# 2

# Rotation Kinematics

Consider a rigid body with a fixed point. Rotation about the fixed point is the only possible motion of the body. We represent the rigid body by a body coordinate frame $B$, that rotates in another coordinate frame $G$, as is shown in Figure 2.1. We develop a rotation calculus based on transformation matrices to determine the orientation of $B$ in $G$, and relate the coordinates of a body point $P$ in both frames.

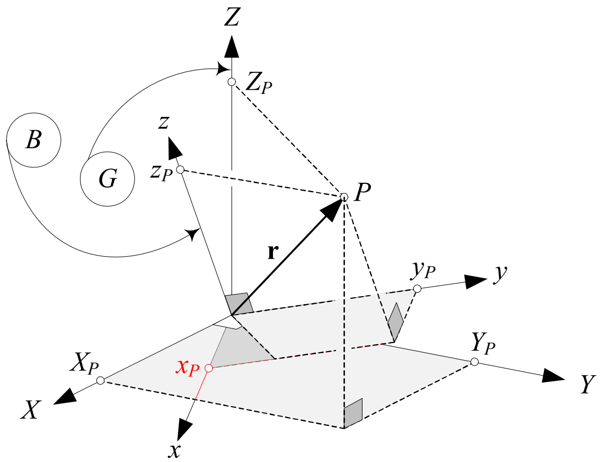


FIGURE 2.1. A rotated body frame $B$ in a fixed global frame $G$, about a fixed point at $O$.

## 2.1 Rotation About Global Cartesian Axes

Consider a rigid body $B$ with a local coordinate frame $Oxyz$ that is originally coincident with a global coordinate frame $OXYZ$. Point $O$ of the body $B$ is fixed to the ground $G$ and is the origin of both coordinate frames. If the rigid body $B$ rotates $\alpha$ degrees about the $Z$-axis of the global coordinate frame, then coordinates of any point $P$ of the rigid body in the local and global coordinate frames are related by the following equation

$$^{G}\mathbf{r} = Q_{Z,\alpha}\,{}^{B}\mathbf{r} \tag{2.1}$$

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where,

$$ {}^{G}\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad {}^{B}\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.2}$$

and

$$ Q_{Z,\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

Similarly, rotation $\beta$ degrees about the $Y$-axis, and $\gamma$ degrees about the $X$-axis of the global frame relate the local and global coordinates of point $P$ by the following equations

$$ {}^{G}\mathbf{r} = Q_{Y,\beta}\, {}^{B}\mathbf{r} \tag{2.4}$$

$$ {}^{G}\mathbf{r} = Q_{X,\gamma}\, {}^{B}\mathbf{r} \tag{2.5}$$

where,

$$ Q_{Y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.6}$$

$$ Q_{X,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}. \tag{2.7}$$

**Proof.** Let $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{I}, \hat{J}, \hat{K})$ be the unit vectors along the coordinate axes of $Oxyz$ and $OXYZ$ respectively. The rigid body has a space fixed point $O$, that is the common origin of $Oxyz$ and $OXYZ$. Figure 2.2 illustrates the top view of the system.

The initial position of a point $P$ is indicated by $P_1$. The position vector $\mathbf{r}_1$ of $P_1$ can be expressed in body and global coordinate frames by

$$ {}^{B}\mathbf{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} \tag{2.8}$$

$$ {}^{G}\mathbf{r}_1 = X_1\hat{I} + Y_1\hat{J} + Z_1\hat{K} \tag{2.9}$$

$$ \begin{aligned} x_1 &= X_1 \\ y_1 &= Y_1 \\ z_1 &= Z_1 \end{aligned} \tag{2.10}$$

where ${}^{B}\mathbf{r}_1$ refers to the position vector $\mathbf{r}_1$ expressed in the body coordinate frame $B$, and ${}^{G}\mathbf{r}_1$ refers to the position vector $\mathbf{r}_1$ expressed in the global coordinate frame $G$.

If the rigid body undergoes a rotation $\alpha$ about the $Z$-axis then, the local frame $Oxyz$, point $P$, and the position vector $\mathbf{r}$ will be seen in a

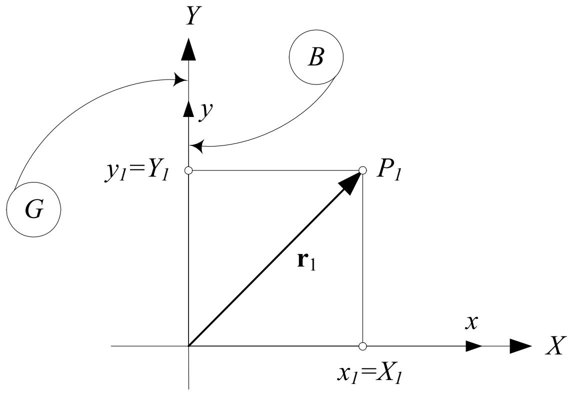


FIGURE 2.2. Position vector of point P when local and global frames are coincident.

second position, as shown in Figure 2.3. Now the position vector $\mathbf{r}_2$ of $P_2$ is expressed in both coordinate frames by

$$^B\mathbf{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} \tag{2.11}$$
$$^G\mathbf{r}_2 = X_2\hat{I} + Y_2\hat{J} + Z_2\hat{K}. \tag{2.12}$$

Using the definition of the inner product and Equation (2.11) we may write

$$X_2 = \hat{I}\cdot\mathbf{r}_2 = \hat{I}\cdot x_2\hat{\imath} + \hat{I}\cdot y_2\hat{\jmath} + \hat{I}\cdot z_2\hat{k} \tag{2.13}$$
$$Y_2 = \hat{J}\cdot\mathbf{r}_2 = \hat{J}\cdot x_2\hat{\imath} + \hat{J}\cdot y_2\hat{\jmath} + \hat{J}\cdot z_2\hat{k} \tag{2.14}$$
$$Z_2 = \hat{K}\cdot\mathbf{r}_2 = \hat{K}\cdot x_2\hat{\imath} + \hat{K}\cdot y_2\hat{\jmath} + \hat{K}\cdot z_2\hat{k} \tag{2.15}$$

or equivalently

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}. \tag{2.16}$$

The elements of the *Z-rotation matrix*, $Q_{Z,\alpha}$, are called the *direction cosines* of $^B\mathbf{r}_2$ with respect to $OXYZ$. Figure 2.4 shows the top view of the initial and final configurations of $\mathbf{r}$ in both coordinate systems $Oxyz$ and $OXYZ$. Analyzing Figure 2.4 indicates that

$$\begin{array}{lll} \hat{I}\cdot\hat{\imath} = \cos\alpha, & \hat{I}\cdot\hat{\jmath} = -\sin\alpha, & \hat{I}\cdot\hat{k} = 0 \\ \hat{J}\cdot\hat{\imath} = \sin\alpha, & \hat{J}\cdot\hat{\jmath} = \cos\alpha, & \hat{J}\cdot\hat{k} = 0 \\ \hat{K}\cdot\hat{\imath} = 0, & \hat{K}\cdot\hat{\jmath} = 0, & \hat{K}\cdot\hat{k} = 1. \end{array} \tag{2.17}$$

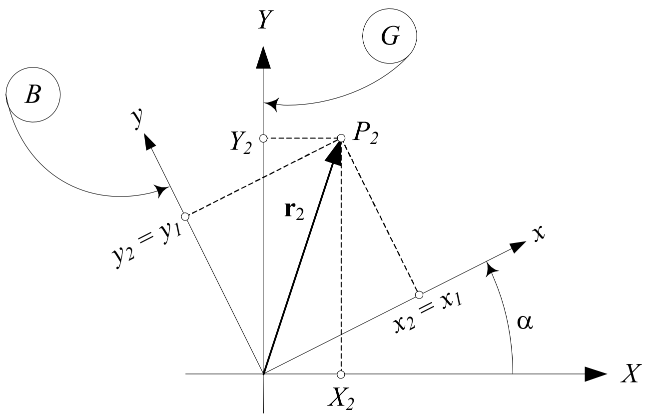


FIGURE 2.3. Position vector of point P when local frames are rotated about the Z-axis.

Combining Equations (2.16) and (2.17) shows that we can find the components of ${}^G\mathbf{r}_2$ by multiplying the $Z$-rotation matrix $Q_{Z,\alpha}$ and the vector ${}^B\mathbf{r}_2$.

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}. \tag{2.18}$$

It can also be shown in the following short notation

$${}^G\mathbf{r}_2 = Q_{Z,\alpha}\,{}^B\mathbf{r}_2 \tag{2.19}$$

where

$$Q_{Z,\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.20}$$

Equation (2.19) says that the vector $\mathbf{r}$ at the second position in the global coordinate frame is equal to $Q_Z$ times the position vector in the local coordinate frame. Hence, we are able to find the global coordinates of a point of a rigid body after rotation about the $Z$-axis, if we have its local coordinates.

Similarly, rotation $\beta$ about the $Y$-axis and rotation $\gamma$ about the $X$-axis are described by the $Y$-*rotation matrix* $Q_{Y,\beta}$ and the $X$-*rotation matrix* $Q_{X,\gamma}$ respectively.

$$Q_{Y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.21}$$

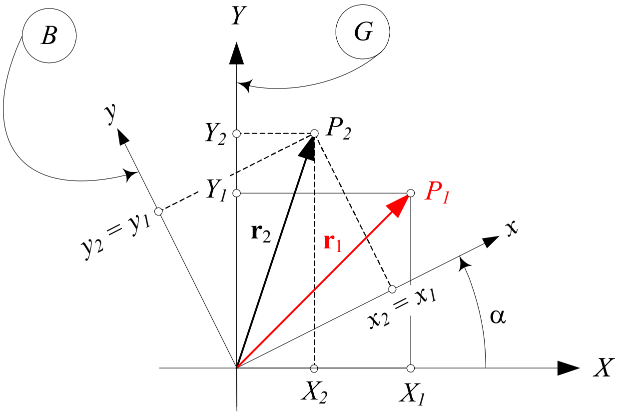


FIGURE 2.4. Position vectors of point P before and after the rotation of the local frame about the Z-axis of the global frame.

$$Q_{X,\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \tag{2.22}$$

The rotation matrices $Q_{Z,\alpha}$, $Q_{Y,\beta}$, and $Q_{X,\gamma}$ are called *basic global rotation matrices*. We usually refer to the first, second and third rotations about the axes of the global coordinate frame by $\alpha$, $\beta$, and $\gamma$ respectively. ■

**Example 3** *Successive rotation about global axes.*

*The final position of the corner* $P(5, 30, 10)$ *of the slab shown in Figure 2.5 after* 30 deg *rotation about the* $Z$*-axis, followed by* 30 deg *about the* $X$*-axis, and then* 90 deg *about the* $Y$*-axis can be found by first multiplying* $Q_{Z,30}$ *by* $[5, 30, 10]^T$ *to get the new global position after first rotation*

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} \tag{2.23}$$

*and then multiplying* $Q_{X,30}$ *and* $[-10.68, 28.48, 10.0]^T$ *to get the position of* $P$ *after the second rotation*

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} \tag{2.24}$$

*and finally multiplying* $Q_{Y,90}$ *and* $[-10.68, 19.66, 22.9]^T$ *to get the final position of* $P$ *after the third rotation. The slab and the point* $P$ *in first, second,*

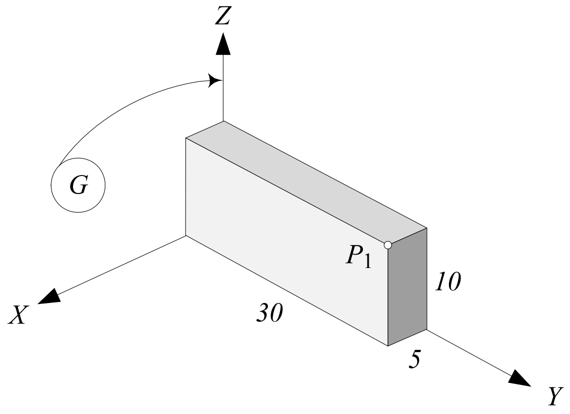


FIGURE 2.5. Corner $P$ of the slab at first position.

*third, and fourth positions are shown in Figure 2.6.*

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix} \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} = \begin{bmatrix} 22.90 \\ 19.66 \\ 10.68 \end{bmatrix} \tag{2.25}$$

**Example 4** *Time dependent global rotation.*

*Consider a rigid body $B$ that is continuously turning about the $Y$-axis of $G$ at a rate of* $0.3\,\text{rad/s}$. *The rotation transformation matrix of the body is:*

$$^{G}Q_B = \begin{bmatrix} \cos 0.3t & 0 & \sin 0.3t \\ 0 & 1 & 0 \\ -\sin 0.3t & 0 & \cos 0.3t \end{bmatrix} \tag{2.26}$$

*Any point of $B$ will move on a circle with radius $R = \sqrt{X^2 + Z^2}$ parallel to $(X, Z)$-plane.*

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= \begin{bmatrix} \cos 0.3t & 0 & \sin 0.3t \\ 0 & 1 & 0 \\ -\sin 0.3t & 0 & \cos 0.3t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} x\cos 0.3t + z\sin 0.3t \\ y \\ z\cos 0.3t - x\sin 0.3t \end{bmatrix} \end{aligned} \tag{2.27}$$

$$\begin{aligned} X^2 + Z^2 &= (x\cos 0.3t + z\sin 0.3t)^2 + (z\cos 0.3t - x\sin 0.3t)^2 \\ &= x^2 + z^2 = R^2 \end{aligned} \tag{2.28}$$

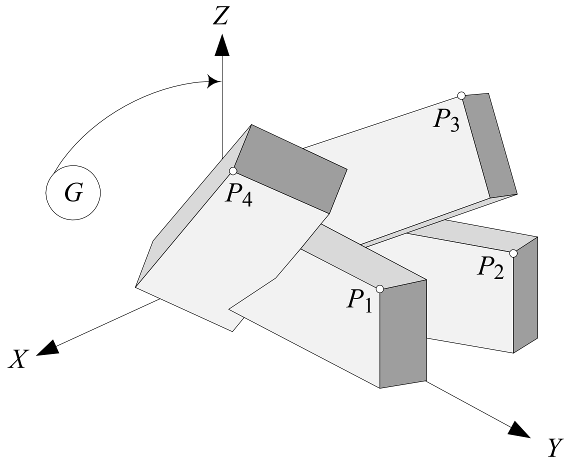


FIGURE 2.6. Corner $P$ and the slab at first, second, third, and final positions.

*Consider a point $P$ at* ${}^B\mathbf{r} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. *After* $t = 1\,\mathrm{s}$, *the point will be seen at:*

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos 0.3 & 0 & \sin 0.3 \\ 0 & 1 & 0 \\ -\sin 0.3 & 0 & \cos 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.955 \\ 0 \\ -0.295 \end{bmatrix} \tag{2.29}$$

*and after* $t = 2\,\mathrm{s}$, *at:*

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos 0.6 & -\sin 0.6 & 0 \\ \sin 0.6 & \cos 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.825 \\ 0.564 \\ 0 \end{bmatrix} \tag{2.30}$$

*We can find the global velocity of the body point $P$ by taking a time derivative of*

$$ {}^G\mathbf{r}_P = Q_{Y,\beta}\,{}^B\mathbf{r}_P \tag{2.31}$$

$$ Q_{Y,\beta} = \begin{bmatrix} \cos 0.3t & 0 & \sin 0.3t \\ 0 & 1 & 0 \\ -\sin 0.3t & 0 & \cos 0.3t \end{bmatrix}. \tag{2.32}$$

*Therefore, the global expression of its velocity vector is:*

$$ {}^G\mathbf{v}_P = \dot{Q}_{Y,\beta}\,{}^B\mathbf{r}_P = 0.3 \begin{bmatrix} z\cos 0.3t - x\sin 0.3t \\ 0 \\ -x\cos 0.3t - z\sin 0.3t \end{bmatrix} \tag{2.33}$$

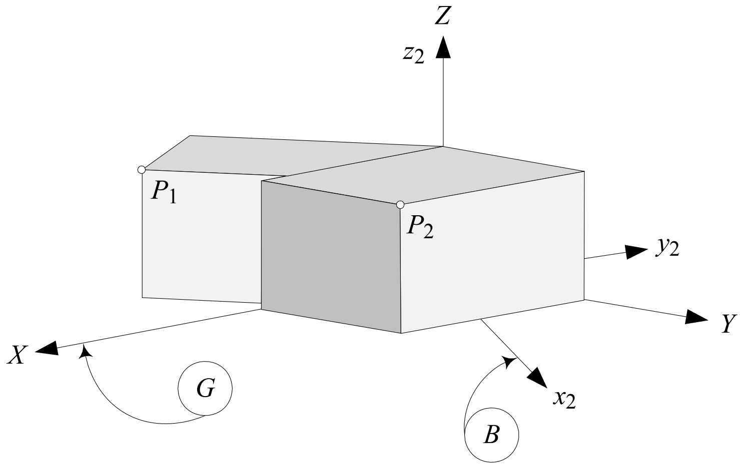


FIGURE 2.7. Positions of point $P$ in Example 5 before and after rotation.

