14

★ Time Optimal Control

The main job of an industrial robot is to move an object on a pre-specified path, rest to rest, repeatedly. To increase productivity, the robot should do its job in minimum time. We introduce a numerical method to solve the time optimal control problem of multi degree of freedom robots.

14.1 ★ Minimum Time and Bang-Bang Control

The most important job of industrial robots is moving between two points rest-to-rest. Minimum time control is what we need to increase industrial robots productivity. The objective of time-optimal control is to transfer the end-effector of a robot from an initial position to a desired destination in minimum time. Consider a system with the following equation of motion:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{Q}(t)\right) \tag{14.1}$$

where $\mathbf{Q}$ is the control input, and $\mathbf{x}$ is the state vector of the system.

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \tag{14.2}$$

The minimum time problem is always subject to bounded input such as:

$$|\mathbf{Q}(t)| \leq \mathbf{Q}_{Max} \tag{14.3}$$

The solution of the time-optimal control problem subject to bounded input is *bang-bang control*. The control in which the input variable takes either the maximum or minimum values is called bang-bang control.

Proof. The goal of minimum time control is to find the trajectory $\mathbf{x}(t)$ and input $\mathbf{Q}(t)$ starting from an initial state $\mathbf{x}_0(t)$ and arriving at the final state $\mathbf{x}_f(t)$ under the condition that the whole trajectory minimizes the following time integral.

$$J = \int_{t_0}^{t_f} dt \tag{14.4}$$

The input command vector $\mathbf{Q}(t)$ usually has the constraint (14.3).

We define a scalar function H, and a vector $\mathbf{p}$

$$H(\mathbf{x}, \mathbf{Q}, \mathbf{p}) = \mathbf{p}^T \mathbf{f}\left(\mathbf{x}(t), \mathbf{Q}(t)\right) \tag{14.5}$$

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that provide the following two equations:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}^T \tag{14.6}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}^T \tag{14.7}$$

Based on the *Pontryagin principle*, the optimal input $\mathbf{Q}(t)$ is the one that minimizes the function H. Such an optimal input is to apply the maximum effort, $\mathbf{Q}_{Max}$ or $-\mathbf{Q}_{Max}$, over the entire time interval. When the control command takes a value at the boundary of its admissible region, it is said to be *saturated*. The function H is called *Hamiltonian*, and the vector $\mathbf{p}$ is called a *co-state.* ■

Example 371 ★ *A linear dynamic system.*

Consider a linear dynamic system given by

$$Q = \ddot{x} \tag{14.8}$$

or

$$\dot{\mathbf{x}} = [A]\,\mathbf{x} + \mathbf{b}Q \tag{14.9}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad [A] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14.10}$$

along with a constraint on the input variable

$$Q \leq 1. \tag{14.11}$$

By defining a co-state vector

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{14.12}$$

the Hamiltonian (14.5) becomes

$$H(\mathbf{x}, \mathbf{Q}, \mathbf{p}) = \mathbf{p}^T \left([A]\,\mathbf{x} + \mathbf{b}Q\right) \tag{14.13}$$

that provides two first-order differential equations

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}^T = [A]\,\mathbf{x} + \mathbf{b}Q \tag{14.14}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}^T = -[A]\,\mathbf{p}. \tag{14.15}$$

Equation (14.15) is

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -p_2 \end{bmatrix} \tag{14.16}$$

which can be integrated to find $\mathbf{p}$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ -C_1 t + C_2 \end{bmatrix}. \tag{14.17}$$

The Hamiltonian is then equal to:

$$H = Qp_2 + p_1 x_2 = (-C_1 t + C_2)\, Q + p_1 x_2 \tag{14.18}$$

The control command Q *only appears in*

$$\mathbf{p}^T \mathbf{b} Q = (-C_1 t + C_2)\, Q \tag{14.19}$$

which can be maximized by

$$Q(t) = \begin{cases} 1 & \text{if} \quad -C_1 t + C_2 \geq 0 \\ -1 & \text{if} \quad -C_1 t + C_2 < 0 \end{cases}. \tag{14.20}$$

This solution implies that $Q(t)$ *has a jump point at* $t = \frac{C_2}{C_1}$. *The jump point, at which the control command suddenly changes from maximum to minimum or from minimum to maximum, is called the* ***switching point****.*

Substituting the control input (14.20) into (14.9) gives us two first-order differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ Q \end{bmatrix}. \tag{14.21}$$

Equation (14.21) can be integrated to find the path $\mathbf{x}(t)$*.*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{1}{2}(t + C_3)^2 + C_4 \\ t + C_3 \end{bmatrix} & \text{if} \quad Q = 1 \\ \begin{bmatrix} -\frac{1}{2}(t - C_3)^2 + C_4 \\ -t + C_3 \end{bmatrix} & \text{if} \quad Q = -1 \end{cases} \tag{14.22}$$

The constants of integration, C_1, C_2, C_3, *and* C_4, *must be calculated based on the following boundary conditions:*

$$\mathbf{x}_0 = \mathbf{x}(t_0) \qquad \mathbf{x}_f = \mathbf{x}(t_f). \tag{14.23}$$

Eliminating t *between equations in (14.22) provides the relationship between the state variables* x_1 *and* x_2*.*

$$x_1 = \begin{cases} \frac{1}{2}x_2^2 + C_4 & \text{if} \quad Q = 1 \\ -\frac{1}{2}x_2^2 + C_4 & \text{if} \quad Q = -1 \end{cases} \tag{14.24}$$

These equations show a series of parabolic curves in the x_1x_2*-plane with* C_4 *as a parameter. The parabolas are shown in Figure 14.1(a) and (b) with the arrows indicating the direction of motion on the paths. The* x_1x_2*-plane is called the* ***phase plane****.*

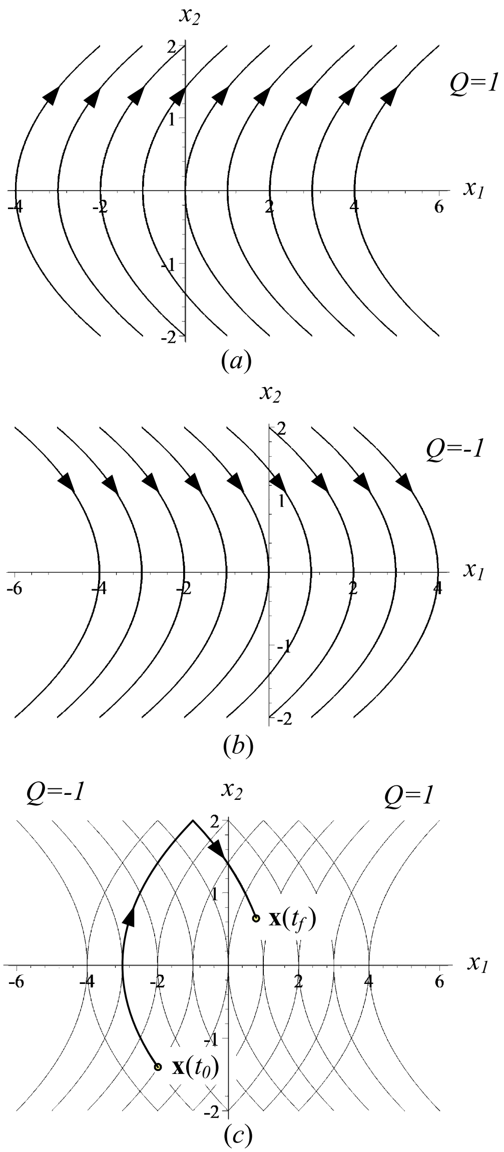

FIGURE 14.1. Optimal path for $Q = \ddot{x}$ in phase plane and the mesh of optimal paths in phase plane.

Considering that there is one switching point in this system, the overall optimal paths are shown in Figure 14.1(c). As an example, assume the state of the system at initial and final times are $\mathbf{x}(t_0)$ *and* $\mathbf{x}(t_f)$ *respectively. The motion starts with* $Q = 1$*, which forces the system to move on the control path* $x_1 = \frac{1}{2}x_2^2 + \left(x_{10} - \frac{1}{2}x_{20}^2\right)$ *up to the intersection point with* $x_1 = -\frac{1}{2}x_2^2 + \left(x_{1f} + \frac{1}{2}x_{2f}^2\right)$*. The intersection is the switching point at which the control input changes to* $Q = -1$*. The switching point is at*

$$x_1 = \frac{1}{4}\left(2x_{10} + 2x_f - x_{20}^2 + x_{2f}^2\right) \tag{14.25}$$

$$x_2 = \sqrt{\left(x_{1f} + \frac{1}{2}x_{2f}^2\right) - \left(x_{10} - \frac{1}{2}x_{20}^2\right)}. \tag{14.26}$$

Example 372 ★ *Robot equations in state equations.*

The vector form of the equations of motion of a robot is

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{Q}. \tag{14.27}$$

We can define a state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \tag{14.28}$$

and transform the equations of motion to an equation in state space

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{Q}(t)\right) \tag{14.29}$$

where

$$\mathbf{f}\left(\mathbf{x}(t), \mathbf{Q}(t)\right) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{D}^{-1}\left(\mathbf{Q} - \mathbf{H} - \mathbf{G}\right) \end{bmatrix}. \tag{14.30}$$

