7

Angular Velocity

Angular velocity of a rotating body B in a global frame G is the instantaneous rotation of the body with respect to G. Angular velocity is a vectorial quantity. Using the analytic description of angular velocity, we introduce the velocity and time derivative of homogenous transformation matrices.

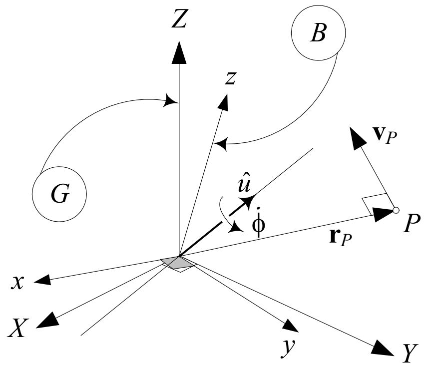

FIGURE 7.1. A rotating rigid body $B(Oxyz)$ with a fixed point O in a global frame $G(OXYZ)$.

7.1 Angular Velocity Vector and Matrix

Consider a rotating rigid body $B(Oxyz)$ with a fixed point O in a reference frame $G(OXYZ)$ as shown in Figure 7.1. The motion of the body can be expressed by a time varying rotation transformation matrix between the global and body frames. The transformation matrix maps the instantaneous coordinates of any fixed point in body frame B into their coordinates in the global frame G.

$$ {}^{G}\mathbf{r}(t) = {}^{G}R_{B}(t)\,{}^{B}\mathbf{r} \tag{7.1}$$

The velocity of a body point in the global frame is

$$ {}^{G}\mathbf{v}(t) = {}^{G}\dot{\mathbf{r}}(t) = {}^{G}\dot{R}_{B}(t)\,{}^{B}\mathbf{r} = {}_{G}\tilde{\omega}_{B}\,{}^{G}\mathbf{r}(t) = {}_{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}(t) \tag{7.2}$$

where ${}_{G}\boldsymbol{\omega}_{B}$ is the *angular velocity vector* of B with respect to G. It is equal to a rotation with *angular rate* $\dot{\phi}$ about an *instantaneous axis of rotation*

R.N. Jazar, *Theory of Applied Robotics,* 2nd ed., DOI 10.1007/978-1-4419-1750-8_7,

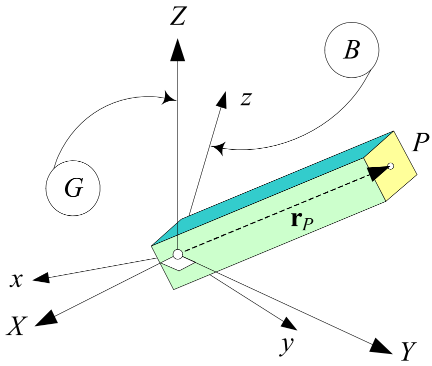

FIGURE 7.2. A body fixed point P at ${}^B\mathbf{r}$ in the rotating body frame B.

$\hat{u}$.

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \dot{\phi}\,\hat{u} \tag{7.3}$$

The angular velocity vector is associated with a skew symmetric matrix ${}_G\tilde{\omega}_B$ called the *angular velocity matrix*

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{7.4}$$

where

$${}_G\tilde{\omega}_B = {}^G\dot{R}_B\,{}^GR_B^T = \dot{\phi}\,\tilde{u}. \tag{7.5}$$

Proof. Consider a rigid body with a fixed point O and an attached frame $B(Oxyz)$ as shown in Figure 7.2. The body frame B is initially coincident with the global frame G. Therefore, the position vector of a body point P is

$${}^G\mathbf{r}(t_0) = {}^B\mathbf{r}. \tag{7.6}$$

The global time derivative of ${}^G\mathbf{r}$ is:

$$\begin{aligned} {}^G\mathbf{v} &= {}^G\dot{\mathbf{r}} = \frac{{}^Gd}{dt}\,{}^G\mathbf{r}(t) = \frac{{}^Gd}{dt}\left[{}^GR_B(t)\,{}^B\mathbf{r}\right] = \frac{{}^Gd}{dt}\left[{}^GR_B(t)\,{}^G\mathbf{r}(t_0)\right] \\ &= {}^G\dot{R}_B(t)\,{}^B\mathbf{r} \end{aligned} \tag{7.7}$$

Eliminating ${}^B\mathbf{r}$ between (7.1) and (7.7) determines the velocity of the point in global frame.

$${}^G\mathbf{v} = {}^G\dot{R}_B(t)\,{}^GR_B^T(t)\,{}^G\mathbf{r}(t) \tag{7.8}$$

We denote the coefficient of ${}^{G}\mathbf{r}(t)$ by $\tilde{\omega}$

$$ {}_{G}\tilde{\omega}_{B} = {}^{G}\dot{R}_{B}\, {}^{G}R_{B}^{T} \tag{7.9}$$

and write the Equation (7.8) as

$$ {}^{G}\mathbf{v} = {}_{G}\tilde{\omega}_{B}\, {}^{G}\mathbf{r}(t) \tag{7.10}$$

or as

$$ {}^{G}\mathbf{v} = {}_{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}(t). \tag{7.11}$$

The time derivative of the orthogonality condition, ${}^{G}R_{B}\, {}^{G}R_{B}^{T} = \mathbf{I}$, introduces an important identity

$$ {}^{G}\dot{R}_{B}\, {}^{G}R_{B}^{T} + {}^{G}R_{B}\, {}^{G}\dot{R}_{B}^{T} = 0 \tag{7.12}$$

which can be utilized to show that ${}_{G}\tilde{\omega}_{B} = [{}^{G}\dot{R}_{B}\, {}^{G}R_{B}^{T}]$ is a skew symmetric matrix, because

$$ {}^{G}R_{B}\, {}^{G}\dot{R}_{B}^{T} = \left[{}^{G}\dot{R}_{B}\, {}^{G}R_{B}^{T}\right]^{T}. \tag{7.13}$$

The vector ${}_{G}^{G}\boldsymbol{\omega}_{B}$ is called the *instantaneous angular velocity* of the body B relative to the global frame G as seen from the G frame.

Since a vectorial equation can be expressed in any coordinate frame, we may use any of the following expressions for the velocity of a body point in body or global frames

$$ {}_{G}^{G}\mathbf{v}_{P} = {}_{G}^{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}_{P} \tag{7.14}$$
$$ {}_{G}^{B}\mathbf{v}_{P} = {}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{r}_{P} \tag{7.15}$$

where ${}_{G}^{G}\mathbf{v}_{P}$ is the global velocity of point P expressed in global frame and ${}_{G}^{B}\mathbf{v}_{P}$ is the global velocity of point P expressed in body frame.

$$ {}_{G}^{G}\mathbf{v}_{P} = {}^{G}R_{B}\, {}_{G}^{B}\mathbf{v}_{P} = {}^{G}R_{B}\left({}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{r}_{P}\right) \tag{7.16}$$

${}_{G}^{G}\mathbf{v}_{P}$ and ${}_{G}^{B}\mathbf{v}_{P}$ can be converted to each other using a rotation matrix.

$$\begin{aligned} {}_{G}^{B}\mathbf{v}_{P} &= {}^{G}R_{B}^{T}\, {}_{G}^{G}\mathbf{v}_{P} = {}^{G}R_{B}^{T}\, {}_{G}\tilde{\omega}_{B}\, {}_{G}^{G}\mathbf{r}_{P} = {}^{G}R_{B}^{T}\, {}^{G}\dot{R}_{B}\, {}^{G}R_{B}^{T}\, {}_{G}^{G}\mathbf{r}_{P} \\ &= {}^{G}R_{B}^{T}\, {}^{G}\dot{R}_{B}\, {}_{G}^{B}\mathbf{r}_{P} \end{aligned} \tag{7.17}$$

showing that

$$ {}_{G}^{B}\tilde{\omega}_{B} = {}^{G}R_{B}^{T}\, {}^{G}\dot{R}_{B} \tag{7.18}$$

which is called the *instantaneous angular velocity* of B relative to the global frame G as seen from the B frame. From the definitions of ${}_{G}\tilde{\omega}_{B}$ and ${}_{G}^{B}\tilde{\omega}_{B}$ we are able to transform the two angular velocity matrices by

$$ {}_{G}\tilde{\omega}_{B} = {}^{G}R_{B}\, {}_{G}^{B}\tilde{\omega}_{B}\, {}^{G}R_{B}^{T} \tag{7.19}$$
$$ {}_{G}^{B}\tilde{\omega}_{B} = {}^{G}R_{B}^{T}\, {}_{G}^{G}\tilde{\omega}_{B}\, {}^{G}R_{B} \tag{7.20}$$

or equivalently

$$\begin{aligned} {}^G\dot{R}_B &= {}_G\tilde{\omega}_B \, {}^G R_B & (7.21) \\ {}^G\dot{R}_B &= {}^G R_B \, {}^B_G\tilde{\omega}_B & (7.22) \end{aligned}$$

$${}_G\tilde{\omega}_B \, {}^G R_B = {}^G R_B \, {}^B_G\tilde{\omega}_B. \tag{7.23}$$

The angular velocity of B in G is negative of the angular velocity of G in B if both are expressed in the same coordinate frame.

$$\begin{aligned} {}^G_G\tilde{\omega}_B &= -{}^G_B\tilde{\omega}_G & (7.24) \\ {}^B_G\tilde{\omega}_B &= -{}^B_B\tilde{\omega}_G. & (7.25) \end{aligned}$$

${}_G\boldsymbol{\omega}_B$ and can always be expressed in the form

$${}_G\boldsymbol{\omega}_B = \omega\hat{u} \tag{7.26}$$

where $\hat{u}$ is a unit vector parallel to ${}_G\boldsymbol{\omega}_B$ and indicates the *instantaneous axis of rotation.*

Using the Rodriguez rotation formula (3.4) we can show that

$$\dot{R}_{\hat{u},\phi} = \dot{\phi}\,\tilde{u}\,R_{\hat{u},\phi} \tag{7.27}$$

and therefore

$$\tilde{\omega} = \dot{\phi}\,\tilde{u} \tag{7.28}$$

or equivalently

$$\begin{aligned} {}_G\tilde{\omega}_B &= \lim_{\phi\to 0} \frac{{}^G d}{dt} R_{\hat{u},\phi} = \lim_{\phi\to 0} \frac{{}^G d}{dt}\left(-\tilde{u}^2\cos\phi + \tilde{u}\sin\phi + \tilde{u}^2 + \mathbf{I}\right) \\ &= \dot{\phi}\,\tilde{u} & (7.29) \end{aligned}$$

and therefore

$$\boldsymbol{\omega} = \dot{\phi}\,\hat{u}. \tag{7.30}$$

■

Example 198 *Rotation of a body point about a global axis.*

The slab shown in Figure 2.5 is turning about the Z-axis with $\dot{\alpha} = 10\,\mathrm{deg}/s$. The global velocity of the corner point $P(5, 30, 10)$, when the slab is at $\alpha = 30\,\mathrm{deg}$, is:

$$\begin{aligned} {}^G\mathbf{v}_P &= {}^G\dot{R}_B(t)\,{}^B\mathbf{r}_P & (7.31) \\ &= \frac{{}^G d}{dt}\left(\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} \\ &= \dot{\alpha}\begin{bmatrix} -\sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} \\ &= \frac{10\pi}{180}\begin{bmatrix} -\sin\frac{\pi}{6} & -\cos\frac{\pi}{6} & 0 \\ \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -4.97 \\ -1.86 \\ 0 \end{bmatrix} \end{aligned}$$

at this moment, the point P is at:

$$
\begin{aligned}
{}^{G}\mathbf{r}_{P} &= {}^{G}R_{B}\,{}^{B}\mathbf{r}_{P} \\
&= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix}
\end{aligned} \tag{7.32}
$$

Example 199 *Rotation of a global point about a global axis.*

The corner P of the slab shown in Figure 2.5, is at ${}^{B}\mathbf{r}_{P} = \begin{bmatrix} 5 & 30 & 10 \end{bmatrix}^{T}$*. When it is turned* $\alpha = 30\deg$ *about the Z-axis, the global position of P is:*

$$
\begin{aligned}
{}^{G}\mathbf{r}_{P} &= {}^{G}R_{B}\,{}^{B}\mathbf{r}_{P} \\
&= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix}
\end{aligned} \tag{7.33}
$$

If the slab is turning with $\dot{\alpha} = 10\deg/\mathrm{s}$*, the global velocity of the point P would be*

$$
\begin{aligned}
{}^{G}\mathbf{v}_{P} &= {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\,{}^{G}\mathbf{r}_{P} \\
&= \frac{10\pi}{180} \begin{bmatrix} -s\frac{\pi}{6} & -c\frac{\pi}{6} & 0 \\ c\frac{\pi}{6} & -s\frac{\pi}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\frac{\pi}{6} & -s\frac{\pi}{6} & 0 \\ s\frac{\pi}{6} & c\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix} \\
&= \begin{bmatrix} -4.97 \\ -1.86 \\ 0 \end{bmatrix}.
\end{aligned} \tag{7.34}
$$

Example 200 *Principal angular velocities.*

The principal rotational matrices about the axes X, Y, and Z are:

$$
R_{X,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \tag{7.35}
$$

$$
R_{Y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{7.36}
$$

$$
R_{Z,\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.37}
$$

and hence, their time derivatives are:

$$
\dot{R}_{X,\gamma} = \dot{\gamma} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\gamma & -\cos\gamma \\ 0 & \cos\gamma & -\sin\gamma \end{bmatrix} \tag{7.38}
$$

$$\dot{R}_{Y,\beta} = \dot{\beta} \begin{bmatrix} -\sin\beta & 0 & \cos\beta \\ 0 & 0 & 0 \\ -\cos\beta & 0 & -\sin\beta \end{bmatrix} \tag{7.39}$$

$$\dot{R}_{Z,\alpha} = \dot{\alpha} \begin{bmatrix} -\sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.40}$$

Therefore, the principal angular velocity matrices about axes X, Y, and Z are

$$_G\tilde{\omega}_X = \dot{R}_{X,\gamma} R^T_{X,\gamma} = \dot{\gamma} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.41}$$

$$_G\tilde{\omega}_Y = \dot{R}_{Y,\beta} R^T_{Y,\beta} = \dot{\beta} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{7.42}$$

$$_G\tilde{\omega}_Z = \dot{R}_{Z,\alpha} R^T_{Z,\alpha} = \dot{\alpha} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.43}$$

which are equivalent to

$$_G\tilde{\omega}_X = \dot{\gamma}\tilde{I} \tag{7.44}$$

$$_G\tilde{\omega}_Y = \dot{\beta}\tilde{J} \tag{7.45}$$

$$_G\tilde{\omega}_Z = \dot{\alpha}\tilde{K} \tag{7.46}$$

and therefore, the principal angular velocity vectors are

$$_G\boldsymbol{\omega}_X = \omega_X \hat{I} = \dot{\gamma}\hat{I} \tag{7.47}$$

$$_G\boldsymbol{\omega}_Y = \omega_Y \hat{J} = \dot{\beta}\hat{J} \tag{7.48}$$

$$_G\boldsymbol{\omega}_Z = \omega_Z \hat{K} = \dot{\alpha}\hat{K}. \tag{7.49}$$

Utilizing the same technique, we can find the following principal angular velocity matrices about the local axes.

$$^B_G\tilde{\omega}_x = R^T_{x,\psi}\, \dot{R}_{x,\psi} = -\dot{\psi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -\dot{\psi}\,\tilde{\imath} \tag{7.50}$$

$$^B_G\tilde{\omega}_y = R^T_{y,\theta} \dot{R}_{y,\theta} = -\dot{\theta} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = -\dot{\theta}\,\tilde{\jmath} \tag{7.51}$$

$$^B_G\tilde{\omega}_z = R^T_{z,\varphi} \dot{R}_{z,\varphi} = -\dot{\varphi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\dot{\varphi}\,\tilde{k} \tag{7.52}$$

Example 201 *Decomposition of an angular velocity vector.*

Every angular velocity vector can be decomposed to three principal angular velocity vectors.

$$\begin{aligned} {}_G\boldsymbol{\omega}_B &= \left({}_G\boldsymbol{\omega}_B \cdot \hat{I}\right)\hat{I} + \left({}_G\boldsymbol{\omega}_B \cdot \hat{J}\right)\hat{J} + \left({}_G\boldsymbol{\omega}_B \cdot \hat{K}\right)\hat{K} \\ &= \omega_X\,\hat{I} + \omega_Y\,\hat{J} + \omega_Z\,\hat{K} = \dot{\gamma}\hat{I} + \dot{\beta}\hat{J} + \dot{\alpha}\hat{K} \\ &= \boldsymbol{\omega}_X + \boldsymbol{\omega}_Y + \boldsymbol{\omega}_Z \end{aligned} \tag{7.53}$$

Example 202 *Combination of angular velocities.*

Starting from a combination of rotations

$${}^0R_2 = {}^0R_1\,{}^1R_2 \tag{7.54}$$

and taking derivative, we find

$${}^0\dot{R}_2 = {}^0\dot{R}_1\,{}^1R_2 + {}^0R_1\,{}^1\dot{R}_2. \tag{7.55}$$

Now, substituting the derivative of rotation matrices with

$${}^0\dot{R}_2 = {}_0\tilde{\omega}_2\,{}^0R_2 \tag{7.56}$$

$${}^0\dot{R}_1 = {}_0\tilde{\omega}_1\,{}^0R_1 \tag{7.57}$$

$${}^1\dot{R}_2 = {}_1\tilde{\omega}_2\,{}^1R_2 \tag{7.58}$$

results in

$$\begin{aligned} {}_0\tilde{\omega}_2\,{}^0R_2 &= {}_0\tilde{\omega}_1\,{}^0R_1\,{}^1R_2 + {}^0R_1\,{}_1\tilde{\omega}_2\,{}^1R_2 \\ &= {}_0\tilde{\omega}_1\,{}^0R_2 + {}^0R_1\,{}_1\tilde{\omega}_2\,{}^0R_1^T\,{}^0R_1\,{}^1R_2 \\ &= {}_0\tilde{\omega}_1\,{}^0R_2 + {}_1^0\tilde{\omega}_2\,{}^0R_2 \end{aligned} \tag{7.59}$$

where

$${}^0R_1\,{}_1\tilde{\omega}_2\,{}^0R_1^T = {}_1^0\tilde{\omega}_2. \tag{7.60}$$

Therefore, we find

$${}_0\tilde{\omega}_2 = {}_0\tilde{\omega}_1 + {}_1^0\tilde{\omega}_2 \tag{7.61}$$

which indicates that the angular velocities may be added relatively

$${}_0\boldsymbol{\omega}_2 = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2. \tag{7.62}$$

