13

Path Planning

Path planning includes three tasks: 1−Defining a geometric curve for the end-effector between two points. 2−Defining a rotational motion between two orientations. 3−Defining a time function for variation of a coordinate between two given values. All of these three definitions are called *path planning*. Figure 13.1 illustrates a path of the tip point of a $2R$ manipulator between points P_1 and P_2 to avoid two obstacles.

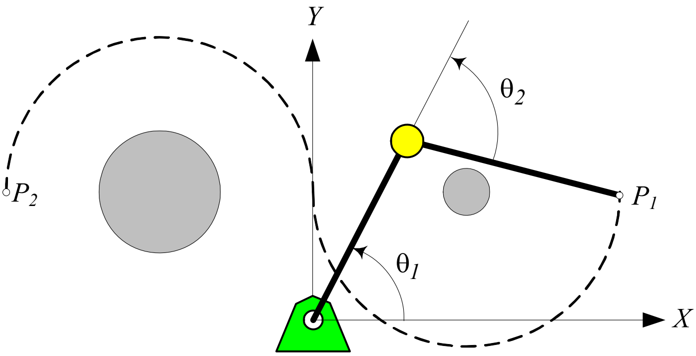

FIGURE 13.1. A path of the tip point of a $2R$ manipulator to avoid two obstacles.

13.1 Cubic Path

A cubic function is the simplest polynomial to determine the time behavior of a variable between two given values, rest-to-rest.

A cubic path in joint space for the joint variable $q(t)$, or in Cartesian space for a Cartesian coordinate $q(t)$, between two points $q(t_0)$ and $q(t_f)$ is

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \tag{13.1}$$

where

$$a_0 = -\frac{q_1 t_0^2 \left(t_0 - 3t_f\right) + q_0 t_f^2 \left(3t_0 - t_f\right)}{\left(t_f - t_0\right)^3} - t_0 t_f \frac{q_0' t_f + q_1' t_0}{\left(t_f - t_0\right)^2} \tag{13.2}$$

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$$\begin{aligned} a_1 &= 6t_0t_f\frac{q_0-q_1}{(t_f-t_0)^3} \\ &\quad +\frac{q_0't_f\left(t_f^2+t_0t_f-2t_0^2\right)+q_1't_0\left(2t_f^2-t_0^2-t_0t_f\right)}{(t_f-t_0)^3} \end{aligned} \tag{13.3}$$

$$\begin{aligned} a_2 &= -\frac{q_0(3t_0+3t_f)+q_1(-3t_0-3t_f)}{(t_f-t_0)^3} \\ &\quad -\frac{q_1'\left(t_0t_f-2t_0^2+t_f^2\right)+q_0'\left(2t_f^2-t_0^2-t_0t_f\right)}{(t_f-t_0)^3} \end{aligned} \tag{13.4}$$

$$a_3=\frac{2q_0-2q_1+q_0'(t_f-t_0)+q_1'(t_f-t_0)}{(t_f-t_0)^3} \tag{13.5}$$

and

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0)=q_0' \\ q(t_f) &= q_f \qquad \dot{q}(t_f)=q_f'. \end{aligned} \tag{13.6}$$

Proof. A cubic polynomial has four coefficients. Therefore, it can satisfy the position and velocity constraints at the initial and final points. For simplicity, we call the value of the variable, the *position*, and the rate of the variable, the *velocity*. Assume that the position and velocity of a variable at the initial time t_0 and at the final time t_f are given as (13.6).

Substituting the boundary conditions in the position and velocity equations of the joint variable

$$q(t) = a_0+a_1t+a_2t^2+a_3t^3 \tag{13.7}$$

$$\dot{q}(t) = a_1+2a_2t+3a_3t^2 \tag{13.8}$$

generates four equations for the coefficients of the path.

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} q_0 \\ q_0' \\ q_f \\ q_f' \end{bmatrix} \tag{13.9}$$

Their solutions are given in (13.2) to (13.5).

In case that $t_0=0$, the coefficients simplify to

$$a_0 = q_0 \tag{13.10}$$

$$a_1 = q_0' \tag{13.11}$$

$$a_2 = \frac{3(q_f-q_0)-\left(2q_0'+q_f'\right)t_f}{t_f^2} \tag{13.12}$$

$$a_3 = \frac{-2(q_f-q_0)+\left(q_0'+q_f'\right)t_f}{t_f^3}. \tag{13.13}$$

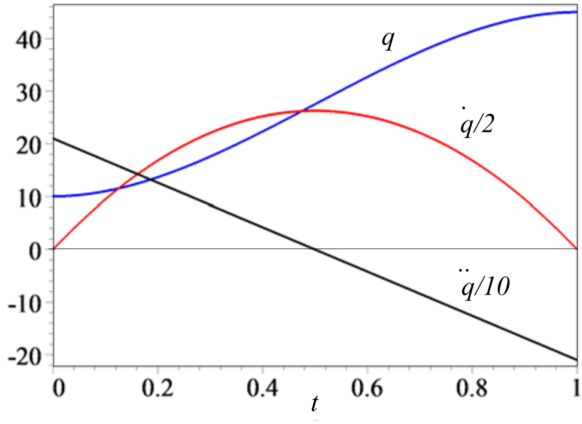

FIGURE 13.2. Kinematics of a rest-to-rest cubic path.

It is also possible to employ a time shift and search for a cubic polynomial of the form

$$q(t) = a_0 + a_1 (t - t_0) + a_2 t (t - t_0)^2 + a_3 (t - t_0)^3 . \tag{13.14}$$

Now, the boundary conditions (13.6) generate a set of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & (t_f - t_0) & (t_f - t_0)^2 & (t_f - t_0)^3 \\ 0 & 1 & 2(t_f - t_0) & 3(t_f - t_0)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q'_0 \\ q_f \\ q'_f \end{bmatrix} \tag{13.15}$$

with the following solutions:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ q'_0 \\ -(t_f - t_0)^{-2} \left(3q_0 - 3q_f - 2t_0 q'_0 - t_0 q'_f + 2t_f q'_0 + t_f q'_f\right) \\ (t_f - t_0)^{-3} \left(2q_0 - 2q_f - t_0 q'_0 - t_0 q'_f + t_f q'_0 + t_f q'_f\right) \end{bmatrix} \tag{13.16}$$

A disadvantage of cubic paths is the acceleration jump at boundaries that introduces infinite jerks. ■

Example 348 *Rest-to-rest cubic path.*

Assume $q(0) = 10\,\text{deg}$, $q(1) = 45\,\text{deg}$, *and* $\dot{q}(0) = \dot{q}(1) = 0$. *The coefficients of the cubic path are*

$$a_0 = 10 \qquad a_1 = 0 \qquad a_2 = 105 \qquad a_3 = -70 \tag{13.17}$$

that generate a path for the variable as

$$q(t) = 10 + 105t^2 - 70t^3\,\text{deg} . \tag{13.18}$$

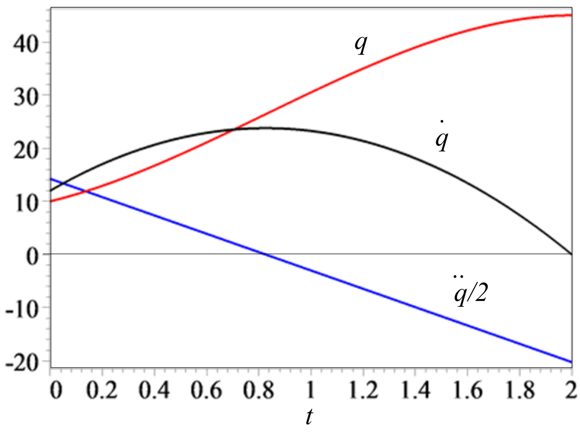

FIGURE 13.3. Kinematics of a to-rest cubic path in joint space.

The path information is shown in Figure 13.2.

Example 349 *To-rest cubic path.*

Assume the angle of a joint starts from $\theta(0) = 10\deg$, $\dot{\theta}(0) = 12\deg/\mathrm{s}$ and ends at $\theta(2) = 45\deg$, $\dot{\theta}(0) = 0$. The coefficients of a cubic path for this motion are:

$$a_0 = 10 \qquad a_1 = 12 \qquad a_2 = \frac{81}{2} \qquad a_3 = \frac{-29}{2} \tag{13.19}$$

The kinematics of this path are

$$\theta(t) = 10 + 12t + 40.5t^2 - 14.5t^3 \deg \tag{13.20}$$

$$\dot{\theta}(t) = 81t - 43.5t^2 + 12 \deg/\mathrm{s} \tag{13.21}$$

$$\ddot{\theta}(t) = 81 - 87t \deg/\mathrm{s}^2 \tag{13.22}$$

and are shown graphically in Figure 13.3.

Example 350 *Rest-to-rest path with a constant velocity in the middle.*

Assume we need a rest-to-rest path with a constant given velocity $\dot{q} = \dot{q}_c$ for $t_1 < t < t_2$ where $t_0 < t_1 < t_2 < t_f$. We show the boundary conditions to be satisfied as

$$\begin{aligned} q(t_0) &= q_0 & \dot{q}(t_0) = q_0' \\ \dot{q}(t) &= q_c' & t_1 < t < t_2 \\ q(t_f) &= q_f & \dot{q}(t_f) = q_f'. \end{aligned} \tag{13.23}$$

The path has three parts: rest-to, constant-velocity, and to-rest. We need an equation for the rest-to part of the motion to achieve the given velocity.

A quadratic path has three coefficients and can be utilized to satisfy three conditions.

$$q_1(t) = a_0 + a_1 t + a_2 t^2 \tag{13.24}$$
$$\dot{q}_1(t) = a_1 + 2a_2 t \tag{13.25}$$

The conditions are the initial position and velocity, and the final constant velocity. Assuming $t_0 = 0$ *the conditions for the rest-to path are*

$$q_1(0) = q_0 \qquad \dot{q}_1(0) = 0 \qquad \dot{q}_1(t_1) = q'_c \tag{13.26}$$

that generate the following equations:

$$q_0 = a_0 \tag{13.27}$$
$$0 = a_1 \tag{13.28}$$
$$q'_c = 2a_2 t_1. \tag{13.29}$$

Therefore, the rest-to path is:

$$q_1(t) = q_0 + \frac{q'_c}{2t_1} t^2 \qquad 0 < t < t_1 \tag{13.30}$$

Given the specific constant velocity q'_c *shows that the path in the middle part is:*

$$\dot{q}_2(t) = q'_c \tag{13.31}$$
$$q_2(t) = q'_c t + C_1 \qquad t_1 < t < t_2 \tag{13.32}$$

The constant of integration can be found by utilizing the position condition at $t = t_1$.

$$q_0 + \frac{q'_c}{2t_1} t_1^2 = q'_c t_1 + C_1 \tag{13.33}$$
$$C_1 = q_0 - \frac{1}{2} t_1 q'_c \tag{13.34}$$

There are four conditions for the to-rest part of the path. Therefore, it can be calculated utilizing a cubic equation

$$q_3(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 \tag{13.35}$$
$$\dot{q}_3(t) = b_1 + 2b_2 t + 3b_3 t^2 \tag{13.36}$$

and the following boundary conditions:

$$q_3(t_f) = q_f \tag{13.37}$$
$$\dot{q}_3(t_f) = 0 \tag{13.38}$$
$$q_3(t_2) = q_2(t_2) = q_2 = q'_c t_2 + q_0 - \frac{1}{2} t_1 q'_c \tag{13.39}$$
$$\dot{q}(t_2) = \dot{q}_2(t_2) = q'_c \tag{13.40}$$

These conditions generate three equations

$$\begin{bmatrix} 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 0 & 1 & 2t_2 & 3t_2^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} q_f \\ 0 \\ q_c' t_2 + q_0 - \frac{1}{2} t_1 q_c' \\ q_c' \end{bmatrix} \tag{13.41}$$

with the following solutions:

$$\begin{aligned} b_0 &= -t_2 t_f^2 \frac{q_c'}{-2t_2 t_f + t_2^2 + t_f^2} + q_2 \frac{t_f^3 - 3t_2 t_f^2}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \\ &\quad + q_f \frac{-t_2^3 + 3t_2^2 t_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \end{aligned} \tag{13.42}$$

$$\begin{aligned} b_1 &= q_c' \frac{2t_2 t_f + t_f^2}{-2t_2 t_f + t_2^2 + t_f^2} + 6q_2 t_2 \frac{t_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \\ &\quad - 6t_2 q_f \frac{t_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \end{aligned} \tag{13.43}$$

$$\begin{aligned} b_2 &= q_c' \frac{-t_2 - 2t_f}{-2t_2 t_f + t_2^2 + t_f^2} + q_2 \frac{-3t_2 - 3t_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \\ &\quad + q_f \frac{3t_2 + 3t_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \end{aligned} \tag{13.44}$$

$$\begin{aligned} b_3 &= \frac{q_c'}{-2t_2 t_f + t_2^2 + t_f^2} + 2 \frac{q_2}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \\ &\quad - 2 \frac{q_f}{-t_2^3 + t_f^3 - 3t_2 t_f^2 + 3t_2^2 t_f} \end{aligned} \tag{13.45}$$

for

$$t_2 < t < t_f. \tag{13.46}$$

A graph of the path for the following values is illustrated in Figure 13.4.

$$\begin{aligned} t_1 &= 0.4\,\mathrm{s} \qquad t_2 = 0.7\,\mathrm{s} \qquad t_f = 1\,\mathrm{s} \\ q_0 &= 0 \qquad q_f = 60\deg \qquad q_c' = 50\deg/\mathrm{s} \end{aligned} \tag{13.47}$$

Example 351 *A quadratic path through three points.*

A quadratic path passing through three points (q_1, t_1), (q_2, t_2), *and* (q_3, t_3) *is:*

$$\begin{aligned} q(t) &= \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} q_1 + \frac{(t - t_3)(t - t_1)}{(t_2 - t_3)(t_2 - t_1)} q_2 \\ &\quad + \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_1)} q_3 \end{aligned} \tag{13.48}$$

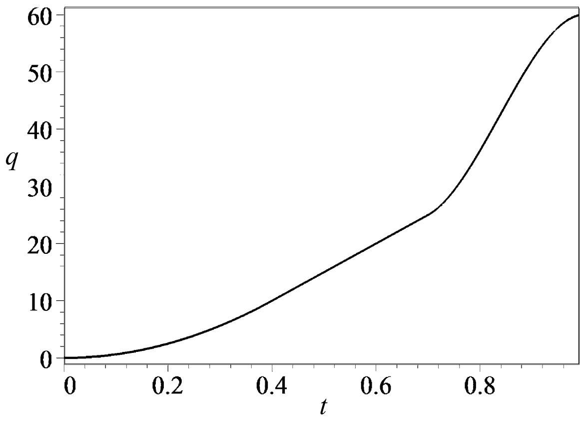

FIGURE 13.4. A piecewise rest-to-rest path with a constant velocity in the middle.

As an example, the path passing through $(10\,\text{deg}, 0)$, $(25\,\text{deg}, 0.5)$, *and* $(45\,\text{deg}, 1)$ *is:*

$$\begin{aligned} q(t) &= \frac{(t-0.5)(t-1)}{-0.5(-1)}10 + \frac{(t-1)\,t}{(-0.5)(0.5)}25 + \frac{t\,(t-0.5)}{1}45 \\ &= -\frac{5}{2}\left(14.0t^2 - 19.0t - 4.0\right) \end{aligned} \tag{13.49}$$

The velocity of the path at both ends are:

$$\dot{q}(0) = 47.5\,\text{deg}\,/\,\text{s} \tag{13.50}$$

$$\dot{q}(1) = -22.5\,\text{deg}\,/\,\text{s} \tag{13.51}$$

13.2 Polynomial Path

The number of required conditions determines the degree of the polynomial for $q = q(t)$. In general, a polynomial path of degree n,

$$q(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \tag{13.52}$$

needs $n+1$ conditions. The conditions may be of two types: positions at a series of points, so that the trajectory will pass through all specified points; or position, velocity, acceleration, and jerk at two points, so that the smoothness of the path can be controlled.

The problem of searching for the coefficients of a polynomial reduces to a set of linear algebraic equations and may be solved numerically. However, the path planning can be simplified by splitting the whole path into a series of segments and utilizing combinations of lower order polynomials for different segments of the path. The polynomials must then be joined together to satisfy all the required boundary conditions.

