

6

Inverse Kinematics

What are the joint variables for a given configuration of a robot? This is the inverse kinematic problem. The determination of the joint variables reduces to solving a set of nonlinear coupled algebraic equations. Although there is no standard and generally applicable method to solve the inverse kinematic problem, there are a few analytic and numerical methods to solve the problem. The main difficulty of inverse kinematic is the multiple solutions such as the one that is shown in Figure 6.1 for a planar 2R manipulator.

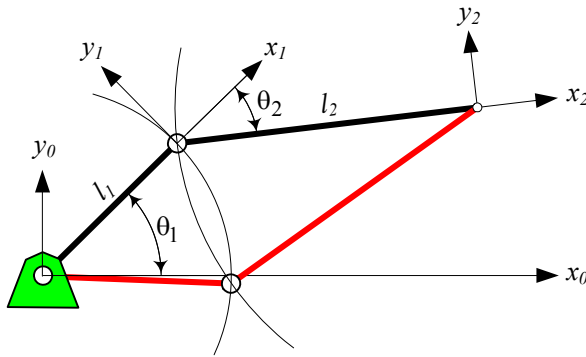


FIGURE 6.1. Multiple solution for inverse kinematic problem of a planar 2R manipulator.

6.1 Decoupling Technique

Determination of joint variables in terms of the end-effector position and orientation is called *inverse kinematics*. Mathematically, inverse kinematics is searching for the elements of vector \mathbf{q}

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3 \quad \cdots \quad q_n]^T \quad (6.1)$$

when a transformation 0T_n is given as a function of the joint variables $q_1, q_2, q_3, \dots, q_n$.

$${}^0T_n = {}^0T_1(q_1) {}^1T_2(q_2) {}^2T_3(q_3) {}^3T_4(q_4) \cdots {}^{n-1}T_n(q_n) \quad (6.2)$$

Computer controlled robots are usually actuated in the joint variable space, however objects to be manipulated are usually expressed in the global Cartesian coordinate frame. Therefore, carrying kinematic information, back and forth, between joint space and Cartesian space, is a need in robotics. To control the configuration of the end-effector to reach an object, the inverse kinematics problem must be solved. Hence, we need to know what the required values of joint variables are, to reach a desired point in a desired orientation.

The result of forward kinematics of a 6 *DOF* robot is a 4×4 transformation matrix

$$\begin{aligned} {}^0T_6 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (6.3)$$

where 12 elements are trigonometric functions of six unknown joint variables. However, because the upper left 3×3 submatrix of (6.3) is a rotation matrix, only three elements of them are independent. This is because of the orthogonality condition (2.197). Hence, only six equations out of the 12 equations of (6.3) are independent.

Trigonometric functions inherently provide multiple solutions. Therefore, multiple configurations of the robot are expected when the six equations are solved for the unknown joint variables.

It is possible to decouple the inverse kinematics problem into two sub-problems, known as *inverse position* and *inverse orientation* kinematics. The practical consequence of such a decoupling is the allowance to break the problem into two independent problems, each with only three unknown parameters. Following the decoupling principle, the overall transformation matrix of a robot can be decomposed to a translation and a rotation.

$$\begin{aligned} {}^0T_6 &= \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} \\ &= {}^0D_6 {}^0R_6 = \begin{bmatrix} \mathbf{I} & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^0R_6 & \mathbf{0} \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (6.4)$$

The translation matrix 0D_6 indicates the position of the end-effector in B_0 and involves only the three joint variables of the manipulator. We can solve ${}^0\mathbf{d}_6$ for the variables that control the wrist position. The rotation matrix 0R_6 indicates the orientation of the end-effector in B_0 and involves only the three joint variables of the wrist. We can solve 0R_6 for the variables that control the wrist orientation.

Proof. Most robots have a wrist made of three revolute joints with intersecting and orthogonal axes at the wrist point. Taking advantage of having

a spherical wrist, we can decouple the kinematics of the wrist and manipulator by decomposing the overall forward kinematics transformation matrix 0T_6 into the wrist orientation and wrist position

$${}^0T_6 = {}^0T_3 {}^3T_6 = \begin{bmatrix} {}^0R_3 & {}^0\mathbf{d}_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^3R_6 & \mathbf{0} \\ 0 & 1 \end{bmatrix} \quad (6.5)$$

where the wrist orientation matrix is:

$${}^3R_6 = {}^0R_3^T {}^0R_6 = {}^0R_3^T \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (6.6)$$

and the wrist position vector is:

$${}^0\mathbf{d}_6 = \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix} \quad (6.7)$$

The wrist position vector ${}^0\mathbf{d}_6 \equiv {}^0\mathbf{d}_3$ includes the manipulator joint variables only. Hence, to solve the inverse kinematics of such a robot, we must solve ${}^0\mathbf{d}_3$ for position of the wrist point, and then solve 3R_6 for orientation of the wrist.

The components of the wrist position vector ${}^0\mathbf{d}_6 = {}^0\mathbf{d}_{wrist}$ provides three equations for the three unknown manipulator joint variables. Solving ${}^0\mathbf{d}_6$, for manipulator joint variables, leads to calculating 3R_6 from (6.6). Then, the wrist orientation matrix 3R_6 can be solved for wrist joint variables.

In case we include the tool coordinate frame in forward kinematics, the decomposition must be done according to the following equation to exclude the effect of tool distance d_7 from the robot's kinematics.

$$\begin{aligned} {}^0T_7 &= {}^0T_3 {}^3T_7 = {}^0T_3 {}^3T_6 {}^6T_7 \\ &= \begin{bmatrix} {}^0R_3 & \mathbf{d}_w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^3R_6 & \mathbf{0} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ 0 & d_7 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (6.8)$$

In this case, inverse kinematics starts from determination of 0T_6 , which can be found by

$$\begin{aligned} {}^0T_6 &= {}^0T_7 {}^6T_7^{-1} \\ &= {}^0T_7 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = {}^0T_7 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (6.9)$$

■

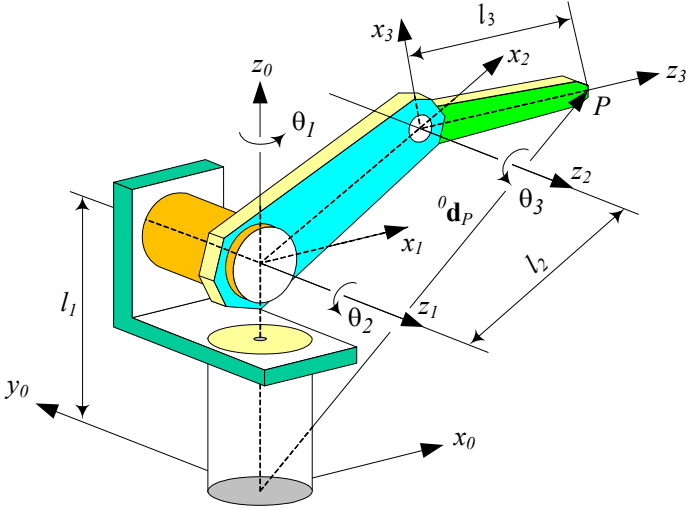


FIGURE 6.2. An R-R-R articulated manipulator.

Example 182 *An articulated manipulator.*

Consider an articulated manipulator as is shown in Figure 6.2. The links of the manipulator are R-R(90), R-R(0), R-R(90), and their associated transformation matrices between coordinate frames are:

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ \sin \theta_1 & 0 & -\cos \theta_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.10}$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.11}$$

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0 \\ \sin \theta_3 & 0 & -\cos \theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.12}$$

The forward kinematics of the manipulator is:

$$\begin{aligned} {}^0T_3 &= {}^0T_1 {}^1T_2 {}^2T_3 \tag{6.13} \\ &= \begin{bmatrix} c\theta_1 c(\theta_2 + \theta_3) & s\theta_1 & c\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2 + \theta_3) & -c\theta_1 & s\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_2 s\theta_1 \\ s(\theta_2 + \theta_3) & 0 & -c(\theta_2 + \theta_3) & l_1 + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and therefore, the tip point P is at:

$$\begin{aligned} {}^0\mathbf{d}_P &= \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = {}^0T_3 \begin{bmatrix} 0 \\ 0 \\ l_3 \end{bmatrix} \\ &= \begin{bmatrix} l_3 \sin(\theta_2 + \theta_3) \cos \theta_1 + l_2 \cos \theta_1 \cos \theta_2 \\ l_3 \sin(\theta_2 + \theta_3) \sin \theta_1 + l_2 \sin \theta_1 \cos \theta_2 \\ l_1 - l_3 \cos(\theta_2 + \theta_3) + l_2 \sin \theta_2 \end{bmatrix} \end{aligned} \quad (6.14)$$

Point P is supposed to be the point at which we attach a spherical wrist. Therefore, ${}^0\mathbf{d}_P$ is the decoupled position vector of the wrist point that will not be affected by the wrist attachment. ${}^0\mathbf{d}_P$ provides three equations for the three joint variables of the manipulator $\theta_1, \theta_2, \theta_3$. The first angle can be found from

$$d_x \sin \theta_1 - d_y \cos \theta_1 = 0 \quad (6.15)$$

that is:

$$\theta_1 = \text{atan2}(d_y, d_x) \quad (6.16)$$

We combine the first and second elements of ${}^0\mathbf{d}_P$ to find:

$$d_x \cos \theta_1 + d_y \sin \theta_1 = l_3 \sin(\theta_2 + \theta_3) + l_2 \cos \theta_2 \quad (6.17)$$

Now, combining this equation and the third element of ${}^0\mathbf{d}_P$ provides:

$$(d_z - l_1 - l_2 \sin \theta_2)^2 + (d_x \cos \theta_1 + d_y \sin \theta_1 - l_2 \cos \theta_2)^2 = l_3^2 \quad (6.18)$$

or

$$\begin{aligned} -2l_2(d_x \cos \theta_1 + d_y \sin \theta_1) \cos \theta_2 + 2l_2(l_1 - d_z) \sin \theta_2 &= \\ l_3^2 - \left((d_x \cos \theta_1 + d_y \sin \theta_1)^2 + l_1^2 - 2l_1 d_z + l_2^2 + d_z^2 \right) \end{aligned} \quad (6.19)$$

that is a trigonometric equation of the form (6.88).

$$a \cos \theta_2 + b \sin \theta_2 = c \quad (6.20)$$

$$\begin{aligned} a &= -2l_2(d_x \cos \theta_1 + d_y \sin \theta_1) \\ b &= 2l_2(l_1 - d_z) \\ c &= l_3^2 - \left((d_x \cos \theta_1 + d_y \sin \theta_1)^2 + l_1^2 - 2l_1 d_z + l_2^2 + d_z^2 \right) \end{aligned} \quad (6.21)$$

We solve this equation for θ_2 . Dividing (6.17) by the third element of ${}^0\mathbf{d}_P$ determines θ_3 .

$$\tan(\theta_2 + \theta_3) = \frac{d_x \cos \theta_1 + d_y \sin \theta_1 - l_2 \cos \theta_2}{l_1 + l_2 \sin \theta_2 - d_z} \quad (6.22)$$

$$\theta_3 = \text{atan2} \left(\frac{d_x \cos \theta_1 + d_y \sin \theta_1 - l_2 \cos \theta_2}{l_1 + l_2 \sin \theta_2 - d_z} \right) - \theta_2 \quad (6.23)$$

Example 183 *Numerical case of an articulated manipulator.*

To check the inverse kinematic equations of Example 182, let us examine an articulated manipulator with the following dimensions

$$\begin{aligned} l_1 &= 1 \text{ m} \\ l_2 &= 1.05 \text{ m} \\ l_3 &= 0.89 \text{ m} \end{aligned} \quad (6.24)$$

when its tip point is at:

$${}^0\mathbf{d}_P = [1 \quad 1.1 \quad 1.2]^T \quad (6.25)$$

Equation (6.16) provides θ_1 .

$$\begin{aligned} \theta_1 &= \text{atan2}(d_y, d_x) = \tan^{-1} \frac{1.1}{1} \\ &= 0.83298 \text{ rad} \approx 47.727 \text{ deg} \end{aligned} \quad (6.26)$$

To determine θ_2 , we should solve Equation (6.20)

$$a \cos \theta_2 + b \sin \theta_2 = c \quad (6.27)$$

where,

$$a = -2l_2(d_x \cos \theta_1 + d_y \sin \theta_1) = -3.941263019 \quad (6.28)$$

$$b = 2l_2(l_1 - d_z) = -0.5302360813 \quad (6.29)$$

$$\begin{aligned} c &= l_3^2 - \left((d_x \cos \theta_1 + d_y \sin \theta_1)^2 + l_1^2 - 2l_1 d_z + l_2^2 + d_z^2 \right) \\ &= -3.232420149, \quad -5.232420149. \end{aligned} \quad (6.30)$$

We find two values for θ_2 for $c = -3.232$

$$\theta_2 = 0.7555416816 \text{ rad} \approx 43.28934959 \text{ deg} \quad (6.31)$$

$$\theta_2 = -0.4880785028 \text{ rad} \approx -27.96483827 \text{ deg} \quad (6.32)$$

and we get no real answer for $c = -5.232$. θ_3 comes from (6.23). If $\theta_2 = 0.755 \text{ rad}$ then we have

$$\begin{aligned} \theta_3 &= \text{atan2} \left(\frac{d_x \cos \theta_1 + d_y \sin \theta_1 - l_2 \cos \theta_2}{l_1 + l_2 \sin \theta_2 - d_z} \right) - \theta_2 \\ &= .1913201914 \text{ rad} \approx 11 \text{ deg} \end{aligned} \quad (6.33)$$

and if $\theta_2 = -0.488 \text{ rad}$ then we have:

$$\theta_3 = -.1913201910 \text{ rad} \approx -11 \text{ deg} \quad (6.34)$$

Example 184 *Inverse kinematics for a 2R planar manipulator.*

Figure 5.9 illustrates a 2R planar manipulator with two R||R links according to the coordinate frames setup shown in the figure. The forward kinematics of the manipulator was found to be

$$\begin{aligned} {}^0T_2 &= {}^0T_1 {}^1T_2 & (6.35) \\ &= \begin{bmatrix} c(\theta_1 + \theta_2) & -s(\theta_1 + \theta_2) & 0 & l_1c\theta_1 + l_2c(\theta_1 + \theta_2) \\ s(\theta_1 + \theta_2) & c(\theta_1 + \theta_2) & 0 & l_1s\theta_1 + l_2s(\theta_1 + \theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The inverse kinematics of planar robots are generally easier to find analytically. The global position of the tip point of the manipulator is at

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} \quad (6.36)$$

therefore

$$X^2 + Y^2 = l_1^2 + l_2^2 + 2l_1l_2 \cos \theta_2 \quad (6.37)$$

and

$$\cos \theta_2 = \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1l_2} \quad (6.38)$$

$$\theta_2 = \cos^{-1} \frac{X^2 + Y^2 - l_1^2 - l_2^2}{2l_1l_2}. \quad (6.39)$$

However, we usually avoid using arcsin and arccos because of the inaccuracy. So, we employ the half angle formula

$$\tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} \quad (6.40)$$

to find θ_2 using an atan2 function

$$\theta_2 = \pm 2 \operatorname{atan2} \sqrt{\frac{(l_1 + l_2)^2 - (X^2 + Y^2)}{(X^2 + Y^2) - (l_1 - l_2)^2}}. \quad (6.41)$$

The \pm is because of the square root, which generates two solutions. These two solutions are called **elbow up** and **elbow down**, as shown in Figure 6.3(a) and (b) respectively.