**Example 5** *Global rotation, local position.*

*If a point $P$ is moved to ${}^G\mathbf{r}_2 = [4, 3, 2]^T$ after a $60\,\text{deg}$ rotation about the $Z$-axis, its position in the local coordinate is:*

$$
\begin{aligned}
{}^B\mathbf{r}_2 &= Q_{Z,60}^{-1}\, {}^G\mathbf{r}_2 \qquad (2.34)\\
\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} &= \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.60 \\ -1.95 \\ 2.0 \end{bmatrix}
\end{aligned}
$$

*The local coordinate frame was coincident with the global coordinate frame before rotation, thus the global coordinates of $P$ before rotation was also ${}^G\mathbf{r}_1 = [4.60, -1.95, 2.0]^T$. Positions of $P$ before and after rotation are shown in Figure 2.7.*

## 2.2 Successive Rotation About Global Cartesian Axes

The final global position of a point $P$ in a rigid body $B$ with position vector $\mathbf{r}$, after a sequence of rotations $Q_1$, $Q_2$, $Q_3$, ..., $Q_n$ about the global axes can be found by

$$ {}^G\mathbf{r} = {}^GQ_B\, {}^B\mathbf{r} \qquad (2.35) $$

where,

$$ {}^GQ_B = Q_n \cdots Q_3 Q_2 Q_1 \qquad (2.36) $$

and, ${}^{G}\mathbf{r}$ and ${}^{B}\mathbf{r}$ indicate the position vector $\mathbf{r}$ in the global and local coordinate frames. ${}^{G}Q_{B}$ is called the *global rotation matrix.* It maps the local coordinates to their corresponding global coordinates.

Since matrix multiplications do not commute the sequence of performing rotations is important. A rotation matrix is *orthogonal;* i.e., its transpose $Q^{T}$ is equal to its inverse $Q^{-1}$.

$$Q^{T}=Q^{-1} \tag{2.37}$$

Rotation about global coordinate axes is conceptually simple because the axes of rotations are fixed in space. Assume we have the coordinates of every point of a rigid body in the global frame that is equal to the local coordinates initially. The rigid body rotates about a global axis, then the proper global rotation matrix gives us the new global coordinate of the points. When we find the coordinates of points of the rigid body after the first rotation, our situation before the second rotation is similar to what we had before the first rotation.

**Example 6** *Successive global rotation matrix.*

*The global rotation matrix after a rotation $Q_{Z,\alpha}$ followed by $Q_{Y,\beta}$ and then $Q_{X,\gamma}$ is:*

$$\begin{aligned} {}^{G}Q_{B} &= Q_{X,\gamma}Q_{Y,\beta}Q_{Z,\alpha} \qquad (2.38)\\ &= \begin{bmatrix} c\alpha c\beta & -c\beta s\alpha & s\beta \\ c\gamma s\alpha + c\alpha s\beta s\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -c\beta s\gamma \\ s\alpha s\gamma - c\alpha c\gamma s\beta & c\alpha s\gamma + c\gamma s\alpha s\beta & c\beta c\gamma \end{bmatrix} \end{aligned}$$

**Example 7** *Successive global rotations, global position.*

*The end point $P$ of the arm shown in Figure 2.8 is located at:*

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ l\cos\theta \\ l\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1\cos 75 \\ 1\sin 75 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.26 \\ 0.97 \end{bmatrix} \tag{2.39}$$

*The rotation matrix to find the new position of the end point after* $-29\,\mathrm{deg}$ *rotation about the $X$-axis, followed by* $30\,\mathrm{deg}$ *about the $Z$-axis, and again* $132\,\mathrm{deg}$ *about the $X$-axis is*

$${}^{G}Q_{B} = Q_{X,132}Q_{Z,30}Q_{X,-29} = \begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix} \tag{2.40}$$

*and its new position is at:*

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 0.87 & -0.44 & -0.24 \\ -0.33 & -0.15 & -0.93 \\ 0.37 & 0.89 & -0.27 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.26 \\ 0.97 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.94 \\ -0.031 \end{bmatrix} \tag{2.41}$$

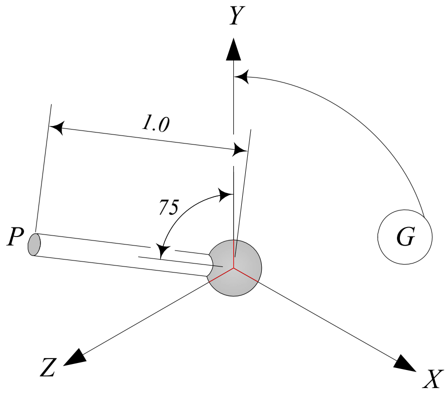


FIGURE 2.8. The arm of Example 7.

**Example 8** *Twelve independent triple global rotations.*

*Consider a rigid body in the final orientation after a series of rotation about global axes. We may transform its body coordinate frame $B$ from the coincident position with a global frame $G$ to any final orientation by only three rotations about the global axes provided that no two consequence rotations are about the same axis. In general, there are* 12 *different independent combinations of triple rotations about the global axes. They are:*

$$\begin{array}{c} 1 - Q_{X,\gamma}Q_{Y,\beta}Q_{Z,\alpha} \\ 2 - Q_{Y,\gamma}Q_{Z,\beta}Q_{X,\alpha} \\ 3 - Q_{Z,\gamma}Q_{X,\beta}Q_{Y,\alpha} \\ 4 - Q_{Z,\gamma}Q_{Y,\beta}Q_{X,\alpha} \\ 5 - Q_{Y,\gamma}Q_{X,\beta}Q_{Z,\alpha} \\ 6 - Q_{X,\gamma}Q_{Z,\beta}Q_{Y,\alpha} \\ 7 - Q_{X,\gamma}Q_{Y,\beta}Q_{X,\alpha} \\ 8 - Q_{Y,\gamma}Q_{Z,\beta}Q_{Y,\alpha} \\ 9 - Q_{Z,\gamma}Q_{X,\beta}Q_{Z,\alpha} \\ 10 - Q_{X,\gamma}Q_{Z,\beta}Q_{X,\alpha} \\ 11 - Q_{Y,\gamma}Q_{X,\beta}Q_{Y,\alpha} \\ 12 - Q_{Z,\gamma}Q_{Y,\beta}Q_{Z,\alpha} \end{array} \tag{2.42}$$

*The expanded form of the* 12 *global axes triple rotations are presented in Appendix A.*

**Example 9** *Order of rotation, and order of matrix multiplication.*

*Changing the order of global rotation matrices is equivalent to changing the order of rotations. The position of a point $P$ of a rigid body $B$ is located at ${}^B\mathbf{r}_P = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Its global position after rotation* $30\deg$ *about $X$-*

*axis and then* 45 deg *about* $Y$*-axis is at*

$$\begin{aligned}\left({}^{G}\mathbf{r}_P\right)_1 &= Q_{Y,45}\,Q_{X,30}\,{}^{B}\mathbf{r}_P \\ &= \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.76 \\ 3.27 \\ -1.64 \end{bmatrix}\end{aligned} \tag{2.43}$$

*and if we change the order of rotations then its position would be at:*

$$\begin{aligned}\left({}^{G}\mathbf{r}_P\right)_2 &= Q_{X,30}\,Q_{Y,45}\,{}^{B}\mathbf{r}_P \\ &= \begin{bmatrix} 0.53 & 0.0 & 0.85 \\ -0.84 & 0.15 & 0.52 \\ -0.13 & -0.99 & 0.081 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.08 \\ 1.02 \\ -1.86 \end{bmatrix}\end{aligned} \tag{2.44}$$

*These two final positions of* $P$ *are* $d = \left|\left({}^{G}\mathbf{r}_P\right)_1 - \left({}^{G}\mathbf{r}_P\right)_2\right| = 4.456$ *apart.*

**Example 10** ★ *Repeated rotation about global axes.*

*If we turn a body frame* $B$ *about* $X$*-axis* $\gamma$ rad, *where,*

$$\alpha = \frac{2\pi}{n} \qquad n \in \mathbb{N} \tag{2.45}$$

*then we need to repeat the rotation* $n$ *times to turn the body back to its original configuration. We can check it by multiplying* $Q_{X,\alpha}$ *by itself until we achieve an identity matrix. So, any body point of* $B$ *will be mapped to the same point in global frame. To show this, we may find that* $Q_{X,\alpha}$ *to the power* $m$ *as:*

$$\begin{aligned}Q^m_{X,\alpha} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ 0 & \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}^m \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos m\frac{2\pi}{n} & -\sin m\frac{2\pi}{n} \\ 0 & \sin m\frac{2\pi}{n} & \cos m\frac{2\pi}{n} \end{bmatrix}\end{aligned} \tag{2.46}$$

*If* $m = n$*, then we have an identity matrix.*

$$Q^n_{X,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos n\frac{2\pi}{n} & -\sin n\frac{2\pi}{n} \\ 0 & \sin n\frac{2\pi}{n} & \cos n\frac{2\pi}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.47}$$

*Repeated rotation about any other global axis provides the same result.*

*Let us now rotate B about two global axes repeatedly, such as turning $\alpha$ about Z-axis followed by a rotation $\gamma$ about X-axis, such that*

$$\alpha = \frac{2\pi}{n_1} \qquad \gamma = \frac{2\pi}{n_2} \qquad \{n_1, n_2\} \in \mathbb{N}. \tag{2.48}$$

*We may guess that repeating the rotations $n = n_1 \times n_2$ times will turn B back to its original configuration.*

$$[Q_{X,\gamma}\, Q_{Z,\alpha}]^{n_1 \times n_2} = [I] \tag{2.49}$$

*As an example consider $\alpha = \frac{2\pi}{3}$ and $\gamma = \frac{2\pi}{4}$. We need 13 times combined rotation to achieve the original configuration.*

$$^{G}Q_B = Q_{X,\gamma}\, Q_{Z,\alpha} = \begin{bmatrix} -0.5 & -0.866\,03 & 0 \\ 0 & 0 & -1.0 \\ 0.866\,03 & -0.5 & 0 \end{bmatrix} \tag{2.50}$$

$$^{G}Q_B^{13} = \begin{bmatrix} 0.9997 & -0.01922 & -0.01902 \\ 0.01902 & 0.99979 & -0.0112 \\ 0.01922 & 0.01086 & 0.9998 \end{bmatrix} \approx \mathbf{I} \tag{2.51}$$

*We may turn B back to its original configuration by lower number of combined rotations if $n_1$ and $n_2$ have a common divisor. For example if $n_1 = n_2 = 4$, we only need to apply the combined rotation three times. In a general case, determination of the required number n to repeat a general combined rotation $^{G}Q_B$ to turn back to the original orientation is an unsolved question.*

$$^{G}Q_B = \prod_{j=1}^{m} Q_{X_i,\alpha_j} \qquad i = 1, 2, 3 \tag{2.52}$$

$$\alpha_j = \frac{2\pi}{n_j} \qquad m, n_j \in \mathbb{N} \tag{2.53}$$

$$^{G}Q_B^{n} = [I] \qquad n = ? \tag{2.54}$$

## 2.3 Global Roll-Pitch-Yaw Angles

The rotation about the $X$-axis of the global coordinate frame is called a *roll*, the rotation about the $Y$-axis of the global coordinate frame is called a *pitch*, and the rotation about the $Z$-axis of the global coordinate frame is called a *yaw*. The global *roll-pitch-yaw rotation matrix* is:

$$\begin{aligned} ^{G}Q_B &= Q_{Z,\gamma} Q_{Y,\beta} Q_{X,\alpha} \\ &= \begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + c\gamma s\alpha s\beta & s\alpha s\gamma + c\alpha c\gamma s\beta \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -c\gamma s\alpha + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha c\beta \end{bmatrix} \end{aligned} \tag{2.55}$$

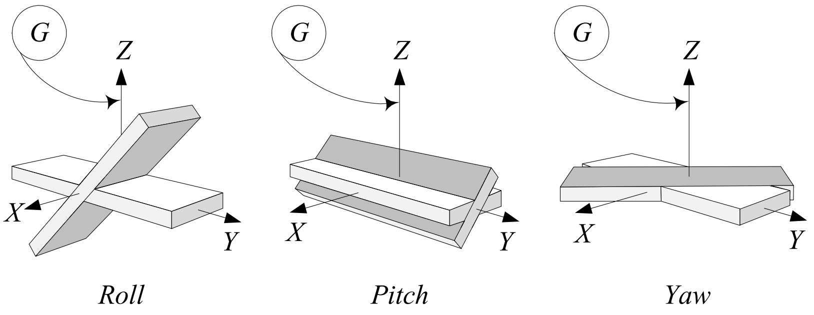


FIGURE 2.9. Global roll, pitch, and yaw rotations.

Figures 2.9 illustrates 45 deg roll, pitch, and yaw rotations about the axes of a global coordinate frame.

Given the roll, pitch, and yaw angles, we can compute the overall rotation matrix using Equation (2.55). Also we are able to compute the equivalent roll, pitch, and yaw angles when a rotation matrix is given. Suppose that $r_{ij}$ indicates the element of row $i$ and column $j$ of the roll-pitch-yaw rotation matrix (2.55), then the roll angle is

$$\alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) \tag{2.56}$$

and the pitch angle is

$$\beta = -\sin^{-1}(r_{31}) \tag{2.57}$$

and the yaw angle is

$$\gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) \tag{2.58}$$

provided that $\cos\beta \neq 0$.

**Example 11** *Determination of roll-pitch-yaw angles.*

*Let us determine the required roll-pitch-yaw angles to make the $x$-axis of the body coordinate $B$ parallel to $\mathbf{u}$, while $y$-axis remains in $(X, Y)$-plane.*

$$\mathbf{u} = \hat{I} + 2\hat{J} + 3\hat{K} \tag{2.59}$$

*Because $x$-axis must be along $\mathbf{u}$, we have*

$$^G\hat{\imath} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{14}}\hat{I} + \frac{2}{\sqrt{14}}\hat{J} + \frac{3}{\sqrt{14}}\hat{K} \tag{2.60}$$

*and because $y$-axis is in $(X, Y)$-plane, we have*

$$^G\hat{\jmath} = \left(\hat{I}\cdot\hat{\jmath}\right)\hat{I} + \left(\hat{J}\cdot\hat{\jmath}\right)\hat{J} = \cos\theta\hat{I} + \sin\theta\hat{J}. \tag{2.61}$$

*The axes* ${}^{G}\hat{\imath}$ *and* ${}^{G}\hat{\jmath}$ *must be orthogonal, therefore,*

$$\begin{aligned} \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} &= 0 && (2.62) \\ \theta &= -26.56\,\deg. && (2.63) \end{aligned}$$

*We may find* ${}^{G}\hat{k}$ *by a cross product.*

$${}^{G}\hat{k} = {}^{G}\hat{\imath} \times {}^{G}\hat{\jmath} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \times \begin{bmatrix} 0.894 \\ -0.447 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.358 \\ 0.717 \\ -0.597 \end{bmatrix} \tag{2.64}$$

*Hence, the transformation matrix* ${}^{G}Q_B$ *is:*

$${}^{G}Q_B = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & 0.894 & 0.358 \\ 2/\sqrt{14} & -0.447 & 0.717 \\ 3/\sqrt{14} & 0 & -0.597 \end{bmatrix} \tag{2.65}$$

*Now it is possible to determine the required roll-pitch-yaw angles to move the body coordinate frame B from the coincidence orientation with G to the final orientation.*

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) = \tan^{-1}\left(\frac{0}{-0.597}\right) = 0 && (2.66) \\ \beta &= -\sin^{-1}(r_{31}) = -\sin^{-1}\left(3/\sqrt{14}\right) \approx -0.93\,\text{rad} && (2.67) \\ \gamma &= \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) = \tan^{-1}\left(\frac{2/\sqrt{14}}{1/\sqrt{14}}\right) \approx 1.1071\,\text{rad} && (2.68) \end{aligned}$$

## 2.4 Rotation About Local Cartesian Axes

Consider a rigid body $B$ with a space fixed point at $O$. The local body coordinate frame $B(Oxyz)$ is coincident with a global coordinate frame $G(OXYZ)$, where the origin of both frames are on the fixed point $O$. If the body undergoes a rotation $\varphi$ about the $z$-axis of its local coordinate frame, as can be seen in the top view shown in Figure 2.10, then coordinates of any point of the rigid body in local and global coordinate frames are related by the following equation

$${}^{B}\mathbf{r} = A_{z,\varphi}\,{}^{G}\mathbf{r}. \tag{2.69}$$

The vectors ${}^{G}\mathbf{r}$ and ${}^{B}\mathbf{r}$ are the position vectors of the point in global and local frames respectively

$$\begin{aligned} {}^{G}\mathbf{r} &= \begin{bmatrix} X & Y & Z \end{bmatrix}^T && (2.70) \\ {}^{B}\mathbf{r} &= \begin{bmatrix} x & y & z \end{bmatrix}^T && (2.71) \end{aligned}$$

and $A_{z,\varphi}$ is the *z-rotation matrix.*

$$A_{z,\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.72}$$

Similarly, rotation $\theta$ about the $y$-axis and rotation $\psi$ about the $x$-axis are described by the *y-rotation matrix* $A_{y,\theta}$ and the *x-rotation matrix* $A_{x,\psi}$ respectively.

$$A_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.73}$$

$$A_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \tag{2.74}$$

**Proof.** Vector $\mathbf{r}$ indicates the position of a point $P$ of the rigid body $B$ where it is initially at $P_1$. Using the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ along the axes of local coordinate frame $B(Oxyz)$, and $(\hat{I}, \hat{J}, \hat{K})$ along the axes of global coordinate frame $G(OXYZ)$, the initial and final position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in both coordinate frames can be expressed by

$$^{B}\mathbf{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} \tag{2.75}$$
$$^{G}\mathbf{r}_1 = X_1\hat{I} + Y_1\hat{J} + Z_1\hat{K} \tag{2.76}$$

$$^{B}\mathbf{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} \tag{2.77}$$
$$^{G}\mathbf{r}_2 = X_2\hat{I} + Y_2\hat{J} + Z_2\hat{K}. \tag{2.78}$$

The vectors $^{B}\mathbf{r}_1$ and $^{B}\mathbf{r}_2$ are the initial and final positions of the vector $\mathbf{r}$ expressed in body coordinate frame $B(Oxyz)$, and $^{G}\mathbf{r}_1$ and $^{G}\mathbf{r}_2$ are the initial and final positions of the vector $\mathbf{r}$ expressed in the global coordinate frame $G(OXYZ)$.