Example 352 *Quintic path.*

Forcing a variable to have specific position, velocity, and acceleration at boundaries introduces six conditions:

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = q_0' \qquad \ddot{q}(t_0) = q_0'' \\ q(t_f) &= q_f \qquad \dot{q}(t_f) = q_f' \qquad \ddot{q}(t_f) = q_f'' \end{aligned} \tag{13.53}$$

A five degree polynomial can satisfy these conditions

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{13.54}$$

and generates a set of six equations:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_0' \\ q_0'' \\ q_f \\ q_f' \\ q_f'' \end{bmatrix} \tag{13.55}$$

A rest-to-rest path with no acceleration at the rest positions with the following conditions:

$$\begin{aligned} q(0) &= 10\,\text{deg} \qquad \dot{q}(0) = 0 \qquad \ddot{q}(0) = 0 \\ q(1) &= 45\,\text{deg} \qquad \dot{q}(1) = 0 \qquad \ddot{q}(1) = 0 \end{aligned} \tag{13.56}$$

can be found by solving a set of equations for the coefficients of the polynomial

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 45 \\ 0 \\ 0 \end{bmatrix} \tag{13.57}$$

which shows

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 350 \\ -525 \\ 210 \end{bmatrix}. \tag{13.58}$$

The path equation is then equal to

$$q(t) = 10 + 350t^3 - 525t^4 + 210t^5. \tag{13.59}$$

which is shown in Figure 13.5.

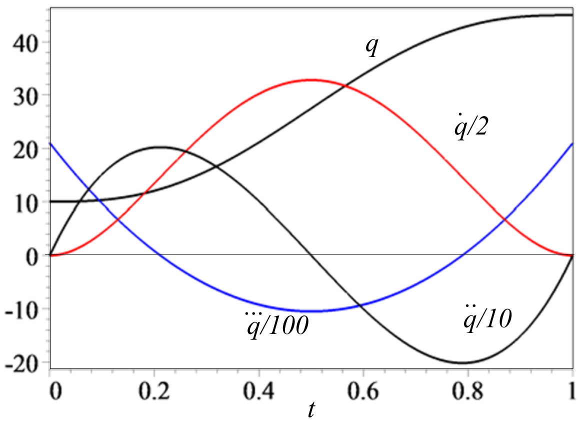

FIGURE 13.5. A quintic rest-to-rest path.

Example 353 *A jerk zero at a start-stop path.*

To make a path start and stop with zero jerk, a seven degree polynomial

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 \tag{13.60}$$

and eight boundary conditions must be employed.

$$\begin{array}{lllll} q(0) = q_0 & \dot{q}(0) = 0 & \ddot{q}(0) = 0 & \dddot{q}(0) = 0 \\ q(1) = q_f & \dot{q}(1) = 0 & \ddot{q}(1) = 0 & \dddot{q}(1) = 0 \end{array} \tag{13.61}$$

Such a zero jerk start-stop path for $q(0) = 10\deg$ *and* $q(1) = 45\deg$, *can be found by solving the following set of equations for the unknown coefficients* $a_0, a_1, \cdots, a_7$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 45 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13.62}$$

which provides

$$q(t) = 10 + 1225t^4 - 2940t^5 + 2450t^6 - 700t^7. \tag{13.63}$$

A graph of this path is illustrated in Figure 13.6.

Figures 13.2, depicts the path of a rest-to-rest motion with no condition on the acceleration and jerk. Figure 13.5 shows an improvement by forcing

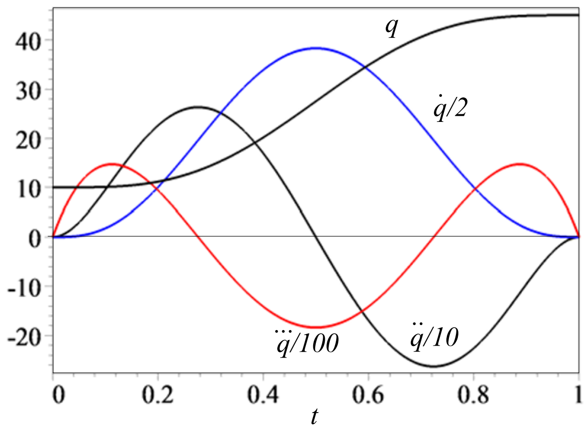

FIGURE 13.6. A jerk zero at start-stop path.

the motion to have zero accelerations at start and stop. In Figure 13.6, the motion is forced to have zero acceleration and zero jerk at start and stop. Hence, it shows the smoothest start and stop. However, increasing the smoothness of the start and stop increases the peak value of acceleration.

Example 354 *Constant acceleration path.*

A constant acceleration path has two segments with positive and negative accelerations. Let's assume the absolute value of the positive and negative accelerations are given.

$$|\ddot{q}(t_0)| = a_c \tag{13.64}$$

The first half of the motion has a positive acceleration that needs a second degree polynomial

$$\dot{q}_1(t_0) = a_c t \tag{13.65}$$

$$q_1(t_0) = \frac{1}{2}a_c t^2 + q_0 \tag{13.66}$$

for

$$0 < t < \frac{1}{2}t_f. \tag{13.67}$$

The constants of integration are found based on the initial conditions.

$$q_1(0) = q_0 \qquad \dot{q}_1(0) = 0 \tag{13.68}$$

For the second half of the path, we may start with a second degree polynomial

$$q_2(t) = a_0 + a_1 t + a_2 t^2 \qquad \frac{1}{2}t_f < t < t_f \tag{13.69}$$

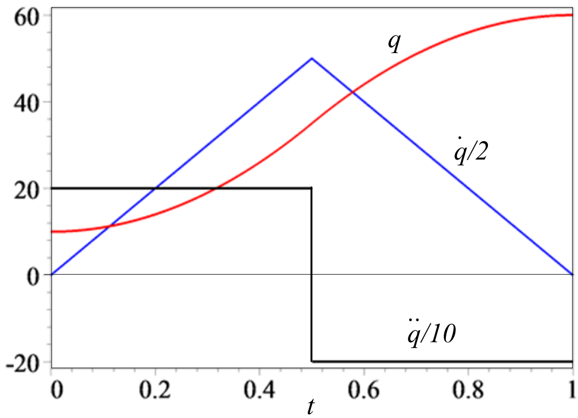

FIGURE 13.7. A constant acceleration path.

and impose the following boundary conditions:

$$\begin{aligned} q_2(t_f) &= q_f \\ \dot{q}_2(t_f) &= 0 \\ q_1(\frac{t_f}{2}) &= q_2(\frac{t_f}{2}) = \frac{1}{8} a_c t_f^2 + q_0 \end{aligned} \tag{13.70}$$

These conditions generate three equations for the unknown coefficients

$$\begin{bmatrix} 1 & t_f & t_f^2 \\ 0 & 1 & 2t_f \\ 0 & 1 & t_f \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_f \\ 0 \\ \frac{1}{8} a_c t_f^2 + q_0 \end{bmatrix} \tag{13.71}$$

with the following solution:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} q_f - t_f \left(q_0 + \frac{1}{8} a_c t_f^2 \right) \\ 2q_0 + \frac{1}{4} a_c t_f^2 \\ -\frac{1}{t_f} \left(q_0 + \frac{1}{8} a_c t_f^2 \right) \end{bmatrix} \tag{13.72}$$

A constant acceleration path is shown in Figure 13.7 for the conditions $q_0 = 10\,\mathrm{deg}$, $q_f = 45\,\mathrm{deg}$, $t_f = 1$, *and* $a_c = 200\,\mathrm{deg}/\mathrm{s}^2$.

Example 355 *Point sequence path.*

A path can be assigned via a series of points that the variable must attain at specific times. The points may also be defined to approximate a trajectory. Consider an example path specified by four points q_0, q_1, q_2, *and* q_3, *such that the points are reached at times* t_0, t_1, t_2, *and* t_3 *respectively. In addition to positions, we usually impose constraint on initial and final velocities and* *accelerations. The conditions for such a sequence of points can be*

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = 0 \qquad \ddot{q}(t_0) = 0 \\ q(t_1) &= q_1 \\ q(t_2) &= q_2 \\ q(t_3) &= q_3 \qquad \dot{q}(t_3) = 0 \qquad \ddot{q}(t_3) = 0. \end{aligned} \tag{13.73}$$

A seven degree polynomial can be utilized to satisfy these eight conditions.

$$\begin{aligned} q(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \\ &\quad + a_5 t^5 + a_6 t^6 + a_7 t^7 \end{aligned} \tag{13.74}$$

The set of equations for the unknown coefficients is

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 & t_0^6 & t_0^7 \\ 0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 & 6t_0^5 & 7t_0^6 \\ 0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 & 30t_0^4 & 42t_0^5 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & t_1^5 & t_1^6 & t_1^7 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & t_2^5 & t_2^6 & t_2^7 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & t_3^5 & t_3^6 & t_3^7 \\ 0 & 1 & 2t_3 & 3t_3^2 & 4t_3^3 & 5t_3^4 & 6t_3^5 & 7t_3^6 \\ 0 & 0 & 2 & 6t_3 & 12t_3^2 & 20t_3^3 & 30t_3^4 & 42t_3^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ 0 \\ q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix} \tag{13.75}$$

that can be simplified to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.4 & 0.4^2 & 0.4^3 & 0.4^4 & 0.4^5 & 0.4^6 & 0.4^7 \\ 1 & 0.7 & 0.7^2 & 0.7^3 & 0.7^4 & 0.7^5 & 0.7^6 & 0.7^7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 20 \\ 30 \\ 45 \\ 0 \\ 0 \end{bmatrix} \tag{13.76}$$

for the following example data.

$$\begin{aligned} q(0) &= 10\,\mathrm{deg} \qquad \dot{q}(0) = 0 \qquad \ddot{q}(0) = 0 \\ q(0.4) &= 20\,\mathrm{deg} \\ q(0.7) &= 30\,\mathrm{deg} \\ q(1) &= 45\,\mathrm{deg} \qquad \dot{q}(1) = 0 \qquad \ddot{q}(1) = 0 \end{aligned} \tag{13.77}$$

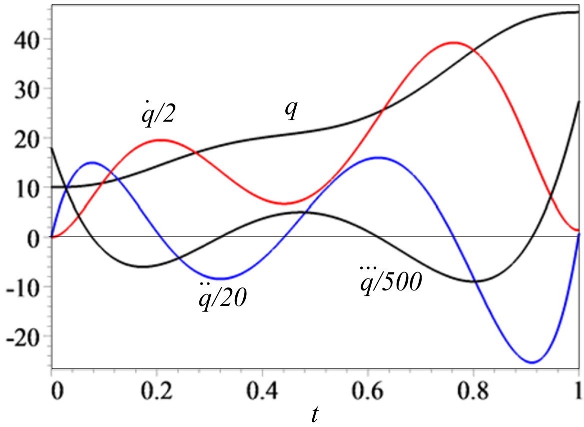

FIGURE 13.8. A point sequence path.

The solution for the coefficients is:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 1500.5 \\ -7053 \\ 12891 \\ -10380 \\ 3076.9 \end{bmatrix} \tag{13.78}$$

These coefficients generate a path as shown in Figure 13.8.

This method provides a continuous and differentiable function for the q variable. Continuity and differentiability of $q = q(t)$ is an advantage that provides a continuous velocity, acceleration, and jerk. However, the number of equations increases by increasing the number of points, which needs larger data storage and increases the calculating time.

Example 356 *Splitting a path into a series of segments.*

Instead of using a single high degree polynomial for the entire trajectory, we may prefer to split the trajectory into some segments and use a series of low degree polynomials.

Consider a path for the following boundary conditions:

$$\begin{array}{lllll} q(t_0) & = & q_0 & \dot{q}(t_0) = 0 & \ddot{q}(t_0) = 0 \\ q(t_4) & = & q_3 & \dot{q}(t_4) = 0 & \ddot{q}(t_4) = 0 \end{array} \tag{13.79}$$

which must also pass through three middle points given below.

$$\begin{aligned} q(t_1) &= q_1 \\ q(t_2) &= q_2 \\ q(t_3) &= q_3 \end{aligned} \tag{13.80}$$

Let's split the entire path into four segments, namely $q_1(t)$, $q_2(t)$, $q_3(t)$, *and* $q_4(t)$.

$$\begin{array}{llll} q_1(t) & \text{for} \quad q(t_0) < q_1(t) < q(t_1) & \text{and} & t_0 < t < t_1 \\ q_2(t) & \text{for} \quad q(t_1) < q_2(t) < q(t_2) & \text{and} & t_1 < t < t_2 \\ q_3(t) & \text{for} \quad q(t_2) < q_3(t) < q(t_3) & \text{and} & t_2 < t < t_3 \\ q_4(t) & \text{for} \quad q(t_3) < q_4(t) < q(t_4) & \text{and} & t_3 < t < t_4 \end{array}$$

The boundary conditions for the first segment are

$$\begin{aligned} q_1(t_0) &= q_0 \qquad \dot{q}_1(t_0) = 0 \qquad \ddot{q}_1(t_0) = 0 \\ q_1(t_1) &= q_1 \end{aligned} \tag{13.81}$$

which can be satisfied by a cubic function.

$$q_1(t) = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2 + a_3 (t - t_0)^3 \tag{13.82}$$

The coefficients can be calculated by solving a set of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & (t_1 - t_0) & (t_1 - t_0)^2 & (t_1 - t_0)^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ 0 \\ 0 \\ q_1 \end{bmatrix} \tag{13.83}$$

that provides

$$a_0 = q_0 \qquad a_1 = 0 \qquad a_2 = 0 \qquad a_3 = \frac{q_1 - q_0}{(t_1 - t_0)^3}. \tag{13.84}$$

The path in the second segment must satisfy the following boundary conditions:

$$\begin{aligned} q_2(t_1) &= q_1 \\ \dot{q}_2(t_1) &= \dot{q}_1(t_1) = a_1 + 2a_2 (t_1 - t_0)^2 + 3a_3 (t_1 - t_0)^2 \\ &= q_0 + 3\frac{q_1 - q_0}{t_1 - t_0} \\ q_2(t_2) &= q_2 \end{aligned} \tag{13.85}$$

A quadratic polynomial will satisfy these conditions:

$$q_2(t) = b_0 + b_1 t + b_2 t^2 \tag{13.86}$$

The coefficients are the solutions of

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 0 & 1 & 2t_1 \\ 1 & t_2 & t_2^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_0 + 3\frac{q_1 - q_0}{t_1 - t_0} \\ q_2 \end{bmatrix} \tag{13.87}$$

that provide

$$\begin{aligned} b_0 &= q_2 \frac{t_1^2}{-2t_1t_2 + t_1^2 + t_2^2} + q_1 \frac{-2t_1t_2 + t_2^2}{-2t_1t_2 + t_1^2 + t_2^2} \\ &\quad -t_1 \frac{t_2}{-t_1 + t_2} \left(q_0 + 3\frac{-q_0 + q_1}{-t_0 + t_1} \right) \end{aligned} \tag{13.88}$$

$$\begin{aligned} b_1 &= 2q_1 \frac{t_1}{-2t_1t_2 + t_1^2 + t_2^2} - 2q_2 \frac{t_1}{-2t_1t_2 + t_1^2 + t_2^2} \\ &\quad + \frac{t_1 + t_2}{-t_1 + t_2} \left(q_0 + 3\frac{-q_0 + q_1}{-t_0 + t_1} \right) \end{aligned} \tag{13.89}$$

$$\begin{aligned} b_2 &= -\frac{q_1}{-2t_1t_2 + t_1^2 + t_2^2} + \frac{q_2}{-2t_1t_2 + t_1^2 + t_2^2} \\ &\quad - \frac{1}{-t_1 + t_2} \left(q_0 + 3\frac{-q_0 + q_1}{-t_0 + t_1} \right). \end{aligned} \tag{13.90}$$

The boundary conditions in the third segment are:

$$\begin{aligned} q_3(t_2) &= q_2 \qquad \dot{q}_3(t_2) = \dot{q}_2(t_2) = b_1 + 2b_2t_2 \\ q_3(t_3) &= q_3 \end{aligned} \tag{13.91}$$

We can satisfy these conditions with a quadratic equation

$$q_3(t) = c_0 + c_1 t + c_2 t^2 \tag{13.92}$$

that provides three equations for the unknown coefficients.

$$\begin{bmatrix} 1 & t_2 & t_2^2 \\ 0 & 1 & 2t_2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} q_2 \\ b_1 + 2b_2t_2 \\ q_3 \end{bmatrix} \tag{13.93}$$

