the stochastic differential equation in a separable Hilbert space H,

$$\begin{cases} dX(t) + AX(t)dt = B\,dW(t), & t \ge 0, \\ X(0) = x. \end{cases} \tag{4.148}$$

Here $A : D(A) \subset H \to H$ is a quasi-m-accretive operator in H, $B \in L(U,H)$, where U is another Hilbert space and $W(t)$ is a cylindrical Wiener process in U defined on a probability space $\{\Omega, \mathscr{F}, \mathbb{P}\}$. This means that

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k,$$

where $\{e_k\}_k$ is an orthonormal basis in U and $\{\beta_k\}_k$ is a sequence of mutually independent Brownian motions on $\{\Omega, \mathscr{F}, \mathbb{P}\}$. Denote by $\mathscr{F}_t$ the σ-algebra generated by $\beta_k(s)$ for $s \le t$, $k \in \mathbf{N}$ (also called *filtration*).

By solution to (4.148) we mean a stochastic process $X = X(t)$ on $\{\Omega, \mathscr{F}, \mathbb{P}\}$ adapted to $\mathscr{F}_t$; that is, $X(t)$ is measurable with respect to the σ-algebra $\mathscr{F}_t$, and satisfies equation

$$X(t) = x - \int_0^t AX(s)ds + \int_0^t B\,dW(s)ds, \qquad \forall t \ge 0,\ \mathbb{P}\text{-a.s.}, \tag{4.149}$$

where the integral $\int_0^t B\,dW(s)$ is considered in the sense of Ito (see Da Prato [28], Da Prato and Zabczyk [29], and Prévot and Roeckner [48]) for the definition and basic existence results for equation (4.149).

A standard way to study the existence for equation (4.148) is to reduce it via substitution

$$y(t) = X(t) - BW(t)$$

to the random differential equation

$$\begin{cases} \dfrac{d}{dt}y(t,\omega) + A(y(t,\omega) + BW(t,\omega)) = 0,\ t \geq 0,\ \mathbb{P}\text{-a.s.},\ \omega \in \Omega, \\ y(0,\omega) = x. \end{cases} \tag{4.150}$$

For almost all $\omega \in \Omega$ (i.e., P-a.s.), (4.150) is a deterministic time-dependent equation in H of the form (4.135); that is,

$$\begin{cases} \dfrac{dy}{dt}(t) + A(t)y(t) = 0, \qquad t \geq 0, \\ y(0) = x, \end{cases}$$

where $A(t)y = A(y + BW(t))$. This fact explains why one cannot expect a complete theory of existence similar to that from the deterministic case. In fact, because the Wiener process $t \to W(t)$ does not have bounded variation, Theorems 4.18 and 4.19 are inapplicable in the present situation. More appropriate for this scope is, however, Theorem 4.17 which requires no regularity in t for $A(t)$.

Then, we assume that V is a reflexive Banach space continuously embedded in H and so we have

$$V \subset H \subset V'$$

algebraically and topologically, where V' is the dual space of V.

Let $A : V \to V'$ satisfy the conditions of Theorem 4.10:

(ℓ) A is a demicontinuous monotone operator and

$$(Au, u) \geq \gamma \|u\|_V^p + C_1, \qquad \forall u \in V,$$

$$\|Au\|_{V'} \leq C_2(1 + \|u\|_V^{p-1}), \qquad \forall u \in V,$$

where $\gamma > 0$ and $p > 1$.

Then, we have the following theorem.

Theorem 4.20. *Assume that A satisfies hypothesis* (ℓ) *and that*

$$BW \in L^p(0,T;V),\ \mathbb{P}\text{-a.s.} \tag{4.151}$$

Then, for each $x \in H$, equation (4.150) *has a unique adapted solution $X = X(t,\omega) \in L^p(0,T;V) \cap C([0,T];H)$, a.e. $\omega \in \Omega$.*

Proof. One simply applies Theorem 4.17 to the operator $A(t)y = A(y+BW(t))$ and check that conditions (i)–(iii) are satisfied under hypotheses (ℓ) and (4.151).