The components of $^{B}\mathbf{r}_2$ can be found if we have the components of $^{G}\mathbf{r}_2$. Using Equation (2.78) and the definition of the inner product, we may write

$$x_2 = \hat{\imath}\cdot\mathbf{r}_2 = \hat{\imath}\cdot X_2\hat{I} + \hat{\imath}\cdot Y_2\hat{J} + \hat{\imath}\cdot Z_2\hat{K} \tag{2.79}$$
$$y_2 = \hat{\jmath}\cdot\mathbf{r}_2 = \hat{\jmath}\cdot X_2\hat{I} + \hat{\jmath}\cdot Y_2\hat{J} + \hat{\jmath}\cdot Z_2\hat{K} \tag{2.80}$$
$$z_2 = \hat{k}\cdot\mathbf{r}_2 = \hat{k}\cdot X_2\hat{I} + \hat{k}\cdot Y_2\hat{J} + \hat{k}\cdot Z_2\hat{K} \tag{2.81}$$

or equivalently

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\imath}\cdot\hat{J} & \hat{\imath}\cdot\hat{K} \\ \hat{\jmath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}. \tag{2.82}$$

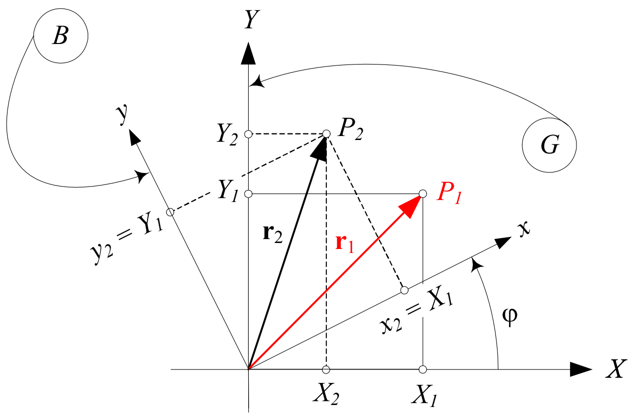


FIGURE 2.10. Position vectors of point $P$ before and after rotation of the local frame about the $z$-axis of the local frame.

The elements of the $z$-rotation matrix $A_{z,\varphi}$ are the *direction cosines* of ${}^G\mathbf{r}_2$ with respect to $Oxyz$. So, the elements of the matrix in Equation (2.82) are:

$$\begin{array}{lll} \hat{\imath}\cdot\hat{I}=\cos\varphi, & \hat{\imath}\cdot\hat{J}=\sin\varphi, & \hat{\imath}\cdot\hat{K}=0 \\ \hat{\jmath}\cdot\hat{I}=-\sin\varphi, & \hat{\jmath}\cdot\hat{J}=\cos\varphi, & \hat{\jmath}\cdot\hat{K}=0 \\ \hat{k}\cdot\hat{I}=0, & \hat{k}\cdot\hat{J}=0, & \hat{k}\cdot\hat{K}=1 \end{array} \tag{2.83}$$

Combining Equations (2.82) and (2.83), we can find the components of ${}^B\mathbf{r}_2$ by multiplying $z$-rotation matrix $A_{z,\varphi}$ and vector ${}^G\mathbf{r}_2$.

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \tag{2.84}$$

It can also be shown in the following short form

$${}^B\mathbf{r}_2 = A_{z,\varphi}\,{}^G\mathbf{r}_2 \tag{2.85}$$

where,

$$A_{z,\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.86}$$

Equation (2.85) says that after rotation about the $z$-axis of the local coordinate frame, the position vector in the local frame is equal to $A_{z,\varphi}$ times the position vector in the global frame. Hence, after rotation about the $z$-axis, we are able to find the coordinates of any point of a rigid body in local coordinate frame, if we have its coordinates in the global frame.

Similarly, rotation $\theta$ about the $y$-axis and rotation $\psi$ about the $x$-axis are described by the $y$-rotation matrix $A_{y,\theta}$ and the $x$-rotation matrix $A_{x,\psi}$ respectively.

$$A_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.87}$$

$$A_{x,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \tag{2.88}$$

We indicate the first, second, and third rotations about the local axes by $\varphi$, $\theta$, and $\psi$ respectively. ■

**Example 12** *Local rotation, local position.*

*If a local coordinate frame Oxyz has been rotated* 60 deg *about the z-axis and a point P in the global coordinate frame OXYZ is at* $(4, 3, 2)$, *its coordinates in the local coordinate frame Oxyz are:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 60 & \sin 60 & 0 \\ -\sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.60 \\ -1.97 \\ 2.0 \end{bmatrix} \tag{2.89}$$

**Example 13** *Local rotation, global position.*

*If a local coordinate frame Oxyz has been rotated* 60 deg *about the z-axis and a point P in the local coordinate frame Oxyz is at* $(4, 3, 2)$, *its position in the global coordinate frame OXYZ is at:*

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos 60 & \sin 60 & 0 \\ -\sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.60 \\ 4.96 \\ 2.0 \end{bmatrix} \tag{2.90}$$

**Example 14** *Successive local rotation, global position.*

*The arm shown in Figure 2.11 has two actuators. The first actuator rotates the arm* $-90$ deg *about y-axis and then the second actuator rotates the arm* 90 deg *about x-axis. If the end point P is at*

$$^B\mathbf{r}_P = \begin{bmatrix} 9.5 & -10.1 & 10.1 \end{bmatrix}^T \tag{2.91}$$

*then its position in the global coordinate frame is at:*

$$\begin{aligned} ^G\mathbf{r}_2 &= \left[A_{x,90}\, A_{y,-90}\right]^{-1}\, {}^B\mathbf{r}_P = A_{y,-90}^{-1}\, A_{x,90}^{-1}\, {}^B\mathbf{r}_P \\ &= A_{y,-90}^T\, A_{x,90}^T\, {}^B\mathbf{r}_P = \begin{bmatrix} 10.1 \\ -10.1 \\ 9.5 \end{bmatrix} \end{aligned} \tag{2.92}$$

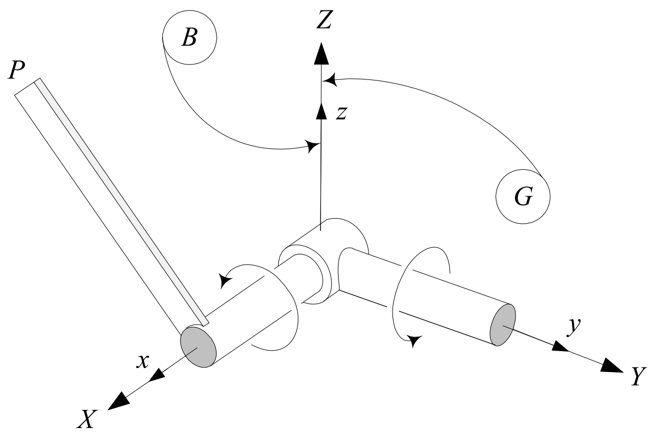


FIGURE 2.11. Arm of Example 14.

## 2.5 Successive Rotation About Local Cartesian Axes

The final global position of a point $P$ in a rigid body $B$ with position vector $\mathbf{r}$, after some rotations $A_1$, $A_2$, $A_3$, ..., $A_n$ about the local axes, can be found by

$$^B\mathbf{r} = {}^BA_G \, {}^G\mathbf{r} \tag{2.93}$$

where,

$$^BA_G = A_n \cdots A_3 A_2 A_1. \tag{2.94}$$

$^BA_G$ is called the *local rotation matrix* and it maps the global coordinates to their corresponding local coordinates.

Rotation about the local coordinate axis is conceptually interesting because in a sequence of rotations, each rotation is about one of the axes of the local coordinate frame, which has been moved to its new global position during the last rotation.

Assume that we have the coordinates of every point of a rigid body in a global coordinate frame. The rigid body and its local coordinate frame rotate about a local axis, then the proper local rotation matrix relates the new global coordinates of the points to the corresponding local coordinates. If we introduce an intermediate space-fixed frame coincident with the new position of the body coordinate frame, we may give the rigid body a second rotation about a local coordinate axis. Now another proper local rotation matrix relates the coordinates in the intermediate fixed frame to the corresponding local coordinates. Hence, the final global coordinates of the points must first be transformed to the intermediate fixed frame and second transformed to the original global axes.

**Example 15** *Successive local rotation, local position.*

*A local coordinate frame $B(Oxyz)$ that initially is coincident with a global coordinate frame $G(OXYZ)$ undergoes a rotation $\varphi = 30\deg$ about the $z$-axis, then $\theta = 30\deg$ about the $x$-axis, and then $\psi = 30\deg$ about the $y$-axis. The local coordinates of a point $P$ located at $X = 5$, $Y = 30$, $Z = 10$ can be found by $\begin{bmatrix} x & y & z \end{bmatrix}^T = A_{y,\psi}A_{x,\theta}A_{z,\varphi}\begin{bmatrix} 5 & 30 & 10 \end{bmatrix}^T$. The local rotation matrix is*

$$ {}^{B}A_G = A_{y,30}A_{x,30}A_{z,30} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \tag{2.95} $$

*and coordinates of $P$ in the local frame is:*

$$ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 18.35 \\ 25.35 \\ 7.0 \end{bmatrix} \tag{2.96} $$

**Example 16** *Successive local rotation.*

*The rotation matrix for a body point $P(x, y, z)$ after rotation $A_{z,\varphi}$ followed by $A_{x,\theta}$ and $A_{y,\psi}$ is:*

$$ \begin{aligned} {}^{B}A_G &= A_{y,\psi}A_{x,\theta}A_{z,\varphi} \\ &= \begin{bmatrix} c\varphi c\psi - s\theta s\varphi s\psi & c\psi s\varphi + c\varphi s\theta s\psi & -c\theta s\psi \\ -c\theta s\varphi & c\theta c\varphi & s\theta \\ c\varphi s\psi + s\theta c\psi s\varphi & s\varphi s\psi - c\varphi s\theta c\psi & c\theta c\psi \end{bmatrix} \end{aligned} \tag{2.97} $$

**Example 17** *Twelve independent triple local rotations.*

*Euler proved that: Any two independent orthogonal coordinate frames with a common origin can be related by a sequence of three rotations about the local coordinate axes, where no two successive rotations may be about the same axis. In general, there are 12 different independent combinations of triple rotation about local axes. They are:*

$$ \begin{array}{c} 1 - A_{x,\psi}A_{y,\theta}A_{z,\varphi} \\ 2 - A_{y,\psi}A_{z,\theta}A_{x,\varphi} \\ 3 - A_{z,\psi}A_{x,\theta}A_{y,\varphi} \\ 4 - A_{z,\psi}A_{y,\theta}A_{x,\varphi} \\ 5 - A_{y,\psi}A_{x,\theta}A_{z,\varphi} \\ 6 - A_{x,\psi}A_{z,\theta}A_{y,\varphi} \\ 7 - A_{x,\psi}A_{y,\theta}A_{x,\varphi} \\ 8 - A_{y,\psi}A_{z,\theta}A_{y,\varphi} \\ 9 - A_{z,\psi}A_{x,\theta}A_{z,\varphi} \\ 10 - A_{x,\psi}A_{z,\theta}A_{x,\varphi} \\ 11 - A_{y,\psi}A_{x,\theta}A_{y,\varphi} \\ 12 - A_{z,\psi}A_{y,\theta}A_{z,\varphi} \end{array} \tag{2.98} $$

*The expanded form of the 12 local axes' triple rotation are presented in Appendix B.*

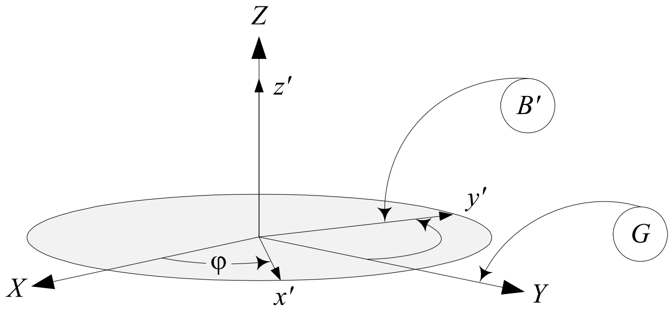


FIGURE 2.12. First Euler angle.

## 2.6 Euler Angles

The rotation about the $Z$-axis of the global coordinate is called *precession*, the rotation about the $x$-axis of the local coordinate is called *nutation*, and the rotation about the $z$-axis of the local coordinate is called *spin*. The *precession-nutation-spin rotation* angles are also called *Euler angles*. Euler angles rotation matrix has many application in rigid body kinematics. To find the Euler angles rotation matrix to go from the global frame $G(OXYZ)$ to the final body frame $B(Oxyz)$, we employ a body frame $B'(Ox'y'z')$ as shown in Figure 2.12 that before the first rotation coincides with the global frame. Let there be at first a rotation $\varphi$ about the $z'$-axis. Because $Z$-axis and $z'$-axis are coincident, we have:

$$^{B'}\mathbf{r} = {}^{B'}A_G \, {}^{G}\mathbf{r} \tag{2.99}$$

$$^{B'}A_G = A_{z,\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.100}$$

Next we consider the $B'(Ox'y'z')$ frame as a new fixed global frame and introduce a new body frame $B''(Ox''y''z'')$. Before the second rotation, the two frames coincide. Then, we execute a $\theta$ rotation about $x''$-axis as shown in Figure 2.13. The transformation between $B'(Ox'y'z')$ and $B''(Ox''y''z'')$ is:

$$^{B''}\mathbf{r} = {}^{B''}A_{B'} \, {}^{B'}\mathbf{r} \tag{2.101}$$

$$^{B''}A_{B'} = A_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{2.102}$$

Finally we consider the $B''(Ox''y''z'')$ frame as a new fixed global frame and consider the final body frame $B(Oxyz)$ to coincide with $B''$ before the

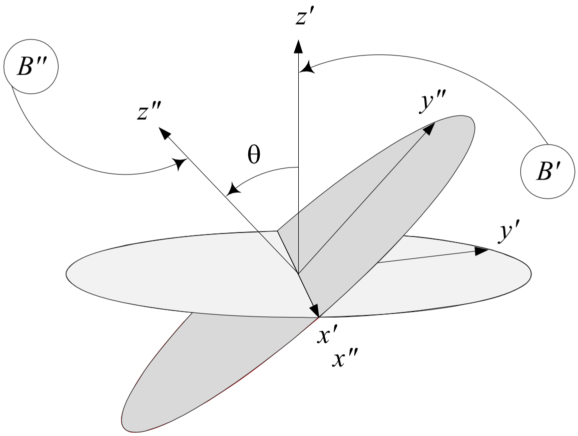


FIGURE 2.13. Second Euler angle.

third rotation. We now execute a $\psi$ rotation about the $z''$-axis as shown in Figure 2.14. The transformation between $B''(Ox''y''z'')$ and $B(Oxyz)$ is:

$$
\begin{aligned}
{}^{B}\mathbf{r} &= {}^{B}A_{B''}\,{}^{B''}\mathbf{r} && (2.103)\\
{}^{B}A_{B''} &= A_{z,\psi} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} && (2.104)
\end{aligned}
$$

By the rule of composition of rotations, the transformation from $G(OXYZ)$ to $B(Oxyz)$ is

$$
{}^{B}\mathbf{r} = {}^{B}A_G\,{}^{G}\mathbf{r} \tag{2.105}
$$

where,

$$
\begin{aligned}
{}^{B}A_G &= A_{z,\psi}A_{x,\theta}A_{z,\varphi} \\
&= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & c\psi s\varphi + c\theta c\varphi s\psi & s\theta s\psi \\ -c\varphi s\psi - c\theta c\psi s\varphi & -s\varphi s\psi + c\theta c\varphi c\psi & s\theta c\psi \\ s\theta s\varphi & -c\varphi s\theta & c\theta \end{bmatrix}
\end{aligned} \tag{2.106}
$$

and therefore,

$$
\begin{aligned}
{}^{G}Q_B &= {}^{B}A_G^{-1} = {}^{B}A_G^{T} = \left[A_{z,\psi}A_{x,\theta}A_{z,\varphi}\right]^T \\
&= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & -c\varphi s\psi - c\theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\varphi s\psi + c\theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}.
\end{aligned} \tag{2.107}
$$

Given the angles of precession $\varphi$, nutation $\theta$, and spin $\psi$, we can compute the overall rotation matrix using Equation (2.106). Also we are able

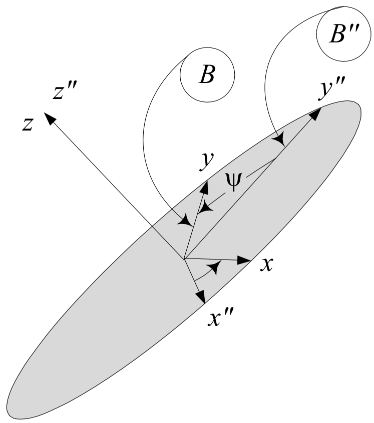


FIGURE 2.14. Third Euler angle.

to compute the equivalent precession, nutation, and spin angles when a rotation matrix is given.

