Chapter 1
Fundamental Functional Analysis

Abstract The aim of this chapter is to provide some standard basic results pertaining to geometric properties of normed spaces, convex functions, Sobolev spaces, and variational theory of linear elliptic boundary value problems. Most of these results, which can be easily found in textbooks or monographs, are given without proof or with a sketch of proof only.

1.1 Geometry of Banach Spaces

Throughout this section X is a real normed space and X^* denotes its dual. The value of a functional $x^* \in X^*$ at $x \in X$ is denoted by either (x^*,x) or $x^*(x)$, as is convenient. The norm of X is denoted by $\|\cdot\|$, and the norm of X^* is denoted by $\|\cdot\|_*$. If there is no danger of confusion we omit the asterisk from the notation $\|\cdot\|_*$ and denote both the norms of X and X^* by the symbol $\|\cdot\|$.

We use the symbol lim or $\to$ to indicate *strong convergence* in X and w-lim or $\rightharpoonup$ for *weak convergence* in X. By w^*-lim or $\rightharpoonup$ we indicate *weak-star* convergence in X^*. The space X^* endowed with the weak-star topology is denoted by X^*_w.

Define on X the mapping $J : X \to 2^{X^*}$:

$$J(x) = \{x^* \in X^*;\ (x^*,x) = \|x\|^2 = \|x^*\|^2\}, \qquad \forall x \in X. \tag{1.1}$$

By the Hahn–Banach theorem we know that for every $x_0 \in X$ there is some $x_0^* \in X^*$ such that $(x_0^*,x_0) = \|x_0\|$ and $\|x_0^*\| \le 1$.

Indeed, the linear functional $f : Y \to \mathbf{R}$ defined by $f(x) = \alpha\|x_0\|$ for $x = \alpha x_0$, where $Y = \{\alpha x_0;\ \alpha \in \mathbf{R}\}$, has a linear continuous extension $x_0^* \in X^*$ on X such that $|(x_0^*,x)| \le \|x\|\ \forall x \in X$. Hence, $(x_0^*,x_0) = \|x_0\|$ and $\|x_0^*\| \le 1$ (in fact, $\|x_0^*\| = 1$). Clearly, $x_0^*\|x_0\| \in J(x_0)$ and so $J(x_0) \ne \emptyset$ for every $x_0 \in X$.

The mapping $J : X \to X^*$ is called the *duality mapping* of the space X. In general, the duality mapping J is multivalued.

V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*,
Springer Monographs in Mathematics, DOI 10.1007/978-1-4419-5542-5_1,

The inverse mapping $J^{-1}: X^* \to X$ defined by $J^{-1}(x^*) = \{x \in X;\ x^* \in J(x)\}$ also satisfies

$$J^{-1}(x^*) = \{x \in X;\ \|x\| = \|x^*\|,\ (x^*, x) = \|x\|^2 = \|x^*\|^2\}.$$

If the space X is reflexive (i.e., $X = X^{**}$), then clearly J^{-1} is just the duality mapping of X^* and so $D(J^{-1}) = X^*$. As a matter of fact, reflexivity plays an important role everywhere in the following and it should be recalled that a normed space is reflexive if and only if its dual X^* is reflexive (see, e.g., Yosida [16], p. 113).

It turns out that the properties of the duality mapping are closely related to the nature of the spaces X and X^*, more precisely to the convexity and smoothing properties of the closed balls in X and X^*.

Recall that the space X is called *strictly convex* if the unity ball B of X is strictly convex, that is the boundary ∂B contains no line segments.

The space X is said to be *uniformly convex* if for each $\varepsilon > 0$, $0 < \varepsilon < 2$, there is $\delta(\varepsilon) > 0$ such that if $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta(\varepsilon))$.

Obviously, every uniformly convex space X is strictly convex. Hilbert spaces as well as the spaces $L^p(\Omega)$, $1 < p < \infty$, are uniformly convex spaces (see, e.g., Köthe [13]). Recall also that, by virtue of the Milman theorem (see, e.g., Yosida [16], p. 127), every uniformly convex Banach space X is reflexive. Conversely, it turns out that every reflexive Banach space X can be renormed such that X and X^* become strictly convex. More precisely, one has the following important result due to Asplund [4].

Theorem 1.1. *Let X be a reflexive Banach space with the norm $\|\cdot\|$. Then there is an equivalent norm $\|\cdot\|_0$ on X such that X is strictly convex in this norm and X^* is strictly convex in the dual norm $\|\cdot\|_0^*$.*

Regarding the properties of the duality mapping associated with strictly or uniformly convex Banach spaces, we have the following.

Theorem 1.2. *Let X be a Banach space. If the dual space X^* is strictly convex, then the duality mapping $J: X \to X^*$ is single-valued and demicontinuous (i.e., it is continuous from X to X^*_w). If the space X^* is uniformly convex, then J is uniformly continuous on every bounded subset of X.*

Proof. Clearly, for every $x \in X$, $J(x)$ is a closed convex subset of X^*. Because $J(x) \subset \partial B$, where B is the open ball of radius $\|x\|$ and center 0, we infer that if X^* is strictly convex, then $J(x)$ consists of a single point. Now, let $\{x_n\} \subset X$ be strongly convergent to x_0 and let x_0^* be any weak-star limit point of $\{J(x_n)\}$. (Because the unit ball of the dual space is w^*-compact (Yosida [16], p. 137) such an x_0^* exists.) We have $(x_0^*, x_0) = \|x_0\|^2 \geq \|x_0^*\|^2$ because the closed ball of radius $\|x_0\|$ in X^* is weak-star closed. Hence $\|x_0\|^2 = \|x_0^*\|^2 - (x_0^*, x_0)$. In other words, $x_0^* = J(x_0)$, and so

$$J(x_n) \rightharpoonup J(x_0),$$

as claimed. □

To prove the second part of the theorem, let us first establish the following lemma.

Lemma 1.1. *Let X be a uniformly convex Banach space. If $x_n \rightharpoonup x$ and* $\limsup_{n\to\infty} \|x_n\| \le \|x\|$, *then $x_n \to x$ as $n \to \infty$.*

Proof. One can assume of course that $x \neq 0$. By hypothesis, $(x^*, x_n) \to (x^*, x)$ for all $x \in X$, and so, by the weak lower semicontinuity of the norm in X,

$$\|x\| \le \liminf_{n\to\infty} \|x_n\| \le \|x\|.$$

Hence, $\lim_{n\to\infty} \|x_n\| = \|x\|$. Now, we set

$$y_n = \frac{x_n}{\|x_n\|}, \qquad y = \frac{x}{\|x\|}\,.$$

Clearly, $y_n \rightharpoonup y$ as $n \to \infty$. Let us assume that $y_n \not\to y$ and argue from this to a contradiction. Indeed, in this case we have a subsequence y_{n_k}, $\|y_{n_k} - y\| \ge \varepsilon$, and so there is $\delta > 0$ such that $\|y_{n_k} + y\| \le 2(1-\delta)$. Letting $n_k \to \infty$ and using once again the fact that the norm $y \to \|y\|$ is weakly lower semicontinuous, we infer that $\|y\| \le 1 - \delta$. The contradiction we have arrived at shows that the initial supposition is false. □

Proof of Theorem 1.2 (continued). Assume now that X^* is uniformly convex. We suppose that there exist subsequences $\{u_n\}, \{v_n\}$ in X such that $\|u_n\|, \|v_n\| \le M$, $\|u_n - v_n\| \to 0$ for $n \to \infty$, $\|J(u_n) - J(v_n)\| \ge \varepsilon > 0$ for all n, and argue from this to a contradiction. We set $x_n = u_n\|u_n\|^{-1}$, $y_n = v_n\|v_n\|^{-1}$. Clearly, we may assume without loss of generality that $\|u_n\| \ge \alpha > 0$ and that $\|v_n\| \ge \alpha > 0$ for all n. Then, as easily seen,

$$\|x_n - y_n\| \to 0 \quad \text{as } n \to \infty$$

and

$$(J(x_n) + J(y_n), x_n) = \|x_n\|^2 + \|y_n\|^2 + (x_n - y_n, J(y_n)) \ge 2 - \|x_n - y_n\|.$$

Hence

$$\frac{1}{2}\,\|J(x_n) + J(y_n)\| \ge 1 - \frac{1}{2}\,\|x_n - y_n\|, \qquad \forall n.$$

Inasmuch as $\|J(x_n)\| = \|J(y_n)\| = 1$ and the space X^* is uniformly convex, this implies that $\lim_{n\to\infty}(J(x_n) - J(y_n)) = 0$. On the other hand, we have

$$J(u_n) - J(v_n) = \|u_n\|(J(x_n) - J(y_n)) + (\|u_n\| - \|v_n\|)J(y_n),$$

so that $\lim_{n\to\infty}(J(u_n) - J(v_n)) = 0$ strongly in X^*. □

Now, let us give some examples of duality mappings.

1. $X = H$ is a Hilbert space identified with its own dual. Then $J = I$, the identity operator in H. If H is not identified with its dual H^*, then the duality mapping $J : H \to H^*$ is the canonical isomorphism Λ of H onto H^*. For instance, if $H = H_0^1(\Omega)$ and $H^* = H^{-1}(\Omega)$ and Ω is a bounded and open subset of $\mathbf{R}^N$, then $J = \Lambda$ is defined by

$$(\Lambda u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \forall u, v \in H_0^1(\Omega). \tag{1.2}$$

In other words, $J = \Lambda$ is the Laplace operator $-\Delta$ under Dirichlet homogeneous boundary conditions in $\Omega \subset \mathbf{R}^N$. Here $H_0^1(\Omega)$ is the Sobolev space $\{u \in L^2(\Omega); \nabla u \in L^2(\Omega);\ u = 0 \text{ on } \partial\Omega\}$. (See Section 1.3 below.)