Thus, one finds a solution $X = X(t,\omega)$ to (4.150) that satisfies (4.76) for $\mathbb{P}$-almost all $\omega \in \Omega$. Taking into account that, as seen earlier, such a solution can be obtained as the limit of solutions y_λ to the approximating equations

$$\begin{cases} \frac{d}{dt} y_\lambda + A_\lambda(y_\lambda + BW) = 0, & t \in (0,T), \\ y_\lambda(0) = x, \end{cases}$$

where A_λ is the Yosida approximation of $A\big|_H$ (the restriction of the operator A to H), we may conclude that X is adapted with respect to the filtration $\{\mathscr{F}_t\}$. One might also prove H-continuity of $t \to X(t,\omega)$ by the methods of Krylov and Rozovski [39] (see also Prévot and Roeckner [48]), which completes the proof. In particular, Theorem 4.20 applies to parabolic stochastic differential equations of the type mentioned in Remark 4.11. □

It should be said, however, that this variational framework covers only a small part of stochastic partial differential equations because most of them cannot be written in this variational setting and so, in general, other arguments should be involved. This is the case, for instance, with the reflection problem for stochastic differential equations in a Hilbert space H. Namely, for the equation

$$\begin{aligned} dX(t) + (AX(t) + F(X(t)) + \partial I_K(X(t)))dt &\ni \sqrt{Q}\,dW(t), \\ X(0) &= x \in K, \end{aligned} \tag{4.152}$$

where K is a closed convex subset of H such that $0 \in \overset{\circ}{K}$ and

(j) $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that A^{-1} is compact and $(Ax,x) \geq \delta |x|^2$, $\forall x \in D(A)$, for some $\delta > 0$.

(jj) $Q : H \to H$ is a linear, bounded, positive, and self-adjoint operator on H such that $Qe^{-tA} = e^{-tA}Q$ for all $t \geq 0$, $Q(H) \subset D(A)$ and $\mathrm{Tr}[AQ] < \infty$.

(jjj) $F : H \to H$ is a Lipschitzian mapping such that, for some $\gamma > 0$, we have

$$(F(x),x) \geq -\gamma |x|^2, \qquad \forall x \in H.$$

(jv) W is a cylindrical Wiener process on H of the form

$$W(t) = \sum_{k=1}^{\infty} \mu_k \beta_k(t) e_k, \qquad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on filtered probability spaces $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ (see [28]) and $\{e_k\}$ is an orthonormal basis in H taken as a system of eigenfunctions for A.

We denote, as usual, by $C([0,T];H)$ the space of all continuous functions from $[0,T]$ to H and by $BV([0,T];H)$ the space of all functions with bounded variation from $[0,T]$ to H. We set $V = D(A^{1/2})$ with the norm $\|\cdot\|$ and denote by V' the dual of V in the pairing induced by the scalar product $(\cdot,\cdot)$ of H. By $C_W([0,T];H)$, $L^2_W([0,T];V)$, $L^2_W([0,T];V')$ we denote the standard spaces of adapted processes on $[0,T]$ (see [28, 29]).

Denote by W_A the stochastic convolution,

$$W_A(t) = \int_0^t e^{-A(t-s)}\sqrt{Q}\,dW(s)$$

and note that (4.152) can be rewritten as

$$\begin{cases} \dfrac{d}{dt}Y(t)+AY(t)+F(Y(t)+W_A(t))+\partial I_K(Y(t)+W_A(t)) \ni 0, \\ \qquad\qquad\qquad\qquad \forall t \in (0,T),\ \mathbb{P}\text{-a.s. } \omega \in \Omega \\ Y(0) = x, \end{cases} \tag{4.152$'$}$$

where $Y(t) = X(t) - W_A(t)$.

Definition 4.5. The adapted process $X \in C_W(0,T];H) \cap L^2_W(0,T;V)$ is said to be a solution to (4.152) if there are functions $Y \in C_W([0,T];H) \cap L^2_W(0,T;V)$ and $\eta \in BV([0,T];H)$ such that $X(t) = Y(t) + W_A(t) \in K$, a.e. in $\Omega \times (0,T)$ and

$$Y(t) + \int_0^t (AY(s) + F(X(s)))ds + \eta(t) = x, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{4.153}$$

$$\int_0^t (d\eta(s), X(s) - Z(s))ds \geq 0, \qquad \forall Z \in C([0,T];K),\ \mathbb{P}\text{-a.s.} \tag{4.154}$$

Here $\int_0^t (d\eta(s), X(s) - Z(s))ds$ is the Stieltjes integral with respect to η.

