12

Robot Dynamics

We find the dynamics equations of motion of robots by two methods: *Newton-Euler* and *Lagrange*. The Newton-Euler method is more fundamental and finds the dynamic equations to determine the required actuators' force and torque to move the robot, as well as the joint forces. Lagrange method provides only the required differential equations that determines the actuators' force and torque.

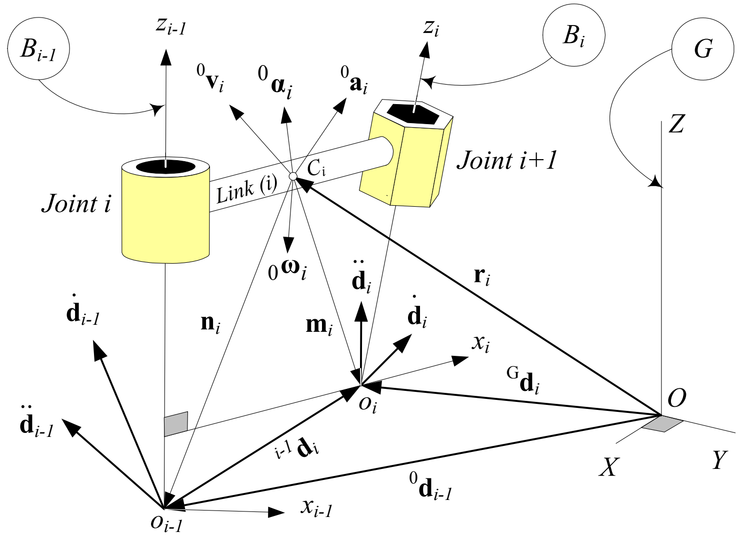

FIGURE 12.1. A link (i) and its vectorial kinematic characteristics.

12.1 Rigid Link Newton-Euler Dynamics

Figure 12.1 illustrates a link (i) of a manipulator and its velocity and acceleration vectorial characteristics. Figure 12.2 illustrates free body diagram of the link (i). The force $\mathbf{F}_{i-1}$ and moment $\mathbf{M}_{i-1}$ are the resultant force and moment that link $(i-1)$ applies to link (i) at joint i. Similarly, $\mathbf{F}_i$ and $\mathbf{M}_i$ are the resultant force and moment that link (i) applies to link $(i+1)$ at joint $i+1$. We measure and show the force systems $(\mathbf{F}_{i-1}\ ,\ \mathbf{M}_{i-1})$ and $(\mathbf{F}_i\ ,\ \mathbf{M}_i)$ at the origin of the coordinate frames B_{i-1} and B_i respectively.

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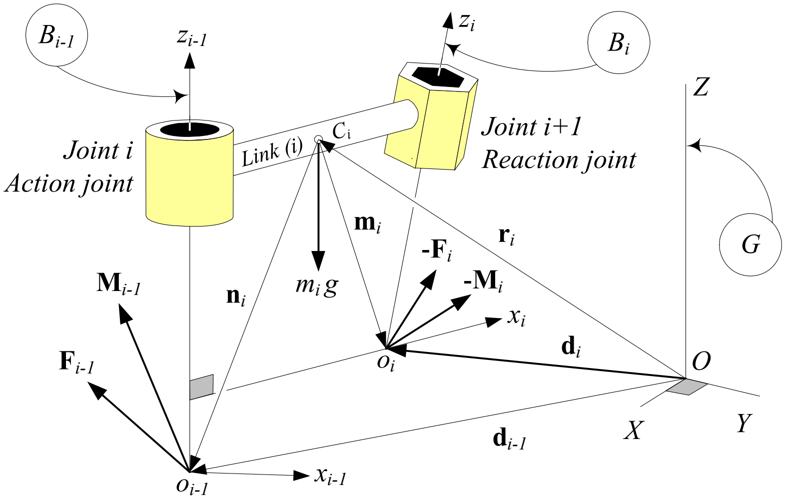

FIGURE 12.2. Force system on link (i).

The sum of the external loads acting on the link (i) are shown by $\sum \mathbf{F}_{e_i}$ and $\sum \mathbf{M}_{e_i}$.

The *Newton-Euler equations of motion* for the link (i) in the global coordinate frame are:

$$ {}^0\mathbf{F}_{i-1} - {}^0\mathbf{F}_i + \sum {}^0\mathbf{F}_{e_i} = m_i \, {}^0\mathbf{a}_i \tag{12.1}$$

$$\begin{aligned} &{}^0\mathbf{M}_{i-1} - {}^0\mathbf{M}_i + \sum {}^0\mathbf{M}_{e_i} \\ &\quad + \left({}^0\mathbf{d}_{i-1} - {}^0\mathbf{r}_i\right) \times {}^0\mathbf{F}_{i-1} - \left({}^0\mathbf{d}_i - {}^0\mathbf{r}_i\right) \times {}^0\mathbf{F}_i = {}^0I_i \, {}_0\boldsymbol{\alpha}_i \end{aligned} \tag{12.2}$$

Proof. The force system at the distal end of a link (i) is made of a force $\mathbf{F}_i$ and a moment $\mathbf{M}_i$ measured at the origin of B_i. The right subscript on $\mathbf{F}_i$ and $\mathbf{M}_i$ is a number indicating the number of coordinate frame B_i.

At joint $i+1$ there is always an *action force* $\mathbf{F}_i$, that link (i) applies to link $(i+1)$, and a *reaction force* $-\mathbf{F}_i$, that the link $(i+1)$ applies to the link (i). Therefore, on link (i) there is always an action force $\mathbf{F}_{i-1}$ coming from link $(i-1)$, and a reaction force $-\mathbf{F}_i$ coming from link $(i+1)$. Action force is called *driving force*, and reaction force is called *driven force*.

Similarly, at joint $i+1$ there is always an *action moment* $\mathbf{M}_i$, that link (i) applies to the link $(i+1)$, and a *reaction moment* $-\mathbf{M}_i$, that link $(i+1)$ applies to the link (i). Hence, on link (i) there is always an action moment $\mathbf{M}_{i-1}$ coming from link $(i-1)$, and a reaction moment $-\mathbf{M}_i$ coming from link $(i+1)$. The action moment is called the *driving moment*, and the reaction moment is called the *driven moment*.

Therefore, there is a driving force system ($\mathbf{F}_{i-1}$, $\mathbf{M}_{i-1}$) at the origin of the coordinate frame B_{i-1}, and a driven force system ($\mathbf{F}_i$, $\mathbf{M}_i$) at the origin of the coordinate frame B_i. The driving force system ($\mathbf{F}_{i-1}$, $\mathbf{M}_{i-1}$) gives motion to link (i) and the driven force system ($\mathbf{F}_i$, $\mathbf{M}_i$) gives motion to link $(i+1)$.

In addition to the action and reaction force systems, there might be some external forces acting on the link (i) that their resultant makes a force system ($\sum \mathbf{F}_{e_i}$, $\sum \mathbf{M}_{e_i}$) at the mass center C_i. In robotic application, weight is usually the only external load on middle links, and reactions from the environment are extra external force systems on the base and end-effector links. The force and moment that the base actuator applies to the first link are $\mathbf{F}_0$ and $\mathbf{M}_0$, and the force and moment that the end-effector applies to the environment are $\mathbf{F}_n$ and $\mathbf{M}_n$. If weight is the only external load on link (i) and it is in $-{}^0\hat{k}_0$ direction, then we have

$$\sum {}^0\mathbf{F}_{e_i} = m_i\,{}^0\mathbf{g} = -m_i\,g\,{}^0\hat{k}_0 \tag{12.3}$$

$$\sum {}^0\mathbf{M}_{e_i} = {}^0\mathbf{r}_i \times m_i\,{}^0\mathbf{g} = -\,{}^0\mathbf{r}_i \times m_i\,g\,{}^0\hat{k}_0 \tag{12.4}$$

where $\mathbf{g}$ is the gravitational acceleration vector.

As shown in Figure 12.2, we indicate the global position of the mass center of the link by ${}^0\mathbf{r}_i$, and the global position of the origin of body frames B_i and B_{i-1} by ${}^0\mathbf{d}_i$ and ${}^0\mathbf{d}_{i-1}$ respectively. The link's velocities ${}^0\mathbf{v}_i$, ${}_0\boldsymbol{\omega}_i$ and accelerations ${}^0\mathbf{a}_i$, ${}_0\boldsymbol{\alpha}_i$ are measured and shown at C_i. The physical properties of the link (i) are specified by its mass m_i and moment of inertia ${}^0I_i$ about the link's mass center C_i.

The Newton's equation of motion determines that the sum of forces applied to the link (i) is equal to the mass of the link times its acceleration at C_i.

$$ {}^0\mathbf{F}_{i-1} - {}^0\mathbf{F}_i + \sum {}^0\mathbf{F}_{e_i} = m_i\,{}^0\mathbf{a}_i \tag{12.5}$$

For the Euler equation, in addition to the action and reaction moments, we must add the moments of the action and reaction forces about C_i. The moment of $-\mathbf{F}_i$ and $\mathbf{F}_{i-1}$ are equal to $-\mathbf{m}_i \times \mathbf{F}_i$ and $\mathbf{n}_i \times \mathbf{F}_{i-1}$ where $\mathbf{m}_i$ is the position vector of o_i from C_i and $\mathbf{n}_i$ is the position vector of o_{i-1} from C_i. Therefore, the link's Euler equation of motion is

$$\begin{aligned} &{}^0\mathbf{M}_{i-1} - {}^0\mathbf{M}_i + \sum {}^0\mathbf{M}_{e_i} \\ &\qquad + {}^0\mathbf{n}_i \times {}^0\mathbf{F}_{i-1} - {}^0\mathbf{m}_i \times {}^0\mathbf{F}_i = {}^0I_i\,{}_0\boldsymbol{\alpha}_i \end{aligned} \tag{12.6}$$

however, $\mathbf{n}_i$ and $\mathbf{m}_i$ can be expressed by

$${}^0\mathbf{n}_i = {}^0\mathbf{d}_{i-1} - {}^0\mathbf{r}_i \tag{12.7}$$

$${}^0\mathbf{m}_i = {}^0\mathbf{d}_i - {}^0\mathbf{r}_i \tag{12.8}$$

$${}_{i-1}^{\;\;\,0}\mathbf{d}_i = {}^0\mathbf{m}_i - {}^0\mathbf{n}_i \tag{12.9}$$

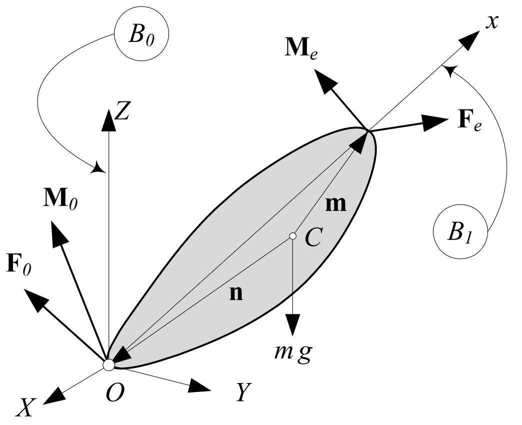

FIGURE 12.3. One link manipulator.

to derive Equation (12.2).

Since there is one translational and one rotational equation of motion for each link of a robot, there are $2n$ vectorial equations of motion for an n link robot. However, there are $2(n+1)$ forces and moments involved. Therefore, one set of force systems (usually $\mathbf{F}_n$ and $\mathbf{M}_n$) must be specified to solve the equations and find the joints' force and moment. ■

Example 323 *One-link manipulator.*

Figure 12.3 depicts a link attached to the ground via a spherical joint at O. The free body diagram of the link is made of an external force and moment at the endpoint, gravity, and the driving force and moment at the joint. The Newton-Euler equations for the link are:

$$\begin{aligned} {}^0\mathbf{F}_0 + {}^0\mathbf{F}_e + mg\,\hat{K} &= m\,{}^0\mathbf{a}_C && (12.10) \\ {}^0\mathbf{M}_0 + {}^0\mathbf{M}_e + {}^0\mathbf{n} \times {}^0\mathbf{F}_0 + {}^0\mathbf{m} \times {}^0\mathbf{F}_e &= {}^0I\,{}_0\boldsymbol{\alpha}_1 && (12.11) \end{aligned}$$

To see the application, let us consider the uniform beam of Figure 12.4(a). Figure 12.4(b) illustrates the FBD of the beam and its relative position vectors $\mathbf{m}$ *and* $\mathbf{n}$*.*

$$ {}^0\mathbf{m} = \begin{bmatrix} \frac{l}{2}\cos\theta \\ \frac{l}{2}\sin\theta \\ 0 \end{bmatrix} \qquad {}^0\mathbf{n} = \begin{bmatrix} -\frac{l}{2}\cos\theta \\ -\frac{l}{2}\sin\theta \\ 0 \end{bmatrix} \tag{12.12}$$

The kinematics of the beam are:

$$\begin{aligned} {}^0\mathbf{r}_1 &= -\,{}^0\mathbf{n}_1 && (12.13) \\ {}^0\mathbf{d}_1 &= -\,{}^0\mathbf{n}_1 + {}^0\mathbf{m}_1 && (12.14) \end{aligned}$$

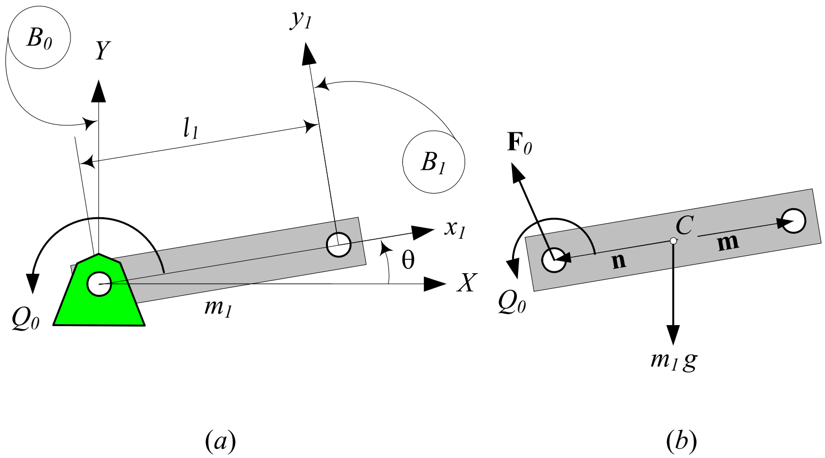

FIGURE 12.4. A turning uniform beam.

where, ${}^0\mathbf{r}_1$ *indicates the position of* C*, and* ${}^0\mathbf{d}_1$ *indicates the position of the tip point, both in* B_0*.*

$$
\begin{aligned}
{}_0\boldsymbol{\omega}_1 &= \dot{\theta}\,\hat{K} && (12.15)\\
{}_0\boldsymbol{\alpha}_1 &= {}_0\dot{\boldsymbol{\omega}}_1 = \ddot{\theta}\,\hat{K} && (12.16)\\
\mathbf{g} &= -g\,\hat{J} && (12.17)
\end{aligned}
$$

$$
\begin{aligned}
{}^0\mathbf{a}_C &= {}_0\boldsymbol{\alpha}_1 \times {}^0\mathbf{r}_1 - {}_0\boldsymbol{\omega}_1 \times \left({}_0\boldsymbol{\omega}_1 \times {}^0\mathbf{r}_1\right)\\
&= \begin{bmatrix} -\frac{l}{2}\ddot{\theta}\sin\theta + \frac{l}{2}\dot{\theta}^2(\cos\theta) \\ \frac{l}{2}\ddot{\theta}\cos\theta + \frac{l}{2}\dot{\theta}^2\sin\theta \\ 0 \end{bmatrix} \qquad (12.18)
\end{aligned}
$$

The forces on the beam are:

$$
{}^0\mathbf{F}_0 = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} \qquad {}^0\mathbf{F}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (12.19)
$$

$$
{}^0\mathbf{M}_0 = \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} \qquad {}^0\mathbf{M}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (12.20)
$$

Let us assume that ${}^1I_1$ *is the mass moment matrix of the beam about its mass center.*

$$
{}^1I_1 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad (12.21)
$$

$$
{}^0R_1 = R_{Z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.22}
$$

$$
\begin{aligned}
{}^0I_1 &= R_{Z,\theta}\,{}^1I_1\,R_{Z,\theta}^T = {}^0R_1 \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} {}^0R_1^T \\
&= \begin{bmatrix} I_x\cos^2\theta + I_y\sin^2\theta & (I_x - I_y)\cos\theta\sin\theta & 0 \\ (I_x - I_y)\cos\theta\sin\theta & I_y\cos^2\theta + I_x\sin^2\theta & 0 \\ 0 & 0 & I_z \end{bmatrix}
\end{aligned} \tag{12.23}
$$

Substituting the above information in Equations (12.10) and (12.11) provides the following equations of motion.

$$
{}^0\mathbf{F}_0 + {}^0\mathbf{F}_e + m_1\mathbf{g} = m_1\,{}^0\mathbf{a}_C \tag{12.24}
$$

$$
\begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}m_1 l\left(\ddot\theta\sin\theta - \dot\theta^2\cos\theta\right) \\ \frac{1}{2}m_1 l\left(\ddot\theta\cos\theta + \dot\theta^2\sin\theta\right) + m_1 g \\ 0 \end{bmatrix} \tag{12.25}
$$

$$
{}^0\mathbf{M}_0 + {}^0\mathbf{M}_e + {}^0\mathbf{n}\times{}^0\mathbf{F}_0 + {}^0\mathbf{m}\times{}^0\mathbf{F}_e = I\,{}_0\boldsymbol{\alpha}_1 \tag{12.26}
$$

$$
\begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} = \begin{bmatrix} \frac{l}{2}F_Z\sin\theta \\ -\frac{l}{2}F_Z\cos\theta \\ I_z\ddot\theta + \frac{l}{2}F_Y\cos\theta - \frac{l}{2}F_X\sin\theta \end{bmatrix} \tag{12.27}
$$

Let us substitute the force components from (12.25) to determine the components of the driving moment ${}^0\mathbf{M}_0$*.*

$$
\begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \left(I_z + \frac{m_1 l^2}{4}\right)\ddot\theta + \frac{1}{2}m_1 g l\cos\theta \end{bmatrix} \tag{12.28}
$$

Example 324 *A four-bar linkage dynamics.*

Figure 12.5(a) illustrates a closed loop four-bar linkage along with the free body diagrams of the links, shown in Figure 12.5(b). The position of the mass centers are given, and therefore the vectors ${}^0\mathbf{n}_i$ *and* ${}^0\mathbf{m}_i$ *for each link are also known. The Newton-Euler equations for the link* (i) *are*

$$
\begin{aligned}
{}^0\mathbf{F}_{i-1} - {}^0\mathbf{F}_i + m_i g\,\hat{J} &= m_i\,{}^0\mathbf{a}_i && (12.29) \\
{}^0\mathbf{M}_{i-1} - {}^0\mathbf{M}_i + {}^0\mathbf{n}_i\times{}^0\mathbf{F}_{i-1} - {}^0\mathbf{m}_i\times{}^0\mathbf{F}_i &= I_i\,{}_0\boldsymbol{\alpha}_i && (12.30)
\end{aligned}
$$

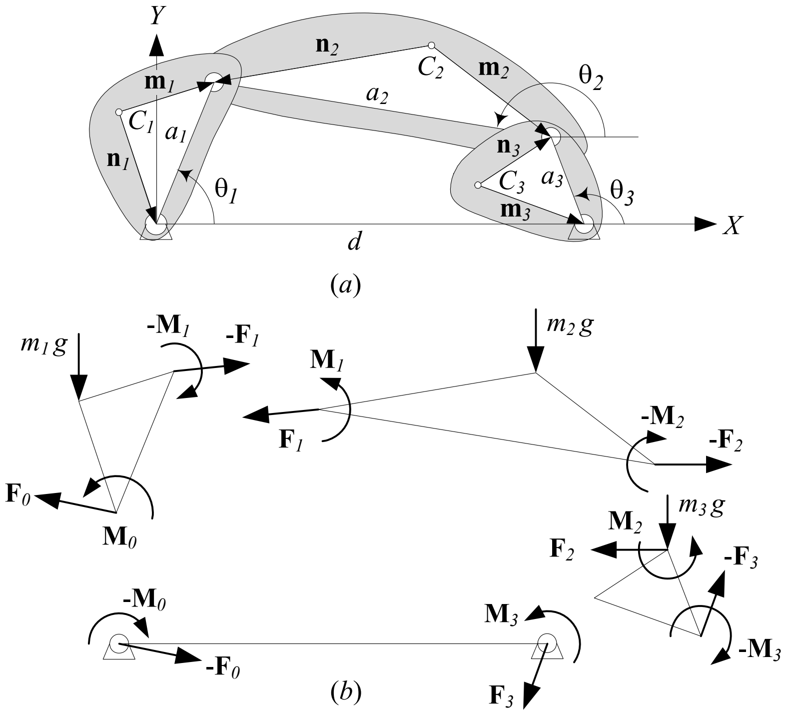

FIGURE 12.5. A four-bar linkage, and free body diagram of each link.