2. $X = L^p(\Omega)$, where $1 < p < \infty$ and Ω is a measurable subset of $\mathbf{R}^N$. Then, the duality mapping of X is given by

$$J(u)(x) = |u(x)|^{p-2} u(x) \|u\|_{L^p(\Omega)}^{2-p}, \quad \text{a.e. } x \in \Omega,\ \forall u \in L^p(\Omega). \tag{1.3}$$

Indeed, it is readily seen that if Φ_p is the mapping defined by the right-hand side of (1.3), we have

$$\int_\Omega \Phi_p(u) u \, dx = \left(\int_\Omega |u|^p dx \right)^{2/p} = \left(\int_\Omega |\Phi_p(u)|^q dx \right)^{2/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Because the duality mapping J of $L^p(\Omega)$ is single-valued (because L^p is uniformly convex for $p > 1$) and $\Phi_p(u) \in J(u)$, we conclude that $J = \Phi_p$, as claimed. If $X = L^1(\Omega)$, then as we show later (Corollary 2.7)

$$J(u) = \{v \in L^\infty(\Omega);\ v(x) \in \operatorname{sign} u(x) \cdot \|u\|_{L^1(\Omega)}, \text{ a.e. } x \in \Omega\}. \tag{1.4}$$

3. Let X be the Sobolev space $W_0^{1,p}(\Omega)$, where $1 < p < \infty$ and Ω is a bounded and open subset of $\mathbf{R}^N$. (See Section 1.3 below.) Then,

$$J(u) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \|u\|_{W_0^{1,p}(\Omega)}^{2-p}. \tag{1.5}$$

In other words, $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$, $(1/p) + (1/q) = 1$, is defined by

$$(J(u), v) = \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \|u\|_{W_0^{1,p}(\Omega)}^{2-p}, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{1.6}$$

We later show that the duality mapping J of the space X can be equivalently defined as the subdifferential (Gâteaux differential if X^* is strictly convex) of the function $x \to 1/2 \|x\|^2$.

1.2 Convex Functions and Subdifferentials

Here we briefly present the basic results pertaining to convex analysis in infinite-dimensional spaces. For further results and complete treatment of the subject we refer the reader to Moreau [14], Rockafellar [15], Brezis [8], Barbu and Precupanu [6] and Zălinescu [17].

Let X be a real Banach space with dual X^*. A *proper convex function* on X is a function $\varphi : X \to (-\infty, +\infty] = \overline{\mathbf{R}}$ that is not identically $+\infty$ and that satisfies the inequality

$$\varphi((1-\lambda)x+\lambda y) \leq (1-\lambda)\varphi(x)+\lambda\varphi(y) \tag{1.7}$$

for all $x, y \in X$ and all $\lambda \in [0,1]$.

The function $\varphi : X \to (-\infty, +\infty]$ is said to be *lower semicontinuous* (l.s.c.) on X if

$$\liminf_{u\to x} \varphi(u) \geq \varphi(x), \qquad \forall x \in X,$$

or, equivalently, every level subset $\{x \in X;\ \varphi(x) \leq \lambda\}$ is closed.

The function $\varphi : X \to]-\infty, +\infty]$ is said to be *weakly lower semicontinuous* if it is lower semicontinuous on the space X endowed with weak topology.

Because every level set of a convex function is convex and every closed convex set is weakly closed (this is an immediate consequence of Mazur's theorem, Yosida [16], p. 109), we may therefore conclude that a proper convex function is lower semicontinuous if and only if it is weakly lower semicontinuous.

Given a lower semicontinuous convex function $\varphi : X \to (-\infty, +\infty] = \overline{\mathbf{R}}$, $\varphi \not\equiv \infty$, we use the following notations:

$$D(\varphi) = \{x \in X;\ \varphi(x) < \infty\} \quad \text{(the effective domain of } \varphi), \tag{1.8}$$

$$\mathrm{Epi}(\varphi) = \{(x,\lambda) \in X \times \mathbf{R};\ \varphi(x) \leq \lambda\} \quad \text{(the epigraph of } \varphi). \tag{1.9}$$

It is readily seen that $\mathrm{Epi}(\varphi)$ is a closed convex subset of $X \times \mathbf{R}$, and as a matter of fact its properties are closely related to those of the function φ.

Now, let us briefly describe some elementary properties of l.s.c., convex functions.

Proposition 1.1. *Let $\varphi : X \to \overline{\mathbf{R}}$ be a proper, l.s.c., and convex function. Then φ is bounded from below by an affine function; that is there are $x_0^* \in X^*$ and $\beta \in \mathbf{R}$ such that*

$$\varphi(x) \geq (x_0^*, x) + \beta, \qquad \forall x \in X. \tag{1.10}$$

Proof. Let $E(\varphi) = \mathrm{Epi}(\varphi)$ and let $x_0 \in X$ and $r \in \mathbf{R}$ be such that $\varphi(x_0) > r$. By the classical separation theorem (see, e.g., Brezis [7]), there is a closed hyperplane $H = \{(x,\lambda) \in X \times \mathbf{R}; -(x_0^*, x) + \lambda = \alpha\}$ that separates $E(\varphi)$ and (x_0, r). This means that

$$-(x_0^*, x) + \lambda \geq \alpha, \qquad \forall x \in E(\varphi) \quad \text{and} \quad -(x_0^*, x_0) + r < \alpha.$$

Hence, for $\lambda = \varphi(x)$, we have

$$-(x_0^*, x) + \varphi(x) \geq -(x_0^*, x_0) + r, \qquad \forall x \in X,$$

which implies (1.10). □

Proposition 1.2. *Let $\varphi : X \to \overline{\mathbf{R}}$ be a proper, convex, and l.s.c. function. Then φ is continuous on* $\operatorname{int} D(\varphi)$.

Proof. Let $x_0 \in \operatorname{int} D(\varphi)$. We prove that φ is continuous at x_0. Without loss of generality, we assume that $x_0 = 0$ and that $\varphi(0) = 0$. Because the set $\{x : \varphi(x) > -\varepsilon\}$ is open it suffices to show that $\{x : \varphi(x) < \varepsilon\}$ is a neighborhood of the origin. We set $C = \{x \in X;\ \varphi(x) \le \varepsilon\} \cap \{x \in X;\ \varphi(-x) \le \varepsilon\}$. Clearly, C is a closed balanced set of X (i.e., $\alpha x \in C$ for $|\alpha| \le 1$ and $x \in C$). Moreover, C is absorbing; that is, for every $x \in X$ there exists $\alpha > 0$ such that $\alpha x \in C$ (because the function $t \to \varphi(tx)$ is convex and finite in a neighborhood of the origin and therefore it is continuous). Because X is a Banach space, the preceding properties of C imply that it is a neighborhood of the origin, as claimed. □

The function $\varphi^* : X^* \to \overline{\mathbf{R}}$ defined by

$$\varphi^*(p) = \sup\{(p,x) - \varphi(x);\ x \in X\} \tag{1.11}$$

is called the *conjugate* of φ.

Proposition 1.3. *Let $\varphi : X \to \overline{\mathbf{R}}$ be l.s.c., convex, and proper. Then φ^* is l.s.c., convex, and proper on the space X^*.*

Proof. As supremum of a set of affine functions, φ^* is convex and l.s.c. Moreover, by Proposition 1.2 we see that $\varphi^* \not\equiv \infty$. □

Proposition 1.4. *Let $\varphi : X \to (-\infty, +\infty]$ be a weakly lower semicontinuous function such that every level set $\{x \in X;\ \varphi(x) \le \lambda\}$ is weakly compact. Then φ attains its infimum on X. In particular, if X is reflexive and φ is an l.s.c. proper convex function on X such that*

$$\lim_{\|x\| \to \infty} \varphi(x) = \infty, \tag{1.12}$$

then there exists $x_0 \in X$ such that $\varphi(x_0) = \inf\{\varphi(x);\ x \in X\}$.

Proof. Let $d = \inf\{\varphi(x);\ x \in X\}$ and let $\{x_n\} \subset X$ such that $d \le \varphi(x_n) \le d + (1/n)$. Then $\{x_n\}$ is weakly compact in X and, therefore, there is $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x$ as $n_k \to \infty$. Because φ is weakly semicontinuous, this implies that $\varphi(x) \le d$. Hence $\varphi(x) = d$, as desired. If X is reflexive, then formula (1.12) implies that $\{x \in X;\ \varphi(x) \le \lambda\}$ are weakly compact. As seen earlier, every convex and l.s.c. function is weakly lower semicontinuous, therefore we can apply the first part. □

Given a function f from a Banach space X to $\mathbf{R}$, the mapping $f' : X \times X \to \mathbf{R}$ defined by

$$f'(x,y) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda y) - f(x)}{\lambda}, \qquad x, y \in X, \tag{1.13}$$

(if it exists) is called the *directional derivative* of f at x in direction y.

The function $f : X \to \mathbf{R}$ is said to be *Gâteaux differentiable* at $x \in X$ if there exists $\nabla f(x) \in X^*$ (the *Gâteaux differential*) such that

$$f'(x,y) = (\nabla f(x), y), \qquad \forall y \in X. \tag{1.14}$$

If the convergence in (1.13) is uniform in y on bounded subsets, then f is said to be *Fréchet differentiable* and ∇f is called the *Fréchet differential* (derivative) of f.

Given an l.s.c., convex, proper function $\varphi : X \to \overline{\mathbf{R}}$, the mapping $\partial\varphi : X \to X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^*;\ \varphi(x) \le \varphi(y) + (x^*, x-y),\ \forall y \in X\} \tag{1.15}$$

is called the *subdifferential* of φ.

In general, $\partial\varphi$ is a multivalued operator from X to X^* not everywhere defined and can be seen as a subset of $X \times X^*$.