Example 373 ★ *Time-optimal control for robots.*

Assume that a robot is initially at

$$\mathbf{x}(\mathbf{t}_0) = \mathbf{x}_0 = \begin{bmatrix} \mathbf{q}_0 \\ \dot{\mathbf{q}}_0 \end{bmatrix} \tag{14.31}$$

and it is supposed to be finally at

$$\mathbf{x}(\mathbf{t}_f) = \mathbf{x}_f = \begin{bmatrix} \mathbf{q}_f \\ \dot{\mathbf{q}}_f \end{bmatrix} \tag{14.32}$$

in the shortest possible time. The torques of the actuators at each joint is assumed to be bounded

$$|Q_i| \le Q_{i_{Max}}. \tag{14.33}$$

The optimal control problem is to minimize the time performance index

$$J = \int_{t_0}^{t_f} dt = t_f - t_0. \tag{14.34}$$

The Hamiltonian H is defined as

$$H(\mathbf{x}, \mathbf{Q}, \mathbf{p}) = \mathbf{p}^T \mathbf{f}\left(\mathbf{x}(t), \mathbf{Q}(t)\right) \tag{14.35}$$

which provides the following two sets of equations:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}^T \tag{14.36}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}^T. \tag{14.37}$$

The optimal control input $\mathbf{Q}_t(t)$ is the one that minimizes the function H. Hamiltonian minimization reduces the time-optimal control problem to a two-point boundary value problem. The boundary conditions are the states of the robot at times t_0 and t_f. Due to nonlinearity of the robots' equations of motion, there is no analytic solution for the boundary value problem. Hence, a numerical technique must be developed.

Example 374 ★ *Euler-Lagrange equation.*

To show that a path $x = x^\star(t)$ is a minimizing path for the functional J

$$J(x) = \int_{t_0}^{t_f} f(x, \dot{x}, t)dt \tag{14.38}$$

with boundary conditions $x(t_0) = x_0$, $x(t_f) = x_f$, we need to show that

$$J(x) \geq J(x^\star) \tag{14.39}$$

for all continuous paths $x(t)$ satisfying the boundary conditions. Any path $x(t)$ satisfying the boundary conditions $x(t_0) = x_0$, $x(t_f) = x_f$, is called admissible. To see that $x^\star(t)$ is the optimal path, we may examine the integral J for every admissible path. An admissible path may be defined by

$$x(t) = x^\star + \epsilon y(t) \tag{14.40}$$

where

$$y(t_0) = y(t_f) = 0 \tag{14.41}$$

and

$$\epsilon \ll 1 \tag{14.42}$$

is a small number. Substituting $x(t)$ in J and subtracting from (14.38) provides ΔJ

$$\begin{aligned} \Delta J &= J\left(x^\star + \epsilon y(t)\right) - J\left(x^\star\right) \qquad (14.43) \\ &= \int_{t_0}^{t_f} f(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t)dt - \int_{t_0}^{t_f} f(x^\star, x^\star, t)dt. \end{aligned}$$

Let us expand $f(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t)$ *about* $(x^\star, \dot{x}^\star)$

$$f(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t) = f(x^\star, \dot{x}^\star, t) + \epsilon\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right)$$
$$+\epsilon^2\left(y^2\frac{\partial^2 f}{\partial x^2} + 2y\dot{y}\frac{\partial^2 f}{\partial x\,\partial \dot{x}} + \dot{y}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right)dt$$
$$+O\left(\epsilon^3\right) \tag{14.44}$$

and find

$$\Delta J = \epsilon V_1 + \epsilon^2 V_2 + O\left(\epsilon^3\right). \tag{14.45}$$

where

$$V_1 = \int_{t_0}^{t_f}\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right)dt \tag{14.46}$$

$$V_2 = \int_{t_0}^{t_f}\left(y^2\frac{\partial^2 f}{\partial x^2} + 2y\dot{y}\frac{\partial^2 f}{\partial x\,\partial \dot{x}} + \dot{y}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right) \tag{14.47}$$

The first integral, V_1 *is called the* ***first variation*** *of* J*, and the second integral,* V_2 *is called the* ***second variation*** *of* J*. All the higher variations are combined and shown as* $O\left(\epsilon^3\right)$*. If* $x^\star$ *is the minimizing curve, then it is necessary that* $\Delta J \geq 0$ *for every admissible* $y(t)$*. If we divide* ΔJ *by* ϵ *and make* $\epsilon \to 0$ *then we find a necessary condition for* $x^\star$ *to be the optimal path as* $V_1 = 0$*. This condition is equivalent to*

$$\int_{t_0}^{t_f}\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right)dt = 0. \tag{14.48}$$

By integrating by parts we may write

$$\int_{t_0}^{t_f}\dot{y}\frac{\partial f}{\partial \dot{x}}dt = \left(y\frac{\partial f}{\partial \dot{x}}\right)_{t_0}^{t_f} - \int_{t_0}^{t_f} y\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)dt. \tag{14.49}$$

Since $y(t_0) = y(t_f) = 0$*, the first term on the right-hand side is zero. Therefore, the minimization integral condition (14.48), for every admissible* $y(t)$*, reduces to*

$$\int_{t_0}^{t_f} y\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right)dt = 0. \tag{14.50}$$

The terms in the parentheses are continuous functions of t*, evaluated on the optimal path* $x^\star$*, and they do not involve* $y(t)$*. So, the only way that the bounded integral of the parentheses* $\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right)$*, multiplied by a nonzero function* $y(t)$*, from* t_0 *and* t_f *to be zero, is that*

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0. \tag{14.51}$$

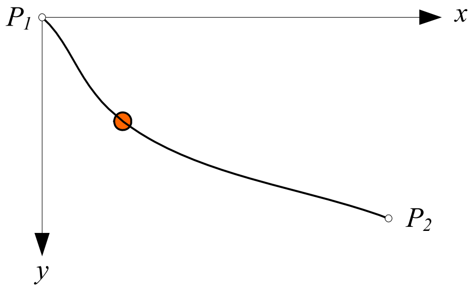

FIGURE 14.2. A curve joining points P_1 and P_2, and a frictionless sliding point.

The Equation (14.51) is a necessary condition for $x = x^{\star}(t)$ to be a solution of the minimization problem (14.38). This differential equation is called the ***Euler-Lagrange*** *equation. It is the same Lagrange equation that we utilized to derive the equations of motion of a robot. The second necessary condition to have $x = x^{\star}(t)$ as a minimizing solution is that the second variation, evaluated on $x^{\star}(t)$, must be negative.*

Example 375 ★ *The Lagrange equation for extremizing $J = \int_1^2 \dot{x}^2 dt$.*

The Lagrange equation for extremizing the functional

$$J = \int_1^2 \dot{x}^2 dt \tag{14.52}$$

is

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = -\ddot{x} = 0 \tag{14.53}$$

that shows the optimal path is

$$x = C_1 t + C_2. \tag{14.54}$$

Considering the boundary conditions $x(1) = 0$, $x(2) = 3$ provides

$$x = 3t - 3. \tag{14.55}$$

Example 376 ★ *Brachistochrone problem.*

We may utilize the Lagrange equation and find the frictionless curve joining two points as shown if Figure 14.2, along which a particle falling from rest due to gravity, travels from the higher to the lower point in the minimum time. This is the well-known ***brachistochrone*** *problem.*

If v is the velocity of the falling point along the curve, then the time required to fall an arc length ds is ds/v. Then, the objective function to find the curve of minimum time is

$$J = \int_1^2 \frac{ds}{v}. \tag{14.56}$$

However,

$$ds = \sqrt{1 + y'^2} dx \tag{14.57}$$

and according to the conservation of energy

$$v = \sqrt{2gy.} \tag{14.58}$$

Therefore, the objective function simplifies to

$$J = \int_1^2 \sqrt{\frac{1 + y'^2}{2gy}} dx. \tag{14.59}$$

Applying the Lagrange equations we find

$$y\left(1 + y'^2\right) = 2r \tag{14.60}$$

where r *is a constant. The optimal curve starting from* $y(0) = 0$ *can be expressed by two parametric equations*

$$x = r(\beta - \sin\beta) \tag{14.61}$$
$$y = r(1 - \cos\beta). \tag{14.62}$$

The optimal curve is a ***cycloid****.*

The name of the problem is derived from the Greek word "βραχιστοζ," meaning "shortest," and "χρονοζ," meaning "time." The brachistochrone problem was originally discussed by Galilei in 1630 *and later solved by Johann and Jacob Bernoulli in* 1696*.*

Example 377 ★ *Lagrange multiplier.*

Assume $f(x)$ *is defined on an open interval* (a, b) *and has continuous first and second order derivatives in some neighborhood of* $x_0 \in (a, b)$*. The point* x_0 *is a local extremum of* $f(x)$ *if*

$$\frac{df(x_0)}{dx} = 0. \tag{14.63}$$

Assume $f(\mathbf{x}) = 0$*,* $\mathbf{x} \in \mathbb{R}^n$ *and* $g_i(\mathbf{x}) = 0$*,* $i = 1, 2, \cdots, m$ *are functions defined on an open region* $\mathbb{R}^n$ *and have continuous first and second order derivatives in* $\mathbb{R}^n$*. The necessary condition that* $\mathbf{x}_0$ *be an extremum of* $f(\mathbf{x})$ *subject to the constraints* $g_i(\mathbf{x}) = 0$ *is that there exist* m ***Lagrange multipliers*** λ_i*,* $i = 1, 2, \cdots, m$ *such that*

$$\nabla\left(s + \sum \lambda_i g_i\right) = 0. \tag{14.64}$$

As an example, we can find the minimum of

$$f = 1 - x_1^2 - x_2^2 \tag{14.65}$$

subject to

$$g = x_1^2 + x_2 - 1 = 0 \tag{14.66}$$

by finding the gradient of $f + \lambda g$

$$\nabla \left(1 - x_1^2 - x_2^2 + \lambda \left(x_1^2 + x_2 - 1\right)\right) = 0. \tag{14.67}$$