The first joint variable θ_1 of an elbow up configuration can geometrically be found from

$$\theta_1 = \operatorname{atan2} \frac{Y}{X} + \operatorname{atan2} \frac{l_2 \sin \theta_2}{l_1 + l_2 \cos \theta_2} \quad (6.42)$$

and for an elbow down configuration from

$$\theta_1 = \operatorname{atan2} \frac{Y}{X} - \operatorname{atan2} \frac{l_2 \sin \theta_2}{l_1 + l_2 \cos \theta_2}. \quad (6.43)$$

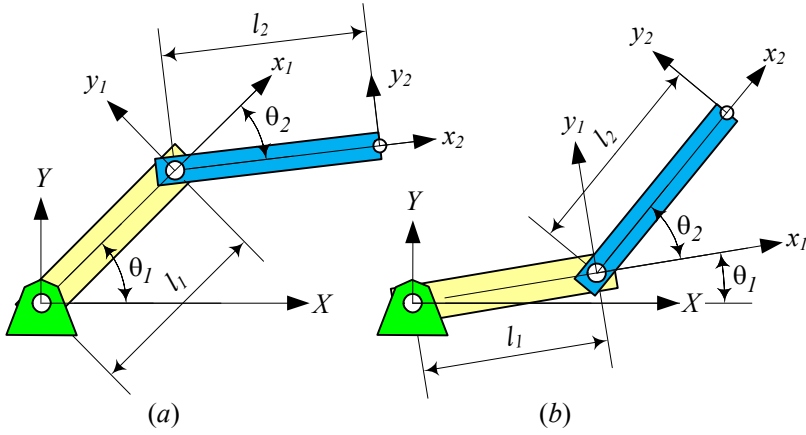


FIGURE 6.3. Illustration of a 2R planar manipulator in two possible configurations: (a) elbow up and (b) elbow down.

θ_1 can also be found from the following alternative equation.

$$\theta_1 = \text{atan2} \frac{-Xl_2 \sin \theta_2 + Y(l_1 + l_2 \cos \theta_2)}{Yl_2 \sin \theta_2 + X(l_1 + l_2 \cos \theta_2)} \tag{6.44}$$

Most of the time, the value of θ_1 should be corrected by adding or subtracting π depending on the sign of X . It is also possible to combine Equations of (6.36) and determine a trigonometric equation for θ_1 .

$$2Xl_1 \cos \theta_1 + 2Yl_1 \sin \theta_1 = X^2 + Y^2 + l_1^2 - l_2^2 \tag{6.45}$$

It is also convenient to use the following equation.

$$l_1 + l_2 \cos \theta_2 = \frac{X^2 + Y^2 + l_1^2 - l_2^2}{2l_1} \tag{6.46}$$

The two different sets of solutions for θ_1 and θ_2 correspond to the elbow up and elbow down configurations.

Example 185 Motion of a 2R manipulator.

Consider a 2R planar manipulator with

$$l_1 = 1 \text{ m} \quad l_2 = 1 \text{ m} \tag{6.47}$$

that its tip point is moving from $P_1(1.2, 1.5)$ to $P_2(-1.2, 1.5)$ on a straight line. The using the inverse kinematic equations (6.39) and (6.42), we can determine the configuration of the manipulator at any point of the path. Figure 6.4 illustrates the manipulator at 42 equally spaced points between P_1 and P_2 . Let us assume that the tip point is moving of the line based on

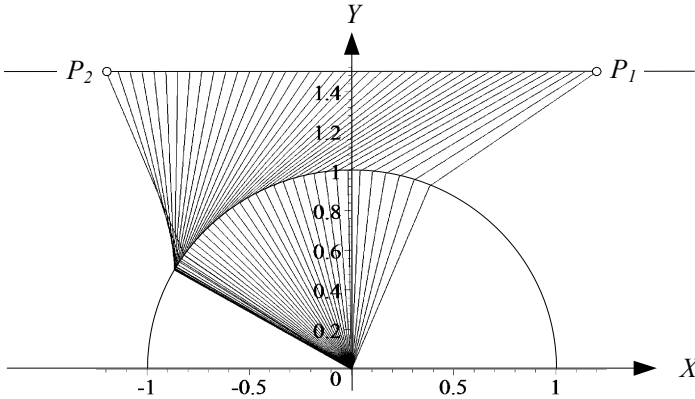


FIGURE 6.4. A 2R planar manipulator with $l_1 = 1$ m, $l_2 = 1$ m moving from P_1 to P_2 on a straight line.

the following time behavior.

$$X = 1.2 - t \quad Y = 1.5 \quad 0 \leq t \leq 2.4 \quad (6.48)$$

The variation of the angles θ_1 and θ_2 are as shown in Figure 6.5.

Example 186 Inverse kinematics of an articulated robot.

The forward kinematics of the articulated robot, illustrated in Figure 6.6, was found in Example 166, where the overall transformation matrix of the end-effector was found, based on the wrist and arm transformation matrices.

$${}^0T_7 = T_{arm}T_{wrist} = {}^0T_3 {}^3T_7$$

The wrist transformation matrix T_{wrist} is described in (5.124) and the manipulator transformation matrix, T_{arm} is found in (5.74). However, according to a new setup coordinate frame, as shown in Figure 6.6, we have a 6R robot with a six links configuration

1	R-R(90)
2	R R(0)
3	R-R(90)
4	R-R(-90)
5	R-R(90)
6	R R(0)

and a displacement T_{Z,d_7} . Therefore, the individual links' transformation

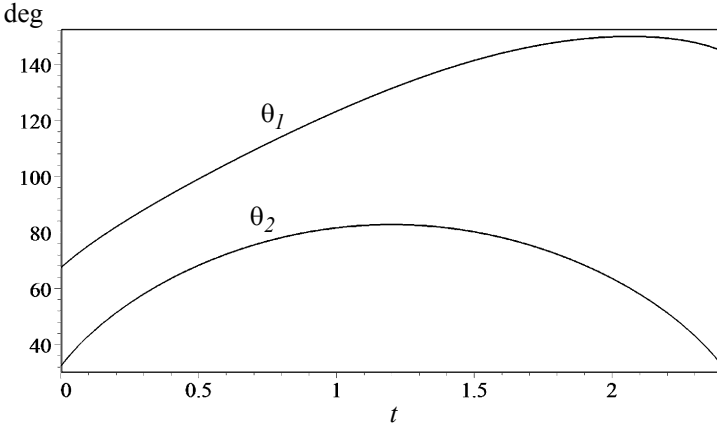


FIGURE 6.5. The variation of the angles θ_1 and θ_2 of the 2R planar manipulator of Figure 6.4.

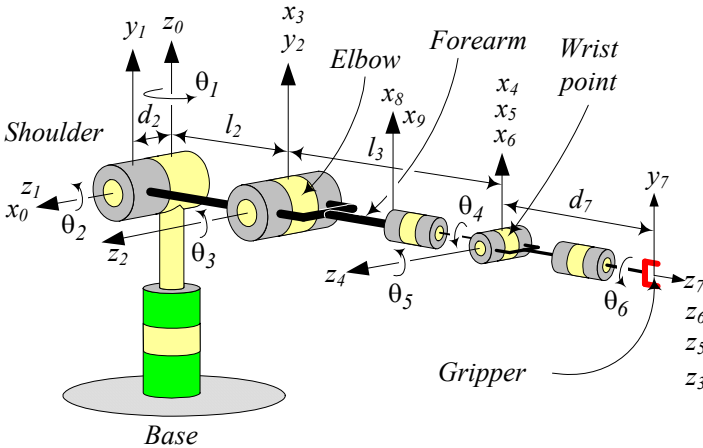


FIGURE 6.6. A 6 DOF articulated manipulator.

matrices are

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ \sin \theta_1 & 0 & -\cos \theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.49)$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.50)$$

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0 \\ \sin \theta_3 & 0 & -\cos \theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.51)$$

$${}^3T_4 = \begin{bmatrix} \cos \theta_4 & 0 & -\sin \theta_4 & 0 \\ \sin \theta_4 & 0 & \cos \theta_4 & 0 \\ 0 & -1 & 0 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.52)$$

$${}^4T_5 = \begin{bmatrix} \cos \theta_5 & 0 & \sin \theta_5 & 0 \\ \sin \theta_5 & 0 & -\cos \theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.53)$$

$${}^5T_6 = \begin{bmatrix} \cos \theta_6 & -\sin \theta_6 & 0 & 0 \\ \sin \theta_6 & \cos \theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.54)$$

$${}^6T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.55)$$

and the tool transformation matrix in the base coordinate frame is

$$\begin{aligned} {}^0T_7 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 {}^6T_7 \\ &= {}^0T_3 {}^3T_6 {}^6T_7 \\ &= \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (6.56)$$

where

$${}^0T_3 = \begin{bmatrix} c\theta_1 c(\theta_2 + \theta_3) & s\theta_1 & c\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_1 c\theta_2 + d_2 s\theta_1 \\ s\theta_1 c(\theta_2 + \theta_3) & -c\theta_1 & s\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_2 s\theta_1 - d_2 c\theta_1 \\ s(\theta_2 + \theta_3) & 0 & -c(\theta_2 + \theta_3) & l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.57)$$

$${}^3T_6 = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & 0 \\ c\theta_5 c\theta_6 s\theta_4 + c\theta_4 s\theta_6 & c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & 0 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.58)$$

and

$$t_{11} = c\theta_1 (c(\theta_2 + \theta_3) (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6) - c\theta_6 s\theta_5 s(\theta_2 + \theta_3)) + s\theta_1 (c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4) \quad (6.59)$$

$$t_{21} = s\theta_1 (c(\theta_2 + \theta_3) (-s\theta_4 s\theta_6 + c\theta_4 c\theta_5 c\theta_6) - c\theta_6 s\theta_5 s(\theta_2 + \theta_3)) - c\theta_1 (c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4) \quad (6.60)$$

$$t_{31} = s(\theta_2 + \theta_3) (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6) + c\theta_6 s\theta_5 c(\theta_2 + \theta_3) \quad (6.61)$$

$$t_{12} = c\theta_1 (s\theta_5 s\theta_6 s(\theta_2 + \theta_3) - c(\theta_2 + \theta_3) (c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6)) + s\theta_1 (c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6) \quad (6.62)$$

$$t_{22} = s\theta_1 (s\theta_5 s\theta_6 s(\theta_2 + \theta_3) - c(\theta_2 + \theta_3) (c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6)) + c\theta_1 (-c\theta_4 c\theta_6 + c\theta_5 s\theta_4 s\theta_6) \quad (6.63)$$

$$t_{32} = -s\theta_5 s\theta_6 c(\theta_2 + \theta_3) - s(\theta_2 + \theta_3) (c\theta_6 s\theta_4 + c\theta_4 c\theta_5 s\theta_6) \quad (6.64)$$

$$t_{13} = s\theta_1 s\theta_4 s\theta_5 + c\theta_1 (c\theta_5 s(\theta_2 + \theta_3) + c\theta_4 s\theta_5 c(\theta_2 + \theta_3)) \quad (6.65)$$

$$t_{23} = -c\theta_1 s\theta_4 s\theta_5 + s\theta_1 (c\theta_5 s(\theta_2 + \theta_3) + c\theta_4 s\theta_5 c(\theta_2 + \theta_3)) \quad (6.66)$$

$$t_{33} = c\theta_4 s\theta_5 s(\theta_2 + \theta_3) - c\theta_5 c(\theta_2 + \theta_3) \quad (6.67)$$

$$t_{14} = d_6 (s\theta_1 s\theta_4 s\theta_5 + c\theta_1 (c\theta_4 s\theta_5 c(\theta_2 + \theta_3) + c\theta_5 s(\theta_2 + \theta_3))) + l_3 c\theta_1 s(\theta_2 + \theta_3) + d_2 s\theta_1 + l_2 c\theta_1 c\theta_2 \quad (6.68)$$

$$t_{24} = d_6 (-c\theta_1 s\theta_4 s\theta_5 + s\theta_1 (c\theta_4 s\theta_5 c(\theta_2 + \theta_3) + c\theta_5 s(\theta_2 + \theta_3))) + s\theta_1 s(\theta_2 + \theta_3) l_3 - d_2 c\theta_1 + l_2 c\theta_2 s\theta_1 \quad (6.69)$$

$$t_{34} = d_6 (c\theta_4 s\theta_5 s(\theta_2 + \theta_3) - c\theta_5 c(\theta_2 + \theta_3)) + l_2 s\theta_2 + l_3 c(\theta_2 + \theta_3). \quad (6.70)$$

Solution of the inverse kinematics problem starts with the wrist position vector \mathbf{d} , which is $[t_{14} \ t_{24} \ t_{34}]^T$ of 0T_7 for $d_7 = 0$

$$\mathbf{d} = \begin{bmatrix} c\theta_1 (l_3 s(\theta_2 + \theta_3) + l_2 c\theta_2) + d_2 s\theta_1 \\ s\theta_1 (l_3 s(\theta_2 + \theta_3) + l_2 c\theta_2) - d_2 c\theta_1 \\ l_3 c(\theta_2 + \theta_3) + l_2 s\theta_2 \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}. \quad (6.71)$$

Theoretically, we must be able to solve Equation (6.71) for the three joint variables θ_1 , θ_2 , and θ_3 . It can be seen that

$$d_x \sin \theta_1 - d_y \cos \theta_1 = d_2 \quad (6.72)$$

which provides

$$\theta_1 = 2 \operatorname{atan2}(d_x \pm \sqrt{d_x^2 + d_y^2 - d_2^2}, d_2 - d_y). \quad (6.73)$$

Equation (6.73) has two solutions for $d_x^2 + d_y^2 > d_2^2$, one solution for $d_x^2 + d_y^2 = d_2^2$, and no real solution for $d_x^2 + d_y^2 < d_2^2$.