and therefore, we have three sets of equations.

$$
\begin{aligned}
{}^0\mathbf{F}_0 - {}^0\mathbf{F}_1 + m_1 g\,\hat{J} &= m_1\,{}^0\mathbf{a}_1 && (12.31)\\
{}^0\mathbf{M}_0 - {}^0\mathbf{M}_1 + {}^0\mathbf{n}_1 \times {}^0\mathbf{F}_0 - {}^0\mathbf{m}_1 \times {}^0\mathbf{F}_1 &= I_1\,{}_0\boldsymbol{\alpha}_1 && (12.32)
\end{aligned}
$$

$$
\begin{aligned}
{}^0\mathbf{F}_1 - {}^0\mathbf{F}_2 + m_2 g\,\hat{J} &= m_2\,{}^0\mathbf{a}_2 && (12.33)\\
{}^0\mathbf{M}_1 - {}^0\mathbf{M}_2 + {}^0\mathbf{n}_2 \times {}^0\mathbf{F}_1 - {}^0\mathbf{m}_2 \times {}^0\mathbf{F}_2 &= I_2\,{}_0\boldsymbol{\alpha}_2 && (12.34)
\end{aligned}
$$

$$
\begin{aligned}
{}^0\mathbf{F}_2 - {}^0\mathbf{F}_3 + m_3 g\,\hat{J} &= m_2\,{}^0\mathbf{a}_2 && (12.35)\\
{}^0\mathbf{M}_2 - {}^0\mathbf{M}_3 + {}^0\mathbf{n}_3 \times {}^0\mathbf{F}_2 - {}^0\mathbf{m}_3 \times {}^0\mathbf{F}_3 &= I_3\,{}_0\boldsymbol{\alpha}_3 && (12.36)
\end{aligned}
$$

Let us assume that there is no friction in joints and the mechanism is planar. Therefore, the force vectors are in the XY plane, and the moments are parallel to Z-axis. So, the equations of motion simplify to

$$
\begin{aligned}
{}^0\mathbf{F}_0 - {}^0\mathbf{F}_1 + m_1 g\,\hat{J} &= m_1\,{}^0\mathbf{a}_1 && (12.37)\\
{}^0\mathbf{M}_0 + {}^0\mathbf{n}_1 \times {}^0\mathbf{F}_0 - {}^0\mathbf{m}_1 \times {}^0\mathbf{F}_1 &= I_1\,{}_0\boldsymbol{\alpha}_1 && (12.38)
\end{aligned}
$$

$$\begin{aligned} {}^0\mathbf{F}_1 - {}^0\mathbf{F}_2 + m_2 g\,\hat{J} &= m_2\,{}^0\mathbf{a}_2 && (12.39)\\ {}^0\mathbf{n}_2 \times {}^0\mathbf{F}_1 - {}^0\mathbf{m}_2 \times {}^0\mathbf{F}_2 &= I_2\,{}_0\boldsymbol{\alpha}_2 && (12.40) \end{aligned}$$

$$\begin{aligned} {}^0\mathbf{F}_2 - {}^0\mathbf{F}_3 + m_3 g\,\hat{J} &= m_2\,{}^0\mathbf{a}_2 && (12.41)\\ {}^0\mathbf{n}_3 \times {}^0\mathbf{F}_2 - {}^0\mathbf{m}_3 \times {}^0\mathbf{F}_3 &= I_3\,{}_0\boldsymbol{\alpha}_3 && (12.42) \end{aligned}$$

where, ${}^0\mathbf{M}_0$ *is the driving torque of the mechanism. The number of equations reduces to* 9 *and the unknowns of the mechanism are:*

$$F_{0x}, F_{0y}, F_{1x}, F_{1y}, F_{2x}, F_{2y}, F_{3x}, F_{3y}, M_0 \tag{12.43}$$

We can rearrange the set of equations in a matrix form

$$[A]\,\mathbf{x} = \mathbf{b} \tag{12.44}$$

where,

$$[A] = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ -n_{1y} & n_{1x} & m_{1y} & -m_{1x} & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -n_{2y} & n_{2x} & m_{2y} & -m_{2x} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -n_{3y} & n_{3x} & m_{3y} & -m_{3x} & 0 \end{bmatrix} \tag{12.45}$$

$$\mathbf{x} = \begin{bmatrix} F_{0x}\\ F_{0y}\\ F_{1x}\\ F_{1y}\\ F_{2x}\\ F_{2y}\\ F_{3x}\\ F_{3y}\\ M_0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} m_1 a_{1x}\\ m_1 a_{1y} - m_1 g\\ I_1 \alpha_1\\ m_2 a_{2x}\\ m_2 a_{2y} - m_2 g\\ I_2 \alpha_2\\ m_3 a_{3x}\\ m_3 a_{3y} - m_3 g\\ I_3 \alpha_3 \end{bmatrix} \tag{12.46}$$

The matrix $[A]$ *describes the geometry of the mechanism, the vector* $\mathbf{x}$ *is the unknown forces, and the vector* $\mathbf{b}$ *indicates the dynamic terms. To solve the dynamics of the four-bar mechanism, we must calculate the accelerations* ${}^0\mathbf{a}_i$ *and* ${}_0\boldsymbol{\alpha}_i$ *and then find the required driving moment* ${}^0\mathbf{M}_0$ *and the joints' forces.*

The force

$$\mathbf{F}_s = \mathbf{F}_3 - \mathbf{F}_0 \tag{12.47}$$

is called the ***shaking force*** *and shows the reaction of the mechanism on the ground.*

Example 325 *A turning uniform beam with a tip mass.*

Let us consider the uniform beam of Figure 12.6(a) with a hanging mass m_2 at the tip point. Figure 12.6(b) illustrates the FBD of the beam. The mass center of the beam is at ${}^1\mathbf{r}_1$

$$\begin{aligned} {}^1\mathbf{r}_1 &= \frac{m_1}{m_1+m_2}\begin{bmatrix} l/2 \\ 0 \\ 0 \end{bmatrix} + \frac{m_2}{m_1+m_2}\begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{m_1+2m_2}{2\left(m_1+m_2\right)}l \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_x \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{12.48}$$

and its relative position vectors **m** *and* **n** *are:*

$$ {}^1\mathbf{n}_1 = -\,{}^1\mathbf{r}_1 = -r_x\hat{\imath} \tag{12.49}$$

$$ {}^1\mathbf{m}_1 = l\hat{\imath} - {}^1\mathbf{r}_1 = \left(l - r_x\right)\hat{\imath} \tag{12.50}$$

$$ {}^0\mathbf{d}_1 = -\,{}^1\mathbf{n}_1 + {}^1\mathbf{m}_1 = l\hat{\imath} \tag{12.51}$$

$$ {}^0\mathbf{m} = \begin{bmatrix} \left(l-r_x\right)\cos\theta \\ \left(l-r_x\right)\sin\theta \\ 0 \end{bmatrix} \qquad {}^0\mathbf{n} = \begin{bmatrix} -r_x\cos\theta \\ -r_x\sin\theta \\ 0 \end{bmatrix} \tag{12.52}$$

The kinematics of the beam are:

$$ {}_0\boldsymbol{\omega}_1 = \dot{\theta}\,\hat{K} \tag{12.53}$$

$$ {}_0\boldsymbol{\alpha}_1 = {}_0\dot{\boldsymbol{\omega}}_1 = \ddot{\theta}\,\hat{K} \tag{12.54}$$

$$ \mathbf{g} = -g\,\hat{J} \tag{12.55}$$

$$\begin{aligned} {}^0\mathbf{a}_C &= {}_0\boldsymbol{\alpha}_1 \times {}^0\mathbf{r}_1 + {}_0\boldsymbol{\omega}_1 \times \left({}_0\boldsymbol{\omega}_1 \times {}^0\mathbf{r}_1\right) \\ &= \begin{bmatrix} -r_x\ddot{\theta}\sin\theta + r_x\dot{\theta}^2\left(\cos\theta\right) \\ r_x\ddot{\theta}\cos\theta + r_x\dot{\theta}^2\sin\theta \\ 0 \end{bmatrix} \end{aligned} \tag{12.56}$$

The forces on the beam are:

$$ {}^0\mathbf{F}_0 = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} \qquad {}^0\mathbf{F}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.57}$$

$$ {}^0\mathbf{M}_0 = \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} \qquad {}^0\mathbf{M}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.58}$$

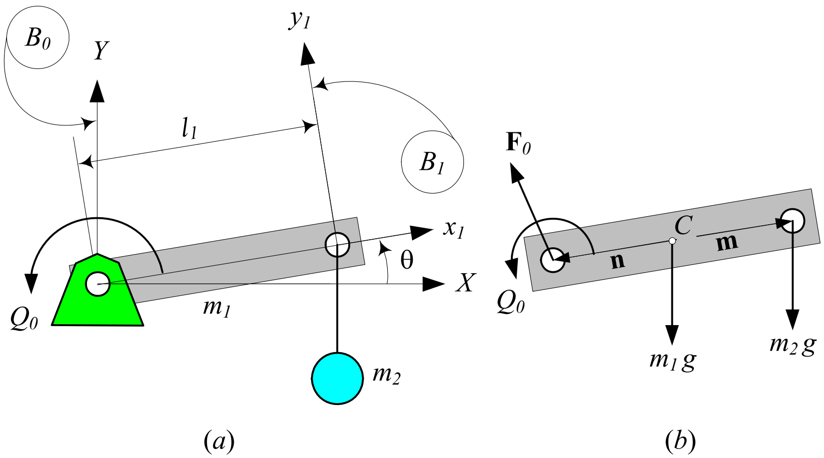

FIGURE 12.6. A uniform beam with a hanging weight m_2 at the tip point.

Let us assume that ${}^1I_1$ *is the mass moment matrix of the beam about its center,*

$$ {}^1I_1 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{12.59} $$

then the mass moment matrix of the manipulator about the common mass center at ${}^1\mathbf{r}_1$ *is:*

$$ {}^1I_1 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{12.60} $$

$$ I_3 = I_z + M_1 \left(r_x - \frac{l}{2} \right)^2 + M_2 \left(l - r_x \right)^2 \tag{12.61} $$

Knowing the transformation matrix ${}^0R_1$*, we can determine* ${}^0I_1$*.*

$$ {}^0R_1 = R_{Z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.62} $$

$$ \begin{aligned} {}^0I_1 &= R_{Z,\theta}\, {}^1I_1\, R^T_{Z,\theta} = {}^0R_1 \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} {}^0R_1^T \\ &= \begin{bmatrix} I_x \cos^2\theta + I_y \sin^2\theta & (I_x - I_y)\cos\theta\sin\theta & 0 \\ (I_x - I_y)\cos\theta\sin\theta & I_y \cos^2\theta + I_x \sin^2\theta & 0 \\ 0 & 0 & I_3 \end{bmatrix} \end{aligned} \tag{12.63} $$

Substituting the above information in Equations (12.10) and (12.11) provides the following equations of motion.

$$ {}^{0}\mathbf{F}_0 + {}^{0}\mathbf{F}_e + m_1 g \hat{K} = m_1 \, {}^{0}\mathbf{a}_C + m_2 \frac{1}{2} \, {}^{0}\mathbf{a}_C \tag{12.64} $$

$$ \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} = \begin{bmatrix} -\left(m_1+m_2\right) r_x \left(\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta\right) \\ \left(m_1+m_2\right) r_x \left(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta\right) + \left(m_2+m_1\right) g \\ 0 \end{bmatrix} \tag{12.65} $$

$$ {}^{0}\mathbf{M}_0 + {}^{0}\mathbf{M}_e + {}^{0}\mathbf{n} \times {}^{0}\mathbf{F}_0 + {}^{0}\mathbf{m} \times {}^{0}\mathbf{F}_e = I \, {}_0\boldsymbol{\alpha}_1 \tag{12.66} $$

$$ \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} = \begin{bmatrix} r_x F_Z \sin\theta \\ -r_x F_Z \cos\theta \\ I_3\ddot{\theta} + r_x F_Y \cos\theta - r_x F_X \sin\theta \end{bmatrix} \tag{12.67} $$

Let us substitute the force components from (12.65) to determine the components of the driving moment ${}^{0}\mathbf{M}_0$.

$$ \begin{bmatrix} Q_X \\ Q_Y \\ Q_Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \left(I_z + \left(m_1+m_2\right) r_x^2\right)\ddot{\theta} + \left(m_1+m_2\right) r_x g \cos\theta \end{bmatrix} \tag{12.68} $$

Substituting r_x *provides the required torque* Q_0.

$$ Q_0 = Q_Z = \left(\frac{1}{4} m_1 l^2 + m_2 l^2 + I_z\right)\ddot{\theta} + \left(\frac{1}{2} m_1 + m_2\right) g l \cos\theta \tag{12.69} $$

Example 326 *2R planar manipulator Newton-Euler dynamics.*

A 2R planar manipulator and its free body diagram are shown in Figure 12.7. The torques of actuators are parallel to the Z-axis and are indicated by Q_0 *and* Q_1. *The Newton-Euler equations of motion for the first link are:*

$$ \begin{aligned} {}^{0}\mathbf{F}_0 - {}^{0}\mathbf{F}_1 + m_1 g \hat{J} &= m_1 \, {}^{0}\mathbf{a}_1 && (12.70) \\ {}^{0}\mathbf{Q}_0 - {}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_1 \times {}^{0}\mathbf{F}_0 - {}^{0}\mathbf{m}_1 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_1 \, {}_0\boldsymbol{\alpha}_1 && (12.71) \end{aligned} $$

and the equations of motion for the second link are:

$$ \begin{aligned} {}^{0}\mathbf{F}_1 + m_2 g \hat{J} &= m_2 \, {}^{0}\mathbf{a}_2 && (12.72) \\ {}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_2 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_2 \, {}_0\boldsymbol{\alpha}_2 && (12.73) \end{aligned} $$

There are four equations for four unknowns $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{Q}_0$, *and* $\mathbf{Q}_1$. *These equations can be set in a matrix form*

$$ [A]\,\mathbf{x} = \mathbf{b} \tag{12.74} $$

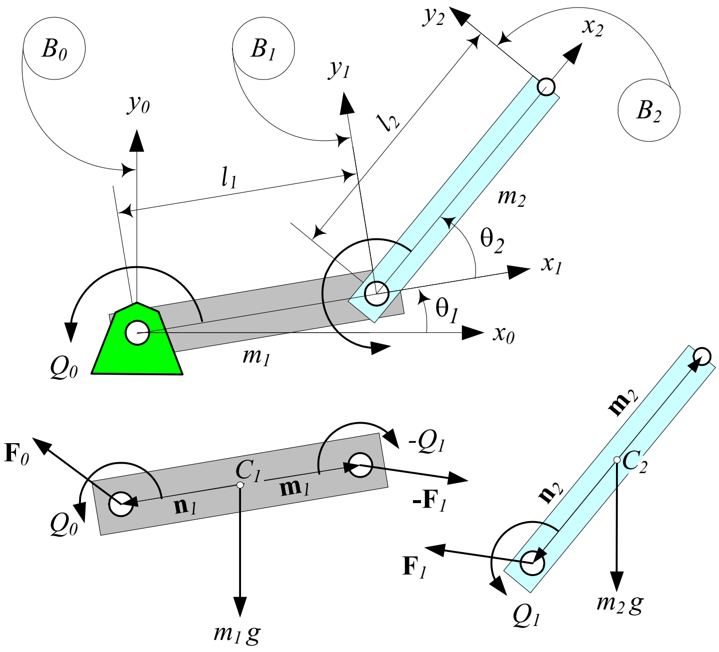

FIGURE 12.7. Free body diagram of a $2R$ palanar manipulator.

where,

$$[A] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ n_{1y} & -n_{1x} & 1 & -m_{1y} & m_{1x} & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & n_{2y} & -n_{2x} & 1 \end{bmatrix} \tag{12.75}$$

$$\mathbf{x} = \begin{bmatrix} F_{0x} \\ F_{0y} \\ Q_0 \\ F_{1x} \\ F_{1y} \\ Q_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} m_1 a_{1x} \\ m_1 a_{1y} - m_1 g \\ {}^0I_1 \alpha_1 \\ m_2 a_{2x} \\ m_2 a_{2y} - m_2 g \\ {}^0I_2 \alpha_2 \end{bmatrix}. \tag{12.76}$$

Example 327 *Equations for joint actuators.*

In robot dynamics, we do not need to find joint forces. Actuator torques are much more important as they are used to control a robot. In Example 326 we identified four equations for the joints' force system of the 2R *manipulator that is shown in Figure 12.7.*

$$
\begin{aligned}
{}^{0}\mathbf{F}_0 - {}^{0}\mathbf{F}_1 + m_1 g\,\hat{J} &= m_1\,{}^{0}\mathbf{a}_1 & (12.77)\\
{}^{0}\mathbf{Q}_0 - {}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_1 \times {}^{0}\mathbf{F}_0 - {}^{0}\mathbf{m}_1 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_1\,{}_{0}\boldsymbol{\alpha}_1 & (12.78)\\
{}^{0}\mathbf{F}_1 + m_2 g\,\hat{J} &= m_2\,{}^{0}\mathbf{a}_2 & (12.79)\\
{}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_2 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_2\,{}_{0}\boldsymbol{\alpha}_2. & (12.80)
\end{aligned}
$$

However, we may eliminate the joint forces $\mathbf{F}_0$, $\mathbf{F}_1$, *and reduce the number of equations to two for the two torques* $\mathbf{Q}_0$ *and* $\mathbf{Q}_1$*. Eliminating* $\mathbf{F}_1$ *between (12.79) and (12.80) provides*

$$
{}^{0}\mathbf{Q}_1 = {}^{0}I_2\,{}_{0}\boldsymbol{\alpha}_2 - {}^{0}\mathbf{n}_2 \times \left(m_2\,{}^{0}\mathbf{a}_2 - m_2 g\,\hat{J}\right) \tag{12.81}
$$

and eliminating $\mathbf{F}_0$ *and* $\mathbf{F}_1$ *between (12.77) and (12.80) gives:*

$$
\begin{aligned}
{}^{0}\mathbf{Q}_0 = {}&{}^{0}\mathbf{Q}_1 + {}^{0}I_1\,{}_{0}\boldsymbol{\alpha}_1 + {}^{0}\mathbf{m}_1 \times \left(m_2\,{}^{0}\mathbf{a}_2 - m_2 g\,\hat{J}\right)\\
&- {}^{0}\mathbf{n}_1 \times \left(m_1\,{}^{0}\mathbf{a}_1 - m_1 g\,\hat{J} + m_2\,{}^{0}\mathbf{a}_2 - m_2 g\,\hat{J}\right) \qquad (12.82)
\end{aligned}
$$

The forces $\mathbf{F}_0$ *and* $\mathbf{F}_1$*, if we are interested, are equal to:*

$$
\begin{aligned}
{}^{0}\mathbf{F}_1 &= m_2\,{}^{0}\mathbf{a}_2 - m_2 g\,\hat{J} & (12.83)\\
{}^{0}\mathbf{F}_0 &= m_1\,{}^{0}\mathbf{a}_1 + m_2\,{}^{0}\mathbf{a}_2 - (m_1 + m_2)\, g\,\hat{J} & (12.84)
\end{aligned}
$$

Example 328 $2R$ *planar manipulator with massive arms and joints.*

In a real situation for a $2R$ *planar manipulators, we generally have a massive motor at joint* 0 *to turn the link* (1) *and a massive motor at joint* 1 *to turn the link* (2)*. We may also carry a massive object by the gripper at the tip point. The motor at joint* 0 *is siting on the ground and its weight will not effect the dynamics of the manipulator. The FBD of the manipulator is similar to Figure 12.8.*

The massive joints will displace the position of C_i *and changes the relative position vectors* $\mathbf{m}$ *and* $\mathbf{n}$*. We will have the same equations of motion (12.70)-(12.72) provided we determine* $\mathbf{m}$ *and* $\mathbf{n}$ *for the new position of* C_i *as suggested in Figure 12.9.*

$$
\begin{aligned}
{}^{0}\mathbf{F}_0 - {}^{0}\mathbf{F}_1 + (m_{11} + m_{12})\, g\,\hat{J} &= m_1\,{}^{0}\mathbf{a}_1 & (12.85)\\
{}^{0}\mathbf{Q}_0 - {}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_1 \times {}^{0}\mathbf{F}_0 - {}^{0}\mathbf{m}_1 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_1\,{}_{0}\boldsymbol{\alpha}_1 & (12.86)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}\mathbf{F}_1 + (m_{21} + m_{22})\, g\,\hat{J} &= (m_{21} + m_{22})\,{}^{0}\mathbf{a}_2 & (12.87)\\
{}^{0}\mathbf{Q}_1 + {}^{0}\mathbf{n}_2 \times {}^{0}\mathbf{F}_1 &= {}^{0}I_2\,{}_{0}\boldsymbol{\alpha}_2. & (12.88)
\end{aligned}
$$

We may show the masses as

$$
\begin{aligned}
m_1 &= m_{11} + m_{12} & (12.89)\\
m_2 &= m_{21} + m_{22} & (12.90)
\end{aligned}
$$

and use the same equations (12.79)-(12.80) with asymmetric mass centers.