If $r_{ij}$ indicates the element of row $i$ and column $j$ of the precession-nutation-spin rotation matrix (2.106), then,

$$\theta = \cos^{-1}(r_{33}) \tag{2.108}$$

$$\varphi = -\tan^{-1}\left(\frac{r_{31}}{r_{32}}\right) \tag{2.109}$$

$$\psi = \tan^{-1}\left(\frac{r_{13}}{r_{23}}\right) \tag{2.110}$$

provided that $\sin\theta \neq 0$.

**Example 18** *Euler angle rotation matrix.*

*The Euler or precession-nutation-spin rotation matrix for* $\varphi = 79.15\,\text{deg}$, $\theta = 41.41\,\text{deg}$, *and* $\psi = -40.7\,\text{deg}$ *would be found by substituting* $\varphi$, $\theta$, *and* $\psi$ *in Equation (2.106).*

$$\begin{aligned} {}^{B}A_{G} &= A_{z,-40.7}A_{x,41.41}A_{z,79.15} \\ &= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \end{aligned} \tag{2.111}$$

**Example 19** *Euler angles of a local rotation matrix.*

*The local rotation matrix after rotation* $30\,\text{deg}$ *about the z-axis, then rotation* $30\,\text{deg}$ *about the x-axis, and then* $30\,\text{deg}$ *about the y-axis is*

$$\begin{aligned} {}^{B}A_{G} &= A_{y,30}A_{x,30}A_{z,30} \\ &= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \end{aligned} \tag{2.112}$$

*and therefore, the local coordinates of a sample point at* $X = 5$, $Y = 30$, $Z = 10$ *are:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.50 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 18.35 \\ 25.35 \\ 7.0 \end{bmatrix} \tag{2.113}$$

*The Euler angles of the corresponding precession-nutation-spin rotation matrix are:*

$$\begin{aligned} \theta &= \cos^{-1}(0.75) = 41.41 \deg \\ \varphi &= -\tan^{-1}\left(\frac{0.65}{-0.125}\right) = 79.15 \deg \\ \psi &= \tan^{-1}\left(\frac{-0.43}{0.50}\right) = -40.7 \deg \end{aligned} \tag{2.114}$$

*Hence,* $A_{y,30}A_{x,30}A_{z,30} = A_{z,\psi}A_{x,\theta}A_{z,\varphi}$ *when* $\varphi = 79.15 \deg$, $\theta = 41.41 \deg$, *and* $\psi = -40.7 \deg$. *In other words, the rigid body attached to the local frame moves to the final configuration by undergoing either three consecutive rotations* $\varphi = 79.15 \deg$, $\theta = 41.41 \deg$, *and* $\psi = -40.7 \deg$ *about* $z$, $x$, *and* $z$ *axes respectively, or three consecutive rotations* $30 \deg$, $30 \deg$, *and* $30 \deg$ *about* $z$, $x$, *and* $y$ *axes.*

**Example 20** *Relative rotation matrix of two bodies.*

*Consider a rigid body* $B_1$ *with an orientation matrix* ${}^{B_1}A_G$ *made by Euler angles* $\varphi = 30 \deg$, $\theta = -45 \deg$, $\psi = 60 \deg$, *and another rigid body* $B_2$ *having* $\varphi = 10 \deg$, $\theta = 25 \deg$, $\psi = -15 \deg$, *with respect to the global frame. To find the relative rotation matrix* ${}^{B_1}A_{B_2}$ *to map the coordinates of second body frame* $B_2$ *to the first body frame* $B_1$, *we need to find the individual rotation matrices first.*

$$\begin{aligned} {}^{B_1}A_G &= A_{z,60}A_{x,-45}A_{z,30} \\ &= \begin{bmatrix} 0.127 & 0.78 & -0.612 \\ -0.927 & -0.127 & -0.354 \\ -0.354 & 0.612 & 0.707 \end{bmatrix} \end{aligned} \tag{2.115}$$

$$\begin{aligned} {}^{B_2}A_G &= A_{z,10}A_{x,25}A_{z,-15} \\ &= \begin{bmatrix} 0.992 & -0.0633 & -0.109 \\ 0.103 & 0.907 & 0.408 \\ 0.0734 & -0.416 & 0.906 \end{bmatrix} \end{aligned} \tag{2.116}$$

*The desired rotation matrix* ${}^{B_1}A_{B_2}$ *may be found by*

$${}^{B_1}A_{B_2} = {}^{B_1}A_G \, {}^{G}A_{B_2} \tag{2.117}$$

*which is equal to:*

$$\begin{aligned} {}^{B_1}A_{B_2} &= {}^{B_1}A_G \, {}^{B_2}A_G^T \\ &= \begin{bmatrix} 0.992 & 0.103 & 0.0734 \\ -0.0633 & 0.907 & -0.416 \\ -0.109 & 0.408 & 0.906 \end{bmatrix} \end{aligned} \tag{2.118}$$

**Example 21** *Euler angles rotation matrix for small angles.*

*The Euler rotation matrix* ${}^{B}A_G = A_{z,\psi}A_{x,\theta}A_{z,\varphi}$ *for very small Euler angles* $\varphi, \theta,$ *and* $\psi$ *is approximated by*

$${}^{B}A_G = \begin{bmatrix} 1 & \gamma & 0 \\ -\gamma & 1 & \theta \\ 0 & -\theta & 1 \end{bmatrix} \tag{2.119}$$

*where,*

$$\gamma = \varphi + \psi. \tag{2.120}$$

*Therefore, in case of small angles of rotation, the angles* $\varphi$ *and* $\psi$ *are indistinguishable.*

**Example 22** *Small second Euler angle.*

*If* $\theta \to 0$ *then the Euler rotation matrix* ${}^{B}A_G = A_{z,\psi}A_{x,\theta}A_{z,\varphi}$ *approaches to*

$$\begin{aligned} {}^{B}A_G &= \begin{bmatrix} c\varphi c\psi - s\varphi s\psi & c\psi s\varphi + c\varphi s\psi & 0 \\ -c\varphi s\psi - c\psi s\varphi & -s\varphi s\psi + c\varphi c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\varphi+\psi) & \sin(\varphi+\psi) & 0 \\ -\sin(\varphi+\psi) & \cos(\varphi+\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{2.121}$$

*and therefore, the angles* $\varphi$ *and* $\psi$ *are indistinguishable even if the value of* $\varphi$ *and* $\psi$ *are finite. Hence, the Euler set of angles in rotation matrix (2.106) is not unique when* $\theta = 0$.

**Example 23** *Euler angles application in motion of rigid bodies.*

*The* $zxz$ *Euler angles are good parameters to describe the configuration of a rigid body with a fixed point. The Euler angles to show the configuration of a top are shown in Figure 2.15 as an example.*

**Example 24** ★ *Angular velocity vector in terms of Euler frequencies.*

*A Eulerian local frame* $E(o, \hat{e}_\varphi, \hat{e}_\theta, \hat{e}_\psi)$ *can be introduced by defining unit vectors* $\hat{e}_\varphi$, $\hat{e}_\theta$, *and* $\hat{e}_\psi$ *as shown in Figure 2.16. Although the Eulerian frame is not necessarily orthogonal, it is very useful in rigid body kinematic analysis.*

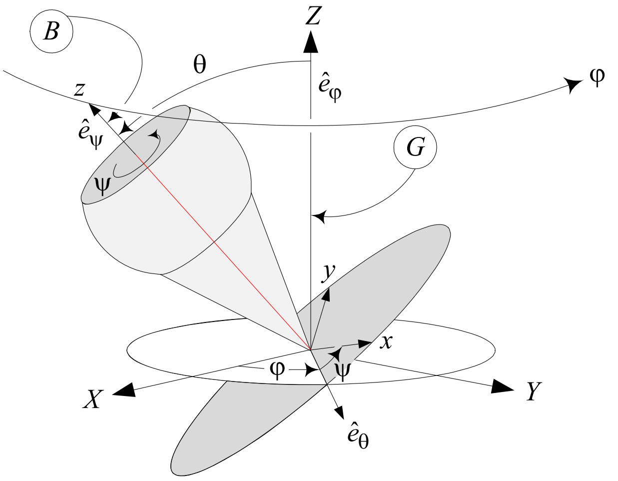


FIGURE 2.15. Application of Euler angles in describing the configuration of a top.

*The angular velocity vector* ${}_G\boldsymbol{\omega}_B$ *of the body frame* $B(Oxyz)$ *with respect to the global frame* $G(OXYZ)$ *can be written in Euler angles frame* $E$ *as the sum of three Euler angle rate vectors.*

$$ {}_G^E\boldsymbol{\omega}_B = \dot{\varphi}\hat{e}_\varphi + \dot{\theta}\hat{e}_\theta + \dot{\psi}\hat{e}_\psi \tag{2.122} $$

*where, the rate of Euler angles,* $\dot{\varphi}$, $\dot{\theta}$, *and* $\dot{\psi}$ *are called Euler frequencies.*

*To find* ${}_G\boldsymbol{\omega}_B$ *in body frame we must express the unit vectors* $\hat{e}_\varphi$, $\hat{e}_\theta$, *and* $\hat{e}_\psi$ *shown in Figure 2.16, in the body frame. The unit vector* $\hat{e}_\varphi = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T = \hat{K}$ *is in the global frame and can be transformed to the body frame after three rotations.*

$$ {}^B\hat{e}_\varphi = {}^BA_G\,\hat{K} = A_{z,\psi}A_{x,\theta}A_{z,\varphi}\hat{K} = \begin{bmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{bmatrix} \tag{2.123} $$

*The unit vector* $\hat{e}_\theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \hat{\imath}'$ *is in the intermediate frame* $Ox'y'z'$ *and needs to get two rotations* $A_{x,\theta}$ *and* $A_{z,\psi}$ *to be transformed to the body frame.*

$$ {}^B\hat{e}_\theta = {}^BA_{Ox'y'z'}\,\hat{\imath}' = A_{z,\psi}\,A_{x,\theta}\,\hat{\imath}' = \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} \tag{2.124} $$

*The unit vector* $\hat{e}_\psi$ *is already in the body frame,* $\hat{e}_\psi = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T = \hat{k}$.

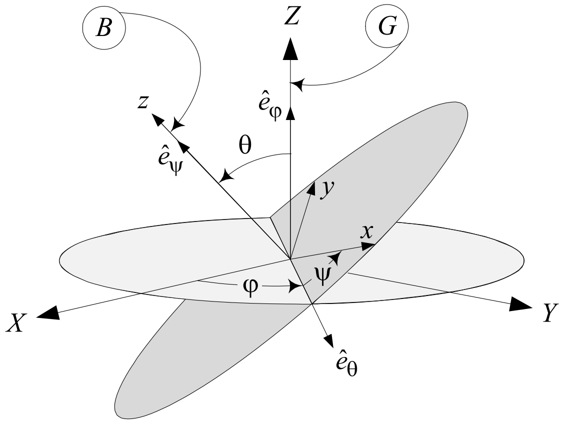


FIGURE 2.16. Euler angles frame $\hat{e}_\varphi, \hat{e}_\theta, \hat{e}_\psi$.

*Therefore,* ${}_G\boldsymbol{\omega}_B$ *is expressed in body coordinate frame as*

$$\begin{aligned} {}^B_G\boldsymbol{\omega}_B &= \dot{\varphi}\begin{bmatrix} \sin\theta\sin\psi \\ \sin\theta\cos\psi \\ \cos\theta \end{bmatrix} + \dot{\theta}\begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)\hat{\imath} + \left(\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)\hat{\jmath} \\ &\quad + \left(\dot{\varphi}\cos\theta + \dot{\psi}\right)\hat{k} \end{aligned} \tag{2.125}$$

*and therefore, components of* ${}_G\boldsymbol{\omega}_B$ *in body frame* $Oxyz$ *are related to the Euler angle frame* $O\varphi\theta\psi$ *by the following relationship:*

$${}^B_G\boldsymbol{\omega}_B = {}^BA_E\,{}^E_G\boldsymbol{\omega}_B \tag{2.126}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{2.127}$$

*Then,* ${}_G\boldsymbol{\omega}_B$ *can be expressed in the global frame using an inverse transformation of Euler rotation matrix (2.106)*

$$\begin{aligned} {}^G_G\boldsymbol{\omega}_B &= {}^BA_G^{-1}\,{}^B_G\boldsymbol{\omega}_B = {}^BA_G^{-1}\begin{bmatrix} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{bmatrix} \\ &= \left(\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi\right)\hat{I} + \left(\dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta\right)\hat{J} \\ &\quad + \left(\dot{\varphi} + \dot{\psi}\cos\theta\right)\hat{K} \end{aligned} \tag{2.128}$$

*and hence, components of* ${}_G\boldsymbol{\omega}_B$ *in global coordinate frame* $OXYZ$ *are related to the Euler angle coordinate frame* $O\varphi\theta\psi$ *by the following relation-*

*ship.*

$$ {}^G_G\boldsymbol{\omega}_B = {}^G Q_E \, {}^E_G\boldsymbol{\omega}_B \tag{2.129} $$

$$ \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = \begin{bmatrix} 0 & \cos\varphi & \sin\theta\sin\varphi \\ 0 & \sin\varphi & -\cos\varphi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix} \tag{2.130} $$

**Example 25** ★ *Euler frequencies based on a Cartesian angular velocity vector.*

*The vector ${}^B_G\boldsymbol{\omega}_B$, that indicates the angular velocity of a rigid body $B$ with respect to the global frame $G$ written in frame $B$, is related to the Euler frequencies by*

$$ {}^B_G\boldsymbol{\omega}_B = {}^B A_E \, {}^E_G\boldsymbol{\omega}_B \tag{2.131} $$

$$ {}^B_G\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix}. \tag{2.132} $$

*The matrix of coefficients is not an orthogonal matrix because,*

$$ {}^B A_E^T \neq {}^B A_E^{-1} \tag{2.133} $$

$$ {}^B A_E^T = \begin{bmatrix} \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.134} $$

$$ {}^B A_E^{-1} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & 0 \\ -\cos\theta\sin\psi & -\cos\theta\cos\psi & 1 \end{bmatrix}. \tag{2.135} $$

*It is because the Euler angles coordinate frame $O\varphi\theta\psi$ is not an orthogonal frame. For the same reason, the matrix of coefficients that relates the Euler frequencies and the components of ${}^G_G\boldsymbol{\omega}_B$*

$$ {}^G_G\boldsymbol{\omega}_B = {}^G Q_E \, {}^E_G\boldsymbol{\omega}_B \tag{2.136} $$

$$ {}^G_G\boldsymbol{\omega}_B = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = \begin{bmatrix} 0 & \cos\varphi & \sin\theta\sin\varphi \\ 0 & \sin\varphi & -\cos\varphi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix} \tag{2.137} $$

*is not an orthogonal matrix. Therefore, the Euler frequencies based on local and global decomposition of the angular velocity vector ${}_G\boldsymbol{\omega}_B$ must solely be found by the inverse of coefficient matrices*

$$ {}^E_G\boldsymbol{\omega}_B = {}^B A_E^{-1} \, {}^B_G\boldsymbol{\omega}_B \tag{2.138} $$

$$ \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & 0 \\ -\cos\theta\sin\psi & -\cos\theta\cos\psi & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2.139} $$

*and*

$$\begin{aligned} {}^E_G\boldsymbol{\omega}_B &= {}^GQ_E^{-1}\,{}^G_G\boldsymbol{\omega}_B && (2.140)\\ \begin{bmatrix}\dot{\varphi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} &= \frac{1}{\sin\theta}\begin{bmatrix}-\cos\theta\sin\varphi & \cos\theta\cos\varphi & 1\\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & 0\\ \sin\varphi & -\cos\varphi & 0\end{bmatrix}\begin{bmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{bmatrix}. && (2.141)\end{aligned}$$

*Using (2.138) and (2.140), it can be verified that the transformation matrix* ${}^BA_G = {}^BA_E\,{}^GQ_E^{-1}$ *would be the same as Euler transformation matrix (2.106).*

*The angular velocity vector can thus be expressed as:*

$$\begin{aligned} {}_G\boldsymbol{\omega}_B &= \begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\hat{I} & \hat{J} & \hat{K}\end{bmatrix}\begin{bmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{bmatrix}\\ &= \begin{bmatrix}\hat{K} & \hat{e}_\theta & \hat{k}\end{bmatrix}\begin{bmatrix}\dot{\varphi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} && (2.142)\end{aligned}$$

**Example 26** ★ *Integrability of the angular velocity components.*

*The integrability condition for an arbitrary total differential of* $f = f(x, y)$

$$df = f_1 dx + f_2 dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{2.143}$$