That leads to

$$\frac{\partial f}{\partial x_1} = -2x_1 + 2\lambda x_1 = 0 \tag{14.68}$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + \lambda = 0. \tag{14.69}$$

To find the three unknowns, x_1*,* x_2*, and* λ*, we employ Equations (14.68), (14.69), and (14.66). There are two sets of solutions as follows:*

$$\begin{array}{lll} x_1 = 0 & x_2 = 1 & \lambda = 2 \\ x_1 = \pm 1/\sqrt{2} & x_2 = 1/2 & \lambda = 1 \end{array} \tag{14.70}$$

Example 378 ★ *Admissible control function.*

The components of the control command $\mathbf{Q}(t)$ *are allowed to be piecewise continuous and the values they can take may be any number within the bounded region of the control space. As an example, consider a 2 DOF system* $\mathbf{Q}(t) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$ *with the restriction* $|Q_i| < 1$*,* $i = 1, 2$*. The control space is a circle in the plane* Q_1Q_2*. The control components may have any piecewise continuous value within the circle. Such controls are called admissible.*

Example 379 *Description of the time optimal control problem.*

The aim of minimum time control is to guide the robot on a path in minimum time to increase the robot's productivity. Except for low order, autonomous, and linear problems, there is no general analytic solution for the time optimal control problems of dynamic systems. The problem of time optimal control is always a bounded input problem. If there exists an admissible time optimal control for a given initial condition and final target, then, at any time, at least one of the control variables attains its maximum or minimum value. Based on Pontryagin's principle, the solution of minimum time problems with bounded inputs is a bang-bang control, indicating that at least one of the input actuators must be saturated at any time. However, finding the switching points at which the saturated input signal is replaced with another saturated signal is not straightforward, and is the main concern of numerical solution methods.

In a general case, the problem reduces to a two-points boundary value problem that is difficult to solve. The corresponding nonsingular, nonlinear two-point boundary value problem must be solved to determine the switching times. A successful approach is to assume that the configuration trajectory of the dynamical system is preplanned, and then reduce the problem to a minimum time motion along the trajectory.

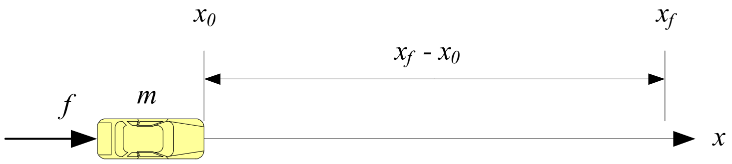

FIGURE 14.3. Rest-to-rest motion of a mass on a straight line time-optimally.

14.2 ★ Floating Time Method

Consider a particle with mass m, as shown in Figure 14.3, is moving according to the following equation of motion:

$$m\ddot{x} = g(x, \dot{x}) + f(t) \tag{14.71}$$

where $g(x, \dot{x})$ is a general nonlinear external force function, and $f(t)$ is the unknown input control force function. The control command $f(t)$ is bounded to

$$|f(t)| \leq F. \tag{14.72}$$

The particle starts from rest at position $x(0) = x_0$ and moves on a straight line to the destination point $x(t_f) = x_f$ at which it stops.

We can solve this rest-to-rest control problem and find the required $f(t)$ to move m from $\dot{x}_0$ to $\dot{x}_f$ in minimum time utilizing the *floating time algorithm*.

Algorithm 14.1. Floating time technique.

1. *Divide the preplanned path of motion* $x(t)$ *into* $s+1$ *intervals and specify all coordinate values* x_i, $(i = 0, 1, 2, 3, ..., s+1)$

2. *Set* $f_0 = +F$ *and calculate*

$$\tau_0 = \sqrt{\frac{2m\,(x_1 - x_0)}{F}} \tag{14.73}$$

3. *Set* $f_{s+1} = -F$ *and calculate*

$$\tau_s = \sqrt{\frac{2m\,(x_s - x_{s-1})}{-F}} \tag{14.74}$$

4. *For* i *from* 1 *to* $s-1$, *calculate* τ_i *such that* $f_i = +F$ and

$$\begin{aligned} f_i &= m\ddot{x}_i - g(x_i, \dot{x}_i) \\ &= \frac{4m}{\tau_i^2 + \tau_{i-1}^2}\left(\frac{\tau_{i-1}}{\tau_i + \tau_{i-1}}x_{i+1} + \frac{\tau_i}{\tau_i + \tau_{i-1}}x_{i-1} + x_i\right) \\ &\quad -g(x_i, \dot{x}_i) \end{aligned} \tag{14.75}$$

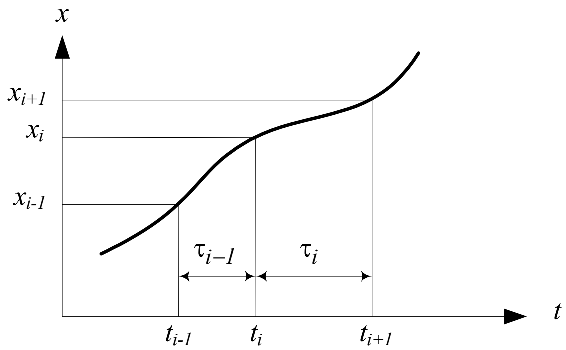

FIGURE 14.4. Time history of motion for the point mass m.

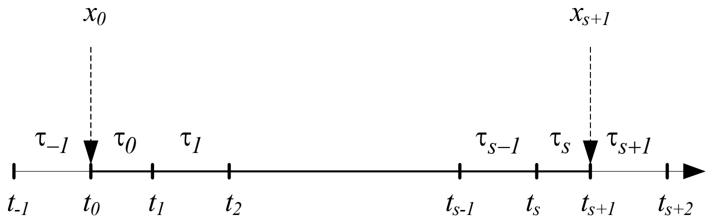

FIGURE 14.5. Introducing two extra points, x_{-1} and x_{s+2}, before the initial and after the final points.

5. *If* $|f_i| \le F$*, then stop, otherwise set* $j = s$*,*
6. *Calculate* τ_{j-1} *such that* $f_j = -F$
7. *If* $|f_{j-1}| \le F$*, then stop, otherwise set* $j = j - 1$ *and return to step 6*

Proof. Assume $g(x_i, \dot{x}_i) = 0$ and $x(t)$, as shown in Figure 14.4, is the time history of motion for the point mass m. We divide the path of motion into $s+1$ arbitrary, and not necessarily equal, segments. Hence, the coordinates x_i, $(i = 0, 1, 2, ..., s+1)$ are known. The *floating-time* $\tau_i = t_i - t_{i-1}$ is defined as the required time to move m from x_i to x_{i+1}.

Utilizing the central difference method, we may define the first and second derivatives at point i by

$$\dot{x}_i = \frac{x_{i+1} - x_{i-1}}{\tau_i + \tau_{i-1}} \tag{14.76}$$

$$\ddot{x}_i = \frac{4}{\tau_i^2 + \tau_{i-1}^2} \left(\frac{\tau_{i-1}}{\tau_i + \tau_{i-1}} x_{i+1} + \frac{\tau_i}{\tau_i + \tau_{i-1}} x_{i-1} - x_i \right). \tag{14.77}$$

These equations indicate that the velocity and acceleration at point i depend on x_i, and two adjacent points x_{i-1}, and x_{i+1}, as well as on the floating times τ_i, and τ_{i-1}. Therefore, two extra points, x_{-1} and x_{s+2}, before the initial point and after the final point are needed to define velocity and acceleration at x_0 and x_{s+1}. These extra points and their corresponding floating times are shown in Figure 14.5.

The rest conditions at the beginning and at the end of motion require

$$x_{-1} = x_1 \qquad x_{s+2} = x_s \tag{14.78}$$

$$\tau_0 = \tau_{-1} \qquad \tau_{s+1} = \tau_s. \tag{14.79}$$

Using Equation (14.77), the equation of motion, $f_i = m\ddot{x}_i$, at the initial point is

$$f_0 = m\ddot{x}_0 = \frac{4m}{2\tau_0^2}\left(x_1 - x_0\right). \tag{14.80}$$

The minimum value of the first floating time τ_0 is found by setting $f_0 = F$.

$$\tau_0 = \sqrt{\frac{2m\left(x_1 - x_0\right)}{F}} \tag{14.81}$$

It is the minimum value of the first floating time because, if τ_0 is less than the value given by (14.81), then f_0 will be greater than F and breaks the constraint (14.72). On the other hand, if τ_0 is greater than the value given by (14.81), then f_0 will be less than F and the input is not saturated yet. The same conditions exist at the final point where the equation of motion is

$$f_{s+1} = m\ddot{x}_{s+1} = \frac{4m}{2\tau_s^2}\left(x_s - x_{s-1}\right). \tag{14.82}$$

The minimum value of the final floating-time, τ_s, is achieved by setting $f_{s+1} = -F$.