This result also holds for any number of angular velocities.

$${}_0\boldsymbol{\omega}_n = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2 + {}_2^0\boldsymbol{\omega}_3 + \cdots + {}_{n-1}^{\;\;0}\boldsymbol{\omega}_n = \sum_{i=1}^{n} {}_{i-1}^{\;\;0}\boldsymbol{\omega}_i \tag{7.63}$$

Example 203 *Angular velocity in terms of Euler frequencies.*

The angular velocity vector can be expressed by Euler frequencies as described in Chapter 2. Therefore,

$$\begin{aligned}
{}_G^B\boldsymbol{\omega}_B &= \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} = \dot{\varphi}\hat{e}_\varphi + \dot{\theta}\hat{e}_\theta + \dot{\psi}\hat{e}_\psi \\
&= \dot{\varphi}\begin{bmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{bmatrix} + \dot{\theta}\begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{aligned} \tag{7.64}$$

and

$$\begin{aligned}
{}_G^G\boldsymbol{\omega}_B &= {}^BR_G^{-1}\, {}_G^B\boldsymbol{\omega}_B = {}^BR_G^{-1}\begin{bmatrix} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \cos\varphi & \sin\theta\sin\varphi \\ 0 & \sin\varphi & -\cos\varphi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{aligned} \tag{7.65}$$

where the inverse of the Euler transformation matrix is:

$${}^BR_G^{-1} = \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & -c\varphi s\psi - c\theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\varphi s\psi + c\theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \tag{7.66}$$

Example 204 *Angular velocity in terms of rotation frequencies.*

Appendices A and B show the 12 *global and* 12 *local axes' triple rotations. Utilizing those equations, we are able to find the angular velocity matrix and vector of a rigid body in terms of rotation frequencies. As an example, consider the Euler angles transformation matrix in case* 9*, of the Appendix B.*

$${}^BR_G = R_{z,\psi}R_{x,\theta}R_{z,\varphi} \tag{7.67}$$

The angular velocity matrix is then equal to

$$\begin{aligned}
{}_B\tilde{\omega}_G &= {}^B\dot{R}_G\, {}^BR_G^T \\
&= \left(\dot{\varphi}\, R_{z,\psi}R_{x,\theta}\frac{dR_{z,\varphi}}{d\varphi} + \dot{\theta}\, R_{z,\psi}\frac{dR_{x,\theta}}{d\theta}R_{z,\varphi} + \dot{\psi}\,\frac{dR_{z,\psi}}{d\psi}R_{x,\theta}R_{z,\varphi}\right) \\
&\quad \times \left(R_{z,\psi}R_{x,\theta}R_{z,\varphi}\right)^T \\
&= \dot{\varphi}\, R_{z,\psi}R_{x,\theta}\frac{dR_{z,\varphi}}{d\varphi}R_{z,\varphi}^TR_{x,\theta}^TR_{z,\psi}^T + \dot{\theta}\, R_{z,\psi}\frac{dR_{x,\theta}}{d\theta}R_{x,\theta}^TR_{z,\psi}^T \\
&\quad + \dot{\psi}\,\frac{dR_{z,\psi}}{d\psi}R_{z,\psi}^T
\end{aligned} \tag{7.68}$$

which, in matrix form, is

$$
\begin{aligned}
{}_B\tilde{\omega}_G &= \dot{\varphi}\begin{bmatrix} 0 & \cos\theta & -\sin\theta\cos\psi \\ -\cos\theta & 0 & \sin\theta\sin\psi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & 0 \end{bmatrix} \\
&\quad +\dot{\theta}\begin{bmatrix} 0 & 0 & \sin\psi \\ 0 & 0 & \cos\psi \\ -\sin\psi & -\cos\psi & 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{7.69}
$$

or

$$
{}_B\tilde{\omega}_G = \begin{bmatrix} 0 & \dot{\psi} + \dot{\varphi}c\theta & \dot{\theta}s\psi - \dot{\varphi}s\theta c\psi \\ -\dot{\psi} - \dot{\varphi}c\theta & 0 & \dot{\theta}c\psi + \dot{\varphi}s\theta s\psi \\ -\dot{\theta}s\psi + \dot{\varphi}s\theta c\psi & -\dot{\theta}c\psi - \dot{\varphi}s\theta s\psi & 0 \end{bmatrix}. \tag{7.70}
$$

The corresponding angular velocity vector is

$$
\begin{aligned}
{}_B\boldsymbol{\omega}_G &= -\begin{bmatrix} \dot{\theta}c\psi + \dot{\varphi}s\theta s\psi \\ -\dot{\theta}s\psi + \dot{\varphi}s\theta c\psi \\ \dot{\psi} + \dot{\varphi}c\theta \end{bmatrix} \\
&= -\begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}.
\end{aligned} \tag{7.71}
$$

However,

$$
{}^B_B\tilde{\omega}_G = -{}^B_G\tilde{\omega}_B \tag{7.72}
$$

$$
{}^B_B\boldsymbol{\omega}_G = -{}^B_G\boldsymbol{\omega}_B \tag{7.73}
$$

and therefore,

$$
{}^B_G\boldsymbol{\omega}_B = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{7.74}
$$

Example 205 ★ *Coordinate transformation of angular velocity.*

Angular velocity ${}^1_1\boldsymbol{\omega}_2$ of coordinate frame B_2 with respect to B_1 and expressed in B_1 can be expressed in base coordinate frame B_0 according to

$$
{}^0R_1\,{}_1\tilde{\omega}_2\,{}^0R_1^T = {}^0_1\tilde{\omega}_2. \tag{7.75}
$$

To show this equation, it is enough to apply both sides on an arbitrary vector ${}^0\mathbf{r}$. Therefore, the left-hand side would be

$$
\begin{aligned}
{}^0R_1\,{}_1\tilde{\omega}_2\,{}^0R_1^T\,{}^0\mathbf{r} &= {}^0R_1\,{}_1\tilde{\omega}_2\,{}^1R_0\,{}^0\mathbf{r} = {}^0R_1\,{}_1\tilde{\omega}_2\,{}^1\mathbf{r} \\
&= {}^0R_1\left({}_1\boldsymbol{\omega}_2 \times {}^1\mathbf{r}\right) = {}^0R_1\,{}_1\boldsymbol{\omega}_2 \times {}^0R_1\,{}^1\mathbf{r} \\
&= {}^0_1\boldsymbol{\omega}_2 \times {}^0\mathbf{r}
\end{aligned} \tag{7.76}
$$

which is equal to the right-hand side after applying on the vector ${}^{0}\mathbf{r}$.

$$ {}_{1}^{0}\tilde{\omega}_{2}\,{}^{0}\mathbf{r} = {}_{1}^{0}\boldsymbol{\omega}_{2} \times {}^{0}\mathbf{r} \tag{7.77}$$

Example 206 ★ *Time derivative of unit vectors.*

Using Equation (7.15) we can define the time derivative of unit vectors of a body coordinate frame $B(\hat{\imath}, \hat{\jmath}, \hat{k})$, *rotating in the global coordinate frame* $G(\hat{I}, \hat{J}, \hat{K})$.

$$ \frac{{}^{G}d\hat{\imath}}{dt} = {}_{G}^{B}\boldsymbol{\omega}_{B} \times \hat{\imath} \tag{7.78}$$
$$ \frac{{}^{G}d\hat{\jmath}}{dt} = {}_{G}^{B}\boldsymbol{\omega}_{B} \times \hat{\jmath} \tag{7.79}$$
$$ \frac{{}^{G}d\hat{k}}{dt} = {}_{G}^{B}\boldsymbol{\omega}_{B} \times \hat{k} \tag{7.80}$$

Example 207 ★ *Angular velocity in terms of quaternion and Euler parameters.*

The angular velocity vector can also be expressed by Euler parameters. Starting from the unit quaternion representation of a finite rotation

$$ {}^{G}\mathbf{r} = e(t)\ {}^{B}\mathbf{r}\ e^{*}(t) = e(t)\ {}^{B}\mathbf{r}\ e^{-1}(t) \tag{7.81}$$

where

$$ e = e_0 + \mathbf{e} \tag{7.82}$$
$$ e^{*} = e^{-1} = e_0 - \mathbf{e} \tag{7.83}$$

we can find

$$ {}^{G}\dot{\mathbf{r}} = \dot{e}\ {}^{B}\mathbf{r}\ e^{*} + e\ {}^{B}\mathbf{r}\ \dot{e}^{*} = \dot{e}\ e^{*}\,{}^{G}\mathbf{r} + {}^{G}\mathbf{r}\ e\ \dot{e}^{*} = 2\dot{e}\ e^{*}\,{}^{G}\mathbf{r} \tag{7.84}$$

and therefore, the angular velocity quaternion is

$$ {}_{G}\boldsymbol{\omega}_{B} = 2\dot{e}\,e^{*}. \tag{7.85}$$

We have used the orthogonality property of unit quaternion

$$ e\,e^{-1} = e\,e^{*} = 1 \tag{7.86}$$

which provides

$$ \dot{e}\,e^{*} + e\,\dot{e}^{*} = 0. \tag{7.87}$$

The angular velocity quaternion can be expanded using quaternion prod-

ucts to find the angular velocity components based on Euler parameters.

$$\begin{aligned}
{}_G\boldsymbol{\omega}_B &= 2\dot{e}\ e^* = 2\left(\dot{e}_0 + \dot{\mathbf{e}}\right)\left(e_0 - \mathbf{e}\right) \\
&= 2\left(\dot{e}_0 e_0 + e_0\dot{\mathbf{e}} - \dot{e}_0\mathbf{e} + \dot{\mathbf{e}}\cdot\mathbf{e} - \dot{\mathbf{e}}\times\mathbf{e}\right) \\
&= 2\begin{bmatrix} 0 \\ e_0\dot{e}_1 - e_1\dot{e}_0 + e_2\dot{e}_3 - e_3\dot{e}_2 \\ e_0\dot{e}_2 - e_2\dot{e}_0 - e_1\dot{e}_3 + e_3\dot{e}_1 \\ e_0\dot{e}_3 + e_1\dot{e}_2 - e_2\dot{e}_1 - e_3\dot{e}_0 \end{bmatrix} \\
&= 2\begin{bmatrix} \dot{e}_0 & -\dot{e}_1 & -\dot{e}_2 & -\dot{e}_3 \\ \dot{e}_1 & \dot{e}_0 & -\dot{e}_3 & \dot{e}_2 \\ \dot{e}_2 & \dot{e}_3 & \dot{e}_0 & -\dot{e}_1 \\ \dot{e}_3 & -\dot{e}_2 & \dot{e}_1 & \dot{e}_0 \end{bmatrix}\begin{bmatrix} e_0 \\ -e_1 \\ -e_2 \\ -e_3 \end{bmatrix}
\end{aligned} \tag{7.88}$$

The scalar component of the angular velocity quaternion is zero because

$$\dot{e}_0 e_0 + \dot{\mathbf{e}}\cdot\mathbf{e} = \dot{e}_0 e_0 + e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 = 0. \tag{7.89}$$

The angular velocity vector can also be defined as a quaternion

$$\overleftrightarrow{{}_G\boldsymbol{\omega}_B} = 2\,\overleftrightarrow{\dot{e}}\ \overleftrightarrow{e^*} \tag{7.90}$$

to be utilized for definition of the derivative of a rotation quaternion

$$\overleftrightarrow{\dot{e}} = \frac{1}{2}\,\overleftrightarrow{{}_G\boldsymbol{\omega}_B}\,\overleftrightarrow{e}. \tag{7.91}$$

A coordinate transformation can transform the angular velocity into a body coordinate frame

$$\begin{aligned}
{}^B_G\boldsymbol{\omega}_B &= e^*\ {}^G_G\boldsymbol{\omega}_B\ e = 2e^*\,\dot{e} \\
&= 2\begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix}\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix}
\end{aligned} \tag{7.92}$$

and therefore,

$$\overleftrightarrow{{}^B_G\boldsymbol{\omega}_B} = 2\,\overleftrightarrow{e^*}\ \overleftrightarrow{\dot{e}} \tag{7.93}$$

$$\overleftrightarrow{\dot{e}} = \frac{1}{2}\,\overleftrightarrow{e}\ \overleftrightarrow{{}^B_G\boldsymbol{\omega}_B}. \tag{7.94}$$

Example 208 ★ *Differential of Euler parameters.*

The rotation matrix ${}^GR_B$ based on Euler parameters is given in Equation (3.82)

$$\begin{aligned}
{}^GR_B &= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2\left(e_1e_2 - e_0e_3\right) & 2\left(e_0e_2 + e_1e_3\right) \\ 2\left(e_0e_3 + e_1e_2\right) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2\left(e_2e_3 - e_0e_1\right) \\ 2\left(e_1e_3 - e_0e_2\right) & 2\left(e_0e_1 + e_2e_3\right) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
\end{aligned} \tag{7.95}$$

and the individual parameters can be found from any set of Equations (3.149) to (3.152). The first set indicates that

$$\begin{aligned} e_0 &= \pm\frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}} \\ e_1 &= \frac{1}{4}\frac{r_{32}-r_{23}}{e_0} \qquad e_2 = \frac{1}{4}\frac{r_{13}-r_{31}}{e_0} \qquad e_3 = \frac{1}{4}\frac{r_{21}-r_{12}}{e_0} \end{aligned} \tag{7.96}$$

and therefore,

$$\dot{e}_0 = \frac{\dot{r}_{11}+\dot{r}_{22}+\dot{r}_{33}}{8e_0} \tag{7.97}$$

$$\dot{e}_1 = \frac{1}{4e_0^2}\left(\left(\dot{r}_{32}-\dot{r}_{23}\right)e_0-\left(r_{32}-r_{23}\right)\dot{e}_0\right) \tag{7.98}$$

$$\dot{e}_2 = \frac{1}{4e_0^2}\left(\left(\dot{r}_{13}-\dot{r}_{31}\right)e_0-\left(r_{13}-r_{31}\right)\dot{e}_0\right) \tag{7.99}$$

$$\dot{e}_3 = \frac{1}{4e_0^2}\left(\left(\dot{r}_{21}-\dot{r}_{12}\right)e_0-\left(r_{21}-r_{12}\right)\dot{e}_0\right). \tag{7.100}$$

We may use the differential of the transformation matrix

$${}^G\dot{R}_B = {}_G\tilde{\omega}_B\,{}^GR_B$$

to show that

$$\dot{e}_0 = \frac{1}{2}\left(-e_1\omega_1-e_2\omega_2-e_3\omega_3\right) \tag{7.101}$$

$$\dot{e}_1 = \frac{1}{2}\left(e_0\omega_1+e_2\omega_3-e_3\omega_2\right) \tag{7.102}$$

$$\dot{e}_2 = \frac{1}{2}\left(e_0\omega_2-e_1\omega_3-e_3\omega_1\right) \tag{7.103}$$

$$\dot{e}_3 = \frac{1}{2}\left(e_0\omega_3+e_1\omega_2-e_2\omega_1\right). \tag{7.104}$$

Similarly we can find $\dot{e}_1$, $\dot{e}_2$, and $\dot{e}_3$, however the final result can be set in a matrix form

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{7.105}$$

or

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e_0 & -e_3 & -e_2 & -e_1 \\ e_1 & e_0 & -e_3 & e_2 \\ e_2 & e_1 & e_0 & -e_3 \\ e_3 & -e_2 & e_1 & e_0 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{7.106}$$

Example 209 ★ *Elements of the angular velocity matrix.*
Utilizing the permutation symbol introduced in (3.144)

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \quad , \quad i,j,k = 1,2,3 \tag{7.107}$$

allows us to find the elements of the angular velocity matrix, $\tilde{\omega}$, when the angular velocity vector, $\boldsymbol{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$, is given.

$$\tilde{\omega}_{ij} = \epsilon_{ijk}\,\omega_k \tag{7.108}$$

7.2 ★ Time Derivative and Coordinate Frames

The time derivative of a vector depends on the coordinate frame in which we are taking the derivative. The time derivative of a vector $\mathbf{r}$ in the global frame is called *G-derivative* and is denoted by

$$\frac{{}^{G}d}{dt}\mathbf{r}$$

while the time derivative of the vector in the body frame is called the *B-derivative* and is denoted by

$$\frac{{}^{B}d}{dt}\mathbf{r}.$$

The left superscript on the derivative symbol indicates the frame in which the derivative is taken, and hence, its unit vectors are considered constant. Therefore, the derivative of ${}^{B}\mathbf{r}_P$ in B and the derivative of ${}^{G}\mathbf{r}_P$ in G are:

$$\frac{{}^{B}d}{dt}\,{}^{B}\mathbf{r}_P = {}^{B}\dot{\mathbf{r}}_P = {}^{B}\mathbf{v}_P = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k} \tag{7.109}$$

$$\frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r}_P = {}^{G}\dot{\mathbf{r}}_P = {}^{G}\mathbf{v}_P = \dot{X}\,\hat{I} + \dot{Y}\,\hat{J} + \dot{Z}\,\hat{K} \tag{7.110}$$

It is also possible to find the G-derivative of ${}^{B}\mathbf{r}_P$ and the B-derivative of ${}^{G}\mathbf{r}_P$. We use Equation (7.15) for the global velocity of a body fixed point P, expressed in body frame to define the mixed derivatives. The G-derivative of a body vector ${}^{B}\mathbf{r}_P$ is denoted by

$${}^{B}_{G}\mathbf{v}_P = \frac{{}^{G}d}{dt}\,{}^{B}\mathbf{r}_P \tag{7.111}$$

and similarly, the B-derivative of a global vector ${}^{G}\mathbf{r}_P$ is denoted by

$${}^{G}_{B}\mathbf{v}_P = \frac{{}^{B}d}{dt}\,{}^{G}\mathbf{r}_P. \tag{7.112}$$

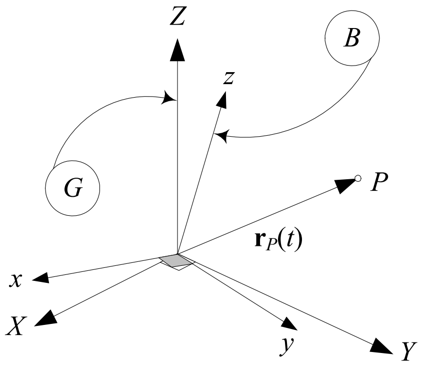

FIGURE 7.3. A moving body point P at ${}^B\mathbf{r}(t)$ in the rotating body frame B.