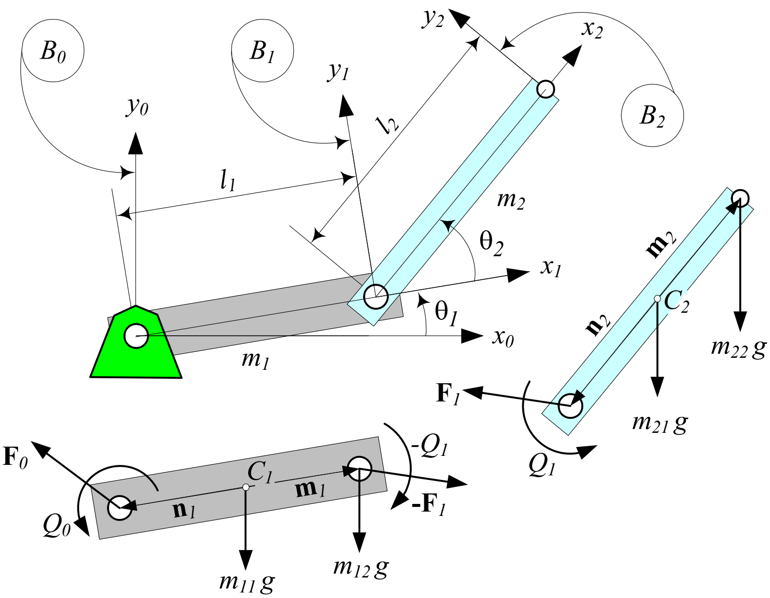

FIGURE 12.8. A $2R$ planar manipulator with massive arms and massive joints.

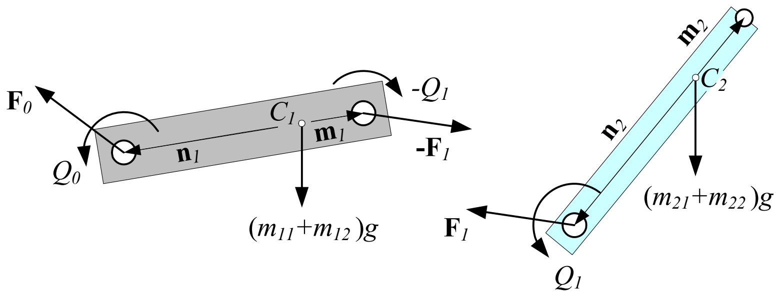

FIGURE 12.9. Determination of the vectors **m** and **n** for new positions of mass center C_i.

Example 329 *2R planar manipulator general equations.*

Let us analyze a general 2R manipulator that has massive arms and carries a payload m_0 as is shown in Figure 12.10.

The equations of motion are:

$$\begin{aligned}
{}^0\mathbf{F}_0 - {}^0\mathbf{F}_1 + m_1 g\,\hat{J} &= m_1\,{}^0\mathbf{a}_1 && (12.91)\\
{}^0\mathbf{Q}_0 - {}^0\mathbf{Q}_1 + {}^0\mathbf{n}_1 \times {}^0\mathbf{F}_0 - {}^0\mathbf{m}_1 \times {}^0\mathbf{F}_1 &= {}^0I_1\,{}_0\boldsymbol{\alpha}_1 && (12.92)\\
{}^0\mathbf{F}_1 + (m_0 + m_2)\,g\,\hat{J} &= m_2\,{}^0\mathbf{a}_2 && (12.93)\\
{}^0\mathbf{Q}_1 + {}^0\mathbf{n}_2 \times {}^0\mathbf{F}_1 + {}^0\mathbf{m}_2 \times m_0 g\,\hat{J} &= {}^0I_2\,{}_0\boldsymbol{\alpha}_2 && (12.94)
\end{aligned}$$

Elimination the joint forces $\mathbf{F}_0$, $\mathbf{F}_1$ provides the following equations for the torques $\mathbf{Q}_0$ and $\mathbf{Q}_1$.

$$ {}^0\mathbf{Q}_1 = {}^0I_2\,{}_0\boldsymbol{\alpha}_2 - {}^0\mathbf{n}_2 \times \left(m_2\,{}^0\mathbf{a}_2 - (m_0 + m_2)\,g\,\hat{J}\right) - {}^0\mathbf{m}_2 \times m_0 g\,\hat{J} \quad (12.95)$$

$$\begin{aligned}
{}^0\mathbf{Q}_0 &= {}^0\mathbf{Q}_1 + {}^0I_1\,{}_0\boldsymbol{\alpha}_1 + {}^0\mathbf{m}_1 \times \left(m_2\,{}^0\mathbf{a}_2 - (m_0 + m_2)\,g\,\hat{J}\right)\\
&\quad - {}^0\mathbf{n}_1 \times \left(m_1\,{}^0\mathbf{a}_1 + m_2\,{}^0\mathbf{a}_2 - (m_0 + m_1 + m_2)\,g\,\hat{J}\right) && (12.96)
\end{aligned}$$

The forces $\mathbf{F}_0$ and $\mathbf{F}_1$ are equal to:

$$\begin{aligned}
{}^0\mathbf{F}_1 &= m_2\,{}^0\mathbf{a}_2 - (m_0 + m_2)\,g\,\hat{J} && (12.97)\\
{}^0\mathbf{F}_0 &= m_1\,{}^0\mathbf{a}_1 + m_2\,{}^0\mathbf{a}_2 - (m_0 + m_1 + m_2)\,g\,\hat{J} && (12.98)
\end{aligned}$$

$$\begin{aligned}
{}^0\mathbf{r}_1 &= -\,{}^0\mathbf{n}_1 && (12.99)\\
{}^0\mathbf{r}_2 &= -\,{}^0\mathbf{n}_1 + {}^0\mathbf{m}_1 - {}^0\mathbf{n}_2 && (12.100)\\
{}^0\mathbf{d}_1 &= -\,{}^0\mathbf{n}_1 + {}^0\mathbf{m}_1 && (12.101)\\
{}_1^0\mathbf{d}_2 &= -\,{}^0\mathbf{n}_2 + {}^0\mathbf{m}_2 && (12.102)\\
{}^0\mathbf{d}_2 &= -\,{}^0\mathbf{n}_1 + {}^0\mathbf{m}_1 - {}^0\mathbf{n}_2 + {}^0\mathbf{m}_2 && (12.103)
\end{aligned}$$

In a general case, the local position vectors of C_i are:

$$\begin{aligned}
{}^0\mathbf{n}_1 &= {}^0R_1\,{}^1\mathbf{n}_1 = -R_{Z,\theta_1}\,c_1\,{}^1\hat{\imath}_1 = \begin{bmatrix} -c_1\cos\theta_1 \\ -c_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.104)\\
{}^0\mathbf{n}_2 &= -\,{}^0R_2\,{}^2\mathbf{n}_2 = -\,{}^0R_1\,{}^1R_2\,{}^2\mathbf{n}_2\\
&= -R_{Z,\theta_1}\,R_{Z,\theta_2}\,c_2\,{}^2\hat{\imath}_2 = \begin{bmatrix} -c_2\cos(\theta_1+\theta_2) \\ -c_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix} && (12.105)
\end{aligned}$$

$$ {}^1\mathbf{n}_2 = -\,{}^1R_2\,{}^2\mathbf{n}_2 = -R_{Z,\theta_2}\,c_2\,{}^2\hat{\imath}_2 = \begin{bmatrix} -c_2\cos\theta_2 \\ -c_2\sin\theta_2 \\ 0 \end{bmatrix} \quad (12.106)$$

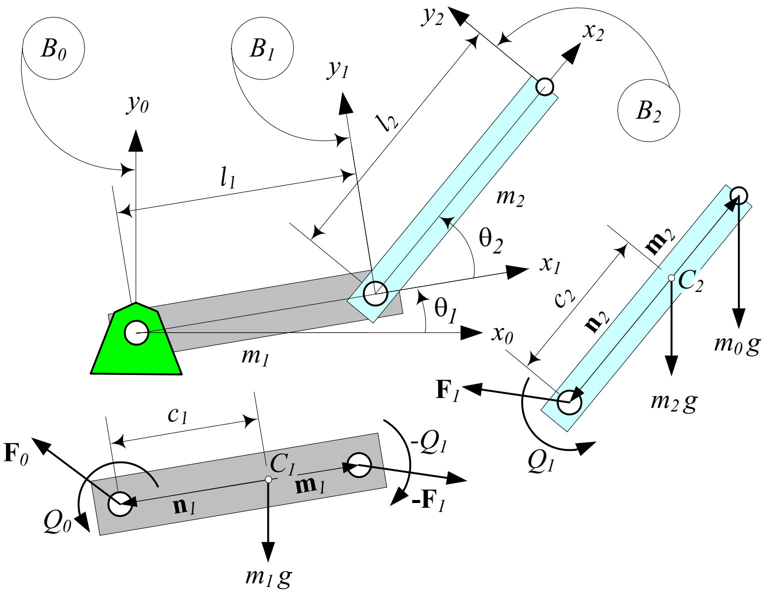

FIGURE 12.10. A $2R$ manipulator that has massive arms and carries a payload m_0.

$$ {}^0\mathbf{m}_1 = {}^0R_1\, {}^1\mathbf{m}_1 = R_{Z,\theta_1}\ (l_1 - c_1)\ {}^1\hat{\imath}_1 = \begin{bmatrix} (l_1 - c_1)\cos\theta_1 \\ (l_1 - c_1)\sin\theta_1 \\ 0 \end{bmatrix} \tag{12.107} $$

$$ \begin{aligned} {}^0\mathbf{m}_2 &= {}^0R_2\, {}^2\mathbf{m}_2 = {}^0R_2\ (l_2 - c_2)\ {}^2\hat{\imath}_2 \\ &= \begin{bmatrix} (l_2 - c_2)\cos(\theta_1 + \theta_2) \\ (l_2 - c_2)\sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \end{aligned} \tag{12.108} $$

where,

$$ {}^0R_1 = R_{Z,\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.109} $$

$$ {}^1R_2 = R_{Z,\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.110} $$

$$ {}^0R_2 = R_{z,\theta_1+\theta_2} = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12.111} $$

The position vectors are as follows.

$$
\begin{aligned}
{}^{0}\mathbf{r}_1 &= -\,{}^{0}\mathbf{n}_1 = {}^{0}R_1\,{}^{1}\mathbf{r}_1 = {}^{0}R_1\,c_1\hat{\imath}_1 = \begin{bmatrix} c_1\cos\theta_1 \\ c_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.112) \\
{}^{0}\mathbf{r}_2 &= -\,{}^{0}\mathbf{n}_1 + {}^{0}\mathbf{m}_1 - {}^{0}\mathbf{n}_2 = {}^{0}\mathbf{d}_1 + {}^{0}R_2\,{}^{2}\mathbf{r}_2 \\
&= \begin{bmatrix} l_1\cos\theta_1 + c_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + c_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix} && (12.113)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}\mathbf{d}_1 &= -\,{}^{0}\mathbf{n}_1 + {}^{0}\mathbf{m}_1 = \begin{bmatrix} l_1\cos\theta_1 \\ l_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.114) \\
{}_{1}^{0}\mathbf{d}_2 &= -\,{}^{0}\mathbf{n}_2 + {}^{0}\mathbf{m}_2 = \begin{bmatrix} l_2\cos(\theta_1+\theta_2) \\ l_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix} && (12.115) \\
{}^{0}\mathbf{d}_2 &= -\,{}^{0}\mathbf{n}_1 + {}^{0}\mathbf{m}_1 - {}^{0}\mathbf{n}_2 + {}^{0}\mathbf{m}_2 \\
&= {}^{0}\mathbf{d}_1 + {}_{1}^{0}\mathbf{d}_2 = \begin{bmatrix} l_2\cos(\theta_1+\theta_2) + l_1\cos\theta_1 \\ l_2\sin(\theta_1+\theta_2) + l_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.116)
\end{aligned}
$$

The links' angular velocity and acceleration are:

$$
\begin{aligned}
{}_{0}\boldsymbol{\omega}_1 &= \dot{\theta}_1\,\hat{K} && (12.117) \\
{}_{0}\boldsymbol{\alpha}_1 &= {}_{0}\dot{\boldsymbol{\omega}}_1 = \ddot{\theta}_1\,\hat{K} && (12.118) \\
{}_{0}\boldsymbol{\omega}_2 &= \left(\dot{\theta}_1 + \dot{\theta}_2\right)\hat{K} && (12.119) \\
{}_{0}\boldsymbol{\alpha}_2 &= {}_{0}\dot{\boldsymbol{\omega}}_2 = \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\hat{K} && (12.120)
\end{aligned}
$$

The translational acceleration of C_i are:

$$
\begin{aligned}
{}^{0}\mathbf{a}_1 &= {}_{0}\boldsymbol{\alpha}_1 \times {}^{0}\mathbf{r}_1 - {}_{0}\boldsymbol{\omega}_1 \times \left({}_{0}\boldsymbol{\omega}_1 \times {}^{0}\mathbf{r}_1\right) \\
&= \begin{bmatrix} -c_1\ddot{\theta}_1\sin\theta_1 + c_1\dot{\theta}_1^2\cos\theta_1 \\ c_1\ddot{\theta}_1\cos\theta_1 + c_1\dot{\theta}_1^2\sin\theta_1 \\ 0 \end{bmatrix} && (12.121)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}\ddot{\mathbf{d}}_1 &= {}_{0}\boldsymbol{\alpha}_1 \times {}^{0}\mathbf{d}_1 - {}_{0}\boldsymbol{\omega}_1 \times \left({}_{0}\boldsymbol{\omega}_1 \times {}^{0}\mathbf{d}_1\right) \\
&= \begin{bmatrix} -l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 \\ l_1\ddot{\theta}_1\cos\theta_1 + l_1\dot{\theta}_1^2\sin\theta_1 \\ 0 \end{bmatrix} && (12.122)
\end{aligned}
$$

$$\begin{aligned} {}^0\ddot{\mathbf{d}}_2 &= \frac{{}^G d^2\, {}^0\mathbf{d}_2}{dt^2} = {}^0\ddot{\mathbf{d}}_1 + {}^0_1\dot{\boldsymbol{\omega}}_2 \times {}^0_1\mathbf{d}_2 - {}^0_1\boldsymbol{\omega}_2 \times \left({}^0_1\boldsymbol{\omega}_2 \times {}^0_1\mathbf{d}_2\right) \\ &= \begin{bmatrix} {}^0\ddot{d}_{2x} \\ {}^0\ddot{d}_{2y} \\ 0 \end{bmatrix} \end{aligned} \tag{12.123}$$

$$\begin{aligned} {}^0\ddot{d}_{2x} &= -l_1\ddot{\theta}_1 \sin\theta_1 - l_2\ddot{\theta}_2 \sin(\theta_1+\theta_2) \\ &\quad +l_1\dot{\theta}_1^2 \cos\theta_1 + l_2\dot{\theta}_2^2 \cos(\theta_1+\theta_2) \end{aligned} \tag{12.124}$$

$$\begin{aligned} {}^0\ddot{d}_{2y} &= l_1\ddot{\theta}_1 \cos\theta_1 + l_2\ddot{\theta}_2 \cos(\theta_1+\theta_2) \\ &\quad +l_1\theta_1^2 \sin\theta_1 + l_2\dot{\theta}_2^2 \sin(\theta_1+\theta_2) \end{aligned} \tag{12.125}$$

$$\begin{aligned} {}^0\mathbf{a}_2 &= {}^0\ddot{\mathbf{d}}_2 + {}_0\boldsymbol{\alpha}_2 \times \left({}^0\mathbf{r}_2 - {}^0\mathbf{d}_2\right) - {}_0\boldsymbol{\omega}_2 \times \left({}_0\boldsymbol{\omega}_2 \times \left({}^0\mathbf{r}_2 - {}^0\mathbf{d}_2\right)\right) \\ &= {}^0\ddot{\mathbf{d}}_2 - {}_0\boldsymbol{\alpha}_2 \times {}^0\mathbf{m}_2 + {}_0\boldsymbol{\omega}_2 \times \left({}_0\boldsymbol{\omega}_2 \times {}^0\mathbf{m}_2\right) \\ &= \begin{bmatrix} {}^0a_{2x} \\ {}^0a_{2y} \\ 0 \end{bmatrix} \end{aligned} \tag{12.126}$$

$$\begin{aligned} {}^0a_{2x} &= \left((l_2 - c_2)\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - l_2\ddot{\theta}_2\right) \sin(\theta_1+\theta_2) \\ &\quad -l_1\ddot{\theta}_1 \sin\theta_1 + l_1\dot{\theta}_1^2 \cos\theta_1 \\ &\quad -\left((l_2 - c_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 - l_2\dot{\theta}_2^2\right) \cos(\theta_1+\theta_2) \end{aligned} \tag{12.127}$$

$$\begin{aligned} {}^0a_{2y} &= -\left((l_2 - c_2)\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) - l_2\ddot{\theta}_2\right) \cos(\theta_1+\theta_2) \\ &\quad +l_1\ddot{\theta}_1 \cos\theta_1 + l_1\dot{\theta}_1^2 \sin\theta_1 \\ &\quad -\left((l_2 - c_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 - l_2\dot{\theta}_2^2\right) \sin(\theta_1+\theta_2) \end{aligned} \tag{12.128}$$

The moment of inertia matrices in the global coordinate frame are:

$$\begin{aligned} {}^0I_1 &= R_{Z,\theta_1}\, {}^1I_1\, R_{Z,\theta_1}^T = {}^0R_1 \begin{bmatrix} I_{x_1} & 0 & 0 \\ 0 & I_{y_1} & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix} {}^0R_1^T \\ &= \begin{bmatrix} I_{x_1}c^2\theta_1 + I_{y_1}s^2\theta_1 & (I_{x_1} - I_{y_1})\, c\theta_1 s\theta_1 & 0 \\ (I_{x_1} - I_{y_1})\, c\theta_1 s\theta_1 & I_{y_1}c^2\theta_1 + I_{x_1}s^2\theta_1 & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix} \end{aligned} \tag{12.129}$$

$$
\begin{aligned}
{}^{0}I_{2} &= {}^{0}R_{2}\,{}^{2}I_{2}\,{}^{0}R_{2}^{T} = {}^{0}R_{2}\begin{bmatrix} I_{x_2} & 0 & 0 \\ 0 & I_{y_2} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix}{}^{0}R_{2}^{T} \\
&= \begin{bmatrix} I_{x_2}c^2\theta_{12} + I_{y_2}s^2\theta_{12} & \left(I_{x_2} - I_{y_2}\right)c\theta_{12}s\theta_{12} & 0 \\ \left(I_{x_2} - I_{y_2}\right)c\theta_{12}s\theta_{12} & I_{y_2}c^2\theta_{12} + I_{x_2}s^2\theta_{12} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix} \qquad (12.130)
\end{aligned}
$$

$$
\theta_{12} = \theta_1 + \theta_2 \qquad (12.131)
$$

Substituting these results in Equations (12.95) and (12.96), and solving for Q_0 and Q_1, provides the dynamic equations for the $2R$ manipulator.