$$\tau_s = \sqrt{\frac{2m\left(x_s - x_{s-1}\right)}{-F}} \tag{14.83}$$

To find the minimum value of τ_1, we develop the equation of motion at x_1

$$f_1 = \frac{4m}{\tau_1^2 + \tau_0^2}\left(\frac{\tau_0}{\tau_1 + \tau_0}x_2 + \frac{\tau_1}{\tau_1 + \tau_0}x_0 - x_1\right) \tag{14.84}$$

which is an equation with two unknowns f_1 and τ_1. We are able to find τ_1 numerically by adjusting τ_1 to provide $f_1 = F$. Applying this procedure we are able to find the minimum floating times τ_{i+1} by applying the maximum force constraint $f_i = F$, and solving the equation of motion for τ_{i+1} numerically. When τ_i is known and the maximum force is applied to find the next floating-time, τ_{i+1}, we are in the *forward path of the floating time algorithm.* In the last step of the forward path, τ_{s-1} is found at x_{s-1}. At this step, all the floating-times τ_i, $(i = 0, 1, 2, ..., s)$ are known, while

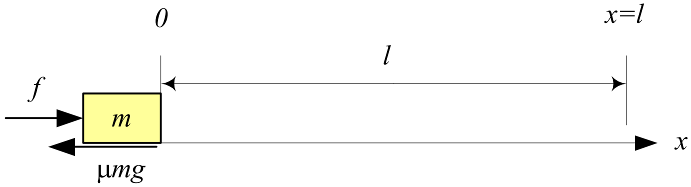

FIGURE 14.6. A rectilinear motion of a rigid mass m under the influence of a control force $f(t)$ and a friction force μmg.

$f_i = F$ for $i = 0, 1, 2, ..., s-1$, and $f_i = -F$ for $i = s$. Then, the force f_s is the only variable that is not calculated during the forward path by using the equation of motion at point x_s. It is actually dictated by the equation of motion at point x_s, because τ_{s-1} is known from the forward path procedure, and τ_s is known from Equation (14.83) to satisfy the final point condition. So, the value of f_s can be found from the equation of motion at $i = s$ and substituting τ_s, τ_{s-1}, x_{s-1}, x_s, and x_{s+1}.

$$f_s = \frac{4m}{\tau_s^2 + \tau_{s-1}^2} \left(\frac{\tau_{s-1}}{\tau_s + \tau_{s-1}} x_{s+1} + \frac{\tau_s}{\tau_s + \tau_{s-1}} x_{s-1} - x_s \right) \tag{14.85}$$

Now, if f_s does not break the constraint $|f(t)| \leq F$, the problem is solved and the minimum time motion is determined. The input signals, f_i, ($i = 0, 1, 2, ..., s+1$), $i \neq s$, are always saturated, and also none of the floating-times τ_i can be reduced any more. However, it is expected that f_s breaks the constraint $|f(t)| \leq F$ because accelerating in $s-1$ steps with $f = F$ produces a large amount of kinetic energy and a huge deceleration is needed to stop the mass m in the final step.

Now we reverse the procedure, and start a *backward path*. According to (14.85), f_s can be adjusted to satisfy the constraint $f_s = -F$ by tuning τ_{s-1}. Now f_{n-1} must be checked for the constraint $|f(t)| \leq F$. This is because τ_{s-2} is already found in the forward path, and τ_{s-1} in the backward path. Hence, the value of f_{n-1} is dictated by the equation of motion at point x_{s-1}. If f_{s-1} does not break the constraint $|f(t)| \leq F$, the problem is solved and the time-optimal motion is achieved. Otherwise, the backward path must be continued to a point where the force constraint is satisfied. The position x_k in the backward path, where $|f_k| \leq F$, is called switching point because $f_j = F$ for $j < k$, $0 \leq j < k$ and $f_j = -F$ for $j > k$, $k < j \leq s+1$. ■

Example 380 ★ *Moving a mass on a rough surface.*

Consider a rectilinear motion of a rigid mass m under the influence of a variable force $f(t)$ and a friction force μmg, as shown in Figure 14.6. The force is bounded by $|f(t)| \leq F$, where $\pm F$ is the limit of available force. It is necessary to find a function $f(t)$ that moves m, from the initial conditions

$x(0) = 0$, $v(0) = 0$ to the final conditions $x(t_f) = l > 0$, $v(t_f) = 0$ in minimum total time $t = t_f$. The motion is described by the following equation of motion and boundary conditions:

$$f = m\ddot{x} - \mu mg \tag{14.86}$$

$$\begin{array}{ll} x(0) = 0 & v(0) = 0 \\ x(t_f) = l & v(t_f) = 0. \end{array} \tag{14.87}$$

Using the theory of optimal control, we know that a time optimal control solution for $\mu = 0$ is a piecewise constant function where the only discontinuity is at the switching point $t = \tau = t_f/2$ and

$$f(t) = \begin{cases} F & \text{if} \quad t < \tau \\ -F & \text{if} \quad t > \tau. \end{cases} \tag{14.88}$$

Therefore, the time optimal control solution for moving a mass m from $x(0) = x_0 = 0$ to $x(t_f) = x_f = l$ on a smooth straight line is a bang-bang control with only one switching time. The input force $f(t)$ is on its maximum, $f = F$, before the switching point $x = (x_f - x_0)/2$ at $\tau = t_f/2$, and $f = -F$ after that. Any asymmetric characteristics, such as friction, will make the problem asymmetric by moving the switching point.

In applying the floating-time algorithm, we assume that a particle of unit mass, $m = 1\,\text{kg}$, slides under Coulomb friction on a rough horizontal surface. The magnitude of the friction force is μmg, where μ is the friction coefficient and $g = 9.81\,\text{m/s}^2$. We apply the floating-time algorithm using the following numerical values:

$$F = 10\,\text{N} \qquad l = 10\,\text{m} \qquad s + 1 = 200 \tag{14.89}$$

Figures 14.7, 14.8, and 14.9 show the results for some different values of μ. Figure 14.7 illustrates the time history of the optimal input force for different values of μ. Each curve is indicated by the value of μ and the corresponding minimum time of motion t_f. Time history of the optimal motions $x(t)$ are shown in Figure 14.8, while the time history of the optimal inputs $f(t)$ are shown in Figure 14.9. The switching times and positions are shown in Figures 14.7 and 14.9, respectively.

If $\mu = 0$ then switching occurs at the midpoint of the motion $x(\tau) = l/2$ and halfway through the time $\tau = t_f/2$. Increasing μ delays both the switching times and the switching positions. The total time of motion also increases by increasing μ.

Example 381 ★ *First and second derivatives in central difference method.*

Using a Taylor series, we expand x at points x_{i-1} and x_{i+1} as an extrapolation of point x_i

$$x_{i+1} = x_i + \dot{x}_i \tau_i + \frac{1}{2}\ddot{x}_i \tau_i^2 + \cdots \tag{14.90}$$

$$x_{i-1} = x_i - \dot{x}_i \tau_{i-1} + \frac{1}{2}\ddot{x}_i \tau_{i-1}^2 - \cdots . \tag{14.91}$$

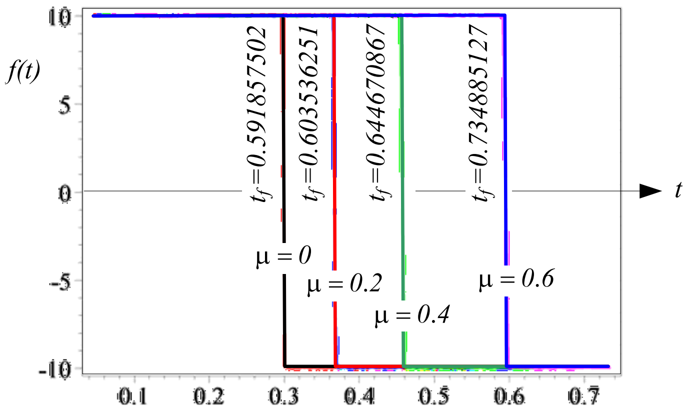

FIGURE 14.7. Time history of the optimal input $f(t)$ for different friction coefficients μ.

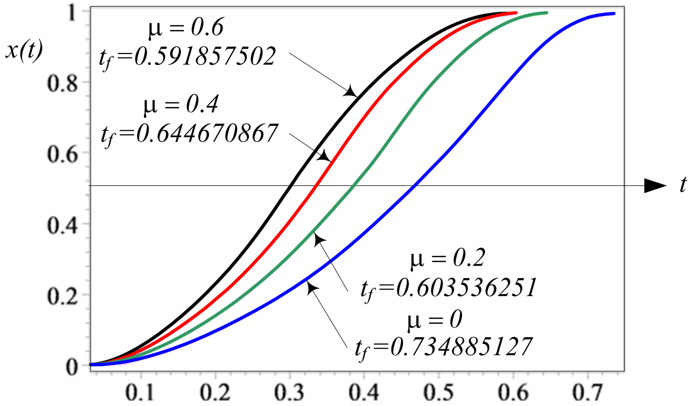

FIGURE 14.8. Time history of the optimal motion $x(t)$ for different friction coefficients μ.