When point P is moving in frame B while B is rotating in G, the G-derivative of ${}^B\mathbf{r}_P\,(t)$ is defined by

$$\frac{{}^Gd}{dt}\,{}^B\mathbf{r}_P\,(t) = {}^B\dot{\mathbf{r}}_P + {}^B_G\boldsymbol{\omega}_B \times {}^B\mathbf{r}_P = {}^B_G\dot{\mathbf{r}}_P \tag{7.113}$$

and the B-derivative of ${}^G\mathbf{r}_P$ is defined by

$$\frac{{}^Bd}{dt}\,{}^G\mathbf{r}_P\,(t) = {}^G\dot{\mathbf{r}}_P - {}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}_P = {}^G_B\dot{\mathbf{r}}_P. \tag{7.114}$$

Proof. Let $G(OXYZ)$ with unit vectors $\hat{I}$, $\hat{J}$, and $\hat{K}$ be the global coordinate frame, and let $B(Oxyz)$ with unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ be a body coordinate frame. The position vector of a moving point P, as shown in Figure 7.3, can be expressed in the body and global frames

$$\begin{aligned} {}^B\mathbf{r}_P\,(t) &= x\,(t)\,\hat{\imath} + y\,(t)\,\hat{\jmath} + z\,(t)\,\hat{k} && (7.115)\\ {}^G\mathbf{r}_P\,(t) &= X\,(t)\,\hat{I} + Y\,(t)\,\hat{J} + Z\,(t)\,\hat{K}. && (7.116)\end{aligned}$$

The time derivative of ${}^B\mathbf{r}_P$ in B and ${}^G\mathbf{r}_P$ in G are

$$\begin{aligned} \frac{{}^Bd}{dt}\,{}^B\mathbf{r}_P &= {}^B\dot{\mathbf{r}}_P = {}^B\mathbf{v}_P = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k} && (7.117)\\ \frac{{}^Gd}{dt}\,{}^G\mathbf{r}_P &= {}^G\dot{\mathbf{r}}_P = {}^G\mathbf{v}_P = \dot{X}\,\hat{I} + \dot{Y}\,\hat{J} + \dot{Z}\,\hat{K} && (7.118)\end{aligned}$$

because the unit vectors of B in Equation (7.115) and the unit vectors of G in Equation (7.116) are considered constant.

Using the definition (7.111), we can find the G-derivative of the position vector ${}^{B}\mathbf{r}_P$ as

$$\begin{aligned}
\frac{{}^{G}d}{dt}\,{}^{B}\mathbf{r}_P &= \frac{{}^{G}d}{dt}\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right) \\
&= \dot{x}\,\hat{\imath}+\dot{y}\,\hat{\jmath}+\dot{z}\,\hat{k}+x\frac{{}^{G}d\hat{\imath}}{dt}+y\frac{{}^{G}d\hat{\jmath}}{dt}+z\frac{{}^{G}d\hat{k}}{dt} \\
&= {}^{B}\dot{\mathbf{r}}_P+{}^{B}_{G}\boldsymbol{\omega}_B\times\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)={}^{B}\dot{\mathbf{r}}_P+{}^{B}_{G}\boldsymbol{\omega}_B\times{}^{B}\mathbf{r}_P \\
&= \frac{{}^{B}d}{dt}\,{}^{B}\mathbf{r}_P+{}^{B}_{G}\boldsymbol{\omega}_B\times{}^{B}\mathbf{r}_P. \qquad (7.119)
\end{aligned}$$

We achieved this result because the x, y, and z components of ${}^{B}\mathbf{r}_P$ are scalar. Scalars are invariant with respect to frame transformations. Therefore, if x is a scalar then,

$$\frac{{}^{G}d}{dt}x=\frac{{}^{B}d}{dt}x=\dot{x}. \qquad (7.120)$$

The B-derivative of ${}^{G}\mathbf{r}_P$ is

$$\begin{aligned}
\frac{{}^{B}d}{dt}\,{}^{G}\mathbf{r}_P &= \frac{{}^{B}d}{dt}\left(X\hat{I}+Y\hat{J}+Z\hat{K}\right) \\
&= \dot{X}\,\hat{I}+\dot{Y}\,\hat{J}+\dot{Z}\,\hat{K}+X\frac{{}^{B}d\hat{I}}{dt}+Y\frac{{}^{B}d\hat{J}}{dt}+Z\frac{{}^{B}d\hat{K}}{dt} \\
&= {}^{G}\dot{\mathbf{r}}_P+{}^{G}_{B}\boldsymbol{\omega}_G\times{}^{G}\mathbf{r}_P \qquad (7.121)
\end{aligned}$$

and therefore,

$$\frac{{}^{B}d}{dt}\,{}^{G}\mathbf{r}_P={}^{G}\dot{\mathbf{r}}_P-{}_{G}\boldsymbol{\omega}_B\times{}^{G}\mathbf{r}_P. \qquad (7.122)$$

The angular velocity of B relative to G is a vector quantity and can be expressed in either frames.

$$\begin{aligned}
{}^{G}_{G}\boldsymbol{\omega}_B &= \omega_X\,\hat{I}+\omega_Y\,\hat{J}+\omega_Z\,\hat{K} \qquad (7.123) \\
{}^{B}_{G}\boldsymbol{\omega}_B &= \omega_x\hat{\imath}+\omega_y\hat{\jmath}+\omega_z\hat{k}. \qquad (7.124)
\end{aligned}$$

■

Example 210 *Rotation of B about Z-axis.*

A body frame B is rotating in G with $\dot{\alpha}$ about the Z-axis. Therefore, a point at ${}^{B}\mathbf{r}$ will be seen at

$$\begin{aligned}
{}^{G}\mathbf{r}_P &= {}^{G}R_B\,{}^{B}\mathbf{r}=R_{Z,\alpha}(t)\,{}^{B}\mathbf{r} \qquad (7.125) \\
&= \begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}x\cos\alpha-y\sin\alpha\\ y\cos\alpha+x\sin\alpha\\ z\end{bmatrix}
\end{aligned}$$

The angular velocity matrix of B *is*

$$ {}_G\tilde{\omega}_B = {}^G\dot{R}_B \, {}^G R_B^T = \dot{\alpha}\tilde{K} \tag{7.126}$$

that gives

$$ {}_G\boldsymbol{\omega}_B = \dot{\alpha}\hat{K}. \tag{7.127}$$

We can find the body expression of ${}_G\tilde{\omega}_B$

$$ {}_G^B\tilde{\omega}_B = {}^G R_B^T \, {}_G^G\tilde{\omega}_B \, {}^G R_B = \dot{\alpha}\tilde{k} \tag{7.128}$$

and therefore,

$$ {}_G^B\boldsymbol{\omega}_B = \dot{\alpha}\hat{k}. \tag{7.129}$$

Now we can find the following derivatives.

$$ \frac{{}^B d}{dt}\, {}^B\mathbf{r} = {}^B\dot{\mathbf{r}} = 0 \tag{7.130}$$

$$\begin{aligned} \frac{{}^G d}{dt}\, {}^G\mathbf{r} &= {}^G\dot{\mathbf{r}} = \frac{{}^G d}{dt}\begin{bmatrix} x\cos\alpha - y\sin\alpha \\ y\cos\alpha + x\sin\alpha \\ z \end{bmatrix} \\ &= \left(-x\dot{\alpha}\sin\alpha - y\dot{\alpha}\cos\alpha\right)\hat{I} + \left(x\dot{\alpha}\cos\alpha - y\dot{\alpha}\sin\alpha\right)\hat{J} + \dot{z}\hat{K} \end{aligned} \tag{7.131}$$

For the mixed derivatives we start with the global velocity expressed in B*.*

$$\begin{aligned} \frac{{}^G d}{dt}\, {}^B\mathbf{r} &= {}_G^B\boldsymbol{\omega}_B \times {}^B\mathbf{r} \\ &= \dot{\alpha}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dot{\alpha}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \\ &= -y\dot{\alpha}\hat{\imath} + x\dot{\alpha}\hat{\jmath} = {}_G^B\dot{\mathbf{r}} \end{aligned} \tag{7.132}$$

We can transform ${}_G^B\dot{\mathbf{r}}$ *to the global frame and find the global expression velocity,* ${}^G\dot{\mathbf{r}}$*.*

$$\begin{aligned} {}^G\dot{\mathbf{r}} &= {}^G R_B \, {}_G^B\dot{\mathbf{r}} \\ &= \dot{\alpha}\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = \dot{\alpha}\begin{bmatrix} -y\cos\alpha - x\sin\alpha \\ x\cos\alpha - y\sin\alpha \\ 0 \end{bmatrix} \\ &= \dot{\alpha}\left(-y\cos\alpha - x\sin\alpha\right)\hat{I} + \dot{\alpha}\left(x\cos\alpha - y\sin\alpha\right)\hat{J} \end{aligned} \tag{7.133}$$

The next derivative is the velocity of body points relative to B *and expressed in* G*.*

$$ \frac{{}^B d}{dt}\, {}^G\mathbf{r} = {}^G\dot{\mathbf{r}} - {}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r} \tag{7.134}$$

$$ = \dot{\alpha}\begin{bmatrix} -y\cos\alpha - x\sin\alpha \\ x\cos\alpha - y\sin\alpha \\ 0 \end{bmatrix} - \dot{\alpha}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} x\cos\alpha - y\sin\alpha \\ y\cos\alpha + x\sin\alpha \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 211 *Time derivative of a moving point in B.*

Consider a local frame B, rotating in G by $\dot{\alpha}$ about the Z-axis, and a moving point at ${}^{B}\mathbf{r}_P(t) = t\hat{\imath}$. Therefore,

$$\begin{aligned} {}^{G}\mathbf{r}_P &= {}^{G}R_B\, {}^{B}\mathbf{r}_P = R_{Z,\alpha}(t)\, {}^{B}\mathbf{r}_P \qquad (7.135) \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\cos\alpha\hat{I} + t\sin\alpha\hat{J}. \end{aligned}$$

The angular velocity matrix is

$$_G\tilde{\omega}_B = {}^{G}\dot{R}_B\, {}^{G}R_B^T = \dot{\alpha}\tilde{K} \qquad (7.136)$$

that gives

$$_G\boldsymbol{\omega}_B = \dot{\alpha}\hat{K}. \qquad (7.137)$$

It can also be verified that

$${}^{B}_{G}\tilde{\omega}_B = {}^{G}R_B^T\, {}^{G}_{G}\tilde{\omega}_B\, {}^{G}R_B = \dot{\alpha}\tilde{k} \qquad (7.138)$$

and therefore,

$${}^{B}_{G}\boldsymbol{\omega}_B = \dot{\alpha}\hat{k}. \qquad (7.139)$$

Now we can find the following derivatives

$$\frac{{}^{B}d}{dt}\, {}^{B}\mathbf{r}_P = {}^{B}\dot{\mathbf{r}}_P = \hat{\imath} \qquad (7.140)$$

$$\begin{aligned} \frac{{}^{G}d}{dt}\, {}^{G}\mathbf{r}_P &= {}^{G}\dot{\mathbf{r}}_P \\ &= (\cos\alpha - t\dot{\alpha}\sin\alpha)\,\hat{I} + (\sin\alpha + t\dot{\alpha}\cos\alpha)\,\hat{J}. \qquad (7.141) \end{aligned}$$

For the mixed derivatives we start with

$$\begin{aligned} \frac{{}^{G}d}{dt}\, {}^{B}\mathbf{r}_P &= \frac{{}^{B}d}{dt}\, {}^{B}\mathbf{r}_P + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\mathbf{r}_P \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \dot{\alpha}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ t\dot{\alpha} \\ 0 \end{bmatrix} \\ &= \hat{\imath} + t\dot{\alpha}\hat{\jmath} = {}^{B}_{G}\dot{\mathbf{r}}_P \qquad (7.142) \end{aligned}$$

which is the global velocity of P expressed in B. We may, however, transform ${}^{B}_{G}\dot{\mathbf{r}}_P$ to the global frame and find the global velocity expressed in G.

$$\begin{aligned} {}^{G}\dot{\mathbf{r}}_P &= {}^{G}R_B\, {}^{B}_{G}\dot{\mathbf{r}}_P \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t\dot{\alpha} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha - t\dot{\alpha}\sin\alpha \\ \sin\alpha + t\dot{\alpha}\cos\alpha \\ 0 \end{bmatrix} \\ &= (\cos\alpha - t\dot{\alpha}\sin\alpha)\,\hat{I} + (\sin\alpha + t\dot{\alpha}\cos\alpha)\,\hat{J} \qquad (7.143) \end{aligned}$$

The next derivative is

$$\begin{aligned}
\frac{{}^{B}d}{dt}\,{}^{G}\mathbf{r}_{P} &= {}^{G}\dot{\mathbf{r}}_{P} - {}_{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}_{P} \\
&= \begin{bmatrix} \cos\alpha - t\dot{\alpha}\sin\alpha \\ \sin\alpha + t\dot{\alpha}\cos\alpha \\ 0 \end{bmatrix} - \dot{\alpha}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} t\cos\alpha \\ t\sin\alpha \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = (\cos\alpha)\,\hat{I} + (\sin\alpha)\,\hat{J} = {}^{G}_{B}\dot{\mathbf{r}}_{P}
\end{aligned} \tag{7.144}$$

which is the velocity of P relative to B and expressed in G. To express this velocity in B we apply a frame transformation.

$$\begin{aligned}
{}^{B}\dot{\mathbf{r}}_{P} &= {}^{G}R_{B}^{T}\,{}^{G}_{B}\dot{\mathbf{r}}_{P} \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{\imath}
\end{aligned} \tag{7.145}$$

Sometimes it is more applied if we transform the vector to the same frame in which we are taking the derivative and then apply the differential operator. Therefore,

$$\begin{aligned}
\frac{{}^{G}d}{dt}\,{}^{B}\mathbf{r}_{P} &= \frac{{}^{G}d}{dt}\left({}^{G}R_{B}\,{}^{B}\mathbf{r}_{P}\right) \\
&= \frac{{}^{G}d}{dt}\begin{bmatrix} t\cos\alpha \\ t\sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha - t\dot{\alpha}\sin\alpha \\ \sin\alpha + t\dot{\alpha}\cos\alpha \\ 0 \end{bmatrix}
\end{aligned} \tag{7.146}$$

and

$$\frac{{}^{B}d}{dt}\,{}^{G}\mathbf{r}_{P} = \frac{{}^{B}d}{dt}\left({}^{G}R_{B}^{T}\,{}^{G}\mathbf{r}_{P}\right) = \frac{{}^{B}d}{dt}\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7.147}$$

Example 212 *Orthogonality of position and velocity vectors.*

If the position vector of a body point in global frame is denoted by $\mathbf{r}$ *then*

$$\frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = 0. \tag{7.148}$$

To show this property we may take a derivative from

$$\mathbf{r}\cdot\mathbf{r} = r^{2} \tag{7.149}$$

and find

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} + \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 2\frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = 0. \tag{7.150}$$

The Equation (7.148) is correct in every coordinate frame and for every constant length vector, as long as the vector and the derivative are expressed in the same coordinate frame.

Example 213 ★ *Derivative transformation formula.*

The global velocity of a fixed point in the body coordinate frame $B\,(Oxyz)$ can be found by Equation (7.2). Now consider a point P that can move in $B\,(Oxyz)$. In this case, the body position vector ${}^{B}\mathbf{r}_{P}$ is not constant and therefore, the global velocity of such a point expressed in B is

$$\frac{{}^{G}d}{dt}\,{}^{B}\mathbf{r}_{P}=\frac{{}^{B}d}{dt}\,{}^{B}\mathbf{r}_{P}+{}_{G}^{B}\boldsymbol{\omega}_{B}\times{}^{B}\mathbf{r}_{P}={}_{G}^{B}\dot{\mathbf{r}}_{P}. \tag{7.151}$$

Sometimes the result of Equation (7.151) is utilized to define transformation of the differential operator from a body to a global coordinate frame

$$\frac{{}^{G}d}{dt}\Box=\frac{{}^{B}d}{dt}\Box+{}_{G}^{B}\boldsymbol{\omega}_{B}\times{}_{G}^{B}\Box={}_{G}^{B}\dot{\Box} \tag{7.152}$$

however, special attention must be paid to the coordinate frame in which the vector $\Box$ and the final result are expressed. The final result is ${}_{G}^{B}\dot{\Box}$ showing the global (G) time derivative expressed in body frame (B). The vector $\Box$ might be any vector such as position, velocity, angular velocity, momentum, angular velocity, or even a time-varying force vector.

The Equation (7.152) is called the ***derivative transformation formula*** *and relates the time derivative of a vector as it would be seen from frame G to its derivative as seen in frame B. The derivative transformation formula (7.152) is more general and can be applied to every vector for derivative transformation between every two relatively moving coordinate frames.*

Example 214 ★ *Differential equation for rotation matrix.*

Equation (7.5) for defining the angular velocity matrix may be written as a first-order differential equation

$$\frac{d}{dt}\,{}^{G}R_{B}-{}^{G}R_{B}\,{}_{G}\tilde{\omega}_{B}=0. \tag{7.153}$$

The solution of the equation confirms the exponential definition of the rotation matrix as

$${}^{G}R_{B}=e^{\tilde{\omega}t} \tag{7.154}$$

or

$$\tilde{\omega}t=\dot{\phi}\,\tilde{u}=\ln\left({}^{G}R_{B}\right). \tag{7.155}$$

Example 215 ★ *Acceleration of a body point in the global frame.*

The angular acceleration vector of a rigid body $B(Oxyz)$ in the global frame $G(OXYZ)$ is denoted by ${}_{G}\boldsymbol{\alpha}_{B}$ and is defined as the global time derivative of ${}_{G}\boldsymbol{\omega}_{B}$.

$${}_{G}\boldsymbol{\alpha}_{B}=\frac{{}^{G}d}{dt}\,{}_{G}\boldsymbol{\omega}_{B} \tag{7.156}$$

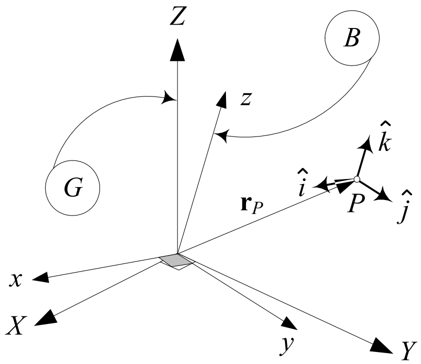

FIGURE 7.4. A body coordinate frame moving with a fixed point in the global coordinate frame.