$$
\begin{aligned}
{}^{0}\mathbf{Q}_{1} &= {}^{0}I_{2\,0}\boldsymbol{\alpha}_{2} - {}^{0}\mathbf{n}_{2} \times \left(m_2\,{}^{0}\mathbf{a}_{2} - \left(m_0 + m_2\right)g\,\hat{J}\right) - {}^{0}\mathbf{m}_{2} \times m_0 g\,\hat{J} \\
&= \begin{bmatrix} 0 \\ 0 \\ {}^{0}Q_{1z} \end{bmatrix} \qquad (12.132)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}Q_{1z} &= \left(I_{z_2} + m_2c_2^2 - m_2l_2c_2 + m_2l_1c_2\cos\theta_2\right)\ddot{\theta}_1 \\
&\quad + \left(I_{z_2} + m_2c_2^2\right)\ddot{\theta}_2 - m_2c_2l_1\dot{\theta}_1^2\sin\theta_2 \\
&\quad - \left(m_2c_2 + m_0l_2\right)g\cos\left(\theta_1 + \theta_2\right) \qquad (12.133)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}\mathbf{Q}_{0} &= {}^{0}\mathbf{Q}_{1} + {}^{0}I_{1\,0}\boldsymbol{\alpha}_{1} + {}^{0}\mathbf{m}_{1} \times \left(m_2\,{}^{0}\mathbf{a}_{2} - \left(m_0 + m_2\right)g\,\hat{J}\right) \\
&\quad - {}^{0}\mathbf{n}_{1} \times \left(m_1\,{}^{0}\mathbf{a}_{1} - m_1 g\,\hat{J} + \left(m_2\,{}^{0}\mathbf{a}_{2} - \left(m_0 + m_2\right)g\,\hat{J}\right)\right) \\
&= \begin{bmatrix} 0 \\ 0 \\ {}^{0}Q_{0z} \end{bmatrix} \qquad (12.134)
\end{aligned}
$$

$$
\begin{aligned}
{}^{0}Q_{0z} &= \left(I_{z_1} + I_{z_2} + m_1c_1^2 + m_2\left(l_1^2 + c_2^2 - l_2c_2 + l_1\left(2c_2 - l_2\right)\cos\theta_2\right)\right)\ddot{\theta}_1 \\
&\quad + \left(I_{z_2} + m_2c_2\left(c_2 + l_1\cos\theta_2\right)\right)\ddot{\theta}_2 - m_2l_1l_2\dot{\theta}_1^2\sin\theta_2 \\
&\quad + m_2l_1c_2\dot{\theta}_2^2\sin\theta_2 - 2m_2l_1\left(l_2 - c_2\right)\dot{\theta}_1\dot{\theta}_2\sin\theta_2 \\
&\quad - \left(m_0l_1 + m_1c_1 + m_2l_1\right)g\cos\theta_1 \\
&\quad - \left(m_0l_2 + m_2c_2\right)g\cos\left(\theta_1 + \theta_2\right) \qquad (12.135)
\end{aligned}
$$

Example 330 ★ *Matrix form of equations of motion.*

Let us rearrange the equations of motion (12.133) and (12.135) in a matrix form.

$$
\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{Q} \qquad (12.136)
$$

$$
\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} {}^{0}Q_{1z} \\ {}^{0}Q_{2z} \end{bmatrix} \qquad (12.137)
$$

$$\mathbf{D}(\mathbf{q}) = \begin{bmatrix} Z_1 - Z_2 + Z_3 \cos\theta_2 & Z_1 \\ Z_1 + Z_4 - Z_2 + Z_5 + Z_6 \cos\theta_2 & Z_1 + Z_3 \cos\theta_2 \end{bmatrix} \tag{12.138}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -Z_3\dot{\theta}_1 \sin\theta_2 & 0 \\ \left(-Z_7\dot{\theta}_1 - Z_8\dot{\theta}_2\right)\sin\theta_2 & \left(Z_3\dot{\theta}_2 - Z_8\dot{\theta}_1\right)\sin\theta_2 \end{bmatrix} \tag{12.139}$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} -Z_9 \cos(\theta_1 + \theta_2) \\ -Z_9 \cos(\theta_1 + \theta_2) + Z_{10}\cos\theta_1 \end{bmatrix} \tag{12.140}$$

$$\begin{aligned}
Z_1 &= I_{z_2} + m_2 c_2^2 && (12.141)\\
Z_2 &= m_2 l_2 c_2 && (12.142)\\
Z_3 &= m_2 l_1 c_2 && (12.143)\\
Z_4 &= I_{z_1} + m_1 c_1^2 && (12.144)\\
Z_5 &= m_2 l_1^2 && (12.145)\\
Z_6 &= m_2 l_1 (2c_2 - l_2) && (12.146)\\
Z_7 &= m_2 l_1 l_2 && (12.147)\\
Z_8 &= m_2 l_1 (l_2 - c_2) && (12.148)\\
Z_9 &= (m_2 c_2 + m_0 l_2)\, g && (12.149)\\
Z_{10} &= (m_0 l_1 + m_1 c_1 + m_2 l_1)\, g && (12.150)
\end{aligned}$$

Example 331 ★ *Joint forces of the general 2R manipulator.*

Substituting the vectorial information of (12.104)-(12.130) in (12.97) and (12.98), we find the joint forces of the general $2R$ manipulator that is shown in Figure 12.10. The manipulator has massive arms with mass center at C_i and carries a payload m_0.

$${}^0\mathbf{F}_1 = m_2\, {}^0\mathbf{a}_2 - (m_0 + m_2)\, g\, \hat{J} = \begin{bmatrix} {}^0F_{1x} \\ {}^0F_{1y} \\ 0 \end{bmatrix} \tag{12.151}$$

$$\begin{aligned}
{}^0F_{1x} = {} & (m_2 (l_2 - c_2) \sin(\theta_1 + \theta_2) - m_2 l_1 \sin\theta_1)\, \ddot{\theta}_1 \\
& -m_2 c_2 \ddot{\theta}_2 \sin(\theta_1 + \theta_2) + m_2 c_2 \dot{\theta}_2^2 \cos(\theta_1 + \theta_2) \\
& + (m_2 (l_2 - c_2) \cos(\theta_1 + \theta_2) + m_2 l_1 \cos\theta_1)\, \dot{\theta}_1^2 \\
& +2 m_2 l_1 (l_2 - c_2)\, \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2) && (12.152)
\end{aligned}$$

$$\begin{aligned}
{}^0F_{1y} = {} & (-m_2 (l_2 - c_2) \cos(\theta_1 + \theta_2) + m_2 l_1 \cos\theta_1)\, \ddot{\theta}_1 \\
& +m_2 c_2 \ddot{\theta}_2 \cos(\theta_1 + \theta_2) + m_2 c_2 \dot{\theta}_2^2 \sin(\theta_1 + \theta_2) \\
& + (-m_2 (l_2 - c_2) \sin(\theta_1 + \theta_2) + m_2 l_1 \sin\theta_1)\, \dot{\theta}_1^2 \\
& -2 m_2 l_1 (l_2 - c_2)\, \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 + \theta_2) \\
& - (m_0 + m_2)\, g && (12.153)
\end{aligned}$$

$$ {}^{0}\mathbf{F}_{0} = m_1\, {}^{0}\mathbf{a}_1 + m_2\, {}^{0}\mathbf{a}_2 - (m_0 + m_1 + m_2)\, g\, \hat{J} = \begin{bmatrix} {}^{0}F_{0x} \\ {}^{0}F_{0y} \\ 0 \end{bmatrix} \tag{12.154} $$

$$ \begin{aligned} {}^{0}F_{0x} \;=\;& \left(m_2\left(l_2 - c_2\right)\sin\left(\theta_1 + \theta_2\right) - \left(m_1 c_1 + m_2 l_1\right)\sin\theta_1\right)\ddot{\theta}_1 \\ & - m_2 c_2 \ddot{\theta}_2 \sin\left(\theta_1 + \theta_2\right) - m_2 c_2 \dot{\theta}_2^2 \cos\left(\theta_1 + \theta_2\right) \\ & + \left(-m_2\left(l_2 - c_2\right)\cos\left(\theta_1 + \theta_2\right) + \left(m_2 l_1 + m_1 c_1\right)\cos\theta_1\right)\dot{\theta}_1^2 \\ & - 2 m_2\left(l_2 - c_2\right)\dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 + \theta_2\right) \end{aligned} \tag{12.155} $$

$$ \begin{aligned} {}^{0}F_{0y} \;=\;& \left(-m_2\left(l_2 - c_2\right)\cos\left(\theta_1 + \theta_2\right) + \left(m_2 l_1 + m_1 c_1\right)\cos\theta_1\right)\ddot{\theta}_1 \\ & + m_2 c_2 \ddot{\theta}_2 \cos\left(\theta_1 + \theta_2\right) + m_2 c_2 \dot{\theta}_2^2 \sin\left(\theta_1 + \theta_2\right) \\ & + \left(-m_2\left(l_2 - c_2\right)\sin\left(\theta_1 + \theta_2\right) + \left(m_2 l_1 + m_1 c_1\right)\sin\theta_1\right)\dot{\theta}_1^2 \\ & - 2 m_2 l_1\left(l_2 - c_2\right)\dot{\theta}_1 \dot{\theta}_2 \sin\left(\theta_1 + \theta_2\right) \\ & - \left(m_0 + m_1 + m_2\right) g \end{aligned} \tag{12.156} $$

12.2 ★ Recursive Newton-Euler Dynamics

An advantage of the Newton-Euler equations of motion in robotic application is that we can calculate the joint forces of one link at a time. Therefore, starting from the end-effector link, we can analyze the links one by one and end up at the base link or vice versa. For such an analysis, we need to reform the Newton-Euler equations of motion to work in the interested link's frame.

The *backward recursive Newton-Euler equations of motion* for the link (i) in its body coordinate frame B_i are:

$$ {}^{i}\mathbf{F}_{i-1} = {}^{i}\mathbf{F}_{i} - \sum {}^{i}\mathbf{F}_{e_i} + m_i\, {}^{i}_{0}\mathbf{a}_i \tag{12.157} $$

$$ \begin{aligned} {}^{i}\mathbf{M}_{i-1} \;=\;& {}^{i}\mathbf{M}_{i} - \sum {}^{i}\mathbf{M}_{e_i} - \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_{i-1} \\ & + \left({}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_{i} + {}^{i}I_i\, {}^{i}_{0}\boldsymbol{\alpha}_i + {}^{i}_{0}\boldsymbol{\omega}_i \times {}^{i}I_i\, {}^{i}_{0}\boldsymbol{\omega}_i \end{aligned} \tag{12.158} $$

$$ {}^{i}\mathbf{n}_i = {}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i \tag{12.159} $$

$$ {}^{i}\mathbf{m}_i = {}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_i \tag{12.160} $$

When the driving force system $({}^{i}\mathbf{F}_{i-1}, {}^{i}\mathbf{M}_{i-1})$ is found in frame B_i, we can transform them to the frame B_{i-1} and apply the Newton-Euler equation for link $(i-1)$.

$$ {}^{i-1}\mathbf{F}_{i-1} = {}^{i-1}T_i\, {}^{i}\mathbf{F}_{i-1} \tag{12.161} $$

$$ {}^{i-1}\mathbf{M}_{i-1} = {}^{i-1}T_i\, {}^{i}\mathbf{M}_{i-1} \tag{12.162} $$

The negative of the converted force system acts as the driven force system $(-\,{}^{i-1}\mathbf{F}_{i-1}, -\,{}^{i-1}\mathbf{M}_{i-1})$ for the link $(i-1)$.

The *forward recursive Newton-Euler equations of motion* for the link (i) in its body coordinate frame B_i are:

$$ {}^{i}\mathbf{F}_i = {}^{i}\mathbf{F}_{i-1} + \sum {}^{i}\mathbf{F}_{e_i} - m_i\,{}^{i}_{0}\mathbf{a}_i \tag{12.163} $$

$$ \begin{aligned} {}^{i}\mathbf{M}_i &= {}^{i}\mathbf{M}_{i-1} + \sum {}^{i}\mathbf{M}_{e_i} + \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_{i-1} \\ &\quad - \left({}^{i}\mathbf{d}_i - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_i - {}^{i}I_i\,{}^{i}_{0}\boldsymbol{\alpha}_i - {}^{i}_{0}\boldsymbol{\omega}_i \times {}^{i}I_i\,{}^{i}_{0}\boldsymbol{\omega}_i. \end{aligned} \tag{12.164} $$

$$ {}^{i}\mathbf{n}_i = {}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i \tag{12.165} $$

$$ {}^{i}\mathbf{m}_i = {}^{i}\mathbf{d}_i - {}^{i}\mathbf{r}_i \tag{12.166} $$

When the reaction force system $({}^{i}\mathbf{F}_i, {}^{i}\mathbf{M}_i)$ is found in frame B_i, we can transform them to frame B_{i+1}.

$$ {}^{i+1}\mathbf{F}_i = {}^{i}T_{i+1}^{-1}\,{}^{i}\mathbf{F}_i \tag{12.167} $$

$$ {}^{i+1}\mathbf{M}_i = {}^{i}T_{i+1}^{-1}\,{}^{i}\mathbf{M}_i \tag{12.168} $$

The negative of the converted force system acts as the action force system $(-\,{}^{i+1}\mathbf{F}_i, -\,{}^{i+1}\mathbf{M}_i)$ for the link $(i+1)$.

Proof. The Euler equation for a rigid link in body coordinate frame is:

$$ \begin{aligned} {}^{B}\mathbf{M} &= \frac{{}^{G}d}{dt}\,{}^{B}\mathbf{L} = {}^{B}\dot{\mathbf{L}} + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\mathbf{L} \\ &= {}^{i}I_i\,{}_{i}\boldsymbol{\alpha}_i + {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{i}I_i\,{}_{i}\boldsymbol{\omega}_i \end{aligned} \tag{12.169} $$

where $\mathbf{L}$ is the angular momentum of the link.

$$ {}^{B}\mathbf{L} = {}^{B}I\,{}^{B}_{G}\boldsymbol{\omega}_B \tag{12.170} $$

We may solve the Newton-Euler equations of motion (12.1) and (12.2) for the action force system

$$ {}^{0}\mathbf{F}_{i-1} = {}^{0}\mathbf{F}_i - \sum {}^{0}\mathbf{F}_{e_i} + m_i\,{}^{0}\mathbf{a}_i \tag{12.171} $$

$$ \begin{aligned} {}^{0}\mathbf{M}_{i-1} &= {}^{0}\mathbf{M}_i - \sum {}^{0}\mathbf{M}_{e_i} - \left({}^{0}\mathbf{d}_{i-1} - {}^{0}\mathbf{r}_i\right) \times {}^{0}\mathbf{F}_{i-1} \\ &\quad + \left({}^{0}\mathbf{d}_i - {}^{0}\mathbf{r}_i\right) \times {}^{0}\mathbf{F}_i + \frac{{}^{0}d}{dt}\,{}^{0}\mathbf{L}_i \end{aligned} \tag{12.172} $$

and then, transform the equations to the coordinate frame B_i attached to the link's (i) to make the recursive form of the Newton-Euler equations of motion.

$$ {}^{i}\mathbf{F}_{i-1} = {}^{0}T_i^{-1}\,{}^{0}\mathbf{F}_{i-1} = {}^{i}\mathbf{F}_i - \sum {}^{i}\mathbf{F}_{e_i} + m_i\,{}^{i}_{0}\mathbf{a}_i \tag{12.173} $$

$$
\begin{aligned}
{}^{i}\mathbf{M}_{i-1} &= {}^{0}T_{i}^{-1}\,{}^{0}\mathbf{M}_{i-1} \\
&= {}^{i}\mathbf{M}_{i} - \sum {}^{i}\mathbf{M}_{e_i} - \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i-1} \qquad (12.174) \\
&\quad + \left({}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i} + \frac{{}^{0}d}{dt}\,{}^{i}\mathbf{L}_{i} \\
&= {}^{i}\mathbf{M}_{i} - \sum {}^{i}\mathbf{M}_{e_i} - \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i-1} \\
&\quad + \left({}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i} + {}^{i}I_{i}\,{}_{0}^{i}\boldsymbol{\alpha}_{i} + {}_{0}^{i}\boldsymbol{\omega}_{i} \times {}^{i}I_{i}\,{}_{0}^{i}\boldsymbol{\omega}_{i}. \qquad (12.175)
\end{aligned}
$$

Starting from link (i) and deriving the equations of motion of the previous link $(i-1)$ is called the *backward Newton-Euler equations of motion.*

We may also start from link (i) and derive the equations of motion of the next link $(i+1)$. This method is called the *forward Newton-Euler equations of motion.* Employing the Newton-Euler equations of motion (12.157) and (12.158), we can write them in a *forward recursive* form in coordinate frame B_i attached to the link (i).

$$
{}^{i}\mathbf{F}_{i} = {}^{i}\mathbf{F}_{i-1} + \sum {}^{i}\mathbf{F}_{e_i} - m_i\,{}_{0}^{i}\mathbf{a}_{i} \qquad (12.176)
$$

$$
\begin{aligned}
{}^{i}\mathbf{M}_{i} &= {}^{i}\mathbf{M}_{i-1} + \sum {}^{i}\mathbf{M}_{e_i} + \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i-1} \\
&\quad - \left({}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_{i}\right) \times {}^{i}\mathbf{F}_{i} - {}^{i}I_{i}\,{}_{0}^{i}\boldsymbol{\alpha}_{i} - {}_{0}^{i}\boldsymbol{\omega}_{i} \times {}^{i}I_{i}\,{}_{0}^{i}\boldsymbol{\omega}_{i}. \qquad (12.177)
\end{aligned}
$$

$$
\begin{aligned}
{}^{i}\mathbf{n}_{i} &= {}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_{i} \qquad (12.178) \\
{}^{i}\mathbf{m}_{i} &= {}^{i}\mathbf{d}_{i} - {}^{i}\mathbf{r}_{i} \qquad (12.179)
\end{aligned}
$$

Using the forward Newton-Euler equations of motion (12.176) and (12.177), we can calculate the reaction force system $({}^{i}\mathbf{F}_{i},\ {}^{i}\mathbf{M}_{i})$ by having the action force system $({}^{i}\mathbf{F}_{i-1},\ {}^{i}\mathbf{M}_{i-1})$. When the reaction force system $({}^{i}\mathbf{F}_{i},\ {}^{i}\mathbf{M}_{i})$ is found in frame B_i, we can transform them to frame B_{i+1}.

$$
\begin{aligned}
{}^{i+1}\mathbf{F}_{i} &= {}^{i}T_{i+1}^{-1}\,{}^{i}\mathbf{F}_{i} \qquad (12.180) \\
{}^{i+1}\mathbf{M}_{i} &= {}^{i}T_{i+1}^{-1}\,{}^{i}\mathbf{M}_{i} \qquad (12.181)
\end{aligned}
$$

The negative of the converted force system acts as the action force system $(-\,{}^{i+1}\mathbf{F}_{i},\ -\,{}^{i+1}\mathbf{M}_{i})$ for the link $(i+1)$ and we can apply the Newton-Euler equation to the link $(i+1)$.

The forward Newton-Euler equations of motion allows us to start from a known action force system $({}^{1}\mathbf{F}_{0},\ {}^{1}\mathbf{M}_{0})$, that the base link applies to the link (1), and calculate the action force of the next link. Therefore, analyzing the links of a robot, one by one, we end up with the force system that the end-effector applies to the environment.