Combining the first two elements of \mathbf{d} gives

$$l_3 \sin(\theta_2 + \theta_3) = \pm \sqrt{d_x^2 + d_y^2 - d_2^2} - l_2 \cos \theta_2 \quad (6.74)$$

then, the third element of \mathbf{d} may be utilized to find

$$l_3^2 = \left(\pm \sqrt{d_x^2 + d_y^2 - d_2^2} - l_2 \cos \theta_2 \right)^2 + (d_z - l_2 \sin \theta_2)^2 \quad (6.75)$$

which can be rearranged to the following form

$$a \cos \theta_2 + b \sin \theta_2 = c \quad (6.76)$$

$$a = 2l_2 \sqrt{d_x^2 + d_y^2 - d_2^2} \quad (6.77)$$

$$b = 2l_2 d_z \quad (6.78)$$

$$c = d_x^2 + d_y^2 + d_z^2 - d_2^2 + l_2^2 - l_3^2. \quad (6.79)$$

with two solutions

$$\theta_2 = \operatorname{atan2}\left(\frac{c}{r}, \pm \sqrt{1 - \frac{c^2}{r^2}}\right) - \operatorname{atan2}(a, b) \quad (6.80)$$

$$r^2 = a^2 + b^2. \quad (6.81)$$

Summing the squares of the elements of \mathbf{d} gives

$$d_x^2 + d_y^2 + d_z^2 = d_2^2 + l_2^2 + l_3^2 + 2l_2 l_3 \sin(2\theta_2 + \theta_3) \quad (6.82)$$

that provides

$$\theta_3 = \arcsin\left(\frac{d_x^2 + d_y^2 + d_z^2 - d_2^2 - l_2^2 - l_3^2}{2l_2 l_3}\right) - 2\theta_2. \quad (6.83)$$

Having θ_1 , θ_2 , and θ_3 means we can find the wrist point in space. However, because the joint variables in 0T_3 and in 3T_6 are independent, we

should find the orientation of the end-effector by solving 3T_6 or 3R_6 for θ_4 , θ_5 , and θ_6 .

$$\begin{aligned}
 {}^3R_6 &= \begin{bmatrix} c\theta_4c\theta_5c\theta_6 - s\theta_4s\theta_6 & -c\theta_6s\theta_4 - c\theta_4c\theta_5s\theta_6 & c\theta_4s\theta_5 \\ c\theta_5c\theta_6s\theta_4 + c\theta_4s\theta_6 & c\theta_4cc\theta_6 - c\theta_5s\theta_4s\theta_6 & s\theta_4s\theta_5 \\ -c\theta_6s\theta_5 & s\theta_5s\theta_6 & c\theta_5 \end{bmatrix} \\
 &= \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{6.84}
 \end{aligned}$$

The angles θ_4 , θ_5 , and θ_6 can be found by examining elements of 3R_6

$$\theta_4 = \text{atan2}(s_{23}, s_{13}) \tag{6.85}$$

$$\theta_5 = \text{atan2}\left(\sqrt{s_{13}^2 + s_{23}^2}, s_{33}\right) \tag{6.86}$$

$$\theta_6 = \text{atan2}(s_{32}, -s_{31}). \tag{6.87}$$

Example 187 ★ *Solution of trigonometric equation $a \cos \theta + b \sin \theta = c$.
The first type of trigonometric equation*

$$a \cos \theta + b \sin \theta = c \tag{6.88}$$

can be solved by introducing two new variables r and ϕ such that

$$a = r \sin \phi \tag{6.89}$$

$$b = r \cos \phi \tag{6.90}$$

and

$$r = \sqrt{a^2 + b^2} \tag{6.91}$$

$$\phi = \text{atan2}(a, b). \tag{6.92}$$

Substituting the new variables show that

$$\sin(\phi + \theta) = \frac{c}{r} \tag{6.93}$$

$$\cos(\phi + \theta) = \pm \sqrt{1 - \frac{c^2}{r^2}}. \tag{6.94}$$

Hence, the solutions of the problem are

$$\theta = \text{atan2}\left(\frac{c}{r}, \pm \sqrt{1 - \frac{c^2}{r^2}}\right) - \text{atan2}(a, b) \tag{6.95}$$

or

$$\theta = \text{atan2}(c, \pm \sqrt{r^2 - c^2}) - \text{atan2}(a, b). \tag{6.96}$$

Therefore, the equation $a \cos \theta + b \sin \theta = c$ has two solutions if $r^2 = a^2 + b^2 > c^2$, one solution if $r^2 = c^2$, and no solution if $r^2 < c^2$.

As an example, let us solve the following equation.

$$1.5 \cos \theta + 2.5 \sin \theta = 2.549 \tag{6.97}$$

Having $a = 1.5$ and $b = 2.5$, we find r and ϕ .

$$r = \sqrt{a^2 + b^2} = 2.915475947 \tag{6.98}$$

$$\phi = \text{atan2}(a, b) = 0.5404195 \text{ rad} \tag{6.99}$$

Therefore,

$$\begin{aligned} \theta &= \text{atan2}(c, \pm \sqrt{r^2 - c^2}) - \text{atan2}(a, b) \\ &= \text{atan2}(2.549, \pm \sqrt{2}) - \phi \\ &= 0.5235718477 \text{ rad}, 1.537181805 \text{ rad} \\ &\approx 30 \text{ deg}, 88.07 \text{ deg} \end{aligned} \tag{6.100}$$

Example 188 ★ *Meaning of the function $\tan^{-1} \frac{y}{x} = \text{atan2}(y, x)$.*

In robotic calculation, specially in solving inverse kinematic problems, we need to find an angle based on the sin and cos functions of the angle. However, \tan^{-1} cannot show the effect of the individual sign for the numerator and denominator. It always represents an angle in the first or fourth quadrant. To overcome this problem and determine the joint angles in the correct quadrant, the atan2 function is introduced as:

$$\text{atan2}(y, x) = \begin{cases} \text{sgn } y \tan^{-1} \left| \frac{y}{x} \right| & \text{if } x > 0, y \neq 0 \\ \frac{\pi}{2} \text{sgn } y & \text{if } x = 0, y \neq 0 \\ \text{sgn } y \left(\pi - \tan^{-1} \left| \frac{y}{x} \right| \right) & \text{if } x < 0, y \neq 0 \\ \pi - \pi \text{sgn } x & \text{if } x \neq 0, y = 0 \end{cases} \tag{6.101}$$

The sgn represents the signum function.

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{6.102}$$

As an example, let us compare the \tan^{-1} and atan2 for four points in four quadrants.

$x = 1, y = 1$	then	$\tan^{-1} \frac{1}{1} = 0.785$	$\text{atan2}(1, 1) = 0.785$
$x = -1, y = 1$	then	$\tan^{-1} \frac{1}{-1} = -0.785$	$\text{atan2}(1, -1) = 2.356$
$x = -1, y = -1$	then	$\tan^{-1} \frac{-1}{-1} = 0.785$	$\text{atan2}(-1, -1) = -2.356$
$x = 1, y = -1$	then	$\tan^{-1} \frac{-1}{1} = -0.785$	$\text{atan2}(-1, 1) = -0.785$

In this text, whether it has been mentioned or not, wherever $\tan^{-1} \frac{y}{x}$ is used, it must be calculated based on $\text{atan2}(y, x)$.

Example 189 ★ *Fundamental properties of arcsin and arccos. The general solution of equations*

$$\sin \varphi = a \quad \cos \theta = b \quad \tan \psi = c \quad (6.103)$$

are:

$$\varphi = \sin^{-1} a = (-1)^k \sin^{-1} a + k\pi \quad (6.104)$$

$$\theta = \cos^{-1} b = \pm \cos^{-1} b + 2k\pi \quad (6.105)$$

$$\psi = \tan^{-1} c = \tan^{-1} c + k\pi \quad c^2 \neq -1 \quad (6.106)$$

Example 190 ★ *General inverse kinematics formulas.*

There are some general trigonometric equations that regularly appear in inverse kinematics problems. The following indicates the most frequently equations and solutions.

1. If

$$\sin \theta = a \quad (6.107)$$

then, we have two answers: θ and $\pi - \theta$.

$$\theta = \operatorname{atan2} \frac{a}{\pm \sqrt{1 - a^2}} \quad (6.108)$$

2. If

$$\cos \theta = b \quad (6.109)$$

then, we have two answers: θ and $-\theta$.

$$\theta = \operatorname{atan2} \frac{\pm \sqrt{1 - b^2}}{b} \quad (6.110)$$

3. If

$$\sin \theta = a \quad \cos \theta = b \quad (6.111)$$

then,

$$\theta = \operatorname{atan2} \frac{a}{b}. \quad (6.112)$$

4. If

$$a \cos \theta + b \sin \theta = 0 \quad (6.113)$$

then, we have two answers: θ and $\theta + \pi$.

$$\theta = \operatorname{atan2} \frac{a}{b} \quad \theta = \operatorname{atan2} \frac{-a}{-b} \quad (6.114)$$

5. If

$$a \cos \theta + b \sin \theta = c \quad (6.115)$$

then,

$$\theta = \operatorname{atan2} \frac{a}{b} + \operatorname{atan2} \frac{\pm \sqrt{a^2 + b^2 - c^2}}{c}. \quad (6.116)$$

6. If

$$a \cos \theta + b \sin \theta = c \quad (6.117)$$

$$a \cos \theta - b \sin \theta = d \quad (6.118)$$

then,

$$a^2 + b^2 = c^2 + d^2 \quad (6.119)$$

$$\theta = \operatorname{atan2} \frac{ac - bd}{ad + bc}. \quad (6.120)$$

7. If

$$\sin \theta \sin \varphi = a \quad \cos \theta \sin \varphi = b \quad (6.121)$$

then, we have two answers: θ and $\theta + \pi$.

$$\theta = \operatorname{atan2} \frac{a}{b} \quad \theta = \operatorname{atan2} \frac{-a}{-b} \quad (6.122)$$

8. If

$$\sin \theta \sin \varphi = a \quad \cos \theta \sin \varphi = b \quad \cos \varphi = c \quad (6.123)$$

then, we have two answers for θ and φ : θ corresponds to φ , and $\theta + \pi$ corresponds to $-\varphi$.

$$\theta = \operatorname{atan2} \frac{a}{b} \quad \theta = \operatorname{atan2} \frac{-a}{-b} \quad (6.124)$$

$$\varphi = \operatorname{atan2} \frac{\sqrt{a^2 + b^2}}{c} \quad \varphi = \operatorname{atan2} \frac{-\sqrt{a^2 + b^2}}{c} \quad (6.125)$$

6.2 Inverse Transformation Technique

Assume we have the transformation matrix 0T_6 indicating the global position and the orientation of the end-effector of a 6 *DOF* robot in the base frame B_0 . Furthermore, assume the geometry and individual transformation matrices ${}^0T_1(q_1)$, ${}^1T_2(q_2)$, ${}^2T_3(q_3)$, ${}^3T_4(q_4)$, ${}^4T_5(q_5)$, and ${}^5T_6(q_6)$ are given as functions of joint variables.

According to forward kinematics,

$$\begin{aligned} {}^0T_6 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \quad (6.126) \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We can solve the inverse kinematics problem by solving the following equations for the unknown joint variables:

$${}^1T_6 = {}^0T_1^{-1} {}^0T_6 \quad (6.127)$$

$${}^2T_6 = {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \quad (6.128)$$

$${}^3T_6 = {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \quad (6.129)$$

$${}^4T_6 = {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \quad (6.130)$$

$${}^5T_6 = {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \quad (6.131)$$

$$\mathbf{I} = {}^5T_6^{-1} {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \quad (6.132)$$

Proof. We multiply both sides of the transformation matrix 0T_6 by ${}^0T_1^{-1}$ to obtain

$$\begin{aligned} {}^0T_1^{-1} {}^0T_6 &= {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \\ &= {}^1T_6. \end{aligned} \quad (6.133)$$

Note that ${}^0T_1^{-1}$ is the mathematical inverse of the 4×4 matrix 0T_1 , and not an inverse transformation. So, ${}^0T_1^{-1}$ must be calculated by a mathematical matrix inversion.

The left-hand side of Equation (6.133) is a function of q_1 . However, the elements of the matrix 1T_6 on the right-hand side are either zero, constant, or functions of q_2, q_3, q_4, q_5 , and q_6 . The zero or constant elements of the right-hand side provides the required algebraic equation to be solved for q_1 .

Then, we multiply both sides of (6.133) by ${}^1T_2^{-1}$ to obtain

$$\begin{aligned} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 &= {}^1T_2^{-1} {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \\ &= {}^2T_6. \end{aligned} \quad (6.134)$$

The left-hand side of this equation is a function of q_2 , while the elements of the matrix 2T_6 , on the right hand side, are either zero, constant, or functions of q_3, q_4, q_5 , and q_6 . Equating the associated element, with constant or zero elements on the right-hand side, provides the required algebraic equation to be solved for q_2 .