The coefficients are:

$$\begin{aligned} c_0 &= -t_2 \frac{t_3}{-t_2 + t_3} (b_1 + 2b_2t_2) + q_3 \frac{t_2^2}{-2t_2t_3 + t_2^2 + t_3^2} \\ &\quad + q_2 \frac{-2t_2t_3 + t_3^2}{-2t_2t_3 + t_2^2 + t_3^2} \end{aligned} \tag{13.94}$$

$$c_1 = \frac{t_2+t_3}{-t_2+t_3}(b_1+2b_2t_2)+2q_2\frac{t_2}{-2t_2t_3+t_2^2+t_3^2} -2q_3\frac{t_2}{-2t_2t_3+t_2^2+t_3^2} \tag{13.95}$$

$$c_2 = -\frac{1}{-t_2+t_3}(b_1+2b_2t_2)-\frac{q_2}{-2t_2t_3+t_2^2+t_3^2} +\frac{q_3}{-2t_2t_3+t_2^2+t_3^2}. \tag{13.96}$$

The boundary conditions for the fourth segment are

$$\begin{aligned} q_4(t_3) &= q_3 \qquad \dot{q}_4(t_3)=\dot{q}_3(t_3)=c_1+2c_2t_3 \\ q_4(t_4) &= q_4 \qquad \dot{q}_4(t_4)=0 \qquad \ddot{q}_4(t_4)=0 \end{aligned} \tag{13.97}$$

which needs a fourth degree polynomial to be satisfied.

$$q_4(t)=d_0+d_1t+d_2t^2+d_3t^3+d_4t^4 \tag{13.98}$$

Substituting the boundary conditions generates a set of four equations for the coefficient.

$$\begin{bmatrix} 1 & t_3 & t_3^2 & t_3^3 & t_3^4 \\ 0 & 1 & 2t_2 & 3t_2^2 & 4t_2^3 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 \\ 0 & 1 & 2t_4 & 3t_4^2 & 4t_4^3 \\ 0 & 0 & 2 & 6t_4 & 12t_4^2 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} q_3 \\ c_1+2c_2t_3 \\ q_4 \\ 0 \\ 0 \end{bmatrix} \tag{13.99}$$

As an example, a set of conditions given by

$$t_0=0 \qquad t_1=0.4 \qquad t_2=0.7 \qquad t_3=0.9 \qquad t_4=1 \tag{13.100}$$

$$\begin{aligned} q(0) &= 10\,\mathrm{deg} \qquad \dot{q}(0)=0 \qquad \ddot{q}(0)=0 \\ q(0.4) &= 20\,\mathrm{deg} \\ q(0.7) &= 30\,\mathrm{deg} \\ q(0.9) &= 35\,\mathrm{deg} \\ q(1) &= 45\,\mathrm{deg} \qquad \dot{q}_i(1)=0 \qquad \ddot{q}(1)=0 \end{aligned} \tag{13.101}$$

provides

$$q_1(t) = 10+156.25t^3 \tag{13.102}$$

$$q_2(t) = -41.56+222.78t-172.2t^2 \tag{13.103}$$

$$q_3(t) = 148.99-321.67t+216.67t^2 \tag{13.104}$$

$$q_4(t) = 198545-827166.6672t +1290500t^2-893500t^3+231666.67t^4 \tag{13.105}$$

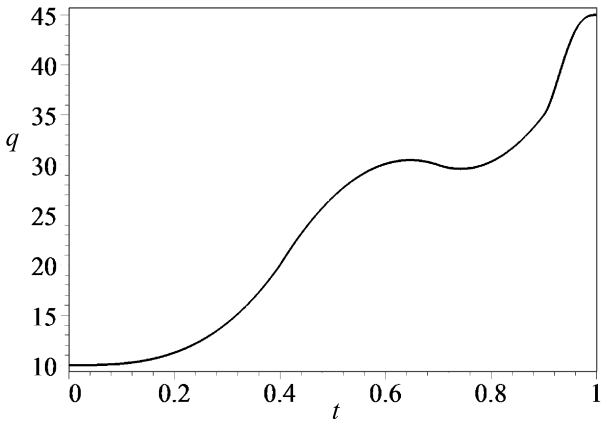

FIGURE 13.9. Spliting a path into a series of segments.

which is shown in Figure 13.9 graphically.

The disadvantage of the segment method is the lack of a smooth overall path and having a discontinuous acceleration. To increase the smoothness of the path, we need to use higher degree polynomials and put constraints on acceleration and possibly jerk.

Equations (13.102)-(13.105) indicate that:

$$\ddot{q}_1(t_1) = 375 \qquad \ddot{q}_2(t_1) = -344.4 \tag{13.106}$$

$$\ddot{q}_2(t_2) = -344.4 \qquad \ddot{q}_3(t_2) = 433.34 \tag{13.107}$$

$$\ddot{q}_3(t_3) = 433.34 \qquad \ddot{q}_4(t_3) = 7900 \tag{13.108}$$

$$\ddot{q}_1(t_1) \neq \ddot{q}_2(t_1) \qquad \ddot{q}_2(t_2) \neq \ddot{q}_3(t_2) \qquad \ddot{q}_3(t_3) \neq \ddot{q}_4(t_3) \tag{13.109}$$

Therefore, the acceleration of the path is not continuos at the connection points and show a finite jump. A jump in acceleration introduces an infinity jerk. Having continuos acceleration is the minimum requirement for smoothness of a path. A piecewise path with continuous acceleration is called ***spline***.

Example 357 ★ *Least-squares polynomial.*

When the number of points to approximate a trajectory is too large, we may use a low degree polynomial to pass close to the points. Least-squares is an applied method to determine the coefficients of a selected polynomial to approximate the path.

Consider a path with N given points,

$$p_i = p(t_i) \qquad i = 1, 2, 3, \cdots, N \tag{13.110}$$

and a polynomial of degree n that is supposed to approximate the path. If $N = n + 1$ then the polynomial passes exactly through all given points. To *work with low degree polynomials, we choose* $n < N + 1$.

$$q = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \tag{13.111}$$

Having the N *points (13.110) and the polynomial (13.111), we define an error* e_i *at* t_i.

$$e_i = p_i - q_i = p_i - a_0 - a_1 t_i - a_2 t_i^2 - \cdots - a_n t_i^n \tag{13.112}$$

Sum of e_i^2 *for all points* p_i *is the total error* e.

$$e = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} \left(p_i - a_0 - a_1 t_i - a_2 t_i^2 - \cdots - a_n t_i^n\right)^2 \tag{13.113}$$

The minimum error e *provides the best approximate polynomial (13.111). At the minimum, all the partial derivatives* $\partial e/\partial a_0$, $\partial e/\partial a_1$, $\cdots$, $\partial e/\partial a_n$ *vanish. These conditions generate* $n+1$ *equations:*

$$\begin{aligned}
\frac{\partial e}{\partial a_0} &= -2\sum_{i=1}^{N} \left(p_i - a_0 - a_1 t_i - a_2 t_i^2 - \cdots - a_n t_i^n\right) = 0 \\
\frac{\partial e}{\partial a_1} &= -2\sum_{i=1}^{N} t_i \left(p_i - a_0 - a_1 t_i - a_2 t_i^2 - \cdots - a_n t_i^n\right) = 0 \\
&\cdots \\
\frac{\partial e}{\partial a_n} &= -2\sum_{i=1}^{N} t_i^n \left(p_i - a_0 - a_1 t_i - a_2 t_i^2 - \cdots - a_n t_i^n\right) = 0
\end{aligned} \tag{13.114}$$

Dividing each equation by -2 *and rearrangement gives* $n+1$ *equations to be simultaneously solved for the coefficients* a_i, $i = 1, 2, \cdots, n$.

$$\begin{aligned}
a_0 N + a_1 \sum_{i=1}^{N} t_i + \cdots + a_n \sum_{i=1}^{N} t_i^n &= \sum_{i=1}^{N} p_i \\
a_0 \sum_{i=1}^{N} t_i + a_1 \sum_{i=1}^{N} t_i^2 + \cdots + a_n \sum_{i=1}^{N} t_i^{n+1} &= \sum_{i=1}^{N} t_i p_i \\
&\cdots \\
a_0 \sum_{i=1}^{N} t_i^n + a_1 \sum_{i=1}^{N} t_i^{n+1} + \cdots + a_n \sum_{i=1}^{N} t_i^{2n} &= \sum_{i=1}^{N} t_i^n p_i
\end{aligned} \tag{13.115}$$

Rearrangement makes a set of linear equations to be solved for a_i, $i = 1, 2, \cdots, n$

$$[A]\,\mathbf{a} = \mathbf{b} \tag{13.116}$$

where,

$$[A] = \begin{bmatrix} N & \sum_{i=1}^{N} t_i & \sum_{i=1}^{N} t_i^2 & \cdots & \sum_{i=1}^{N} t_i^n \\ \sum_{i=1}^{N} t_i & \sum_{i=1}^{N} t_i^2 & \sum_{i=1}^{N} t_i^3 & \cdots & \sum_{i=1}^{N} t_i^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{N} t_i^n & \sum_{i=1}^{N} t_i^{n+1} & \sum_{i=1}^{N} t_i^{n+2} & \cdots & \sum_{i=1}^{N} t_i^{2n} \end{bmatrix} \tag{13.117}$$

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \sum_{i=1}^{N} p_i \\ \sum_{i=1}^{N} t_i p_i \\ \sum_{i=1}^{N} t_i^2 p_i \\ \cdots \\ \sum_{i=1}^{N} t_i^n p_i \end{bmatrix}. \tag{13.118}$$

13.3 ★ Non-Polynomial Path Planning

A path of motion in either joint or Cartesian spaces may be defined based on different mathematical functions. Harmonic and cycloid functions are the most common paths.

$$q(t) = a_0 + a_1 \cos a_2 t + a_3 \sin a_2 t \tag{13.119}$$

$$q(t) = a_0 + a_1 t - a_2 \sin a_3 t \tag{13.120}$$

However, we may also use other function approximate methods such as Fourier,

$$q(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx)\right] \tag{13.121}$$

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} q(t) dt \tag{13.122}$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} q(t) \cos(nx)\, dt \tag{13.123}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} q(t) \sin(nx)\, dt \tag{13.124}$$

Legendre,

$$q_n(t) = \sum_{i=0}^{n} L_i(t) q(t_i) \tag{13.125}$$

$$L_i(t) = \prod_{j=0, j\neq i}^{n} \frac{t - t_j}{t_i - t_j} \qquad i = 0, 1, 2, \cdots, n \tag{13.126}$$

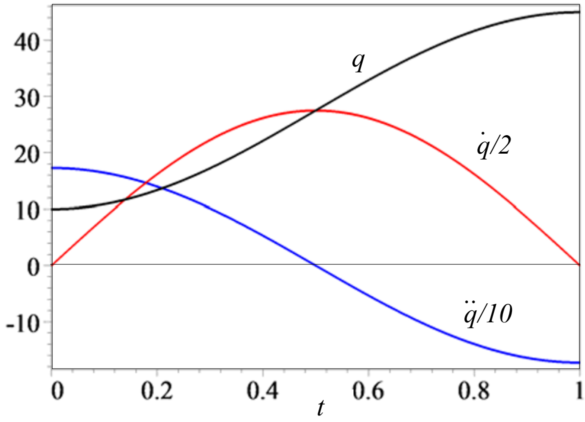

FIGURE 13.10. A harmonic path.

Chebyshev.

$$q_{n+1}(t) = 2tq_n(t) - q_{n-1}(t) \tag{13.127}$$
$$q_0(t) = 1 \qquad q_1(t) = t \tag{13.128}$$

Example 358 *Harmonic path.*

Consider a harmonic path between two points $q(t_0)$ and $q(t_f)$

$$q(t) = a_0 + a_1 \cos a_2 t + a_3 \sin a_2 t \tag{13.129}$$

with the rest-to-rest boundary conditions.

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = 0 \\ q(t_f) &= q_f \qquad \dot{q}(t_f) = 0 \end{aligned} \tag{13.130}$$

Applying the conditions to the harmonic equation (13.129) provides the following solution:

$$q(t) = \frac{1}{2}\left(q_f + q_0 - (q_f - q_0)\cos\frac{\pi(t - t_0)}{t_f - t_0}\right). \tag{13.131}$$

A plot of the solution is depicted in Figure 13.10 for the following numerical values:

$$\begin{aligned} t_0 &= 0 \qquad & t_f = 1 \\ q_0 &= 10 \deg \qquad & q_0' = 0 \\ q_f &= 45 \deg \qquad & q_f' = 0. \end{aligned} \tag{13.132}$$

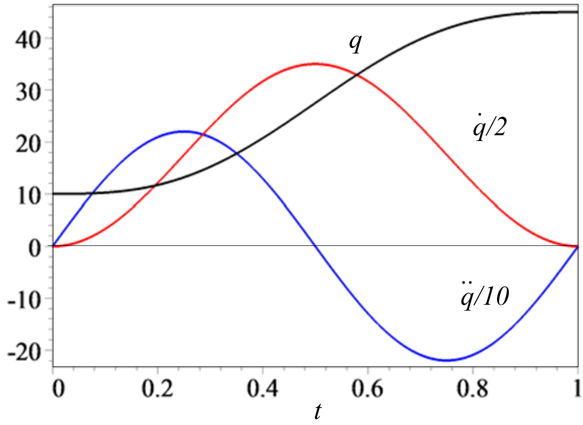

FIGURE 13.11. A cycloid path.

Example 359 *A cycloid path.*

A cycloid path between two points $q(t_0)$ and $q(t_f)$ with rest-to-rest boundary conditions

$$\begin{aligned} q(t_0) &= q_0 \qquad \dot{q}(t_0) = 0 \\ q(t_f) &= q_f \qquad \dot{q}(t_f) = 0 \end{aligned} \tag{13.133}$$

is:

$$q(t) = q_0 + \frac{q_f - q_0}{\pi}\left(\frac{\pi(t-t_0)}{t_f - t_0} - \frac{1}{2}\sin\frac{2\pi(t-t_0)}{t_f - t_0}\right) \tag{13.134}$$

A plot of the cycloid path is illustrated in Figure 13.11 for the following numerical values:

$$\begin{aligned} t_0 &= 0 \qquad t_f = 1 \\ q_0 &= 10\,\text{deg} \qquad q_0' = 0 \\ q_f &= 45\,\text{deg} \qquad q_f' = 0. \end{aligned} \tag{13.135}$$

Comparing Figure 13.11 with 13.5 indicates that the main kinematic characteristics of a cycloid path are similar to quintic rest-to-rest path.

13.4 Manipulator Motion by Joint Path

Having the joint variables as functions of time, and employing the forward kinematics of manipulators, allows us to calculate the path of motion for the end-effector.

Example 360 *2R manipulator motion based on joints' path.*

Assume that we have calculated the paths of the two joints of a $2R$ *planar manipulator according to cubic functions, and they are:*

$$\theta_1(t) = 10 + 105t^2 - 70t^3 \deg \tag{13.136}$$
$$\theta_2(t) = 10 + 350t^3 - 525t^4 + 210t^5 \deg \tag{13.137}$$

The joints' paths satisfy the following conditions:

$$\begin{aligned} \theta_1(0) &= 10\deg & \dot\theta_1(0) &= 0 \\ \theta_1(1) &= 45\deg & \dot\theta_1(1) &= 0 \end{aligned} \tag{13.138}$$

$$\begin{aligned} \theta_2(0) &= 10\deg & \dot\theta_2(0) &= 0 & \ddot\theta_2(0) &= 0 \\ \theta_2(1) &= 45\deg & \dot\theta_2(1) &= 0 & \ddot\theta_2(1) &= 0 \end{aligned} \tag{13.139}$$

The forward kinematics of a $2R$ *manipulator are found in Example 141 as below.*

$$\begin{aligned} {}^0T_2 &= {}^0T_1\ {}^1T_2 \\ &= \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1 c\theta_1 + l_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{13.140}$$

The fourth column of ${}^0T_2$ *indicates the Cartesian position of the tip point of the manipulator in the base frame. Therefore, the* X *and* Y *components of the tip point are:*

$$X = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \tag{13.141}$$
$$Y = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \tag{13.142}$$

Substituting θ_1 *and* θ_2 *from (13.136) and (13.137) provides the time variation of the position of the tip point. These variations, for* $l_1 = l_2 = 1\,\mathrm{m}$ *are shown in Figure 13.12, while the configurations of the manipulator at initial and final positions are shown in Figure 13.13.*

As long as the joint variables are defined and given as functions of time, it is immaterial which joint turns first. The joint variables are relative coordinates and the final configuration of the robot would be the same. They can even turn together.