*is:*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \tag{2.144}$$

*The angular velocity components* $\omega_x$, $\omega_y$, *and* $\omega_z$ *along the body coordinate axes* $x$, $y$, *and* $z$ *can not be integrated to obtain the associated angles because*

$$\omega_x dt = \sin\theta\sin\psi\, d\varphi + \cos\psi\, d\theta \tag{2.145}$$

*and*

$$\frac{\partial(\sin\theta\sin\psi)}{\partial\theta} \neq \frac{\partial\cos\psi}{\partial\varphi}. \tag{2.146}$$

*However, the integrability condition (2.144) is satisfied by the Euler frequencies. From (2.139), we have:*

$$\begin{aligned} d\varphi &= \frac{\sin\psi}{\sin\theta}(\omega_x\, dt) + \frac{\cos\psi}{\sin\theta}(\omega_y\, dt) && (2.147)\\ d\theta &= \cos\psi\,(\omega_x\, dt) - \sin\psi\,(\omega_y\, dt) && (2.148)\\ d\psi &= \frac{-\cos\theta\sin\psi}{\sin\theta}(\omega_x\, dt) + \frac{-\cos\theta\cos\psi}{\sin\theta}(\omega_y\, dt) + \frac{(\omega_z\, dt)}{\sin\theta} && (2.149)\end{aligned}$$

*For example, the second equation indicates that*

$$\cos\psi = \frac{\partial\theta}{\partial(\omega_x\, dt)} \qquad -\sin\psi = \frac{\partial\theta}{\partial(\omega_y\, dt)} \tag{2.150}$$

*and therefore,*

$$\frac{\partial\left(\cos\psi\right)}{\partial\left(\omega_y\,dt\right)} = -\sin\psi\frac{\partial\psi}{\partial\left(\omega_y\,dt\right)} = \frac{\sin\psi\cos\theta\cos\psi}{\sin\theta} \tag{2.151}$$

$$\frac{\partial\left(-\sin\psi\right)}{\partial\left(\omega_x\,dt\right)} = -\cos\psi\frac{\partial\psi}{\partial\left(\omega_x\,dt\right)} = \frac{\sin\psi\cos\theta\cos\psi}{\sin\theta} \tag{2.152}$$

*It can be checked that $d\varphi$ and $d\psi$ are also integrable.*

**Example 27** ★ *Cardan angles and frequencies.*

*The system of Euler angles is singular at $\theta = 0$, and as a consequence, $\varphi$ and $\psi$ become coplanar and indistinguishable. From 12 angle systems of Appendix B, each with certain names, characteristics, advantages, and disadvantages, the rotations about three different axes such as ${}^{B}A_{G} = A_{z,\psi}A_{y,\theta}A_{x,\varphi}$ are called Cardan or Bryant angles. The Cardan angle system is not singular at $\theta = 0$, and has some application in mechatronics and attitude analysis of satellites in a central force field.*

$$ {}^{B}A_{G} = \begin{bmatrix} c\theta c\psi & c\varphi s\psi + s\theta c\psi s\varphi & s\varphi s\psi - c\varphi s\theta c\psi \\ -c\theta s\psi & c\varphi c\psi - s\theta s\varphi s\psi & c\psi s\varphi + c\varphi s\theta s\psi \\ s\theta & -c\theta s\varphi & c\theta c\varphi \end{bmatrix} \tag{2.153}$$

*The angular velocity $\boldsymbol{\omega}$ of a rigid body can either be expressed in terms of the components along the axes of $B(Oxyz)$, or in terms of the Cardan frequencies along the axes of the non-orthogonal Cardan frame. The angular velocity in terms of Cardan frequencies is*

$$ {}_{G}\boldsymbol{\omega}_{B} = \dot{\varphi}A_{z,\psi}A_{y,\theta}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \dot{\theta}A_{z,\psi}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.154}$$

*therefore,*

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \sin\psi & 0 \\ -\cos\theta\sin\psi & \cos\psi & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{2.155}$$

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \frac{\cos\psi}{\cos\theta} & -\frac{\sin\psi}{\cos\theta} & 0 \\ \sin\psi & \cos\psi & 0 \\ -\tan\theta\cos\psi & \tan\theta\sin\psi & 1 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \tag{2.156}$$

*In case of small Cardan angles, we have*

$$ {}^{B}A_{G} = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \varphi \\ \theta & -\varphi & 1 \end{bmatrix} \tag{2.157}$$

*and*

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & \psi & 0 \\ -\psi & 1 & 0 \\ \theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{2.158}$$

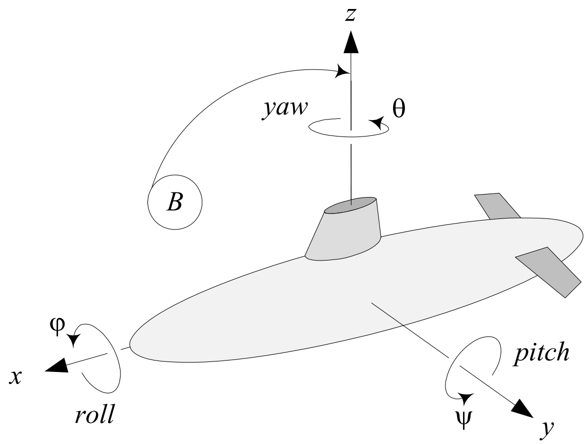


FIGURE 2.17. Local roll-pitch-yaw angles.

## 2.7 Local Roll-Pitch-Yaw Angles

Rotation about the $x$-axis of the local frame is called *roll* or *bank*, rotation about $y$-axis of the local frame is called *pitch* or *attitude*, and rotation about the $z$-axis of the local frame is called *yaw, spin,* or *heading.* The local roll-pitch-yaw angles are shown in Figure 2.17.

The local *roll-pitch-yaw rotation matrix* is:

$$\begin{aligned} {}^{B}A_{G} &= A_{z,\psi}A_{y,\theta}A_{x,\varphi} \\ &= \begin{bmatrix} c\theta c\psi & c\varphi s\psi + s\theta c\psi s\varphi & s\varphi s\psi - c\varphi s\theta c\psi \\ -c\theta s\psi & c\varphi c\psi - s\theta s\varphi s\psi & c\psi s\varphi + c\varphi s\theta s\psi \\ s\theta & -c\theta s\varphi & c\theta c\varphi \end{bmatrix} \end{aligned} \tag{2.159}$$

Note the difference between roll-pitch-yaw and Euler angles, although we show both utilizing $\varphi$, $\theta$, and $\psi$.

**Example 28** ★ *Angular velocity and local roll-pitch-yaw rate.*

*Using the roll-pitch-yaw frequencies, the angular velocity of a body $B$ with respect to the global reference frame is*

$$\begin{aligned} {}_{G}\boldsymbol{\omega}_{B} &= \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \\ &= \dot{\varphi}\hat{e}_{\varphi} + \dot{\theta}\hat{e}_{\theta} + \dot{\psi}\hat{e}_{\psi}. \end{aligned} \tag{2.160}$$

*Relationships between the components of ${}_{G}\boldsymbol{\omega}_{B}$ in body frame and roll-pitch-yaw components are found when the local roll unit vector $\hat{e}_{\varphi}$ and pitch unit vector $\hat{e}_{\theta}$ are transformed to the body frame. The roll unit vector $\hat{e}_{\varphi} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{T}$ transforms to the body frame after rotation $\theta$ and then* *rotation $\psi$.*

$$^{B}\hat{e}_{\varphi} = A_{z,\psi}A_{y,\theta}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi \\ -\cos\theta\sin\psi \\ \sin\theta \end{bmatrix} \tag{2.161}$$

*The pitch unit vector $\hat{e}_{\theta} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ transforms to the body frame after rotation $\psi$.*

$$^{B}\hat{e}_{\theta} = A_{z,\psi}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\psi \\ \cos\psi \\ 0 \end{bmatrix} \tag{2.162}$$

*The yaw unit vector $\hat{e}_{\psi} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is already along the local z-axis. Hence, ${}_G\boldsymbol{\omega}_B$ can be expressed in body frame $Oxyz$ as*

$$\begin{aligned} {}^{B}_{G}\boldsymbol{\omega}_B &= \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \dot{\varphi}\begin{bmatrix} \cos\theta\cos\psi \\ -\cos\theta\sin\psi \\ \sin\theta \end{bmatrix} + \dot{\theta}\begin{bmatrix} \sin\psi \\ \cos\psi \\ 0 \end{bmatrix} + \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\psi & \sin\psi & 0 \\ -\cos\theta\sin\psi & \cos\psi & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \end{aligned} \tag{2.163}$$

*and therefore, ${}_G\boldsymbol{\omega}_B$ in global frame $OXYZ$ in terms of local roll-pitch-yaw frequencies is:*

$$\begin{aligned} {}^{G}_{G}\boldsymbol{\omega}_B &= \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = {}^{B}A_G^{-1}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{B}A_G^{-1}\begin{bmatrix} \dot{\theta}\sin\psi + \dot{\varphi}\cos\theta\cos\psi \\ \dot{\theta}\cos\psi - \dot{\varphi}\cos\theta\sin\psi \\ \dot{\psi} + \dot{\varphi}\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \dot{\varphi} + \dot{\psi}\sin\theta \\ \dot{\theta}\cos\varphi - \dot{\psi}\cos\theta\sin\varphi \\ \dot{\theta}\sin\varphi + \dot{\psi}\cos\theta\cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\varphi & -\cos\theta\sin\varphi \\ 0 & \sin\varphi & \cos\theta\cos\varphi \end{bmatrix}\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \end{aligned} \tag{2.164}$$

## 2.8 Local Axes Versus Global Axes Rotation

The global rotation matrix ${}^{G}Q_B$ is equal to the inverse of the local rotation matrix ${}^{B}A_G$ and vice versa,

$${}^{G}Q_B = {}^{B}A_G^{-1} \quad , \quad {}^{B}A_G = {}^{G}Q_B^{-1} \tag{2.165}$$

where

$$\begin{aligned} {}^{G}Q_B &= A_1^{-1}A_2^{-1}A_3^{-1}\cdots A_n^{-1} \quad (2.166) \\ {}^{B}A_G &= Q_1^{-1}Q_2^{-1}Q_3^{-1}\cdots Q_n^{-1}. \quad (2.167) \end{aligned}$$

Also, premultiplication of the global rotation matrix is equal to postmultiplication of the local rotation matrix.

**Proof.** Consider a sequence of global rotations and their resultant global rotation matrix ${}^{G}Q_{B}$ to transform a position vector ${}^{B}\mathbf{r}$ to ${}^{G}\mathbf{r}$.

$$ {}^{G}\mathbf{r} = {}^{G}Q_{B}\,{}^{B}\mathbf{r} \tag{2.168} $$

The global position vector ${}^{G}\mathbf{r}$ can also be transformed to ${}^{B}\mathbf{r}$ using a local rotation matrix ${}^{B}A_{G}$.

$$ {}^{B}\mathbf{r} = {}^{B}A_{G}\,{}^{G}\mathbf{r} \tag{2.169} $$

Combining Equations (2.168) and (2.169) leads to

$$ {}^{G}\mathbf{r} = {}^{G}Q_{B}\,{}^{B}A_{G}\,{}^{G}\mathbf{r} \tag{2.170} $$
$$ {}^{B}\mathbf{r} = {}^{B}A_{G}\,{}^{G}Q_{B}\,{}^{B}\mathbf{r} \tag{2.171} $$

and hence,

$$ {}^{G}Q_{B}\,{}^{B}A_{G} = {}^{B}A_{G}\,{}^{G}Q_{B} = \mathbf{I}. \tag{2.172} $$

Therefore, the global and local rotation matrices are the inverse of each other.

$$ \begin{aligned} {}^{G}Q_{B} &= {}^{B}A_{G}^{-1} \\ {}^{G}Q_{B}^{-1} &= {}^{B}A_{G} \end{aligned} \tag{2.173} $$

Assume that ${}^{G}Q_{B} = Q_{n}\cdots Q_{3}Q_{2}Q_{1}$ and ${}^{B}A_{G} = A_{n}\cdots A_{3}A_{2}A_{1}$ then,

$$ {}^{G}Q_{B} = {}^{B}A_{G}^{-1} = A_{1}^{-1}A_{2}^{-1}A_{3}^{-1}\cdots A_{n}^{-1} \tag{2.174} $$
$$ {}^{B}A_{G} = {}^{G}Q_{B}^{-1} = Q_{1}^{-1}Q_{2}^{-1}Q_{3}^{-1}\cdots Q_{n}^{-1} \tag{2.175} $$

and Equation (2.172) becomes

$$ Q_{n}\cdots Q_{2}Q_{1}A_{n}\cdots A_{2}A_{1} = A_{n}\cdots A_{2}A_{1}Q_{n}\cdots Q_{2}Q_{1} = \mathbf{I} \tag{2.176} $$

and therefore,

$$ \begin{aligned} Q_{n}\cdots Q_{3}Q_{2}Q_{1} &= A_{1}^{-1}A_{2}^{-1}A_{3}^{-1}\cdots A_{n}^{-1} \\ A_{n}\cdots A_{3}A_{2}A_{1} &= Q_{1}^{-1}Q_{2}^{-1}Q_{3}^{-1}\cdots Q_{n}^{-1} \end{aligned} \tag{2.177} $$

or

$$ Q_{1}^{-1}Q_{2}^{-1}Q_{3}^{-1}\cdots Q_{n}^{-1}Q_{n}\cdots Q_{3}Q_{2}Q_{1} = \mathbf{I} \tag{2.178} $$
$$ A_{1}^{-1}A_{2}^{-1}A_{3}^{-1}\cdots A_{n}^{-1}A_{n}\cdots A_{3}A_{2}A_{1} = \mathbf{I}. \tag{2.179} $$

Hence, the effect of in order rotations about the global coordinate axes is equivalent to the effect of the same rotations about the local coordinate axes performed in the reverse order. ■

**Example 29** *Global position and postmultiplication of rotation matrix.*

*The local position of a point $P$ after rotation is at ${}^B\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. If the local rotation matrix to transform ${}^G\mathbf{r}$ to ${}^B\mathbf{r}$ is given as*

$$ {}^BA_{z,\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.180} $$

*then we may find the global position vector ${}^G\mathbf{r}$ by postmultiplication ${}^BA_{z,\varphi}$ by the local position vector ${}^B\mathbf{r}^T$,*

$$ \begin{aligned} {}^G\mathbf{r}^T &= {}^B\mathbf{r}^T \, {}^BA_{z,\varphi} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.13 & 2.23 & 3.0 \end{bmatrix} \end{aligned} \tag{2.181} $$

*instead of premultiplication of ${}^BA_{z,\varphi}^{-1}$ by ${}^B\mathbf{r}$.*

$$ \begin{aligned} {}^G\mathbf{r} &= {}^BA_{z,\varphi}^{-1} \, {}^B\mathbf{r} \\ &= \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.13 \\ 2.23 \\ 3 \end{bmatrix} \end{aligned} \tag{2.182} $$

## 2.9 General Transformation

Consider a general situation in which two coordinate frames, $G(OXYZ)$ and $B(Oxyz)$ with a common origin $O$, are employed to express the components of a vector $\mathbf{r}$. There is always a *transformation matrix* ${}^GR_B$ to map the components of $\mathbf{r}$ from the reference frame $B(Oxyz)$ to the other reference frame $G(OXYZ)$.

$$ {}^G\mathbf{r} = {}^GR_B \, {}^B\mathbf{r} \tag{2.183} $$

In addition, the inverse map, ${}^B\mathbf{r} = {}^GR_B^{-1} \, {}^G\mathbf{r}$, can be done by ${}^BR_G$

$$ {}^B\mathbf{r} = {}^BR_G \, {}^G\mathbf{r} \tag{2.184} $$

where,

$$ \left| {}^GR_B \right| = \left| {}^BR_G \right| = 1 \tag{2.185} $$

and

$$ {}^BR_G = {}^GR_B^{-1} = {}^GR_B^T. \tag{2.186} $$

**Proof.** Decomposition of the unit vectors of $G(OXYZ)$ along the axes of $B(Oxyz)$

$$\hat{I} = (\hat{I}\cdot\hat{\imath})\hat{\imath} + (\hat{I}\cdot\hat{\jmath})\hat{\jmath} + (\hat{I}\cdot\hat{k})\hat{k} \tag{2.187}$$
$$\hat{J} = (\hat{J}\cdot\hat{\imath})\hat{\imath} + (\hat{J}\cdot\hat{\jmath})\hat{\jmath} + (\hat{J}\cdot\hat{k})\hat{k} \tag{2.188}$$
$$\hat{K} = (\hat{K}\cdot\hat{\imath})\hat{\imath} + (\hat{K}\cdot\hat{\jmath})\hat{\jmath} + (\hat{K}\cdot\hat{k})\hat{k} \tag{2.189}$$

introduces the transformation matrix ${}^{G}R_{B}$ to map the local frame to the global frame

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = {}^{G}R_{B} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \tag{2.190}$$

where,

$$\begin{aligned} {}^{G}R_{B} &= \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\hat{I},\hat{\imath}) & \cos(\hat{I},\hat{\jmath}) & \cos(\hat{I},\hat{k}) \\ \cos(\hat{J},\hat{\imath}) & \cos(\hat{J},\hat{\jmath}) & \cos(\hat{J},\hat{k}) \\ \cos(\hat{K},\hat{\imath}) & \cos(\hat{K},\hat{\jmath}) & \cos(\hat{K},\hat{k}) \end{bmatrix}. \end{aligned} \tag{2.191}$$

Each column of ${}^{G}R_{B}$ is decomposition of a unit vector of the local frame $B(Oxyz)$ in the global frame $G(OXYZ)$.

$${}^{G}R_{B} = \begin{bmatrix} | & | & | \\ {}^{G}\hat{\imath} & {}^{G}\hat{\jmath} & {}^{G}\hat{k} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \hat{\mathbf{r}}_{V_1} & \hat{\mathbf{r}}_{V_2} & \hat{\mathbf{r}}_{V_3} \\ | & | & | \end{bmatrix} \tag{2.192}$$

Similarly, each row of ${}^{G}R_{B}$ is decomposition of a unit vector of the global frame $G(OXYZ)$ in the local frame $B(Oxyz)$.

$${}^{G}R_{B} = \begin{bmatrix} - & {}^{B}\hat{I}^{T} & - \\ - & {}^{B}\hat{J}^{T} & - \\ - & {}^{B}\hat{K}^{T} & - \end{bmatrix} = \begin{bmatrix} - & \hat{\mathbf{r}}_{H_1} & - \\ - & \hat{\mathbf{r}}_{H_2} & - \\ - & \hat{\mathbf{r}}_{H_3} & - \end{bmatrix} \tag{2.193}$$

The elements of ${}^{G}R_{B}$ are direction cosines of the axes of $G(OXYZ)$ in frame $B(Oxyz)$. This set of nine direction cosines then completely specifies the orientation of the frame $B(Oxyz)$ in the frame $G(OXYZ)$, and can be used to map the coordinates of any point $(x, y, z)$ to its corresponding coordinates $(X, Y, Z)$.