An element $x^* \in \partial\varphi(x)$ (if any) is called a *subgradient* of φ in x. We denote as usual by $D(\partial\varphi)$ the set of all $x \in X$ for which $\partial\varphi(x) \ne \emptyset$.

Let us pause briefly to give some simple examples.

1. $\varphi(x) = 1/2\,\|x\|^2$. Then, $\partial\varphi = J$ (the duality mapping of the space X). Indeed, if $x^* \in J(x)$, then

$$(x^*, x-y) = \|x\|^2 - (x^*, y) \ge \frac{1}{2}\,(\|x\|^2 - \|y\|^2), \qquad \forall x \in X.$$

Hence $x^* \in \partial\varphi(x)$. Now, let $x^* \in \partial\varphi(x)$; that is,

$$\frac{1}{2}(\|x\|^2 - \|y\|^2) \le (x^* - y, x), \qquad \forall y \in X. \tag{1.16}$$

We take $y = \lambda x,\ 0 < \lambda < 1$, in (1.16), getting

$$(x^*, x) \ge \frac{1}{2}\,\|x\|^2(1+\lambda).$$

Hence, $(x^*, x) \ge \|x\|^2$. If $y = \lambda x$, where $\lambda > 1$, we get that $(x^*, x) \le \|x\|^2$. Hence, $(x^*, x) = \|x\|^2$ and $\|x^*\| \ge \|x\|$. On the other hand, taking $y = x + \lambda u$ in (1.16), where $\lambda > 0$ and u is arbitrary in X, we get

$$\lambda(x^*, u) \le \frac{1}{2}\,(\|x + \lambda u\|^2 - \|x\|^2),$$

which yields

$$(x^*, u) \le \|x\|\;\|u\|.$$

Hence, $\|x^*\| \le \|x\|$. We have therefore proven that $(x^*, x) = \|x\|^2 = \|x^*\|^2$ as claimed.

2. Let K be a closed convex subset of X. The function $I_K : X \to \overline{\mathbf{R}}$ defined by

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K, \end{cases} \tag{1.17}$$

is called the *indicator function* of K, and its dual function H,

$$H_K(p) = \sup\{(p,u);\ u \in K\}, \qquad \forall p \in X^*,$$

is called the *support function* of K. It is readily seen that $D(\partial I_K) = K$, $\partial I_K(x) = 0$ for $x \in \operatorname{int} K$ (if nonempty) and that

$$\partial I_K(x) = N_K(x) = \{x^* \in X^*;\ (x^*, x-u) \geq 0,\ \forall u \in K\}, \qquad \forall x \in K. \tag{1.18}$$

For every $x \in \partial K$ (the boundary of K), $N_K(x)$ is the *normal cone* at K in x.

3. Let φ be convex and Gâteaux differentiable at x. Then $\partial\varphi(x) = \nabla\varphi(x)$. Indeed, because φ is convex, we have

$$\varphi(x + \lambda(y-x)) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$. Hence,

$$\frac{\varphi(x+\lambda(y-x)) - \varphi(x)}{\lambda} \leq \varphi(y) - \varphi(x),$$

and letting λ tend to zero, we see that $\nabla\varphi(x) \in \partial\varphi(x)$. Now, let w be an arbitrary element of $\partial\varphi(x)$. We have

$$\varphi(x) - \varphi(y) \leq (w, x-y), \qquad \forall y \in X.$$

Equivalently,

$$\frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda} \geq (w,y), \qquad \forall \lambda > 0,\ y \in X,$$

and this implies that $(\nabla\varphi(x) - w, y) \geq 0$ for all $y \in X$. Hence, $w = \nabla\varphi(x)$.

By the definition of $\partial\varphi$ it is easily seen that $\varphi(x) = \inf\{\varphi(u);\ u \in X\}$ iff $0 \in \partial\varphi(x)$. There is a close relationship between $\partial\varphi$ and $\partial\varphi^*$. More precisely, we have the following.

Proposition 1.5. *Let X be a reflexive Banach space and let $\varphi : X \to \overline{\mathbf{R}}$ be an l.s.c., convex, proper function. Then the following conditions are equivalent.*

(i) $x^* \in \partial\varphi(x)$,
(ii) $\varphi(x) + \varphi^*(x^*) = (x^*, x)$,
(iii) $x \in \partial\varphi^*(x^*)$.

In particular, $\partial\varphi^* = (\partial\varphi)^{-1}$ and $(\varphi^*)^* = \varphi$.

Proof. By definition of φ^*, we see that

$$\varphi^*(x^*) \geq (x^*, x) - \varphi(x), \qquad \forall x \in X,$$

with equality if and only if $0 \in \partial_x(-(x^*,x) + \varphi(x))$. Hence, (i) and (ii) are equivalent. Now, if (ii) holds, then x^* is a minimum point for the function $\varphi^*(p) - (x,p)$ and so

$x \in \partial\varphi^*(x^*)$. Hence, (ii) $\Rightarrow$ (iii). Because conditions (i) and (ii) are equivalent for φ^*, we may equivalently express (iii) as $\varphi^*(x^*) + (\varphi^*)^*(x) = (x^*, x)$. Thus, to prove (ii) it suffices to show that $(\varphi^*)^* = \varphi$. It is readily seen that $(\varphi^*)^* = \varphi^{**} \le \varphi$. We suppose now that there exists $x_0 \in X$ such that $\varphi^{**}(x_0) < \varphi(x_0)$, and we argue from this to a contradiction. We have, therefore, $(x_0, \varphi^{**}(x_0)) \notin \mathrm{Epi}(\varphi)$ and so, by the separation theorem, it follows that there are $x_0^* \in X^*$ and $\alpha \in \mathbf{R}$ such that $(x_0^*, x_0) + \alpha\varphi^{**}(x_0) > \sup\{(x_0^*, x) + \alpha\lambda;\ (x, \lambda) \in \mathrm{Epi}(\varphi)\}$. After some calculation, it follows that $\alpha < 0$. Then, dividing this inequality by $-\alpha$, we get that

$$
\begin{aligned}
-\left(x_0^*, \frac{x_0}{\alpha}\right) - \varphi^{**}(x_0) &> \sup\left\{\left(x_0^*, -\frac{x}{\alpha}\right) - \lambda;\ (x, \lambda) \in \mathrm{Epi}(\varphi)\right\} \\
&= \sup\left\{\left(-\frac{x_0^*}{\alpha}, x\right) - \varphi(x);\ x \in D(\varphi)\right\} = \varphi^*\left(-\frac{x_0^*}{\alpha}\right),
\end{aligned}
$$

which clearly contradicts the definition of φ^{**}. $\square$

We mention without proof the following density result. (See, e.g., [2].)

Proposition 1.6. *Let $\varphi : X \to \overline{\mathbf{R}}$ be an l.s.c., convex, and proper function. Then $D(\partial\varphi)$ is a dense subset of $D(\varphi)$.*

Proposition 1.7. *Let φ be an l.s.c., proper, convex function on X. Then* $\mathrm{int}\, D(\varphi) \subset D(\partial\varphi)$.

Proof. Let $x_0 \in \mathrm{int}\, D(\varphi)$ and let $V = B(x_0, r) = \{x;\ \|x - x_0\| < r\}$ be such that $V \subset D(\varphi)$. We know by Proposition 1.2 that φ is continuous on V and this implies that the set $C = \{(x, \lambda) \in V \times \mathbf{R};\ \varphi(x) < \lambda\}$ is an open convex set of $X \times \mathbf{R}$. Thus, there is a closed hyperplane, $H = \{(x, \lambda) \in X \times \mathbf{R};\ (x_0^*, x) + \lambda = \alpha\}$, that separates $(x_0, \varphi(x_0))$ from $\overline{C}$. Hence, $(x_0^*, x_0) + \varphi(x_0) < \alpha$ and

$$(x_0^*, x) + \lambda \ge \alpha, \qquad \forall (x, \lambda) \in \overline{C}.$$

This yields

$$\varphi(x_0) - \varphi(x) < -(x_0^*, x_0 - x), \qquad \forall x \in V.$$

But, for every $u \in X$, there exists $0 < \lambda < 1$ such that $x = \lambda x_0 + (1 - \lambda)u \in V$. Substituting this x in the preceding inequality and using the convexity of φ, we obtain that

$$\varphi(x_0) \le \varphi(u) + (x_0^*, x_0 - u), \qquad \forall u \in X.$$

Hence, $x_0 \in D(\partial\varphi)$ and $x_0^* \in \partial\varphi(x_0)$. $\square$

There is a close connection between the range of subdifferential $\partial\varphi$ of a lower semicontinuous convex function $\varphi : X \to \overline{\mathbf{R}}$ and its behavior for $\|x\| \to \infty$. Namely, one has

Proposition 1.8. *The following two conditions are equivalent.*

(j) $R(\partial\varphi) = X^*$, *and $\partial\varphi^* = (\partial\varphi)^{-1}$ is bounded on bounded subsets,*
(jj) $\lim_{\|x\| \to \infty} \varphi(x)/\|x\| = +\infty$.

Proof. (jj) $\Rightarrow$ (j). If (jj) holds, then by Proposition 1.4 it follows that for each $f \in X^*$ the equation $f \in \partial\varphi(x)$ or, equivalently, $0 \in \partial(\varphi(x) - f(x))$, has at least one solution $x \in D(\partial\varphi)$. Moreover, if $\{f\}$ remains in a bounded subset of X^*, the same is true of $(\partial\varphi)^{-1}f$.