Moving a robot by applying a set of joint paths is not always a proper method. In case the joint variables are not monotonic in time and are fluctuating, defining a joint path is more complicated. Furthermore, it is not easy to move the end-effector of a robot on a desired geometric path by defining joint paths.

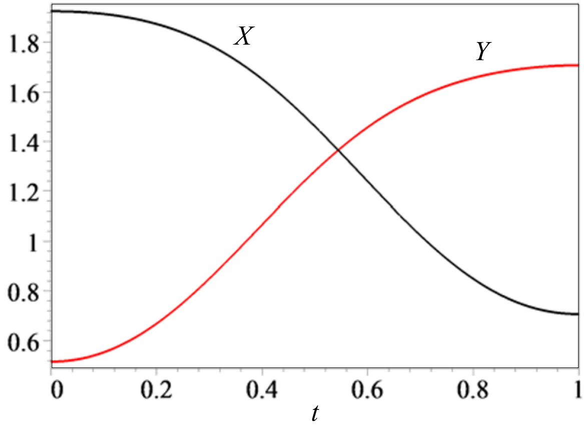

FIGURE 13.12. X and Y components of the tip point position of a 2R planar manipulator.

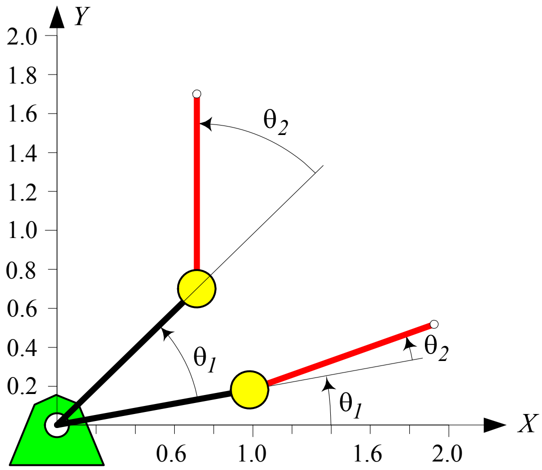

FIGURE 13.13. Configuration of a $2R$ manipulator at initial and final positions.

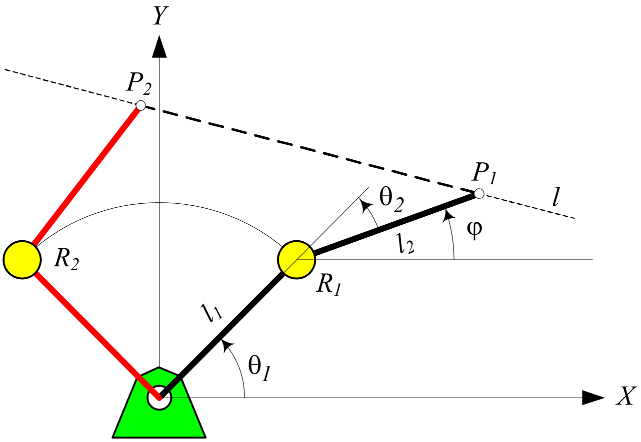

FIGURE 13.14. A $2R$ robot moveing along a given line.

Example 361 *A $2R$ robot moving along a line.*
Let us consider a $2R$ manipulator with

$$l_1 = l_2 = 0.25 \tag{13.143}$$

that its tip point is supposed to move on a given line $Y = f(X)$ as is shown in Figure 13.14.

$$Y = -0.25998X + 0.3705 \tag{13.144}$$

Assume that the first angle is moving between $45\deg$ *and* $135\deg$ *in* $10\sec$

$$45\deg < \theta_1 < 135\deg \tag{13.145}$$

based on a cubic path.

$$\theta_1 = \frac{\pi}{4} + \frac{3\pi}{200}t^2 - \frac{\pi}{1000}t^3 \qquad 0 < t < 10\sec \tag{13.146}$$

The elbow joint R will move on a circle and at the beginning is at:

$$X_{R_1} = 0.25\cos\frac{\pi}{4} = 0.176\,78 \tag{13.147}$$

$$Y_{R_1} = 0.25\sin\frac{\pi}{4} = 0.176\,78 \tag{13.148}$$

Point P_1 must be on the line (13.144) at a distance $d = 0.25$ from R_1.

$$\begin{aligned} d &= \sqrt{(X - 0.176\,78)^2 + (Y - 0.176\,78)^2} \\ &= \sqrt{(X - 0.176\,78)^2 + (-0.25998X + 0.3705 - 0.176\,78)^2} \\ &= 0.25 \end{aligned} \tag{13.149}$$

Therefore, P_1 is at:

$$X_{P_1} = 0.411\,22 \qquad Y_{P_1} = 0.263\,59 \tag{13.150}$$

and initial values of angles φ and θ_2 are:

$$\begin{aligned} \varphi &= \arctan \frac{Y_{P_1} - Y_{R_1}}{X_{P_1} - X_{R_1}} = \arctan \frac{0.263\,59 - 0.176\,78}{0.411\,22 - 0.176\,78} \\ &= 0.354\,63 \,\mathrm{rad} \approx 20.319 \deg \end{aligned} \tag{13.151}$$

$$\begin{aligned} \theta_2 &= \theta_1 - \varphi = \frac{\pi}{4} - 0.354\,63 \\ &= 0.430\,77 \,\mathrm{rad} \approx 24.681 \deg \end{aligned} \tag{13.152}$$

The elbow joint R at the final position is at:

$$X_{R_2} = 0.25 \cos \frac{3\pi}{4} = -0.176\,78 \tag{13.153}$$

$$Y_{R_2} = 0.25 \sin \frac{3\pi}{4} = 0.176\,78 \tag{13.154}$$

Point P_2 must be on the line (13.144) at a distance $d = 0.25$ from R_2.

$$\begin{aligned} d &= \sqrt{(X + 0.176\,78)^2 + (Y - 0.176\,78)^2} \\ &= \sqrt{(X + 0.176\,78)^2 + (-0.25998X + 0.3705 - 0.176\,78)^2} \\ &= 0.25 \end{aligned} \tag{13.155}$$

Therefore, P_2 is at:

$$X_{P_2} = -2.818\,8 \times 10^{-2} \qquad Y_{P_2} = 0.377\,83 \tag{13.156}$$

and final values of angles φ and θ_2 are:

$$\begin{aligned} \varphi &= \arctan \frac{Y_{P_2} - Y_{R_2}}{X_{P_2} - X_{R_2}} = \arctan \frac{0.377\,83 - 0.176\,78}{-2.818\,8 \times 10^{-2} + 0.176\,78} \\ &= 0.934\,32 \,\mathrm{rad} \approx 53.533 \deg \end{aligned} \tag{13.157}$$

$$\begin{aligned} \theta_2 &= \theta_1 - \varphi = \frac{3\pi}{4} - 0.934\,32 \\ &= 1.421\,9 \,\mathrm{rad} \approx 81.469 \deg \end{aligned} \tag{13.158}$$

To determine θ_2 during the motion, we should follow the same procedure. Let us find the position of the elbow joint R as a function of θ_1.

$$X_R = 0.25 \cos \theta_1 \qquad Y_R = 0.25 \sin \theta_1 \tag{13.159}$$

The tip point P must be on the line (13.144) at a distance $d = 0.25$ from the elbow joint R.

$$\begin{aligned} d &= \sqrt{(X_P - 0.25\cos\theta_1)^2 + (Y_P - 0.25\sin\theta_1)^2} \\ &= \sqrt{(X_P - 0.25\cos\theta_1)^2 + (-0.25998X_P + 0.3705 - 0.25\sin\theta_1)^2} \\ &= 0.25 \end{aligned} \tag{13.160}$$

Solution of this equation for X_P and substitution in (13.144) provides the coordinates (X_P, Y_P) of the tip point P during the motion. Then, the angle φ and θ_2 would be:

$$\varphi = \arctan\frac{Y_P - Y_R}{X_P - X_R} = \arctan\frac{Y_P - 0.25\sin\theta_1}{X_P - 0.25\cos\theta_1} \tag{13.161}$$

$$\theta_2 = \theta_1 - \varphi = \theta_1 - \arctan\frac{Y_P - 0.25\sin\theta_1}{X_P - 0.25\cos\theta_1} \tag{13.162}$$

Therefore, to make the point P moving along the line (13.144), while θ_1 is varying as (13.146), the angle θ_2 must vary according to (13.162).

13.5 Cartesian Path

Cartesian path planning is mathematically similar to joint space path planning. Having the coordinates of the start and stop point of the end-effector as

$$P_0 = P_0(X_0, Y_0, Z_0) \qquad P_1 = P_1(X_1, Y_1, Z_1) \tag{13.163}$$

we can connect the points by a geometric space curve

$$Z = Z(X) \qquad Y = Y(X) \tag{13.164}$$

where,

$$X(t_0) = X_0 \qquad X(t_f) = X_f. \tag{13.165}$$

Then, we may define a time path for one of the coordinates, say X, between P_0 and P_f to determine the kinematic behavior of the other coordinates on the geometric path (13.164).

A point-to-point path can also be planned by connecting the points, or designing a path to pass close to but not necessarily through the points. A practical method is to design a path utilizing straight lines with constant velocity, and deform the corners to have a smooth transition.

The path connecting points $\mathbf{r}_0$ to $\mathbf{r}_2$, and passing close to the corner $\mathbf{r}_1$

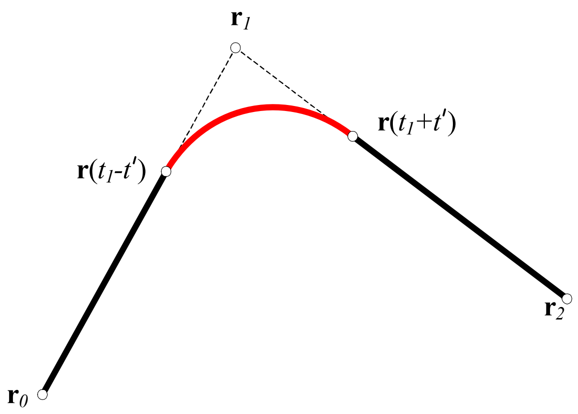

FIGURE 13.15. Transition parabola between two line segments as a path in Cartesian space.

on a transition curve, can be designed by a piecewise motion.

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_1 - \frac{t_1 - t}{t_1 - t_0}(\mathbf{r}_1 - \mathbf{r}_0) & t_0 \le t \le t_1 - t' \\
\mathbf{r}(t) &= \mathbf{r}_1 - \frac{(t - t' - t_1)^2}{4t'(t_1 - t_0)}(\mathbf{r}_1 - \mathbf{r}_0) & \\
&\quad + \frac{(t + t' - t_1)^2}{4t'(t_2 - t_1)}(\mathbf{r}_2 - \mathbf{r}_1) & t_1 - t' \le t \le t_1 + t' \\
\mathbf{r}(t) &= \mathbf{r}_1 - \frac{t_1 - t}{t_2 - t_1}(\mathbf{r}_2 - \mathbf{r}_1) & t_1 + t' \le t \le t_2
\end{aligned} \tag{13.166}$$

The path starts from $\mathbf{r}_0$ at time t_0 and moves with constant velocity $\mathbf{v}_1 = \frac{\mathbf{r}_1 - \mathbf{r}_0}{t_1 - t_0}$ along a line until a point at switching time $t_1 - t'$. At this time, the path switches to a constant acceleration parabola. At another switching point at time $t_1 + t'$, the path switches to the second line and moves with constant velocity $\mathbf{v}_2 = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$ toward the destination at point $\mathbf{r}_2$. The time $t_1 - t_0$ is the required time to move from $\mathbf{r}_0$ to $\mathbf{r}_1$ and $t_2 - t_1$ is the required time to move from $\mathbf{r}_1$ to $\mathbf{r}_2$, if there were no transition path. The path is shown in Figure 13.15 schematically.

Proof. The first line segment starts from a point $\mathbf{r}_0$ at time t_0 and, without any deformation, it arrives at point $\mathbf{r}_1$ at time t_1 via a constant velocity. The second line ends with a constant velocity at point $\mathbf{r}_2$ at time t_2 and, without deformation, it would start from point $\mathbf{r}_1$ at time t_1.

$$\mathbf{r}(t) = \begin{cases} \mathbf{r}_1 - \dfrac{t_1 - t}{t_1 - t_0}(\mathbf{r}_1 - \mathbf{r}_0) & t_0 \le t \le t_1 \\ \mathbf{r}_1 - \dfrac{t_1 - t}{t_2 - t_1}(\mathbf{r}_2 - \mathbf{r}_1) & t_1 \le t \le t_2 \end{cases} \tag{13.167}$$

We introduce an interval time t' before arriving at $\mathbf{r}_1$ to switch from the line to a transition curve. The transition curve is then between times $t_1 - t'$ and $t_1 + t'$. The simplest transition curve is a parabola, which, at the end points, has the same speed as the lines.

The boundary positions of the transition curve on the first and second lines are respectively at

$$\mathbf{r}(t_1 - t') = \mathbf{r}_1 - \frac{t'}{t_1 - t_0}\boldsymbol{\delta}_1 \tag{13.168}$$

$$\mathbf{r}(t_1 + t') = \mathbf{r}_1 + \frac{t'}{t_2 - t_1}\boldsymbol{\delta}_2 \tag{13.169}$$

where,

$$\boldsymbol{\delta}_1 = \mathbf{r}_1 - \mathbf{r}_0 \tag{13.170}$$

$$\boldsymbol{\delta}_2 = \mathbf{r}_2 - \mathbf{r}_1. \tag{13.171}$$

The velocity at the beginning and final points of the transition curve are respectively equal to:

$$\dot{\mathbf{r}}(t_1 - t') = \frac{1}{t_1 - t_0}\boldsymbol{\delta}_1 \tag{13.172}$$

$$\dot{\mathbf{r}}(t_1 + t') = \frac{1}{t_2 - t_1}\boldsymbol{\delta}_2 \tag{13.173}$$

Assume the acceleration of motion along the transition curve is constant

$$\ddot{\mathbf{r}}(t) = \ddot{\mathbf{r}}_c = const \tag{13.174}$$

and therefore, the transition curve after integration is equal to

$$\mathbf{r}(t) = \mathbf{r}(t_1 - t') + (t - t_1 + t')\dot{\mathbf{r}}(t_1 - t') + \frac{1}{2}(t - t_1 + t')^2\ddot{\mathbf{r}}_c. \tag{13.175}$$

Substituting (13.168) and (13.172) provides

$$\mathbf{r}(t) = \mathbf{r}_1 + \frac{t - t_1}{t_1 - t_0}\boldsymbol{\delta}_1 + \frac{1}{2}\ddot{\mathbf{r}}_c(t - t_1 + t')^2. \tag{13.176}$$

The transition curve $\mathbf{r}(t)$ must be at the end point when $t = t_1 + t'$

$$\mathbf{r}(t_1 + t') = \mathbf{r}_1 + \frac{t'}{t_2 - t_1}\boldsymbol{\delta}_2 = \mathbf{r}_1 + \frac{t'}{t_1 - t_0}\boldsymbol{\delta}_1 + 2\ddot{\mathbf{r}}_c\, t_1^2 \tag{13.177}$$

therefore, the acceleration on the curve must be

$$\ddot{\mathbf{r}}_c = \frac{1}{2t'}\left(\frac{\boldsymbol{\delta}_2}{t_2 - t_1} - \frac{\boldsymbol{\delta}_1}{t_1 - t_0}\right). \tag{13.178}$$

Hence, the curve equation becomes

$$\mathbf{r}(t) = \mathbf{r}_1 - \boldsymbol{\delta}_1 \frac{(t - t' - t_1)^2}{4t'(t_1 - t_0)} + \boldsymbol{\delta}_2 \frac{(t + t' - t_1)^2}{4t'(t_2 - t_1)} \tag{13.179}$$

showing that the path between $\mathbf{r}_0$ and $\mathbf{r}_2$ has a piecewise character given in (13.166).

A Cartesian path followed by the manipulator, plus the time profile along the path, specify the position and orientation of the end frame. Issues in Cartesian path planning include attaining a specific target from an initial starting point, avoiding obstacles, and staying within manipulator capabilities. A path is modeled by n points called *control points.* The control points are connected via straight lines and the transient parabolas will be implemented to exclude the sharp corners.