Alternatively, using the method of unit vector decomposition to develop the matrix ${}^{B}R_{G}$ leads to:

$$
\begin{aligned}
{}^{B}\mathbf{r} &= {}^{B}R_{G}\,{}^{G}\mathbf{r} = {}^{G}R_{B}^{-1}\,{}^{G}\mathbf{r} \qquad (2.194)\\
{}^{B}R_{G} &= \begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\imath}\cdot\hat{J} & \hat{\imath}\cdot\hat{K} \\ \hat{\jmath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} \\
&= \begin{bmatrix} \cos(\hat{\imath},\hat{I}) & \cos(\hat{\imath},\hat{J}) & \cos(\hat{\imath},\hat{K}) \\ \cos(\hat{\jmath},\hat{I}) & \cos(\hat{\jmath},\hat{J}) & \cos(\hat{\jmath},\hat{K}) \\ \cos(\hat{k},\hat{I}) & \cos(\hat{k},\hat{J}) & \cos(\hat{k},\hat{K}) \end{bmatrix} \qquad (2.195)
\end{aligned}
$$

and shows that the inverse of a transformation matrix is equal to the transpose of the transformation matrix.

$$
{}^{G}R_{B}^{-1} = {}^{G}R_{B}^{T} \qquad (2.196)
$$

A matrix with condition (2.196) is called *orthogonal.* Orthogonality of $R$ comes from this fact that it maps an orthogonal coordinate frame to another orthogonal coordinate frame.

The transformation matrix $R$ has only three *independent* elements. The constraint equations among the elements of $R$ will be found by applying the orthogonality condition (2.196).

$$
{}^{G}R_{B}\cdot{}^{G}R_{B}^{T} = [I] \qquad (2.197)
$$

$$
\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.198)
$$

Therefore, the dot product of any two different rows of ${}^{G}R_{B}$ is zero, and the dot product of any row of ${}^{G}R_{B}$ with the same row is one.

$$
\begin{aligned}
r_{11}^2 + r_{12}^2 + r_{13}^2 &= 1 \\
r_{21}^2 + r_{22}^2 + r_{23}^2 &= 1 \\
r_{31}^2 + r_{32}^2 + r_{33}^2 &= 1 \\
r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} &= 0 \\
r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} &= 0 \\
r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} &= 0 \qquad (2.199)
\end{aligned}
$$

These relations are also true for columns of ${}^{G}R_{B}$, and evidently for rows and columns of ${}^{B}R_{G}$. The orthogonality condition can be summarized in the following equation:

$$
\hat{\mathbf{r}}_{H_i}\cdot\hat{\mathbf{r}}_{H_j} = \hat{\mathbf{r}}_{H_i}^{T}\,\hat{\mathbf{r}}_{H_j} = \sum_{i=1}^{3} r_{ij}r_{ik} = \delta_{jk} \qquad (j,k=1,2,3) \qquad (2.200)
$$

where $r_{ij}$ is the element of row $i$ and column $j$ of the transformation matrix $R$, and $\delta_{jk}$ is the *Kronecker's delta.*

$$\delta_{jk} = 1 \text{ if } j = k, \text{ and } \delta_{jk} = 0 \text{ if } j \neq k \tag{2.201}$$

Equation (2.200) gives six independent relations satisfied by nine direction cosines. It follows that there are only three independent direction cosines. The independent elements of the matrix $R$ cannot obviously be in the same row or column, or any diagonal.

The determinant of a transformation matrix is equal to one,

$$\left|{}^G R_B\right| = 1 \tag{2.202}$$

because of Equation (2.197), and noting that

$$\begin{aligned}\left|{}^G R_B \cdot {}^G R_B^T\right| &= \left|{}^G R_B\right| \cdot \left|{}^G R_B^T\right| = \left|{}^G R_B\right| \cdot \left|{}^G R_B\right| \\ &= \left|{}^G R_B\right|^2 = 1.\end{aligned} \tag{2.203}$$

Using linear algebra and row vectors $\hat{\mathbf{r}}_{H_1}, \hat{\mathbf{r}}_{H_2}$, and $\hat{\mathbf{r}}_{H_3}$ of ${}^G R_B$, we know that

$$\left|{}^G R_B\right| = \hat{\mathbf{r}}_{H_1}^T \cdot \left(\hat{\mathbf{r}}_{H_2} \times \hat{\mathbf{r}}_{H_3}\right) \tag{2.204}$$

and because the coordinate system is right handed, we have $\hat{\mathbf{r}}_{H_2} \times \hat{\mathbf{r}}_{H_3} = \hat{\mathbf{r}}_{H_1}$ so $\left|{}^G R_B\right| = \hat{\mathbf{r}}_{H_1}^T \cdot \hat{\mathbf{r}}_{H_1} = 1.$ ■

**Example 30** *Elements of transformation matrix.*

*The position vector* $\mathbf{r}$ *of a point* $P$ *may be expressed in terms of its components with respect to either* $G\,(OXYZ)$ *or* $B\,(Oxyz)$ *frames. Body and a global coordinate frames are shown in Figure 2.18. If* ${}^G\mathbf{r} = 100\hat{I} - 50\hat{J} + 150\hat{K}$*, and we are looking for components of* $\mathbf{r}$ *in the* $Oxyz$ *frame, then we have to find the proper rotation matrix* ${}^B R_G$ *first.*

*The row elements of* ${}^B R_G$ *are the direction cosines of the* $Oxyz$ *axes in the* $OXYZ$ *coordinate frame. The* $x$*-axis lies in the* $XZ$ *plane at* $40\deg$ *from the* $X$*-axis, and the angle between* $y$ *and* $Y$ *is* $60\deg$*. Therefore,*

$$\begin{aligned}{}^B R_G &= \begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\imath}\cdot\hat{J} & \hat{\imath}\cdot\hat{K} \\ \hat{\jmath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} = \begin{bmatrix} \cos 40 & 0 & \sin 40 \\ \hat{\jmath}\cdot\hat{I} & \cos 60 & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} \\ &= \begin{bmatrix} 0.766 & 0 & 0.643 \\ \hat{\jmath}\cdot\hat{I} & 0.5 & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix}\end{aligned} \tag{2.205}$$

*and by using* ${}^B R_G\, {}^G R_B = {}^B R_G\, {}^B R_G^T = I$

$$\begin{bmatrix} 0.766 & 0 & 0.643 \\ r_{21} & 0.5 & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 0.766 & r_{21} & r_{31} \\ 0 & 0.5 & r_{32} \\ 0.643 & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.206}$$

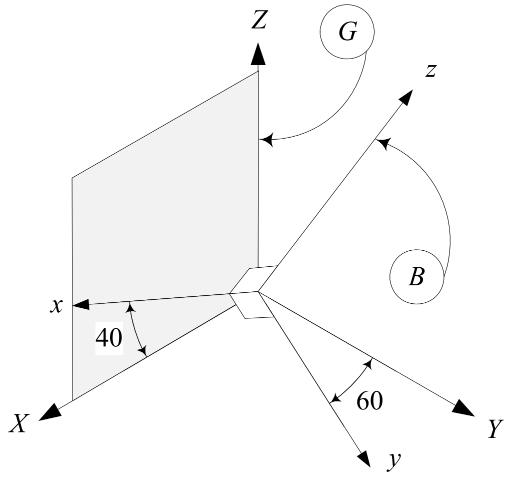


FIGURE 2.18. Body and global coordinate frames of Example 30.

*we obtain a set of equations to find the missing elements.*

$$\begin{aligned} 0.766\, r_{21} + 0.643\, r_{23} &= 0 \\ 0.766\, r_{31} + 0.643\, r_{33} &= 0 \\ r_{21}^2 + r_{23}^2 + 0.25 &= 1 \\ r_{21}r_{31} + 0.5r_{32} + r_{23}r_{33} &= 0 \\ r_{31}^2 + r_{32}^2 + r_{33}^2 &= 1 \end{aligned} \tag{2.207}$$

*Solving these equations provides the following transformation matrix:*

$${}^{B}R_G = \begin{bmatrix} 0.766 & 0 & 0.643 \\ 0.557 & 0.5 & -0.663 \\ -0.322 & 0.866 & 0.383 \end{bmatrix} \tag{2.208}$$

*and then we can find the components of* ${}^{B}\mathbf{r}$.

$$\begin{aligned} {}^{B}\mathbf{r} &= {}^{B}R_G\, {}^{G}\mathbf{r} = \begin{bmatrix} 0.766 & 0 & 0.643 \\ 0.557 & 0.5 & -0.663 \\ -0.322 & 0.866 & 0.383 \end{bmatrix} \begin{bmatrix} 100 \\ -50 \\ 150 \end{bmatrix} \\ &= \begin{bmatrix} 173.05 \\ -68.75 \\ -18.05 \end{bmatrix} \end{aligned} \tag{2.209}$$

**Example 31** *Global position, using* ${}^{B}\mathbf{r}$ *and* ${}^{B}R_G$.

*The position vector* $\mathbf{r}$ *of a point* $P$ *may be described in either* $G\,(OXYZ)$ *or* $B\,(Oxyz)$ *frames. If* ${}^{B}\mathbf{r} = 100\hat{\imath} - 50\hat{\jmath} + 150\hat{k}$, *and the following* ${}^{B}R_G$ *is* *the transformation matrix to map* ${}^{G}\mathbf{r}$ *to* ${}^{B}\mathbf{r}$

$$\begin{aligned}{}^{B}\mathbf{r} &= {}^{B}R_{G}\,{}^{G}\mathbf{r} \\ &= \begin{bmatrix} 0.766 & 0 & 0.643 \\ 0.557 & 0.5 & -0.663 \\ -0.322 & 0.866 & 0.383 \end{bmatrix} {}^{G}\mathbf{r} \end{aligned} \tag{2.210}$$

*then the components of* ${}^{G}\mathbf{r}$ *in* $G\,(OXYZ)$ *would be*

$$\begin{aligned}{}^{G}\mathbf{r} &= {}^{G}R_{B}\,{}^{B}\mathbf{r} = {}^{B}R_{G}^{T}\,{}^{B}\mathbf{r} \\ &= \begin{bmatrix} 0.766 & 0.557 & -0.322 \\ 0 & 0.5 & 0.866 \\ 0.643 & -0.663 & 0.383 \end{bmatrix} \begin{bmatrix} 100 \\ -50 \\ 150 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 104.9 \\ 154.9 \end{bmatrix}. \end{aligned} \tag{2.211}$$

**Example 32** *Two points transformation matrix.*

*The global position vector of two points,* $P_1$ *and* $P_2$*, of a rigid body* $B$ *are:*

$${}^{G}\mathbf{r}_{P_1} = \begin{bmatrix} 1.077 \\ 1.365 \\ 2.666 \end{bmatrix} \qquad {}^{G}\mathbf{r}_{P_2} = \begin{bmatrix} -0.473 \\ 2.239 \\ -0.959 \end{bmatrix} \tag{2.212}$$

*The origin of the body* $B\,(Oxyz)$ *is fixed on the origin of* $G\,(OXYZ)$*, and the points* $P_1$ *and* $P_2$ *are lying on the local* $x$*-axis and* $y$*-axis respectively.*

*To find* ${}^{G}R_{B}$*, we use the local unit vectors* ${}^{G}\hat{\imath}$ *and* ${}^{G}\hat{\jmath}$

$${}^{G}\hat{\imath} = \frac{{}^{G}\mathbf{r}_{P_1}}{\left|{}^{G}\mathbf{r}_{P_1}\right|} = \begin{bmatrix} 0.338 \\ 0.429 \\ 0.838 \end{bmatrix} \tag{2.213}$$

$${}^{G}\hat{\jmath} = \frac{{}^{G}\mathbf{r}_{P_2}}{\left|{}^{G}\mathbf{r}_{P_2}\right|} = \begin{bmatrix} -0.191 \\ 0.902 \\ -0.387 \end{bmatrix} \tag{2.214}$$

*to obtain* ${}^{G}\hat{k}$

$$\begin{aligned}{}^{G}\hat{k} &= \hat{\imath} \times \hat{\jmath} = \tilde{\imath}\,\hat{\jmath} \\ &= \begin{bmatrix} 0 & -0.838 & 0.429 \\ 0.838 & 0 & -0.338 \\ -0.429 & 0.338 & 0 \end{bmatrix} \begin{bmatrix} -0.191 \\ 0.902 \\ -0.387 \end{bmatrix} \\ &= \begin{bmatrix} -0.922 \\ -0.029 \\ 0.387 \end{bmatrix} \end{aligned} \tag{2.215}$$

*where* $\tilde{\imath}$ *is the skew-symmetric matrix corresponding to* $\hat{\imath}$*, and* $\tilde{\imath}\,\hat{\jmath}$ *is an alternative for* $\hat{\imath} \times \hat{\jmath}$*.*

*Hence, the transformation matrix using the coordinates of two points* ${}^{G}\mathbf{r}_{P_1}$ *and* ${}^{G}\mathbf{r}_{P_2}$ *would be*

$$\begin{aligned} {}^{G}R_{B} &= \left[ \begin{array}{ccc} {}^{G}\hat{\imath} & {}^{G}\hat{\jmath} & {}^{G}\hat{k} \end{array} \right] \\ &= \left[ \begin{array}{ccc} 0.338 & -0.191 & -0.922 \\ 0.429 & 0.902 & -0.029 \\ 0.838 & -0.387 & 0.387 \end{array} \right]. \end{aligned} \tag{2.216}$$

**Example 33** *Length invariant of a position vector.*

*Describing a vector in different frames utilizing rotation matrices does not affect the length and direction properties of the vector. Therefore, length of a vector is an invariant*

$$|\mathbf{r}| = \left|{}^{G}\mathbf{r}\right| = \left|{}^{B}\mathbf{r}\right|. \tag{2.217}$$

*The length invariant property can be shown by*

$$\begin{aligned} |\mathbf{r}|^2 &= {}^{G}\mathbf{r}^{T}\,{}^{G}\mathbf{r} = \left[{}^{G}R_{B}\,{}^{B}\mathbf{r}\right]^{T}\,{}^{G}R_{B}\,{}^{B}\mathbf{r} = {}^{B}\mathbf{r}^{T}\,{}^{G}R_{B}^{T}\,{}^{G}R_{B}\,{}^{B}\mathbf{r} \\ &= {}^{B}\mathbf{r}^{T}\,{}^{B}\mathbf{r}. \end{aligned} \tag{2.218}$$

**Example 34** *Skew symmetric matrices for* $\hat{\imath}$, $\hat{\jmath}$, *and* $\hat{k}$.

*The definition of skew symmetric matrix* $\tilde{a}$ *corresponding to a vector* $\mathbf{a}$ *is defined by*

$$\tilde{a} = \left[ \begin{array}{ccc} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{array} \right]. \tag{2.219}$$

*Hence,*

$$\tilde{\imath} = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right] \tag{2.220}$$

$$\tilde{\jmath} = \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right] \tag{2.221}$$

$$\tilde{k} = \left[ \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]. \tag{2.222}$$

**Example 35** *Inverse of Euler angles rotation matrix.*

*Precession-nutation-spin or Euler angle rotation matrix (2.106)*

$$\begin{aligned} {}^{B}R_{G} &= A_{z,\psi}A_{x,\theta}A_{z,\varphi} \\ &= \left[ \begin{array}{ccc} c\varphi c\psi - c\theta s\varphi s\psi & c\psi s\varphi + c\theta c\varphi s\psi & s\theta s\psi \\ -c\varphi s\psi - c\theta c\psi s\varphi & -s\varphi s\psi + c\theta c\varphi c\psi & s\theta c\psi \\ s\theta s\varphi & -c\varphi s\theta & c\theta \end{array} \right] \end{aligned} \tag{2.223}$$