Using the forward or backward recursive Newton-Euler equations of motion depends on the measurement and sensory system of the robot. ■

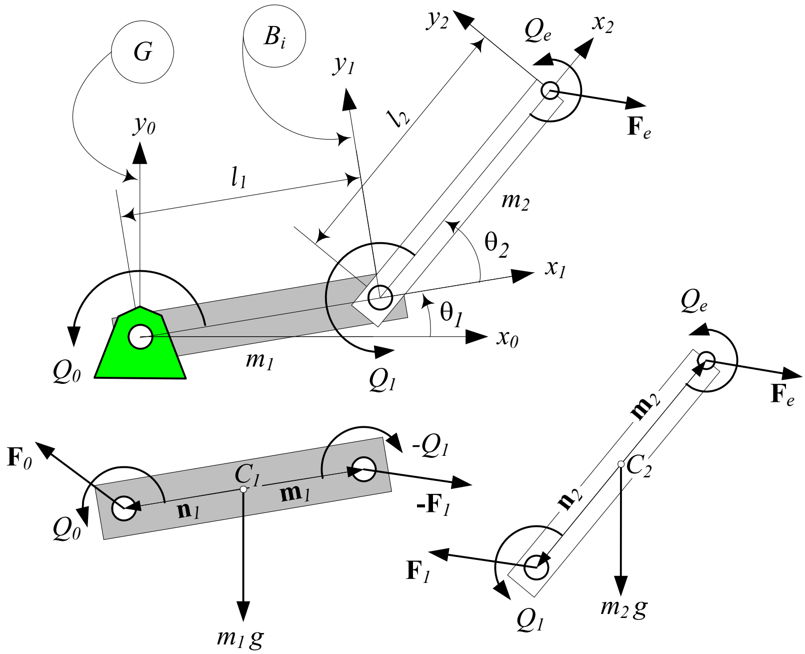

FIGURE 12.11. A $2R$ planar manipulator carrying a load at the endpoint.

Example 332 ★ *Recursive dynamics of a $2R$ planar manipulator.*

Consider the $2R$ planar manipulator shown in Figure 12.11. The manipulator is carrying a force system at the endpoint. We use this manipulator to show how we can, step by step, develop the dynamic equations for a robot.

The backward recursive Newton-Euler equations of motion for the first link are

$$\begin{aligned} {}^1\mathbf{F}_0 &= {}^1\mathbf{F}_1 - \sum {}^1\mathbf{F}_{e_1} + m_1 \, {}^1_0\mathbf{a}_1 \\ &= {}^1\mathbf{F}_1 - m_1 \, {}^1\mathbf{g} + m_1 \, {}^1_0\mathbf{a}_1 \end{aligned} \tag{12.182}$$

$$\begin{aligned} {}^1\mathbf{M}_0 &= {}^1\mathbf{M}_1 - \sum {}^1\mathbf{M}_{e_1} - \left({}^1\mathbf{d}_0 - {}^1\mathbf{r}_1\right) \times {}^1\mathbf{F}_0 \\ &\quad + \left({}^1\mathbf{d}_1 - {}^1\mathbf{r}_1\right) \times {}^1\mathbf{F}_1 + {}^1I_1 \, {}^1_0\boldsymbol{\alpha}_1 + {}^1_0\boldsymbol{\omega}_1 \times {}^1I_1 \, {}^1_0\boldsymbol{\omega}_1 \\ &= {}^1\mathbf{M}_1 - {}^1\mathbf{n}_1 \times {}^1\mathbf{F}_0 + {}^1\mathbf{m}_1 \times {}^1\mathbf{F}_1 \\ &\quad + {}^1I_1 \, {}^1_0\boldsymbol{\alpha}_1 + {}^1_0\boldsymbol{\omega}_1 \times {}^1I_1 \, {}^1_0\boldsymbol{\omega}_1 \end{aligned} \tag{12.183}$$

and the backward recursive equations of motion for the second link are:

$$\begin{aligned} {}^2\mathbf{F}_1 &= {}^2\mathbf{F}_2 - \sum {}^2\mathbf{F}_{e_2} + m_2 \, {}^2_0\mathbf{a}_2 \\ &= -m_2 \, {}^2\mathbf{g} - {}^2\mathbf{F}_e + m_2 \, {}^2_0\mathbf{a}_2 \end{aligned} \tag{12.184}$$

$$\begin{aligned}
{}^{2}\mathbf{M}_1 &= {}^{2}\mathbf{M}_2 - \sum {}^{2}\mathbf{M}_{e_2} - \left({}^{2}\mathbf{d}_1 - {}^{2}\mathbf{r}_2\right) \times {}^{2}\mathbf{F}_1 \\
&\quad + \left({}^{2}\mathbf{d}_2 - {}^{2}\mathbf{r}_2\right) \times {}^{2}\mathbf{F}_2 + {}^{2}I_2 \, {}_0^2\boldsymbol{\alpha}_2 + {}_0^2\boldsymbol{\omega}_2 \times {}^{2}I_2 \, {}_0^2\boldsymbol{\omega}_2 \\
&= -{}^{2}\mathbf{M}_e - {}^{2}\mathbf{m}_2 \times {}^{2}\mathbf{F}_e - {}^{2}\mathbf{n}_2 \times {}^{2}\mathbf{F}_1 \\
&\quad + {}^{2}I_2 \, {}_0^2\boldsymbol{\alpha}_2 + {}_0^2\boldsymbol{\omega}_2 \times {}^{2}I_2 \, {}_0^2\boldsymbol{\omega}_2
\end{aligned} \tag{12.185}$$

The manipulator consists of two $R\|R(0)$ links, therefore their transformation matrices ${}^{i-1}T_i$ are of class (5.32). Substituting $d_i = 0$ and $a_i = l_i$, produces the following transformation matrices.

$$ {}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.186}$$

$$ {}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.187}$$

The homogeneous moments of inertia matrices are:

$$ {}^{1}I_1 = \frac{m_1 l_1^2}{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad {}^{2}I_2 = \frac{m_2 l_2^2}{12}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12.188}$$

The homogeneous moment of inertia matrix is obtained by appending a zero row and column to the I matrix.

The position vectors involved are:

$$ {}^{1}\mathbf{n}_1 = \begin{bmatrix} -l_1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{2}\mathbf{n}_2 = \begin{bmatrix} -l_2/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.189}$$

$$ {}^{1}\mathbf{m}_1 = \begin{bmatrix} l_1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{2}\mathbf{m}_2 = \begin{bmatrix} l_2/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.190}$$

$$ {}^{1}\mathbf{r}_1 = -{}^{1}\mathbf{n}_1 \qquad {}^{2}\mathbf{r}_2 = -{}^{2}\mathbf{n}_1 + {}^{2}\mathbf{m}_2 - {}^{2}\mathbf{n}_2 \tag{12.191}$$

The angular velocities and accelerations are:

$$ {}_0^1\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} \qquad {}_0^2\boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \\ 0 \end{bmatrix} \tag{12.192}$$

$$
{}_0^1\boldsymbol{\alpha}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix} \qquad {}_0^2\boldsymbol{\alpha}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \\ 0 \end{bmatrix} \tag{12.193}
$$

The translational acceleration of C_1 is

$$
\begin{aligned}
{}_0^1\mathbf{a}_1 &= {}_0^1\boldsymbol{\alpha}_1 \times \left(-{}^1\mathbf{m}_1\right) + {}_0^1\boldsymbol{\omega}_1 \times \left({}_0^1\boldsymbol{\omega}_1 \times \left(-{}^1\mathbf{m}_1\right)\right) + {}_0^1\ddot{\mathbf{d}}_1 \\
&= \begin{bmatrix} -\frac{1}{2}l_1\dot{\theta}_1^2 \\ \frac{1}{2}l_1\ddot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{12.194}
$$

because

$$
{}^1\ddot{\mathbf{d}}_1 = 2\ {}^1\mathbf{a}_1. \tag{12.195}
$$

The translational acceleration of C_2 is

$$
\begin{aligned}
{}_0^2\mathbf{a}_2 &= {}_0^2\boldsymbol{\alpha}_2 \times \left(-{}^2\mathbf{m}_2\right) + {}_0^2\boldsymbol{\omega}_2 \times \left({}_0^2\boldsymbol{\omega}_2 \times \left(-{}^2\mathbf{m}_2\right)\right) + {}_0^2\ddot{\mathbf{d}}_2 \\
&= \begin{bmatrix} -\frac{1}{2}l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ \frac{1}{2}l_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{12.196}
$$

because

$$
{}^2\ddot{\mathbf{d}}_2 = 2\ {}^2\mathbf{a}_2. \tag{12.197}
$$

The gravitational acceleration vector in the links' frame are:

$$
{}^1\mathbf{g} = {}^0T_1^{-1}\ {}^0\mathbf{g} = \begin{bmatrix} -g\sin\theta_2 \\ g\cos\theta_2 \\ 0 \\ 0 \end{bmatrix} \tag{12.198}
$$

$$
{}^2\mathbf{g} = {}^0T_2^{-1}\ {}^0\mathbf{g} = \begin{bmatrix} -g\sin\left(\theta_1 + \theta_2\right) \\ g\cos\left(\theta_1 + \theta_2\right) \\ 0 \\ 0 \end{bmatrix} \tag{12.199}
$$

The external load is usually given in the global coordinate frame. We must transform them to the interested link's frame to apply the recursive equations of motion. Therefore, the external force system expressed in B_2 is:

$$
{}^2\mathbf{F}_e = {}^0T_2^{-1}\,{}^0\mathbf{F}_e = \begin{bmatrix} F_{ex}\cos\left(\theta_1 + \theta_2\right) + F_{ey}\sin\left(\theta_1 + \theta_2\right) \\ F_{ey}\cos\left(\theta_1 + \theta_2\right) - F_{ex}\sin\left(\theta_1 + \theta_2\right) \\ 0 \\ 0 \end{bmatrix} \tag{12.200}
$$

$$
{}^{2}\mathbf{M}_{e} = {}^{0}T_{2}^{-1}\, {}^{0}\mathbf{M}_{e} = \begin{bmatrix} 0 \\ 0 \\ M_{e} \\ 0 \end{bmatrix} \tag{12.201}
$$

Now, we start from the final link and calculate its action force system. The backward Newton equation for link (2) *is*

$$
{}^{2}\mathbf{F}_{1} = -m_{2}\, {}^{2}\mathbf{g} - {}^{2}\mathbf{F}_{e} + m_{2}\, {}^{2}_{0}\mathbf{a}_{2} = \begin{bmatrix} {}^{2}F_{1x} \\ {}^{2}F_{1y} \\ 0 \\ 0 \end{bmatrix} \tag{12.202}
$$

$$
\begin{aligned}
{}^{2}F_{1x} &= -\frac{1}{2} l_{2} m_{2} \left(\dot{\theta}_{1} + \dot{\theta}_{2}\right)^{2} - F_{ex} \cos\left(\theta_{1} + \theta_{2}\right) \\
&\quad - \left(F_{ey} - g m_{2}\right) \sin\left(\theta_{1} + \theta_{2}\right) && (12.203) \\
{}^{2}F_{1y} &= \frac{1}{2} l_{2} m_{2} \left(\ddot{\theta}_{1} + \ddot{\theta}_{2}\right) + F_{ex} \sin\left(\theta_{1} + \theta_{2}\right) \\
&\quad - \left(F_{ey} + g m_{2}\right) \cos\left(\theta_{1} + \theta_{2}\right) && (12.204)
\end{aligned}
$$

and the backward Euler equation for link (2) *is*

$$
\begin{aligned}
{}^{2}\mathbf{M}_{1} &= -\, {}^{2}\mathbf{M}_{e} - {}^{2}\mathbf{m}_{2} \times {}^{2}\mathbf{F}_{e} - {}^{2}\mathbf{n}_{2} \times {}^{2}\mathbf{F}_{1} \\
&\quad + {}^{2}I_{2}\, {}^{2}_{0}\boldsymbol{\alpha}_{2} + {}^{2}_{0}\boldsymbol{\omega}_{2} \times {}^{2}I_{2}\, {}^{2}_{0}\boldsymbol{\omega}_{2} \\
&= \begin{bmatrix} 0 \\ 0 \\ {}^{2}M_{1z} \\ 0 \end{bmatrix} && (12.205)
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{2}M_{1z} &= -M_{e} + l_{2} F_{ex} \sin\left(\theta_{1} + \theta_{2}\right) - l_{2} F_{ey} \cos\left(\theta_{1} + \theta_{2}\right) \\
&\quad + \frac{1}{3} l_{2}^{2} m_{2} \left(\ddot{\theta}_{1} + \ddot{\theta}_{2}\right) - \frac{1}{2} g l_{2} m_{2} \cos\left(\theta_{1} + \theta_{2}\right). && (12.206)
\end{aligned}
$$

Finally the action force on link (1) *is*

$$
\begin{aligned}
{}^{1}\mathbf{F}_{0} &= {}^{1}\mathbf{F}_{1} - m_{1}\, {}^{1}\mathbf{g} + m_{1}\, {}^{1}_{0}\mathbf{a}_{1} \\
&= {}^{1}T_{2}\, {}^{2}\mathbf{F}_{1} - m_{1}\, {}^{1}\mathbf{g} + m_{1}\, {}^{1}_{0}\mathbf{a}_{1} = \begin{bmatrix} {}^{1}F_{0x} \\ {}^{1}F_{0y} \\ 0 \\ 0 \end{bmatrix} && (12.207)
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{1}F_{0x} &= -F_{ex} \cos\theta_{1} - \left(F_{ey} - g m_{1}\right) \sin\theta_{1} \\
&\quad - \frac{1}{2} l_{2} m_{2} \left(\ddot{\theta}_{1} + \ddot{\theta}_{2}\right) \sin\theta_{2} - \frac{1}{2} l_{2} m_{2} \left(\dot{\theta}_{1} + \dot{\theta}_{2}\right)^{2} \cos\theta_{2} \\
&\quad + g m_{2} \sin\left(2\theta_{2} + \theta_{1}\right) - \frac{1}{2} l_{1} m_{1} \dot{\theta}_{1}^{2} && (12.208)
\end{aligned}
$$

$$\begin{aligned}
{}^{1}F_{0y} &= F_{ex}\sin\theta_1 - (F_{ey}+gm_1)\cos\theta_1 \\
&\quad +\frac{1}{2}l_2m_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\cos\theta_2 - \frac{1}{2}l_2m_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\sin\theta_2 \\
&\quad -gm_2\cos\left(2\theta_2+\theta_1\right) + \frac{1}{2}l_1m_1\ddot{\theta}_1
\end{aligned} \tag{12.209}$$

and the action moment on link (1) *is*

$$\begin{aligned}
{}^{1}\mathbf{M}_0 &= {}^{1}\mathbf{M}_1 - {}^{1}\mathbf{n}_1 \times {}^{1}\mathbf{F}_0 + {}^{1}\mathbf{m}_1 \times {}^{1}\mathbf{F}_1 \\
&\quad + {}^{1}I_1\ {}^{1}_{0}\boldsymbol{\alpha}_1 + {}^{1}_{0}\boldsymbol{\omega}_1 \times {}^{1}I_1\ {}^{1}_{0}\boldsymbol{\omega}_1 \\
&= {}^{1}T_2\ {}^{2}\mathbf{M}_1 - {}^{1}\mathbf{n}_1 \times {}^{1}\mathbf{F}_0 + {}^{1}\mathbf{m}_1 \times {}^{1}T_2\ {}^{2}\mathbf{F}_1 \\
&\quad + {}^{1}I_1\ {}^{1}_{0}\boldsymbol{\alpha}_1 + {}^{1}_{0}\boldsymbol{\omega}_1 \times {}^{1}I_1\ {}^{1}_{0}\boldsymbol{\omega}_1 \\
&= \begin{bmatrix} 0 \\ 0 \\ {}^{1}M_{0z} \\ 0 \end{bmatrix}
\end{aligned} \tag{12.210}$$

where,

$$\begin{aligned}
{}^{1}M_{0z} &= -M_e + \frac{1}{3}l_2^2m_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + \frac{1}{3}l_1^2m_1\ddot{\theta}_1 \\
&\quad -\left(F_{ey}l_2 + \frac{1}{2}gl_2m_2\right)\cos\left(\theta_1+\theta_2\right) \\
&\quad -\frac{1}{2}l_1m_1g\cos\theta_1 + F_{ex}l_2\sin\left(\theta_1+\theta_2\right).
\end{aligned} \tag{12.211}$$

Example 333 ★ *Actuator's force and torque.*

Applying a backward recursive force analysis ends up with a set of known force systems at joints. Each joint is driven by a motor known as an actuator that applies a force in a P joint, or a torque in an R joint. When the joint i is prismatic, the force of the driving actuator is along the z_{i-1}-axis

$$F_m = {}^{0}\hat{k}_{i-1}^{T}\ {}^{0}\mathbf{F}_i \tag{12.212}$$

showing that the $\hat{k}_{i-1}$ component of the joint force $\mathbf{F}_i$ is supported by the actuator. The $\hat{\imath}_{i-1}$ and $\hat{\jmath}_{i-1}$ components of $\mathbf{F}_i$ must be supported by the bearings of the joint. Similarly, when the joint i is revolute, the torque of the driving actuator is along the z_{i-1}-axis

$$M_m = {}^{0}\hat{k}_{i-1}^{T}\ {}^{0}\mathbf{M}_i \tag{12.213}$$

showing that the $\hat{k}_{i-1}$ component of the joint torque $\mathbf{M}_i$ is supported by the actuator. The $\hat{\imath}_{i-1}$ and $\hat{\jmath}_{i-1}$ components of $\mathbf{M}_i$ must be supported by the bearings of the joint.

12.3 Robot Lagrange Dynamics

The Lagrange equation of motion provides a systematic approach to obtain the dynamics equations for robots. The Lagrangean is defined as the difference between the kinetic and potential energies

$$\mathcal{L} = K - V. \tag{12.214}$$

The Lagrange equation of motion for a robotic system can be found by applying the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \qquad i = 1, 2, \cdots n \tag{12.215}$$

where q_i is the coordinates by which the energies are expressed, and Q_i is the corresponding generalized nonpotential force that drives q_i.

The equations of motion for an n link serial manipulator can be set in a matrix form

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{Q} \tag{12.216}$$

or

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{Q} \tag{12.217}$$

or in a summation form

$$\sum_{j=1}^{n} D_{ij}(q)\,\ddot{q}_j + \sum_{k=1}^{n}\sum_{m=1}^{n} H_{ikm}\dot{q}_k\dot{q}_m + G_i = Q_i. \tag{12.218}$$

D_{ij} is an $n \times n$ inertial-type symmetric matrix

$$D_{ij} = \sum_{k=1}^{n}\left(\mathbf{J}_{Dk}^T\; m_k\; \mathbf{J}_{Dk} + \frac{1}{2}\mathbf{J}_{Rk}^T\; {}^0I_k\; \mathbf{J}_{Rk}\right) \tag{12.219}$$

H_{ikm} is the velocity coupling vector

$$H_{ijk} = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial D_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{jk}}{\partial q_i}\right) \tag{12.220}$$

and G_i is the gravitational vector

$$G_i = \sum_{j=1}^{n} m_j \mathbf{g}^T\, \mathbf{J}_{Dj}^{(i)}. \tag{12.221}$$

Proof. Kinetic energy of link (i) is:

$$K_i = \frac{1}{2}\,{}^0\mathbf{v}_i^T\, m_i\; {}^0\mathbf{v}_i + \frac{1}{2}\,{}_0\boldsymbol{\omega}_i^T\; {}^iI_i\; {}_0\boldsymbol{\omega}_i \tag{12.222}$$

where, m_i is the mass of the link, ${}^iI_i$ is the moment of inertia matrix of the link in the link's frame B_i, ${}^0\mathbf{v}_i$ is the global velocity of the link at its mas center C, and ${}_0\boldsymbol{\omega}_i$ is the global angular velocity of the link.

The translational and angular velocity vectors can be expressed based on the joint coordinate velocities, utilizing the *Jacobian of the link* $\mathbf{J}_i$

$$\dot{\mathbf{X}}_i = \begin{bmatrix} {}^0\mathbf{v}_i \\ {}^0\boldsymbol{\omega}_i \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{Di} \\ \mathbf{J}_{Ri} \end{bmatrix} \dot{\mathbf{q}} = \mathbf{J}_i\,\dot{\mathbf{q}}. \tag{12.223}$$

The link's Jacobian $\mathbf{J}_i$ is a $6 \times n$ matrix that transforms the instantaneous joint coordinate velocities into the instantaneous link's translational and angular velocities. The jth column of $\mathbf{J}_i$ is made of $\mathbf{c}_{Di}^{(j)}$ and $\mathbf{c}_{Ri}^{(j)}$, where for $j \leq i$

$$\mathbf{c}_{Di}^{(j)} = \begin{cases} \hat{k}_{j-1} \times {}_{j-1}^{\ \ 0}\mathbf{r}_i & \text{for a R joint} \\ \hat{k}_{j-1} & \text{for a P joint} \end{cases} \tag{12.224}$$

and

$$\mathbf{c}_{Ri}^{(j)} = \begin{cases} \hat{k}_{j-1} & \text{for a R joint} \\ 0 & \text{for a P joint} \end{cases} \tag{12.225}$$

and ${}_{j-1}^{\ \ 0}\mathbf{r}_i$ is the position of C of the link (i) in the coordinate frame B_{j-1} expressed in the base frame. The columns of $\mathbf{J}_i$ are zero for $j > i$.