An alternative method is applying an interpolating or approximating method, such as least-squared, to design a continuous path over the control points, or close to them. ■

Example 362 *A path in* 2*D Cartesian space.*

Consider a line in the XY *plane connecting* $(1,0)$ *and* $(1,1)$*, and another line connecting* $(1,1)$ *and* $(0,1)$*. Assume that the time is zero at* $(1,0)$*, is* $t = 1\,\text{sec}$ *at* $(1,1)$*, and is* $t = 2\,\text{sec}$ *at* $(0,1)$*. For an interval time* $t' = 0.1\,\text{sec}$*, the position vector at control points are*

$$\mathbf{r}_0 = \hat{\imath} \tag{13.180}$$
$$\mathbf{r}_1 = \hat{\imath} + \hat{\jmath} \tag{13.181}$$
$$\mathbf{r}_2 = \hat{\jmath} \tag{13.182}$$

$$\mathbf{r}(t_1 - t') = \mathbf{r}_1 - \frac{t'}{t_1}\boldsymbol{\delta}_1 = \hat{\imath} + \left(1 - \frac{t'}{t_1}\right)\hat{\jmath} \tag{13.183}$$
$$\mathbf{r}(t_1 + t') = \mathbf{r}_1 + \frac{t'}{t_2}\boldsymbol{\delta}_2 = \left(1 - \frac{t'}{t_2}\right)\hat{\imath} + \hat{\jmath} \tag{13.184}$$

where

$$\boldsymbol{\delta}_1 = \mathbf{r}_1 - \mathbf{r}_0 = \hat{\jmath} \tag{13.185}$$
$$\boldsymbol{\delta}_2 = \mathbf{r}_2 - \mathbf{r}_1 = -\hat{\imath}. \tag{13.186}$$

The path of motion is then expressed by the following piecewise function as shown in Figure 13.16:

$$\mathbf{r}(t) = \begin{cases} \hat{\imath} + t\hat{\jmath} & 0 \le t \le 0.9 \\ \left(1 - \frac{(t-0.9)^2}{0.4}\right)\hat{\imath} + \left(1 - \frac{(t-1.1)^2}{0.4}\right)\hat{\jmath} & 0.9 \le t \le 1.1 \\ (2-t)\,\hat{\imath} + \hat{\jmath} & 1.1 \le t \le 2 \end{cases} \tag{13.187}$$

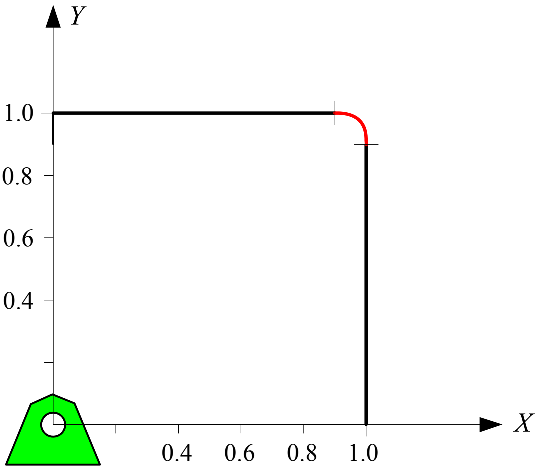

FIGURE 13.16. A transition parabola connecting two lines.

The velocity of motion along the path is also a piecewise function given below.

$$\dot{\mathbf{r}}(t) = \begin{cases} \hat{\jmath} & 0 \le t \le 0.9 \\ \frac{t-0.9}{0.2}\hat{\imath} - \frac{t-1.1}{0.2}\hat{\jmath} & 0.9 \le t \le 1.1 \\ -\hat{\imath} & 1.1 \le t \le 2 \end{cases} \tag{13.188}$$

Example 363 *A 2R manipulator following a line.*

Assume the 2R manipulator in Figure 13.14 has

$$l_1 = l_2 = 0.25 \tag{13.189}$$

and its tip point is supposed to move on a given line $Y = f(X)$.

$$Y = -0.25998X + 0.3705 \tag{13.190}$$

The manipulator moves form P_1 to P_2 in $10\,\mathrm{sec}$.

$$X_{P_1} = 0.411\,22 \qquad Y_{P_1} = 0.263\,59 \tag{13.191}$$

$$X_{P_2} = -2.818\,8 \times 10^{-2} \qquad Y_{P_2} = 0.377\,83 \tag{13.192}$$

Let us define a rest-to-rest cubic path for X.

$$X = 0.41122 - 0.01149096t^2 + 0.000766064t^3 \tag{13.193}$$

We determine the equation of Y as a function of t by substituting $X = X(t)$ in the line equation (13.190).

$$Y = -1.9916 \times 10^{-4}t^3 + 2.9874 \times 10^{-3}t^2 + 0.26359 \tag{13.194}$$

13.6 ★ Rotational Path

Consider an end-effector frame to have a rotation matrix ${}^{G}R_0$ at an initial orientation at time t_0. The end-effector must be at a final orientation ${}^{G}R_f$ at time t_f. The *rotational path* is defined by the angle-axis rotation matrix $R_{\hat{u},\phi}$

$$R_{{}^0\hat{u},\phi} = {}^0R_f = {}^GR_0^T \, {}^GR_f \tag{13.195}$$

that transforms the end-effector frame from the final orientation ${}^GR_f$ to the initial orientation ${}^GR_0$. The axis of rotation ${}^0\hat{u}$ is defined by a unit vector expressed in the initial frame. Therefore, the desired rotation matrix for going from initial to the final orientation, would be

$$R^T_{{}^0\hat{u},\phi} = {}^GR_f^T \, {}^GR_0. \tag{13.196}$$

Keeping ${}^0\hat{u}$ constant, we can define an angular path for φ to vary $R^T_{{}^0\hat{u},\phi}$ from ${}^GR_0$ to ${}^GR_f$ at t_f.

To control a rotation, we may define a series of control orientations ${}^GR_1$, ${}^GR_2$, $\cdots$, ${}^GR_n$ between the initial and final orientations, and rotate the end-effector frame through the control orientations. When there is a control orientation ${}^GR_1$ between the initial and final orientations, then the initial orientation ${}^GR_0$ transforms to the control orientation ${}^GR_1$ using an angle-axis rotation $R_{{}^0\hat{u},\phi_0}$, and then it transforms from the control orientation ${}^GR_1$ to the final orientation using a second-angle axis rotation $R_{{}^1\hat{u},\phi_1}$.

$$R_{{}^0\hat{u},\phi_0} = {}^GR_0^T \, {}^GR_1 \tag{13.197}$$

$$R_{{}^1\hat{u},\phi_1} = {}^GR_1^T \, {}^GR_f \tag{13.198}$$

Proof. According to the Rodriguez rotation formula (3.4),

$${}^0R_f = R_{{}^0\hat{u},\phi} = \mathbf{I}\cos\phi + {}^0\hat{u}\,{}^0\hat{u}^T \operatorname{vers}\phi + {}^0\tilde{u}\sin\phi \tag{13.199}$$

the angle and axis that transforms a frame B_f to another frame B_0 are found from

$$\cos\phi = \frac{1}{2}\left(\operatorname{tr}\left({}^0R_f\right) - 1\right) \tag{13.200}$$

$${}^0\tilde{u} = \frac{1}{2\sin\phi}\left({}^0R_f - {}^0R_f^T\right). \tag{13.201}$$

If ${}^GR_0$ is the rotation matrix from B_0 to the global frame G, and ${}^GR_f$ is the rotation matrix from B_f to G, then

$${}^GR_f = {}^GR_0 \, {}^0R_f \tag{13.202}$$

and therefore,

$${}^0R_f = R_{{}^0\hat{u},\phi} = {}^GR_0^T \, {}^GR_f. \tag{13.203}$$

We define a linearly time dependent rotation matrix by varying the angle of rotation about the axis of rotation

$$\begin{aligned} {}^0R_f(t) &= R_{{}^0\hat{u},\left(\frac{t-t_0}{t_f-t_0}\right)\phi} \\ &= \begin{bmatrix} r_{11}(t) & r_{12}(t) & r_{13}(t) \\ r_{21}(t) & r_{22}(t) & r_{23}(t) \\ r_{31}(t) & r_{32}(t) & r_{33}(t) \end{bmatrix} \qquad t_0 \le t \le t_f \end{aligned} \tag{13.204}$$

where, t_0 is the time when the end-effector frame is at orientation ${}^GR_0$ and t_f is the time at which the end-effector frame is at orientation ${}^GR_f$, and

$$\begin{aligned} r_{11}(t) &= u_1^2 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + \cos\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{21}(t) &= u_1u_2 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + u_3\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{31}(t) &= u_1u_3 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi - u_2\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \end{aligned} \tag{13.205}$$

$$\begin{aligned} r_{12}(t) &= u_1u_2 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi - u_3\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{22}(t) &= u_2^2 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + \cos\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{32}(t) &= u_2u_3 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + u_1\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \end{aligned} \tag{13.206}$$

$$\begin{aligned} r_{13}(t) &= u_1u_3 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + u_2\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{23}(t) &= u_2u_3 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi - u_1\sin\left(\frac{t-t_0}{t_f-t_0}\right)\phi \\ r_{33}(t) &= u_3^2 \operatorname{vers}\left(\frac{t-t_0}{t_f-t_0}\right)\phi + \cos\left(\frac{t-t_0}{t_f-t_0}\right)\phi. \end{aligned} \tag{13.207}$$

The matrix ${}^0R_f(t)$ can turn the final frame about the axis of rotation ${}^0\hat{u}$ onto the initial frame, and therefore,

$${}^GR_f = {}^GR_0\,{}^0R_f(t). \tag{13.208}$$

If there is a control orientation frame ${}^GR_1$ between the initial and final orientations, then

$${}^GR_1 = {}^GR_0\,{}^0R_1 \tag{13.209}$$

$${}^GR_f = {}^GR_1\,{}^1R_f \tag{13.210}$$

and therefore,

$$R_{{}^0\hat{u},\phi_0} = {}^0R_1 = {}^GR_0^T\, {}^GR_1 \tag{13.211}$$
$$R_{{}^1\hat{u},\phi_1} = {}^1R_f = {}^GR_1^T\, {}^GR_f. \tag{13.212}$$

The rotation matrices ${}^0R_1$ and ${}^1R_f$ may be defined as linearly time varying rotation matrices by

$$\begin{aligned} {}^0R_1(t) &= R_{{}^0\hat{u},(\frac{t-t_0}{t_1-t_0})\phi_0} && t_0 \le t \le t_1 \\ {}^1R_f(t) &= R_{{}^1\hat{u},(\frac{t-t_1}{t_f-t_1})\phi_1} && t_1 \le t \le t_f. \end{aligned} \tag{13.213}$$

Using these variable matrices, we can turn the end-effector frame from the initial orientation ${}^GR_0$ about ${}^0\hat{u}$ to achieve the control orientation ${}^GR_1$, and then turn the end-effector frame about ${}^1\hat{u}$ to achieve the final orientation ${}^GR_f$.

Following the parabola transition technique of section 13.5, we may define an orientation path connecting ${}^GR_0$ and ${}^GR_f$, and passing close to the corner orientation ${}^GR_1$ on a transient rotation path. The path starts from ${}^GR_0$ at time t_0 and turns with constant angular velocity along an axis until $t = t_1 - t'$. At this time, the path switches to a rotational parabolic path with constant angular acceleration. At another switching orientation at time $t = t_1 + t'$, the path switches to the second path and turns with constant velocity toward the destination orientation ${}^GR_f$. The time $t_1 - t_0$ is the required time to move from ${}^GR_0$ to ${}^GR_1$, and $t_2 - t_1$ is the required time to move from ${}^GR_1$ to ${}^GR_f$ if there were no transition path.

We introduce an interval time t' before arriving at orientation ${}^GR_1$ to switch from the first path segment to a transition path. The transition path is then between times $t_1 - t'$ and $t_1 + t'$. At the second switching orientation, the transition path ends at the same angular velocity as the third path segment.

The boundary positions of the transition path between the first and third segments are respectively

$$\begin{aligned} {}^GR_1(t_1 - t') &= {}^GR_0\ {}^0R_1(t_1 - t') \\ &= {}^GR_0\ R_{{}^0\hat{u},(1-\frac{t'}{t_1-t_0})\phi_0} \qquad t = t_1 - t' \end{aligned} \tag{13.214}$$
$$\begin{aligned} {}^GR_f(t_1 + t') &= {}^GR_1\ {}^1R_f(t_1 + t') \\ &= {}^GR_1\ R_{{}^1\hat{u},(\frac{t'}{t_f-t_1})\phi_1} \qquad t = t_1 + t'. \end{aligned} \tag{13.215}$$

The transition path is then equal to

$$\begin{aligned} R_t(t) &= {}^GR_0\ {}^0R_1\left(\frac{t_1 - t' - t}{2t'} - \frac{(t - t' - t_1)^2}{4t'\,(t_1 - t_0)}\right)\ {}^1R_f\left(\frac{(t + t' - t_1)^2}{4t'\,(t_f - t_1)}\right) \\ &= {}^GR_0\ R_{{}^0\hat{u},(\frac{t_1-t'-t}{2t'} - \frac{(t-t'-t_1)^2}{4t'(t_1-t_0)})\phi_0}\ R_{{}^1\hat{u},(\frac{(t+t'-t_1)^2}{4t'\left(t_f-t_1\right)})\phi_1} \end{aligned} \tag{13.216}$$

$t_1 - t' \le t \le t_1 + t'$

and the entire path is:

$$\begin{array}{ll} R(t) = {}^0R_1(t) = R_{0\hat{u},(\frac{t-t_0}{t_1-t_0})\phi_0} & t_0 \le t \le t_1 - t' \\ R(t) = R_t(t) & t_1 - t' \le t \le t_1 + t' \\ R(t) = {}^1R_f(t) = R_{1\hat{u},(\frac{t-t_1}{t_f-t_1})\phi_1} & t_1 + t' \le t \le t_2 \end{array} \tag{13.217}$$

■

Example 364 *Rotation about Z-axis.*

Consider a body B which is initially coincident with the global coordinate frame G at $t = 0$. So, its initial transformation matrix is an identity.

$$ {}^GR_1 = \mathbf{I} \tag{13.218}$$

B is suppose to be at ${}^GR_2$ after 10 sec.

$$ {}^GR_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13.219}$$

The axis of rotation ${}^2R_1$ is the Z-axis, and the angle of rotation is π. The transformation matrix between the initial and final orientations of B_1 and B_2 is:

$$ {}^2R_1 = {}^GR_1^T \, {}^GR_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13.220}$$

Let us define a cubic rest-to-rest path for the angle of rotation α.

$$ \alpha = \frac{3\pi}{100}t^2 - \frac{\pi}{500}t^3 \tag{13.221}$$

The angular path of B between B_1 an B_2 is:

$$\begin{aligned} {}^2R_1 &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{3\pi}{100}t^2 - \frac{\pi}{500}t^3 & -\sin\frac{3\pi}{100}t^2 - \frac{\pi}{500}t^3 & 0 \\ \sin\frac{3\pi}{100}t^2 - \frac{\pi}{500}t^3 & \cos\frac{3\pi}{100}t^2 - \frac{\pi}{500}t^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{13.222}$$

Example 365 *Rotation about X-axis.*

A body B is initially at

$$ {}^GR_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{10} & -\sin\frac{\pi}{10} \\ 0 & \sin\frac{\pi}{10} & \cos\frac{\pi}{10} \end{bmatrix} \tag{13.223}$$

The body is supposed to be at ${}^{G}R_2$ *in* 10 sec.

$$ {}^{G}R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ 0 & \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \tag{13.224} $$

The axis of rotation ${}^{2}R_1$ *is the* X*-axis, and the angle of rotation is* $\frac{2}{5}\pi = \frac{\pi}{2} - \frac{\pi}{10}$. *We define a cubic rest-to-rest path for the angle of rotation* γ

$$ \gamma = \frac{\pi}{10} + \frac{3\pi t^2}{250} - \frac{\pi t^3}{1250} \tag{13.225} $$

to determine the angular path of B *between* G *an* B_2 *is:*

$$ {}^{G}R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \tag{13.226} $$

At any time t*, the body* B *with respect to* B_1 *is at* ${}^{1}R_2$.

$$ \begin{aligned} {}^{1}R_2 &= {}^{1}R_G \, {}^{G}R_2 \qquad (13.227) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.951\cos\gamma - 0.309\sin\gamma & -0.309\cos\gamma - 0.951\sin\gamma \\ 0 & 0.309\cos\gamma + 0.951\sin\gamma & 0.951\cos\gamma - 0.309\sin\gamma \end{bmatrix} \end{aligned} $$

13.7 Manipulator Motion by End-Effector Path

Cartesian path planning is the most natural application of path planning. Considering the pick and place motion as the main job of industrial robot, we have to determine a desired geometric path for the end-effector in the 3-dimenssional Cartesian space of the base frame. We may then define a time path for one of the coordinates, say X, and determine the time history of the other coordinates by using the geometric path. Having the time functions of the coordinates of the end-effector, we can determine the velocity, acceleration and jerk behavior of the end-effector.