*must be inverted to be a transformation matrix to map body coordinates to global coordinates.*

$$\begin{aligned} {}^{G}R_{B} &= {}^{B}R_{G}^{-1} = A_{z,\varphi}^{T} A_{x,\theta}^{T} A_{z,\psi}^{T} \\ &= \begin{bmatrix} c\varphi c\psi - c\theta s\varphi s\psi & -c\varphi s\psi - c\theta c\psi s\varphi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\varphi s\psi + c\theta c\varphi c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \end{aligned} \quad (2.224)$$

*The transformation matrix (2.223) is called a local Euler rotation matrix, and (2.224) is called a global Euler rotation matrix.*

**Example 36** ★ *Group property of transformations.*

*A set $S$ together with a binary operation $\otimes$ defined on elements of $S$ is called a group $(S, \otimes)$ if it satisfies the following four axioms.*

1. ***Closure:*** *If $s_1, s_2 \in S$, then $s_1 \otimes s_2 \in S$.*

2. ***Identity:*** *There exists an identity element $s_0$ such that $s_0 \otimes s = s \otimes s_0 = s$ for $\forall s \in S$.*

3. ***Inverse:*** *For each $s \in S$, there exists a unique inverse $s^{-1} \in S$ such that $s^{-1} \otimes s = s \otimes s^{-1} = s_0$.*

4. ***Associativity:*** *If $s_1, s_2, s_3 \in S$, then $(s_1 \otimes s_2) \otimes s_3 = s_1 \otimes (s_2 \otimes s_3)$.*

   *Three dimensional coordinate transformations make a group if we define the set of rotation matrices by*

$$S = \left\{ R \in \mathbb{R}^{3\times3} : RR^{T} = R^{T}R = \mathbf{I}, |R| = 1 \right\}. \quad (2.225)$$

   *Therefore, the elements of the set $S$ are transformation matrices $R_i$, the binary operator $\otimes$ is matrix multiplication, the identity matrix is $\mathbf{I}$, and the inverse of element $R$ is $R^{-1} = R^{T}$.*

   *$S$ is also a continuous group because*

5. *The binary matrix multiplication is a continuous operation, and*

6. *The inverse of any element in $S$ is a continuous function of that element.*

*Therefore, $S$ is a* ***differentiable manifold****. A group that is a differentiable manifold is called a* ***Lie group****.*

**Example 37** ★ *Transformation with determinant $-1$.*

*An orthogonal matrix with determinant $+1$ corresponds to a rotation as described in Equation (2.202). In contrast, an orthogonal matrix with determinant $-1$ describes a* ***reflection****. Moreover it transforms a right-handed coordinate system into a left-handed, and vice versa. This transformation does not correspond to any possible physical action on rigid bodies.*

**Example 38** *Alternative proof for transformation matrix.*
*Starting with an identity*

$$\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = 1 \tag{2.226}$$

*we may write*

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix}. \tag{2.227}$$

*Since matrix multiplication can be performed in any order we find*

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = {}^G R_B \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \tag{2.228}$$

*where,*

$${}^G R_B = \begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \end{bmatrix}. \tag{2.229}$$

*Following the same method we can show that*

$${}^B R_G = \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \begin{bmatrix} \hat{I} & \hat{J} & \hat{K} \end{bmatrix}. \tag{2.230}$$

## 2.10 Active and Passive Transformation

Rotation of a local frame when the position vector ${}^G\mathbf{r}$ of a point $P$ is fixed in global frame and does not rotate with the local frame, is called *passive transformation.* Alternatively, rotation of a local frame when the position vector ${}^B\mathbf{r}$ of a point $P$ is fixed in the local frame and rotates with the local frame, is called *active transformation.* Surprisingly, the passive and active transformations are mathematically equivalent. In other words, the rotation matrix for a rotated frame and rotated vector (active transformation) is the same as the rotation matrix for a rotated frame and fixed vector (passive transformation).

**Proof.** Consider a rotated local frame $B(Oxyz)$ with respect to a fixed global frame $G(OXYZ)$, as shown in Figure 2.19. $P$ is a fixed point in the global frame, and so is its global position vector ${}^G\mathbf{r}$. Position vector of $P$ can be decomposed in either a local or global coordinate frame, denoted by ${}^B\mathbf{r}$ and ${}^G\mathbf{r}$ respectively.

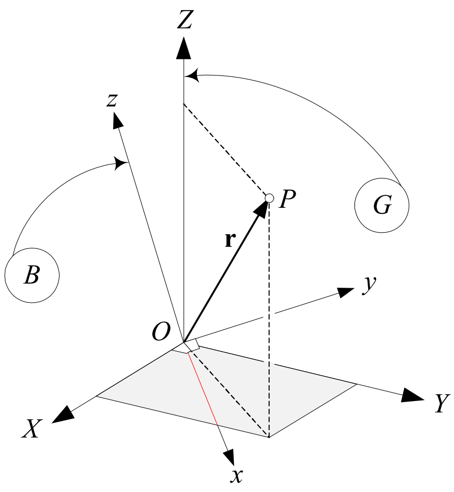


FIGURE 2.19. A position vector **r**, in a local and a global frame.

The transformation from ${}^{G}\mathbf{r}$ to ${}^{B}\mathbf{r}$ is equivalent to the required rotation of the body frame $B(Oxyz)$ to be coincided with the global frame $G(OXYZ)$. This is a passive transformation because the local frame cannot move the vector ${}^{G}\mathbf{r}$. In a passive transformation, we usually have the coordinates of $P$ in a global frame and we need its coordinates in a local frame; hence, we use the following equation:

$$ {}^{B}\mathbf{r} = {}^{B}R_{G}\,{}^{G}\mathbf{r}. \tag{2.231}$$

We may alternatively assume that $B(Oxyz)$ was coincident with $G(OXYZ)$ and the vector $\mathbf{r} = {}^{B}\mathbf{r}$ was fixed in $B(Oxyz)$, before $B(Oxyz)$ and ${}^{B}\mathbf{r}$ move to the new position in $G(OXYZ)$. This is an active transformation and there is a rotation matrix to map the coordinates of ${}^{B}\mathbf{r}$ in the local frame to the coordinates of ${}^{G}\mathbf{r}$ in global frame. In an active transformation, we usually have the coordinates of $P$ in the local frame and we need its coordinates in the global frame; hence, we use the following equation:

$$ {}^{G}\mathbf{r} = {}^{G}R_{B}\,{}^{B}\mathbf{r}. \tag{2.232}$$

■

**Example 39** *Active and passive rotation about X-axis.*

*Consider a local and global frames B and G that are coincident. A body point P is at* ${}^{B}\mathbf{r}$.

$$ {}^{B}\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{2.233}$$

*A rotation of* 45 deg *about* $X$*-axis will move the point to* ${}^G\mathbf{r}$.

$$\begin{aligned} {}^G\mathbf{r} &= R_{X,90}\, {}^B\mathbf{r} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{aligned} \tag{2.234}$$

*Now assume that* $P$ *is fixed in* $G$*. When* $B$ *rotates* 90 deg *about* $X$*-axis, the coordinates of* $P$ *in the local frame will change such that*

$$\begin{aligned} {}^B\mathbf{r} &= R_{X,-90}\, {}^G\mathbf{r} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{-\pi}{2} & -\sin\frac{-\pi}{2} \\ 0 & \sin\frac{-\pi}{2} & \cos\frac{-\pi}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}. \end{aligned} \tag{2.235}$$

## 2.11 Summary

The objectives of this chapter are:

1−To learn how to determine the transformation matrix between two Cartesian coordinate frames $B$ and $G$ with a common origin by applying rotations about principal axes.

2−To decompose a given transformation matrix to a series of required principal rotations.

Two Cartesian coordinate frames $B$ and $G$ with a common origin are related by nine directional cosines of a frame in the other. The conversion of coordinates in the two frames can be cast in a matrix transformation

$$ {}^{G}\mathbf{r} = {}^{G}R_{B}\, {}^{B}\mathbf{r} \tag{2.236} $$

$$ \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \tag{2.237} $$

where,

$$ {}^{G}R_{B} = \begin{bmatrix} \cos(\hat{I},\hat{\imath}) & \cos(\hat{I},\hat{\jmath}) & \cos(\hat{I},\hat{k}) \\ \cos(\hat{J},\hat{\imath}) & \cos(\hat{J},\hat{\jmath}) & \cos(\hat{J},\hat{k}) \\ \cos(\hat{K},\hat{\imath}) & \cos(\hat{K},\hat{\jmath}) & \cos(\hat{K},\hat{k}) \end{bmatrix}. \tag{2.238} $$

The transformation matrix ${}^{G}R_{B}$ is orthogonal; so its determinant is one, and its inverse is equal to its transpose.

$$ \left|{}^{G}R_{B}\right| = 1 \tag{2.239} $$

$$ {}^{G}R_{B}^{-1} = {}^{G}R_{B}^{T} \tag{2.240} $$

The orthogonality condition generates six equations between the elements of ${}^{G}R_{B}$ that shows only three elements of ${}^{G}R_{B}$ are independent.

Any relative orientation of $B$ in $G$ can be achieved by three consecutive principal rotations about the coordinate axes in either the $B$ or $G$ frame. If $B$ is the body coordinate frame, and $G$ is the globally fixed frame, the global principal rotation transformation matrices are:

$$ R_{X,\gamma} = {}^{G}R_{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \tag{2.241} $$

$$ R_{Y,\beta} = {}^{G}R_{B} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.242} $$

$$ R_{Z,\alpha} = {}^{G}R_{B} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.243} $$

and the body principal rotation transformation matrices are:

$$R_{x,\psi} = {}^{B}R_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \tag{2.244}$$

$$R_{y,\theta} = {}^{B}R_G = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.245}$$

$$R_{z,\varphi} = {}^{B}R_G = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.246}$$

The global and local rotation transformations are inverse of each other.

$$R_{X,\gamma} = R_{x,\gamma}^T \tag{2.247}$$

$$R_{Y,\beta} = R_{y,\beta}^T \tag{2.248}$$

$$R_{Z,\alpha} = R_{z,\alpha}^T \tag{2.249}$$

## 2.12 Key Symbols

| | |
|---|---|
| $\mathbf{a}$ | a general vector |
| $\tilde{a}$ | skew symmetric matrix of the vector $\mathbf{a}$ |
| $A$ | transformation matrix of rotation about a local axis |
| $B$ | body coordinate frame, local coordinate frame |
| $c$ | cos |
| $d$ | distance between two points |
| $\hat{e}_\varphi, \hat{e}_\theta, \hat{e}_\psi$ | coordinate axes of $E$, local roll-pitch-yaw coordinate axes |
| $E$ | Eulerian local frame |
| $f, f_1, f_2$ | a function of $x$ and $y$ |
| $G$ | global coordinate frame, fixed coordinate frame |
| $\mathbf{I} = [I]$ | identity matrix |
| $\hat{\imath}, \hat{\jmath}, \hat{k}$ | local coordinate axes unit vectors |
| $\tilde{\imath}, \tilde{\jmath}, \tilde{k}$ | skew symmetric matrices of the unit vector $\hat{\imath}, \hat{\jmath}, \hat{k}$ |
| $\hat{I}, \hat{J}, \hat{K}$ | global coordinate axes unit vectors |
| $l$ | length |
| $m$ | number of repeating rotation |
| $n$ | fraction of $2\pi$, number of repeating rotation |
| $\mathbb{N}$ | the set of natural numbers |
| $O$ | common origin of $B$ and $G$ |
| $O\varphi\theta\psi$ | Euler angle frame |
| $P$ | a body point, a fixed point in $B$, a partial derivative |
| $Q$ | transformation matrix of rotation about a global axis, a partial derivative |
| $\mathbf{r}$ | position vector |
| $r_{ij}$ | the element of row $i$ and column $j$ of a matrix |
| $R$ | rotation transformation matrix, radius of a circle |
| $\mathbb{R}$ | the set of real numbers |
| $s$ | sin, a member of $S$ |
| $S$ | a set |
| $t$ | time |
| $\mathbf{u}$ | a general axis |
| $\mathbf{v}$ | velocity vector |
| $x, y, z$ | local coordinate axes |
| $X, Y, Z$ | global coordinate axes |
| Greek | |
| $\alpha, \beta, \gamma$ | rotation angles about global axes |
| $\delta_{ij}$ | Kronecker's delta |
| $\varphi, \theta, \psi$ | rotation angles about local axes, Euler angles |
| $\dot{\varphi}, \dot{\theta}, \dot{\psi}$ | Euler frequencies |
| $\omega_x, \omega_y, \omega_z$ | angular velocity components |
| $\boldsymbol{\omega}$ | angular velocity vector |

| Symbol | |
|---|---|
| $[\quad]^{-1}$ | inverse of the matrix $[\quad]$ |
| $[\quad]^{T}$ | transpose of the matrix $[\quad]$ |
| $\otimes$ | a binary operation |
| $(S, \otimes)$ | a group |

# Exercises

1. Notation and symbols.

   Describe the meaning of these notations.

   a- ${}^{G}\mathbf{r}$ b- ${}^{G}\mathbf{r}_P$ c- ${}^{B}\mathbf{r}_P$ d- ${}^{G}R_B$ e- ${}^{G}R_B^T$ f- ${}^{B}R_G$

   g- ${}^{B}R_G^{-1}$ h- ${}^{G}\mathbf{d}_B$ i- ${}^{2}\mathbf{d}_1$ j- $Q_X$ k- $Q_{Y,\beta}$ l- $Q_{Y,45}^{-1}$

   m- $\hat{k}$ n- $\hat{J}$ o- $A_{z,\varphi}^T$ p- $\hat{e}_\psi$ q- $\tilde{\imath}$ r- $\mathbf{I}$

2. Body point and global rotations.

   The point $P$ is at ${}^{B}\mathbf{r}_P = [1, 2, 1]^T$ in a body coordinate $B(Oxyz)$. Find the final global position of $P$ after

   (a) A rotation of 30 deg about the $X$-axis, followed by a 45 deg rotation about the $Z$-axis

   (b) A rotation of 30 deg about the $Z$-axis, followed by a 45 deg rotation about the $X$-axis.

   (c) ★ Point $P$ will move on a sphere. Let us name the initial global position of $P$ by $P_1$, the second position by $P_2$, and the third position by $P_3$. Determine the angles of $\angle P_1OP_2$, $\angle P_2OP_3$, $\angle P_3OP_1$.

   (d) ★ Determine the area of the triangle made by points ${}^{G}\mathbf{r}_P = [1, 2, 1]^T$, and the global position of $P$ after rotations $a$ and $b$.

3. ★ Alternative motions to reach an orientation.

   The coordinates of a body point $P$ in $B$ and $G$ frames are:

$$
{}^{B}\mathbf{r}_P = \begin{bmatrix} 1.23 \\ 4.56 \\ 7.89 \end{bmatrix} \qquad {}^{G}\mathbf{r}_P = \begin{bmatrix} 4.56 \\ 7.89 \\ 1.23 \end{bmatrix}
$$

   Determine

   (a) If it is possible to transform ${}^{B}\mathbf{r}_P$ to ${}^{G}\mathbf{r}_P$?

   (b) A transformation matrix between ${}^{B}\mathbf{r}_P$ and ${}^{G}\mathbf{r}_P$.

   (c) Euler angles to transform ${}^{B}\mathbf{r}_P$ to ${}^{G}\mathbf{r}_P$.

   (d) Global roll-pitch-yaw to transform ${}^{B}\mathbf{r}_P$ to ${}^{G}\mathbf{r}_P$.

   (e) Body roll-pitch-yaw to transform ${}^{B}\mathbf{r}_P$ to ${}^{G}\mathbf{r}_P$.

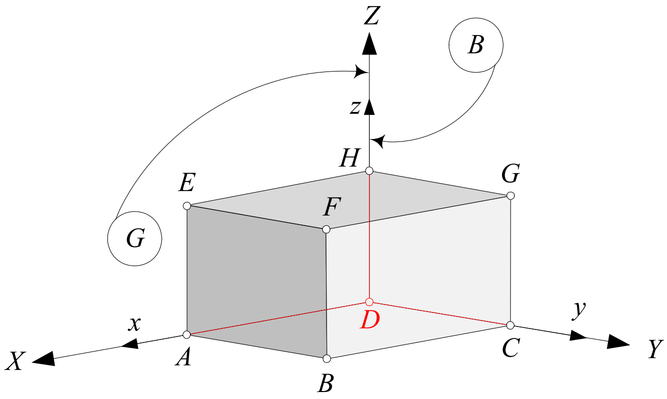


FIGURE 2.20. A cube at its initial position.

4. Body point after global rotation.

   Find the position of a point $P$ in the local coordinate, if it is moved to ${}^{G}\mathbf{r}_{P} = [1, 3, 2]^{T}$ after

   (a) A rotation of 60 deg about $Z$-axis,

   (b) A rotation of 60 deg about $X$-axis,

   (c) A rotation of 60 deg about $Y$-axis,

   (d) Rotations of 60 deg about $Z$-axis, 60 deg about $X$-axis and 60 deg about $Y$-axis.