Using this definition, the acceleration of a fixed body point in the global frame is

$$\begin{aligned} {}^G\mathbf{a}_P &= \frac{{}^G d}{dt}\left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}_P\right) \\ &= {}_G\boldsymbol{\alpha}_B \times {}^G\mathbf{r}_P + {}_G\boldsymbol{\omega}_B \times \left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}_P\right). \end{aligned} \tag{7.157}$$

Example 216 ★ *Alternative definition of angular velocity vector.*

The angular velocity vector of a rigid body $B(\hat{\imath}, \hat{\jmath}, \hat{k})$ in global frame $G(\hat{I}, \hat{J}, \hat{K})$ can also be defined by

$${}_G^B\boldsymbol{\omega}_B = \hat{\imath}(\frac{{}^G d\hat{\jmath}}{dt} \cdot \hat{k}) + \hat{\jmath}(\frac{{}^G d\hat{k}}{dt} \cdot \hat{\imath}) + \hat{k}(\frac{{}^G d\hat{\imath}}{dt} \cdot \hat{\jmath}). \tag{7.158}$$

Proof. *Consider a body coordinate frame B moving with a fixed point in the global coordinate frame G. The fixed point of the body is taken as the origin of both coordinate frames, as shown in Figure 7.4. In order to describe the motion of the body, it is sufficient to describe the motion of the local unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. Let $\mathbf{r}_P$ be the position vector of a body point P. Then, ${}^B\mathbf{r}_P$ is a vector with constant components.*

$${}^B\mathbf{r}_P = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{7.159}$$

When the body moves, it is only the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ that vary relative to the global coordinate frame. Therefore, the vector of differential displacement is

$$d\mathbf{r}_P = x\,d\hat{\imath} + y\,d\hat{\jmath} + z\,d\hat{k} \tag{7.160}$$

which can also be expressed by

$$d\mathbf{r}_P = (d\mathbf{r}_P \cdot \hat{\imath})\,\hat{\imath} + (d\mathbf{r}_P \cdot \hat{\jmath})\,\hat{\jmath} + \left(d\mathbf{r}_P \cdot \hat{k}\right)\hat{k}. \tag{7.161}$$

Substituting (7.160) in the right-hand side of (7.161) results in

$$\begin{aligned} d\mathbf{r}_P &= \left(x\hat{\imath}\cdot d\hat{\imath}+y\hat{\imath}\cdot d\hat{\jmath}+z\hat{\imath}\cdot d\hat{k}\right)\hat{\imath} \\ &\quad+\left(x\hat{\jmath}\cdot d\hat{\imath}+y\hat{\jmath}\cdot d\hat{\jmath}+z\hat{\jmath}\cdot d\hat{k}\right)\hat{\jmath} \\ &\quad+\left(x\hat{k}\cdot d\hat{\imath}+y\hat{k}\cdot d\hat{\jmath}+z\hat{k}\cdot d\hat{k}\right)\hat{k}. \end{aligned} \tag{7.162}$$

Utilizing the unit vectors' relationships

$$\hat{\jmath}\cdot d\hat{\imath} = -\hat{\imath}\cdot d\hat{\jmath} \tag{7.163}$$
$$\hat{k}\cdot d\hat{\jmath} = -\hat{\jmath}\cdot d\hat{k} \tag{7.164}$$
$$\hat{\imath}\cdot d\hat{k} = -\hat{k}\cdot d\hat{\imath} \tag{7.165}$$
$$\hat{\imath}\cdot d\hat{\imath} = \hat{\jmath}\cdot d\hat{\jmath}=\hat{k}\cdot d\hat{k}=0 \tag{7.166}$$
$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k}=\hat{k}\cdot\hat{\imath}=0 \tag{7.167}$$
$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1 \tag{7.168}$$

the $d\mathbf{r}_P$ *reduces to*

$$\begin{aligned} d\mathbf{r}_P &= \left(z\hat{\imath}\cdot d\hat{k}-y\hat{\jmath}\cdot d\hat{\imath}\right)\hat{\imath} \\ &\quad+\left(x\hat{\jmath}\cdot d\hat{\imath}-z\hat{k}\cdot d\hat{\jmath}\right)\hat{\jmath} \\ &\quad+\left(y\hat{k}\cdot d\hat{\jmath}-x\hat{\imath}\cdot d\hat{k}\right)\hat{k}. \end{aligned} \tag{7.169}$$

This equation can be rearranged to be expressed as a vector product

$$d\mathbf{r}_P = \left((\hat{k}\cdot d\hat{\jmath})\hat{\imath}+(\hat{\imath}\cdot d\hat{k})\hat{\jmath}+(\hat{\jmath}\cdot d\hat{\imath})\hat{k}\right)\times\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right) \tag{7.170}$$

or

$${}^B_G\mathbf{v}_P = \left((\hat{k}\cdot\frac{{}^Gd\hat{\jmath}}{dt})\hat{\imath}+(\hat{\imath}\cdot\frac{{}^Gd\hat{k}}{dt})\hat{\jmath}+(\hat{\jmath}\cdot\frac{{}^Gd\hat{\imath}}{dt})\hat{k}\right)\times\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right). \tag{7.171}$$

Comparing this result with

$$\dot{\mathbf{r}}_P = {}_G\boldsymbol{\omega}_B\times\mathbf{r}_P$$

shows that

$${}^B_G\boldsymbol{\omega}_B = \hat{\imath}\left(\frac{{}^Gd\hat{\jmath}}{dt}\cdot\hat{k}\right)+\hat{\jmath}\left(\frac{{}^Gd\hat{k}}{dt}\cdot\hat{\imath}\right)+\hat{k}\left(\frac{{}^Gd\hat{\imath}}{dt}\cdot\hat{\jmath}\right). \tag{7.172}$$

■

Example 217 ★ *Alternative proof for angular velocity definition (7.158).*

The angular velocity definition presented in Equation (7.158) can also be shown by direct substitution for ${}^{G}R_{B}$ in the angular velocity matrix ${}_{G}^{B}\tilde{\omega}_{B}$

$$ {}_{G}^{B}\tilde{\omega}_{B} = {}^{G}R_{B}^{T}\, {}^{G}\dot{R}_{B}. \tag{7.173}$$

Therefore,

$$\begin{aligned} {}_{G}^{B}\tilde{\omega}_{B} &= \begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\imath}\cdot\hat{J} & \hat{\imath}\cdot\hat{K} \\ \hat{\jmath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} \cdot \frac{{}^{G}d}{dt} \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\imath}\cdot\frac{{}^{G}d\hat{\imath}}{dt} & \hat{\imath}\cdot\frac{{}^{G}d\hat{\jmath}}{dt} & \hat{\imath}\cdot\frac{{}^{G}d\hat{k}}{dt} \\ \hat{\jmath}\cdot\frac{{}^{G}d\hat{\imath}}{dt} & \hat{\jmath}\cdot\frac{{}^{G}d\hat{\jmath}}{dt} & \hat{\jmath}\cdot\frac{{}^{G}d\hat{k}}{dt} \\ \hat{k}\cdot\frac{{}^{G}d\hat{\imath}}{dt} & \hat{k}\cdot\frac{{}^{G}d\hat{\jmath}}{dt} & \hat{k}\cdot\frac{{}^{G}d\hat{k}}{dt} \end{bmatrix} \end{aligned} \tag{7.174}$$

which shows that

$$ {}_{G}^{B}\boldsymbol{\omega}_{B} = \begin{bmatrix} \frac{{}^{G}d\hat{\jmath}}{dt}\cdot\hat{k} \\ \frac{{}^{G}d\hat{k}}{dt}\cdot\hat{\imath} \\ \frac{{}^{G}d\hat{\imath}}{dt}\cdot\hat{\jmath} \end{bmatrix}. \tag{7.175}$$

Example 218 ★ *Second derivative.*

In general, ${}^{G}d\,\mathbf{r}/dt$ is a variable vector in $G(OXYZ)$ and in any other coordinate frame such as $B\,(oxyz)$. Therefore, it can be differentiated in either coordinate frames G or B. However, the order of differentiating is important. In general,

$$\frac{{}^{B}d}{dt}\frac{{}^{G}d\mathbf{r}}{dt} \neq \frac{{}^{G}d}{dt}\frac{{}^{B}d\mathbf{r}}{dt}. \tag{7.176}$$

As an example, consider a rotating body coordinate frame about the Z-axis, and a variable vector as

$$ {}^{G}\mathbf{r} = t\hat{I}. \tag{7.177}$$

Therefore,

$$\frac{{}^{G}d\mathbf{r}}{dt} = {}^{G}\dot{\mathbf{r}} = \hat{I} \tag{7.178}$$

and hence,

$$\begin{aligned} {}^{B}\left(\frac{{}^{G}d\mathbf{r}}{dt}\right) &= {}_{G}^{B}\dot{\mathbf{r}} = R_{Z,\varphi}^{T}\left[\hat{I}\right] = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \cos\varphi\hat{\imath} - \sin\varphi\hat{\jmath} \end{aligned} \tag{7.179}$$

which provides

$$\frac{{}^{B}d}{dt}\frac{{}^{G}d\mathbf{r}}{dt} = -\dot{\varphi}\sin\varphi\hat{\imath} - \dot{\varphi}\cos\varphi\hat{\jmath} \tag{7.180}$$

and

$${}^{G}\left(\frac{{}^{B}d}{dt}\frac{{}^{G}d\mathbf{r}}{dt}\right) = -\dot{\varphi}\hat{J}. \tag{7.181}$$

Now

$${}^{B}\mathbf{r} = R^{T}_{Z,\varphi}\left[t\hat{I}\right] = t\cos\varphi\hat{\imath} - t\sin\varphi\hat{\jmath} \tag{7.182}$$

that provides

$$\frac{{}^{B}d\mathbf{r}}{dt} = \left(-t\dot{\varphi}\sin\varphi + \cos\varphi\right)\hat{\imath} - \left(\sin\varphi + t\dot{\varphi}\cos\varphi\right)\hat{\jmath} \tag{7.183}$$

and

$$\begin{aligned}
{}^{G}\left(\frac{{}^{B}d\mathbf{r}}{dt}\right) &= {}^{G}_{B}\dot{\mathbf{r}} = R_{Z,\varphi}\left(\left(-t\dot{\varphi}\sin\varphi + \cos\varphi\right)\hat{\imath} - \left(\sin\varphi + t\dot{\varphi}\cos\varphi\right)\hat{\jmath}\right) \\
&= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -t\dot{\varphi}\sin\varphi + \cos\varphi \\ -\sin\varphi - t\dot{\varphi}\cos\varphi \\ 0 \end{bmatrix} \\
&= \hat{I} - t\dot{\varphi}\hat{J}
\end{aligned} \tag{7.184}$$

which shows

$$\frac{{}^{G}d}{dt}\frac{{}^{B}d\mathbf{r}}{dt} = -\left(\dot{\varphi} + t\ddot{\varphi}\right)\hat{J} \neq \frac{{}^{B}d}{dt}\frac{{}^{G}d\mathbf{r}}{dt}. \tag{7.185}$$

7.3 Rigid Body Velocity

Consider a rigid body with an attached local coordinate frame $B\,(oxyz)$ moving freely in a fixed global coordinate frame $G(OXYZ)$, as shown in Figure 7.5. The rigid body can rotate in the global frame, while the origin of the body frame B can translate relative to the origin of G. The coordinates of a body point P in local and global frames are related by the following equation:

$${}^{G}\mathbf{r}_P = {}^{G}R_B\,{}^{B}\mathbf{r}_P + {}^{G}\mathbf{d}_B \tag{7.186}$$

where ${}^{G}\mathbf{d}_B$ indicates the position of the moving origin o relative to the fixed origin O.

The velocity of the point P in G is

$$\begin{aligned}
{}^{G}\mathbf{v}_P &= {}^{G}\dot{\mathbf{r}}_P = {}^{G}\dot{R}_B\,{}^{B}\mathbf{r}_P + {}^{G}\dot{\mathbf{d}}_B \\
&= {}_{G}\tilde{\omega}_B\,{}^{G}_{B}\mathbf{r}_P + {}^{G}\dot{\mathbf{d}}_B = {}_{G}\tilde{\omega}_B\left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B \\
&= {}_{G}\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B.
\end{aligned} \tag{7.187}$$

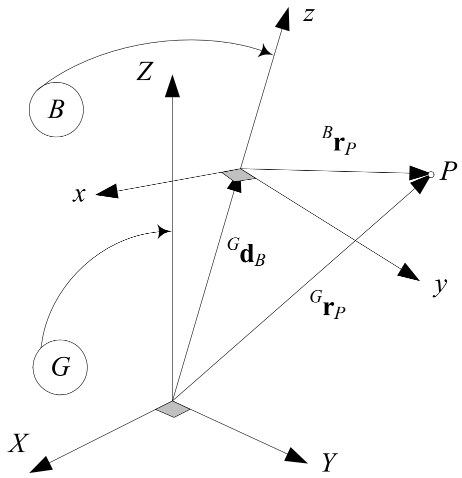

FIGURE 7.5. A rigid body with an attached coordinate frame $B\,(oxyz)$ moving freely in a global coordinate frame $G(OXYZ)$.

Proof. Direct differentiating shows

$$\begin{aligned} {}^{G}\mathbf{v}_P &= \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r}_P = {}^{G}\dot{\mathbf{r}}_P = \frac{{}^{G}d}{dt}\left({}^{G}R_B\,{}^{B}\mathbf{r}_P + {}^{G}\mathbf{d}_B\right) \\ &= {}^{G}\dot{R}_B\,{}^{B}\mathbf{r}_P + {}^{G}\dot{\mathbf{d}}_B. \end{aligned} \tag{7.188}$$

The local position vector ${}^{B}\mathbf{r}_P$ can be substituted from (7.186) to obtain

$$\begin{aligned} {}^{G}\mathbf{v}_P &= {}^{G}\dot{R}_B\,{}^{G}R_B^T\left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B \\ &= {}_{G}\tilde{\omega}_B\left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B \\ &= {}_{G}\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B. \end{aligned} \tag{7.189}$$

It may also be written using relative position vector

$$ {}^{G}\mathbf{v}_P = {}_{G}\boldsymbol{\omega}_B \times {}^{G}_{B}\mathbf{r}_P + {}^{G}\dot{\mathbf{d}}_B. \tag{7.190}$$

■

Example 219 *Geometric interpretation of rigid body velocity.*

Figure 7.6 illustrates a body point P of a moving rigid body. The global velocity of the point P

$$ {}^{G}\mathbf{v}_P = {}_{G}\boldsymbol{\omega}_B \times {}^{G}_{B}\mathbf{r}_P + {}^{G}\dot{\mathbf{d}}_B \tag{7.191}$$

is a vector addition of rotational and translational velocities, both expressed in the global frame. At the moment, the body frame is assumed to be coincident with the global frame, and the body frame has a velocity ${}^{G}\dot{\mathbf{d}}_B$ with

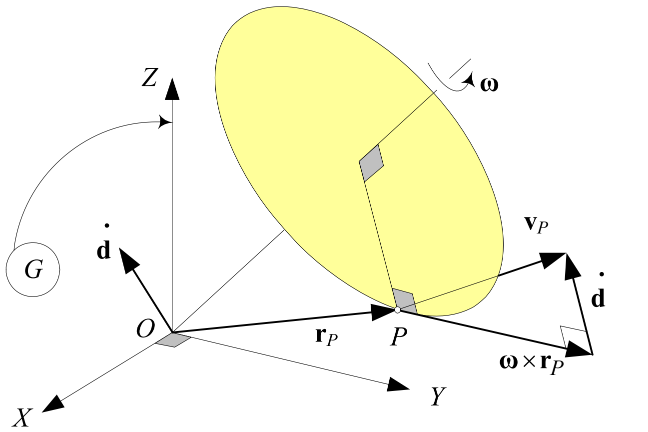

FIGURE 7.6. Geometric interpretation of rigid body velocity.

respect to the global frame. The translational velocity ${}^{G}\dot{\mathbf{d}}_B$ *is a common property for every point of the body, but the rotational velocity* ${}_{G}\boldsymbol{\omega}_B \times {}^{G}_{B}\mathbf{r}_P$ *differs for different points of the body.*

Example 220 *Velocity of a moving point in a moving body frame.*

Assume that point P in Figure 7.5 is moving in the frame B, indicating by a time varying position vector ${}^{B}\mathbf{r}_P(t)$. The global velocity of P is a composition of the velocity of P in B, rotation of B relative to G, and velocity of B relative to G.

$$\begin{aligned}\frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r}_P &= \frac{{}^{G}d}{dt}\left({}^{G}\mathbf{d}_B + {}^{G}R_B\,{}^{B}\mathbf{r}_P\right) \\ &= \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{d}_B + \frac{{}^{G}d}{dt}\left({}^{G}R_B\,{}^{B}\mathbf{r}_P\right) \\ &= {}^{G}\dot{\mathbf{d}}_B + {}^{G}_{B}\dot{\mathbf{r}}_P + {}_{G}\boldsymbol{\omega}_B \times {}^{G}_{B}\mathbf{r}_P \end{aligned} \tag{7.192}$$

Example 221 *Velocity of a body point in multiple coordinate frames.*

Consider three frames, B_0, B_1 and B_2, as shown in Figure 7.7. The velocity of point P should be measured and expressed in a coordinate frame. If the point is stationary in a frame, say B_2, then the time derivative of ${}^{2}\mathbf{r}_P$ in B_2 is zero. If the frame B_2 is moving relative to the frame B_1, then, the time derivative of ${}^{1}\mathbf{r}_P$ is a combination of the rotational component due to rotation of B_2 relative to B_1 and the velocity of B_2 relative to B_1. In forward velocity kinematics of robots, the velocities must be measured in the base frame B_0. Therefore, the velocity of point P in the base frame is a combination of the velocity of B_2 relative to B_1 and the velocity of B_1 relative to B_0.

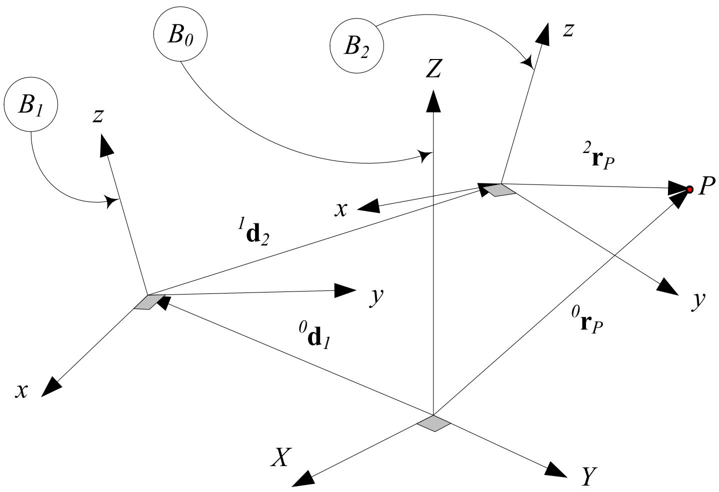

FIGURE 7.7. A rigid body coordinate frame B_2 is moving in a frame B_1 that is moving in the base coordinate frame B_0.