Inverse kinematics will determine the kinematics of joint variables. Substituting the joint variables' position, velocity, and acceleration in the dynamic equations of motion provide the required actuators' torque or force to move the end-effector on the desired path with the planned kinematics.

The geometric Cartesian path is an applied method of path planning in robotics, because it can control the level of force and jerk inserted by the hand of a robot to the carrying object. Path planning in Cartesian space also determines the geometric constraints of the external world. However, a Cartesian path needs inverse kinematics to determine the time history of the joint variables.

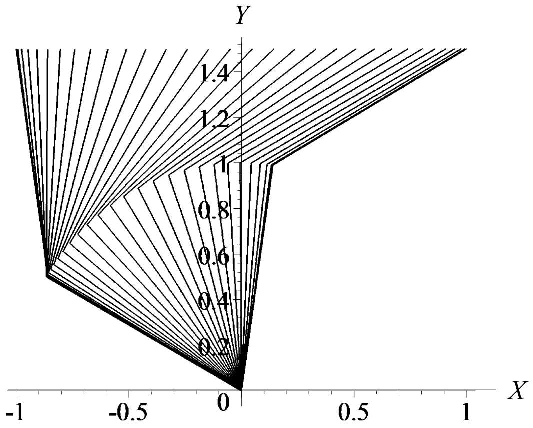

FIGURE 13.17. Illustration of a $2R$ panipulator when the tip point moves on a straight line $y = 1.5$.

Example 366 *Joint path for a designed Cartesian path.*

Consider a rest-to-rest Cartesian path from point $(1, 1.5)$ *to point* $(-1, 1.5)$ *on a straight line* $Y = 1.5$. *A cubic polynomial can satisfy the position and velocity constraints at initial and final points.*

$$\begin{aligned} X(0) &= X_0 = 1 \qquad \dot{X}(0) = \dot{X}_0 = 0 \\ X(1) &= X_f = -1 \qquad \dot{X}(1) = \dot{X}_f = 0 \end{aligned} \tag{13.228}$$

The coefficients of the polynomial are

$$a_0 = 1 \qquad a_1 = 0 \qquad a_2 = -6 \qquad a_3 = 4 \tag{13.229}$$

and the Cartesian path is:

$$X = 1 - 6t^2 + 4t^3 \tag{13.230}$$

$$Y = 1.5 \tag{13.231}$$

The inverse kinematics of a $2R$ *planar manipulator is calculated in Example 184 as*

$$\theta_2 = \pm 2 \operatorname{atan2} \sqrt{\frac{(l_1 + l_2)^2 - (X^2 + Y^2)}{(X^2 + Y^2) - (l_1 - l_2)^2}} \tag{13.232}$$

$$\theta_1 = \operatorname{atan2} \frac{X (l_1 + l_2 \cos\theta_2) + Y l_2 \sin\theta_2}{Y (l_1 + l_2 \cos\theta_2) - X l_2 \sin\theta_2} \tag{13.233}$$

where the sign $(\pm)$ *indicates the elbow-up and elbow-down configurations of the manipulator. Depending on the initial configuration at point* $(1, 1.5)$, *the manipulator is supposed to stay in that configuration. Let's consider an elbow-up configuration. Therefore, we accept only those values of the joint valuables that belong to the elbow-up configuration. Substituting (13.230) and (13.231) in (13.232) and (13.233) provides the path in joint space.*

$$\theta_2 = \pm 2\,\mathrm{atan2}\sqrt{\frac{\left(l_1+l_2\right)^2-\left(4t^3-6t^2+2.5\right)^2}{\left(4t^3-6t^2+1\right)^2-\left(l_1-l_2\right)^2}} \tag{13.234}$$

$$\theta_1 = \mathrm{atan2}\,\frac{\left(1-6t^2+4t^3\right)\left(l_1+l_2\cos\theta_2\right)+1.5l_2\sin\theta_2}{1.5\left(l_1+l_2\cos\theta_2\right)-\left(1-6t^2+4t^3\right)l_2\sin\theta_2} \tag{13.235}$$

A graphical illustration of the manipulator at every $1/30\,th$ *of the total time is shown in Figure 13.17.*

Example 367 *A* $2R$ *manipulator on a line.*
Consider the $2R$ *manipulator of Figure 13.14 with*

$$l_1 = l_2 = 0.25\,\mathrm{m} \tag{13.236}$$

that its tip point is supposed to move on a given line

$$Y = -0.25998X + 0.3705 \tag{13.237}$$

between P_1 *and* P_2 *in* $10\,\mathrm{sec}$.

$$X_{P_1} = 0.41122 \qquad Y_{P_1} = 0.26359 \tag{13.238}$$

$$X_{P_2} = -0.0282 \qquad Y_{P_2} = 0.37783 \tag{13.239}$$

Defining a rest-to-rest cubic path for X*, we determine the Cartesian path of the tip point.*

$$X = 0.41122 - 0.0131826t^2 + 0.00087884t^3 \tag{13.240}$$

$$Y = -0.00022848t^3 + 0.003427t^2 + 0.26359 \tag{13.241}$$

The kinematics of the tip point are shown in Figures 13.18 to 13.20.
Employing the inverse kinematics of equations (6.39) and (6.42), we find the variation of the joint angles as are shown in Figure 13.21.
Let us divide the total time of the motion in $n = 40$ *equal intervals. The configuration of the manipulator at each time step are shown in Figure 13.22.*

Example 368 *A* $2R$ *manipulator on a line with no end acceleration.*
Consider the $2R$ *manipulator of Figure 13.14 with*

$$l_1 = l_2 = 0.25\,\mathrm{m} \tag{13.242}$$

that its tip point is supposed to move on a given line

$$Y = -0.25998X + 0.3705 \tag{13.243}$$

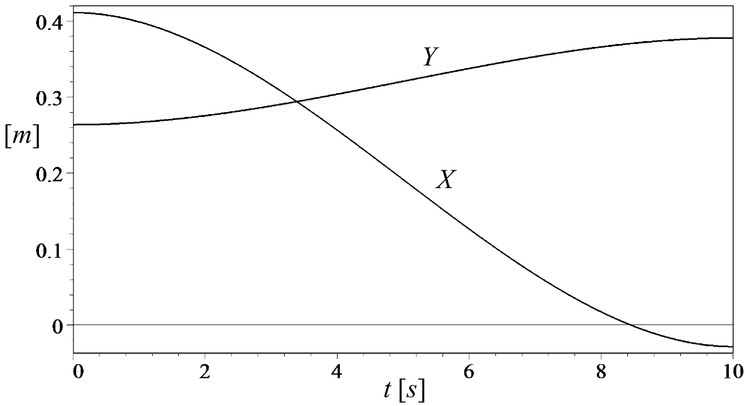

FIGURE 13.18. Cartesian coordinates of the tip point versus time.

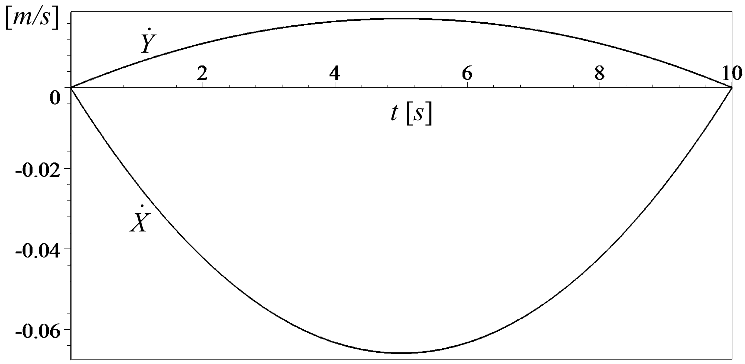

FIGURE 13.19. Components of the tip point velocity versus time.

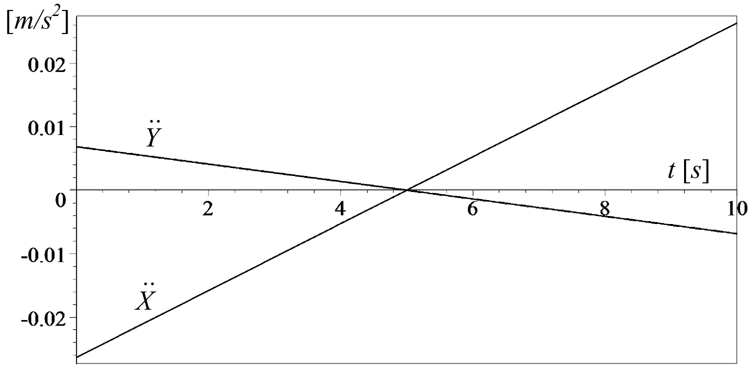

FIGURE 13.20. Components of the tip point acceleration versus time.

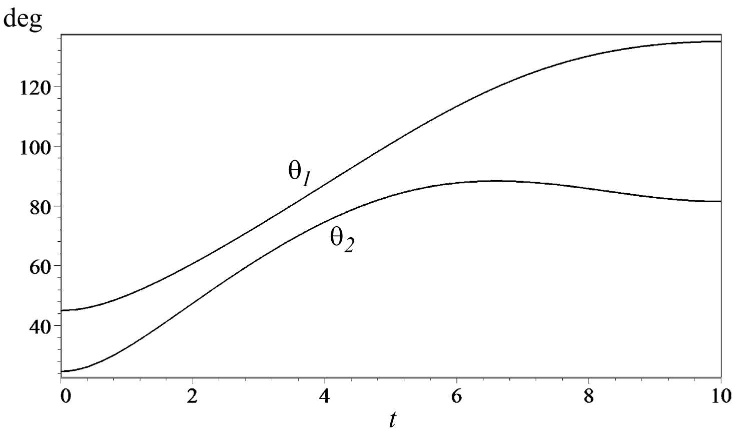

FIGURE 13.21. The variation of joint angles of the 2R manipulator.

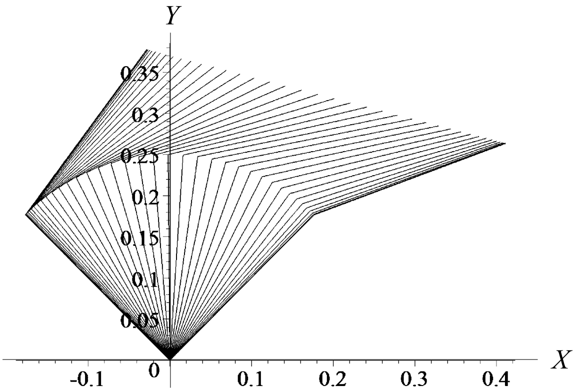

FIGURE 13.22. The configuration of the $2R$ manipulator at 42 equal time steps.

between P_1 *and* P_2 *in* 10 sec.

$$X_{P_1} = 0.41122 \qquad Y_{P_1} = 0.26359 \tag{13.244}$$

$$X_{P_2} = -0.0282 \qquad Y_{P_2} = 0.37783 \tag{13.245}$$

Let us define a quintic path for X *to apply a zero acceleration at both ends.*

$$X = 0.41122 - 0.0043942t^3 + 0.00065913t^4 - 0.0000263652t^5 \tag{13.246}$$

Substituting X *in the line equation (13.243), we also determine the variation of* Y*.*

$$Y = 0.26359 + 0.0011424t^3 - 0.00017136t^4 + 0.0000068544t^5 \tag{13.247}$$

Using the Cartesian components (13.246) and (13.247), we determine the kinematics of the tip point as are shown in Figures 13.23 to 13.20.

Using Equations (6.39) and (6.42), we find the variation of the joint angles as are shown in Figure 13.26.

Example 369 *A* $2R$ *manipulator on a line with no end acceleration.*

Consider the $2R$ *manipulator of Figure 13.27 with equal arms' length.*

$$l_1 = l_2 = 0.25\,\mathrm{m} \tag{13.248}$$

The tip point is supposed to move from P_1 *to* P_2 *in* 10 sec.

$$X_{P_1} = 0.41122 \qquad Y_{P_1} = 0.26359 \tag{13.249}$$

$$X_{P_2} = -0.0282 \qquad Y_{P_2} = 0.37783 \tag{13.250}$$

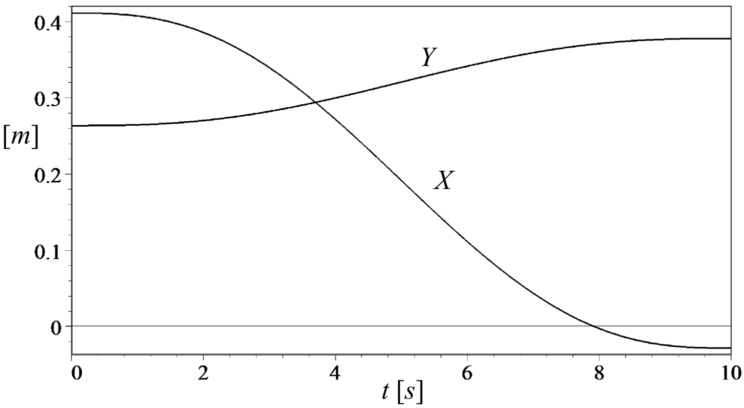

FIGURE 13.23. Cartesian coordinates of the tip point versus time on a no end acceleration path.

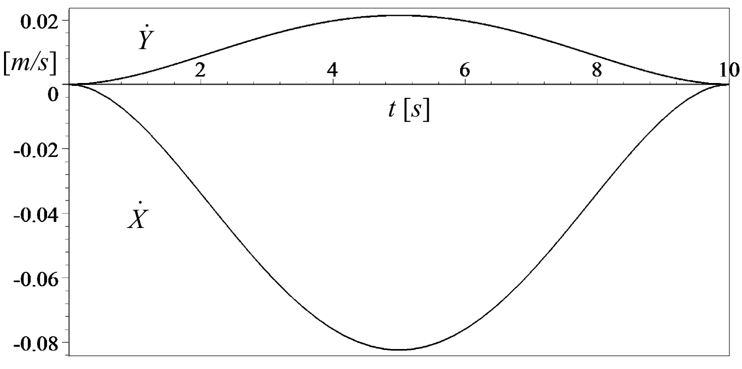

FIGURE 13.24. Components of the tip point velocity versus time on a no end acceleration path.

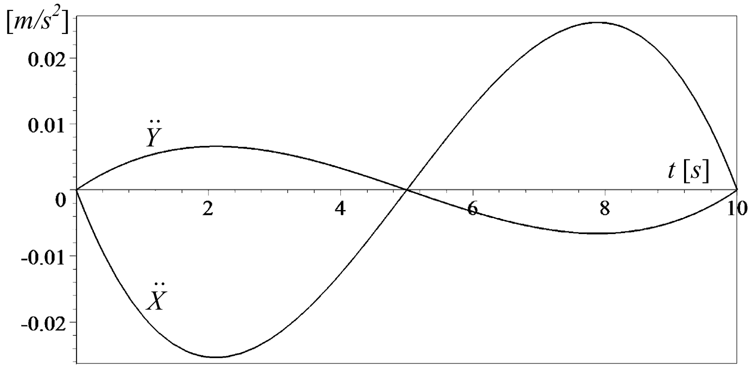

FIGURE 13.25. Components of the tip point acceleration versus time on a no end acceleration path.

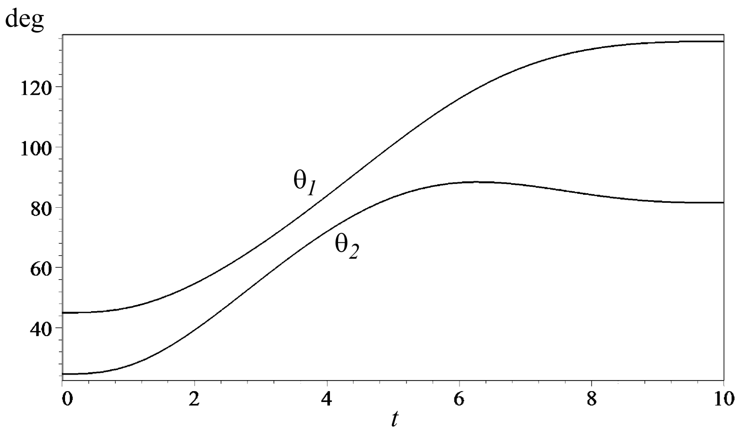

FIGURE 13.26. The variation of joint angles of the $2R$ manipulator on a no end acceleration path.