(j) $\Rightarrow$ (jj). By Proposition 1.5 we have

$$\varphi(x) \geq (x^*, x) - \varphi^*(x^*), \qquad \forall x^* \in X^*,\ \forall x \in X.$$

This yields, for $x^* = \rho J(x)\|x\|^{-1}$,

$$\varphi(x) \geq \rho\|x\| - \varphi^*(\rho J(x)\|x\|^{-1}), \qquad \forall \rho > 0,\ \forall x \in X.$$

Because φ^* and $\partial\varphi^*$ are bounded on bounded subsets, the latter implies (jj). □

1.3 Sobolev Spaces and Linear Elliptic Boundary Value Problems

Throughout this section, until further notice, we assume that Ω is an open subset of $\mathbf{R}^N$. To begin with, let us briefly recall the notion of *distribution*. Let $f = f(x)$ be a complex-valued function defined on Ω. By the *support* of f, abbreviated $\operatorname{supp} f$, we mean the closure of the set $\{x \in \Omega;\ f(x) \neq 0\}$ or, equivalently, the smallest closed set of Ω outside of which f vanishes identically. We denote by $C^k(\Omega)$, $0 \leq k \leq \infty$, the set of all complex-valued functions defined in Ω that have continuous partial derivatives of order up to and including k (of any order $< \infty$ if $k = \infty$). Let $C_0^k(\Omega)$ denote the set of all functions $\varphi \in C^k(\Omega)$ with compact support in Ω.

It is readily seen that $C_0^\infty(\Omega)$ is a linear space. We may introduce in $C_0^\infty(\Omega)$ a convergence as follows. We say that the sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ is convergent to φ, denoted $\varphi_k \Rightarrow \varphi$, if

(a) There is a compact $K \subset \Omega$ such that $\operatorname{supp}\varphi_k \subset K$ for all $k = 1, \ldots$.
(b) $\lim_{k\to\infty} D^\alpha\varphi_k = D^\alpha\varphi$ uniformly on K for all $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Here $D^\alpha = D_{x_1}^{\alpha} \cdots D_{x_N}^{\alpha_n}$, $D_{x_i} = \partial/\partial x_i$, $i = 1, \ldots, n$. Equipped in this way, the space $C_0^\infty(\Omega)$ is denoted by $\mathscr{D}(\Omega)$. As a matter of fact, $\mathscr{D}(\Omega)$ can be redefined as a locally convex, linear topological space with a suitable chosen family of seminorms.

Definition 1.1. A linear continuous functional u on $\mathscr{D}(\Omega)$ is called a *distribution* on Ω.

In other words, a *distribution* is a linear functional u on $C_0^\infty(\Omega)$ having the property that $\lim_{k\to\infty} u(\varphi_k) = 0$ for every sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that $\varphi_k \Rightarrow 0$.

The set of all distributions on Ω is a linear space, denoted by $\mathscr{D}'(\Omega)$.

The distribution is a natural extension of the notion of locally summable function on Ω for if $f \in L^1_{\text{loc}}(\Omega)$, then the linear functional u_f on $C_0^\infty(\Omega)$ defined by

$$u_f(\varphi) = \int_\Omega f(x)\varphi(x)dx, \qquad \forall \varphi \in C_0^\infty(\Omega)$$

is a distribution on Ω; that is, $u_f \in \mathscr{D}'(\Omega)$. Moreover, the map $f \to u_f$ is injective from $L^1_{\text{loc}}(\Omega)$ to $\mathscr{D}'(\Omega)$.

Given $u \in \mathscr{D}'(\Omega)$, by definition, the derivative of order $\alpha = (\alpha_1, ..., \alpha_n)$, $D^\alpha u$, of u, is the distribution

$$(D^\alpha u)(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi), \qquad \forall \varphi \in \mathscr{D}(\Omega), \quad \text{where } |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Let Ω be an open subset of $\mathbf{R}^N$ and let m be a positive integer. Denote by $H^m(\Omega)$ the set of all real valued functions $u \in L^2(\Omega)$ such that distributional derivatives $D^\alpha u$ of u of order $|\alpha| \le m$ all belong to $L^2(\Omega)$. In other words,

$$H^m(\Omega) = \{u \in L^2(\Omega);\ D^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}. \tag{1.19}$$

This is the *Sobolev space* of order m on Ω. It is easily seen that $H^m(\Omega)$ is a linear space by $(u_1 + u_2)(x) = u_1(x) + u_2(x)$, $(\lambda u)(x) = \lambda u(x)$, $\forall \lambda \in \mathbf{R}$, a.e., $x \in \Omega$, under the convention that two $L^2(\Omega)$ functions u_1, u_2 represent the same element of $H^m(\Omega)$ if $u_1(x) = u_2(x)$, a.e., $x \in \Omega$. In other words, we do not distinguish two functions in $H^m(\Omega)$ that coincide almost everywhere. In this context we say that $u \in H^m(\Omega)$ is continuous, differentiable, or absolutely continuous if there exists a function $\bar{u} \in H^m(\Omega)$ which has these properties and coincides almost everywhere with u on Ω.

We present below a few basic properties of Sobolev spaces and refer to the books of Brezis [7], Adams [1] and Barbu [5] for proofs.

Proposition 1.9. *$H^m(\Omega)$ is a Hilbert space with the scalar product*

$$\langle u, v\rangle_m = \sum_{|\alpha| \le m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx, \qquad \forall u, v \in H^m(\Omega). \tag{1.20}$$

If $\Omega = (a,b)$, $-\infty < a < b < \infty$, then $H^1(\Omega)$ reduces to the subspace of absolutely continuous functions on the interval $[a,b]$ with derivative in $L^2(a,b)$.

Proposition 1.10. *$H^1(a,b)$ coincides with the space of absolutely continuous functions $u : [a,b] \to \mathbf{R}$ having the property that $u' \in L^2(a,b)$. Moreover, for each function $u \in H^1(a,b)$ the derivative $D^1 u$ in the sense of distributions coincides with the ordinary derivative u' that exists almost everywhere.*

More generally, for an integer $m \ge 1$ and $1 \le p \le \infty$, one defines the Sobolev space

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega),\ |\alpha| \le m\} \tag{1.21}$$

with the norm

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u(x)|^p dx \right)^{1/p}. \tag{1.22}$$

For $0 < m < 1$, the space $W^{m,p}(\Omega)$ is defined by (see Adams [1], p. 214)

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \frac{u(x) - u(y)}{|x-y|^{m+(N/p)}} \in L^p(\Omega \times \Omega) \right\}$$

with the natural norm. For $m > 1$, $m = s + a$, $s = [m]$, $0 < a < 1$, define

$$W^{m,p}(\Omega) = \{u \in W^{s,p}(\Omega);\ D^\alpha u \in W^{a,p}(\Omega);\ |\alpha| \le s\}.$$

It turns out that, if $u \in W^{1,p}(a,b)$, then there is an absolutely continuous function $\bar{u}$ with $\bar{u}' \in L^p(a,b)$ such that $\bar{u}(x) = u(x)$ and $\bar{u}'(x) = (D^1 u)(x)$, a.e., $x \in (a,b)$. Conversely, any absolutely continuous function u with u' in $L^p(a,b)$ belongs to $W^{1,p}(a,b)$ and u' coincides, a.e. on (a,b), with the distributional derivative $D^1 u$ of u.

Proposition 1.10 and its counterpart in $W^{1,p}(\Omega)$ show that, in one dimension, the Sobolev spaces are just the classical spaces of absolutely continuous functions with derivatives in $L^p(\Omega)$.

It turns out, via regularization, that $C_0^\infty(\mathbf{R}^N)$ is dense in $H^1(\mathbf{R}^N)$.

We recall that an open subset Ω of $\mathbf{R}^N$ and its boundary $\partial\Omega$ are said to be of class C^1 if for each $x \in \partial\Omega$ there are a neighborhood U of x and a one-to-one mapping φ of $Q = \{x = (x', x_N) \in \mathbf{R}^N;\ \|x'\| < 1,\ |x_N| < 1\}$ onto U such that

$$\varphi \in C^1(\overline{Q}), \qquad \varphi^{-1} \in C^1(\overline{U}), \qquad \varphi(Q_+) = U \cap \Omega, \qquad \varphi(Q_0) = U \cap \partial\Omega,$$

where $Q_+ = \{(x', x_N) \in Q;\ x_N > 0\}$, $Q_0 = \{(x', 0);\ \|x'\| < 1\}$.

We are now ready to formulate the extension theorem for the elements of the space $H^1(\Omega)$, a result upon which most of the properties of this space are built.

Theorem 1.3. *Let Ω be an open subset of $\mathbf{R}^N$ that is of class C^1. Assume that either $\partial\Omega$ is compact or $\Omega = \mathbf{R}^N_+$. Then, there is a linear operator $P : H^1(\Omega) \to H^1(\mathbf{R}^N)$ and a positive constant C independent of u, such that*

$$(Pu)(x) = u(x), \quad \text{a.e. } x \in \Omega, \qquad \forall u \in H^1(\Omega), \tag{1.23}$$

$$\|Pu\|_{L^2(\mathbf{R}^N)} \le C\|u\|_{L^2(\Omega)}, \qquad \forall u \in H^1(\Omega), \tag{1.24}$$

$$\|Pu\|_{H^1(\mathbf{R}^N)} \le C\|u\|_{H^1(\Omega)}, \qquad \forall u \in H^1(\Omega). \tag{1.25}$$

Theorem 1.3 follows from the next extension result.

Let $u \in H^1(Q_+)$ and let $u^* : Q \to \mathbf{R}$ be the extension of u to Q

$$u^*(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N \ge 0 \\ u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then $u^* \in H^1(Q)$ and $\|u^*\|_{L^2(Q)} \le 2\|u\|_{L^2(Q_+)}$, $\|u^*\|_{H^1(Q)} \le 2\|u\|_{H^1(Q_+)}$. The general result follows by a specific argument involving partition of unity (see, e.g., Brezis [7] or Barbu [5]).