Theorem 4.21 below is an existence result for equation (4.152) (equivalently, (4.152)$'$) and is given only to illustrate how the previous methods work in the case of stochastic infinite-dimensional equations.

Theorem 4.21. *Under the above hypotheses there is a unique strong solution to equation* (4.152).

Proof. Existence. We start with the approximating equation

$$\begin{cases} dX_\varepsilon + (AX_\varepsilon + F(X_\varepsilon) + \beta_\varepsilon(X_\varepsilon))dt = \sqrt{Q}\,dW, \\ X_\varepsilon(0) = x, \end{cases} \tag{4.155}$$

where β_ε is the Yosida approximation of ∂I_K,

$$\beta_\varepsilon(x) = \frac{1}{\varepsilon}(x - \Pi_K(x)), \qquad \forall x \in H,\ \varepsilon > 0,$$

and Π_K is the projection on K.

Equation (4.155) has a unique strong solution $X_\varepsilon \in C_W([0,T];H)$ such that $Y_\varepsilon := X_\varepsilon - W_A$ belongs to $L^2_W(0,T;H)$. As seen above, we can rewrite (4.155) as

$$\begin{cases} \dfrac{dY_\varepsilon}{dt} + AY_\varepsilon + F(X_\varepsilon) + \beta_\varepsilon(X_\varepsilon) = 0, \\ Y_\varepsilon(0) = x, \end{cases} \tag{4.156}$$

which is considered here for a fixed $\omega \in \Omega$. Because $0 \in \overset{\circ}{K}$, there is $\rho > 0$ such that $(\beta_\varepsilon(x), x - \rho\theta) \geq 0$, $\forall \theta \in H$, $|\theta| = 1$. This yields $\rho|\beta_\varepsilon(x)| \leq (\beta_\varepsilon(x), x)$, $\forall x \in H$.

Step 1. There exists $C = C(\omega) > 0$ such that

$$|Y_\varepsilon(t)|^2 + \int_0^t \|Y_\varepsilon(s)\|^2 ds + \int_0^t |\beta_\varepsilon(X_\varepsilon(s))| ds \leq C. \tag{4.157}$$

Indeed, multiplying (4.156) scalarly in H by $Y_\varepsilon(s)$ and integrating over $(0,t)$ yields

$$\begin{aligned} &\frac{1}{2}|Y_\varepsilon(t)|^2 + \int_0^t \|Y_\varepsilon(s)\|^2 ds + \rho \int_0^t |\beta_\varepsilon(X_\varepsilon(s))| ds \\ &\leq \frac{1}{2}|x|^2 + \gamma \int_0^t |X_\varepsilon(s)|^2 ds + \int_0^t (F(X_\varepsilon(s)) + \beta_\varepsilon(X_\varepsilon(s)), W_A(s)) ds. \end{aligned} \tag{4.158}$$

In order to estimate the last term in formula (4.158), we choose a decomposition $0 < t_1 < \cdots < t_N = t$ of $[0,t]$ such that, for $t, s \in [t_{i-1}, t_i]$, we have

$$|W_A(t) - W_A(s)| \leq \frac{\rho}{2}.$$

This is possible because W_A is $\mathbb{P}$-a.s. continuous in H, and so we may assume that

$$\sup_{t\in[0,T]} |W_A(t+h) - W_A(t)| \leq \delta(h) \to 0 \quad \text{as } h \to 0,$$

because by (jj) it follows that W_A is $\mathbb{P}$-a.s. continuous in H (see Da Prato [28]).

Then, we write

$$\begin{aligned} \int_0^t (\beta_\varepsilon(X_\varepsilon(s)), W_A(s)) ds = &\sum_{i=1}^N \int_{t_{i-1}}^{t_i} (\beta_\varepsilon(X_\varepsilon(s)), W_A(s) - W_A(t_i)) ds \\ &+ \sum_{i=1}^N \left(W_A(t_i), \int_{t_{i-1}}^{t_i} \beta_\varepsilon(X_\varepsilon(s)) ds \right). \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_0^t (\beta_\varepsilon(X_\varepsilon(s)), W_A(s)) ds \leq \frac{\rho}{2} \int_0^t |\beta_\varepsilon(X_\varepsilon(s))| ds \\ &+ \left| \sum_{i=1}^N \left(W_A(t_i), \int_{t_{i-1}}^{t_i} (AY_\varepsilon(s) + F(X_\varepsilon(s))) ds + Y_\varepsilon(t_i) - Y_\varepsilon(t_{i-1}) \right) \right|. \end{aligned}$$

Now, using the estimate

$$\left(W_A(t_i), \int_{t_{i-1}}^{t_i} AY_\varepsilon(s)ds\right) \le C\int_{t_{i-1}}^{t_i} \|Y_\varepsilon(s)\|^2 ds,$$

we get (4.157).