The global coordinate of the body point P is

$$\begin{aligned} {}^0\mathbf{r}_P &= {}^0\mathbf{d}_1 + {}^0_1\mathbf{d}_2 + {}^0_2\mathbf{r}_P & (7.193) \\ &= {}^0\mathbf{d}_1 + {}^0R_1\,{}^1\mathbf{d}_2 + {}^0R_2\,{}^2\mathbf{r}_P. & (7.194) \end{aligned}$$

Therefore, the velocity of point P can be found by combining the relative velocities

$$\begin{aligned} {}^0\dot{\mathbf{r}}_P &= {}^0\dot{\mathbf{d}}_1 + \left({}^0\dot{R}_1\,{}^1\mathbf{d}_2 + {}^0R_1\,{}^1\dot{\mathbf{d}}_2\right) + {}^0\dot{R}_2\,{}^2\mathbf{r}_P \\ &= {}^0\dot{\mathbf{d}}_1 + {}^0_0\boldsymbol{\omega}_1 \times {}^0_1\mathbf{d}_2 + {}^0R_1\,{}^1\dot{\mathbf{d}}_2 + {}^0_0\boldsymbol{\omega}_2 \times {}^0_2\mathbf{r}_P & (7.195) \end{aligned}$$

Most of the time, it is better to use a relative velocity method and write

$$ {}^0_0\mathbf{v}_P = {}^0_0\mathbf{v}_1 + {}^0_1\mathbf{v}_2 + {}^0_2\mathbf{v}_P \tag{7.196}$$

because

$$\begin{aligned} {}^0_0\mathbf{v}_1 &= {}^0_0\dot{\mathbf{d}}_1 & (7.197) \\ {}^0_1\mathbf{v}_2 &= {}^0_0\boldsymbol{\omega}_1 \times {}^0_1\mathbf{d}_2 + {}^0R_1\,{}^1\dot{\mathbf{d}}_2 & (7.198) \\ {}^0_2\mathbf{v}_P &= {}^0_0\boldsymbol{\omega}_2 \times {}^0_2\mathbf{r}_P & (7.199) \end{aligned}$$

and therefore,

$$ {}^0\mathbf{v}_P = {}^0\dot{\mathbf{d}}_1 + {}^0_0\boldsymbol{\omega}_1 \times {}^0_1\mathbf{d}_2 + {}^0R_1\,{}^1\dot{\mathbf{d}}_2 + {}^0_0\boldsymbol{\omega}_2 \times {}^0_2\mathbf{r}_P. \tag{7.200}$$

Example 222 *Velocity vectors are free vectors.*

Velocity vectors are free, so to express them in different coordinate frames we need only to premultiply them by a rotation matrix. Hence, considering ${}_j^k\mathbf{v}_i$ *as the velocity of the origin of the* B_i *coordinate frame with respect to the origin of the frame* B_j *expressed in frame* B_k*, we can write*

$$ {}_j^k\mathbf{v}_i = -{}_i^k\mathbf{v}_j \tag{7.201}$$

and

$$ {}_j^k\mathbf{v}_i = {}^kR_m \, {}_j^m\mathbf{v}_i \tag{7.202}$$

and therefore,

$$ \frac{{}^id}{dt}\, {}_i^i\mathbf{r}_P = {}^i\mathbf{v}_P = {}_j^i\mathbf{v}_P + {}_i^i\boldsymbol{\omega}_j \times {}_j^i\mathbf{r}_P. \tag{7.203}$$

Example 223 ★ *Zero velocity points.*

To answer whether there is a point with zero velocity at each time, we may utilize Equation (7.187) and write

$$ {}_G\tilde{\omega}_B \left({}^G\mathbf{r}_0 - {}^G\mathbf{d}_B\right) + {}^G\dot{\mathbf{d}}_B = 0 \tag{7.204}$$

to search for ${}^G\mathbf{r}_0$ *which refers to a point with zero velocity*

$$ {}^G\mathbf{r}_0 = {}^G\mathbf{d}_B - {}_G\tilde{\omega}_B^{-1} \, {}^G\dot{\mathbf{d}}_B \tag{7.205}$$

however, the skew symmetric matrix ${}_G\tilde{\omega}_B$ *is singular and has no inverse. In other words, there is no general solution for Equation (7.204).*

If we restrict ourselves to planar motions, say XY*-plane, then* ${}_G\boldsymbol{\omega}_B = \omega\hat{K}$ *and* ${}_G\tilde{\omega}_B^{-1} = 1/\omega$*. Hence, in* $2D$ *space there is a point at any time with zero velocity at position* ${}^G\mathbf{r}_0$ *given by*

$$ {}^G\mathbf{r}_0(t) = {}^G\mathbf{d}_B(t) - \frac{1}{\omega} \, {}^G\dot{\mathbf{d}}_B(t). \tag{7.206}$$

The zero velocity point is called the ***pole*** *or* ***instantaneous center of rotation****. The position of the pole is generally a function of time and the path of its motion is called a* ***centroid****.*

Example 224 ★ *Eulerian and Lagrangian view points.*

When a variable quantity is measured within the stationary global coordinate frame, it is called absolute or the ***Lagrangian*** *viewpoint. On the other hand, when the variable is measured within a moving body coordinate frame, it is called relative or the* ***Eulerian*** *viewpoint.*

In $2D$ *planar motion of a rigid body, there is always a pole of zero velocity at*

$$ {}^G\mathbf{r}_0 = {}^G\mathbf{d}_B - \frac{1}{\omega} \, {}^G\dot{\mathbf{d}}_B. \tag{7.207}$$

The position of the pole in the body coordinate frame can be found by substituting for ${}^G\mathbf{r}$ *from (7.186)*

$$ {}^GR_B \, {}^B\mathbf{r}_0 + {}^G\mathbf{d}_B = {}^G\mathbf{d}_B - {}_G\tilde{\omega}_B^{-1} \, {}^G\dot{\mathbf{d}}_B \tag{7.208}$$

and solving for the position of the zero velocity point in the body coordinate frame ${}^{B}\mathbf{r}_0$.

$$\begin{aligned} {}^{B}\mathbf{r}_0 &= -{}^{G}R_B^T \; {}_G\tilde{\omega}_B^{-1} \; {}^{G}\dot{\mathbf{d}}_B = -{}^{G}R_B^T \; \left[{}^{G}\dot{R}_B \, {}^{G}R_B^T\right]^{-1} \; {}^{G}\dot{\mathbf{d}}_B \\ &= -{}^{G}R_B^T \; \left[{}^{G}R_B \; {}^{G}\dot{R}_B^{-1}\right] {}^{G}\dot{\mathbf{d}}_B = -{}^{G}\dot{R}_B^{-1} \, {}^{G}\dot{\mathbf{d}}_B \end{aligned} \tag{7.209}$$

Therefore, ${}^{G}\mathbf{r}_0$ *indicates the path of motion of the pole in the global frame, while* ${}^{B}\mathbf{r}_0$ *indicates the same path in the body frame. The* ${}^{G}\mathbf{r}_0$ *refers to Lagrangian centroid and* ${}^{B}\mathbf{r}_0$ *refers to Eulerian centroid.*

Example 225 ★ *Screw axis and screw motion.*

The screw axis may be defined as a line for a moving rigid body B *whose points* P *have velocity parallel to the angular velocity vector* ${}_G\boldsymbol{\omega}_B = \omega\hat{u}$. *Such points satisfy*

$${}^{G}\mathbf{v}_P = {}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) + {}^{G}\dot{\mathbf{d}}_B = p \, {}_G\boldsymbol{\omega}_B. \tag{7.210}$$

where, p *is a scalar. Since* ${}_G\boldsymbol{\omega}_B$ *is perpendicular to* ${}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r} - {}^{G}\mathbf{d}\right)$, *a dot product of Equation (7.210) by* ${}_G\boldsymbol{\omega}_B$ *yields*

$$p = \frac{1}{\omega^2}\left({}_G\boldsymbol{\omega}_B \cdot {}^{G}\dot{\mathbf{d}}_B\right). \tag{7.211}$$

Introducing a parameter k *to indicate different points of the line, the equation of the screw axis is defined by*

$${}^{G}\mathbf{r}_P = {}^{G}\mathbf{d}_B + \frac{1}{\omega^2}\left({}_G\boldsymbol{\omega}_B \times {}^{G}\dot{\mathbf{d}}_B\right) + k \, {}_G\boldsymbol{\omega}_B \tag{7.212}$$

because if we have $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, *and* $\mathbf{a} \cdot \mathbf{b} = 0$, *then* $\mathbf{x} = -\mathbf{a}^{-2}(\mathbf{a} \times \mathbf{b}) + k\mathbf{a}$. *In our case,*

$${}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right) = p \, {}_G\boldsymbol{\omega}_B - {}^{G}\dot{\mathbf{d}}_B \tag{7.213}$$

$\left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right)$ *is perpendicular to* ${}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r}_P - {}^{G}\mathbf{d}_B\right)$, *and hence is perpendicular to* $(p \, {}_G\boldsymbol{\omega}_B - {}^{G}\dot{\mathbf{d}}_B)$ *too.*

Therefore, there exists at any time a line s *in space, parallel to* ${}_G\boldsymbol{\omega}_B$, *which is the locus of points whose velocity is parallel to* ${}_G\boldsymbol{\omega}_B$.

If $\mathbf{s}$ *is the position vector of a point on* s, *then*

$${}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{s} - {}^{G}\mathbf{d}_B\right) = p \, {}_G\boldsymbol{\omega}_B - {}^{G}\dot{\mathbf{d}}_B \tag{7.214}$$

and the velocity of any point out of $\mathbf{s}$ *is*

$${}^{G}\mathbf{v} = {}_G\boldsymbol{\omega}_B \times \left({}^{G}\mathbf{r} - {}^{G}\mathbf{s}\right) + p \, {}_G\boldsymbol{\omega}_B \tag{7.215}$$

which expresses that at any time the velocity of a body point can be decomposed into perpendicular and parallel components to the angular velocity vector ${}_G\boldsymbol{\omega}_B$. *Therefore, the motion of any point of a rigid body is a screw. The parameter* p *is the ratio of translation velocity to rotation velocity, and is called* ***pitch***. *In general,* $\mathbf{s}$, ${}_G\boldsymbol{\omega}_B$, *and* p *may be functions of time.*

7.4 ★ Velocity Transformation Matrix

Consider the motion of a rigid body B in the global coordinate frame G, as shown in Figure 7.5. Assume the body frame $B(oxyz)$ is coincident at some initial time t_0 with the global frame $G(OXYZ)$. At any time $t \neq t_0$, B is not necessarily coincident with G and therefore, the homogeneous transformation matrix ${}^GT_B(t)$ is time varying.

The global position vector ${}^G\mathbf{r}_P(t)$ of a point P of the rigid body is a function of time, but its local position vector ${}^B\mathbf{r}_P$ is a constant, which is equal to ${}^G\mathbf{r}_P(t_0)$.

$$ {}^B\mathbf{r}_P \equiv {}^G\mathbf{r}_P(t_0) \tag{7.216} $$

The velocity of point P on the rigid body B as seen in the reference frame G is obtained by differentiating the position vector ${}^G\mathbf{r}(t)$ in the reference frame G

$$ {}^G\mathbf{v}_P = \frac{d}{dt}\,{}^G\mathbf{r}_P(t) = {}^G\dot{\mathbf{r}}_P \tag{7.217} $$

where ${}^G\dot{\mathbf{r}}_P$ denotes the differentiation of the quantity ${}^G\mathbf{r}_P(t)$ in the reference frame G.

The velocity of a body point in global coordinate frame can be found by applying a homogeneous transformation matrix

$$ {}^G\mathbf{v}(t) = {}^GV_B\,{}^G\mathbf{r}(t) \tag{7.218} $$

where ${}^GV_B$ is the *velocity transformation matrix.*

$$ \begin{aligned} {}^GV_B &= {}^G\dot{T}_B\,{}^GT_B^{-1} \\ &= \begin{bmatrix} {}^G\dot{R}_B\,{}^GR_B^T & {}^G\dot{\mathbf{d}}_B - {}^G\dot{R}_B\,{}^GR_B^T\,{}^G\mathbf{d}_B \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}_G\tilde{\omega}_B & {}^G\dot{\mathbf{d}}_B - {}_G\tilde{\omega}_B\,{}^G\mathbf{d}_B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} {}_G\tilde{\omega}_B & {}^G\mathbf{v}_B \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{7.219} $$

Proof. Based on homogeneous coordinate transformation, we have

$$ {}^G\mathbf{r}_P(t) = {}^GT_B(t)\,{}^B\mathbf{r}_P = {}^GT_B(t)\,{}^G\mathbf{r}_P(t_0) \tag{7.220} $$

and therefore,

$$ \begin{aligned} {}^G\mathbf{v}_P &= \frac{{}^Gd}{dt}\left[{}^GT_B\,{}^B\mathbf{r}_P\right] = {}^G\dot{T}_B\,{}^B\mathbf{r}_P = \begin{bmatrix} \frac{{}^Gd}{dt}\,{}^GR_B & \frac{{}^Gd}{dt}\,{}^G\mathbf{d}_B \\ 0 & 0 \end{bmatrix} {}^B\mathbf{r}_P \\ &= \begin{bmatrix} {}^G\dot{R}_B & {}^G\dot{\mathbf{d}}_B \\ 0 & 0 \end{bmatrix} {}^B\mathbf{r}_P \end{aligned} \tag{7.221} $$

Substituting for ${}^{B}\mathbf{r}_P$ from Equation (7.220), gives

$$\begin{aligned}
{}^{G}\mathbf{v}_P &= {}^{G}\dot{T}_B\, {}^{G}T_B^{-1}\, {}^{G}\mathbf{r}_P(t) \\
&= \begin{bmatrix} {}^{G}\dot{R}_B & {}^{G}\dot{\mathbf{d}}_B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} {}^{G}R_B^T & -{}^{G}R_B^T\, {}^{G}\mathbf{d}_B \\ 0 & 1 \end{bmatrix} {}^{G}\mathbf{r}_P(t) \\
&= \begin{bmatrix} {}^{G}\dot{R}_B\, {}^{G}R_B^T & {}^{G}\dot{\mathbf{d}}_B - {}^{G}\dot{R}_B\, {}^{G}R_B^T\, {}^{G}\mathbf{d}_B \\ 0 & 0 \end{bmatrix} {}^{G}\mathbf{r}_P(t) \\
&= \begin{bmatrix} {}_{G}\tilde{\omega}_B & {}^{G}\dot{\mathbf{d}}_B - {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{d}_B \\ 0 & 0 \end{bmatrix} {}^{G}\mathbf{r}_P(t).
\end{aligned} \tag{7.222}$$

Thus, the velocity of any point P of the rigid body B in the reference frame G can be obtained by premultiplying the position vector of the point P in G with the *velocity transformation matrix*, ${}^{G}V_B$,

$${}^{G}\mathbf{v}_P(t) = {}^{G}V_B\, {}^{G}\mathbf{r}_P(t) \tag{7.223}$$

where,

$$\begin{aligned}
{}^{G}V_B &= {}^{G}\dot{T}_B\, {}^{G}T_B^{-1} = \begin{bmatrix} {}_{G}\tilde{\omega}_B & {}^{G}\dot{\mathbf{d}}_B - {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{d}_B \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} {}_{G}\tilde{\omega}_B & {}^{G}\mathbf{v}_B \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{7.224}$$

and

$${}_{G}\tilde{\omega}_B = {}^{G}\dot{R}_B\, {}^{G}R_B^T \tag{7.225}$$

$$\begin{aligned}
{}^{G}\mathbf{v}_B &= {}^{G}\dot{\mathbf{d}}_B - {}^{G}\dot{R}_B\, {}^{G}R_B^T\, {}^{G}\mathbf{d}_B = {}^{G}\dot{\mathbf{d}}_B - {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{d}_B \\
&= {}^{G}\dot{\mathbf{d}}_B - {}_{G}\boldsymbol{\omega}_B \times {}^{G}\mathbf{d}_B.
\end{aligned} \tag{7.226}$$

The *velocity transformation matrix* ${}^{G}V_B$ may be assumed as a matrix operator that provides the global velocity of any point attached to $B(oxyz)$. It consists of the angular velocity matrix ${}_{G}\tilde{\omega}_B$ and the frame velocity ${}^{G}\dot{\mathbf{d}}_B$ both described in the global frame $G(OXYZ)$. The matrix ${}^{G}V_B$ depends on six parameters: the three components of the angular velocity vector ${}_{G}\boldsymbol{\omega}_B$ and the three components of the frame velocity ${}^{G}\dot{\mathbf{d}}_B$. Sometimes it is convenient to introduce a 6×1 vector called *velocity transformation vector* to simplify numerical calculations.

$${}_{G}\mathbf{t}_B = \begin{bmatrix} {}^{G}\mathbf{v}_B \\ {}_{G}\boldsymbol{\omega}_B \end{bmatrix} = \begin{bmatrix} {}^{G}\dot{\mathbf{d}}_B - {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{d}_B \\ {}_{G}\boldsymbol{\omega}_B \end{bmatrix} \tag{7.227}$$

In analogy to the two representations of the angular velocity, the velocity of body B in reference frame G can be represented either as the velocity transformation matrix ${}^{G}V_B$ in (7.224) or as the velocity transformation vector ${}_{G}\mathbf{t}_B$ in (7.227). The velocity transformation vector represents a noncommensurate vector since the dimension of ${}_{G}\boldsymbol{\omega}_B$ and ${}^{G}\mathbf{v}_B$ differ. ■

Example 226 ★ *Velocity transformation matrix based on coordinate transformation matrix.*

The velocity transformation matrix can be found based on a coordinate transformation matrix. Starting from

$$^{G}\mathbf{r}(t) = {}^{G}T_{B}\,{}^{B}\mathbf{r} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} {}^{B}\mathbf{r} \tag{7.228}$$

and taking the derivative, shows that

$$^{G}\mathbf{v} = \frac{^{G}d}{dt}\left[{}^{G}T_{B}\,{}^{B}\mathbf{r}\right] = {}^{G}\dot{T}_{B}\,{}^{B}\mathbf{r} = \begin{bmatrix} {}^{G}\dot{R}_{B} & {}^{G}\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} {}^{B}\mathbf{r} \tag{7.229}$$

however,

$$^{B}\mathbf{r} = {}^{G}T_{B}^{-1}\,{}^{G}\mathbf{r} \tag{7.230}$$

and therefore,

$$\begin{aligned} {}^{G}\mathbf{v} &= \begin{bmatrix} {}^{G}\dot{R}_{B} & {}^{G}\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} {}^{G}T_{B}^{-1}\,{}^{G}\mathbf{r} \\ &= \begin{bmatrix} {}^{G}\dot{R}_{B} & {}^{G}\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} {}^{G}R_{B}^{T} & -{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} {}^{G}\mathbf{r} \\ &= \begin{bmatrix} {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T} & {}^{G}\dot{\mathbf{d}} - {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d} \\ 0 & 0 \end{bmatrix} {}^{G}\mathbf{r} \\ &= {}^{G}V_{B}\,{}^{G}\mathbf{r}. \end{aligned} \tag{7.231}$$