Now, we mention without proof an important property of the space $H^1(\Omega)$ known as the *Sobolev embedding theorem.*

Theorem 1.4. *Let Ω be an open subset of $\mathbf{R}^N$ of class C^1 with compact boundary $\partial\Omega$, or $\Omega = \mathbf{R}^N_+$, or $\Omega = \mathbf{R}^N$. Then, if $N > 2$,*

$$H^1(\Omega) \subset L^{p^*}(\Omega) \quad \text{for } \frac{1}{p^*} = \frac{1}{2} - \frac{1}{N}. \tag{1.26}$$

If $N = 2$, then $H^1(\Omega) \subset L^p(\Omega)$ for all $p \in [2, \infty[$.

The inclusion relation (1.26) should be considered of course in the algebraic and topological sense; that is,

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{H^1(\Omega)} \tag{1.27}$$

for some positive constant C independent of u.

Theorem 1.4 has a natural extension to the Sobolev space $W^{m,p}(\Omega)$ for any $m > 0$. More precisely, we have (see Adams [1], p. 217)

Theorem 1.5. *Under the assumptions of Theorem* 1.4, *we have*

$$\begin{aligned}
&W^{m,p}(\Omega) \subset L^{p^*}(\Omega) && \text{if } 1 \leq p < \frac{N}{m},\ \frac{1}{p^*} = \frac{1}{p} - \frac{m}{N},\\
&W^{m,p}(\Omega) \subset L^q(\Omega) && \text{for all } q \geq p \text{ if } p = \frac{N}{m},\\
&W^{m,p}(\Omega) \subset L^\infty(\Omega) && \text{if } p > \frac{N}{m}.
\end{aligned}$$

Remark 1.1. If Ω is a bounded and open subset of $\mathbf{R}^N$ of class C^1, then the following norm on $W^{1,p}(\Omega)$,

$$\|u\|_{1,p} = |\nabla u|_{L^p(\Omega)} + |u|_{L^q(\Omega)},$$

where $1 \leq q \leq p^*$ if $1 \leq p < N$, $1 \leq q < \infty$ if $p = N$ and $1 \leq q \leq \infty$ if $p > N$,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

is equivalent with the norm (1.22) for $m = 1$ (see, e.g., Brezis [7], p. 170).

We note also the following compactness embedding result.

Theorem 1.6. *Let Ω be an open and bounded subset of $\mathbf{R}^N$ that is of class C^1. Then, the injection of the space $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

The "trace" to $\partial\Omega$ of a Function $u \in H^1(\Omega)$

If Ω is an open C^1 subset of $\mathbf{R}^N$ with the boundary $\partial\Omega$, then each $u \in C(\overline{\Omega})$ is well defined on $\partial\Omega$. We call the restriction of u to $\partial\Omega$ the *trace* of u to $\partial\Omega$ and it is denoted by $\gamma_0(u)$. If $u \in L^2(\Omega)$, then $\gamma_0(u)$ is no longer well defined. We have, however, the following.

Lemma 1.2. *Let Ω be an open subset of class C^1 with compact boundary $\partial\Omega$ or $\Omega = \mathbf{R}^N_+$. Then, there is $C > 0$ such that*

$$\|\gamma_0(u)\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}, \qquad \forall u \in C_0^\infty(\mathbf{R}^N). \tag{1.28}$$

Taking into account that for domains Ω of class C^1 the space $\{u|_\Omega;\ u \in C_0^\infty(\mathbf{R}^N)\}$ is dense in $H^1(\Omega)$ (see, e.g., Adams [1], p. 54, or Brezis [7], p. 162), a natural way to define the *trace* of a function $u \in H^1(\Omega)$ is the following.

Definition 1.2. Let Ω be of class C^1 with compact boundary or $\Omega = \mathbf{R}^N_+$. Let $u \in H^1(\Omega)$. Then $\gamma_0(u) = \lim_{j\to\infty} \gamma_0(u_j)$ in $L^2(\partial\Omega)$, where $\{u_j\} \subset C_0^\infty(\mathbf{R}^N)$ is such that $u_j \to u$ in $H^1(\Omega)$.

It turns out that the definition is consistent; that is, $\gamma_0(u)$ is independent of $\{u_j\}$. Indeed, if $\{u_j\}$ and $\{\bar{u}_j\}$ are two sequences in $C_0^\infty(\mathbf{R}^N)$ convergent to u in $H^1(\Omega)$, then, by (1.28),

$$\|\gamma_0(u_j - \bar{u}_j)\|_{L^2(\partial\Omega)} \leq C\|u_j - \bar{u}_j\|_{H^1(\Omega)} \to 0 \quad \text{as } j \to \infty.$$

Moreover, it follows by Lemma 1.2 that the map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is continuous. As a matter of fact, it turns out that the trace operator $u \to \gamma_0(u)$ is continuous from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ and so it is completely continuous from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

In general (see Adams [1], p. 114), we have $W^{m,p}(\Omega) \subset L^q(\partial\Omega)$ if $mp < N$ and

$$p \leq q \leq \frac{(N-1)p}{(N-mp)}\,.$$

Definition 1.3. Let Ω be any open subset of $\mathbf{R}^N$. The space $H_0^1(\Omega)$ is the closure (the completion) of $C_0^1(\Omega)$ in the norm of $H^1(\Omega)$.

It follows that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and in general it is a proper subspace of $H^1(\Omega)$. It is clear that $H_0^1(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_1 = \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx + \int_\Omega uv\,dx$$

with the corresponding norm

$$\|u\|_1 = \left(\int_\Omega (|\nabla u(x)|^2 + u^2(x))dx\right)^{1/2}.$$

Roughly speaking, $H_0^1(\Omega)$ is the subspace of functions $u \in H^1(\Omega)$ that are zero on $\partial\Omega$. More precisely, we have the following.

Proposition 1.11. *Let Ω be an open set of class C^1 and let $u \in H^1(\Omega)$. Then, the following conditions are equivalent.*

(i) $u \in H_0^1(\Omega)$.
(ii) $\gamma_0(u) \equiv 0$.

Proposition 1.12 below is called the *Poincaré inequality*.

Proposition 1.12. *Let Ω be an open and bounded subset of $\mathbf{R}^N$. Then there is $C > 0$ independent of u such that*

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in H_0^1(\Omega).$$

In particular, Proposition 1.12 shows that if Ω is bounded, then the scalar product

$$((u,v)) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx$$

and the corresponding norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2}$$

define an equivalent Hilbertian structure on $H_0^1(\Omega)$.

We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$; that is, the space of all linear continuous functionals on $H_0^1(\Omega)$. Equivalently,

$$H^{-1}(\Omega) = \{u \in \mathscr{D}'(\Omega);\ |u(\varphi)| \le C_u \|\varphi\|_{H^1(\Omega)}, \qquad \forall \varphi \in C_0^\infty(\Omega)\}.$$

The space $H^{-1}(\Omega)$ is endowed with the dual norm

$$\|u\|_{-1} = \sup\{|u(\varphi)|;\ \|\varphi\| \le 1\}, \qquad \forall u \in H^{-1}(\Omega).$$

By Riesz's theorem, we know that $H^{-1}(\Omega)$ is isometric to $H_0^1(\Omega)$. Note also that

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

in the algebraic and topological sense. In other words, the injections of $L^2(\Omega)$ into $H^{-1}(\Omega)$ and of $H_0^1(\Omega)$ into $L^2(\Omega)$ are continuous. Note also that the above injections are dense.

There is an equivalent definition of $H^{-1}(\Omega)$ given in Theorem 1.7 below.

Theorem 1.7. *The space $H^{-1}(\Omega)$ coincides with the set of all distributions $u \in \mathscr{D}'(\Omega)$ of the form*

$$u = f_0 + \sum_{i=1}^N \frac{\partial f_i}{\partial x_i} \quad \text{in } \mathscr{D}'(\Omega), \text{ where } f_i \in L^2(\Omega),\ i = 1,\dots,N.$$

The space $W_0^{1,p}(\Omega)$, $p \ge 1$, is similarly defined as the closure of $C_0^1(\Omega)$ into $W^{1,p}(\Omega)$ norm. The dual of $W_0^{1,p}(\Omega)$ is the space

$$W^{-1,q}(\Omega), \qquad \frac{1}{p}+\frac{1}{q}=1$$

defined as in Theorem 1.7 with $f_0, f_1, ..., f_N \in L^q$.

Variational Theory of Elliptic Boundary Value Problems

We begin by recalling an abstract existence result, the *Lax–Milgram lemma*, which is the foundation upon which all the results of this section are built. Before presenting it, we need to clarify certain concepts.

Let V be a real Hilbert space and let V^* be the topological dual space of V. For each $v^* \in V^*$ and $v \in V$ we denote by (v^*, v) the value $v^*(v)$ of functional v^* at v. The functional $a : V \times V \to \mathbf{R}$ is said to be *bilinear* if for each $u \in V$, $v \to a(u,v)$ is linear and for each $v \in V$, $u \to a(u,v)$ is linear on V. The functional a is said to be *continuous* if there exists $M > 0$ such that

$$|a(u,v)| \le M\|u\|_V \, \|v\|_V, \qquad \forall u, v \in V.$$

The functional a is said to be *coercive* if

$$a(u,u) \ge \omega\|u\|_V^2, \qquad \forall u \in V,$$

for some $\omega > 0$, and *symmetric* if

$$a(u,v) = a(v,u), \qquad \forall u, v \in V.$$

Lemma 1.3 (Lax–Milgram). *Let $a : V \times V \to \mathbf{R}$ be a bilinear, continuous, and coercive functional. Then, for each $f \in V^*$, there is a unique $u^* \in V$ such that*

$$a(u^*, v) = (f, v), \qquad \forall v \in V. \tag{1.29}$$

Moreover, the map $f \to u^*$ is Lipschitzian from V^* to V with Lipschitz constant $\le \omega^{-1}$. If a is symmetric, then u^* minimizes the function $u \to (1/2)a(u,u) - (f,u)$ on V; that is,

$$\frac{1}{2}a(u^*,u^*) - (f,u^*) = \min\left\{\frac{1}{2}a(u,u) - (f,u);\ u \in V\right\}. \tag{1.30}$$

If a is symmetric, then the Lax–Milgram lemma is a simple consequence of Riesz's representation theorem. Indeed, in this case $(u,v) \to a(u,v)$ is an equivalent scalar product on V and so, by the Riesz theorem, the functional $v \to (f,v)$ can be represented as (1.29) for some $u^* \in V$. In the general case we proceed as follows. For each $u \in V$, the functional $v \to a(u,v)$ is linear and continuous on V and we denote it by $Au \in V^*$. Then, the equation

$$a(u,v) = (f,v), \qquad \forall v \in V$$

can be rewritten as $Au = f$. Then, the conclusion follows because $R(A)$ is simultaneously closed and dense in V^*.