Following this procedure, we can find the joint variables q_3, q_4, q_5 , and q_6 by using the following equalities respectively.

$$\begin{aligned} &{}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \\ &= {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \\ &= {}^3T_6. \end{aligned} \quad (6.135)$$

$$\begin{aligned} &{}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \\ &= {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \\ &= {}^4T_6. \end{aligned} \quad (6.136)$$

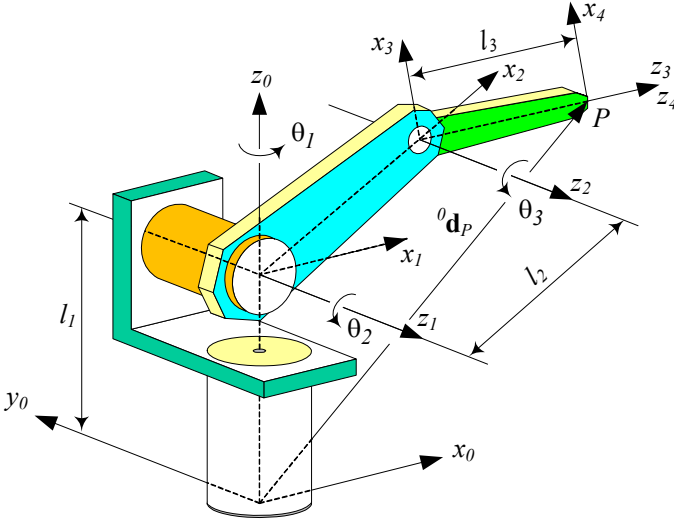


FIGURE 6.7. An articulated manipulator.

$$\begin{aligned}
 & {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \\
 &= {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \quad (6.137) \\
 &= {}^5T_6.
 \end{aligned}$$

$$\begin{aligned}
 & {}^5T_6^{-1} {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 \\
 &= {}^5T_6^{-1} {}^4T_5^{-1} {}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6) \\
 &= \mathbf{I}. \quad (6.138)
 \end{aligned}$$

The inverse transformation technique may sometimes be called *Pieper technique*. ■

Example 191 *Articulated manipulator and numerical case.*

Consider the articulated manipulator shown in Figure 6.7. The transformation matrices between its coordinate frames are:

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ \sin \theta_1 & 0 & -\cos \theta_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.139)$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.140)$$

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0 \\ \sin \theta_3 & 0 & -\cos \theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.141)$$

The forward kinematics of the manipulator is:

$$\begin{aligned} {}^0T_3 &= {}^0T_1 {}^1T_2 {}^2T_3 \\ &= \begin{bmatrix} c\theta_1 c(\theta_2 + \theta_3) & s\theta_1 & c\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2 + \theta_3) & -c\theta_1 & s\theta_1 s(\theta_2 + \theta_3) & l_2 c\theta_2 s\theta_1 \\ s(\theta_2 + \theta_3) & 0 & -c(\theta_2 + \theta_3) & l_1 + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (6.142)$$

Point P is supposed to be the point at which we attach a spherical wrist. Therefore, we attach a takht coordinate frame B_4 at P that is at a constant distance l_3 from B_3 .

$${}^3T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.143)$$

So, the overall forward kinematics of the manipulator is:

$$\begin{aligned} {}^0T_4 &= {}^0T_3 {}^3T_4 = \\ &= \begin{bmatrix} c\theta_1 c(\theta_2 + \theta_3) & s\theta_1 & c\theta_1 s(\theta_2 + \theta_3) & l_3 s(\theta_2 + \theta_3) c\theta_1 + l_2 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2 + \theta_3) & -c\theta_1 & s\theta_1 s(\theta_2 + \theta_3) & l_3 s(\theta_2 + \theta_3) s\theta_1 + l_2 c\theta_2 s\theta_1 \\ s(\theta_2 + \theta_3) & 0 & -c(\theta_2 + \theta_3) & l_1 - l_3 c(\theta_2 + \theta_3) + l_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (6.144)$$

Using the following dimensions

$$l_1 = 1 \text{ m} \quad l_2 = 1.05 \text{ m} \quad l_3 = 0.89 \text{ m} \quad (6.145)$$

when its tip point is at:

$${}^0\mathbf{d}_P = [1 \quad 1.1 \quad 1.2]^T \quad (6.146)$$

the forward kinematics reduces to:

$${}^0T_4 = \begin{bmatrix} \cos(\theta_2 + \theta_3) \cos \theta_1 & \sin \theta_1 & \sin(\theta_2 + \theta_3) \cos \theta_1 & 1 \\ \cos(\theta_2 + \theta_3) \sin \theta_1 & -\cos \theta_1 & \sin(\theta_2 + \theta_3) \sin \theta_1 & 1.1 \\ \sin(\theta_2 + \theta_3) & 0 & -\cos(\theta_2 + \theta_3) & 1.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.147)$$

Let us multiply both sides by ${}^0T_1^{-1}$ to have:

$${}^0T_1^{-1} {}^0T_4 = {}^0T_1^{-1} ({}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4) = {}^1T_4 \quad (6.148)$$

where,

$$\begin{aligned}
 {}^0T_1^{-1} {}^0T_4 &= {}^1T_4 = \begin{bmatrix} \cos \theta_1 & \sin \theta_1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \sin \theta_1 & -\cos \theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^0T_4 & (6.149) \\
 &= \begin{bmatrix} \cos(\theta_2 + \theta_3) & 0 & \sin(\theta_2 + \theta_3) & \cos \theta_1 + 1.1 \sin \theta_1 \\ \sin(\theta_2 + \theta_3) & 0 & -\cos(\theta_2 + \theta_3) & 0.2 \\ 0 & 1 & 0 & \sin \theta_1 - 1.1 \cos \theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 {}^1T_2 {}^2T_3 {}^3T_4 &= & (6.150) \\
 \begin{bmatrix} c(\theta_2 + \theta_3) & 0 & s(\theta_2 + \theta_3) & 1.2s(\theta_2 + \theta_3) + 1.1c\theta_2 \\ s(\theta_2 + \theta_3) & 0 & -c(\theta_2 + \theta_3) & 1.1s\theta_2 - 1.2c(\theta_2 + \theta_3) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

The last column of the left hand side of (6.148) is only a function of θ_1 while the right hand side is a function of θ_2 and θ_3 . Equating the element r_{24} of both sides of (6.148) provides an equation to determine θ_1 .

$$\sin \theta_1 - 1.1 \cos \theta_1 = 0 \quad (6.151)$$

$$\begin{aligned}
 \theta_1 &= \text{atan2}(1.1, 1) = \tan^{-1} \frac{1.1}{1} \\
 &= 0.8329812667 \text{ rad} \approx 47.72631098 \text{ deg} & (6.152)
 \end{aligned}$$

Substituting $\theta_1 = 0.83298$ rad in (6.149) provides a matrix 1T_4 with a numerical values in the last column.

$${}^1T_4 = \begin{bmatrix} \cos(\theta_2 + \theta_3) & 0 & \sin(\theta_2 + \theta_3) & 1.4866 \\ \sin(\theta_2 + \theta_3) & 0 & -\cos(\theta_2 + \theta_3) & 0.2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.153)$$

We multiply both sides of (6.153) by ${}^1T_2^{-1}$ to have:

$${}^1T_2^{-1} {}^1T_4 = {}^1T_2^{-1} ({}^1T_2 {}^2T_3 {}^3T_4) = {}^2T_4 \quad (6.154)$$

where,

$$\begin{aligned}
 {}^1T_2^{-1} {}^1T_4 &= {}^2T_4 = \begin{bmatrix} \cos \theta_2 & \sin \theta_2 & 0 & -1.05 \\ -\sin \theta_2 & \cos \theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^1T_4 & (6.155) \\
 &= \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 1.4866 \cos \theta_2 + 0.2 \sin \theta_2 - 1.05 \\ \sin \theta_3 & 0 & -\cos \theta_3 & 0.2 \cos \theta_2 - 1.4866 \sin \theta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

and

$${}^2T_3 {}^3T_4 = \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0.89 \sin \theta_3 \\ \sin \theta_3 & 0 & -\cos \theta_3 & -0.89 \cos \theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6.156)$$

Squaring the elements r_{14} and r_{24} of the left hand sides of (6.154), provides an equation to determine θ_2 .

$$\begin{aligned} & (1.4866 \cos \theta_2 + 0.2 \sin \theta_2 - 1.05)^2 \\ & + (0.2 \cos \theta_2 - 1.4866 \sin \theta_2)^2 \\ = & (0.89 \sin \theta_3)^2 + (-0.89 \cos \theta_3)^2 \end{aligned} \quad (6.157)$$

$$3.941 \cos \theta_2 + .53 \sin \theta_2 = 5.232 \quad (6.158)$$

This equation has the following solutions:

$$\theta_2 = .7555518221 \text{ rad} \approx 43.28993061 \text{ deg} \quad (6.159)$$

$$\theta_2 = -.4880908073 \text{ rad} \approx -27.96554327 \text{ deg} \quad (6.160)$$

Having θ_2 , we can calculate θ_3 from the last column of (6.156) and (6.155).

$$\theta_3 = \text{atan2} \left(\frac{1.4866 \cos \theta_2 + 0.2 \sin \theta_2 - 1.05}{0.2 \cos \theta_2 - 1.4866 \sin \theta_2} \right) \quad (6.161)$$

If $\theta_2 = .755$ rad then we have:

$$\theta_3 = \text{atan2}(-0.19437) = -0.19198 \text{ rad} \approx -11 \text{ deg} \quad (6.162)$$

If $\theta_2 = -.488$ rad then we have:

$$\theta_3 = \text{atan2}(0.19437) = 0.19198 \text{ rad} \approx 11 \text{ deg} \quad (6.163)$$

Example 192 Inverse kinematics for a spherical robot.

Transformation matrices of the spherical robot shown in Figure 6.8 are

$$\begin{aligned} {}^0T_1 &= \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^1T_2 &= \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^2T_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^3T_4 &= \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^4T_5 &= \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^5T_6 &= \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (6.164)$$

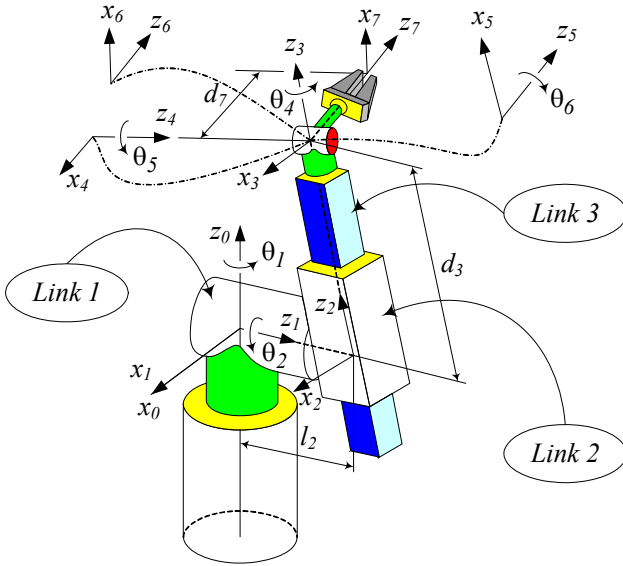


FIGURE 6.8. A spherical robot, made of a spherical manipulator attached to a spherical wrist.

Therefore, the position and orientation of the end-effector for a set of joint variables, which solves the forward kinematics problem, can be found by matrix multiplication

$$\begin{aligned}
 {}^0T_6 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \\
 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.165}
 \end{aligned}$$

where the elements of 0T_6 are the same as the elements of the matrix in Equation (5.159).

Multiplying both sides of the (6.165) by ${}^0T_1^{-1}$ provides

$$\begin{aligned}
 {}^0T_1^{-1} {}^0T_6 &= \begin{bmatrix} \cos \theta_1 & \sin \theta_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.166}
 \end{aligned}$$

where

$$f_{1i} = r_{1i} \cos \theta_1 + r_{2i} \sin \theta_1 \tag{6.167}$$

$$f_{2i} = -r_{3i} \tag{6.168}$$

$$f_{3i} = r_{2i} \cos \theta_1 - r_{1i} \sin \theta_1 \tag{6.169}$$

$$i = 1, 2, 3, 4.$$

Based on the given transformation matrices, we find that

$$\begin{aligned} {}^1T_6 &= {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \\ &= \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.170}$$

$$f_{11} = -c\theta_2 s\theta_4 s\theta_6 + c\theta_6 (-s\theta_2 s\theta_5 + c\theta_2 c\theta_4 c\theta_5) \tag{6.171}$$

$$f_{21} = -s\theta_2 s\theta_4 s\theta_6 + c\theta_6 (c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2) \tag{6.172}$$

$$f_{31} = c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 \tag{6.173}$$

$$f_{12} = -c\theta_2 c\theta_6 s\theta_4 - s\theta_6 (-s\theta_2 s\theta_5 + c\theta_2 c\theta_4 c\theta_5) \tag{6.174}$$

$$f_{22} = -c\theta_6 s\theta_2 s\theta_4 - s\theta_6 (c\theta_2 s\theta_5 + c\theta_4 c\theta_5 s\theta_2) \tag{6.175}$$

$$f_{32} = c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 \tag{6.176}$$

$$f_{13} = c\theta_5 s\theta_2 + c\theta_2 c\theta_4 s\theta_5 \tag{6.177}$$

$$f_{23} = -c\theta_2 c\theta_5 + c\theta_4 s\theta_2 s\theta_5 \tag{6.178}$$

$$f_{33} = s\theta_4 s\theta_5 \tag{6.179}$$

$$f_{14} = d_3 s\theta_2 \tag{6.180}$$

$$f_{24} = -d_3 c\theta_2 \tag{6.181}$$

$$f_{34} = l_2. \tag{6.182}$$

The only constant element of the matrix (6.170) is $f_{34} = l_2$, therefore,

$$r_{24} \cos \theta_1 - r_{14} \sin \theta_1 = l_2. \tag{6.183}$$

This kind of trigonometric equation frequently appears in robotic inverse kinematics, which has a systematic method of solution. We assume

$$r_{14} = r \cos \phi \tag{6.184}$$

$$r_{24} = r \sin \phi \tag{6.185}$$

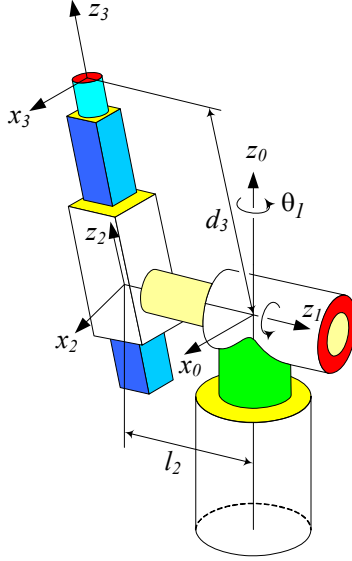


FIGURE 6.9. Left shoulder configuration of a spherical robot.

where

$$r = \sqrt{r_{14}^2 + r_{24}^2} \tag{6.186}$$

$$\phi = \tan^{-1} \frac{r_{24}}{r_{14}} \tag{6.187}$$

and therefore, Equation (6.183) becomes

$$\frac{l_2}{r} = \sin \phi \cos \theta_1 - \cos \phi \sin \theta_1 = \sin(\phi - \theta_1) \tag{6.188}$$

showing that

$$\pm \sqrt{1 - (l_2/r)^2} = \cos(\phi - \theta_1). \tag{6.189}$$

Hence, the solution of Equation (6.183) for θ_1 is

$$\theta_1 = \tan^{-1} \frac{r_{24}}{r_{14}} - \tan^{-1} \frac{l_2}{\pm \sqrt{r^2 - l_2^2}}. \tag{6.190}$$

The $(-)$ sign corresponds to a left shoulder configuration of the robots as shown in Figure 6.9, and the $(+)$ sign corresponds to the right shoulder configuration.