Example 227 ★ *Inverse of a velocity transformation matrix.*

Transformation from a body frame to the global frame is given by Equation (4.67)

$$^{G}T_{B}^{-1} = \begin{bmatrix} {}^{G}R_{B}^{T} & -{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix}. \tag{7.232}$$

Following the same principle, we may introduce the inverse velocity transformation matrix by

$$\begin{aligned} {}^{B}V_{G} &= {}^{G}V_{B}^{-1} \\ &= \begin{bmatrix} \left({}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\right)^{-1} & -\left({}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\right)^{-1}\left({}^{G}\dot{\mathbf{d}} - {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d}\right) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{-1} & -{}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{-1}\left({}^{G}\dot{\mathbf{d}} - {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{T}\,{}^{G}\mathbf{d}\right) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{-1} & -{}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{-1}\,{}^{G}\dot{\mathbf{d}} + {}^{G}\mathbf{d} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{7.233}$$

to have

$$^{G}V_{B}\,{}^{G}V_{B}^{-1} = \mathbf{I}. \tag{7.234}$$

Therefore, having the velocity vector of a body point ${}^G\mathbf{v}_P$ *and the velocity transformation matrix* ${}^GV_B$ *we can find the global position of the point by*

$$ {}^G\mathbf{r}_P = {}^GV_B^{-1}\,{}^G\mathbf{v}_P. \tag{7.235}$$

Example 228 ★ *Velocity transformation matrix in body frame.*

The velocity transformation matrix ${}^GV_B$ *defined in the global frame* G *is described by*

$$ {}^GV_B = \begin{bmatrix} {}^G\dot{R}_B\,{}^GR_B^T & {}^G\dot{\mathbf{d}} - {}^G\dot{R}_B\,{}^GR_B^T\,{}^G\mathbf{d} \\ 0 & 0 \end{bmatrix} \tag{7.236}$$

However, the velocity transformation matrix can be expressed in the body coordinate frame B *as well*

$$\begin{aligned} {}^B_GV_B &= {}^GT_B^{-1}\,{}^G\dot{T}_B \qquad (7.237)\\ &= \begin{bmatrix} {}^GR_B^T & -{}^GR_B^T\,{}^G\mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^G\dot{R}_B & {}^G\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^GR_B^T\,{}^G\dot{R}_B & {}^GR_B^T\,{}^G\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} {}^B_G\boldsymbol{\omega}_B & {}^B\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where ${}^B_G\boldsymbol{\omega}_B$ *is the angular velocity vector of* B *with respect to* G *expressed in* B*, and* ${}^B\dot{\mathbf{d}}$ *is the velocity of the origin of* B *in* G *expressed in* B*.*

It is also possible to use a matrix multiplication to find the velocity transformation matrix in the body coordinate frame.

$$ {}^B_G\mathbf{v}_P = {}^GT_B^{-1}\,{}^G\mathbf{v}_P = {}^GT_B^{-1}\,{}^G\dot{T}_B\,{}^B\mathbf{r}_P = {}^B_GV_B\,{}^B\mathbf{r}_P \tag{7.238}$$

Using the definition of (7.219) and (7.237) we are able to transform the velocity transformation matrices between the B *and* G *frames.*

$$ {}^GV_B = {}^GT_B\,{}^B_GV_B\,{}^GT_B^{-1}. \tag{7.239}$$

It can also be useful if we define the time derivative of the transformation matrix by

$$ {}^G\dot{T}_B = {}^GV_B\,{}^GT_B \tag{7.240}$$

or

$$ {}^G\dot{T}_B = {}^GT_B\,{}^B_GV_B. \tag{7.241}$$

Similarly, we may define a velocity transformation matrix from link (i) to $(i-1)$ *by*

$$ {}^{i-1}V_i = \begin{bmatrix} {}^{i-1}\dot{R}_i\,{}^{i-1}R_i^T & {}^{i-1}\dot{\mathbf{d}} - {}^{i-1}\dot{R}_i\,{}^{i-1}R_i^T\,{}^{i-1}\mathbf{d} \\ 0 & 0 \end{bmatrix} \tag{7.242}$$

and

$$ {}^i_{i-1}V_i = \begin{bmatrix} {}^{i-1}R_i^T\,{}^{i-1}\dot{R}_i & {}^{i-1}R_i^T\,{}^{i-1}\dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix}. \tag{7.243}$$

Example 229 *Motion with a fixed point.*

When a point of a rigid body is fixed to the global frame, it is convenient to set the origins of the moving coordinate frame $B(Oxyz)$ and the global coordinate frame $G(OXYZ)$ on the fixed point. Under these conditions,

$$ {}^{G}\mathbf{d}_{B} = 0 \qquad , \qquad {}^{G}\dot{\mathbf{d}}_{B} = 0 \tag{7.244}$$

and Equation (7.222) reduces to

$$ {}^{G}\mathbf{v}_{P} = {}_{G}\tilde{\omega}_{B}\, {}^{G}\mathbf{r}_{P}(t) = {}_{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}_{P}(t). \tag{7.245}$$

Example 230 *Velocity in spherical coordinates.*

The homogeneous transformation matrix from the spherical coordinates $S(Or\theta\varphi)$ to Cartesian coordinates $G(OXYZ)$ is found as

$$\begin{aligned} {}^{G}T_{S} &= R_{Z,\varphi}\, R_{Y,\theta}\, D_{Z,r} = \begin{bmatrix} {}^{G}R_{B} & {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\varphi & -\sin\varphi & \cos\varphi\sin\theta & r\cos\varphi\sin\theta \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi & r\sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta & r\cos\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{7.246}$$

Time derivative of ${}^{G}T_{S}$ shows that

$$\begin{aligned} {}^{G}\dot{T}_{S} &= {}^{G}V_{S}\, {}^{G}T_{S} = \begin{bmatrix} {}_{G}\tilde{\omega}_{S} & {}^{G}\mathbf{v}_{S} \\ 0 & 0 \end{bmatrix} {}^{G}T_{S} \\ &= \begin{bmatrix} 0 & -\dot{\varphi} & \dot{\theta}\cos\varphi & \dot{r}\cos\varphi\sin\theta \\ \dot{\varphi} & 0 & \dot{\theta}\sin\varphi & \dot{r}\sin\theta\sin\varphi \\ -\dot{\theta}\cos\varphi & -\dot{\theta}\sin\varphi & 0 & \dot{r}\cos\theta \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^{G}T_{B}. \end{aligned} \tag{7.247}$$

Example 231 ★ *Velocity analysis of a planar $R\|R$ manipulator.*

Figure 7.8 illustrates an $R\|R$ planar manipulator with joint variables θ_1 and θ_2. The links (1) and (2) are both $R\|R(0)$ and therefore the transformation matrices ${}^{0}T_{1}$, ${}^{1}T_{2}$, and ${}^{0}T_{2}$ are:

$$ {}^{0}T_{1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.248}$$

$$ {}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.249}$$

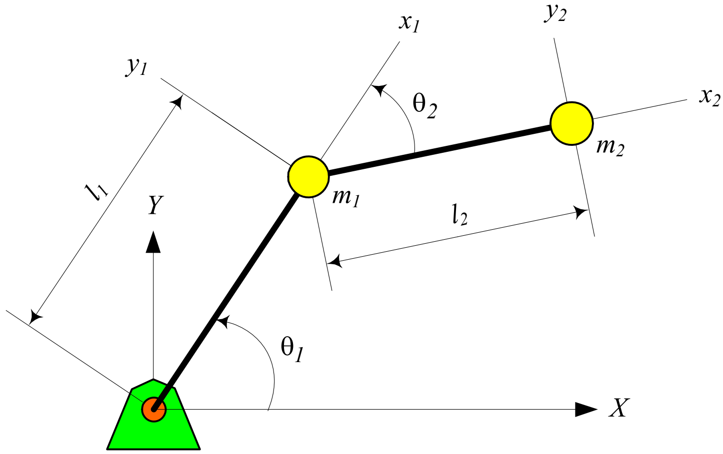

FIGURE 7.8. An R‖R planar manipulator.

$$\begin{aligned} {}^{0}T_{2} &= {}^{0}T_{1}\,{}^{1}T_{2} && (7.250)\\ &= \begin{bmatrix} c\left(\theta_1+\theta_2\right) & -s\left(\theta_1+\theta_2\right) & 0 & l_2 c\left(\theta_1+\theta_2\right)+l_1 c\theta_1 \\ s\left(\theta_1+\theta_2\right) & c\left(\theta_1+\theta_2\right) & 0 & l_2 s\left(\theta_1+\theta_2\right)+l_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The points M_1 and M_2 are at:

$${}^{0}\mathbf{r}_{M_1} = \begin{bmatrix} l_1\cos\theta_1 \\ l_1\sin\theta_1 \\ 0 \\ 1 \end{bmatrix} \qquad {}^{1}\mathbf{r}_{M_2} = \begin{bmatrix} l_2\cos\theta_2 \\ l_2\sin\theta_2 \\ 0 \\ 1 \end{bmatrix} \tag{7.251}$$

$${}^{0}\mathbf{r}_{M_2} = {}^{0}T_{1}\,{}^{1}\mathbf{r}_{M_2} = \begin{bmatrix} l_2\cos\left(\theta_1+\theta_2\right)+l_1\cos\theta_1 \\ l_2\sin\left(\theta_1+\theta_2\right)+l_1\sin\theta_1 \\ 0 \\ 1 \end{bmatrix} \tag{7.252}$$

To determine the velocity of M_2, we calculate ${}^{0}\dot{T}_{2}$. However, ${}^{0}\dot{T}_{2}$ can be calculated by direct differentiation of ${}^{0}T_{2}$.

$$\begin{aligned} {}^{0}\dot{T}_{2} &= \frac{d}{dt}\,{}^{0}T_{2} && (7.253)\\ &= \begin{bmatrix} -\dot{\theta}_{12}s\theta_{12} & -\dot{\theta}_{12}c\theta_{12} & 0 & -l_2\dot{\theta}_{12}s\theta_{12}-\dot{\theta}_1 l_1 s\theta_1 \\ \dot{\theta}_{12}c\theta_{12} & -\dot{\theta}_{12}s\theta_{12} & 0 & l_2\dot{\theta}_{12}c\theta_{12}+\dot{\theta}_1 l_1 c\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \theta_{12} &= \theta_1+\theta_2 \qquad \dot{\theta}_{12}=\dot{\theta}_1+\dot{\theta}_2 && (7.254) \end{aligned}$$

We may also use the chain rule to calculate ${}^0\dot{T}_2$

$$ {}^0\dot{T}_2 = \frac{d}{dt}\left({}^0T_1\,{}^1T_2\right) = {}^0\dot{T}_1\,{}^1T_2 + {}^0T_1\,{}^1\dot{T}_2 \tag{7.255}$$

where,

$$ {}^0\dot{T}_1 = \dot{\theta}_1 \begin{bmatrix} -\sin\theta_1 & -\cos\theta_1 & 0 & -l_1\sin\theta_1 \\ \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.256}$$

$$ {}^1\dot{T}_2 = \dot{\theta}_2 \begin{bmatrix} -\sin\theta_2 & -\cos\theta_2 & 0 & -l_2\sin\theta_2 \\ \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.257}$$

Having ${}^0\dot{T}_1$ *and* ${}^1\dot{T}_2$*, we can find the velocity transformation matrices* ${}^0V_1$ *and* ${}^1V_2$ *by using* ${}^0T_1^{-1}$ *and* ${}^1T_2^{-1}$*.*

$$ {}^0T_1^{-1} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & -l_1 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.258}$$

$$ {}^1T_2^{-1} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & -l_2 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.259}$$

$$ {}^0V_1 = {}^0\dot{T}_1\,{}^0T_1^{-1} = \dot{\theta}_1 \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.260}$$

$$ {}^1V_2 = {}^1\dot{T}_2\,{}^1T_2^{-1} = \dot{\theta}_2 \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.261}$$

Therefore, the velocity of points M_1 *and* M_2 *in* B_0 *and* B_1 *are:*

$$ {}^0\mathbf{v}_{M_1} = {}^0V_1\,{}^0\mathbf{r}_{M_1} = \dot{\theta}_1 \begin{bmatrix} -l_1\sin\theta_1 \\ l_1\cos\theta_1 \\ 0 \\ 0 \end{bmatrix} \tag{7.262}$$

$$ {}^1\mathbf{v}_{M_2} = {}^1V_2\,{}^1\mathbf{r}_{M_2} = \dot{\theta}_2 \begin{bmatrix} -l_2\sin\theta_2 \\ l_2\cos\theta_2 \\ 0 \\ 0 \end{bmatrix} \tag{7.263}$$

To determine the velocity of the tip point M_2 in the base frame, we can use the velocity vector addition.

$$\begin{aligned} {}^0\mathbf{v}_{M_2} &= {}^0\mathbf{v}_{M_1} + {}^0_1\mathbf{v}_{M_2} = {}^0\mathbf{v}_{M_1} + {}^0T_1\,{}^1\mathbf{v}_{M_2} \\ &= \begin{bmatrix} -\left(\dot\theta_1+\dot\theta_2\right) l_2 \sin(\theta_1+\theta_2) - \dot\theta_1 l_1 \sin\theta_1 \\ \left(\dot\theta_1+\dot\theta_2\right) l_2 \cos(\theta_1+\theta_2) + \dot\theta_1 l_1 \cos\theta_1 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{7.264}$$

We can also determine ${}^0\mathbf{v}_{M_2}$ by using the velocity transformation matrix ${}^0V_2$

$${}^0\mathbf{v}_{M_2} = {}^0V_2\,{}^0\mathbf{r}_{M_2} \tag{7.265}$$

where ${}^0V_2$ is:

$${}^0V_2 = {}^0\dot T_2\,{}^0T_2^{-1} = \begin{bmatrix} 0 & -\dot\theta_1-\dot\theta_2 & 0 & \dot\theta_2 l_1\sin\theta_1 \\ \dot\theta_1+\dot\theta_2 & 0 & 0 & -\dot\theta_2 l_1\cos\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.266}$$

$$\begin{aligned} {}^0T_2^{-1} &= {}^2T_1\,{}^1T_0 = {}^1T_2^{-1}\,{}^0T_1^{-1} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & -l_2-l_1\cos\theta_2 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_1\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7.267}$$

We can also determine the velocity transformation matrix ${}^0V_2$ using their addition rule ${}^0V_2 = {}^0V_1 + {}^0_1V_2$,

$$\begin{aligned} {}^0V_2 &= {}^0V_1 + {}^0_1V_2 \\ &= \begin{bmatrix} 0 & -\dot\theta_1-\dot\theta_2 & 0 & \dot\theta_2 l_1\sin\theta_1 \\ \dot\theta_1+\dot\theta_2 & 0 & 0 & -\dot\theta_2 l_1\cos\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7.268}$$

where,

$${}^0_1V_2 = {}^0T_1\,{}^1V_2\,{}^0T_1^{-1} = \begin{bmatrix} 0 & -\dot\theta_2 & 0 & \dot\theta_2 l_1\sin\theta_1 \\ \dot\theta_2 & 0 & 0 & -\dot\theta_2 l_1\cos\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.269}$$

Therefore, ${}^{0}\mathbf{v}_{M_2}$ *would be:*

$$\begin{aligned} {}^{0}\mathbf{v}_{M_2} &= {}^{0}V_2 \, {}^{0}\mathbf{r}_{M_2} \\ &= \begin{bmatrix} -\left(\dot{\theta}_1+\dot{\theta}_2\right) l_2 \sin\left(\theta_1+\theta_2\right) - \dot{\theta}_1 l_1 \sin\theta_1 \\ \left(\dot{\theta}_1+\dot{\theta}_2\right) l_2 \cos\left(\theta_1+\theta_2\right) + \dot{\theta}_1 l_1 \cos\theta_1 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{7.270}$$

7.5 Derivative of a Homogeneous Transformation Matrix

The velocity transformation matrix can be found directly from the homogeneous link transformation matrix. According to forward kinematics, there is a 4×4 homogeneous transformation matrix to move between every two coordinate frames.

$${}^{G}T_B = \begin{bmatrix} {}^{G}R_B & {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.271}$$

When the elements of the transformation matrix are time varying, its derivative is

$$\frac{{}^{G}dT}{dt} = {}^{G}\dot{T}_B = \begin{bmatrix} \frac{dr_{11}}{dt} & \frac{dr_{12}}{dt} & \frac{dr_{13}}{dt} & \frac{dr_{14}}{dt} \\ \frac{dr_{21}}{dt} & \frac{dr_{22}}{dt} & \frac{dr_{23}}{dt} & \frac{dr_{24}}{dt} \\ \frac{dr_{31}}{dt} & \frac{dr_{32}}{dt} & \frac{dr_{33}}{dt} & \frac{dr_{34}}{dt} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.272}$$

The time derivative of the transformation matrix can be arranged to be proportional to the transformation matrix

$${}^{G}\dot{T}_B = {}^{G}V_B \, {}^{G}T_B \tag{7.273}$$

where ${}^{G}V_B$ is a 4×4 homogeneous matrix called velocity transformation matrix or *velocity operator matrix* and is equal to

$$\begin{aligned} {}^{G}V_B &= {}^{G}\dot{T}_B \, {}^{G}T_B^{-1} \\ &= \begin{bmatrix} {}^{G}\dot{R}_B \, {}^{G}R_B^T & {}^{G}\dot{\mathbf{d}} - {}^{G}\dot{R}_B \, {}^{G}R_B^T \, {}^{G}\mathbf{d} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{7.274}$$

The homogeneous matrix and its derivative based on the velocity transformation matrix are useful in forward velocity kinematics. The ${}^{i-1}\dot{T}_i$ for two links connected by a revolute joint is

$$
{}^{i-1}\dot{T}_i = \dot{\theta}_i \begin{bmatrix} -\sin\theta_i & -\cos\theta_i\cos\alpha_i & \cos\theta_i\sin\alpha_i & -a_i\sin\theta_i \\ \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.275}
$$

and for two links connected by a prismatic joint is:

$$
{}^{i-1}\dot{T}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{d}_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.276}
$$

The associated velocity transformation matrix for a revolute joint is

$$
{}^{i-1}V_i = \dot{\theta}_i\ \Delta_R = \dot{\theta}_i \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.277}
$$

and for a prismatic joint is

$$
{}^{i-1}V_i = \dot{d}_i\ \Delta_P = \dot{d}_i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.278}
$$