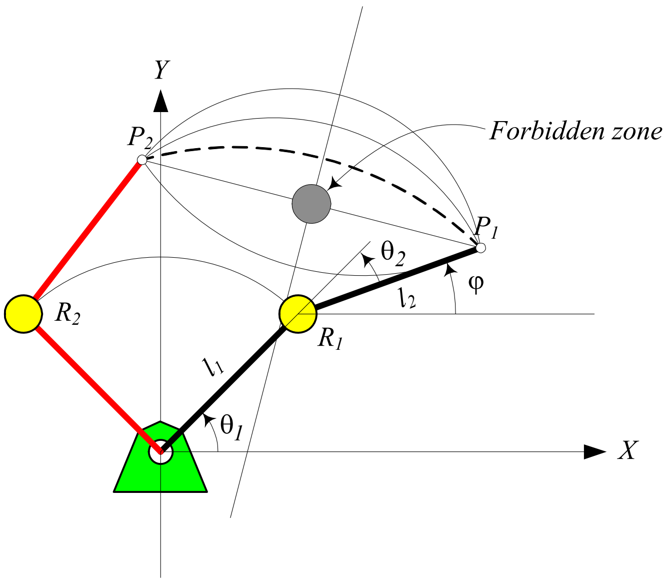

FIGURE 13.27. A few circular paths between P_1 and P_2 to go around forbidden zone at P_3.

However, there is a circular forbidden zone at point P_3, where the tip point cannot pass.

$$X_{P_3} = 0.19151 \qquad Y_{P_3} = 0.32071 \tag{13.251}$$

$$(X - X_{P_3})^2 + (Y - Y_{P_3})^2 = 0.025^2 \tag{13.252}$$

To find a path between P_1 and P_2 to go around P_3, let us choose a circular arc with a center on the bisector of P_1P_2. Figure 13.27 depicts a few optional paths. The arc must be in the working space of the manipulator, which is a circle ring about the base point.

$$(l_1 - l_2)^2 < X^2 + Y^2 < (l_1 + l_2)^2 \tag{13.253}$$

$$0 < X^2 + Y^2 < 0.5^2 \tag{13.254}$$

The center of the circular path should be on the following line.

$$Y - Y_{P_3} = 3.8464\,(X - X_{P_3}) \tag{13.255}$$

Let us pick a point P_C to be the center of the circular path at:

$$X_C = 0.1 \qquad Y_C = -0.06 \tag{13.256}$$

Therefore, the equation of the path is:

$$(X - X_C)^2 + (Y - Y_C)^2 = 0.45^2 \tag{13.257}$$

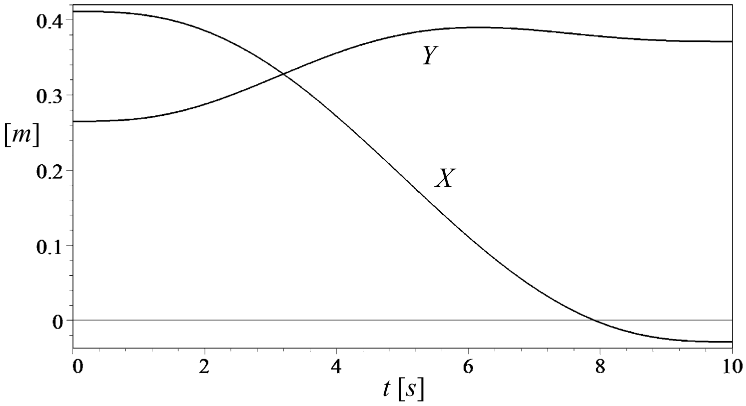

FIGURE 13.28. Cartesian coordinates of the tip point versus time on a circular path.

This path is shown in Figure 13.27 with a dashed line.

We use a quintic time-path for X to apply a zero acceleration at both ends.

$$\begin{aligned} X &= 0.41122 - 0.0043942t^3 + 0.00065913t^4 \\ &\quad -0.0000263652t^5 \end{aligned} \tag{13.258}$$

Substituting X in the path equation (13.257), we determine the time-path of Y.

$$Y = Y_C + \sqrt{0.45^2 - (X - X_C)^2} \tag{13.259}$$

The kinematics of the tip point are shown in Figures 13.28 to 13.30. Equations (6.39) and (6.42), provides the joint angles as are shown in Figure 13.31. The configuration of the manipulator at 42 *equal time steps are shown in Figure 13.32.*

Example 370 *Articulated manipulator on a line.*

Figure 13.33 illustrates an articulated manipulator. Assume that

$$l_1 = 0.5\,\mathrm{m} \qquad l_2 = 1.0\,\mathrm{m} \qquad l_3 = 1.0\,\mathrm{m}. \tag{13.260}$$

The tip point of the manipulator is supposed to move from point P_1 to P_2 in 10 sec.

$$\mathbf{r}_{P_1} = \begin{bmatrix} 1.5 \\ 0.0 \\ 1.0 \end{bmatrix} \qquad \mathbf{r}_{P_2} = \begin{bmatrix} -1.0 \\ 1.0 \\ 1.5 \end{bmatrix} \tag{13.261}$$

Using a quintic path for X, we find the following function to express the time variation of X.

$$X = 1.5 - 0.025t^3 + 0.00375t^4 - 0.00015t^5 \tag{13.262}$$

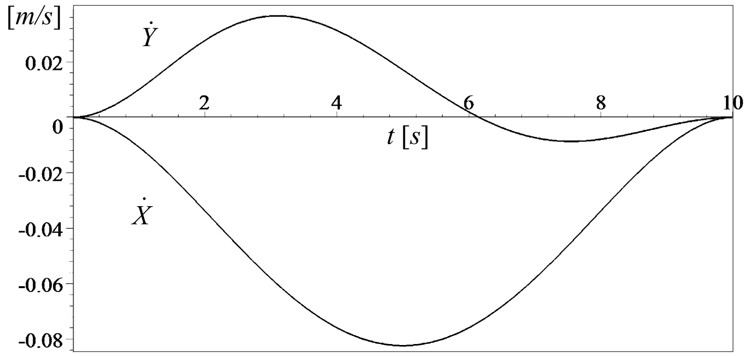

FIGURE 13.29. Components of the tip point velocity versus time on a circular path.

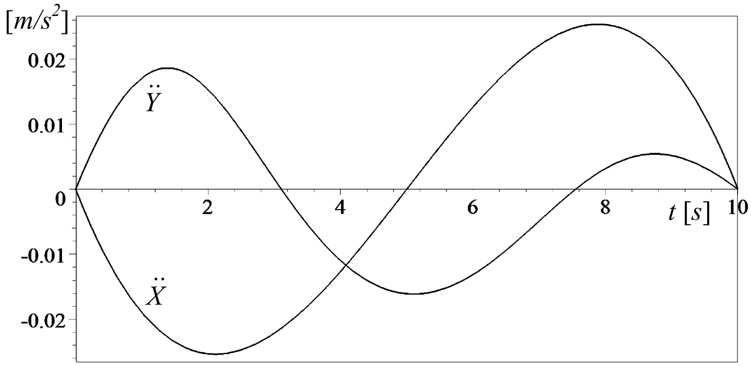

FIGURE 13.30. Components of the tip point acceleration versus time on a circular path.

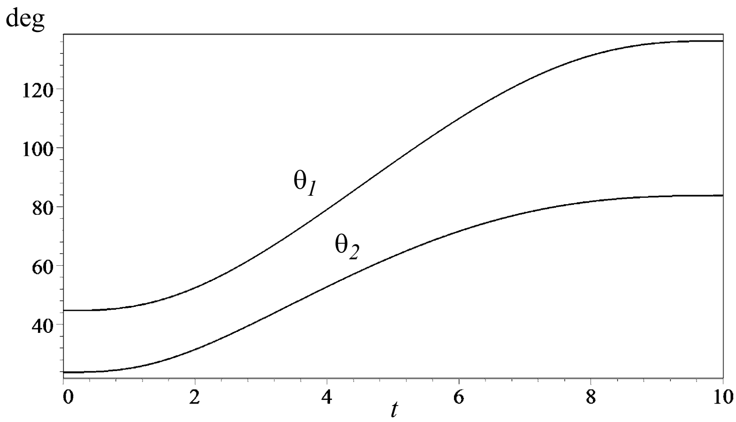

FIGURE 13.31.

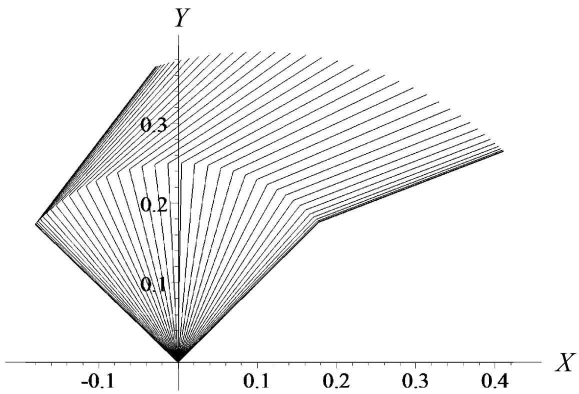

FIGURE 13.32. The configuration of the $2R$ manipulator at 42 equal time steps on a circular path.

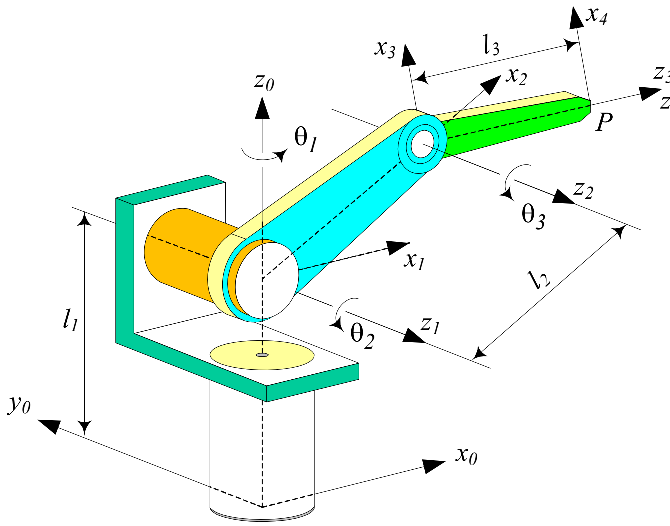

FIGURE 13.33. An articulated manipulator.

Let us connect P_1 and P_2 by a straight line and determine the time variation of Y and Z.

$$\begin{aligned} Y &= Y_{P_1} + \frac{Y_{P_2} - Y_{P_1}}{X_{P_2} - X_{P_1}}(X - X_{P_1}) \\ &= 0.010t^3 - 0.0015t^4 + 0.00006t^5 \end{aligned} \tag{13.263}$$

$$\begin{aligned} Z &= Z_{P_1} + \frac{Z_{P_2} - Z_{P_1}}{X_{P_2} - X_{P_1}}(X - X_{P_1}) \\ &= 1 + 0.005t^3 - 0.00075t^4 + 0.00003t^5 \end{aligned} \tag{13.264}$$

Using the inverse kinematic equations, we can determine the time history of joint variables of the manipulator as are shown in Figure 13.34.

$$\theta_3 = \arccos\left(\frac{l_1 - Z + l_2 \sin\theta_2}{l_3}\right) - \theta_2 \tag{13.265}$$

$$\theta_2 = 2\arctan\frac{-C_2 + \sqrt{C_2^2 - C_1 C_3}}{C_1} \tag{13.266}$$

$$\theta_1 = \begin{cases} \arctan\dfrac{Y}{X} & X \geq 0 \\ \arctan\dfrac{Y}{X} + \pi & X < 0 \end{cases} \tag{13.267}$$

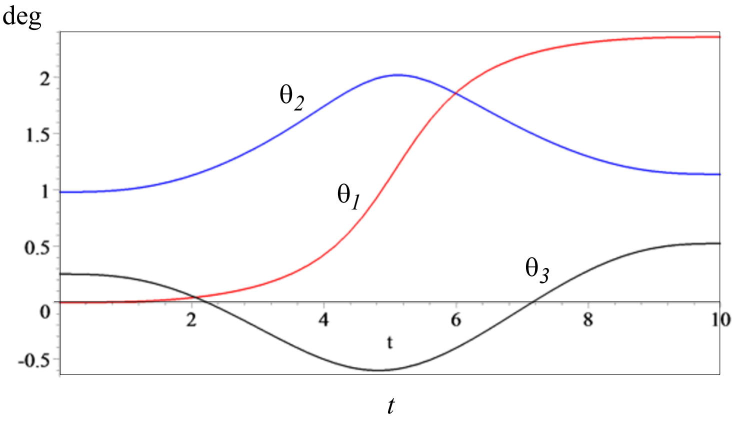

FIGURE 13.34. The time history of joint variables of an articulated manipulator.

$$C_1 = l_1^2 - 2l_1 Z + l_2^2 + \frac{2l_2 X}{\cos\theta_1} - l_3^2 + \frac{X^2}{\cos^2\theta_1} + Z^2 \quad (13.268)$$

$$C_2 = 2l_1 l_2 - 2l_2 Z \quad (13.269)$$

$$C_3 = l_1^2 - 2l_1 Z + l_2^2 - \frac{2l_2 X}{\cos\theta_1} - l_3^2 + \frac{X^2}{\cos^2\theta_1} + Z^2 \quad (13.270)$$

13.8 Summary

A serial robot may be assumed as a variable geometrical chain of links that relates the configuration of its end-effector to the Cartesian coordinate frame in which the base frame is attached. Forward kinematics are mathematical-geometrical relations that provide the end-effector configuration by having the joint coordinates. On the other hand, the inverse kinematics are mathematical-geometrical relations that provide joint coordinates for a given end-effector configuration.

The Cartesian path of motion for the end-effector must be expressed as a function of time to find the links' velocity and acceleration. The first applied path function that can provide a rest-to-rest motion is a cubic path for a variable $q_i(t)$ between two given points $q_i(t_0)$ and $q_i(t_f)$

$$q_i(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \tag{13.271}$$

By increasing the requirements, such as zero acceleration or jerk at some points on the path, we need to employ higher polynomials to satisfy the conditions. An n degree polynomial can satisfy $n+1$ conditions. It is also possible to split a multiple conditional path into some intervals with fewer conditions. The interval paths must then be connected to satisfy their boundary conditions.

A path of motion may also be defined based on different mathematical functions. Harmonic and cycloid functions are the most common paths. Non-polynomial equations introduce some advantages, due to simpler expression, and some disadvantages due to nonlinearity.

When a path of motion either in joint or Cartesian coordinates space is defined, forward and inverse kinematics must be utilized to find the path of motion in the other space.

Rotational maneuver of the end-effector about the wrist point needs a rotational path. A rotational path may mathematically be defined similar to a Cartesian path utilizing the Rodriguez formula and rotation matrices.

13.9 Key Symbols

a_c	constant acceleration
a_i, b_i, c_i	coefficient of path equation
B	body coordinate frame
C	constant of integral
G, B_0	global coordinate frame, Base coordinate frame
l	length
q	dependent variable coordinate, joint variable Cartesian variable
$\mathbf{r}$	position vectors, homogeneous position vector
r_{ij}	the element of row i and column j of a matrix
R	rotation transformation matrix
t	dependent variable, time
t_0	initial time
t_f	final time
$\hat{u}$	axis of rotation
x, y, z	local coordinate axes
X, Y, Z	global coordinate axes
Greek	
$\boldsymbol{\delta}$	difference of position vectors
θ	rotary joint angle, joint variable
ϕ	angle of rotation
Symbol	
$[\quad]^{-1}$	inverse of the matrix $[\quad]$
$[\quad]^T$	transpose of the matrix $[\quad]$
$\equiv$	equivalent
$\vdash$	orthogonal
(i)	link number i

Exercises

1. Notation and symbols.

   Describe their meaning.

   a- t_0 b- t_f c- $q_i(t)$ d- t'

2. Rest-to-rest cubic path.

   Find a cubic path for a joint coordinate to satisfy the following conditions:

   (a)
   $$q(0) = -10\,\text{deg},\ q(1) = 45\,\text{deg},\ \dot{q}(0) = \dot{q}(1) = 0$$

   (b)
   $$q(0) = 0\,\text{deg},\ q(1) = 50\,\text{deg},\ \dot{q}(0) = \dot{q}(1) = 0$$

   (c)
   $$q(0) = 10\,\text{deg},\ q(1) = 60\,\text{deg},\ \dot{q}(0) = \dot{q}(10) = 0$$

3. To-rest path.

   Find a quadratic path to satisfy the following conditions:

   $$q(0) = -10\,\text{deg},\ q(1) = 45\,\text{deg},\ \dot{q}(1) = 0.$$

   Calculate the initial velocity of the path using the quadratic path. Then, find a cubic path to satisfy the same boundary conditions as the quadratic path. Compare the maximum accelerations of the two paths.