We now prove that the sequence $\{Y_\varepsilon\}$ is equicontinuous in $C([0,T];H)$. Let $h>0$, then we have

$$\begin{aligned}&\frac{d}{dt}(Y_\varepsilon(t+h)-Y_\varepsilon(t))+A(Y_\varepsilon(t+h)-Y_\varepsilon(t))\\&+F(X_\varepsilon(t+h))-F(X_\varepsilon(t))+\beta_\varepsilon(X_\varepsilon(t+h))-\beta_\varepsilon(X_\varepsilon(t))=0.\end{aligned}$$

By the monotonicity of β_ε and because F is Lipschitz continuous, we have

$$|Y_\varepsilon(t+h)-Y_\varepsilon(t)| \le C\delta(h), \qquad \forall t\in[0,T],\ h>0,\ \varepsilon>0.$$

So, $\{Y_\varepsilon\}$ is equi-continuous. To apply the Ascoli-Arzelà theorem, we have to prove that, for each $t\in[0,T]$, the set $\{Y_\varepsilon(t)\}_{\varepsilon>0}$ is pre-compact in H. To prove this, choose for any $\varepsilon>0$ a sequence $\{f_n^\varepsilon\}\subset L^2(0,T;V)$ such that

$$|f_n^\varepsilon-\beta_\varepsilon(Y_\varepsilon+W_A)|_{L^1(0,T;H)} \le \frac{1}{n}, \qquad n\in\mathbf{N}.$$

On the other hand, for each $n\in\mathbf{N}$, the set

$$M_n := \left\{\int_0^t e^{-A(t-s)} f_n^\varepsilon ds + e^{-At}x : \varepsilon>0\right\}$$

is compact in H because $\{f_n^\varepsilon\}$ is bounded in $L^2(0,T;H)$ for each $n\in\mathbf{N}$. This implies that, for any $\delta>0$, there are $N(n)\in\mathbf{N}$ and $\{u_i^n\}_{i=1,\dots,N(n)}\subset H$ such that

$$\bigcup_{i=1}^{N(n)} B(u_i^n,\delta)\supset M_n.$$

Therefore,

$$\left\{Y_\varepsilon(t) := \int_0^t e^{-A(t-s)} f_n^\varepsilon ds + e^{-At}x : \varepsilon>0\right\} \subset \bigcup_{i=1}^{N(n)} B(u_i^n,\delta+n^{-1}).$$

Hence, the set $\{Y_\varepsilon(t)\}_{\varepsilon>0}$ is precompact in H, as claimed. Then, by the Ascoli–Arzelà theorem we infer that on a subsequence, $Y_\varepsilon\to Y$ strongly in $C([0,T];H)$ and weakly in $L^2(0,T;V)$. Moreover, thanks to Helly's theorem (see [9]), we have that there is $\eta\in BV([0,T];H)$ such that, for $\varepsilon\to 0$,

$$\int_0^t \beta_\varepsilon(X_\varepsilon(s))ds \to \eta(t) \quad \text{weakly in } H, \qquad \forall t\in[0,T],$$

which implies that

$$\int_0^t (\beta_\varepsilon(X_\varepsilon(s)), Z(s))ds \to \int_0^t (d\eta(s), Z(s))ds, \qquad \forall Z \in C([0,T];K).$$

Letting $\varepsilon \to 0$ into the identity

$$Y_\varepsilon(t) + \int_0^t (AY_\varepsilon(s + F(Y_\varepsilon(s)))ds + \int_0^t \beta_\varepsilon(Y_\varepsilon(s) + W_A(s)))ds = x,$$

we see that (Y, η) satisfy (4.153).