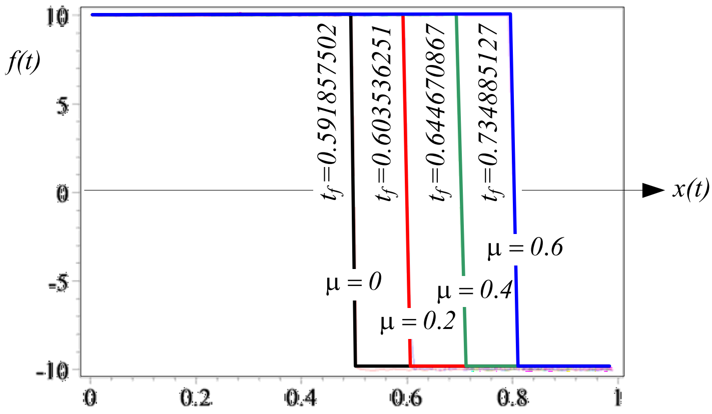

FIGURE 14.9. Position history of the optimal force $f(t)$ for different friction coefficients μ.

Accepting the first two terms and calculating $x_{i+1} - x_{i-1}$ *provides*

$$\dot{x}_i = \frac{x_{i+1} - x_{i-1}}{\tau_i + \tau_{i-1}}. \tag{14.92}$$

Now, accepting the first three terms of the Taylor series and calculating $x_{i+1} + x_{i-1}$ *provides*

$$\ddot{x}_i = \frac{4m}{\tau_i^2 + \tau_{i-1}^2}\left(\frac{\tau_{i-1}}{\tau_i + \tau_{i-1}}x_{i+1} + \frac{\tau_i}{\tau_i + \tau_{i-1}}x_{i-1} - x_i\right). \tag{14.93}$$

Example 382 ★ *Convergence.*

The floating time algorithm presents an iterative method hence, convergence criteria must be identified. In addition, a condition must be defined to terminate the iteration. In the forward path, we calculate the floating-time τ_i *by adjusting it to a value that provides* $f_i = F$*. The floating-time* τ_i *converges to the minimum possible value, as long as* $\partial\ddot{x}_i/\partial\tau_i < 0$ *and* $\partial\ddot{x}_i/\partial\tau_{i-1} > 0$*. Figure 14.10 illustrates the behavior of* $\ddot{x}_i$ *as a function of* τ_i *and* τ_{i-1}*. Using the Equation (14.77), the required conditions are fulfilled within a basin of convergence,*

$$Z_1 x_{s+1} + Z_2 x_s + Z_3 x_{s-1} < 0 \tag{14.94}$$

$$Z_4 x_{s+1} + Z_5 x_s + Z_6 x_{s-1} > 0 \tag{14.95}$$

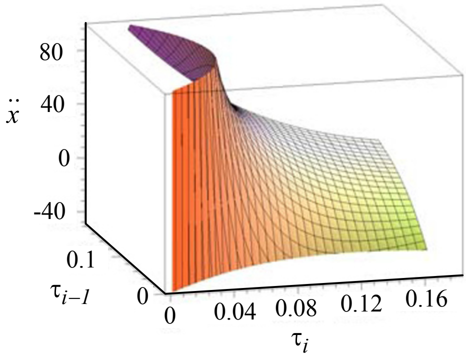

FIGURE 14.10. Behavior of $\ddot{x}_i$ as a function of τ_i and τ_{i-1}.

where,

$$Z_1 = \frac{8\left(6\tau_i^4\tau_{i-1} + 8\tau_i^3\tau_{i-1}^2 + 6\tau_i^2\tau_{i-1}^3\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3} \tag{14.96}$$

$$Z_2 = \frac{8\left(\tau_{i-1}^5 - 3\tau_i^5 - 8\tau_i^3\tau_{i-1}^2 - 9\tau_i^4\tau_{i-1} + 3\tau_i\tau_{i-1}^4\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3} \tag{14.97}$$

$$Z_3 = \frac{8\left(-\tau_{i-1}^5 + 3\tau_i^5 + 3\tau_i^4\tau_{i-1} - 3\tau_i\tau_{i-1}^4 - 6\tau_i^2\tau_{i-1}^3\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3} \tag{14.98}$$

$$Z_4 = \frac{8\left(3\tau_{i-1}^5 - \tau_i^5 - 6\tau_i^3\tau_{i-1}^2 - 3\tau_i^4\tau_{i-1} + 3\tau_i\tau_{i-1}^4\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3} \tag{14.99}$$

$$Z_5 = \frac{8\left(-3\tau_{i-1}^5 + \tau_i^5 + 3\tau_i^4\tau_{i-1} - 9\tau_i\tau_{i-1}^4 - 8\tau_i^2\tau_{i-1}^3\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3} \tag{14.100}$$

$$Z_6 = \frac{8\left(6\tau_i^3\tau_{i-1}^2 + 8\tau_i^2\tau_{i-1}^3 + 6\tau_i\tau_{i-1}^4\right)}{\left(\tau_i^2+\tau_{i-1}^2\right)^3\left(\tau_i+\tau_{i-1}\right)^3}. \tag{14.101}$$

The convergence conditions guarantee that $\ddot{x}_i$ decreases with an increase in τ_i, and increases with an increase in τ_{i-1}. Therefore, if either τ_i or τ_{i-1} is fixed, we are able to find the other floating time by setting $f_i = F$. Convergence conditions for backward path are changed to

$$Z_1x_{s+1} + Z_2x_s + Z_3x_{s-1} > 0 \tag{14.102}$$

$$Z_4x_{s+1} + Z_5x_s + Z_6x_{s-1} < 0. \tag{14.103}$$

A termination criterion may be defined by

$$||f_i| - F| \leq \epsilon. \tag{14.104}$$

where ϵ is a user-specified number. The termination criterion provides a good method to make sure that the maximum deviation is within certain bounds.

Example 383 ★ *Analytic calculating of floating times.*

The rest condition at the beginning of the motion of an m on a straight line requires

$$x_{-1} = x_1 \tag{14.105}$$

$$\tau_0 = \tau_{-1}. \tag{14.106}$$

The first floating time τ_0 is found by setting $f_0 = F$ and developing the equation of motion $f_i = m\ddot{x}_i$ at point x_0.

$$\tau_0 = \sqrt{\frac{2m\left(x_1 - x_0\right)}{F}} \tag{14.107}$$

Now the equation of motion at point x_1 is

$$f_1 = \frac{4m}{\tau_1^2 + \tau_0^2}\left(\frac{\tau_0}{\tau_1 + \tau_0}x_2 + \frac{\tau_1}{\tau_1 + \tau_0}x_0 + x_1\right). \tag{14.108}$$

Substituting τ_0 from (14.107) into (14.108) and applying $f_1 = F$ provides the following equation

$$F = \frac{4mF}{2mx_1 - 2mx_0 + F\tau_1^2}\left(\frac{\tau_0 x_2 + \tau_1 x_0}{\tau_1 + \sqrt{\frac{2m(x_1 - x_0)}{F}}} + x_1\right) \tag{14.109}$$

that must be solved for τ_1. Then substituting τ_1 from (14.109) into the equation of motion at x_2, and setting $f_2 = F$ leads to a new equation to find τ_2. This procedure can similarly be applied to the other steps. However, calculating the floating times in closed form is not straightforward and getting more complicated step by step, hence, a numerical solution is needed. The equations for calculating τ_i are nonlinear and therefore have multiple solutions. Each positive solution must be examined for the constraint $f_i = F$. Negative solutions are not acceptable.

Example 384 ★ *Brachistochrone and path planning.*

The floating-time method is sometimes applicable for path planning problems. As an illustrative example, we considered the well-known brachistochrone problem. As Johann Bernoulli says: "A material particle moves without friction along a curve. This curve connects point A with point B (point A is placed above point B). No forces affect it, except the gravitational attraction. The time of travel from A to B must be the smallest. This brings up the question: what is the form of this curve?"

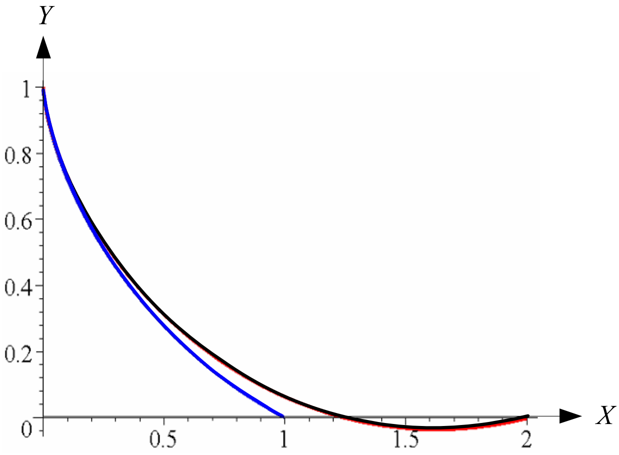

FIGURE 14.11. Time optimal path for a falling unit mass from $A(0,1)$ to two different destinations.

The classical solution of the brachistochrone problem is a cycloid and its parametric equation is

$$x = r(\beta - \sin\beta) \tag{14.110}$$

$$y = r(1 - \cos\beta). \tag{14.111}$$

where r is the radius of the corresponding cycloid and β is the angle of rotation of r. When $\beta = 0$ the particle is at the beginning point $A(0,0)$. The particle is at the second point B when $\beta = \beta_B$. The value of β_B can be obtained from

$$x_B = r(\beta_B - \sin\beta_B) \tag{14.112}$$

$$y_B = r(1 - \cos\beta_B). \tag{14.113}$$

The total time of the motion is

$$t_f = \beta_B \sqrt{\frac{r}{g}} \tag{14.114}$$

In a path-planning problem, except for the boundaries, the path of motion is not known. Hence, the position of x_i in Equations (14.76) and (14.77) are not given. Knowing the initial and final positions, we fix the x_i coordinates while keeping the y_i coordinates free. We will obtain the optimal path of motion by applying the known input force and searching for the optimum y_i that minimizes the floating times.