4. Constant velocity path.

   Calculate a path to satisfy the following conditions:

   $$q(0) = -10\,\text{deg},\ q(10) = 45\,\text{deg},\ \dot{q}(0) = \dot{q}(10) = 0$$

   and move with constant velocity $\dot{q} = 25\,\text{deg}\,/\,\text{sec}$ between $12\,\text{deg}$ and $35\,\text{deg}$.

5. Constant acceleration path.

   Calculate a path with constant acceleration $\ddot{q} = 25\,\text{deg}\,/\,\text{sec}^2$ between $12\,\text{deg}$ and $35\,\text{deg}$, and satisfy the following conditions:

   $$q(0) = -10\,\text{deg},\ q(10) = 45\,\text{deg},\ \dot{q}(0) = \dot{q}(10) = 0.$$

6. Zero jerk path.

   Find a path to satisfy the following boundary conditions:

$$q(0) = 0, \; q(1) = 66 \deg, \; \dot{q}(0) = \dot{q}(1) = 0$$

   and have zero jerk at the beginning, middle, and end points.

7. Control points.

   Find a path to satisfy the conditions

$$q(0) = 10 \deg, \; q(1) = 95 \deg, \; \dot{q}(0) = \dot{q}(1) = 0$$

   and pass through the following control points:

$$q(0.25) = 30 \deg, \; q(0.5) = 65 \deg$$

8. A jerk zero at start-middle-stop path.

   To make a path have jerk as close to zero as possible, an eight degree polynomial

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_7 t^8$$

   and nine boundary conditions can be employed. Find the path.

$$\begin{array}{llll} q(0) = 0 & \dot{q}(0) = 0 & \ddot{q}(0) = 0 & \dddot{q}(0) = 0 \\ & & & \dddot{q}(0.5) = 0 \\ q(1) = 120 \deg & \dot{q}(1) = 0 & \ddot{q}(1) = 0 & \dddot{q}(1) = 0 \end{array}$$

9. Point sequence path.

   The conditions for a sequence of points are given here. Find a path to satisfy the conditions given below.

   (a)

$$\begin{array}{lll} q(0) = 5 \deg & \dot{q}(0) = 0 & \ddot{q}(0) = 0 \\ q(0.4) = 35 \deg & & \\ q(0.75) = 65 \deg & & \\ q(1) = 100 \deg & \dot{q}(1) = 0 & \ddot{q}(1) = 0. \end{array}$$

   (b)

$$\begin{array}{lll} q(0) = 5 \deg & \dot{q}(0) = 0 & \ddot{q}(0) = 0 \\ q(2) = 15 \deg & & \\ q(4) = 35 \deg & & \\ q(7.5) = 65 \deg & & \\ q(10) = 100 \deg & \dot{q}(10) = 0 & \ddot{q}(10) = 0. \end{array}$$

10. Splitting a path into a series of segments.

    Using the splitting method, find a path for the following conditions:

$$\begin{array}{lll} q(0) = 5\,\text{deg} & \dot{q}(0) = 0 & \ddot{q}(0) = 0 \\ q(2) = 15\,\text{deg} & & \\ q(4) = 35\,\text{deg} & & \\ q(7.5) = 65\,\text{deg} & & \\ q(10) = 100\,\text{deg} & \dot{q}(10) = 0 & \ddot{q}(10) = 0. \end{array}$$

    by breaking the entire path into four segments.

$$\begin{array}{lccccc} q_1(t) & \text{for} & q(0) < q_1(t) < q(2) & \text{and} & 0 < t < 2 \\ q_2(t) & \text{for} & q(2) < q_2(t) < q(4) & \text{and} & 2 < t < 4 \\ q_3(t) & \text{for} & q(4) < q_3(t) < q(7.5) & \text{and} & 4 < t < 7.5 \\ q_4(t) & \text{for} & q(7.5) < q_4(t) < q(10) & \text{and} & 7.5 < t < 10 \end{array}$$

11. ★ Extra conditions.

    To have a smooth overall path in the splitting method, we may add extra conditions to match the segments. Solve Exercise 10 having a zero jerk transition between segments.

12. ★ Least-squared path.

    Using the least squared method, find the best polynomial path of degree n to approximate a path given by the following points.

$$\begin{array}{l} q(0) = 5\,\text{deg} \\ q(1) = 7\,\text{deg} \\ q(2) = 15\,\text{deg} \\ q(3) = 21\,\text{deg} \\ q(4) = 35\,\text{deg} \\ q(7.5) = 65\,\text{deg} \\ q(9) = 85\,\text{deg} \\ q(10) = 100\,\text{deg} \end{array}$$

    (a) $n = 2$.

    (b) $n = 3$.

    (c) $n = 4$.

    (d) $n = 5$.

13. ★ Least-squared path and boundary conditions.

    Using the least squared method, find the best polynomial path of degree n to approximate a path given by the following points and conditions.

$$\begin{array}{lll} q(0) = 5\,\text{deg} & \dot{q}(0) = 0 & \ddot{q}(0) = 0 \\ q(2) = 15\,\text{deg} & & \\ q(4) = 35\,\text{deg} & & \\ q(7.5) = 65\,\text{deg} & & \\ q(10) = 100\,\text{deg} & \dot{q}(10) = 0 & \ddot{q}(10) = 0. \end{array}$$

(a) $n = 2$.

(b) $n = 3$.

(c) $n = 4$.

(d) $n = 5$.

14. $2R$ manipulator motion to follow a joint path.

    Find the path of the endpoint of a $2R$ manipulator, with $l_1 = l_2 = 1\,\text{m}$, if the joint variables follow the given paths:

$$\begin{aligned} \theta_1(t) &= 10 + 156.25t^3 \\ \theta_2(t) &= -41.56 + 222.78t - 172.2t^2 \end{aligned}$$

15. $3R$ planar manipulator motion to follow a joint path.

    Find the path of the endpoint of a $3R$ manipulator, with

$$\begin{aligned} l_1 &= 1\,\text{m} \\ l_2 &= 0.65\,\text{m} \\ l_3 &= 0.35\,\text{m} \end{aligned}$$

    if the joint variables follow these given paths:

$$\begin{aligned} \theta_1(t) &= -41.56 + 222.78t - 172.2t^2 \\ \theta_2(t) &= 148.99 - 321.67t + 216.67t^2 \\ \theta_3(t) &= 198545 - 827166.6672t \\ &\quad +1290500t^2 - 893500t^3 + 231666.67t^4 \end{aligned}$$

    Calculate the maximum acceleration and jerk of the endpoint.

16. R$\vdash$R$\|$R articulated arm motion.

    Find the Cartesian trajectory of the endpoint of an articulated manipulator, shown in Figure 5.22, if the geometric parameters are

$$\begin{aligned} d_1 &= 1\,\text{m} \\ d_2 &= 0 \\ l_2 &= 1\,\text{m} \\ l_3 &= 1\,\text{m} \end{aligned}$$

and the joints' paths are:

$$\begin{aligned}
\theta_1(t) &= -41.56 + 222.78t - 172.2t^2 \\
\theta_2(t) &= 148.99 - 321.67t + 216.67t^2 \\
\theta_3(t) &= 198545 - 827166.6672t \\
&\quad +1290500t^2 - 893500t^3 + 231666.67t^4
\end{aligned}$$

17. Cartesian paths.

    Connect the following points with a straight line. Determine the Cartesian coordinates as functions of time for rest-to-rest paths in $t = 1\,\mathrm{s}$.

    (a)
    $$P_1 = (1.5, 1.5) \qquad P_2 = (-0.5, 1.5)$$

    (b)
    $$P_1 = (0, 0) \qquad P_2 = (1, 1.5)$$

    (c)
    $$P_1 = (-1.5, 1) \qquad P_2 = (0.5, 1.5)$$

    (d)
    $$P_1 = (-1.5, 1, 0) \qquad P_2 = (0.5, 1.5, 1)$$

    (e)
    $$P_1 = (-1, 0, -1) \qquad P_2 = (-0.5, 1.5.1)$$

18. Cartesian path for a $2R$ manipulator.

    Consider a $2R$ planar manipulator.

    (a) Calculate a cubic rest-to-rest path in Cartesian space to join the following points with a straight line.
    $$P_1 = (1.5, 1) \qquad P_2 = (-0.5, 1.5)$$

    (b) Calculate and plot the joint coordinates of the manipulator, with $l_1 = l_2 = 1\,\mathrm{m}$, that follows the Cartesian path.

    (c) Calculate the maximum angular acceleration of the joint variables.

19. Cartesian path for a $3R$ manipulator.

    Consider a $3R$ articulated manipulator with $l_1 = l_2 = l_3 = 1\,\mathrm{m}$.

    (a) Calculate a cubic rest-to-rest path in Cartesian space to join the following points with a straight line.
    $$P_1 = (-1.5, 1, 0) \qquad P_2 = (0.5, 1.5, 1)$$

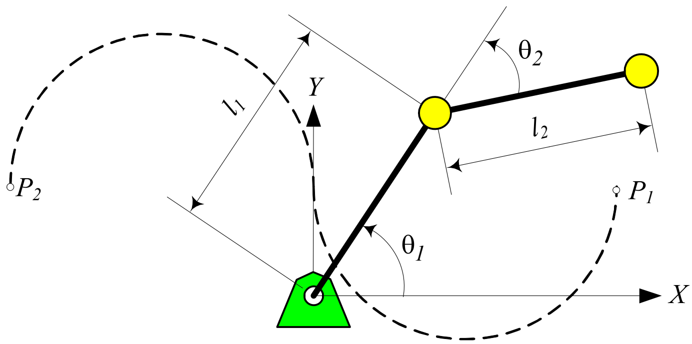

FIGURE 13.35. A $2R$ manipulator moves on a path made of two semi-circles.

(b) Calculate and plot the joint coordinates of the manipulator that follows the Cartesian path.

(c) Calculate the maximum angular velocity and acceleration of the joint variables.

20. ★ Joint path for a given Cartesian path.

Assume that the endpoint of a $2R$ manipulator moves with constant speed $v = 1\,\mathrm{m/\,sec}$ from P_1 to P_2, on a path made of two semi-circles, as shown in Figure 13.35. The center of the circles are at (0.75 m, 0.5 m) and (−0.75 m, 0.5 m).

(a) Calculate and plot the joints' path if $l_1 = l_2 = 1\,\mathrm{m}$.

(b) Calculate the value and positions of the maximum angular velocity in joint variables.

(c) Calculate the value and positions of the maximum angular acceleration in joint variables.

(d) Calculate the value and positions of the maximum angular jerk in joint variables.

21. ★ Obstacle avoidance and path planning.

Let us determine a path between $P_1 = (1.5, 1)$ and $P_2 = (-1, 1)$ to avoid the obstacle shown in Figure 13.36. The path may be made of two straight lines with a transition circular path in the middle. The radius of the circle is $r = 0.5\,\mathrm{m}$ and the center of the circle is at the lower point of the obstacle. The lines connect to the circle smoothly.

The endpoint of the $2R$ manipulator, with $l_1 = l_2 = 1\,\mathrm{m}$, starts at rest from P_1 and moves along the first line with constant acceleration. The endpoint keeps its speed constant $v = 1\,\mathrm{m/\,sec}$ on the circular path and then moves with constant acceleration on the final line segment to stop at P_2.

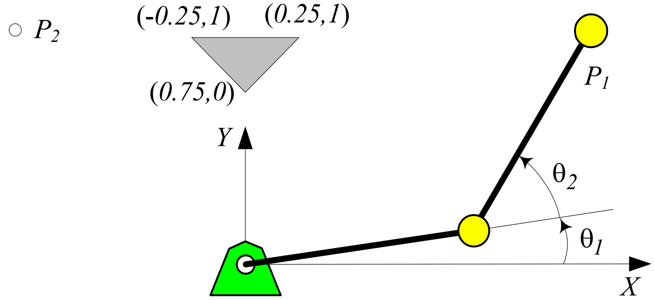

FIGURE 13.36. An obstacle in the Cartesian space of motion for a 2R manipulator.

(a) Calculate and plot the joints' paths.

(b) Find the value and position of the maximum angular velocity for both joints' variable.

(c) Find the value and position of the maximum angular acceleration for both joints' variable.

(d) Find the value and position of the maximum angular jerk for both joints' variable.

22. ★ Joint path for a given Cartesian path.

(a) Connect the points $P_1 = (1.1, 0.8, 0.5)$ and $P_2 = (-1, 1, 0.35)$ with a straight line.

(b) Find a rest-to-rest cubic path and plot the Cartesian coordinates X, Y, and Z as functions of time.

(c) Calculate the joints' path for an articulated manipulator, shown in Figure 5.22, if the geometric parameters are:

$$\begin{aligned} d_1 &= 1\,\text{m} \\ d_2 &= 0 \\ l_2 &= 1\,\text{m} \\ l_3 &= 1\,\text{m} \end{aligned}$$

(d) Find the value and position of the maximum angular velocity, acceleration, and jerk for the joints' variable.

23. ★ Transition parabola.

In Exercise 21, connect the points $P_1 = (1.5, 1)$ and $P_2 = (-1, 1)$ with two straight lines, using $P_0 = (0, 0.6)$ as a corner. Design a parabolic transition path to avoid the corner if the total time of motion is 12 sec and

(a) the interval time is $t' = 1\,\text{sec}$.

(b) the interval time is $t' = 2\,\text{sec}$.

(c) the interval time is $t' = 5\,\text{sec}$.

(d) the interval time is $t' = 8\,\text{sec}$.

(e) the interval time is $t' = 10\,\text{sec}$.

24. ★ Rotational path.

Consider a body frame B that turns 90 deg about Z-axis. Determine the rotation transformation matrix ${}^{G}R_{B}(t)$ such that

(a) the rotation takes place in $t = 1\,\text{s}$ and the angular velocity is constant.

(b) the rotation takes place in $t = 1\,\text{s}$ and the rotation is rest-to-rest.

25. ★ Combined rotational path.

Consider a body frame B that turns 90 deg about Z-axis and 60 deg about X-axis.

(a) Determine the rotation transformation matrix ${}^{G}R_{B}(t)$ such that the body first turns about Z-axis in $t_1 = 1\,\text{s}$ rest-to-rest, and then turns about X-axis in $t_2 = 1\,\text{s}$ rest-to-rest.

(b) Multiply the rotation matrices of $R_Z(t)$ and $R_X(t)$. Now ${}^{G}R_{B}(t)$ has only one time variable. Where would B be after $t = 1\,\text{s}$?

(c) Multiply the rotation matrices of R_Z and R_X and determine ${}^{G}R_{B}$. Determine the angle and axis of rotation of ${}^{G}R_{B}$. Define a rest-to-rest path for the angle of rotation to move B from initial to final orientation in $t = 1\,\text{s}$.

26. ★ Euler angles rotational path.

Assume that the spherical wrist of a 6 DOF robot starts from rest position and turns about the axes of the final coordinate frame B_6 in order z-x-z for $\varphi = 15\,\text{deg}$, $\theta = 38\,\text{deg}$, and $\psi = 77\,\text{deg}$. The frame B_6 is installed at the wrist point.

(a) Design a rest-to-rest cubic rotational path for the angles φ, θ, and ψ, if each rotation takes 1 sec.

(b) Find the axis and angle of rotation, $(\hat{u}, \phi)$, that moves the wrist from the initial to the final orientation.

(c) Design a cubic rotational path for the axis-angle rotation if it takes 3 sec.

(d) Calculate the Euler angles path $\varphi(t)$, $\theta(t)$, and $\psi(t)$ for this motion.

(e) Calculate and compare the maximum angular velocity, acceleration, and jerk for φ, θ, and ψ in the first and second motions in part a and c.

(f) Calculate the maximum angular velocity, acceleration, and jerk of ϕ in the second motion in part c.