Proof. Since any transformation matrix can be decomposed into a rotation and translation

$$
\begin{aligned}
[T] &= \begin{bmatrix} R_{\hat{u},\phi} & \mathbf{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\hat{u},\phi} & 0 \\ 0 & 1 \end{bmatrix} \\
&= [D]\,[R]
\end{aligned} \tag{7.279}
$$

we can find $\dot{T}$ as

$$
\begin{aligned}
\dot{T} &= \begin{bmatrix} \dot{R}_{\hat{u},\phi} & \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \dot{\mathbf{d}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R}_{\hat{u},\phi} & 0 \\ 0 & 1 \end{bmatrix} - \mathbf{I} \\
&= \left[\mathbf{I} + \dot{D}\right]\left[\mathbf{I} + \dot{R}\right] - \mathbf{I} = [V]\,[T]
\end{aligned} \tag{7.280}
$$

where $[V]$ is the velocity transformation matrix described as

$$
\begin{aligned}
[V] &= \dot{T}\,T^{-1} = \begin{bmatrix} \dot{R}_{\hat{u},\phi} & \dot{\mathbf{d}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{\hat{u},\phi} & -R^T_{\hat{u},\phi}\,\mathbf{d} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \dot{R}_{\hat{u},\phi}\,R^T_{\hat{u},\phi} & \dot{\mathbf{d}} - \dot{R}_{\hat{u},\phi}\,R^T_{\hat{u},\phi}\,\mathbf{d} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{\omega} & \dot{\mathbf{d}} - \tilde{\omega}\,\mathbf{d} \\ 0 & 1 \end{bmatrix}.
\end{aligned} \tag{7.281}
$$

The transformation matrix between two neighbor coordinate frames of a robot is described in Equation (5.11) based on the DH parameters,

$$ {}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.282} $$

Direct differentiating shows that in case two links are connected via a revolute joint, then θ_i is the only variable of DH matrix, and therefore,

$$ \begin{aligned} {}^{i-1}\dot{T}_i &= \dot{\theta}_i \begin{bmatrix} -\sin\theta_i & -\cos\theta_i\cos\alpha_i & \cos\theta_i\sin\alpha_i & -a_i\sin\theta_i \\ \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \dot{\theta}_i \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^{i-1}T_i = \dot{\theta}_i \; \Delta_R \; {}^{i-1}T_i. \end{aligned} \tag{7.283} $$

which shows that the *revolute velocity transformation matrix* is

$$ {}^{i-1}V_i = \dot{\theta}_i \; \Delta_R = \dot{\theta}_i \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.284} $$

However, if the two links are connected via a prismatic joint, d_i is the only variable of the DH matrix, and therefore,

$$ {}^{i-1}\dot{T}_i = \dot{d}_i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}^{i-1}T_i = \dot{d}_i \; \Delta_P \; {}^{i-1}T_i \tag{7.285} $$

which shows that the *prismatic velocity transformation matrix* is

$$ {}^{i-1}V_i = \dot{d}_i \; \Delta_P = \dot{d}_i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.286} $$

The Δ_R and Δ_P are revolute and prismatic *velocity coefficient matrices* with some application in velocity analysis of robots. ■

Example 232 *Differential of a transformation matrix.*

Assume a transformation matrix is given as

$$T = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.287}$$

subject to a differential rotation and differential translation given by

$$d\phi\hat{u} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \end{bmatrix} \tag{7.288}$$

$$d\mathbf{d} = \begin{bmatrix} 0.6 & 0.4 & 0.2 \end{bmatrix}. \tag{7.289}$$

Then, the differential transformation matrix dT is:

$$\begin{aligned} dT &= [\mathbf{I} + dD]\,[\mathbf{I} + dR] - \mathbf{I} \\ &= \begin{bmatrix} 0 & -0.3 & 0.2 & 0.6 \\ 0.3 & 0 & -0.1 & 0.4 \\ -0.2 & 0.1 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7.290}$$

Example 233 *Differential rotation and translation.*

Assume the angle of rotation about the axis $\hat{u}$ is too small and indicated by $d\phi$, then the differential rotation matrix is

$$\mathbf{I} + dR_{\hat{u},\phi} = \mathbf{I} + R_{\hat{u},d\phi} = \begin{bmatrix} 1 & -u_3 d\phi & u_2 d\phi & 0 \\ u_3 d\phi & 1 & -u_1 d\phi & 0 \\ -u_2 d\phi & +u_1 d\phi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.291}$$

because when $\phi << 1$, then,

$$\sin\phi \simeq d\phi \tag{7.292}$$

$$\cos\phi \simeq 1 \tag{7.293}$$

$$\operatorname{vers}\phi \simeq 0. \tag{7.294}$$

Differential translation $d\mathbf{d} = d(d_x\hat{I} + d_y\hat{J} + d_z\hat{K})$ is shown by a differential translation matrix

$$\mathbf{I} + dD = \begin{bmatrix} 1 & 0 & 0 & dd_x \\ 0 & 1 & 0 & dd_y \\ 0 & 0 & 1 & dd_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.295}$$

and therefore,

$$\begin{aligned} dT &= [\mathbf{I} + dD]\,[\mathbf{I} + dR] - \mathbf{I} \\ &= \begin{bmatrix} 0 & -d\phi u_3 & d\phi u_2 & dd_x \\ d\phi u_3 & 0 & -d\phi u_1 & dd_y \\ -d\phi u_2 & d\phi u_1 & 0 & dd_z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{7.296}$$

Example 234 *Combination of principal differential rotations.*
The differential rotation about X, Y, Z are

$$R_{X,d\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -d\gamma & 0 \\ 0 & d\gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.297}$$

$$R_{Y,d\beta} = \begin{bmatrix} 1 & 0 & d\beta & 0 \\ 0 & 1 & 0 & 0 \\ -d\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.298}$$

$$R_{Z,d\alpha} = \begin{bmatrix} 1 & -d\alpha & 0 & 0 \\ d\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.299}$$

therefore, the combination of the principal differential rotation matrices about axes X, Y, and Z is:

$$\begin{aligned} & [\mathbf{I} + R_{X,d\gamma}] [\mathbf{I} + R_{Y,d\beta}] [\mathbf{I} + R_{Z,d\alpha}] \\ = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -d\gamma & 0 \\ 0 & d\gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d\beta & 0 \\ 0 & 1 & 0 & 0 \\ -d\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -d\alpha & 0 & 0 \\ d\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 1 & -d\alpha & d\beta & 0 \\ d\alpha & 1 & -d\gamma & 0 \\ -d\beta & d\gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ = & [\mathbf{I} + R_{Z,d\alpha}] [\mathbf{I} + R_{Y,d\beta}] [\mathbf{I} + R_{X,d\gamma}] \end{aligned} \tag{7.300}$$

The combination of differential rotations is commutative.

Example 235 *Derivative of Rodriguez formula.*
Based on the Rodriguez formula, the angle-axis rotation matrix is

$$R_{\hat{u},\phi} = \mathbf{I} \cos\phi + \hat{u}\hat{u}^T \operatorname{vers}\phi + \tilde{u}\sin\phi \tag{7.301}$$

therefore, the time rate of the Rodriguez formula is

$$\dot{R}_{\hat{u},\phi} = -\dot{\phi}\sin\phi\,\mathbf{I} + \hat{u}\hat{u}^T\,\dot{\phi}\sin\phi + \tilde{u}\dot{\phi}\cos\phi = \dot{\phi}\tilde{u}R_{\hat{u},\phi}. \tag{7.302}$$

Example 236 ★ *Velocity of frame B_i in B_0.*
The velocity of the frame B_i attached to the link (i) with respect to the base coordinate frame B_0 can be found by differentiating ${}^0\mathbf{d}_i$ in the base

frame.

$$\begin{aligned}
{}^{0}\mathbf{v}_i &= \frac{{}^{0}d}{dt}\,{}^{0}\mathbf{d}_i = \frac{{}^{0}d}{dt}\left({}^{0}T_i\,{}^{i}\mathbf{d}_i\right) \\
&= {}^{0}\dot{T}_1\,{}^{1}T_2\cdots{}^{i-1}T_i\,{}^{i}\mathbf{d}_i + {}^{0}T_1\,{}^{1}\dot{T}_2\,{}^{2}T_3\cdots{}^{i-1}T_i\,{}^{i}\mathbf{d}_i \\
&\quad + {}^{0}T_1\cdots{}^{i-1}\dot{T}_i\,{}^{i}\mathbf{d}_i \\
&= \left[\sum_{j=1}^{i}\frac{\partial\,{}^{0}T_i}{\partial q_j}\dot{q}_j\right]{}^{i}\mathbf{d}_i
\end{aligned} \tag{7.303}$$

However, the partial derivatives $\partial\,{}^{i-1}T_i/\partial q_i$ can be found by utilizing the velocity coefficient matrices Δ_i, which is either Δ_R or Δ_P.

$$\frac{\partial\,{}^{i-1}T_i}{\partial q_i} = \Delta_i\;{}^{i-1}T_i. \tag{7.304}$$

Hence,

$$\frac{\partial\,{}^{0}T_i}{\partial q_j} = \begin{cases} {}^{0}T_1\,{}^{1}T_2\cdots{}^{j-2}T_{j-1}\;\Delta_j\;{}^{j-1}T_j\;\cdots{}^{i-1}T_i & \text{for } j\le i \\ 0 & \text{for } j>i. \end{cases} \tag{7.305}$$

Example 237 *V reduces to $\tilde{\omega}$, and T reduces to R if $\mathbf{d}=0$.*

Consider a B and G coordinate frames with a common origin. In this cares, $\mathbf{d}=0$ and (7.279) will be

$$\begin{aligned}
[T] &= \begin{bmatrix} R_{\hat{u},\phi} & \mathbf{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_{\hat{u},\phi} & 0 \\ 0 & 1 \end{bmatrix} \\
&= [\mathbf{I}]\,[R] = [R]
\end{aligned} \tag{7.306}$$

and, $\dot{T}$ is:

$$\dot{T} = \dot{R} \tag{7.307}$$

Therefore, is the velocity transformation matrix $[V]$ is equivalent to $\tilde{\omega}$.

$$[V] = \dot{T}\,T^{-1} = \dot{R}\,R^{T} = \tilde{\omega} \tag{7.308}$$

Example 238 *DH matrix between two co-origin coordinate frames.*

If two neighbor coordinate frames have the same origin, then a_i and d_i of DH transformation matrix (5.11) are zero. It simplifies the DH matrix to:

$${}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & 0 \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.309}$$

We can eliminate the last column and row of this matrix, and show it by a rotation transformation matrix .

$${}^{i-1}R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{7.310}$$

When a_i and d_i are zero, the two links are connected by a revolute joint. So, θ_i is the only variable of DH matrix, and therefore,

$$\begin{aligned} {}^{i-1}\dot{R}_i &= \dot{\theta}_i \begin{bmatrix} -\sin\theta_i & -\cos\theta_i\cos\alpha_i & \cos\theta_i\sin\alpha_i \\ \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i \\ 0 & 0 & 0 \end{bmatrix} = {}_{i-1}\boldsymbol{\omega}_i \, {}^{i-1}R_i \\ &= \dot{\theta}_i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^{i-1}R_i = \dot{\theta}_i \, {}^{i-1}\tilde{k}_{i-1} \, {}^{i-1}R_i. \end{aligned} \tag{7.311}$$

which shows that the revolute angular velocity matrix is:

$${}_{i-1}\boldsymbol{\omega}_i = \dot{\theta}_i \, {}^{i-1}\tilde{k}_{i-1} = \dot{\theta}_i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.312}$$

7.6 Summary

The transformation matrix ${}^G R_B$ is time dependent if a body coordinate frame B rotates continuously with respect to frame G with a common origin.

$$ {}^G\mathbf{r}(t) = {}^G R_B(t)\, {}^B\mathbf{r} \tag{7.313} $$

Then, the global velocity of a point in B is

$$ {}^G\dot{\mathbf{r}}(t) = {}^G\mathbf{v}(t) = {}^G\dot{R}_B(t)\, {}^B\mathbf{r} = {}_G\tilde{\omega}_B\, {}^G\mathbf{r}(t) \tag{7.314} $$

where ${}_G\tilde{\omega}_B$ is the skew symmetric angular velocity matrix

$$ {}_G\tilde{\omega}_B = {}^G\dot{R}_B\, {}^G R_B^T = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{7.315} $$

The matrix ${}_G\tilde{\omega}_B$ is associated with the angular velocity vector ${}_G\boldsymbol{\omega}_B = \dot{\phi}\,\hat{u}$, which is equal to an angular rate $\dot{\phi}$ about the instantaneous axis of rotation $\hat{u}$. Angular velocities of connected links of a robot may be added relatively to find the angular velocity of the link (n) in the base frame B_0.

$$ {}_0\boldsymbol{\omega}_n = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2 + {}_2^0\boldsymbol{\omega}_3 + \cdots + {}_{n-1}^{\;\;0}\boldsymbol{\omega}_n = \sum_{i=1}^{n} {}_{i-1}^{\;\;0}\boldsymbol{\omega}_i \tag{7.316} $$

To work with angular velocities of relatively moving links, we need to follow the rules of relative derivatives in body and global coordinate frames.

$$ \frac{{}^B d}{dt}\, {}^B\mathbf{r}_P = {}^B\dot{\mathbf{r}}_P = {}^B\mathbf{v}_P = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k} \tag{7.317} $$

$$ \frac{{}^G d}{dt}\, {}^G\mathbf{r}_P = {}^G\dot{\mathbf{r}}_P = {}^G\mathbf{v}_P = \dot{X}\,\hat{I} + \dot{Y}\,\hat{J} + \dot{Z}\,\hat{K} \tag{7.318} $$

$$ \frac{{}^G d}{dt}\, {}^B\mathbf{r}_P(t) = {}^B\dot{\mathbf{r}}_P + {}_G^B\boldsymbol{\omega}_B \times {}^B\mathbf{r}_P = {}_G^B\dot{\mathbf{r}}_P \tag{7.319} $$

$$ \frac{{}^B d}{dt}\, {}^G\mathbf{r}_P(t) = {}^G\dot{\mathbf{r}}_P - {}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}_P = {}_B^G\dot{\mathbf{r}}_P. \tag{7.320} $$

The global velocity of a point P in a moving frame B at

$$ {}^G\mathbf{r}_P = {}^G R_B\, {}^B\mathbf{r}_P + {}^G\mathbf{d}_B \tag{7.321} $$

is

$$ \begin{aligned} {}^G\mathbf{v}_P &= {}^G\dot{\mathbf{r}}_P = {}_G\tilde{\omega}_B \left({}^G\mathbf{r}_P - {}^G\mathbf{d}_B\right) + {}^G\dot{\mathbf{d}}_B \\ &= {}_G\boldsymbol{\omega}_B \times \left({}^G\mathbf{r}_P - {}^G\mathbf{d}_B\right) + {}^G\dot{\mathbf{d}}_B. \end{aligned} \tag{7.322} $$

The velocity relationship for a body B having a continues rigid motion in G may also be expressed by a homogeneous velocity transformation matrix ${}^{G}V_{B}$

$$ {}^{G}\mathbf{v}(t) = {}^{G}V_{B}\,{}^{G}\mathbf{r}(t) \tag{7.323} $$

where, ${}^{G}V_{B}$ includes both, the translational and rotational velocities of B in G.

$$ {}^{G}V_{B} = {}^{G}\dot{T}_{B}\,{}^{G}T_{B}^{-1} = \begin{bmatrix} {}_{G}\tilde{\omega}_{B} & {}^{G}\dot{\mathbf{d}}_{B} - {}_{G}\tilde{\omega}_{B}\,{}^{G}\mathbf{d}_{B} \\ 0 & 0 \end{bmatrix}. \tag{7.324} $$

7.7 Key Symbols

$\mathbf{a}$	turn vector of end-effector frame
B	body coordinate frame
c	cos
d	differential, prismatic joint variable
d_x, d_y, d_z	elements of $\mathbf{d}$
$\mathbf{d}$	translation vector, displacement vector
D	displacement transformation matrix
e	rotation quaternion
e_0, e_1, e_2, e_3	Euler parameters, components of e
G, B_0	global coordinate frame, Base coordinate frame
$\hat{\imath}, \hat{\jmath}, \hat{k}$	local coordinate axes unit vectors
$\tilde{\imath}, \tilde{\jmath}, \tilde{k}$	skew symmetric matrices of the unit vector $\hat{\imath}, \hat{\jmath}, \hat{k}$
$\hat{I}, \hat{J}, \hat{K}$	global coordinate axes unit vectors
$\mathbf{I} = [I]$	identity matrix
$\mathbf{J}$	Jacobian
l	length
p	pitch of a screw
q	joint coordinate,
$\mathbf{q}$	joints coordinate vector
$\mathbf{r}$	position vectors, homogeneous position vector
r_i	the element i of $\mathbf{r}$
r_{ij}	the element of row i and column j of a matrix
R	rotation transformation matrix
s	sin
$\mathbf{s}$	location vector of a screw
sgn	signum function
$SSRMS$	space station remote manipulator system
T	homogeneous transformation matrix
T_{arm}	manipulator transformation matrix
T_{wrist}	wrist transformation matrix
$\mathbf{T}$	a set of nonlinear algebraic equations of $\mathbf{q}$
$\mathbf{v}$	velocity vector
V	velocity transformation matrix
$\hat{u}$	unit vector along the axis of $\boldsymbol{\omega}$
$\tilde{u}$	skew symmetric matrix of the vector $\hat{u}$
u_1, u_2, u_3	components of $\hat{u}$
x, y, z	local coordinate axes
X, Y, Z	global coordinate axes
Greek	
α, β, γ	angles of rotation about the axes of global frame
δ	Kronecker function, small increment of a parameter
ϵ	small test number to terminate a procedure

θ	rotary joint angle
θ_{ijk}	$\theta_i + \theta_j + \theta_k$
φ, θ, ψ	angles of rotation about the axes of body frame
ϕ	angle of rotation about $\hat{u}$
$\boldsymbol{\omega}$	angular velocity vector
$\tilde{\omega}$	skew symmetric matrix of the vector $\boldsymbol{\omega}$
$\omega_1, \omega_2, \omega_3$	components of $\boldsymbol{\omega}$

Symbol	
$[\quad]^{-1}$	inverse of the matrix $[\quad]$
$[\quad]^T$	transpose of the matrix $[\quad]$
$\vdash$	orthogonal
(i)	link number i
$\parallel$	parallel
$\perp$	perpendicular
e^*	conjugate of e
Δ_P	prismatic velocity coefficient matrices
Δ_R	revolute velocity coefficient matrices

Exercises

1. Notation and symbols.

   Describe the meaning of

a- ${}_G\boldsymbol{\omega}_B$	b- ${}_B\boldsymbol{\omega}_G$	c- ${}^G_G\boldsymbol{\omega}_B$	d- ${}^B_G\boldsymbol{\omega}_B$	e- ${}^B_B\boldsymbol{\omega}_G$	f- ${}^G_B\boldsymbol{\omega}_G$
g- ${}^0_2\boldsymbol{\omega}_1$	h- ${}^2_2\boldsymbol{\omega}_1$	i- ${}^3_2\boldsymbol{\omega}_1$	j- ${}^G\dot{R}_B$	k- ${}^0_2\tilde{\omega}_1$	l- ${}^k_j\boldsymbol{\omega}_i$
m- ${}^G\mathbf{r}_P(t)$	n- ${}^G_2\mathbf{v}_P$	o- Δ_R	p- Δ_P	q- $\frac{{}^Gd}{dt}$	r- $\frac{{}^Bd}{dt}$
s- $\frac{{}^Gd}{dt}\,{}^G\mathbf{r}_P$	t- $\frac{{}^Gd}{dt}\,{}^B\mathbf{r}_P$	u- $\frac{{}^Bd}{dt}\,{}^B\mathbf{r}_P$	v- ${}^G\dot{\mathbf{r}}_P$	w- ${}^G\dot{\mathbf{d}}_P$	x- ${}^GV_B$

2. Local position, global velocity.

   A body is turning about a global principal axis at a constant angular. Find the global velocity of a point at ${}^B\mathbf{r}$.

$$
{}^B\mathbf{r} = \begin{bmatrix} 5 & 30 & 10 \end{bmatrix}^T
$$

   (a) The axis is Z-axis, the angular rate is $\dot{\alpha} = 2\,\mathrm{rad/s}$ when $\alpha = 30\deg$.