Weak Solutions to the Dirichlet Problem

Consider the Dirichlet problem

$$\begin{cases} -\Delta u + c(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.31}$$

where Ω is an open set of $\mathbf{R}^N$, $c \in L^\infty(\Omega)$, and $f \in H^{-1}(\Omega)$ is given.

Definition 1.4. The function u is said to be a *weak* or *variational solution* to the Dirichlet problem (1.31) if $u \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) dx + \int_\Omega c(x)u(x)\varphi(x)dx = (f, \varphi) \tag{1.32}$$

for all $\varphi \in H_0^1(\Omega)$ (equivalently, for all $\varphi \in C_0^\infty(\Omega)$).

In (1.32), ∇u is taken in the sense of distributions and (f, φ) is the value of the functional $f \in H^{-1}(\Omega)$ into $\varphi \in H_0^1(\Omega)$. If $f \in L^2(\Omega) \subset H^{-1}(\Omega)$, then

$$(f, \varphi) = \int_\Omega f(x)\varphi(x)dx.$$

By the Lax–Milgram lemma, applied to the functional

$$a(u,v) = \int_\Omega (\nabla u(x) \cdot \nabla v(x) + c\,uv)dx, \qquad u, v \in V = H_0^1(\Omega),$$

we obtain the following.

Theorem 1.8. *Let Ω be a bounded open set of $\mathbf{R}^N$ and let $c \in L^\infty(\Omega)$ be such that $c(x) \geq 0$, a.e. $x \in \Omega$. Then, for each $f \in H^{-1}(\Omega)$ the Dirichlet problem (1.31) has a unique weak solution $u^* \in H_0^1(\Omega)$. Moreover, u^* minimizes on $H_0^1(\Omega)$ the functional*

$$\frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + c(x)u^2(x))dx - (f, u) \tag{1.33}$$

and the map $f \to u^*$ is Lipschitzian from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$.

Weak Solutions to the Neumann Problem

Consider the boundary value problem

$$\begin{cases} -\Delta u + cu = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{1.34}$$

where $c \in L^\infty(\Omega),\ c(x) \geq \rho > 0$, and $f \in L^2(\Omega),\ g \in L^2(\partial\Omega)$.

Definition 1.5. The function $u \in H^1(\Omega)$ is said to be a *weak solution* to the Neumann problem (1.34) if

$$\int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega cuv\,dx = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,d\sigma, \qquad \forall v \in H^1(\Omega). \tag{1.35}$$

Because for each $v \in H^1(\Omega)$ the trace $\gamma_0(v)$ is in $L^2(\partial\Omega)$, the integral $\int_{\partial\Omega} gv\,d\sigma$ is well defined and so (1.35) makes sense.

Theorem 1.9. *Let Ω be an open subset of $\mathbf{R}^N$. Then, for each $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, problem* (1.34) *has a unique weak solution $u \in H^1(\Omega)$ that minimizes the functional*

$$u \to \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + c(x)u^2(x))dx - \int_\Omega f(x)u(x)dx - \int_{\partial\Omega} gu\,d\sigma \qquad \text{on } H^1(\Omega).$$

Proof. One applies the Lax–Milgram lemma on the space $V = H^1(\Omega)$ to the functional $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + cuv)dx,\ \forall u,v \in H^1(\Omega)$, and $(\widetilde{f}, v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,d\sigma$. □

Regularity of the Weak Solutions

We briefly recall here the regularity of the weak solutions to the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.36}$$

By Theorem 1.8 we know that if Ω is a bounded and open subset of $\mathbf{R}^N$ and $f \in L^2(\Omega)$, then problem (1.36) has a unique solution $u \in H_0^1(\Omega)$. It turns out that if $\partial\Omega$ is smooth enough, then this solution is actually in $H^2(\Omega) \cap H_0^1(\Omega)$. More precisely, we have the following theorem.

Theorem 1.10. *Let Ω be a bounded and open subset of $\mathbf{R}^N$ of class C^2. Let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be the weak solution to* (1.36)*. Then, $u \in H^2(\Omega)$ and*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{1.37}$$

where C is independent of f.

To prove the theorem, one first shows that $u \in H^2(\Omega')$ for each open subset $\Omega' \subset \Omega$ compactly embedded in Ω (interior regularity). The most delicate part (boundary regularity) follows by the *method of tangential quotients* due to L. Nirenberg. In short, the idea is to reduce problem (1.36) to an elliptic Dirichlet problem on $\mathbf{R}^N_+$ and to estimate separately the tangential quotients $(\nabla u)_h$, $h = (h_1, ..., h_{N-1}, 0)$ and the normal quotient $(\nabla u)_h$, $h = (0, ..., 0, h_N)$ in order to show that $v \in H^2(\mathbf{R}^N_+)$. For details we refer to Brezis' book [7]. (See also [5].)

In particular, Theorem 1.10 implies that if $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ is the elliptic operator $A = -\Delta$ in $\mathscr{D}'(\Omega)$; that is,

$$(Au, \varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx, \qquad \forall \varphi \in H^1_0(\Omega),$$

then

$$\{u \in H^1_0(\Omega);\ Au \in L^2(\Omega)\} \subset H^2(\Omega)$$

and

$$\|u\|_{H^2(\Omega)} \le C\|Au\|_{L^2(\Omega)}, \qquad \forall u \in H^1_0(\Omega) \cap H^2(\Omega).$$

Theorem 1.10 remains true if Ω is an open, convex, and bounded subset of $\mathbf{R}^N$. For the proof which uses some specific geometrical properties of Ω we refer the reader to Grisvard [10]. More generally, we have the following.

Theorem 1.11. *If Ω is of class C^{m+2} and $f \in H^m(\Omega)$, then the weak solution u to problem* (1.36) *belongs to $H^{m+2}(\Omega)$ and*

$$\|u\|_{m+2} \le C\|f\|_m, \qquad \forall f \in H^m(\Omega).$$

If $m > N/2$, then $u \in C^2(\overline{\Omega})$. In particular, if Ω is of class C^∞ and $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.

We conclude this section with a regularity result for the weak solution $u \in H^1(\Omega)$ to Neumann's problem

$$\begin{cases} u - \Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.38}$$

Theorem 1.12. *Under the assumptions of Theorem* 1.10 *the weak solution $u \in H^1(\Omega)$ to problem* (1.38) *belongs to $H^2(\Omega)$ and*

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}, \qquad \forall f \in L^2(\Omega). \tag{1.39}$$

Theorem 1.10 remains true in $L^p(\Omega)$ for $p > 1$. Namely, we have (Agmon, Douglis and Nirenberg [2])

Theorem 1.13. *Let Ω be a bounded open subset of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and let $1 < p < \infty$. Then, for each $f \in L^p(\Omega)$, the boundary value problem*

$$-\Delta u = f \quad in\ \ \Omega, \qquad u = 0 \quad on\ \ \partial\Omega$$

has a unique weak solution $u \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$. Moreover, one has

$$\|u\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

where C is independent of f.

The Space $BV(\Omega)$

Let Ω be an open subset of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$.

A function $f \in L^1(\Omega)$ is said to be of bounded variation on Ω if its gradient Df in the sense of distributions is an $\mathbf{R}^N$-valued measure on Ω; that is,

$$\|Df\| := \sup\left\{\int_\Omega f \operatorname{div}\psi\, d\xi : \psi \in C_0^\infty(\Omega;\mathbf{R}^N),\ |\psi|_\infty \le 1\right\} < +\infty,$$

or, equivalently,

$$\|Df\| = \int_\Omega |Df(x)|dx,$$

where $|Df|$ is the total variation of measure Df.

The space of all functions of bounded variation on Ω is denoted by $BV(\Omega)$. It is a Banach space with the norm

$$\|f\|_{BV(\Omega)} = |f|_{L^1(\Omega)} + \|Df\|.$$

Let $f \in BV(\Omega)$. Then there is a Radon measure μ_f on $\overline{\Omega}$ and a μ_f-measurable function $\sigma_f : \Omega \to \mathbf{R}^N$ such that $|\sigma_f(x)| = 1$, μ_f, a.e., and

$$\int_\Omega f \operatorname{div}\psi\, d\xi = -\int_\Omega \psi\cdot\sigma_f\, d\mu_f, \qquad \forall\psi \in C_0^1(\Omega;\mathbf{R}^N). \tag{1.40}$$

For each $f \in BV(\Omega)$ there is the trace $\gamma(f)$ on $\partial\Omega$ (assumed sufficiently smooth) defined by

$$\int_\Omega f \operatorname{div}\psi\, d\xi = -\int_\Omega \psi\cdot\sigma_f\, d\mu_f + \int_{\partial\Omega}\gamma(f)\psi\cdot\nu\, dH^{N-1}, \quad \forall\psi \in C^1(\overline{\Omega};\mathbf{R}^N), \tag{1.41}$$

where ν is the outward normal and dH^{N-1} is the Hausdorff measure on $\partial\Omega$. We have that $|\gamma(f)|_N \in L^1(\partial\Omega; dH^{N-1})$.