The elements f_{14} and f_{24} of matrix (6.170) are functions of θ_1 and θ_2 only.

$$f_{14} = d_3 \sin \theta_2 = r_{14} \cos \theta_1 + r_{24} \sin \theta_1 \tag{6.191}$$

$$f_{24} = -d_3 \cos \theta_2 = -r_{34} \tag{6.192}$$

Hence, it is possible to use them and find θ_2

$$\theta_2 = \tan^{-1} \frac{r_{14} \cos \theta_1 + r_{24} \sin \theta_1}{r_{34}} \quad (6.193)$$

where θ_1 must be substituted from (6.190).

In the next step, we find the third joint variable d_3 from

$${}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 = {}^2T_6 \quad (6.194)$$

where

$${}^1T_2^{-1} = \begin{bmatrix} \cos \theta_2 & \sin \theta_2 & 0 & 0 \\ 0 & 0 & 1 & -l_2 \\ \sin \theta_2 & -\cos \theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.195)$$

and

$${}^2T_6 = \begin{bmatrix} -s\theta_4 s\theta_6 + c\theta_4 c\theta_5 c\theta_6 & -c\theta_6 s\theta_4 - c\theta_4 c\theta_5 s\theta_6 & c\theta_4 s\theta_5 & 0 \\ c\theta_4 s\theta_6 + c\theta_5 c\theta_6 s\theta_4 & c\theta_4 c\theta_6 - c\theta_5 s\theta_4 s\theta_6 & s\theta_4 s\theta_5 & 0 \\ -c\theta_6 s\theta_5 & s\theta_5 s\theta_6 & c\theta_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6.196)$$

Employing the elements of the matrices on both sides of Equation (6.194) shows that the element (3,4) can be utilized to find d_3 .

$$d_3 = r_{34} \cos \theta_2 + r_{14} \cos \theta_1 \sin \theta_2 + r_{24} \sin \theta_1 \sin \theta_2 \quad (6.197)$$

Since there is no other element in Equation (6.194) to be a function of another single variable, we move to the next step and evaluate θ_4 from

$${}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 = {}^4T_6 \quad (6.198)$$

because ${}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 = {}^3T_6$ provides no new equation. Evaluating 4T_6

$${}^4T_6 = \begin{bmatrix} \cos \theta_5 \cos \theta_6 & -\cos \theta_5 \sin \theta_6 & \sin \theta_5 & 0 \\ \cos \theta_6 \sin \theta_5 & -\sin \theta_5 \sin \theta_6 & -\cos \theta_5 & 0 \\ \sin \theta_6 & \cos \theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.199)$$

and the left-hand side of (6.198) utilizing

$${}^2T_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.200)$$

and

$${}^3T_4^{-1} = \begin{bmatrix} \cos \theta_4 & \sin \theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\sin \theta_4 & \cos \theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.201)$$

shows that

$${}^3T_4^{-1} {}^2T_3^{-1} {}^1T_2^{-1} {}^0T_1^{-1} {}^0T_6 = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.202)$$

where

$$g_{1i} = -r_{3i}c\theta_4s\theta_2 + r_{2i}(c\theta_1s\theta_4 + c\theta_2c\theta_4s\theta_1) + r_{1i}(-s\theta_1s\theta_4 + c\theta_1c\theta_2c\theta_4) \quad (6.203)$$

$$g_{2i} = d_3\delta_{4i} - r_{31}c\theta_2 - r_{11}c\theta_1s\theta_2 - r_{21}s\theta_1s\theta_2 \quad (6.204)$$

$$g_{3i} = r_{31}s\theta_2s\theta_4 + r_{21}(c\theta_1c\theta_4 - c\theta_2s\theta_1s\theta_4) + r_{11}(-c\theta_4s\theta_1 - c\theta_1c\theta_2s\theta_4) \quad (6.205)$$

$$i = 1, 2, 3, 4.$$

The symbol δ_{4i} indicates the Kronecker delta and is:

$$\delta_{4i} = \begin{cases} 1 & \text{if } i = 4 \\ 0 & \text{if } i \neq 4 \end{cases} \quad (6.206)$$

Therefore, we can find θ_4 by equating the element (3, 3), θ_5 by equating the elements (1, 3) or (2, 3), and θ_6 by equating the elements (3, 1) or (3, 2). Starting from element (3, 3)

$$r_{13}(-c\theta_4s\theta_1 - c\theta_1c\theta_2s\theta_4) + r_{23}(c\theta_1c\theta_4 - c\theta_2s\theta_1s\theta_4) + r_{33}s\theta_2s\theta_4 = 0 \quad (6.207)$$

we find θ_4

$$\theta_4 = \tan^{-1} \frac{-r_{13}s\theta_1 + r_{23}c\theta_1}{c\theta_2(r_{13}c\theta_1 + r_{23}s\theta_1) - r_{33}s\theta_2} \quad (6.208)$$

which, based on the second value of θ_1 , can also be equal to

$$\theta_4 = \frac{\pi}{2} + \tan^{-1} \frac{-r_{13}s\theta_1 + r_{23}c\theta_1}{c\theta_2(r_{13}c\theta_1 + r_{23}s\theta_1) - r_{33}s\theta_2}. \quad (6.209)$$

Now we use elements (1, 3) and (2, 3),

$$\sin \theta_5 = r_{23}(\cos \theta_1 \sin \theta_4 + \cos \theta_2 \cos \theta_4 \sin \theta_1) - r_{33} \cos \theta_4 \sin \theta_2 + r_{13}(\cos \theta_1 \cos \theta_2 \cos \theta_4 - \sin \theta_1 \sin \theta_4) \quad (6.210)$$

$$-\cos \theta_5 = -r_{33} \cos \theta_2 - r_{13} \cos \theta_1 \sin \theta_2 - r_{23} \sin \theta_1 \sin \theta_2 \quad (6.211)$$

to find θ_5

$$\theta_5 = \tan^{-1} \frac{\sin \theta_5}{\cos \theta_5}. \tag{6.212}$$

Finally, θ_6 can be found from the elements (3, 1) and (3, 2)

$$\begin{aligned} \sin \theta_6 &= r_{31} \sin \theta_2 \sin \theta_4 + r_{21} (\cos \theta_1 \cos \theta_4 - \cos \theta_2 \sin \theta_1 \sin \theta_4) \\ &\quad + r_{11} (-\cos \theta_4 \sin \theta_1 - \cos \theta_1 \cos \theta_2 \sin \theta_4) \end{aligned} \tag{6.213}$$

$$\begin{aligned} \cos \theta_6 &= r_{32} \sin \theta_2 \sin \theta_4 + r_{22} (\cos \theta_1 \cos \theta_4 - \cos \theta_2 \sin \theta_1 \sin \theta_4) \\ &\quad + r_{12} (-\cos \theta_4 \sin \theta_1 - \cos \theta_1 \cos \theta_2 \sin \theta_4) \end{aligned} \tag{6.214}$$

$$\theta_6 = \tan^{-1} \frac{\sin \theta_6}{\cos \theta_6}. \tag{6.215}$$

Example 193 *Inverse of parametric Euler angles transformation matrix. The global rotation matrix based on Euler angles has been found in Equation (2.107).*

$$\begin{aligned} {}^G R_B &= [A_{z,\psi} A_{x,\theta} A_{z,\varphi}]^T = R_{Z,\varphi} R_{X,\theta} R_{Z,\psi} \\ &= \begin{bmatrix} c\varphi c\psi - \theta s\varphi s\psi & -c\varphi s\psi - \theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + \theta c\varphi s\psi & -s\varphi s\psi + \theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \end{aligned} \tag{6.216}$$

Premultiplying ${}^G R_B$ by $R_{Z,\varphi}^{-1}$, gives

$$\begin{aligned} &\begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^G R_B \\ &= \begin{bmatrix} r_{11}c\varphi + r_{21}s\varphi & r_{12}c\varphi + r_{22}s\varphi & r_{13}c\varphi + r_{23}s\varphi \\ r_{21}c\varphi - r_{11}s\varphi & r_{22}c\varphi - r_{12}s\varphi & r_{23}c\varphi - r_{13}s\varphi \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\ &= \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \cos \theta \sin \psi & \cos \theta \cos \psi & -\sin \theta \\ \sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta \end{bmatrix}. \end{aligned} \tag{6.217}$$

Equating the elements (1, 3) of both sides

$$r_{13} \cos \varphi + r_{23} \sin \varphi = 0 \tag{6.218}$$

gives

$$\varphi = \text{atan2}(r_{13}, -r_{23}). \tag{6.219}$$

Having φ helps us to find ψ by using elements (1, 1) and (1, 2)

$$\cos \psi = r_{11} \cos \varphi + r_{21} \sin \varphi \quad (6.220)$$

$$-\sin \psi = r_{12} \cos \varphi + r_{22} \sin \varphi \quad (6.221)$$

therefore,

$$\psi = \operatorname{atan2} \frac{-r_{12} \cos \varphi - r_{22} \sin \varphi}{r_{11} \cos \varphi + r_{21} \sin \varphi}. \quad (6.222)$$

In the next step, we may postmultiply ${}^G R_B$ by $R_{Z,\psi}^{-1}$, to provide

$$\begin{aligned} & {}^G R_B \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r_{11}c\psi - r_{12}s\psi & r_{12}c\psi + r_{11}s\psi & r_{13} \\ r_{21}c\psi - r_{22}s\psi & r_{22}c\psi + r_{21}s\psi & r_{23} \\ r_{31}c\psi - r_{32}s\psi & r_{32}c\psi + r_{31}s\psi & r_{33} \end{bmatrix} \\ &= \begin{bmatrix} \cos \varphi & -\cos \theta \sin \varphi & \sin \theta \sin \varphi \\ \sin \varphi & \cos \theta \cos \varphi & -\cos \varphi \sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}. \end{aligned} \quad (6.223)$$

The elements (3, 1) on both sides make an equation to find ψ .

$$r_{31} \cos \psi - r_{31} \sin \psi = 0 \quad (6.224)$$

Therefore, it is possible to find ψ from the following equation:

$$\psi = \operatorname{atan2}(r_{31}, r_{31}). \quad (6.225)$$

Finally, θ can be found using elements (3, 2) and (3, 3)

$$r_{32}c\psi + r_{31}s\psi = \sin \theta \quad (6.226)$$

$$r_{33} = \cos \theta \quad (6.227)$$

which give

$$\theta = \operatorname{atan2} \frac{r_{32} \cos \psi + r_{31} \sin \psi}{r_{33}}. \quad (6.228)$$

Example 194 Inverse of given Euler angles transformation matrix.

Assume the global rotation matrix based on Euler angles is given as:

$$\begin{aligned} & {}^G R_B = [A_{z,\psi} A_{x,\theta} A_{z,\varphi}]^T = R_{Z,\varphi} R_{X,\theta} R_{Z,\psi} \\ &= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & -c\varphi s\psi - c\theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\varphi s\psi + c\theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \\ &= \begin{bmatrix} 0.12683 & -0.78033 & 0.61237 \\ 0.92678 & -0.12683 & -0.35355 \\ 0.35355 & 0.61237 & 0.70711 \end{bmatrix} \end{aligned} \quad (6.229)$$

Premultiplying ${}^G R_B$ by $R_{Z,\varphi}^{-1}$, gives

$$\begin{aligned}
 & \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^G R_B \\
 = & \begin{bmatrix} 0.126c\varphi + 0.926s\varphi & -0.780c\varphi - 0.126s\varphi & 0.612c\varphi - 0.353s\varphi \\ 0.926c\varphi - 0.126s\varphi & 0.780s\varphi - 0.126c\varphi & -0.353c\varphi - 0.612s\varphi \\ 0.35355 & 0.61237 & 0.70711 \end{bmatrix} \\
 = & \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \cos \theta \sin \psi & \cos \theta \cos \psi & -\sin \theta \\ \sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta \end{bmatrix}. \tag{6.230}
 \end{aligned}$$

Equating the elements (1, 3) of both sides

$$0.61237 \cos \varphi - 0.35355 \sin \varphi = 0 \tag{6.231}$$

gives

$$\varphi = \text{atan2} \left(\frac{0.61237}{0.35355} \right) = 1.0472 \text{ rad} = 60 \text{ deg}. \tag{6.232}$$

Having φ helps us to find ψ by using elements (1, 1) and (1, 2)

$$\cos \psi = 0.126 \cos \varphi + 0.926 \sin \varphi \tag{6.233}$$

$$-\sin \psi = -0.78 \cos \varphi - 0.126 \sin \varphi \tag{6.234}$$

therefore,

$$\begin{aligned}
 \psi &= \text{atan2} \frac{0.78 \cos \varphi + 0.126 \sin \varphi}{0.126 \cos \varphi + 0.926 \sin \varphi} \\
 &= \text{atan2} \frac{0.49912}{0.86494} = 0.523 \text{ rad} = 30 \text{ deg}. \tag{6.235}
 \end{aligned}$$

Although we can find θ from elements (2, 3) and (3, 3), let us postmultiply ${}^G R_B$ by $R_{Z,\psi}^{-1}$, to follow the inverse transformation technique.

$$\begin{aligned}
 & {}^G R_B \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 = & \begin{bmatrix} 0.126c\psi + 0.78s\psi & 0.126s\psi - 0.78c\psi & 0.61237 \\ 0.926c\psi + 0.126s\psi & 0.926s\psi - 0.126c\psi & -0.35355 \\ 0.353c\psi - 0.612s\psi & 0.612c\psi + 0.353s\psi & 0.70711 \end{bmatrix} \\
 = & \begin{bmatrix} \cos \varphi & -\cos \theta \sin \varphi & \sin \theta \sin \varphi \\ \sin \varphi & \cos \theta \cos \varphi & -\cos \varphi \sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \tag{6.236}
 \end{aligned}$$

The elements (3, 1) on both sides make an equation to find ψ .

$$0.35355 \cos \psi - 0.61237 \sin \psi = 0 \tag{6.237}$$

Therefore, it is also possible to find ψ from the following equation:

$$\psi = \text{atan2} \left(\frac{0.353\ 55}{0.612\ 37} \right) = 0.523 \text{ rad} = 30 \text{ deg} \quad (6.238)$$