The kinetic energy K of the whole robot is then

$$\begin{aligned} K &= \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n}\left({}^0\mathbf{v}_i^T\, m_i\ {}^0\mathbf{v}_i + \frac{1}{2}\,{}_0\boldsymbol{\omega}_i^T\ {}^0I_i\ {}_0\boldsymbol{\omega}_i\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(\left(\mathbf{J}_{Di}\,\dot{\mathbf{q}}_i\right)^T\ m_i\ \left(\mathbf{J}_{Di}\,\dot{\mathbf{q}}_i\right) + \frac{1}{2}\left(\mathbf{J}_{Ri}\,\dot{\mathbf{q}}_i\right)^T\ {}^0I_i\ \left(\mathbf{J}_{Ri}\,\dot{\mathbf{q}}_i\right)\right) \\ &= \frac{1}{2}\dot{\mathbf{q}}_i^T\left(\sum_{i=1}^{n}\left(\mathbf{J}_{Di}^T\ m_i\ \mathbf{J}_{Di} + \frac{1}{2}\,\mathbf{J}_{Ri}^T\ {}^0I_i\ \mathbf{J}_{Ri}\right)\right)\dot{\mathbf{q}}_i \end{aligned} \tag{12.226}$$

where ${}^0I_i$ is the inertia matrix of the link (i) about its C and expressed in the base frame.

$${}^0I_i = {}^0R_i\ {}^iI_i\ {}^0R_i^T \tag{12.227}$$

The kinetic energy may be written in a more convenient form as

$$K = \frac{1}{2}\dot{\mathbf{q}}_i^T\, D\,\dot{\mathbf{q}}_i \tag{12.228}$$

where D is an $n \times n$ matrix called the *manipulator inertia matrix.*

$$D = \sum_{i=1}^{n}\left(\mathbf{J}_{Di}^T\ m_i\ \mathbf{J}_{Di} + \frac{1}{2}\,\mathbf{J}_{Ri}^T\ {}^0I_i\ \mathbf{J}_{Ri}\right) \tag{12.229}$$

The potential energy of the link (i) is due to gravity

$$V_i = -m_i \, {}^0\mathbf{g} \cdot {}^0\mathbf{r}_i \tag{12.230}$$

and therefore, the total potential energy of the manipulator is:

$$V = \sum_{i=1}^{n} V_i = -\sum_{i=1}^{n} m_i \, {}^0\mathbf{g}^T \, {}^0\mathbf{r}_i \tag{12.231}$$

where ${}^0\mathbf{g}$ is the gravitational acceleration vector expressed in the base frame.

The Lagrangean of the manipulator is:

$$\begin{aligned}
\mathcal{L} &= K - V = \frac{1}{2}\dot{\mathbf{q}}_i^T \, D \, \dot{\mathbf{q}}_i + \sum_{i=1}^{n} m_i \, {}^0\mathbf{g}^T \, {}^0\mathbf{r}_i \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} \, \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} m_i \, {}^0\mathbf{g}^T \, {}^0\mathbf{r}_i
\end{aligned} \tag{12.232}$$

Based on the Lagrangean $\mathcal{L}$ we can find

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_i} &= \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} D_{jk} \, \dot{q}_j \dot{q}_k\right) + \sum_{j=1}^{n} m_j \, {}^0\mathbf{g}^T \, \frac{\partial \, {}^0\mathbf{r}_j}{\partial q_i} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial D_{jk}}{\partial q_i} \, \dot{q}_j \dot{q}_k + \sum_{j=1}^{n} m_j \, {}^0\mathbf{g}^T \, \mathbf{J}_{Dj}^{(i)}
\end{aligned} \tag{12.233}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{j=1}^{n} D_{ij} \, \dot{q}_j \tag{12.234}$$

and

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} &= \sum_{j=1}^{n} D_{ij} \, \ddot{q}_j + \sum_{j=1}^{n} \frac{dD_{ij}}{dt} \, \dot{q}_j \\
&= \sum_{j=1}^{n} D_{ij} \, \ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial D_{ij}}{\partial q_k} \, \dot{q}_k \, \dot{q}_j .
\end{aligned} \tag{12.235}$$

The generalized force of the Lagrange equations are

$$Q_i = M_i + \mathbf{J}^T \, \mathbf{F}_e \tag{12.236}$$

where M_i is the ith actuator force at joint i, and $\mathbf{F}_e = \begin{bmatrix} -\mathbf{F}_{en}^T & -\mathbf{M}_{en}^T \end{bmatrix}^T$ is the external force system applied on the end-effector.

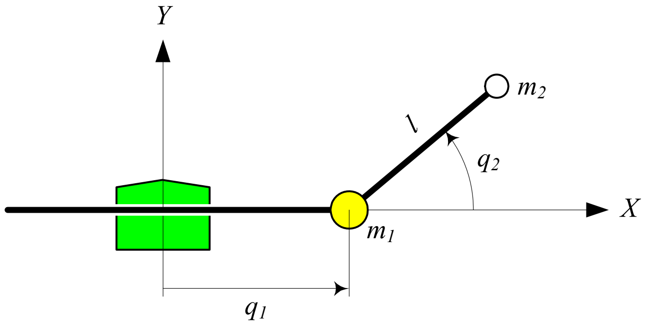

FIGURE 12.12. A prismatic-revolute planar manipulator.

Finally, the Lagrange equations of motion for an n-link manipulator are

$$\sum_{j=1}^{n} D_{ij}(q)\,\ddot{q}_j + H_{ikm}\dot{q}_k\dot{q}_m + G_i = Q_i \tag{12.237}$$

where

$$H_{ijk} = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial D_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{jk}}{\partial q_i}\right) \tag{12.238}$$

$$G_i = \sum_{j=1}^{n} m_j \mathbf{g}^T\,\mathbf{J}_{Dj}^{(i)}. \tag{12.239}$$

We can show the equations of motion for a manipulator in a more concise form to simplify matrix calculations.

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{Q} \tag{12.240}$$

The term $\mathbf{G}(\mathbf{q})$ is called the *gravitational force vector* and the term $\mathbf{H}(\mathbf{q},\dot{\mathbf{q}})$ is called the *velocity coupling vector*. The velocity coupling vector may sometimes be written in the form

$$\mathbf{H}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}. \tag{12.241}$$

■

Example 334 *A prismatic-revolute planar manipulator.*

Figure 12.12 illustrates a planar manipulator with massless links and two massive points m_1 and m_2. To determine the equations of motion, we begin *with calculating the kinetic energy.*

$$\begin{aligned} K_1 &= \frac{1}{2} m_1 \dot{q}_1^2 & (12.242) \\ K_2 &= \frac{1}{2} m_2 \dot{X}_2^2 + \frac{1}{2} m_2 \dot{Y}_2^2 \\ &= \frac{1}{2} m_2 \left(\frac{d}{dt} \left(q_1 + l \cos q_2 \right) \right)^2 + \frac{1}{2} m_2 \left(\frac{d}{dt} \left(l \sin q_2 \right) \right)^2 \\ &= \frac{1}{2} m_2 \left(\dot{q}_1 - l \dot{q}_2 \sin q_2 \right)^2 + \frac{1}{2} m_2 \left(l \dot{q}_2 \cos q_2 \right) \\ &= \frac{1}{2} m_2 \left(\dot{q}_1^2 + l^2 \dot{q}_2^2 - 2 l \dot{q}_1 \dot{q}_2 \sin q_2 \right) & (12.243) \end{aligned}$$

The potential energy of the manipulator is:

$$V = m_2 g Y_2 = m_2 g l \sin q_2 \tag{12.244}$$

Therefore, the Lagrangean is:

$$\begin{aligned} \mathcal{L} &= K - V = K_1 + K_2 - V \\ &= \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \left(\dot{q}_1^2 + l^2 \dot{q}_2^2 - 2 l \dot{q}_1 \dot{q}_2 \sin q_2 \right) - m_2 g l \sin q_2 & (12.245) \end{aligned}$$

Applying the Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \qquad i = 1, 2 \tag{12.246}$$

provides the following equations of motion.

$$\begin{aligned} \left(m_1 + m_2 \right) \ddot{q}_1 - m_2 l \ddot{q}_2 \sin q_2 - m_2 l \dot{q}_2^2 \cos q_2 &= Q_1 & (12.247) \\ m_2 l^2 \ddot{q}_1 - m_2 l \ddot{q}_1 \sin q_2 + m_2 g l \cos q_2 &= Q_2 & (12.248) \end{aligned}$$

We can rearrange these equations to the form of (12.217)

$$\mathbf{D}(\mathbf{q}) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \mathbf{G}(\mathbf{q}) = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{12.249}$$

where,

$$\mathbf{D}(\mathbf{q}) = \begin{bmatrix} m_1 + m_2 & -m_2 l \sin q_2 \\ m_2 l^2 & -m_2 l \sin q_2 \end{bmatrix} \tag{12.250}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_2 l \dot{q}_2 \cos q_2 \\ 0 & 0 \end{bmatrix} \tag{12.251}$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} 0 \\ m_2 g l \cos q_2 \end{bmatrix}. \tag{12.252}$$

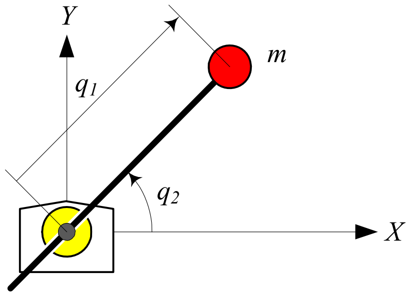

FIGURE 12.13.

Example 335 *A planar polar manipulator.*

Figure 12.13 illustrates a planar polar manipulator with massless link and a massive point m.

The kinetic energy of the manipulator is:

$$\begin{aligned} K &= \frac{1}{2}m_2\dot{X}_2^2 + \frac{1}{2}m_2\dot{Y}_2^2 \\ &= \frac{1}{2}m\left(\frac{d}{dt}\left(q_1\cos q_2\right)\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}\left(q_1\sin q_2\right)\right)^2 \\ &= \frac{1}{2}m\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2\right) \end{aligned} \tag{12.253}$$

The potential energy of the manipulator is:

$$V = mgY_2 = mgq_1\sin q_2 \tag{12.254}$$

and therefore, the Lagrangean of the manipulator is:

$$\mathcal{L} = K - V = \frac{1}{2}m\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2\right) - mgq_1\sin q_2 \tag{12.255}$$

Applying the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = Q_i \qquad i = 1,2 \tag{12.256}$$

provides the following equations of motion.

$$m\ddot{q}_1 - mq_1\dot{q}_2^2 + mg\sin q_2 = Q_1 \tag{12.257}$$

$$mq_1^2\ddot{q}_2 + 2mq_1\dot{q}_1\dot{q}_2 + mgq_1\cos q_2 = Q_2 \tag{12.258}$$

Let us rearrange these equations to the matrix form of (12.217).

$$\mathbf{D}(\mathbf{q})\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \mathbf{G}(\mathbf{q}) = \begin{bmatrix}Q_1\\ Q_2\end{bmatrix} \tag{12.259}$$

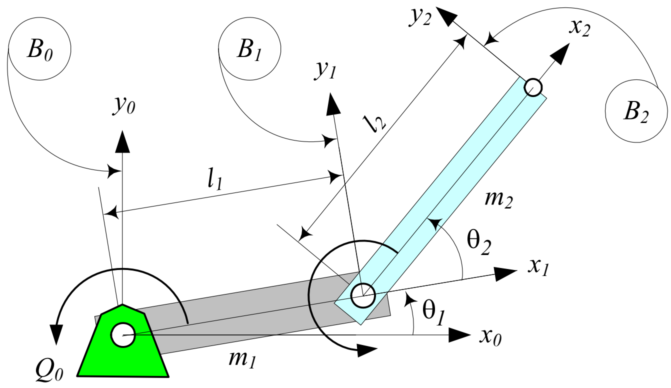

FIGURE 12.14. A $2R$ planar manipulator with massive links.

$$\mathbf{D}(\mathbf{q}) = \begin{bmatrix} m & 0 \\ 0 & mq_1^2 \end{bmatrix} \tag{12.260}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -mq_1\dot{q}_2 \\ mq_1\dot{q}_2 & mq_1\dot{q}_1 \end{bmatrix} \tag{12.261}$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} mg\sin q_2 \\ mgq_1\cos q_2 \end{bmatrix} \tag{12.262}$$

Example 336 *Lagrange equation for $2R$ manipulators with massive arms.*

A $2R$ planar manipulator is shown in Figure 12.14. Its homogeneous transformation matrices are given in Equations (5.29) and (5.30).

$${}^0R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.263}$$

$${}^1R_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.264}$$

Assuming that the links are made of homogeneous material in a bar shape, the position vectors of the mass center C_i are:

$${}^i\mathbf{r}_i = \begin{bmatrix} -l_i/2 \\ 0 \\ 0 \end{bmatrix} \qquad i = 1, 2 \tag{12.265}$$

and the inertia matrices are:

$${}^iI_i = \frac{1}{12}m_i l_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.266}$$

Therefore,

$$\begin{aligned}
{}^0I_1 &= {}^0R_1\ {}^1I_1\ {}^0R_1^T \\
&= \frac{1}{12}m_1 l_1^2 \begin{bmatrix} \sin^2\theta_1 & -\cos\theta_1\sin\theta_1 & 0 \\ -\cos\theta_1\sin\theta_1 & \cos^2\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{12.267}$$

$$\begin{aligned}
{}^0I_2 &= {}^0R_2\ {}^2I_2\ {}^0R_2^T \\
&= \frac{1}{12}m_2 l_2^2 \begin{bmatrix} \sin^2\theta_{12} & -\cos\theta_{12}\sin\theta_{12} & 0 \\ -\cos\theta_{12}\sin\theta_{12} & \cos^2\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{12.268}$$

The gravity is assumed to be in $-\hat{\jmath}_0$ direction

$$\mathbf{g} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \tag{12.269}$$

and the link Jacobian matrices are

$$\mathbf{J}_{D1} = \begin{bmatrix} -\frac{1}{2}l_1\sin\theta_1 & 0 \\ \frac{1}{2}l_1\cos\theta_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{12.270}$$

$$\mathbf{J}_{R1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{12.271}$$

$$\mathbf{J}_{D2} = \begin{bmatrix} -l_1\sin\theta_1 - \frac{1}{2}l_2\sin\theta_{12} & -\frac{1}{2}l_2\sin\theta_{12} \\ l_1\cos\theta_1 + \frac{1}{2}l_2\cos\theta_{12} & \frac{1}{2}l_2\cos\theta_{12} \\ 0 & 0 \end{bmatrix} \tag{12.272}$$

$$\mathbf{J}_{R2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{12.273}$$

We can calculate the manipulator inertia matrix by substituting ${}^0I_i$, $\mathbf{J}_{Di}$, and $\mathbf{J}_{Ri}$ in Equation (12.229)

$$\begin{aligned}
D &= \sum_{i=1}^{2}\left(\mathbf{J}_{Di}^T\ m_i\ \mathbf{J}_{Di} + \frac{1}{2}\mathbf{J}_{Ri}^T\ {}^0I_i\ \mathbf{J}_{Ri}\right) \qquad (12.274) \\
&= \mathbf{J}_{D1}^T\ m_1\ \mathbf{J}_{D1} + \frac{1}{2}\mathbf{J}_{R1}^T\ {}^0I_1\ \mathbf{J}_{R1} + \mathbf{J}_{D2}^T\ m_2\ \mathbf{J}_{D2} + \frac{1}{2}\mathbf{J}_{R2}^T\ {}^0I_2\ \mathbf{J}_{R2} \\
&= \begin{bmatrix} \frac{1}{3}m_1l_1^2 + m_2\left(l_1^2 + l_1l_2c\theta_2 + \frac{1}{3}l_2^2\right) & m_2\left(\frac{1}{2}l_1l_2c\theta_2 + \frac{1}{3}l_2^2\right) \\ m_2\left(\frac{1}{2}l_1l_2c\theta_2 + \frac{1}{3}l_2^2\right) & \frac{1}{3}m_2l_2^2 \end{bmatrix}
\end{aligned}$$

The velocity coupling vector $\mathbf{H}$ *has two elements that are*

$$\begin{aligned} H_1 &= \sum_{j=1}^{1}\sum_{k=1}^{1}\left(\frac{\partial D_{1j}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{jk}}{\partial q_1}\right)\dot{q}_j\dot{q}_k \\ &= -m_2 l_1 l_2\left(\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\right)\dot{\theta}_2\sin\theta_2 \end{aligned} \tag{12.275}$$

$$\begin{aligned} H_2 &= \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial D_{2j}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{jk}}{\partial q_2}\right)\dot{q}_j\dot{q}_k \\ &= \frac{1}{2}m_2 l_1 l_2\dot{\theta}_1^2\sin\theta_2. \end{aligned} \tag{12.276}$$

The elements of the gravitational force vector $\mathbf{G}$ *are:*

$$G_1 = \frac{1}{2}m_1 g l_1\cos\theta_1 + m_2 g l_1\cos\theta_1 + \frac{1}{2}m_2 g l_2\cos\theta_{12} \tag{12.277}$$

$$G_2 = \frac{1}{2}m_2 g l_2\cos\theta_{12} \tag{12.278}$$

Now we can assemble the equations of motion for the $2R$ *planar manipulator. Assuming no external force on the end-effector, the equations of motion are*

$$\begin{aligned} Q_1 &= \left(\frac{1}{3}m_1 l_1^2 + m_2\left(l_1^2 + l_1 l_2 c\theta_2 + \frac{1}{3}l_2^2\right)\right)\ddot{\theta}_1 \\ &\quad + m_2 l_2\left(\frac{1}{2}l_1 c\theta_2 + \frac{1}{3}l_2\right)\ddot{\theta}_2 - m_2 l_1 l_2\left(\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\right)\dot{\theta}_2\sin\theta_2 \\ &\quad + \left(\frac{1}{2}m_1 + m_2\right)g l_1\cos\theta_1 + \frac{1}{2}m_2 g l_2\cos\theta_{12} \end{aligned} \tag{12.279}$$

$$\begin{aligned} Q_2 &= m_2\left(\frac{1}{2}l_1 l_2 c\theta_2 + \frac{1}{3}l_2^2\right)\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2\ddot{\theta}_2 \\ &\quad + \frac{1}{2}m_2 l_1 l_2\dot{\theta}_1^2\sin\theta_2 + \frac{1}{2}m_2 g l_2\cos\theta_{12}. \end{aligned} \tag{12.280}$$

Example 337 ★ *Christoffel operator.*

The symbol $\Gamma^i_{j,k}$ *is called the Christoffel symbol or Christoffel operator with the following definition:*

$$\Gamma^i_{j,k} = \frac{1}{2}\left(\frac{\partial D_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{jk}}{\partial q_i}\right) \tag{12.281}$$

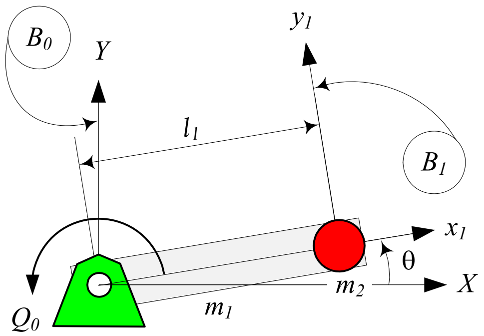

FIGURE 12.15. A uniform beam with a hanging weight m_2 at the tip point.

The velocity coupling vector H_{ijk} is a Christoffel symbol.

$$
\begin{aligned}
H_{ijk} &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial D_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial D_{jk}}{\partial q_i}\right) \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial D_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{jk}}{\partial q_i}\right)
\end{aligned}
\tag{12.282}
$$

Using Christoffel symbol, we can write the equations of motion of a robot as:

$$
\sum_{j=1}^{n} D_{ij}(q)\,\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma^{i}_{j,k}\,\dot{q}_k\dot{q}_m + G_i = Q_i \tag{12.283}
$$

Example 338 ★ *No gravity and no external force.*

Assume there is no gravity and there is no external force applied on the end-effector of a robot. In these conditions, the Lagrangean of the manipulator simplifies to

$$
\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}\,\dot{q}_i\dot{q}_j \tag{12.284}
$$

and the equations of motion reduce to

$$
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}\left(\ddot{q}_i + \Gamma^{j}_{l,m}\dot{q}_l\dot{q}_m\right). \tag{12.285}
$$

Example 339 *Lagrange equation of a one link manipulator.*

To show the advantage and simplicity of the Lagrange method when compared to Newton-Euler method, let us consider derive the equation of motion of the uniform beam of Figure 12.15 with a mass m_2 at the tip point. This is the same system of Figure 12.6(a).