We denote by $BV^0(\Omega)$ the space of all $BV(\Omega)$ functions with vanishing trace on $\partial\Omega$. By the Poincaré inequality it follows that, on $BV^0(\Omega)$, $\|Df\|$ is a norm equivalent with $\|f\|_{BV^0(\Omega)}$.

Theorem 1.14. *Let $1 \le p \le N/(N-1)$ and Ω be a bounded open subset. Then, we have $BV(\Omega) \subset L^p(\Omega)$ with continuous and compact embedding. Moreover, the function $u \to \|Du\|$ is lower semicontinuous in $L^p(\Omega)$.*

We refer the reader to Ambrosio, Fusco and Pallara [3] for proofs and other basic results on functions with bounded variations.

Weak compactness in $L^1(\Omega)$

Let Ω be a measurable subset of $\mathbf{R}^N$. Contrary to what happens in $L^p(\Omega)$ spaces with $1 < p < \infty$ that are reflexive, a bounded subset $\mathscr{M}$ of $L^1(\Omega)$ is not necessarily weakly compact. This happens, however, under some additional conditions on $\mathscr{M}$.

Theorem 1.15. (Dunford–Pettis) *Let $\mathscr{M}$ be a bounded subset of $L^1(\Omega)$ having the property that the family of integrals $\{\int_E u(x)dx;\ E \subset \Omega$ measurable, $u \in \mathscr{M}\}$ is uniformly absolutely continuous; that is, for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ independent of u, such that $\int_E |u(x)|dx \le \varepsilon$ for $m(E) < \delta(\varepsilon)$ (m is the Lebesgue measure). Then the set $\mathscr{M}$ is weakly sequentially compact in $L^1(\Omega)$.*

For the proof, we refer the reader to Edwards [9], p. 270.

Theorem 1.15 remains true, of course, in $(L^1(\Omega))^m$, $m \in \mathbf{N}$.

1.4 Infinite-Dimensional Sobolev Spaces

Let X be a real (or complex) Banach space and let $[a,b]$ be a fixed interval on the real axis. A function $x : [a,b] \to X$ is said to be *finitely valued* if it is constant on each of a finite number of disjoint measurable sets $A_k \subset [a,b]$ and equal to zero on $[a,b] \setminus \cup_k A_k$. The function x is said to be *strongly measurable* on $[a,b]$ if there is a sequence $\{x_n\}$ of finite-valued functions that converges strongly in X and almost everywhere on $[a,b]$ to x. The function x is said to be *Bochner integrable* if there exists a sequence $\{x_n\}$ of finitely valued functions on $[a,b]$ to X that converges almost everywhere to x such that

$$\lim_{n\to\infty} \int_a^b \|x_n(t) - x(t)\| dt = 0.$$

A necessary and sufficient condition guaranteeing that $x : [a,b] \to X$ is Bochner integrable is that x is strongly measurable and that $\int_a^b \|x(t)\| dt < \infty$. The space of all Bochner integrable functions $x : [a,b] \to X$ is a Banach space with the norm

$$\|x\|_1 = \int_a^b \|x(t)\| dt,$$

and is denoted by $L^1(a,b;X)$.

More generally, the space of all (classes of) strongly measurable functions x from $[a,b]$ to X such that

$$\|x\|_p = \left(\int_a^b \|x(t)\|^p dt \right)^{1/p} < \infty$$

for $1 \le p < \infty$ and $\|x\|_\infty = \operatorname{ess\,sup}_{t\in[a,b]} \|x(t)\| < \infty$, is denoted by $L^p(a,b;X)$. This is a Banach space in the norm $\|\cdot\|_p$.

If X is reflexive, then the dual of $L^p(a,b;X)$ is the space $L^q(a,b;X^*)$, where $p<\infty$, $1/p+1/q=1$ (see Edward [9]). Recall also that a function $x:[a,b]\to X$ is said to be *weakly measurable* if for any $x^*\in X^*$, the function $t\to(x^*,x(t))$ is measurable. According to the Pettis theorem, if X is separable then every weakly measurable function is strongly measurable, and so these two notions coincide.

An X-valued function x defined on $[a,b]$ is said to be *absolutely continuous* on $[a,b]$ if for each $\varepsilon>0$ there exists $\delta(\varepsilon)$ such that $\sum_{n=1}^N\|x(t_n)-x(s_n)\|\le\varepsilon$, whenever $\sum_{n=1}^N|t_n-s_n|\le\delta(\varepsilon)$ and $(t_n,s_n)\cap(t_m,s_m)=\emptyset$ for $m\ne n$. Here, (t_n,s_n) is an arbitrary subinterval of (a,b).

A classical result in real analysis says that any real-valued absolutely continuous function is almost everywhere differentiable and it is expressed as the indefinite integral of its derivative. It should be mentioned that this result fails for X-valued absolutely continuous functions if X is a general Banach space.

However, if the space X is reflexive, we have (see, e.g., Komura [12]):

Theorem 1.16. *Let X be a reflexive Banach space. Then every X-valued absolutely continuous function x on $[a,b]$ is almost everywhere differentiable on $[a,b]$ and*

$$x(t)=x(a)+\int_a^t\frac{d}{ds}x(s)ds,\qquad\forall t\in[a,b],\tag{1.42}$$

where $(dx/dt):[a,b]\to X$ is the derivative of x; that is,

$$\frac{d}{dt}x(t)=\lim_{\varepsilon\to0}\frac{x(t+\varepsilon)-x(t)}{\varepsilon}.$$

Let us denote, as above, by $\mathscr{D}(a,b)$ the space of all infinitely differentiable real-valued functions on $[a,b]$ with compact support in (a,b), and by $\mathscr{D}'(a,b;X)$ the space of all continuous operators from $\mathscr{D}(a,b)$ to X. An element u of $\mathscr{D}'(a,b;X)$ is called an X-valued distribution on (a,b). If $u\in\mathscr{D}'(a,b;X)$ and j is a natural number, then

$$u^{(j)}(\varphi)=(-1)^ju(\varphi^{(j)}),\qquad\forall\varphi\in\mathscr{D}(a,b),$$

defines another distribution $u^{(j)}$, which is called the derivative of order j of u.

We note that every element $u\in L^1(a,b;X)$ defines uniquely the distribution (again denoted u)

$$u(\varphi)=\int_a^b u(t)\varphi(t)dt,\qquad\forall\varphi\in\mathscr{D}(a,b),\tag{1.43}$$

and so $L^1(a,b;X)$ can be regarded as a subspace of $\mathscr{D}'(a,b;X)$. In all that follows, we identify a function $u\in L^1(a,b;X)$ with the distribution u defined by (1.43).

Let k be a natural number and $1\le p\le\infty$. We denote by $W^{k,p}([a,b];X)$ the space of all X-valued distributions $u\in\mathscr{D}'(a,b;X)$ such that

$$u^{(j)}\in L^p(a,b;X)\quad\text{for } j=0,1,\dots,k.\tag{1.44}$$

Here, $u^{(j)}$ is the derivative of order j of u in the sense of distributions.

We denote by $A^{1,p}([a,b];X)$, $1 \le p \le \infty$, the space of all absolutely continuous functions u from $[a,b]$ to X having the property that they are a.e. differentiable on (a,b) and $(du/dt) \in L^p(a,b;X)$. If the space X is reflexive, it follows by Theorem 1.16 that $u \in A^{1,p}([a,b];X)$ if and only if u is absolutely continuous on $[a,b]$ and $(du/dt) \in L^p(a,b;X)$.

It turns out that the space $W^{1,p}$ can be identified with $A^{1,p}$. More precisely, we have (see Brezis [7]) the following theorem.

Theorem 1.17. *Let X be a Banach space and let $u \in L^p(a,b;X)$, $1 \le p \le \infty$. Then the following conditions are equivalent.*

(i) $u \in W^{1,p}([a,b];X)$.

(ii) *There is $u^0 \in A^{1,p}([a,b];X)$ such that $u(t) = u^0(t)$, a.e., $t \in (a,b)$. Moreover, $u' = du^0/dt$, a.e. in (a,b).*

Proof. For simplicity, we assume that $[a,b] = [0,T]$.

Let $u \in W^{1,p}([0,T];X)$; that is, $u \in L^p(0,T;X)$ and $u' \in L^p(0,T;X)$, and define the regularization u_n of u,

$$u_n(t) = n \int_0^T u(s)\rho((t-s)n)ds, \qquad \forall t \in [0,T], \tag{1.45}$$

where $\rho \in \mathscr{D}(\mathbf{R})$ is such that $\int \rho(s)ds = 1$, $\rho(t) = \rho(-t)$, $\operatorname{supp}\rho \subset [-1,1]$. It is well known that $u_n \to u$ in $L^p(0,T;X)$ for $n \to \infty$. Note also that u_n is infinitely differentiable. Let $\varphi \in \mathscr{D}(0,T)$ be arbitrary but fixed. Then, by (1.45), we see that

$$\begin{aligned}\int_0^T \frac{du_n}{dt}(t)\varphi(t)dt &= -\int_0^T u_n(t)\frac{d\varphi}{dt}(t)dt = -\int_0^T u(t)\frac{d\varphi_n}{dt}(t)dt\\ &= u'(\varphi_n) = \int_0^T u_n'\varphi\,dt \quad \text{if } \operatorname{supp}\varphi \subset \left(\frac{1}{n}, T-\frac{1}{n}\right).\end{aligned}$$

Hence,

$$\frac{du_n}{dt} = u_n', \quad \text{a.e. in } \left(\frac{1}{n}, T-\frac{1}{n}\right).$$

On the other hand, letting n tend to ∞ in the equation

$$u_n(t) - u_n(s) = \int_s^t \frac{du_n}{d\tau}(\tau)d\tau,$$

we get

$$u(t) - u(s) = \int_s^t u'(\tau)\,d\tau, \quad \text{a.e. } t,s \in (0,T),$$

because $(u')_n \to u'$ in $L^p(0,T;X)$. The latter equation implies that u admits an extension to an absolutely continuous function u^0 on $[0,T]$ that satisfies the equation

$$u^0(t) - u^0(0) = \int_0^t u'(\tau)d\tau, \qquad \forall t \in [0,T].$$

Hence, (i) $\Rightarrow$ (ii).