Consider the points $B_1(1,0)$ and $B_2(2,0)$ as two different destinations of motion for a unit mass falling from point $A(0,1)$. Figure 14.11 illustrates *the optimal path of motion for the two destinations, obtained by the floating-time method for* $s = 100$. *The total time of motion is* $t_{f_1} = 0.61084\,\mathrm{s}$, *and* $t_{f_2} = 0.8057\,\mathrm{s}$ *respectively. In this calculation, the gravitational acceleration is assumed* $g = 10\,\mathrm{m/\,s^2}$ *in* $-Y$ *direction.*

An analytic solution shows that $\beta_{B_1} = 1.934563\,\mathrm{rad}$ *and* $\beta_{B_2} = 2.554295\,\mathrm{rad}$. *The corresponding total times are* $t_{f_1} = 0.6176\,\mathrm{s}$, *and* $t_{f_2} = 0.8077\,\mathrm{s}$ *respectively. By increasing* s, *the calculated minimum time would be closer to the analytical results, and the evaluated path would be closer to a cycloid.*

A more interesting and more realistic problem of brachistochrone can be brachistochrone with friction and brachistochrone with linear drag. Although there are analytical solutions for these two cases, no analytical solution has been developed for brachistochrone with nonlinear (say second degree) drag. Applying the floating-time algorithm for this kind of problem can be an interesting challenge.

14.3 ★ Time-Optimal Control for Robots

Robots are multiple *DOF* dynamical systems. In case of a robot with n *DOF*, the control force $\mathbf{f}$ and the output position $\mathbf{x}$ are vectors.

$$\mathbf{f} = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^T \tag{14.115}$$

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \tag{14.116}$$

The constraint on the input force vector can be shown by

$$|\mathbf{f}_i| \leq \mathbf{F} \tag{14.117}$$

where the elements of the limit vector $\mathbf{F} \in \mathbb{R}^n$ may be different. The floating time algorithm is applied similar to the algorithm 14.2, however, at each step all the elements of the force vector $\mathbf{f}$ must be examined for their constraints. To attain the time optimal control, at least one element of the input vector $\mathbf{f}$ must be saturated at each step, while all the other elements are within their limits.

Algorithm 14.2. Floating time technique for the n *DOF* systems.

1. *Divide the preplanned path of motion* $\mathbf{x}(t)$ *into* $s+1$ *intervals and specify all coordinate vectors* $\mathbf{x}_i$, $(i = 0, 1, 2, 3, ..., s+1)$.

2. *Develop the equations of motion at* $\mathbf{x}_0$ *and calculate* τ_0 *for which only one component of the force vector* $\mathbf{f}_0$ *is saturated on its higher limit, while all the other components are within their limits.*

$$\begin{array}{lll} f_{0_k} = F_k & , \quad k \in \{0, 1, 2, \cdots, n\} & \\ f_{0_r} \leq F_r & , \quad r = 0, 1, 2, \cdots, n & , \quad r \neq k \end{array}$$

3. *Develop the equations of motion at* $\mathbf{x}_{s+1}$ *and calculate* τ_s for which only one component of the force vector $\mathbf{f}_{s+1}$ *is saturated on its higher limit,* while all the other components are within their limits.

$$\begin{array}{lllll} f_{s+1_k} = -F_k & , & k \in \{0, 1, 2, \cdots, n\} & & \\ f_{s+1_r} \leq F_r & , & r = 0, 1, 2, \cdots, n & , & r \neq k \end{array}$$

4. *For* i *from* 1 *to* $s-1$, *calculate* τ_i *such that only one component of the force vector* $\mathbf{f}_i$ is saturated on its higher limit, *while all the other components are within their limits.*

$$\begin{array}{lllll} f_{i_k} = -F_k & , & k \in \{0, 1, 2, \cdots, n\} & & \\ f_{i_r} \leq F_r & , & r = 0, 1, 2, \cdots, n & , & r \neq k \end{array}$$

5. *If* $|\mathbf{f}_s| \leq \mathbf{F}$, *then stop, otherwise set* $j = s$.

6. *Calculate* τ_{j-1} *such that only one component of the force vector* $\mathbf{f}_j$ *is saturated on its lower limit, while all the other components are within their limits.*

7. *If* $|\mathbf{f}_{j-1}| \leq F$, *then stop, otherwise set* $j = j - 1$ *and return to step 6.*

Example 385 ★ $2R$ *manipulator on a straight line.*

Consider a $2R$ *planar manipulator that its endpoint moves rest-to-rest from point* $(1, 1.5)$ *to point* $(-1, 1.5)$ *on a straight line* $Y = 1.5$. *Figure 14.12 illustrates a* $2R$ *planar manipulator with rigid arms. The manipulator has two rotary joints, whose angular positions are defined by the coordinates* θ *and* φ. *The joint axes are both parallel to the* Z*-axis of the global coordinate frame, and the robot moves in the* XY*-plane. Gravity acts in the* $-Y$ *direction and the lengths of the arms are* l_1 *and* l_2.

We express the equations of motion for $2R$ *robotic manipulators in the following form:*

$$P = A\ddot{\theta} + B\ddot{\varphi} + C\dot{\theta}\dot{\varphi} + D\dot{\varphi}^2 + M \tag{14.118}$$

$$Q = E\ddot{\theta} + F\ddot{\varphi} + G\dot{\theta}^2 + N \tag{14.119}$$

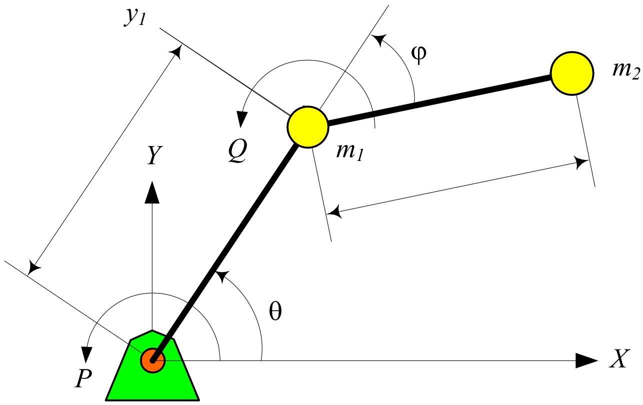

FIGURE 14.12. A 2R planar manipulator with rigid arms.

where P and Q are the actuator torques and

$$\begin{aligned}
A &= A(\varphi) = m_1 l_1^2 + m_2 \left(l_1^2 + l_2^2 + 2l_1 l_2 \cos\varphi\right) \\
B &= B(\varphi) = m_2 \left(l_2^2 + l_1 l_2 \cos\varphi\right) \\
C &= C(\varphi) = -2m_2 l_1 l_2 \sin\varphi \\
D &= D(\varphi) = m_2 l_1 l_2 \sin\varphi \\
E &= E(\varphi) = B \\
F &= m_2 l_2^2 \\
G &= G(\varphi) = -D \\
M &= M(\theta,\varphi) = (m_1 + m_2) g l_1 \cos\theta + m_2 g l_2 \cos(\theta+\varphi) \\
N &= N(\theta,\varphi) = m_2 g l_2 \cos(\theta+\varphi). \qquad (14.120)
\end{aligned}$$

Following Equations (14.76) and (14.77), we define two functions to discretize the velocity and acceleration.

$$v(\dot{x}_i) = \frac{x_{i+1} - x_{i-1}}{\tau_i + \tau_{i-1}} \qquad (14.121)$$

$$a(\ddot{x}_i) = \frac{4}{\tau_i^2 + \tau_{i-1}^2} \left(\frac{\tau_{i-1}}{\tau_i + \tau_{i-1}} x_{i+1} + \frac{\tau_i}{\tau_i + \tau_{i-1}} x_{i-1} - x_i \right). \qquad (14.122)$$

Then, the equations of motion at each instant may be written as

$$P_i(t) = A_i a(\ddot{\theta}) + B_i a(\ddot{\varphi}) + C_i v(\dot{\theta}) v(\dot{\varphi}) + D_i v^2(\dot{\varphi}) + M_i \qquad (14.123)$$

$$Q_i(t) = E_i a(\ddot{\theta}) + F_i a(\ddot{\varphi}) + G_i v^2(\dot{\theta}) + N_i \qquad (14.124)$$

where P_i and Q_i are the required actuator torques at instant i. Actuators are assumed to be bounded by

$$|P_i(t)| \le P_M \qquad |Q_i(t)| \le Q_M. \qquad (14.125)$$

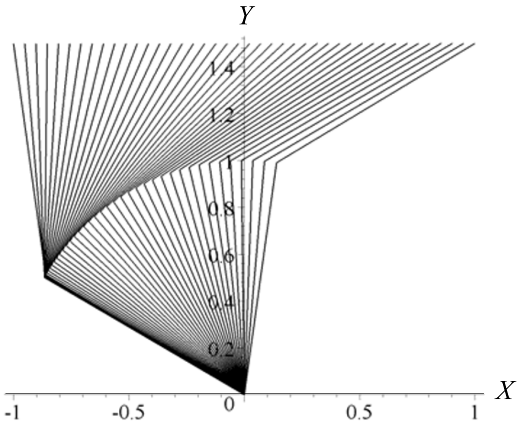

FIGURE 14.13. A 2R planar manipulator, moving from point $(1, 1.5)$ to point $(-1, 1.5)$ on a straight line $Y = 1.5$.