The beam is uniform with a mass center at ${}^0\mathbf{r}_1$ *while the tip mass is at* ${}^0\mathbf{d}_1$*, both in* B_0*.*

$$
{}^0\mathbf{r}_1 = {}^0R_1\,{}^1\mathbf{r}_1 = \begin{bmatrix} \frac{l}{2}\cos\theta \\ \frac{l}{2}\sin\theta \\ 0 \end{bmatrix} \tag{12.286}
$$

$$
{}^0\mathbf{d}_1 = {}^0R_1\,{}^1\mathbf{d}_1 = \begin{bmatrix} l\cos\theta \\ l\sin\theta \\ 0 \end{bmatrix} \tag{12.287}
$$

$$
{}^0R_1 = R_{Z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.288}
$$

The angular velocity of the beam is:

$$
{}_0\boldsymbol{\omega}_1 = \dot{\theta}\,\hat{K} \tag{12.289}
$$

and therefore, the velocity of C *and* m_2 *are:*

$$
{}^0\mathbf{v}_1 = {}_0\boldsymbol{\omega}_1 \times {}^0\mathbf{r}_1 = \begin{bmatrix} -\frac{l}{2}\dot{\theta}\sin\theta \\ \frac{l}{2}\dot{\theta}\cos\theta \\ 0 \end{bmatrix} \tag{12.290}
$$

$$
{}^0\dot{\mathbf{d}}_1 = {}_0\boldsymbol{\omega}_1 \times {}^0\mathbf{d}_1 = \begin{bmatrix} -l\dot{\theta}\sin\theta \\ l\dot{\theta}\cos\theta \\ 0 \end{bmatrix} \tag{12.291}
$$

The kinetic energy of the manipulator is:

$$
\begin{aligned}
K_2 &= \frac{1}{2}m_2\,{}^0\dot{\mathbf{d}}_1\cdot{}^0\dot{\mathbf{d}}_1 + \frac{1}{2}m_1\,{}^0\mathbf{v}_1\cdot{}^0\mathbf{v}_1 + \frac{1}{2}\,{}_0\boldsymbol{\omega}_1^T\,{}^0I_1\,{}_0\boldsymbol{\omega}_1 \\
&= \frac{1}{8}l^2\dot{\theta}^2\left(m_1+4m_2\right) + \frac{1}{2}I_z\dot{\theta}^2
\end{aligned} \tag{12.292}
$$

$$
\begin{aligned}
{}^0I_1 &= R_{Z,\theta}\,{}^1I_1\,R_{Z,\theta}^T = {}^0R_1 \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} {}^0R_1^T \\
&= \begin{bmatrix} I_x\cos^2\theta + I_y\sin^2\theta & (I_x - I_y)\cos\theta\sin\theta & 0 \\ (I_x - I_y)\cos\theta\sin\theta & I_y\cos^2\theta + I_x\sin^2\theta & 0 \\ 0 & 0 & I_z \end{bmatrix}
\end{aligned} \tag{12.293}
$$

The potential energy of the manipulator is:

$$\begin{aligned} V &= m_1 g Y_1 + m_2 g Y_2 = m_1 g r_Y + m_2 g d_Y \\ &= m_1 g \frac{l}{2} \sin\theta + m_2 g l \sin\theta \end{aligned} \tag{12.294}$$

and therefore, the Lagrangean of the manipulator is:

$$\begin{aligned} \mathcal{L} &= K - V = \frac{1}{8} l^2 \dot{\theta}^2 (m_1 + 4m_2) + \frac{1}{2} I_z \dot{\theta}^2 \\ &\quad - m_1 g \frac{l}{2} \sin\theta - m_2 g l \sin\theta \end{aligned} \tag{12.295}$$

Applying the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_0 \tag{12.296}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{4} l^2 (m_1 + 4m_2)\dot{\theta} + I_z \dot{\theta} \tag{12.297}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \left(\frac{1}{4} m_1 l^2 + m_2 l^2 + I_z\right)\ddot{\theta} \tag{12.298}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_1 g \frac{l}{2} \cos\theta - m_2 g l \cos\theta \tag{12.299}$$

determines the equation of motion.

$$Q_0 = \left(\frac{1}{4} m_1 l^2 + m_2 l^2 + I_z\right)\ddot{\theta} + \left(\frac{1}{2} m_1 + m_2\right) g l \cos\theta \tag{12.300}$$

It is the same equation as (12.69).

Example 340 *General model of $2R$ planar manipulator.*

Consider a general $2R$ manipulator with massive arms and joints while carrying a payload m_0 as is shown in Figure 12.16.

The first motor that drives link (1), is on the ground. The second motor with mass m_{12} drives link (2) and is mounted on link (1). The mass of first and second links are m_{11} and m_{21} respectively.

In a general case, the global position vectors of the mass centers C_i and massive joints are:

$${}^0\mathbf{r}_1 = {}^0R_1\, {}^1\mathbf{r}_1 = R_{Z,\theta_1}\, c_1\, {}^1\hat{\imath}_1 = \begin{bmatrix} c_1 \cos\theta_1 \\ c_1 \sin\theta_1 \\ 0 \end{bmatrix} \tag{12.301}$$

$$\begin{aligned} {}^0\mathbf{r}_2 &= {}^0\mathbf{d}_1 + {}^0R_2\, {}^2\mathbf{r}_2 = {}^0\mathbf{d}_1 + R_{Z,\theta_1} R_{Z,\theta_2}\, c_2\, {}^2\hat{\imath}_2 \\ &= \begin{bmatrix} l_1 \cos\theta_1 + c_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + c_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \end{aligned} \tag{12.302}$$

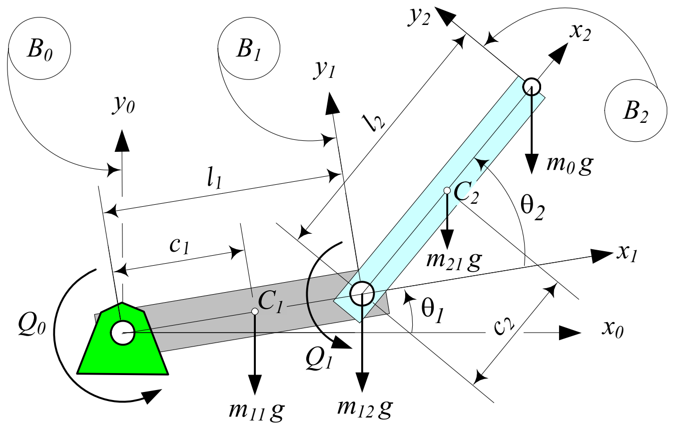

FIGURE 12.16. A $2R$ manipulator with massive arms and a carrying payload m_0.

$$
\begin{aligned}
{}^0\mathbf{d}_1 &= {}^0R_1\,{}^1\mathbf{r}_1 = R_{Z,\theta_1}\,l_1\,{}^1\hat{\imath}_1 = \begin{bmatrix} l_1\cos\theta_1 \\ l_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.303) \\
{}^0\mathbf{d}_2 &= {}^0\mathbf{d}_1 + {}^0R_2\,{}^2\mathbf{d}_2 = \begin{bmatrix} l_2\cos(\theta_1+\theta_2) + l_1\cos\theta_1 \\ l_2\sin(\theta_1+\theta_2) + l_1\sin\theta_1 \\ 0 \end{bmatrix} && (12.304)
\end{aligned}
$$

where,

$$
\begin{aligned}
{}^0R_1 &= R_{Z,\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (12.305) \\
{}^1R_2 &= R_{Z,\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (12.306)
\end{aligned}
$$

$$
{}^0R_2 = {}^0R_1\,{}^1R_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (12.307)
$$

The links' angular velocity are:

$$
\begin{aligned}
{}_0\boldsymbol{\omega}_1 &= \dot{\theta}_1\,\hat{K} && (12.308) \\
{}_0\boldsymbol{\omega}_2 &= \left(\dot{\theta}_1 + \dot{\theta}_2\right)\hat{K} && (12.309)
\end{aligned}
$$

The mass moment matrices in the global coordinate frame are:

$$\begin{aligned} {}^0I_1 &= R_{Z,\theta_1}\, {}^1I_1\, R^T_{Z,\theta_1} = {}^0R_1 \begin{bmatrix} I_{x_1} & 0 & 0 \\ 0 & I_{y_1} & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix} {}^0R_1^T \\ &= \begin{bmatrix} I_{x_1}c^2\theta_1 + I_{y_1}s^2\theta_1 & (I_{x_1} - I_{y_1})\, c\theta_1 s\theta_1 & 0 \\ (I_{x_1} - I_{y_1})\, c\theta_1 s\theta_1 & I_{y_1}c^2\theta_1 + I_{x_1}s^2\theta_1 & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix} \end{aligned} \tag{12.310}$$

$$\begin{aligned} {}^0I_2 &= {}^0R_2\, {}^2I_2\, {}^0R_2^T = {}^0R_2 \begin{bmatrix} I_{x_2} & 0 & 0 \\ 0 & I_{y_2} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix} {}^0R_2^T \\ &= \begin{bmatrix} I_{x_2}c^2\theta_{12} + I_{y_2}s^2\theta_{12} & (I_{x_2} - I_{y_2})\, c\theta_{12} s\theta_{12} & 0 \\ (I_{x_2} - I_{y_2})\, c\theta_{12} s\theta_{12} & I_{y_2}c^2\theta_{12} + I_{x_2}s^2\theta_{12} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix} \end{aligned} \tag{12.311}$$

$$\theta_{12} = \theta_1 + \theta_2 \tag{12.312}$$

The velocity of C_i and the masses are:

$${}^0\mathbf{v}_1 = \frac{{}^0d}{dt}\, {}^0\mathbf{r}_1 = \begin{bmatrix} -c_1\dot\theta_1 \sin\theta_1 \\ c_1\dot\theta_1 \cos\theta_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} {}^0\mathbf{v}_2 &= \frac{{}^0d}{dt}\, {}^0\mathbf{r}_2 \\ &= \begin{bmatrix} -l_1\dot\theta_1 \sin\theta_1 - c_2\left(\dot\theta_1 + \dot\theta_2\right) \sin(\theta_1 + \theta_2) \\ l_1\dot\theta_1 \cos\theta_1 + c_2\left(\dot\theta_1 + \dot\theta_2\right) \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \end{aligned} \tag{12.313}$$

$${}^0\dot{\mathbf{d}}_1 = \begin{bmatrix} -l_1\dot\theta_1 \sin\theta_1 \\ l_1\dot\theta_1 \cos\theta_1 \\ 0 \end{bmatrix} \tag{12.314}$$

$${}^0\dot{\mathbf{d}}_2 = \begin{bmatrix} -l_1\dot\theta_1 \sin\theta_1 - l_2\left(\dot\theta_1 + \dot\theta_2\right) \sin(\theta_1 + \theta_2) \\ l_1\dot\theta_1 \cos\theta_1 + l_2\left(\dot\theta_1 + \dot\theta_2\right) \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \tag{12.315}$$

To calculate Lagrangian $\mathcal{L} = K - V$, we determine the energies of the *manipulator. The kinetic energy of the manipulator is:*

$$\begin{aligned} K &= \frac{1}{2}m_{12}\,{}^0\dot{\mathbf{d}}_1 \cdot {}^0\dot{\mathbf{d}}_1 + \frac{1}{2}m_{11}\,{}^0\mathbf{v}_1 \cdot {}^0\mathbf{v}_1 \\ &\quad + \frac{1}{2}m_0\,{}^0\dot{\mathbf{d}}_2 \cdot {}^0\dot{\mathbf{d}}_2 + \frac{1}{2}m_{21}\,{}^0\mathbf{v}_2 \cdot {}^0\mathbf{v}_2 \\ &\quad + \frac{1}{2}\,{}_0\boldsymbol{\omega}_1^T\,{}^0I_1\,{}_0\boldsymbol{\omega}_1 + \frac{1}{2}\,{}_0\boldsymbol{\omega}_2^T\,{}^0I_2\,{}_0\boldsymbol{\omega}_2 \end{aligned} \tag{12.316}$$

which after substituting (12.305)-(12.315) would be:

$$\begin{aligned} K &= \frac{1}{2}\left(m_{11}c_1^2 + m_{12}l_1^2 + I_{z_1}\right)\dot{\theta}_1^2 \\ &\quad + \frac{1}{2}m_{21}\left(-l_1\dot{\theta}_1 \sin\theta_1 - c_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\left(\theta_1 + \theta_2\right)\right)^2 \\ &\quad + \frac{1}{2}m_{21}\left(l_1\dot{\theta}_1 \cos\theta_1 + c_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos\left(\theta_1 + \theta_2\right)\right)^2 \\ &\quad + \frac{1}{2}m_0\left(-l_1\dot{\theta}_1 \sin\theta_1 - l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\left(\theta_1 + \theta_2\right)\right)^2 \\ &\quad + \frac{1}{2}m_0\left(l_1\dot{\theta}_1 \cos\theta_1 + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos\left(\theta_1 + \theta_2\right)\right)^2 \\ &\quad + \frac{1}{2}I_{z_2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \end{aligned} \tag{12.317}$$

The potential energy of the manipulator is:

$$\begin{aligned} V &= m_{11}gc_1 \sin\theta_1 + m_{12}gl_1 \sin\theta_1 \\ &\quad + m_{21}g\left(l_1 \sin\theta_1 + c_2 \sin\left(\theta_1 + \theta_2\right)\right) \\ &\quad + m_0 g\left(l_1 \sin\theta_1 + l_2 \sin\left(\theta_1 + \theta_2\right)\right) \end{aligned} \tag{12.318}$$

Applying the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = Q_0 \tag{12.319}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = Q_1 \tag{12.320}$$

determines the general equations of motion.

$$\begin{aligned} &\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} Q_o \\ Q_1 \end{bmatrix} \end{aligned} \tag{12.321}$$

$$\begin{aligned} D_{11} &= 2l_1\left(m_{21}c_2 + m_0 l_2\right)\cos\theta_2 + I_{z_1} + I_{z_2} \\ &\quad + m_{11}c_1^2 + m_{12}l_1^2 + m_{21}\left(c_2^2 + l_1^2\right) + m_0\left(l_1^2 + l_2^2\right) \end{aligned} \tag{12.322}$$

$$D_{12} = l_1\left(m_{21}c_2 + m_0 l_2\right)\cos\theta_2 + I_{z_2} + m_0 l_2^2 + m_{21}c_2^2 \tag{12.323}$$

$$D_{21} = l_1 (m_{21}c_2 + m_0 l_2) \cos\theta_2 + I_{z_2} + m_{21}c_2^2 + m_0 l_2^2 \tag{12.324}$$
$$D_{22} = I_{z_2} + m_{21}c_2^2 + m_0 l_2^2 \tag{12.325}$$

$$C_{11} = -l_1 (m_{21}c_2 + m_0 l_2) \dot\theta_2 \sin\theta_2 \tag{12.326}$$
$$C_{12} = -l_1 (m_{21}c_2 + m_0 l_2) \left(\dot\theta_1 + \dot\theta_2\right) \sin\theta_2 \tag{12.327}$$

$$C_{21} = l_1 (m_{21}c_2 + m_0 l_2) \dot\theta_1 \sin\theta_2 \tag{12.328}$$
$$C_{22} = 0 \tag{12.329}$$

$$\begin{aligned} G_1 &= ((m_{21} + m_{12} + m_0) l_1 + m_{11}c_1) \cos\theta_1 \\ &\quad + (m_{21}c_2 + m_0 l_2) \cos(\theta_1 + \theta_2) \end{aligned} \tag{12.330}$$
$$G_2 = (m_{21}c_2 + m_0 l_2) \cos(\theta_1 + \theta_2) \tag{12.331}$$

Example 341 *Special cases of* $2R$ *planar manipulator.*

Figure 12.16 illustrates a general $2R$ *manipulator with massive arms and joints and a carrying payload* m_0*. The second motor has a mass* m_{12} *and is mounted on link* (1)*. The mass of first and second links are* m_{11} *and* m_{21} *respectively and their mass centers are at* c_1 *and* c_2*. The general equations of motion for the* $2R$ *planar manipulator are given in Equations (12.321).*

$$\begin{aligned} &\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} Q_o \\ Q_1 \end{bmatrix} \end{aligned} \tag{12.332}$$

In modeling a special $2R$ *planar manipulator, we may use the equations for simpler models as classified below.*

1. *Massless arms.*

   When the mass of the links of the manipulator are much less than the masses of its motors and the carrying load, we may use a massless arm model. The equations of motion for a massless arms $2R$ *planar manipulator are calculated by substituting* $m_{11} = 0$, $m_{21} = 0$ *in Equations (12.321).*

$$\begin{aligned} D_{11} &= 2m_0 l_1 l_2 \cos\theta_2 + I_{z_1} + I_{z_2} \\ &\quad + m_{12} l_1^2 + m_0 \left(l_1^2 + l_2^2\right) \end{aligned} \tag{12.333}$$
$$D_{12} = m_0 l_1 l_2 \cos\theta_2 + I_{z_2} + m_0 l_2^2 \tag{12.334}$$

$$D_{21} = m_0 l_1 l_2 \cos\theta_2 + I_{z_2} + m_0 l_2^2 \tag{12.335}$$
$$D_{22} = I_{z_2} + m_0 l_2^2 \tag{12.336}$$

$$C_{11} = -m_0 l_1 l_2 \dot{\theta}_2 \sin\theta_2 \quad (12.337)$$
$$C_{12} = -m_0 l_1 l_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \sin\theta_2 \quad (12.338)$$

$$C_{21} = m_0 l_1 l_2 \dot{\theta}_1 \sin\theta_2 \quad (12.339)$$
$$C_{22} = 0 \quad (12.340)$$

$$G_1 = (m_{12} + m_0)\, l_1 \cos\theta_1 + m_0 l_2 \cos(\theta_1 + \theta_2) \quad (12.341)$$
$$G_2 = m_0 l_2 \cos(\theta_1 + \theta_2) \quad (12.342)$$

2. *Massless joints.*

When the mass of the links of the manipulator are much more than the masses of its motors and the carrying load, we may use a massless joints model. The equations of motion for a massless joints 2R planar manipulator are calculated by substituting $m_{12} = 0$, $m_0 = 0$ in Equations (12.321).

$$\begin{aligned} D_{11} &= 2m_{21} l_1 c_2 \cos\theta_2 + I_{z_1} + I_{z_2} \\ &\quad + m_{11} c_1^2 + m_{21}\left(c_2^2 + l_1^2\right) \end{aligned} \quad (12.343)$$
$$D_{12} = m_{21} c_2 (l_1 \cos\theta_2 + c_2) + I_{z_2} \quad (12.344)$$

$$D_{21} = m_{21} c_2 (l_1 \cos\theta_2 + c_2) + I_{z_2} \quad (12.345)$$
$$D_{22} = I_{z_2} + m_{21} c_2^2 \quad (12.346)$$

$$C_{11} = -m_{21} l_1 c_2 \dot{\theta}_2 \sin\theta_2 \quad (12.347)$$
$$C_{12} = -m_{21} l_1 c_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \sin\theta_2 \quad (12.348)$$

$$C_{21} = m_{21} l_1 c_2 \dot{\theta}_1 \sin\theta_2 \quad (12.349)$$
$$C_{22} = 0 \quad (12.350)$$

$$G_1 = (m_{21} l_1 + m_{11} c_1) \cos\theta_1 + m_{21} c_2 \cos(\theta_1 + \theta_2) \quad (12.351)$$
$$G_2 = m_{21} c_2 \cos(\theta_1 + \theta_2) \quad (12.352)$$

If the links of the manipulator are uniform and symmetric, then $c_1 = l_1/2$, $c_2 = l_2/2$, and the equations are simplified to:

$$\begin{aligned} D_{11} &= m_{21} l_1 l_2 \cos\theta_2 + I_{z_1} + I_{z_2} \\ &\quad + \frac{1}{4} m_{11} l_1^2 + m_{21}\left(\frac{1}{4} l_2^2 + l_1^2\right) \end{aligned} \quad (12.353)$$
$$D_{12} = \frac{1}{2} m_{21} l_2 \left(l_1 + \frac{1}{2} l_2\right) \cos\theta_2 + I_{z_2} \quad (12.354)$$

$$D_{21} = \frac{1}{2}m_{21}l_1l_2\cos\theta_2 + I_{z_2} + \frac{1}{4}m_{21}l_2^2 \tag{12.355}$$
$$D_{22} = I_{z_2} + \frac{1}{4}m_{21}l_2^2 \tag{12.356}$$

$$C_{11} = -\frac{1}{2}m_{21}l_1l_2\dot{\theta}_2\sin\theta_2 \tag{12.357}$$
$$C_{12} = -\frac{1}{2}m_{21}l_1l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\theta_2 \tag{12.358}$$

$$C_{21} = \frac{1}{4}m_{21}l_1l_2\dot{\theta}_1\sin\theta_2 \tag{12.359}$$
$$C_{22} = 0 \tag{12.360}$$

$$\begin{aligned} G_1 &= \left(m_{21} + \frac{1}{2}m_{11}\right)l_1\cos\theta_1 \\ &\quad + \frac{1}{2}m_{21}l_2\cos(\theta_1+\theta_2) \end{aligned} \tag{12.361}$$
$$G_2 = \frac{1}{2}m_{21}l_2\cos(\theta_1+\theta_2) \tag{12.362}$$

Example 342 ★ *Equations of motion of an articulated manipulator.*

Figure 12.17 illustrates an articulated manipulator with massive links and a massive load at the tip point. Points C_i, $i = 1, 2, 3$ indicate the mass centers of the links with masses m_i, $i = 1, 2, 3$. The the tip point has a mass of m_0. A top view of the manipulator is shown in Figure 12.18.