Conversely, assume now that $u \in A^{1,p}([0,T];X)$. Then,

$$u'(\varphi) = -\int_0^T u(t)\varphi'(t)dt = -\lim_{\varepsilon\to 0}\int_0^T u(t)\,\frac{\varphi(t)-\varphi(t-\varepsilon)}{\varepsilon}\,dt$$
$$= -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{T-\varepsilon}(u(t)-u(t+\varepsilon))\varphi(t)dt - \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{T-\varepsilon}^T u(t)\varphi(t)dt$$
$$+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon} u(t)\varphi(t-\varepsilon)dt, \qquad \forall\varphi\in\mathscr{D}(0,T).$$

Hence
$$u'(\varphi) = \int_0^T \frac{du}{dt}(t)\varphi(t), \qquad \forall\varphi\in\mathscr{D}(0,T).$$

This shows that $u' \in L^p(0,T;X)$ and $u' = du/dt$. □

Theorem 1.18. *Let X be a reflexive Banach space and let $u \in L^p(a,b;X)$, $1 < p \le \infty$. Then the following two conditions are equivalent.*

(i) $u \in W^{1,p}([a,b];X)$.

(ii) *There is $C > 0$ such that*

$$\int_a^{b-h}\|u(t+h)-u(t)\|^p dt \le C|h|^p, \qquad \forall h\in[0,b-a]$$

with the usual modification in the case $p = \infty$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 1.17, we know that

$$u(t+h)-u(t) = \int_t^{t+h}\frac{du^0}{ds}(s)ds, \qquad \forall t, t+h\in[a,b],$$

where $u^0 \in A^{1,p}([a,b];X)$ that is, $(du^0/dt) \in L^p(a,b;X)$. This yields via the Hölder inequality and Fubini theorem

$$\int_a^{b-h}\|u(t+h)-u(t)\|^p dt \le |h|^{p-1}\int_a^{b-h}dt\int_t^{t+h}\left\|\frac{du^0}{ds}\right\|^p ds \le |h|^p\int_a^b\left\|\frac{du^0}{ds}\right\|^p ds$$

and this implies estimate (ii).

(ii) $\Rightarrow$ (i). Let u_n be the regularization of u. A simple straightforward computation involving formula (1.45) reveals that $\{u_n'\}$ is bounded in $L^p(a,b;X)$. Because $u_n \to u$ in $L^p(a,b;X)$, $u_n' \to u'$ in $\mathscr{D}'(a,b;X)$, and $\{u_n'\}$ is weakly compact in $L^p(a,b;X)$, which is reflexive, we infer that $u' \in L^p(a,b;X)$, as claimed. □

Remark 1.2. If $u \in W^{1,1}([a,b];X)$, then it follows as above that

$$\int_a^{b-h}\|u(t+h)-u(t)\|dt \le C|h|, \qquad \forall h\in[0,b-a].$$

However, this inequality does not characterize the functions u in $W^{1,1}([a,b];X)$, but the functions u with bounded variation on $[a,b]$.

Let V be a reflexive Banach space and H be a real Hilbert space such that $V \subset H \subset V'$ in the algebraic and topological senses. Here, V' is the dual space of V and H is identified with its own dual. Denote by $|\cdot|$ and $\|\cdot\|$ the norms of H and V, respectively, and by $(\cdot,\cdot)$ the duality between V and V'. If $v_1, v_2 \in H$, then (v_1, v_2) is the scalar product in H of v_1 and v_2.

Denote by $W_p([a,b];V)$, $1 < p < \infty$, the space

$$W_p([a,b];V) = \{u \in L^p(a,b;V);\ u' \in L^q(a,b;V')\}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.46}$$

where u' is the derivative of u in the sense of $\mathscr{D}'(a,b;V)$. By Theorem 1.17, we know that every $u \in W_p([a,b];V)$ can be identified with an absolutely continuous function $u^0 : [a,b] \to V'$. However, we have a more precise result.

Theorem 1.19. *Let $u \in W_p([a,b];V)$. Then there is a continuous function $u^0 : [a,b] \to H$ such that $u(t) = u^0(t)$, a.e., $t \in (a,b)$. Moreover, if $u, v \in W_p([a,b];V)$, then the function $t \to (u(t), v(t))$ is absolutely continuous on $[a,b]$ and*

$$\frac{d}{dt}(u(t),v(t)) = (u'(t),v(t)) + (u(t),v'(t)), \quad \text{a.e. } t \in (a,b). \tag{1.47}$$

Proof. Let $u, v \in W_p([a,b];V)$ and $\psi(t) = (u(t), v(t))$. As we have seen in Theorem 1.17, we may assume that $u, v \in AC([a,b];V')$ and

$$\lim_{\varepsilon \downarrow 0} \int_a^{b-\varepsilon} \left\| \frac{u(t+\varepsilon) - u(t)}{\varepsilon} - u'(t) \right\|_{V'}^q dt = 0,$$

$$\lim_{\varepsilon \downarrow 0} \int_a^{b-\varepsilon} \left\| \frac{v(t+\varepsilon) - v(t)}{\varepsilon} - v'(t) \right\|_{V'}^q dt = 0.$$

Then, we have, by the Hölder inequality,

$$\lim_{\varepsilon \downarrow 0} \int_a^{b-\varepsilon} \left| \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon} - (u'(t), v(t)) - (u(t), v'(t)) \right| dt = 0.$$

Hence, $\psi \in W^{1,1}([a,b];\mathbf{R})$ and $(d\psi/dt)(t) = (u'(t),v(t)) + (u(t),v'(t))$, a.e. $t \in (a,b)$, as claimed.

Now, in equation (1.47) we take $v = u$ and integrate from s to t. We get

$$\frac{1}{2}(|u(t)|^2 - |u(s)|^2) = \int_s^t (u'(\tau), u(\tau))d\tau.$$

Hence, the function $t \to |u(t)|$ is continuous. On the other hand, for every $v \in V$ the function $t \to (u(t), v)$ is continuous. Because $|u(t)|$ is bounded on $[a,b]$, this implies that for every $v \in H$ the function $t \to (u(t), v)$ is continuous; that is, $u(t)$ is H-weakly continuous. Then, from the obvious equation

$$|u(t)-u(s)|^2 = |u(t)|^2 + |u(s)|^2 - 2(u(t),u(s)), \qquad \forall t,s \in [a,b]$$

it follows that $\lim_{s\to t}|u(t)-u(s)| = 0$, as claimed. □

The spaces $W^{1,p}([a,b];X)$, as well as $W_p([a,b];V)$, play an important role in the theory of differential equations in infinite-dimensional spaces.

The following compactness result, which is a sharpening of the Arzelà–Ascoli theorem, is particularly useful in this context.

Theorem 1.20 (Aubin). *Let X_0, X_1, X_2 be Banach spaces such that $X_0 \subset X_1 \subset X_2$, X_i reflexive for $i = 0,1,2$, and the injection of X_0 into X_1 is compact. Let $1 < p_i < \infty$, $i = 0,1$. Then the space*

$$W = L^{p_0}(a,b;X_0) \cap W^{1,p_1}([a,b];X_2)$$

is compactly embedded in $L^{p_0}(a,b;X_1)$.

The proof relies on the following property of the spaces X_i (see Lions [11], p. 58). For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u\|_{X_1} \le \varepsilon \|u\|_{X_0} + C_\varepsilon \|u\|_{X_2}, \qquad \forall u \in X_0.$$

References

1. D. Adams, *Sobolev Spaces*, Academic Press, San Diego, 1975.
2. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, *Comm. Pure Appl. Math.*, **12** (1959), pp. 623–727.
3. L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variations and Free Discontinuous Processes*, Oxford University Press, Oxford, UK, 2000.
4. E. Asplund, Average norms, *Israel J. Math.,* **5** (1967), pp. 227–233.
5. V. Barbu, *Partial Differential Equations and Boundary Value Problems*, Kluwer, Dordrecht, 1998.
6. V. Barbu, T. Precupanu, *Convexity and Optimization in Banach Spaces*, D. Reidel, Dordrecht, 1986.
7. H. Brezis, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, 1983.
8. H. Brezis, *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans un Espace de Hilbert*, North-Holland, Amsterdam, 1973.
9. R.E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, New York, 1965.
10. P. Grisvard, *Elliptic Problems in Non Smooth Domains*, Pitman Advanced Publishing Program, Boston, 1984.
11. J.L. Lions, *Quelques Méthodes de Resolution des Problèmes aux Limites Nonlinéaires*, Dunod-Gauthier Villars, Paris, 1969.
12. Y. Komura, Nonlinear semigroups in Hilbert spaces, *J. Math. Soc. Japan*, **19** (1967), pp. 508–520.
13. G. Köthe, *Topological Vector Spaces*, Springer-Verlag, Berlin, 1969.
14. J.J. Moreau, *Fonctionnelles Convexes*, Seminaire sur les équations aux dérivées partielles, Collège de France, Paris, 1966–1967.
15. R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1969.
16. K. Yosida, *Functional Analysis,* Springer-Verlag, New York, 1980.
17. C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.