To move the manipulator time optimally along a known trajectory, motors must exert torques with a known time history. Using the floating-time method, the motion starts at point $i = 0$ *and ends at point* $i = s + 1$*. Introducing two extra points at* $i = -1$ *and* $i = n + 2$*, and applying the rest boundary conditions, we find*

$$\begin{array}{lll} \tau_0 = \tau_1 & \theta_0 = \theta_2 & \varphi_0 = \varphi_2 \\ \tau_s = \tau_{s+1} & \theta_s = \theta_{s+2} & \varphi_s = \varphi_{s+2}. \end{array} \tag{14.126}$$

All inputs of the manipulator at instant i *are controlled by the common floating-times* τ_i *and* τ_{i-1}*. In the forward path, when one of the inputs saturates at instant* i*, while the others are less than their limits, the minimum* τ_i *is achieved. Any reduction in* τ_i *increases the saturated input and breaks one of the constraints (14.125). The same is true in the backward path when we search for* τ_{i-1}*.*

Consider the following numerical values and the path of motion illustrated in Figure 14.13.

$$m_1 = m_2 = 1\,\mathrm{kg} \quad l_1 = l_2 = 1\,\mathrm{m} \quad P_M = Q_M = 100\,\mathrm{N\,m} \tag{14.127}$$

To apply the floating time algorithm, the path of motion in Cartesian space must first be transformed into joint space using inverse kinematics. Then, the path of motion in joint space must be discretized to an arbitrary interval, say 200*, and the algorithm* 14.2 *should be applied.*

Figure 14.14 depicts the actuators' torque for minimum time motion after applying the floating time algorithm. In this maneuver, there exists one switching point, where the grounded actuator switches from maximum to minimum. The ungrounded actuator never saturates, but as expected, one

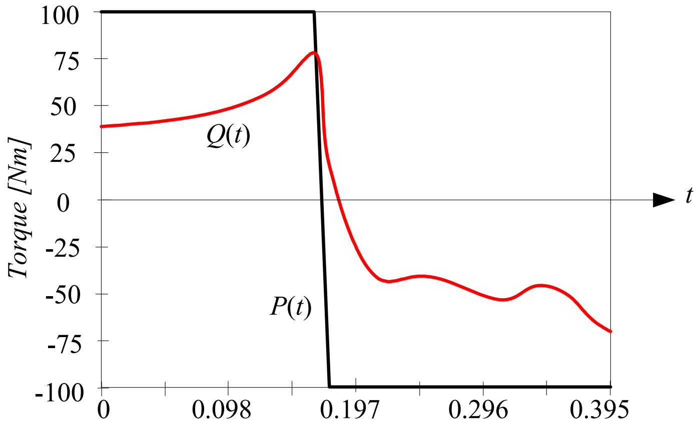

FIGURE 14.14. Time optimal control inputs for a 2R manipulator moving on line $Y = 1.5$, $-1 < X < 1$.

of the inputs is always saturated. Calculating the floating times allows us to calculate the kinematics information of motion in joint coordinate space. Time histories of the joint coordinates can be utilized to determine the kinematics of the end-effector in Cartesian space.

Example 386 ★ *Multiple switching points.*

The $2R$ *manipulator shown in Figure 14.12 is made to follow the path illustrated in Figure 14.15. The floating time algorithm is run for the following data:*

$$\begin{array}{lll} m_1 = m_2 = 1\,\mathrm{kg} & l_1 = l_2 = 1\,\mathrm{m} & P_M = Q_M = 100\,\mathrm{N\,m} \\ X(0) = 1.9\,\mathrm{m} & X(t_f) = 0.5\,\mathrm{m} & Y = 0 \end{array} \tag{14.128}$$

which leads to the solution shown in Figure 14.16. As shown in the Figure, there are three switching points for this motion. It is seen that the optimal motion starts while the grounded actuator is saturated and the ungrounded actuator applies a positive torque within its limits. At the first switching point, the ungrounded actuator reaches its negative limit. The grounded actuator shows a change from positive to negative until it reaches its negative limit when the second switching occurs. Between the second and third switching points, the grounded actuator is saturated. Finally, when the ungrounded actuator touches its negative limit for the second time, the third switching occurs.

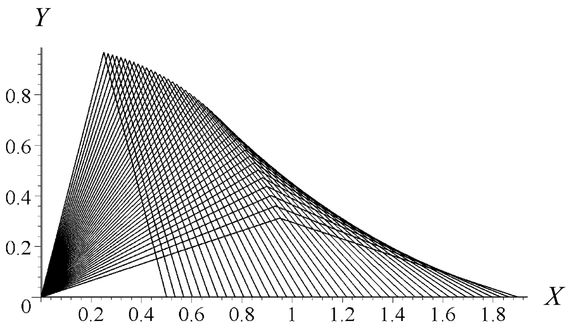

FIGURE 14.15. Illustration of motion of a 2R planar manipulator on line $y = 0$.

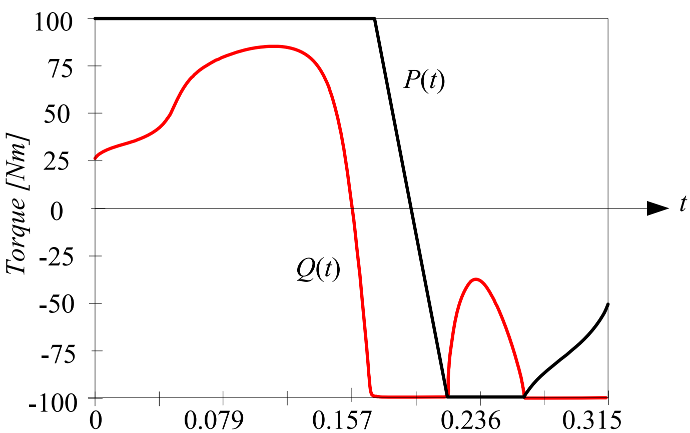

FIGURE 14.16. Time optimal control inputs for a 2R manipulator moving on line $Y = 0$, $0.5 < X < 1.9$.

14.4 Summary

Practically, every actuator can provide only a bounded output. When an actuator is working on its limit, we call it saturated. Time optimal control of an n DOF robot has a simple solution: At every instant of time, at least one actuator must be saturated while the others are within their limits. Floating-time is an applied method to find the saturated actuator, the switching points, and the output of the non-saturated actuators. Switching points are the points that the saturated actuator switches with another one.

The floating-time method is based on discrete equations of motion, utilizing variable time increments. Then, following a recursive algorithm, it calculates the required output for the robot's actuators to follow a given path of motion.

14.5 Key Symbols

A	coefficient matrix of variables
b	coefficient vector of control commands
B	body coordinate frame
c	cos, air resistance coefficient
C	constant of integral
$\mathbf{D}, \mathbf{G}, \mathbf{H}$	coefficient matrices of robot equation of motion
f	function, force, control command
F	force, control command
G, B_0	global coordinate frame, Base coordinate frame
H	Hamiltonian
J	objective function
l	length
m	mass
$\mathbf{p}$	momentum vector
P, Q	torque, control command
$\mathbf{r}$	position vectors, homogeneous position vector
R	rotation transformation matrix
s	sin, arc length, number of increments
t	time
V	variation
x, y, z	local coordinate axes
$\mathbf{x}$	vector of joint states
X, Y, Z	global coordinate axes
Greek	
β	cycloid angular variable
δ	Kronecker function, variation of a variable
ϵ	small number
θ	rotary joint angle
λ	Lagrange multiplier
μ	coefficient of friction
τ	floating time increment
$\triangle$	difference
Symbol	
$[\ \ ]^{-1}$	inverse of the matrix $[\ \ ]$
$[\ \ ]^{T}$	transpose of the matrix $[\ \ ]$
$\mathbf{q}^{\star}$	a guess value for $\mathbf{q}$
$\mathbb{R}$	set of real numbers

Exercises

1. Notation and symbols.

   Describe their meaning.

   a- τ_0 b- τ_i c- f_i d- x_i e- $\ddot{x}_i$

   f- $\dot{x}_i$ g- τ_{-1} h- τ_s i- x_{s+1} j- f_s

2. ★ Time optimal control of a 2 *DOF* system.

   Consider a dynamical system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_2 + 5Q \\ \dot{x}_2 &= 2x_1 - 3x_2\end{aligned}$$

   that must start from an arbitrary initial condition and finish at $x_1 = x_2 = 0$, with a bounded control input $|Q| \leq 1$.

   Show that the functions

$$\begin{aligned}f_1 &= -3x_1 + 2x_2 + 5Q \\ f_2 &= -2x_1 - 3x_2 \\ f_3 &= 1\end{aligned}$$

   along with the Hamiltonian function H

$$H = -1 + p_1 \left(-3x_1 + 2x_2 + 5Q\right) + p_2 \left(2x_1 - 3x_2\right)$$

   and the co-state variables p_1 and p_2 can solve the problem.

3. ★ Nonlinear objective function.

   Consider a one-dimensional control problem

$$\dot{x} = -x + Q$$

   where Q is the control command. The variable $x = x(t)$ must satisfy the boundary conditions

$$\begin{aligned}x(0) &= a \\ x(t_f) &= b\end{aligned}$$

   and minimize the objective function J.

$$J = \frac{1}{2} \int_0^{t_f} Q^2 dt$$

Show that the functions

$$\begin{aligned} f_1 &= \frac{1}{2}Q^2 \\ f_2 &= -x + Q \\ f_3 &= 0 \end{aligned}$$

along with the Hamiltonian function H

$$H = \frac{1}{2}p_0 Q^2 + p_1 (-x + Q)$$

and the co-state variables p_0 and p_1 can solve the problem.