Finally, θ can be found using elements (3, 2) and (3, 3)

$$0.612\ 37 \cos \psi + 0.353\ 55 \sin \psi = \sin \theta \quad (6.239)$$

$$0.707\ 11 = \cos \theta \quad (6.240)$$

which give

$$\theta = \text{atan2} \frac{0.707\ 11}{0.707\ 11} = 1 \text{ rad} = 45 \text{ deg} . \quad (6.241)$$

Example 195 ★ *Inverse kinematics and nonstandard DH frames.*

Consider a 3 DOF planar manipulator shown in Figure 5.4. The nonstandard DH transformation matrices of the manipulator are

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.242)$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_1 \\ \sin \theta_2 & \cos \theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.243)$$

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & l_2 \\ \sin \theta_3 & \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6.244)$$

$${}^3T_4 = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \quad (6.245)$$

The solution of the inverse kinematics problem is a mathematical problem and none of the standard or nonstandard DH methods for defining link frames provide any simplicity. To calculate the inverse kinematics, we start

with calculating the forward kinematics transformation matrix 0T_4

$$\begin{aligned}
 {}^0T_4 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 & (6.246) \\
 &= \begin{bmatrix} \cos \theta_{123} & -\sin \theta_{123} & 0 & l_1 \cos \theta_1 + l_2 \cos \theta_{12} + l_3 \cos \theta_{123} \\ \sin \theta_{123} & \cos \theta_{123} & 0 & l_1 \sin \theta_1 + l_2 \sin \theta_{12} + l_3 \sin \theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

where we used the following short notation to simplify the equation.

$$\theta_{ijk} = \theta_i + \theta_j + \theta_k \quad (6.247)$$

Examining the matrix 0T_4 indicates that

$$\theta_{123} = \text{atan2}(r_{21}, r_{11}). \quad (6.248)$$

The next equation

$$\begin{aligned}
 {}^0T_4 {}^3T_4^{-1} &= {}^0T_1 {}^1T_2 {}^2T_3 & (6.249) \\
 \begin{bmatrix} r_{11} & r_{12} & 0 & r_{14} - l_3 r_{11} \\ r_{21} & r_{22} & 0 & r_{24} - l_3 r_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & l_1 c\theta_1 + l_2 c\theta_{12} \\ s\theta_{123} & c\theta_{123} & 0 & l_1 s\theta_1 + l_2 s\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

shows that

$$\theta_2 = \arccos \frac{f_1^2 + f_2^2 - l_1^2 - l_2^2}{2l_1 l_2} \quad (6.250)$$

$$\theta_1 = \text{atan2}(f_2 f_3 - f_1 f_4, f_1 f_3 + f_2 f_4) \quad (6.251)$$

where

$$\begin{aligned}
 f_1 &= r_{14} - l_3 r_{11} = c\theta_1 (l_2 c\theta_2 + l_1) - s\theta_1 (l_2 s\theta_2) \\
 &= c\theta_1 f_3 - s\theta_1 f_4
 \end{aligned} \quad (6.252)$$

$$\begin{aligned}
 f_2 &= r_{24} - l_3 r_{21} = s\theta_1 (l_2 c\theta_2 + l_1) + c\theta_1 (l_2 s\theta_2) \\
 &= s\theta_1 f_3 + c\theta_1 f_4.
 \end{aligned} \quad (6.253)$$

Finally, the angle θ_3 is

$$\theta_3 = \theta_{123} - \theta_1 - \theta_2. \quad (6.254)$$

6.3 ★ Iterative Technique

The inverse kinematics problem can be interpreted as searching for the solution q_k of a set of nonlinear algebraic equations

$$\begin{aligned} {}^0T_n &= \mathbf{T}(\mathbf{q}) & (6.255) \\ &= {}^0T_1(q_1) {}^1T_2(q_2) {}^2T_3(q_3) {}^3T_4(q_4) \cdots {}^{n-1}T_n(q_n) \\ &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

or

$$r_{ij} = r_{ij}(q_k) \quad k = 1, 2, \dots, n. \quad (6.256)$$

where n is the number of *DOF*. However, maximum $m = 6$ out of 12 equations of (6.255) are independent and can be utilized to solve for joint variables q_k . The functions $\mathbf{T}(\mathbf{q})$ are transcendental, which are given explicitly based on forward kinematic analysis.

Numerous methods are available to find the zeros of Equation (6.255). However, the methods are, in general, *iterative*. The most common method is known as the *Newton-Raphson method*.

In the *iterative technique*, to solve the kinematic equations

$$\mathbf{T}(\mathbf{q}) = 0 \quad (6.257)$$

for variables \mathbf{q} , we start with an initial guess

$$\mathbf{q}^\star = \mathbf{q} + \delta\mathbf{q} \quad (6.258)$$

for the joint variables. Using the forward kinematics, we can determine the configuration of the end-effector frame for the guessed joint variables.

$$\mathbf{T}^\star = \mathbf{T}(\mathbf{q}^\star) \quad (6.259)$$

The difference between the configuration calculated with the forward kinematics and the desired configuration represents an *error*, called *residue*, which must be minimized.

$$\delta\mathbf{T} = \mathbf{T} - \mathbf{T}^\star \quad (6.260)$$

A first order Taylor expansion of the set of equations is:

$$\begin{aligned} \mathbf{T} &= \mathbf{T}(\mathbf{q}^\star + \delta\mathbf{q}) \\ &= \mathbf{T}(\mathbf{q}^\star) + \frac{\partial\mathbf{T}}{\partial\mathbf{q}}\delta\mathbf{q} + O(\delta\mathbf{q}^2) \end{aligned} \quad (6.261)$$

Assuming $\delta\mathbf{q} \ll \mathbf{I}$ allows us to work with a set of linear equations

$$\delta\mathbf{T} = \mathbf{J}\delta\mathbf{q} \quad (6.262)$$

where \mathbf{J} is the Jacobian matrix of the set of equations

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \frac{\partial T_i}{\partial q_j} \end{bmatrix} \quad (6.263)$$

that implies

$$\delta \mathbf{q} = \mathbf{J}^{-1} \delta \mathbf{T}. \quad (6.264)$$

Therefore, the unknown variables \mathbf{q} are:

$$\mathbf{q} = \mathbf{q}^\star + \mathbf{J}^{-1} \delta \mathbf{T} \quad (6.265)$$

We may use the values obtained by (6.265) as a new approximation to repeat the calculations and find newer values. Repeating the methods can be summarized in the following iterative equation to converge to the exact value of the variables.

$$\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + \mathbf{J}^{-1}(\mathbf{q}^{(i)}) \delta \mathbf{T}(\mathbf{q}^{(i)}) \quad (6.266)$$

This iteration technique can be set in an algorithm for easier numerical calculations.

Algorithm 6.1. Inverse kinematics iteration technique.

1. *Set the initial counter $i = 0$.*
2. *Find or guess an initial estimate $\mathbf{q}^{(0)}$.*
3. *Calculate the residue $\delta \mathbf{T}(\mathbf{q}^{(i)}) = \mathbf{J}(\mathbf{q}^{(i)}) \delta \mathbf{q}^{(i)}$.*
If every element of $\mathbf{T}(\mathbf{q}^{(i)})$ or its norm $\|\mathbf{T}(\mathbf{q}^{(i)})\|$ is less than a tolerance, $\|\mathbf{T}(\mathbf{q}^{(i)})\| < \epsilon$ then terminate the iteration. The $\mathbf{q}^{(i)}$ is the desired solution.
4. *Calculate $\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + \mathbf{J}^{-1}(\mathbf{q}^{(i)}) \delta \mathbf{T}(\mathbf{q}^{(i)})$.*
5. *Set $i = i + 1$ and return to step 3.*

The tolerance ϵ can equivalently be set up on variables

$$\mathbf{q}^{(i+1)} - \mathbf{q}^{(i)} < \epsilon \quad (6.267)$$

or on Jacobian

$$\mathbf{J} - \mathbf{I} < \epsilon. \quad (6.268)$$

Example 196 ★ *Inverse kinematics for a 2R planar manipulator.*

In Example 184 we have seen that the tip point of a 2R planar manipulator can be described by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} l_1 c\theta_1 + l_2 c(\theta_1 + \theta_2) \\ l_1 s\theta_1 + l_2 s(\theta_1 + \theta_2) \end{bmatrix}. \quad (6.269)$$

To solve the inverse kinematics of the manipulator and find the joint coordinates for a known position of the tip point, we define

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad (6.270)$$

$$\mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix} \quad (6.271)$$

therefore, the Jacobian of the equations is:

$$\begin{aligned} \mathbf{J}(\mathbf{q}) &= \begin{bmatrix} \frac{\partial T_i}{\partial q_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \theta_1} & \frac{\partial X}{\partial \theta_2} \\ \frac{\partial Y}{\partial \theta_1} & \frac{\partial Y}{\partial \theta_2} \end{bmatrix} \\ &= \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \end{aligned} \quad (6.272)$$

The inverse of the Jacobian is

$$\mathbf{J}^{-1} = \frac{-1}{l_1 l_2 s\theta_2} \begin{bmatrix} -l_2 c(\theta_1 + \theta_2) & -l_2 s(\theta_1 + \theta_2) \\ l_1 c\theta_1 + l_2 c(\theta_1 + \theta_2) & l_1 s\theta_1 + l_2 s(\theta_1 + \theta_2) \end{bmatrix} \quad (6.273)$$

and therefore, the iterative formula (6.266) is set up as

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i+1)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(i)} + \mathbf{J}^{-1} \left(\begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix}^{(i)} \right). \quad (6.274)$$

Let's assume

$$l_1 = l_2 = 1 \quad (6.275)$$

$$\mathbf{T} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (6.276)$$

and start from a guess value

$$\mathbf{q}^{(0)} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(0)} = \begin{bmatrix} \pi/3 \\ -\pi/3 \end{bmatrix} \quad (6.277)$$

for which

$$\begin{aligned} \delta \mathbf{T} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \cos \pi/3 + \cos(\pi/3 + -\pi/3) \\ \sin \pi/3 + \sin(\pi/3 + -\pi/3) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix}. \end{aligned} \quad (6.278)$$

The Jacobian and its inverse for these values are

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \quad (6.279)$$

$$\mathbf{J}^{-1} = \begin{bmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{bmatrix} \quad (6.280)$$

and therefore,

$$\begin{aligned} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(1)} &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^{(0)} + \mathbf{J}^{-1} \delta \mathbf{T} \\ &= \begin{bmatrix} \pi/3 \\ -\pi/3 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3}\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.6245 \\ -1.7792 \end{bmatrix}. \end{aligned} \quad (6.281)$$

Based on the iterative technique, we can find the following values and find the solution in a few iterations.

Iteration 1.

$$\mathbf{J} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \quad (6.282)$$

$$\delta \mathbf{T} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2}\sqrt{3} + 1 \end{bmatrix} \quad (6.283)$$

$$\mathbf{q}^{(1)} = \begin{bmatrix} 1.6245 \\ -1.7792 \end{bmatrix} \quad (6.284)$$

Iteration 2.

$$\mathbf{J} = \begin{bmatrix} -0.844 & 0.154 \\ 0.934 & 0.988 \end{bmatrix} \quad (6.285)$$

$$\delta \mathbf{T} = \begin{bmatrix} 6.516 \times 10^{-2} \\ 0.15553 \end{bmatrix} \quad (6.286)$$

$$\mathbf{q}^{(2)} = \begin{bmatrix} 1.583 \\ -1.582 \end{bmatrix} \quad (6.287)$$

Iteration 3.

$$\mathbf{J} = \begin{bmatrix} -1.00 & -.433 \times 10^{-3} \\ .988 & .999 \end{bmatrix} \quad (6.288)$$

$$\delta \mathbf{T} = \begin{bmatrix} .119 \times 10^{-1} \\ -.362 \times 10^{-3} \end{bmatrix} \quad (6.289)$$

$$\mathbf{q}^{(3)} = \begin{bmatrix} 1.570795886 \\ -1.570867014 \end{bmatrix} \quad (6.290)$$

Iteration 4.

$$\mathbf{J} = \begin{bmatrix} -1.000 & 0.0 \\ 0.99850 & 1.0 \end{bmatrix} \quad (6.291)$$

$$\delta\mathbf{T} = \begin{bmatrix} -.438 \times 10^{-6} \\ .711 \times 10^{-4} \end{bmatrix} \quad (6.292)$$

$$\mathbf{q}^{(4)} = \begin{bmatrix} 1.570796329 \\ -1.570796329 \end{bmatrix} \quad (6.293)$$

The result of the fourth iteration $\mathbf{q}^{(4)}$ is close enough to the exact value $\mathbf{q} = [\pi/2 \quad -\pi/2]^T$.

6.4 ★ Comparison of the Inverse Kinematics Techniques

6.4.1 ★ Existence and Uniqueness of Solution

It is clear that when the desired tool frame position ${}^0\mathbf{d}_7$ is outside the working space of the robot, there can not be any real solution for the joint variables of the robot. In this condition, the overall resultant of the terms under square root signs would be negative. Furthermore, even when the tool frame position ${}^0\mathbf{d}_7$ is within the working space, there may be some tool orientations ${}^0\mathbf{R}_7$ that are not achievable without breaking joint constraints and violating one or more joint variable limits. Therefore, existing solutions for inverse kinematics problem generally depends on the geometric configuration of the robot.

The normal case is when the number of joints is six. Then, provided that no *DOF* is redundant and the configuration assigned to the end-effectors of the robot lies within the workspace, the inverse kinematics solution exists in finite numbers. The different solutions correspond to possible configurations to reach the same end-effector configuration.

Generally speaking, when the solution of the inverse kinematics of a robot exists, they are not unique. Multiple solutions appear because a robot can reach to a point within the working space in different configurations. Every set of solutions is associated to a particular configuration. The elbow-up and elbow-down configuration of the $2R$ manipulator in Example 184 is a simple example.

The multiplicity of the solution depends on the number of joints of the manipulator and their type. The fact that a manipulator has multiple solutions may cause problems since the system has to be able to select one of them. The criteria on which to base a decision may vary, but a very reasonable choice consists of choosing the closest solution to the current configuration.

When the number of joints is less than six, no solution exists unless freedom is reduced in the same time in the task space, for example, by constraining the tool orientation to certain directions.

When the number of joints exceeds six, the structure becomes redundant and an infinite number of solutions exists to reach the same end-effector configuration within the robot workspace. Redundancy of the robot architecture is an interesting feature for systems installed in a highly constrained environment. From the kinematic point of view, the difficulty lies in formulating the environment constraints in mathematical form, to ensure the uniqueness of the solution to the inverse kinematic problem.