5. Invariant of a vector.

   A point was at ${}^{B}\mathbf{r}_{P} = [1, 2, z]^{T}$. After a rotation of 60 deg about $X$-axis, followed by a 30 deg rotation about $Z$-axis, it is at:

$$
{}^{G}\mathbf{r}_{P} = \begin{bmatrix} X \\ Y \\ 2.933 \end{bmatrix}
$$

   Find $z$, $X$, and $Y$.

6. Global rotation of a cube.

   Figure 2.20 illustrates the original position of a cube with a fixed point at $D$ and edges of length $l = 1$.

   Determine,

   (a) Coordinates of the corners after rotation of 30 deg about $X$-axis.

   (b) Coordinates of the corners after rotation of 30 deg about $Y$-axis.

(c) Coordinates of the corners after rotation of 30 deg about $Z$-axis.

(d) Coordinates of the corners after rotation of 30 deg about $X$-axis, then 30 deg about $Y$-axis, and then 30 deg about $Z$-axis.

7. Constant length vector.

   Show that the length of a vector will not change by rotation.

$$\left|{}^{G}\mathbf{r}\right| = \left|{}^{G}R_{B}\,{}^{B}\mathbf{r}\right|$$

   Show that the distance between two body points will not change by rotation.

$$\left|{}^{B}\mathbf{p}_1 - {}^{B}\mathbf{p}_2\right| = \left|{}^{G}R_{B}\,{}^{B}\mathbf{p}_1 - {}^{G}R_{B}\,{}^{B}\mathbf{p}_2\right|$$

8. Repeated global rotations.

   Rotate ${}^{B}\mathbf{r}_P = [2,2,3]^T$, 60 deg about $X$-axis, followed by 30 deg about $Z$-axis. Then, repeat the sequence of rotations for 60 deg about $X$-axis, followed by 30 deg about $Z$-axis. After how many rotations will point $P$ be back to its initial global position?

9. ★ Repeated global rotations.

   How many rotations of $\alpha = \pi/m$ deg about $X$-axis, followed by $\beta = \pi/n$ deg about $Z$-axis are needed to bring a body point to its initial global position, if $m, n \in \mathbb{N}$?

10. Triple global rotations.

    Verify the equations in Appendix A.

11. ★ Special triple rotation.

    Assume that the first triple rotation in Appendix A brings a body point back to its initial global position. What are the angles $\alpha \neq 0$, $\beta \neq 0$, and $\gamma \neq 0$?

12. ★ Combination of triple rotations.

    Any triple rotation in Appendix A can move a body point to its new global position. Assume $\alpha_1, \beta_1$, and $\gamma_1$ for the case $1{-}Q_{X,\gamma_1}Q_{Y,\beta_1}Q_{Z,\alpha_1}$ are given. What can $\alpha_2$, $\beta_2$, and $\gamma_2$ be (in terms of $\alpha_1$, $\beta_1$, and $\gamma_1$) to get the same global position using the case $2 - Q_{Y,\gamma_2}Q_{Z,\beta_2}Q_{X,\alpha_2}$?

13. Global roll-pitch-yaw rotation matrix.

    Calculate the global and local roll-pitch-yaw rotation matrices $Q$ and $A$ for 30 deg rotation about the principal axes. Do the matrices transpose each other? Calculate the local rotation matrix $A$ by rotation about $z$ then $y$ then $x$. Is the transpose of the new matrix $A$ transpose of the global roll-pitch-yaw matrix?

14. Global roll-pitch-yaw rotation angles.

    Calculate the role, pitch, and yaw angles for the following rotation matrix:

$$^{B}R_{G} = \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}$$

15. ★ Back to the initial orientation and Appendix $A$.

    Assume we turn a rigid body $B$ using the first set of Appendix $A$. How can we turn it back to its initial orientation by applying

    (a) The first set of Appendix $A$,

    (b) The second set of Appendix $A$,

    (c) The third set of Appendix $A$.

    (d) ★ Assume that we have turned a rigid body $B$ by $\alpha_1 = 30$ deg, $\beta_1 = 30$ deg, $\gamma_1 = 30$ deg using the first set of Appendix $A$. We want to turn $B$ back to its original orientation. Which one of the second or third set of Appendix $A$ does it faster? Let us assume that the fastest set is the one with minimum sum of $s = \alpha_2 + \beta_2 + \gamma_2$.

16. ★ Back to the original orientation and Appendix $B$.

    Assume we turn a rigid body $B$ using the first set of Appendix $A$. How can we turn it back to its initial orientation by applying

    (a) The first set of Appendix $B$,

    (b) The second set of Appendix $B$,

    (c) The third set of Appendix $B$.

    (d) ★ Assume that we have turned a rigid body $B$ by $\alpha = 30$ deg, $\beta = 30$ deg, $\gamma = 30$ deg using the first set of Appendix $A$. We want to turn $B$ back to its original orientation. Which one of the first, second, or third set of Appendix $B$ does it faster? Let us assume that the fastest set is the one with minimum sum of $s = \varphi + \theta + \psi$.

17. Two local rotations.

    Find the global coordinates of a body point at $^{B}\mathbf{r}_{P} = [2, 2, 3]^{T}$ after

    (a) A rotation of 60 deg about $x$-axis followed by 60 deg about $z$-axis,

    (b) A rotation of 60 deg about $z$-axis followed by 60 deg about $x$-axis,

(c) A rotation of 60 deg about $z$-axis followed by 60 deg about $x$-axis, and a rotation of 60 deg about $z$-axis.

18. Local rotation of a cube.

    Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

    Determine,

    (a) Coordinates of the corners after rotation of 30 deg about $x$-axis.

    (b) Coordinates of the corners after rotation of 30 deg about $y$-axis.

    (c) Coordinates of the corners after rotation of 30 deg about $z$-axis.

    (d) Coordinates of the corners after rotation of 30 deg about $x$-axis, then 30 deg about $y$-axis, and then 30 deg about $z$-axis.

19. Global and local rotation of a cube.

    Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

    Determine,

    (a) Coordinates of the corners after rotation of 30 deg about $x$-axis followed by rotation of 30 deg about $X$-axis.

    (b) Coordinates of the corners after rotation of 30 deg about $y$-axis followed by rotation of 30 deg about $X$-axis.

    (c) Coordinates of the corners after rotation of 30 deg about $z$-axis followed by rotation of 30 deg about $X$-axis.

    (d) Coordinates of the corners after rotation of 30 deg about $x$-axis, then 30 deg about $X$-axis, and then 30 deg about $x$-axis.

    (e) Coordinates of the corners after rotation of 30 deg about $x$-axis, then 30 deg about $Y$-axis, and then 30 deg about $z$-axis.

20. Body point, local rotation.

    What is the global coordinates of a body point at ${}^B\mathbf{r}_P = [2, 2, 3]^T$, after

    (a) A rotation of 60 deg about the $x$-axis,

    (b) A rotation of 60 deg about the $y$-axis,

    (c) A rotation of 60 deg about the $z$-axis.

21. Unknown rotation angle 1.

    Transform ${}^B\mathbf{r}_P = [2, 2, 3]^T$ to ${}^G\mathbf{r}_P = [2, Y_P, 0]^T$ by a rotation about $x$-axis and determine $Y_P$ and the angle of rotation.

22. Unknown rotation angle 2.

    Consider a point $P$ at ${}^B\mathbf{r}_P = [2, \sqrt{3}, \sqrt{2}]^T$. Determine

    (a) The required principal global rotations in order $X, Y, Z$, to move $P$ to ${}^G\mathbf{r}_P = [\sqrt{2}, 2, \sqrt{3}]^T$,

    (b) The required principal global rotations in order $Z, Y, Z$, to move $P$ to ${}^G\mathbf{r}_P = [\sqrt{2}, 2, \sqrt{3}]^T$,

    (c) The required principal global rotations in order $Z, X, Z$, to move $P$ to ${}^G\mathbf{r}_P = [\sqrt{2}, 2, \sqrt{3}]^T$.

23. Triple local rotations.

    Verify the equations in Appendix B.

24. Combination of local and global rotations.

    Find the final global position of a body point at ${}^B\mathbf{r}_P = [10, 10, -10]^T$ after

    (a) A rotation of 45 deg about the $x$-axis followed by 60 deg about the $Z$-axis,

    (b) A rotation of 45 deg about the $z$-axis followed by 60 deg about the $Z$-axis,

    (c) A rotation of 45 deg about the $x$-axis followed by 45 deg about the $Z$-axis and 60 deg about the $X$-axis.

25. Combination of global and local rotations.

    Find the final global position of a body point at ${}^B\mathbf{r}_P = [10, 10, -10]^T$ after

    (a) A rotation of 45 deg about the $X$-axis followed by 60 deg about the $z$-axis,

    (b) A rotation of 45 deg about the $Z$-axis followed by 60 deg about the $z$-axis,

    (c) A rotation of 45 deg about the $X$-axis followed by 45 deg about the $x$-axis and 60 deg about the $z$-axis.

26. Repeated local rotations.

    Rotate ${}^B\mathbf{r}_P = [2, 2, 3]^T$, 60 deg about the $x$-axis, followed by 30 deg about the $z$-axis. Then repeat the sequence of rotations for 60 deg about the $x$-axis, followed by 30 deg about the $z$-axis. After how many rotations will point $P$ move back to its initial global position?

27. ★ Repeated local rotations.

    How many rotations of $\alpha = \pi/m$ deg about the $x$-axis, followed by $\beta = \pi/n$ deg about the $z$-axis are needed to bring a body point to its initial global position if $m, n \in \mathbb{N}$?

28. ★ Remaining rotation.

    Find the result of the following sequence of rotations:

    $$^{G}R_{B} = A_{y,\theta}^{T} A_{z,\psi}^{T} A_{y,-\theta}^{T}$$

29. Angles from rotation matrix.

    Find the angles $\varphi$, $\theta$, and $\psi$ if the rotation transformation matrices of Appendix $B$ are given.

30. Euler angles from rotation matrix.

    (a) Check if the following matrix $^{G}R_{B}$ is a rotation transformation.

    $$^{G}R_{B} = \begin{bmatrix} 0.53 & -0.84 & 0.13 \\ 0.0 & 0.15 & 0.99 \\ -0.85 & -0.52 & 0.081 \end{bmatrix}$$

    (b) Find the Euler angles for $^{G}R_{B}$.

    (c) Find the local roll-pitch-yaw angles for $^{G}R_{B}$.

31. Equivalent Euler angles to two rotations.

    Find the Euler angles corresponding to the rotation matrices

    (a) $^{B}R_{G} = A_{y,45} A_{x,30}$,

    (b) $^{B}R_{G} = A_{x,45} A_{y,30}$,

    (c) $^{B}R_{G} = A_{y,45} A_{z,30}$.

32. Equivalent Euler angles to three rotations.

    Find the Euler angles corresponding to the rotation matrix

    (a) $^{B}R_{G} = A_{z,60} A_{y,45} A_{x,30}$,

    (b) $^{B}R_{G} = A_{z,60} A_{y,45} A_{z,30}$,

    (c) $^{B}R_{G} = A_{x,60} A_{y,45} A_{x,30}$.

33. ★ A cube rotation and forbidden space of $z < 0$.

    Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

    Assume that none of the corners is allowed to have a negative $z$-components at any time.

(a) Present a series of global principal rotations to make the line $FH$ parallel to $z$-axis.

(b) Present a series of global principal rotations to make the line $DB$ on the $z$-axis and point $A$ in $(Z, Y)$-plane.

(c) Present a series of local principal rotations to make the line $FH$ parallel to $z$-axis.

(d) Present a series of local principal rotations to make the line $DB$ on the $z$-axis and point $A$ in $(Z, Y)$-plane.

34. ★ Local and global positions, Euler angles.

Find the conditions between the Euler angles

(a) To transform ${}^{G}\mathbf{r}_{P} = [1, 1, 0]^{T}$ to ${}^{B}\mathbf{r}_{P} = [0, 1, 1]^{T}$,

(b) To transform ${}^{G}\mathbf{r}_{P} = [1, 1, 0]^{T}$ to ${}^{B}\mathbf{r}_{P} = [1, 0, 1]^{T}$.

35. ★ Equivalent Euler angles to a triple rotations.

Find the Euler angles for the rotation matrix of the case

$$4 - A_{z,\psi'} A_{y,\theta'} A_{x,\varphi'}$$

in Appendix B.

36. ★ Integrability of Euler frequencies.

Show that $d\varphi$ and $d\psi$ are integrable, if $\varphi$ and $\psi$ are first and third Euler angles.

37. ★ Cardan angles for Euler angles.

(a) Find the Cardan angles for a given set of Euler angles.

(b) Find the Euler angles for a given set of Cardan angles.

38. ★ Cardan frequencies for Euler frequencies.

(a) Find the Euler frequencies in terms of Cardan frequencies.

(b) Find the Cardan frequencies in terms of Euler frequencies.

39. ★ Transformation matrix and three rotations.

Figure 2.20 illustrates the original position of a cube with a fixed point at $D$ and edges of length $l = 1$.

Assume a new orientation in which points $D$ and $F$ are on $Z$-axis and point $A$ is in $(X, Z)$-plane. Determine

(a) Transformation matrix between initial and new orientations.

(b) Euler angles to move the cube to its new orientation.

(c) Global roll-pitch-yaw angles to move the cube to its new orientation.

(d) Local roll-pitch-yaw angles to move the cube to its new orientation.

40. ★ Alternative maneuvers.

Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

Assume a new orientation in which points $D$ and $F$ are on $Z$-axis and point $A$ is in $(X, Z)$-plane. Determine

(a) Angles for maneuver $Y - X - Z$ as first-second-third rotations.

(b) Angles for maneuver $Y - Z - X$ as first-second-third rotations.

(c) Angles for maneuver $y - x - z$ as first-second-third rotations.

(d) Angles for maneuver $y - z - x$ as first-second-third rotations.

(e) Angles for maneuver $y - Z - x$ as first-second-third rotations.

(f) Angles for maneuver $Y - z - X$ as first-second-third rotations.

(g) Angles for maneuver $x - X - x$ as first-second-third rotations.

41. Elements of rotation matrix.

The elements of rotation matrix ${}^G R_B$ are

$${}^G R_B = \begin{bmatrix} \cos(\hat{I}, \hat{\imath}) & \cos(\hat{I}, \hat{\jmath}) & \cos(\hat{I}, \hat{k}) \\ \cos(\hat{J}, \hat{\imath}) & \cos(\hat{J}, \hat{\jmath}) & \cos(\hat{J}, \hat{k}) \\ \cos(\hat{K}, \hat{\imath}) & \cos(\hat{K}, \hat{\jmath}) & \cos(\hat{K}, \hat{k}) \end{bmatrix}.$$

Find ${}^G R_B$ if ${}^G\mathbf{r}_{P_1} = [0.7071, -1.2247, 1.4142]^T$ is a point on the $x$-axis, and ${}^G\mathbf{r}_{P_2} = [2.7803, 0.38049, -1.0607]^T$ is a point on the $y$-axis.

42. Linearly independent vectors.

A set of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ are considered linearly independent if the equation

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \cdots + k_n\mathbf{a}_n = 0$$

in which $k_1$, $k_2$, $\cdots$, $k_n$ are unknown coefficients, has only one solution

$$k_1 = k_2 = \cdots = k_n = 0.$$

Verify that the unit vectors of a body frame $B(Oxyz)$, expressed in the global frame $G(OXYZ)$, are linearly independent.

43. Product of orthogonal matrices.

A matrix $R$ is called orthogonal if $R^{-1} = R^T$ where $\left(R^T\right)_{ij} = R_{ji}$. Prove that the product of two orthogonal matrices is also orthogonal.

44. Vector identity.

    The formula $(a+b)^2 = a^2 + b^2 + 2ab$ for scalars, is equivalent to

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{b}$$

    for vectors. Show that this formula is equal to

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\,{}^G R_B\,\mathbf{a}\cdot\mathbf{b}$$

    if **a** is a vector in local frame and **b** is a vector in global frame.

45. Rotation as a linear operation.

    Show that

$$R\,(\mathbf{a}\times\mathbf{b}) = R\mathbf{a}\times R\mathbf{b}$$

    where $R$ is a rotation matrix and **a** and **b** are two vectors defined in a coordinate frame.

46. Scalar triple product.

    Show that for three arbitrary vectors **a**, **b**, and **c** we have

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$

47. ★ Euler angles and minimization distances.

    Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

    Assume a new orientation in which points $D$ and $F$ are on $Z$-axis and point $A$ is in $(X, Y)$-plane. Determine

    (a) Transformation matrix between initial and new orientations.

    (b) Euler angles to move the cube to its new orientation.

    (c) Choose three non coplanar corners and determine their position using Euler transformation matrix with unknown Euler angles. Define the distance between the initial and final positions of the points as $d_1$, $d_2$ and $d_3$. Is it possible to determine the Euler angles by minimizing a sum of distances objective function $J = d_1^2 + d_2^2 + d_3^2$?

48. ★ Continues rotation.

    Figure 2.20 illustrates the initial position of a cube with a fixed point at $D$ and edges of length $l = 1$.

    Assume that the cube is turning about $x$-axis with angular speed of $\omega_1$ and at the same time it is turning about $Z$-axis with angular speed of $\omega_2$. Determine the path of motion of point $F$. What is the path for $\omega_1 = \omega_2$, $\omega_1 = 2\omega_2$, $\omega_1 = 3\omega_2$,$\omega_1 = 4\omega_2$?