4. ★ Time optimal control to origin.

Consider a dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + Q \end{aligned}$$

that must start from an arbitrary initial condition and finish at the origin of the phase plane, $x_1 = x_2 = 0$, with a bounded control $|Q| \leq 1$. Find the control command to do this motion in minimum time.

5. ★ A linear dynamical system.

Consider a linear dynamical system

$$\dot{\mathbf{x}} = [A]\,\mathbf{x} + \mathbf{b}Q$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad [A] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

subject to a bounded constraint on the control command

$$Q \leq 1.$$

Find the time optimal control command Q to move from the system from $\mathbf{x}_0$ to $\mathbf{x}_1$.

(a)

$$\mathbf{x}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$\mathbf{x}_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

6. ★ Constraint minimization.

   Find the local minima and maxima of

$$f(x) = x_1^2 + x_2^2 + x_3^2$$

   subject to the constraints

   (a)

$$g_1 = x_1 + x_2 + x_3 - 3 = 0$$

   (b)

$$\begin{aligned} g_1 &= x_1^2 + x_2^2 + x_3^2 - 5 = 0 \\ g_2 &= x_1^2 + x_2^2 + x_3^2 - 2x_1 - 3 = 0. \end{aligned}$$

7. ★ A control command with different limits.

   Consider a rectilinear motion of a point mass $m = 1\,\text{kg}$ under the influence of a control force $f(t)$ on a smooth surface. The force is bounded to $F_1 \le f(t) \le F_2$. The mass is supposed to move from the initial conditions $x(0) = 0$, $v(0) = 0$ to the final conditions $x(t_f) = 10\,\text{m}$, $v(t_f) = 0$ in minimum total time $t = t_f$. Use the floating time algorithm to find the required control command $f(t) = m\ddot{x}$ and the switching time for

   (a)

$$F_1 = 10\,\text{N} \qquad F_2 = 10\,\text{N}$$

   (b)

$$F_1 = 8\,\text{N} \qquad F_2 = 10\,\text{N}$$

   (c)

$$F_1 = 10\,\text{N} \qquad F_2 = 8\,\text{N}.$$

8. ★ A control command with different limits.

   Find the time optimal control command $|f(t)| \le 20\,\text{N}$ to move the mass $m = 2\,\text{kg}$ from rest at point P_1 to P_2, and return to stop at point P_3, as shown in Figure 14.17. The value of μ is:

   (a)

$$\mu = 0$$

   (b)

$$\mu = 0.2$$

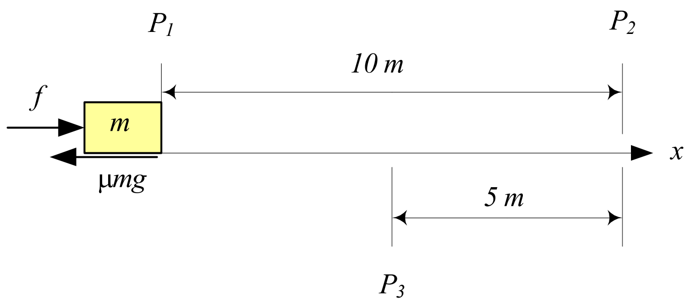

FIGURE 14.17. A rectilinear motion of a mass m from rest at point P_1 to P_2, and a return to stop at point P_3.

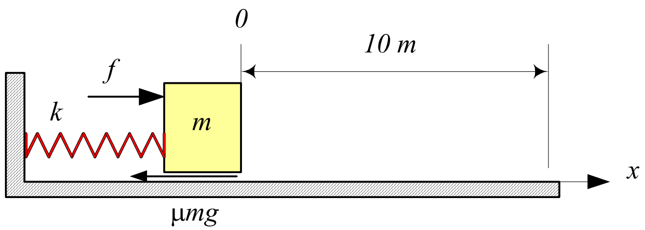

FIGURE 14.18. A rectilinear motion of a mass m on a rough surface and attached to a wall with a spring.

9. ★ Resistive media.

   Consider the mass $m = 2\,\text{kg}$ in Figure 14.17 that is supposed to move from P_1 to P_2 rest-to-rest in minimum time. The control command is limited to $|f(t)| \leq 20\,\text{N}$. However, there is an air resistant proportional to the velocity $c\dot{x}$. Determine the optimal $f(t)$, if

   (a)
   $$\mu = 0 \qquad c = 0.1$$

   (b)
   $$\mu = 0.2 \qquad c = 0.1$$

10. ★ Motion of a mass under friction and spring forces.

    Find the optimal control command $|f(t)| \leq 100\,\text{N}$ to move the mass $m = 1\,\text{kg}$ rest-to-rest from $x(0) = 0$ to $x(t_f) = 10\,\text{m}$. The mass is moving on a rough surface with coefficient μ and is attached to a wall by a linear spring with stiffness k, as shown in Figure 14.18. The value of μ and k are

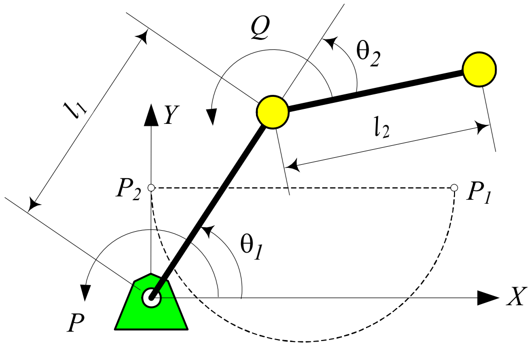

FIGURE 14.19. A 2R manipulator moves between two points on a line and a semi-circle.

(a)

$$\mu = 0.1 \qquad k = 2\,\mathrm{N/\,m}$$

(b)

$$\mu = 0.5 \qquad k = 5\,\mathrm{N/\,m}.$$

11. ★ Convergence conditions.

    Verify Equations (14.96) to (14.101) for the convergence condition of the floating-time algorithm.

12. ★ $2R$ manipulator moving on a line and a circle.

    Calculate the actuators' torque for the $2R$ manipulator, shown in Figure 14.19, such that the end-point moves time optimally from $P_1(1.5\,\mathrm{m},\ 0.5\,\mathrm{m})$ to $P_2(0,\ 0.5\,\mathrm{m})$. The manipulator has the following characteristics:

$$\begin{aligned} m_1 &= m_2 = 1\,\mathrm{kg} \\ l_1 &= l_2 = 1\,\mathrm{m} \\ |P(t)| &\leq 100\,\mathrm{N\,m} \\ |Q(t)| &\leq 80\,\mathrm{N\,m} \end{aligned}$$

    The path of motion is:

    (a) a straight line

    (b) a semi-circle with a center at $(0.75\,\mathrm{m},\ 0.5\,\mathrm{m})$.

13. ★ Time optimal control for a polar manipulator.

    Figure 14.20 illustrates a polar manipulator that is controlled by a torque Q and a force P. The base actuator rotates the manipulator

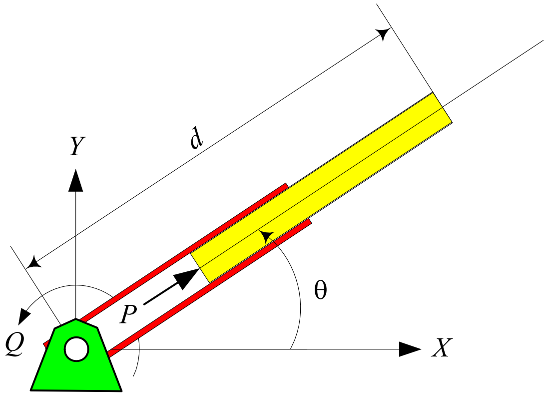

FIGURE 14.20. A polar manipulator, controlled by a torque Q and a force P.

and a force P slides the second link on the first link. Find the optimal controls to move the endpoint from $P_1(1.5\,\mathrm{m},\ 1\,\mathrm{m})$ to $P_2(-1,\ 0.5\,\mathrm{m})$ for the following data:

$$m_1 = 5\,\mathrm{kg} \qquad m_2 = 3\,\mathrm{kg}$$
$$|Q(t)| \le 100\,\mathrm{N\,m} \qquad |P(t)| \le 80\,\mathrm{N\,m}$$

14. ★ Control of an articulated manipulator.

Find the time optimal control of an articulated manipulator, shown in Figure 5.22, to move from $P_1 = (1.1, 0.8, 0.5)$ to $P_2 = (-1, 1, 0.35)$ on a straight line. The geometric parameters of the manipulator are given below. Assume the links are made of uniform bars.

$d_1 = 1\,\mathrm{m}$	$d_2 = 0$	
$l_2 = 1\,\mathrm{m}$		$l_3 = 1\,\mathrm{m}$
$m_1 = 25\,\mathrm{kg}$	$m_2 = 12\,\mathrm{kg}$	$m_3 = 8\,\mathrm{kg}$
$\lvert Q_1(t)\rvert \le 180\,\mathrm{N\,m}$	$\lvert Q_2(t)\rvert \le 100\,\mathrm{N\,m}$	$\lvert Q_3(t)\rvert \le 50\,\mathrm{N\,m}$