6.4.2 ★ *Inverse Kinematics Techniques*

The inverse kinematics problem of robots can be solved by several methods, such as *decoupling*, *inverse transformation*, *iterative*, *screw algebra*, *dual matrices*, *dual quaternions*, and *geometric techniques*. The decoupling and inverse transform technique using 4×4 homogeneous transformation matrices suffers from the fact that the solution does not clearly indicate how to select the appropriate solution from multiple possible solutions for a particular configuration. Thus, these techniques rely on the skills and intuition of the engineer. The iterative solution method often requires a vast amount of computation and moreover, it does not guarantee convergence to the correct solution. It is especially weak when the robot is close to the singular and degenerate configurations. The iterative solution method also lacks a method for selecting the appropriate solution from multiple possible solutions.

Although the set of nonlinear trigonometric equations is typically not possible to be solved analytically, there are some robot structures that are *solvable* analytically. The sufficient condition of solvability is when the 6 *DOF* robot has three consecutive revolute joints with axes intersecting in one point. The other property of inverse kinematics is ambiguity of a solution in singular points. However, when closed-form solutions to the arm equation can be found, they are seldom unique.

Example 197 ★ *Iteration technique and n - m relationship.*

1— *Iteration method when $n = m$.*

When the number of joint variables n is equal to the number of independent equations generated in forward kinematics m , then provided that the Jacobian matrix remains non singular, the linearized equation

$$\delta \mathbf{T} = \mathbf{J} \delta \mathbf{q} \quad (6.294)$$

has a unique set of solutions and therefore, the Newton-Raphson technique may be utilized to solve the inverse kinematics problem.

The cost of the procedure depends on the number of iterations to be performed, which depends upon different parameters such as the distance be-

tween the estimated and effective solutions, and the condition number of the Jacobian matrix at the solution. Since the solution to the inverse kinematics problem is not unique, it may generate different configurations according to the choice of the estimated solution. No convergence may be observed if the initial estimate of the solution falls outside the convergence domain of the algorithm.

2— Iteration method when $n > m$.

When the number of joint variables n is more than the number of independent equations m , then the problem is an overdetermined case for which no solution exists in general because the number of joints is not enough to generate an arbitrary configuration for the end-effector. A solution can be generated, which minimizes the position error.

3— Iteration method when $n < m$.

When the number of joint variables n is less than the number of independent equations m , then the problem is a redundant case for which an infinite number of solutions are generally available.

6.5 ★ Singular Configuration

Generally speaking, for any robot, redundant or not, it is possible to discover some configurations, called *singular configurations*, in which the number of *DOF* of the end-effector is inferior to the dimension in which it generally operates. Singular configurations happen when:

1. Two axes of prismatic joints become parallel
2. Two axes of revolute joints become identical.

At singular positions, the end-effector loses one or more degrees of freedom, since the kinematic equations become linearly dependent or certain solutions become undefined. Singular positions must be avoided as the velocities required to move the end-effector become theoretically infinite.

The singular configurations can be determined from the Jacobian matrix. The Jacobian matrix \mathbf{J} relates the infinitesimal displacements of the end-effector

$$\delta\mathbf{X} = [\delta X_1, \dots, \delta X_m] \quad (6.295)$$

to the infinitesimal joint variables

$$\delta\mathbf{q} = [\delta q_1, \dots, \delta q_n] \quad (6.296)$$

and has thus dimension $m \times n$, where n is the number of joints, and m is the number of end-effector *DOF*.

When n is larger than m and \mathbf{J} has full rank, then there are $m - n$ redundancies in the system to which $m - n$ arbitrary variables correspond.

The Jacobian matrix \mathbf{J} also determines the relationship between end-effector velocities $\dot{\mathbf{X}}$ and joint velocities $\dot{\mathbf{q}}$

$$\dot{\mathbf{X}} = \mathbf{J}\dot{\mathbf{q}}. \quad (6.297)$$

This equation can be interpreted as a linear mapping from an m -dimensional vector space \mathbf{X} to an n -dimensional vector space \mathbf{q} . The subspace $\mathbb{R}(\mathbf{J})$ is the *range space* of the linear mapping, and represents all the possible end-effector velocities that can be generated by the n joints in the current configuration. \mathbf{J} has full row-rank, which means that the system does not present any singularity in that configuration, then the range space $\mathbb{R}(\mathbf{J})$ covers the entire vector space \mathbf{X} . Otherwise, there exists at least one direction in which the end-effector cannot be moved.

The null space $\mathbb{N}(\mathbf{J})$ represents the solutions of $\mathbf{J}\dot{\mathbf{q}} = 0$. Therefore, any vector $\dot{\mathbf{q}} \in \mathbb{N}(\mathbf{J})$ does not generate any motion for the end-effector.

If the manipulator has full rank, the dimension of the null space is then equal to the number $m - n$ of redundant *DOF*. When \mathbf{J} is degenerate, the dimension of $\mathbb{R}(\mathbf{J})$ decreases and the dimension of the null space increases by the same amount. Therefore,

$$\dim \mathbb{R}(\mathbf{J}) + \dim \mathbb{N}(\mathbf{J}) = n. \quad (6.298)$$

Configurations in which the Jacobian no longer has full rank, corresponds to singularities of the robot, which are generally of two types:

1. *Workspace boundary singularities* are those occurring when the manipulator is fully stretched out or folded back on itself. In this case, the end effector is near or at the workspace boundary.
2. *Workspace interior singularities* are those occurring away from the boundary. In this case, generally two or more axes line up.

Mathematically, singularity configurations can be found by calculating the conditions that make

$$|\mathbf{J}| = 0 \quad (6.299)$$

or

$$\left| \mathbf{J}\mathbf{J}^T \right| = 0. \quad (6.300)$$

Identification and avoidance of singularity configurations are very important in robotics. Some of the main reasons are:

1. Certain directions of motion may be unattainable.
2. Some of the joint velocities are infinite.
3. Some of the joint torques are infinite.

4. There will not exist a unique solution to the inverse kinematics problem.

Detecting the singular configurations using the Jacobian determinant may be a tedious task for complex robots. However, for robots having a spherical wrist, it is possible to split the singularity detection problem into two separate problems:

1. Arm singularities resulting from the motion of the manipulator arms.
2. Wrist singularities resulting from the motion of the wrist.

6.6 Summary

Inverse kinematics refers to determining the joint variables of a robot for a given position and orientation of the end-effector frame. The forward kinematics of a 6 *DOF* robot generates a 4×4 transformation matrix

$$\begin{aligned} {}^0T_6 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 {}^4T_5 {}^5T_6 \\ &= \begin{bmatrix} {}^0R_6 & {}^0\mathbf{d}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (6.301)$$

where only six elements out of the 12 elements of 0T_6 are independent. Therefore, the inverse kinematics reduces to finding the six independent elements for a given 0T_6 matrix.

Decoupling, inverse transformation, and iterative techniques are three applied methods for solving the inverse kinematics problem. In decoupling technique, the inverse kinematics of a robot with a spherical wrist can be decoupled into two subproblems: inverse position and inverse orientation kinematics. Practically, the tools transformation matrix 0T_7 is decomposed into three submatrices 0T_3 , 3T_6 , and 6T_7 .

$${}^0T_6 = {}^0T_3 {}^3T_6 {}^6T_7 \quad (6.302)$$

The matrix 0T_3 positions the wrist point and depends on the three manipulator joints' variables. The matrix 3T_6 is the wrist transformation matrix and the 6T_7 is the tools transformation matrix.

In inverse transformation technique, we extract equations with only one unknown from the following matrix equations, step by step.

$${}^1T_6 = {}^0T_1^{-1} {}^0T_6 \quad (6.303)$$

$${}^2T_6 = {}^1T_2^{-1} {}^1T_6 \quad (6.304)$$

$${}^3T_6 = {}^2T_3^{-1} {}^2T_6 \quad (6.305)$$

$${}^4T_6 = {}^3T_4^{-1} {}^3T_6 \quad (6.306)$$

$${}^5T_6 = {}^4T_5^{-1} {}^4T_6 \quad (6.307)$$

$$\mathbf{I} = {}^5T_6^{-1} {}^5T_6 \quad (6.308)$$

The iterative technique is a numerical method seeking to find the joint variable vector \mathbf{q} for a set of equations $\mathbf{T}(\mathbf{q}) = 0$.

6.7 Key Symbols

$\mathbf{0}$	null vector
a, b, c	coefficients of trigonometric equation
\mathbf{a}	turn vector of end-effector frame
A	local rotation transformation matrix
B	body coordinate frame
c	cos
d	joint distance
d_x, d_y, d_z	elements of \mathbf{d}
\mathbf{d}	translation vector, displacement vector
\mathbf{d}_{wrist}	wrist position vector
D	displacement transformation matrix
DH	Denavit-Hartenberg
DOF	degree of freedom
f_{ij}	the element of row i and column j of a matrix
g_{ij}	the element of row i and column j of a matrix
G, B_0	global coordinate frame, Base coordinate frame
$\mathbf{I} = [I]$	identity matrix
J	Jacobian
l	length
m	number of independent equations
n	number of links of a robot, number of joint variables
P	point
r, ϕ	parameters of trigonometric equation
\mathbf{r}	position vectors, homogeneous position vector
q	joint variable
\mathbf{q}	joint variables vector
r_i	the element i of \mathbf{r}
r_{ij}	the element of row i and column j of a matrix
R	rotation transformation matrix
s	sin
sgn	signum function
$SSRMS$	space station remote manipulator system
T	homogeneous transformation matrix
T_{arm}	manipulator transformation matrix
T_{wrist}	wrist transformation matrix
\mathbf{T}	a set of nonlinear algebraic equations of \mathbf{q}
x, y, z	local coordinate axes, local coordinates
X, Y, Z	global coordinate axes, global coordinates

Greek

δ	Kronecker function, small increment of a parameter
ϵ	small test number to terminate a procedure
θ	rotary joint angle
θ_{ijk}	$\theta_i + \theta_j + \theta_k$

Symbol

$[\]^{-1}$	inverse of the matrix $[\]$
$[\]^T$	transpose of the matrix $[\]$
\equiv	equivalent
\vdash	orthogonal
(i)	link number i
\parallel	parallel sign
\perp	perpendicular
\times	vector cross product
\mathbf{q}^\star	a guess value for \mathbf{q}
dim	dimension
\mathbb{N}	null space
\mathbb{R}	range space

Exercises

1. Notation and symbols.

Describe the meaning of:

$$\text{a- } \text{atan2}(a, b) \quad \text{b- } {}^0T_n \quad \text{c- } \mathbf{T}(\mathbf{q}) \quad \text{d- } \mathbf{q} \quad \text{e- } \mathbf{J}$$

2. $3R$ planar manipulator inverse kinematics.

Figure 5.21 illustrates an R||R||R planar manipulator. The forward kinematics of the manipulator generates the following matrices. Solve the inverse kinematics and find $\theta_1, \theta_2, \theta_3$ for given coordinates x_0, y_0 of the tip point and a given value of φ .

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & l_3 \cos \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & l_3 \sin \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & l_1 \cos \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & l_1 \sin \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. $2R$ manipulator tip point on a horizontal path.

Consider an elbow up planar $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1(1, 1.5)$ to $P_2(-1, 1.5)$.

- Divide the Cartesian path in 10 equal sections and determine the joint variables at the 11 points.
- ★ Calculate the joint variable θ_1 at P_1 and at P_2 . Divide the range of θ_1 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_1 .
- ★ Calculate the joint variable θ_2 at P_1 and at P_2 . Divide the range of θ_2 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_2 .

4. $2R$ manipulator tip point on a tilted path.

Consider an elbow up planar $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1(1, 1.5)$ to $P_2(-1, 1)$.

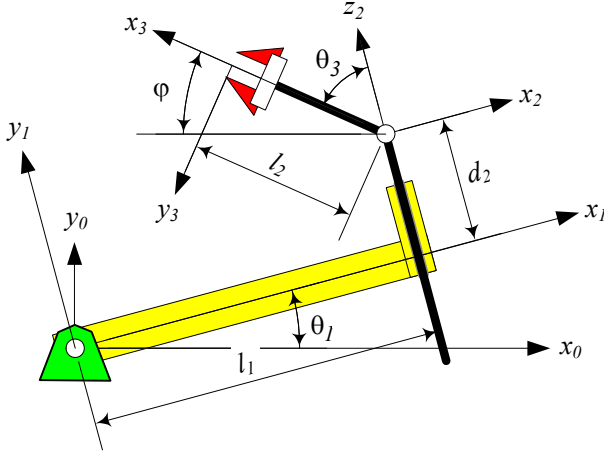


FIGURE 6.10. A planar manipulator.

- (a) Divide the Cartesian path in 10 equal sections and determine the joint variables at the 11 points.
 - (b) ★ Calculate the joint variable θ_1 at P_1 and at P_2 . Divide the range of θ_1 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_1 .
 - (c) ★ Calculate the joint variable θ_2 at P_1 and at P_2 . Divide the range of θ_1 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_2 .
5. $2R$ manipulator motion on a horizontal path.

Consider an elbow up planar $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1 (1, 1.5)$ to $P_2 (-1, 1.5)$ according to the following functions of time.

$$X = 1 - 6t^2 + 4t^3 \quad Y = 1.5$$

- (a) Calculate and plot θ_1 and θ_2 as functions of time if the time of motion is $0 \leq t \leq 1$.
 - (b) ★ Calculate and plot $\dot{\theta}_1$ and $\dot{\theta}_2$ as functions of time.
 - (c) ★ Calculate and plot $\ddot{\theta}_1$ and $\ddot{\theta}_2$ as functions of time.
 - (d) ★ Calculate and plot $\dddot{\theta}_1$ and $\dddot{\theta}_2$ as functions of time.
6. A planar manipulator.

Figure 6.10 illustrates a three DOF planar manipulator.

- (a) Determine the transformation matrices between coordinate frames.

- (b) Solve the forward kinematics and determine the coordinates X , Y , and φ of the end-effector frame B_3 for a given set of joint variables θ_1, d_2, θ_3 .
- (c) Solve the inverse kinematics and determine the joint variables θ_1, d_2, θ_3 for a given set of end-effector coordinates X, Y , and φ .

7. $2R$ manipulator motion on a horizontal path.