Finally, by the monotonicity of β_ε we have (recall that $\beta_\varepsilon(Z(s)) = 0$),

$$(\beta_\varepsilon(Y_\varepsilon(s) + W_A(s)), Y_\varepsilon(s) + W_A(s) - Z(s)) \geq 0, \qquad \forall Z \in C([0,T];K),$$

and so (4.154) holds.

Uniqueness. Assume that $(Y_1, \eta_1), (Y_2, \eta_2)$ are two solutions. Then, we have

$$\int_0^t (d(\eta_1(s) - \eta_2(s)), Y_1(s) - Y_2(s))ds \geq 0, \qquad \forall t \in [0,T].$$

This yields

$$\int_0^t \Big(d(Y_1(s) - Y_2(s)) + \int_0^s (A(Y_1(\tau) - Y_2(\tau)) + F(X_1(\tau) - F(X_2(\tau)))d\tau, Y_1(s) - Y_2(s)) \Big) \leq 0$$

and, by integration, we obtain that

$$\frac{1}{2}|Y_1(t) - Y_2(t)|^2 + \int_0^t (A(Y_1 - Y_2) + F(X_1) - F(X_2), Y_1 - Y_2)ds \leq 0,$$

$\forall t \in [0,T]$, which implies via Gronwall's lemma that $Y_1 = Y_2$.

In particular, the latter implies that the sequence $\{\varepsilon\}$ founded before is independent of ω and so, there is indeed a unique pair satisfying Definition 4.5. (For proof details, we refer to Barbu and Da Prato [6].) □

Remark 4.12. The above argument can be formalized to treat more general equations of the form (4.152)′ and, in particular, the so-called variational inequalities with singular inputs (see Barbu and Răşcanu [7]). In the literature, such a problem is also called the Skorohod problem (see, e.g., Cépa [20]).

Bibliographical Remarks

The existence theory for the Cauchy problem associated with nonlinear m-accretive operators in Banach spaces begins with the influential pioneering papers of Komura [37, 38] and Kato [32] in Hilbert spaces. The theory was subsequently extended in a more general setting by several authors mentioned below.

The main result of Section 4.1 is due to Crandall and Evans [23] (see also Crandall [22]), and Theorem 4.3 has been previously proved by Crandall and Liggett [24]. The existence and uniqueness of integral solutions for problem (4.1) (see Theorem 4.18) is due to Bénilan [10]. Theorems 4.5 and 4.6 were established in a particular case in Banach space by Komura [37] (see also Kato [32]) and later extended in Banach spaces with uniformly convex duals by Crandall and Pazy [25, 26]. Note that the generation theorem, 4.3 remains true for m-accretive operators satisfying the extended range condition (Kobayashi [35])

$$\liminf_{h\downarrow 0} \frac{1}{h} d(x, R(I+\lambda A)) = 0, \qquad \forall x \in \overline{D(A)},$$

$d(x,K)$ is the distance from x to K.

The basic properties of continuous semigroups of contractions have been established by Komura [38], Kato [33], and Crandall and Pazy [25, 26]. For other significant results of this theory, we refer the reader to the author's book [5]. (See also Showalter [50].) The results of Section 4.4 are due to Brezis [13, 14]. Other results related to the smoothing effect of nonlinear semigroups are given in the book by Barbu [5].

Convergence results of the type presented in Section 4.2 were obtained by Brezis and Pazy [16], Kobayashi and Myadera [36], and Goldstein [30].

Time-dependent differential equations of subdifferential type under conditions given here (Section 4.3) were studied by Moreau [41], Peralba [47], Kenmochi [34], and Attouch and Damlamian [3].

Other special problems related to evolutions generated by nonlinear accretive operators are treated in Vrabie's book [54]. We mention in this context a characterization of compact semigroups of nonlinear contractions and evolutions generated by operators of the form $A+F$, where A is m-accretive and F is upper semicontinuous and compact. For other results such as asymptotic behavior and existence of periodic and almost periodic solutions to problem (4.1), we refer the reader to the monographs of Haraux [31] and Moroşanu [42].

We have omitted from our presentation the invariance and viability results related to nonlinear contraction semigroups on closed subsets. We mention in this context the books of Aubin and Cellina [4], Pavel [43, 44] and the recent monograph of Cârjă, Necula, and Vrabie [19], which contains detailed results and complete references on this subject.

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