   (b) The axis is Y-axis, the angular rate is $\dot{\beta} = 2\,\mathrm{rad/s}$ when $\beta = 30\deg$.

   (c) The axis is X-axis, the angular rate is $\dot{\gamma} = 2\,\mathrm{rad/s}$ when $\gamma = 30\deg$.

3. Parametric angular velocity, global principal rotations.

   A body B is turning in a global frame G. The rotation transformation matrix can be decomposed into principal axes. Determine the angular velocity ${}_G\tilde{\omega}_B$ and ${}_G\boldsymbol{\omega}_B$.

   (a) ${}^GR_B$ is the result of a rotation α about Z-axis followed by β about Y-axis.

   (b) ${}^GR_B$ is the result of a rotation β about Y-axis followed by α about Z-axis.

   (c) ${}^GR_B$ is the result of a rotation α about Z-axis followed by γ about X-axis.

   (d) ${}^GR_B$ is the result of a rotation γ about X-axis followed by α about Z-axis.

   (e) ${}^GR_B$ is the result of a rotation γ about X-axis followed by β about Y-axis.

(f) ${}^{G}R_{B}$ is the result of a rotation β about Y-axis followed by γ about X-axis.

4. Numeric angular velocity, global principal rotations.

   A body B is turning in a global frame G. The rotation transformation matrix can be decomposed into principal axes. Determine the angular velocity ${}_{G}\tilde{\omega}_{B}$ and ${}_{G}\boldsymbol{\omega}_{B}$ for Exercises 3.$(a) - (f)$ using $\dot{\alpha} = 2\,\mathrm{rad/\,s}$, $\dot{\beta} = 2\,\mathrm{rad/\,s}$, $\dot{\gamma} = 2\,\mathrm{rad/\,s}$ and $\alpha = 30\,\mathrm{deg}$, $\beta = 30\,\mathrm{deg}$, $\gamma = 30\,\mathrm{deg}$.

5. Global position, constant angular velocity.

   A body is turning about the a global principal axis at a constant angular rate. Find the global position of a point at ${}^{B}\mathbf{r}$ after $t = 3\,\mathrm{sec}$ if the body and global coordinate frames were coincident at $t = 0\,\mathrm{sec}$.

$${}^{B}\mathbf{r} = \begin{bmatrix} 5 & 30 & 10 \end{bmatrix}^{T}$$

   (a) The axis is Z-axis, the angular rate is $\dot{\alpha} = 2\,\mathrm{rad/\,s}$.

   (b) The axis is Y-axis, the angular rate is $\dot{\beta} = 2\,\mathrm{rad/\,s}$.

   (c) The axis is X-axis, the angular rate is $\dot{\gamma} = 2\,\mathrm{rad/\,s}$.

6. Turning about x-axis.

   Find the angular velocity matrix when the body coordinate frame is turning about a body axis.

   (a) The axis is x-axis, the angular rate is $\dot{\varphi} = 2\,\mathrm{rad/\,s}$, and the angle is $\varphi = 45\,\mathrm{deg}$.

   (b) The axis is x-axis, the angular rate is $\dot{\theta} = 2\,\mathrm{rad/\,s}$, and the angle is $\theta = 45\,\mathrm{deg}$.

   (c) The axis is x-axis, the angular rate is $\dot{\psi} = 2\,\mathrm{rad/\,s}$, and the angle is $\psi = 45\,\mathrm{deg}$.

7. Combined rotation and angular velocity.

   Find the rotation matrix for a body frame that turns about the global axes at with given rates, and calculate the angular velocity of B in G.

   (a) The axes are Z, then X, and then Y. The angles are $30\,\mathrm{deg}$ about Z-axis, $30\,\mathrm{deg}$ about the X-axis, and $90\,\mathrm{deg}$ about the Y-axis. The angular rates are $\dot{\alpha} = 20\,\mathrm{deg}/\mathrm{sec}$, $\dot{\beta} = -40\,\mathrm{deg}/\mathrm{sec}$, and $\dot{\gamma} = 55\,\mathrm{deg}/\mathrm{sec}$ about the Z, X, and Y axes respectively.

   (b) The axes are X, then Y, and then Z. The angles are $30\,\mathrm{deg}$ about X-axis, $30\,\mathrm{deg}$ about the Y-axis, and $90\,\mathrm{deg}$ about the Z-axis. The angular rates are $\dot{\alpha} = 20\,\mathrm{deg}/\mathrm{sec}$, $\dot{\beta} = -40\,\mathrm{deg}/\mathrm{sec}$, and $\dot{\gamma} = 55\,\mathrm{deg}/\mathrm{sec}$ about the X, Y, and Z axes respectively.

(c) The axes are Y, then Z, and then X. The angles are 30 deg about X-axis, 30 deg about the Y-axis, and 90 deg about the Z-axis. The angular rates are $\dot{\alpha} = 20\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = -40\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 55\,\text{deg}\,/\,\text{sec}$ about the X, Y, and Z axes respectively.

8. ★ Global triple angular velocity matrix.

   Determine the angular velocity ${}_G\tilde{\omega}_B$ and ${}_G\boldsymbol{\omega}_B$ for the global triple rotations of Appendix A.

9. ★ Local triple angular velocity matrix.

   Determine the angular velocity ${}_G\tilde{\omega}_B$ and ${}_G\boldsymbol{\omega}_B$ for the local triple rotations of Appendix B.

10. Angular velocity, expressed in body frame.

    A point P is at $\mathbf{r}_P = (1, 2, 1)$ in a body coordinate $B(Oxyz)$.

    (a) Find ${}^B_G\tilde{\omega}_B$ when the body frame is turned 30 deg about the X-axis at a rate $\dot{\gamma} = 75\,\text{deg}\,/\,\text{sec}$, followed by 45 deg about the Z-axis at a rate $\dot{\alpha} = 25\,\text{deg}\,/\,\text{sec}$.

    (b) Find ${}^B_G\tilde{\omega}_B$ when the body frame is turned 45 deg about the Z-axis at a rate $\dot{\alpha} = 25\,\text{deg}\,/\,\text{sec}$, followed by 30 deg about the X-axis at a rate $\dot{\gamma} = 75\,\text{deg}\,/\,\text{sec}$.

11. Global roll-pitch-yaw angular velocity.

    Calculate the angular velocity ${}_G\tilde{\omega}_B$ for a global roll-pitch-yaw rotation of

    (a) $\alpha = 30\,\text{deg}$, $\beta = 30\,\text{deg}$, and $\gamma = 30\,\text{deg}$ with $\dot{\alpha} = 20\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = -20\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 20\,\text{deg}\,/\,\text{sec}$.

    (b) $\alpha = 30\,\text{deg}$, $\beta = 30\,\text{deg}$, and $\gamma = 30\,\text{deg}$ with $\dot{\alpha} = 0\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = -20\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 20\,\text{deg}\,/\,\text{sec}$.

    (c) $\alpha = 30\,\text{deg}$, $\beta = 30\,\text{deg}$, and $\gamma = 30\,\text{deg}$ with $\dot{\alpha} = 20\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = 0\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 20\,\text{deg}\,/\,\text{sec}$.

    (d) $\alpha = 30\,\text{deg}$, $\beta = 30\,\text{deg}$, and $\gamma = 30\,\text{deg}$ with $\dot{\alpha} = 20\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = -20\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 0\,\text{deg}\,/\,\text{sec}$.

    (e) $\alpha = 30\,\text{deg}$, $\beta = 30\,\text{deg}$, and $\gamma = 30\,\text{deg}$ with $\dot{\alpha} = 0\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = 0\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 20\,\text{deg}\,/\,\text{sec}$.

12. Roll-pitch-yaw angular velocity.

    Find ${}^B_G\tilde{\omega}_B$ and ${}_G\tilde{\omega}_B$ for the global role, pitch, and yaw rates equal to $\dot{\alpha} = 20\,\text{deg}\,/\,\text{sec}$, $\dot{\beta} = -20\,\text{deg}\,/\,\text{sec}$, and $\dot{\gamma} = 20\,\text{deg}\,/\,\text{sec}$ respectively, and having the following rotation matrix:

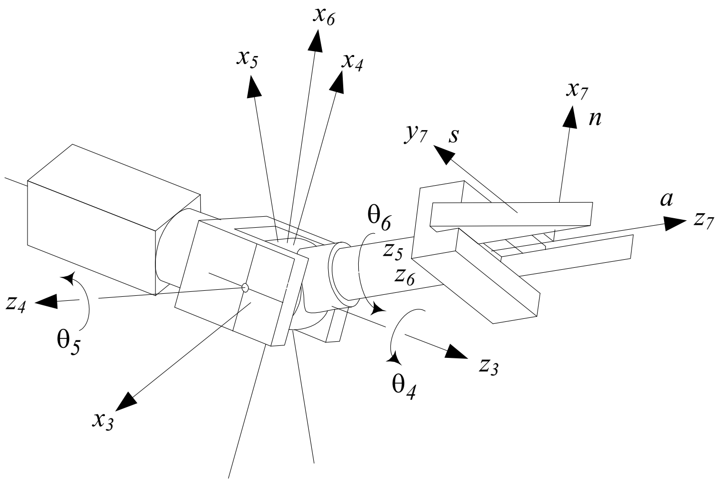

FIGURE 7.9. An Eulerian wrist.

(a)

$$
{}^{B}R_{G} = \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}
$$

(b)

$$
{}^{G}R_{B} = \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}
$$

13. Eulerian spherical wrist.

Figure 7.9 illustrates an Eulerian wrist in motion. Assume B_3 is a globally fixed frame at the wrist point. Determine the angular velocity ${}_3\tilde{\omega}_7$ of the end-effector frame B_7 for the following cases.

(a) Only the first motor is turning with $\dot{\theta}_4$ about z_3.

(b) Only the second motor is turning with $\dot{\theta}_5$ about z_4.

(c) Only the third motor is turning with $\dot{\theta}_6$ about z_5.

(d) The first motor is turning with $\dot{\theta}_4$ about z_3 and the second motor is turning with $\dot{\theta}_5$ about z_4.

(e) The first motor is turning with $\dot{\theta}_4$ about z_3 and the third motor is turning with $\dot{\theta}_6$ about z_5.

(f) The first motor is turning with $\dot{\theta}_4$ about z_3 and the second motor is turning with $\dot{\theta}_5$ about z_4.

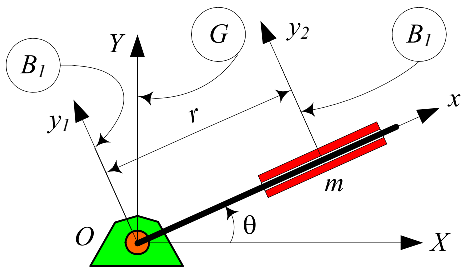

FIGURE 7.10. A slider on a rotating bar.

(g) The first, second, and third motors are turning with $\dot{\theta}_4$, $\dot{\theta}_5$, $\dot{\theta}_6$ about z_3, z_4, and z_5.

14. Angular velocity from Rodriguez formula.

We may find the time derivative of ${}^G R_B = R_{\hat{u},\phi}$ by

$$ {}^G\dot{R}_B = \frac{d}{dt}\,{}^G R_B = \dot{\phi}\frac{d}{d\phi}\,{}^G R_B. $$

Use the Rodriguez rotation formula and find ${}_G\tilde{\omega}_B$ and ${}_G^B\tilde{\omega}_B$.

15. Skew symmetric matrix

Show that any square matrix can be expressed as the sum of a symmetric and skew symmetric matrix.

$$ \begin{aligned} A &= B + C \\ B &= \frac{1}{2}\left(A + A^T\right) \\ C &= \frac{1}{2}\left(A - A^T\right) \end{aligned} $$

16. ★ A rotating slider.

Figure 7.10 illustrates a slider link on a rotating arm. Calculate

$$ \begin{array}{ccccc} \dfrac{{}^G d\hat{\imath}}{dt} & , & \dfrac{{}^G d\hat{\jmath}}{dt} & , & \dfrac{{}^G d\hat{k}}{dt} \\ \dfrac{{}^G d^2\hat{\imath}}{dt^2} & , & \dfrac{{}^G d^2\hat{\jmath}}{dt^2} & , & \dfrac{{}^G d^2\hat{k}}{dt^2} \end{array} $$

and find ${}^B\mathbf{v}$ and ${}^B\mathbf{a}$ of m at mass center C of the slider to find ${}_G^B\mathbf{a}_m = \frac{{}^G d}{dt}\,{}^B\mathbf{v}_m$ using the rule of mixed derivative.

$$ \frac{{}^G d}{dt}\left(\frac{{}^B d}{dt}\mathbf{r}\right) = \frac{{}^B d}{dt}\left(\frac{{}^B d}{dt}\mathbf{r}\right) + {}_G^B\boldsymbol{\omega}_B \times \left(\frac{{}^B d}{dt}\mathbf{r}\right) $$

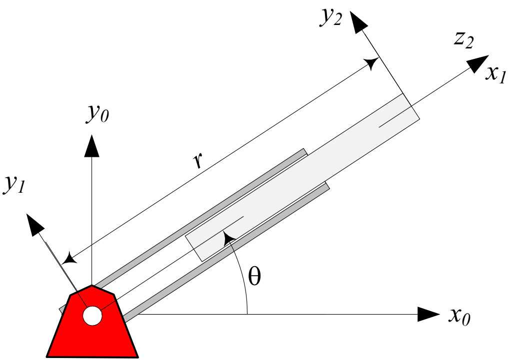

FIGURE 7.11. A planar polar manipulator.

17. ★ Differentiating in local and global frames.

    Consider a local point at ${}^B\mathbf{r}_P$. The local frame B is rotating in G by $\dot{\alpha}$ about the Z-axis. Calculate $\frac{{}^Bd}{dt}\,{}^B\mathbf{r}_P$, $\frac{{}^Gd}{dt}\,{}^G\mathbf{r}_P$, $\frac{{}^Bd}{dt}\,{}^G\mathbf{r}_P$, and $\frac{{}^Gd}{dt}\,{}^B\mathbf{r}_P$.

    (a) ${}^B\mathbf{r}_P = t\hat{\imath} + \hat{\jmath}$

    (b) ${}^B\mathbf{r}_P = t\hat{\imath} + t\hat{\jmath}$

    (c) ${}^B\mathbf{r}_P = t^2\hat{\imath} + \hat{\jmath}$

    (d) ${}^B\mathbf{r}_P = t\hat{\imath} + t^2\hat{\jmath}$

    (e) ${}^B\mathbf{r}_P = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$

    (f) ${}^B\mathbf{r}_P = t\hat{\imath} + t^2\hat{\jmath} + t\hat{k}$

    (g) ${}^B\mathbf{r}_P = \hat{\imath}\sin t$

    (h) ${}^B\mathbf{r}_P = \hat{\imath}\sin\hat{\imath} + \hat{\jmath}\cos t + \hat{k}$

18. ★ Velocity analysis of a polar manipulator.

    Figure 7.11 illustrates a planar polar manipulator with joint variables θ and d.

    Determine ${}^0T_1$, ${}^1T_2$, ${}^0T_2$, ${}^0V_1$, ${}^1V_2$, ${}^0V_2$, and velocity of the tip point of the manipulator.

19. ★ Skew symmetric identity for angular velocity.

    Show that
$$R\tilde{\omega}R^T = \widetilde{R\boldsymbol{\omega}}.$$

20. ★ Transformation of angular velocity exponents.

    Show that
$${}_G^B\tilde{\omega}_B^n = {}^GR_B^T\,{}_G\tilde{\omega}_B^n\,{}^GR_B.$$

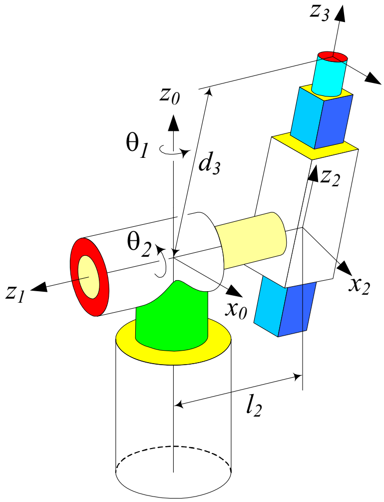

FIGURE 7.12.

21. ★ An angular velocity matrix identity.

    Show that

$$\tilde{\omega}^{2k+1} = (-1)^k \, \omega^{2k} \, \tilde{\omega}$$

    and

$$\tilde{\omega}^{2k} = (-1)^k \, \omega^{2(k-1)} \, \left(\omega^2 \, \mathbf{I} - \boldsymbol{\omega}\boldsymbol{\omega}^T\right).$$

22. ★ Velocity analysis of a spherical manipulator.

    Figure 7.12 illustrates a spherical manipulator with joint variables θ_1, θ_2, and d.

    Determine ${}^0V_1$, ${}^1V_2$, ${}^2V_3$, ${}^0V_2$, ${}^0V_3$, and velocity of the tip point of the manipulator.