Consider a planar elbow up $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1(1, 1.5)$ to $P_2(-1, 1.5)$ with a constant speed.

$$X = 1 - vt \quad Y = 1.5$$

- (a) Calculate v and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 1$.
- (b) Calculate v and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 5$.
- (c) Calculate v and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 10$.
- (d) ★ Plot θ_1 and θ_2 as functions of v at point $(0, 1.5)$.

8. $2R$ manipulator motion on a horizontal path.

Consider a planar elbow up $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1(1, 1.5)$ to $P_2(-1, 1.5)$ with a constant acceleration.

$$X = 1 - \frac{1}{2}at^2 \quad Y = 1.5$$

- (a) Calculate a and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 1$.
- (b) Calculate a and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 5$.
- (c) Calculate a and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 10$.
- (d) ★ Plot θ_1 and θ_2 as functions of a at point $(0, 1.5)$.

9. ★ $2R$ manipulator kinematics on a tilted path.

Consider a planar elbow up $2R$ manipulator with $l_1 = l_2 = 1$. The tip point is moving on a straight line from $P_1(1, 1.5)$ to $P_2(-1, 1.5)$ with a constant speed.

$$X = 1 - vt \quad Y = 1.5$$

- (a) Calculate and plot θ_1 and θ_2 if the time of motion is $0 \leq t \leq 1$.
- (b) Calculate and plot $\dot{\theta}_1$ and $\dot{\theta}_2$ as functions of time.
- (c) Calculate and plot $\ddot{\theta}_1$ and $\ddot{\theta}_2$ as functions of time.
- (d) Calculate and plot $\ddot{\theta}_1$ and $\ddot{\theta}_2$ as functions of time.

10. Acceptable lengths of a $2R$ planar manipulator.

The tip point of a $2R$ planar manipulator is at $(1, 1.1)$.

- Assume $l_1 = 1$. Plot θ_1 and θ_2 versus l_2 and determine the range of possible l_2 for elbow up configuration.
- Assume $l_2 = 1$. Plot θ_1 and θ_2 versus l_1 and determine the range of possible l_1 for elbow up configuration.
- Assume $l_1 = 1$. Plot θ_1 and θ_2 versus l_2 and determine the range of possible l_2 for elbow down configuration.
- Assume $l_2 = 1$. Plot θ_1 and θ_2 versus l_1 and determine the range of possible l_1 for elbow down configuration.

11. $3R$ manipulator tip point on a straight path.

Consider a $3R$ articulated manipulator such as Figure 6.2 with $l_1 = 0.5$, $l_2 = l_3 = 1$. The tip point is moving on a straight line from $P_1(1.5, 0, 1)$ to $P_2(-1, 1, 1.5)$.

- Divide the Cartesian path into 10 equal sections and determine the joint variables at the 11 points.
- Calculate the joint variable θ_1 at P_1 and at P_2 . Divide the range of θ_1 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_2 and θ_3 .
- Calculate the joint variable θ_2 at P_1 and at P_2 . Divide the range of θ_2 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_1 and θ_3 .
- Calculate the joint variable θ_3 at P_1 and at P_2 . Divide the range of θ_3 into 10 equal sections and determine the coordinates of the tip point at the 11 values of θ_1 and θ_2 .

12. $3R$ manipulator motion on a straight path.

Consider a $3R$ articulated manipulator such as Figure 6.2 with $l_1 = 0.5$, $l_2 = l_3 = 1$. The tip point is moving on a straight line from $P_1(1.5, 0, 1)$ to $P_2(-1, 1, 1.5)$ according to the following functions of time.

$$\begin{aligned} X &= 1.5 - 0.025t^3 + 0.00375t^4 - 0.00015t^5 \\ Y &= 0.01t^3 - 0.0015t^4 + 0.00006t^5 \\ Z &= 1 + 0.005t^3 - 0.00075t^4 + 0.00003t^5 \end{aligned}$$

- Calculate and plot θ_1 , θ_2 and θ_3 if the time of motion is $0 \leq t \leq 1$.
- ★ Calculate and plot $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$ as functions of time.

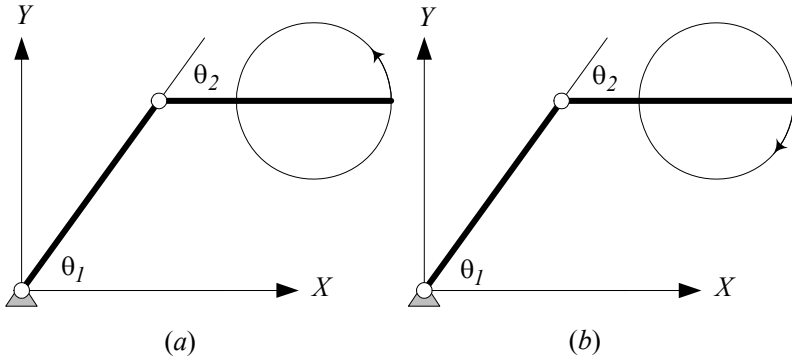


FIGURE 6.11. An elbow up $2R$ manipulator on a circular path.

- (c) ★ Calculate and plot $\ddot{\theta}_1$, $\ddot{\theta}_2$ and $\ddot{\theta}_3$ as functions of time.
 (d) ★ Calculate and plot $\dddot{\theta}_1$, $\dddot{\theta}_2$ and $\dddot{\theta}_3$ as functions of time.

13. An elbow up $2R$ manipulator on a circular path.

The $2R$ manipulator of Figure 6.11 has $l_2 = l_1 = 1$. The tip point of the manipulator is supposed to move on a circular path with a radius $R = 1/3$. Assume the manipulator starts moving when the second link is horizontal.

- (a) Plot θ_1 and θ_2 if the tip point is moving counterclockwise as shown in Figure 6.11(a).
 (b) Plot θ_1 and θ_2 if the tip point is moving clockwise as shown in Figure 6.11(b).

14. Acceptable lengths of a $3R$ manipulator.

The tip point of a $3R$ articulated manipulator is at $(1, 1.1, 0.5)$.

- (a) Assume $l_1 = l_3 = 1$. Plot θ_1 , θ_2 and θ_3 versus l_2 and determine the range of possible l_2 .
 (b) Assume $l_2 = l_3 = 1$. Plot θ_1 , θ_2 and θ_3 versus l_1 and determine the range of possible l_1 .
 (c) Assume $l_2 = l_1 = 1$. Plot θ_1 , θ_2 and θ_3 versus l_3 and determine the range of possible l_3 .

15. An elbow down $2R$ manipulator on a circular path.

The $2R$ manipulator of Figure 6.12 has $l_2 = l_1 = 1$. The tip point of the manipulator is supposed to move on a circular path with a radius $R = 1/3$. Assume the manipulator starts moving when the first link is horizontal.

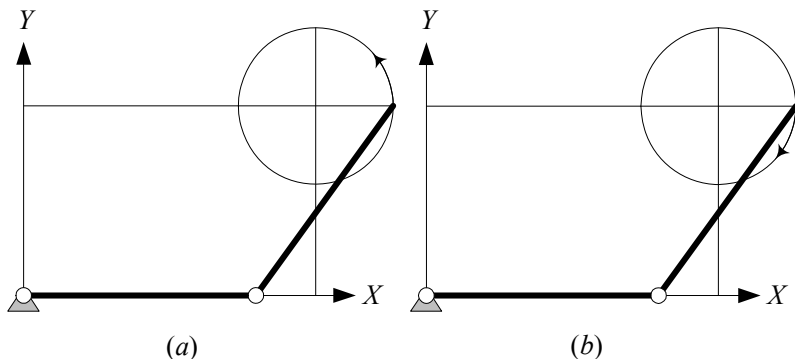


FIGURE 6.12. An elbow down 2R manipulator on a circular path.

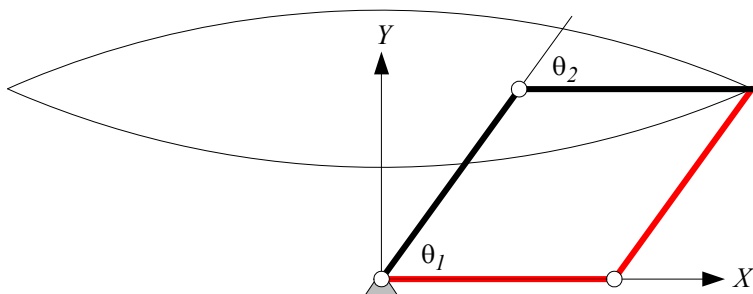


FIGURE 6.13. A 2R manipulator on a circular path.

- (a) Plot θ_1 and θ_2 if the tip point is moving counterclockwise as shown in Figure 6.12(a).
- (b) Plot θ_1 and θ_2 if the tip point is moving clockwise as shown in Figure 6.12(b).

16. A 2R manipulator on a circular path.

The 2R manipulator of Figure 6.13 has $l_2 = l_1 = 1$. The tip point of the manipulator is supposed to move on a circular path with a radius $R = 4$ and a center on Y -axis.

- (a) Assume the elbow up manipulator starts moving on the upper circular path when the second link is horizontal. Plot θ_1 and θ_2 until the first link becomes horizontal at the end of the path.
- (b) Assume the elbow down manipulator starts moving on the upper circular path when the first link is horizontal. Plot θ_1 and θ_2 until the first link becomes horizontal at the end of the path.

- (c) Assume the elbow up manipulator starts moving on the lower circular path when the second link is horizontal. Plot θ_1 and θ_2 until the first link becomes horizontal at the end of the path.
- (d) Assume the elbow down manipulator starts moving on the lower circular path when the first link is horizontal. Plot θ_1 and θ_2 until the first link becomes horizontal at the end of the path.

17. Spherical wrist inverse kinematics.

Figure 5.26 illustrates a spherical wrist with following transformation matrices. Assume that the frame B_3 is the base frame. Solve the inverse kinematics and find $\theta_4, \theta_5, \theta_6$ for a given 3T_6 .

$${}^3T_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4T_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^5T_6 = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

18. ★ Roll-Pitch-Yaw spherical wrist kinematics.

Attach the required *DH* coordinate frames to the Roll-Pitch-Yaw spherical wrist of Figure 5.30, similar to 5.28, and determine the forward and inverse kinematics of the wrist.

19. ★ Pitch-Yaw-Roll spherical wrist kinematics.

Attach the required coordinate *DH* frames to the Pitch-Yaw-Roll spherical wrist of Figure 5.31, similar to 5.28, and determine the forward and inverse kinematics of the wrist.

20. *SCARA* robot inverse kinematics.

Consider the R||R||R||P robot shown in Figure 5.23 with the following transformation matrices. Solve the inverse kinematics and find $\theta_1, \theta_2, \theta_3$ and d for a given 0T_4 .

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & l_1 \cos \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & l_1 \sin \theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
{}^2T_3 &= \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^3T_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_4 &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 \\
&= \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & l_1c\theta_1 + l_2c\theta_{12} \\ s\theta_{123} & c\theta_{123} & 0 & l_1s\theta_1 + l_2s\theta_{12} \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\theta_{123} &= \theta_1 + \theta_2 + \theta_3 & \theta_{12} &= \theta_1 + \theta_2
\end{aligned}$$

21. R†R||R articulated arm inverse kinematics.

Figure 5.22 illustrates 3 *DOF* R†R||R manipulator. Use the following transformation matrices and solve the inverse kinematics for θ_1 , θ_2 , θ_3 .

$$\begin{aligned}
{}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^1T_2 &= \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & l_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & l_2 \sin \theta_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^2T_3 &= \begin{bmatrix} \cos \theta_3 & 0 & \sin \theta_3 & 0 \\ \sin \theta_3 & 0 & -\cos \theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

22. Kinematics of a *PRRR* manipulator.

A *PRRR* manipulator is shown in Figure 6.14.

- Set up the links's coordinate frame according to standard *DH* rules.
- Determine the class of each link.
- Find the links' transformation matrices.
- Calculate the forward kinematics of the manipulator.
- Solve the inverse kinematics problem for the manipulator.

23. ★ Space station remote manipulator system inverse kinematics.

Shuttle remote manipulator system (*SSRMS*) is shown in Figure 5.24 schematically. The forward kinematics of the robot provides the

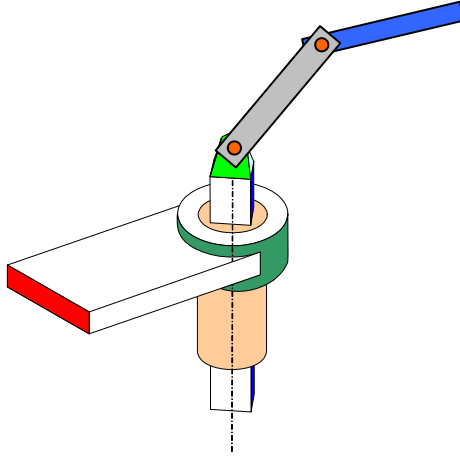


FIGURE 6.14. A PRRR manipulator.

following transformation matrices. Solve the inverse kinematics for the *SSRMS*.

$$\begin{aligned}
 {}^0T_1 &= \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^1T_2 &= \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^2T_3 &= \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3s\theta_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^3T_4 &= \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_4c\theta_4 \\ s\theta_4 & c\theta_4 & 0 & a_4s\theta_4 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^4T_5 &= \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^5T_6 &= \begin{bmatrix} c\theta_6 & 0 & -s\theta_6 & 0 \\ s\theta_6 & 0 & c\theta_6 & 0 \\ 0 & -1 & 0 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^6T_7 &= \begin{bmatrix} c\theta_7 & -s\theta_7 & 0 & 0 \\ s\theta_7 & c\theta_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

Hint: This robot is a one degree redundant robot. It has 7 joints which is one more than the required 6 *DOF* to reach a point at a desired orientation. To solve the inverse kinematics of this robot, we need to introduce one extra condition among the joint variables, or assign a value to one of the joint variables.

- (a) Assume $\theta_1 = 0$ and 1T_7 is given. Determine $\theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$.

- (b) Assume $\theta_2 = 0$ and 1T_7 is given. Determine $\theta_1, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$.
- (c) Assume $\theta_3 = 0$ and 1T_7 is given. Determine $\theta_1, \theta_2, \theta_4, \theta_5, \theta_6, \theta_7$.
- (d) Assume $\theta_5 = 0$ and 1T_7 is given. Determine $\theta_1, \theta_2, \theta_3, \theta_4, \theta_6, \theta_7$.
- (e) Assume $\theta_6 = 0$ and 1T_7 is given. Determine $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_7$.
- (f) Assume $\theta_7 = 0$ and 1T_7 is given. Determine $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$.
- (g) Determine $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$ such that f is minimized.

$$f = \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7$$