The link (1) of the manipulator is an $R\vdash R(90)$ with an extra displacement l_1 along z_1. To determine the transformation matrix ${}^0R_1$ we can begin from a coincident configuration of B_1 and B_0 and move B_1 to its current configuration by a sequence of proper rotations and displacements.

$$\begin{aligned} {}^1T_0 &= D_{z_1,l_1}\,R_{x_1,\pi/2}\,R_{z_1,\theta_1} \\ &= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{12.363}$$

$${}^0T_1 = {}^1T_0^{-1} = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & -l_1\sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & l_1\cos\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.364}$$

The second and third links are $R\|R(0)$, $R\vdash R(90)$, and their associated

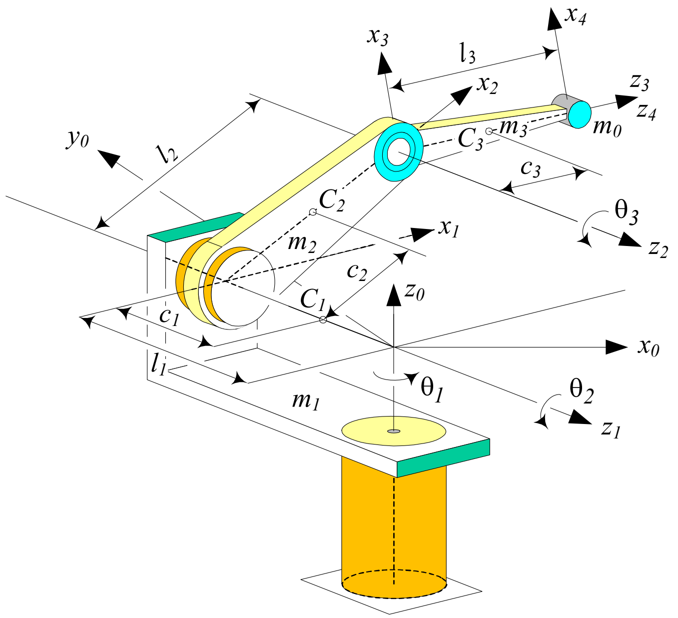

FIGURE 12.17. An articulated manipulator with massive links and a massive load at the tip point.

transformation matrices between coordinate frames are:

$$ {}^{1}T_{2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.365} $$

$$ {}^{2}T_{3} = \begin{bmatrix} \cos\theta_3 & 0 & \sin\theta_3 & 0 \\ \sin\theta_3 & 0 & -\cos\theta_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.366} $$

The global position vectors of the mass centers C_i and joints are:

$$ {}^{0}\mathbf{r}_{1} = {}^{0}T_{1}\,{}^{1}\mathbf{r}_{1} = {}^{0}T_{1}\begin{bmatrix} 0 \\ 0 \\ c_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -(l_1-c_1)\sin\theta_1 \\ (l_1-c_1)\cos\theta_1 \\ 0 \\ 1 \end{bmatrix} \tag{12.367} $$

$$ {}^{0}\mathbf{d}_{1} = {}^{0}T_{1}\,{}^{1}\mathbf{d}_{1} = {}^{0}T_{1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 \\ l_1\cos\theta_1 \\ 0 \\ 1 \end{bmatrix} \tag{12.368} $$

$$ {}^{0}\mathbf{r}_2 = {}^{0}\mathbf{d}_1 + {}^{0}T_2\, {}^{2}\mathbf{r}_2 = \begin{bmatrix} -2l_1\sin\theta_1 + (c_2+l_2)\cos\theta_1\cos\theta_2 \\ 2l_1\cos\theta_1 + (c_2+l_2)\cos\theta_2\sin\theta_1 \\ (c_2+l_2)\sin\theta_2 \\ 2 \end{bmatrix} \tag{12.369} $$

$$ {}^{0}\mathbf{d}_2 = {}^{0}\mathbf{d}_1 + {}^{0}T_2\, {}^{2}\mathbf{d}_2 = \begin{bmatrix} 2l_2\cos\theta_1\cos\theta_2 - 2l_1\sin\theta_1 \\ 2l_1\cos\theta_1 + 2l_2\cos\theta_2\sin\theta_1 \\ 2l_2\sin\theta_2 \\ 2 \end{bmatrix} \tag{12.370} $$

$$ \begin{aligned} {}^{0}\mathbf{r}_3 &= {}^{0}\mathbf{d}_2 + {}^{0}T_3\, {}^{3}\mathbf{r}_3 \\ &= \begin{bmatrix} c_3\cos\theta_1\left(\sin(\theta_2+\theta_3) + 3l_2\cos\theta_2\right) - 3l_1\sin\theta_1 \\ c_3\sin\theta_1\left(\sin(\theta_2+\theta_3) + 3l_2\cos\theta_2\right) + 3l_1\cos\theta_1 \\ 3l_2\sin\theta_2 - c_3\cos(\theta_2+\theta_3) \\ 3 \end{bmatrix} \end{aligned} \tag{12.371} $$

$$ \begin{aligned} {}^{0}\mathbf{d}_3 &= {}^{0}\mathbf{d}_2 + {}^{0}T_3\, {}^{3}\mathbf{d}_3 \\ &= \begin{bmatrix} l_3\cos\theta_1\left(\sin(\theta_2+\theta_3) + 3l_2\cos\theta_2\right) - 3l_1\sin\theta_1 \\ l_3\sin\theta_1\left(\sin(\theta_2+\theta_3) + 3l_2\cos\theta_2\right) + 3l_1\cos\theta_1 \\ 3l_2\sin\theta_2 - l_3\cos(\theta_2+\theta_3) \\ 3 \end{bmatrix} \end{aligned} \tag{12.372} $$

The links' angular velocity are:

$$ {}_{0}\boldsymbol{\omega}_1 = \dot\theta_1\,\hat{k}_0 \qquad {}_{1}\boldsymbol{\omega}_2 = \dot\theta_2\,\hat{k}_1 \qquad {}_{2}\boldsymbol{\omega}_3 = \dot\theta_3\,\hat{k}_2 \tag{12.373} $$

$$ \begin{aligned} {}_{0}\tilde\omega_2 &= {}_{0}\tilde\omega_1 + {}_{1}^{0}\tilde\omega_2 = {}_{0}\tilde\omega_1 + {}^{0}R_1\, {}_{1}\tilde\omega_2\, {}^{0}R_1^T \\ &= \begin{bmatrix} 0 & -\dot\theta_1 & -\dot\theta_2\cos\theta_1 \\ \dot\theta_1 & 0 & -\dot\theta_2\sin\theta_1 \\ \dot\theta_2\cos\theta_1 & \dot\theta_2\sin\theta_1 & 0 \end{bmatrix} \end{aligned} \tag{12.374} $$

$$ \begin{aligned} {}_{0}\tilde\omega_3 &= {}_{0}\tilde\omega_2 + {}_{2}^{0}\tilde\omega_3 = {}_{0}\tilde\omega_2 + {}^{0}R_2\, {}_{2}\tilde\omega_3\, {}^{0}R_2^T \\ &= \begin{bmatrix} 0 & -\dot\theta_1 & -\left(\dot\theta_2+\dot\theta_3\right)\cos\theta_1 \\ \dot\theta_1 & 0 & -\left(\dot\theta_2+\dot\theta_3\right)\sin\theta_1 \\ \left(\dot\theta_2+\dot\theta_3\right)\cos\theta_1 & \left(\dot\theta_2+\dot\theta_3\right)\sin\theta_1 & 0 \end{bmatrix} \end{aligned} \tag{12.375} $$

The mass moment matrices in the global coordinate frame are:

$$ {}^{0}I_1 = {}^{0}R_1 \begin{bmatrix} I_{x_1} & 0 & 0 \\ 0 & I_{y_1} & 0 \\ 0 & 0 & I_{z_1} \end{bmatrix} {}^{0}R_1^T \tag{12.376} $$

$$ = \begin{bmatrix} I_{x_1}\cos^2\theta_1 + I_{z_1}\sin^2\theta_1 & (I_{x_1} - I_{z_1})\cos\theta_1\sin\theta_1 & 0 \\ (I_{x_1} - I_{z_1})\cos\theta_1\sin\theta_1 & I_{z_1}\cos^2\theta_1 + I_{x_1}\sin^2\theta_1 & 0 \\ 0 & 0 & I_{y_1} \end{bmatrix} $$

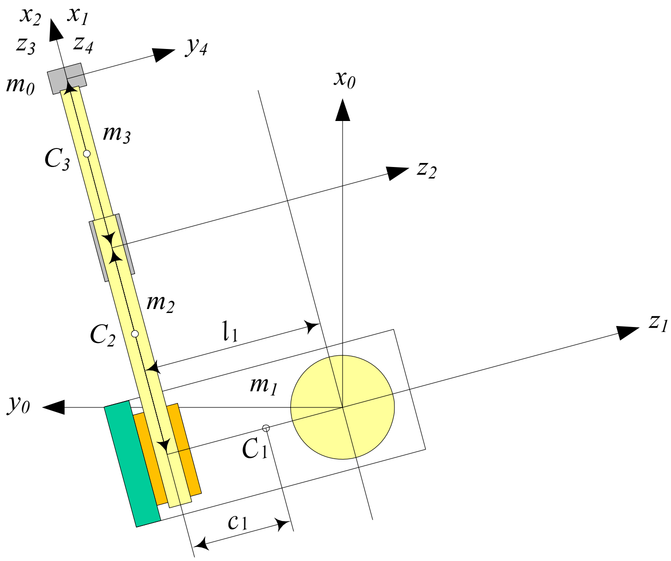

FIGURE 12.18. A top view of an articulated manipulator with massive links and a massive load at the tip point.

$$
{}^0I_2 = {}^0R_2\ {}^2I_2\ {}^0R_2^T = {}^0R_2 \begin{bmatrix} I_{x_2} & 0 & 0 \\ 0 & I_{y_2} & 0 \\ 0 & 0 & I_{z_2} \end{bmatrix} {}^0R_2^T \tag{12.377}
$$

$$
{}^0I_3 = {}^0R_3\ {}^3I_3\ {}^0R_3^T = {}^0R_3 \begin{bmatrix} I_{x_3} & 0 & 0 \\ 0 & I_{y_3} & 0 \\ 0 & 0 & I_{z_3} \end{bmatrix} {}^0R_3^T \tag{12.378}
$$

The velocity of C_i and the joints are:

$$
{}^0\mathbf{v}_1 = \frac{{}^0d}{dt}\,{}^0\mathbf{r}_1 = \begin{bmatrix} -\left(l_1 - c_1\right)\dot{\theta}_1 \cos\theta_1 \\ -\left(l_1 - c_1\right)\dot{\theta}_1 \sin\theta_1 \\ 0 \end{bmatrix} \tag{12.379}
$$

$$
{}^0\mathbf{v}_2 = \frac{{}^0d}{dt}\,{}^0\mathbf{r}_2 \tag{12.380}
$$

$$
{}^0\mathbf{v}_3 = \frac{{}^0d}{dt}\,{}^0\mathbf{r}_3 \tag{12.381}
$$

$$
{}^0\dot{\mathbf{d}}_1 = \begin{bmatrix} -l_1\dot{\theta}_1 \cos\theta_1 \\ -l_1\dot{\theta}_1 \sin\theta_1 \\ 0 \end{bmatrix} \tag{12.382}
$$

$$
\begin{aligned}
{}^{0}\dot{\mathbf{d}}_{2} &= \frac{{}^{0}d}{dt}\,{}^{0}\mathbf{d}_{2} && (12.383)\\
{}^{0}\dot{\mathbf{d}}_{3} &= \frac{{}^{0}d}{dt}\,{}^{0}\mathbf{d}_{3} && (12.384)
\end{aligned}
$$

The kinetic energy of the manipulator is:

$$
\begin{aligned}
K &= \frac{1}{2}m_{1}\,{}^{0}\mathbf{v}_{1}\cdot{}^{0}\mathbf{v}_{1}+\frac{1}{2}m_{21}\,{}^{0}\mathbf{v}_{2}\cdot{}^{0}\mathbf{v}_{2}+\frac{1}{2}m_{3}\,{}^{0}\mathbf{v}_{3}\cdot{}^{0}\mathbf{v}_{3}\\
&\quad+\frac{1}{2}m_{0}\,{}^{0}\dot{\mathbf{d}}_{3}\cdot{}^{0}\dot{\mathbf{d}}_{3}+\frac{1}{2}\,{}_{0}\boldsymbol{\omega}_{1}^{T}\,{}^{0}I_{1}\,{}_{0}\boldsymbol{\omega}_{1}\\
&\quad+\frac{1}{2}\,{}_{0}\boldsymbol{\omega}_{2}^{T}\,{}^{0}I_{2}\,{}_{0}\boldsymbol{\omega}_{2}+\frac{1}{2}\,{}_{0}\boldsymbol{\omega}_{3}^{T}\,{}^{0}I_{3}\,{}_{0}\boldsymbol{\omega}_{3} && (12.385)
\end{aligned}
$$

The potential energy of the manipulator is:

$$
V = m_{2}gr_{2z}+m_{3}gr_{3z}+m_{0}gd_{3z} \tag{12.386}
$$

Using the Lagrangian of the manipulator $\mathcal{L} = K - V$, and applying the Lagrange equation

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_{1}}\right)-\frac{\partial\mathcal{L}}{\partial\theta_{1}} &= Q_{0} && (12.387)\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_{2}}\right)-\frac{\partial\mathcal{L}}{\partial\theta_{2}} &= Q_{1} && (12.388)\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_{3}}\right)-\frac{\partial\mathcal{L}}{\partial\theta_{3}} &= Q_{2} && (12.389)
\end{aligned}
$$

we determines the equations of motion.

$$
\begin{aligned}
&\begin{bmatrix} D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_{1}\\ \ddot{\theta}_{2}\\ \ddot{\theta}_{3}\end{bmatrix}+\begin{bmatrix} C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}\begin{bmatrix}\dot{\theta}_{1}\\ \dot{\theta}_{2}\\ \dot{\theta}_{3}\end{bmatrix}\\
&\qquad+\begin{bmatrix} G_{1}\\ G_{2}\\ G_{3}\end{bmatrix}=\begin{bmatrix} Q_{o}\\ Q_{1}\\ Q_{2}\end{bmatrix} && (12.390)
\end{aligned}
$$

12.4 ★ Lagrange Equations and Link Transformation Matrices

The matrix form of the equations of motion for a robot, based on the Lagrange equations, is

$$
\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}}+\mathbf{H}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{G}(\mathbf{q})=\mathbf{Q} \tag{12.391}
$$

which can also be written in a summation form.

$$\sum_{j=1}^{n} D_{ij}(q)\,\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} H_{ikm}\dot{q}_k\dot{q}_m + G_i = Q_i \tag{12.392}$$

The matrix $\mathbf{D}(\mathbf{q})$ is an $n \times n$ inertial-type symmetric matrix

$$D_{ij} = \sum_{r=\max i,j}^{n} \operatorname{tr}\left(\frac{\partial\,{}^0T_r}{\partial q_i}\,{}^r\bar{I}_r\,\frac{\partial\,{}^0T_r}{\partial q_j}^T\right) \tag{12.393}$$

and H_{ikm} is the velocity coupling term

$$H_{ijk} = \sum_{r=\max i,j,k}^{n} \operatorname{tr}\left(\frac{\partial^2\,{}^0T_r}{\partial q_j \partial q_k}\,{}^r\bar{I}_r\,\frac{\partial\,{}^0T_r}{\partial q_i}^T\right) \tag{12.394}$$

and G_i is the gravitational vector.

$$G_i = -\sum_{r=i}^{n} m_r \mathbf{g}^T\,\frac{\partial\,{}^0T_r}{\partial q_i}\,{}^r\mathbf{r}_r \tag{12.395}$$

Proof. Position vector of a point P of the link (i) at ${}^i\mathbf{r}_P$ in the body coordinate B_i, can be transformed to the base frame by

$${}^0\mathbf{r}_P = {}^0T_i\,{}^i\mathbf{r}_P. \tag{12.396}$$

Therefore, its velocity and square of velocity in the base frame are

$${}^0\dot{\mathbf{r}}_P = \sum_{j=1}^{i} \frac{\partial\,{}^0T_i}{\partial q_j}\,\dot{q}_j\,{}^i\mathbf{r}_P \tag{12.397}$$

and

$$\begin{aligned}
{}^0\dot{\mathbf{r}}_P^2 &= {}^0\dot{\mathbf{r}}_P \cdot {}^0\dot{\mathbf{r}}_P = \operatorname{tr}\left({}^0\dot{\mathbf{r}}_P\,{}^0\dot{\mathbf{r}}_P^T\right) \\
&= \operatorname{tr}\left(\sum_{j=1}^{i} \frac{\partial\,{}^0T_i}{\partial q_j}\,\dot{q}_j\,{}^i\mathbf{r}_P \sum_{k=1}^{i}\left[\frac{\partial\,{}^0T_i}{\partial q_k}\,\dot{q}_k\,{}^i\mathbf{r}_P\right]^T\right) \\
&= \operatorname{tr}\left(\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial\,{}^0T_i}{\partial q_j}\,{}^i\mathbf{r}_P\,{}^i\mathbf{r}_P^T\left[\frac{\partial\,{}^0T_i}{\partial q_k}\right]^T \dot{q}_j\,\dot{q}_k\right).
\end{aligned} \tag{12.398}$$

The kinetic energy of point P having a small mass dm is then equal to

$$\begin{aligned}
dK_P &= \frac{1}{2}\operatorname{tr}\left(\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial\,{}^0T_i}{\partial q_j}\,{}^i\mathbf{r}_P\,{}^i\mathbf{r}_P^T\,\frac{\partial\,{}^0T_i}{\partial q_k}^T\,\dot{q}_j\,\dot{q}_k\right) dm \\
&= \frac{1}{2}\operatorname{tr}\left(\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial\,{}^0T_i}{\partial q_j}\left({}^i\mathbf{r}_P\,dm\,{}^i\mathbf{r}_P^T\right)\frac{\partial\,{}^0T_i}{\partial q_k}^T\,\dot{q}_j\,\dot{q}_k\right)
\end{aligned} \tag{12.399}$$

and the kinetic energy of the link (i) is:

$$\begin{aligned} K_i &= \int_{B_i} dK_P \\ &= \frac{1}{2}\operatorname{tr}\left(\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial\,{}^0T_i}{\partial q_j}\left(\int_{B_i}{}^i\mathbf{r}_P\,{}^i\mathbf{r}_P^T\,dm\right)\frac{\partial\,{}^0T_i}{\partial q_k}^T\dot{q}_j\,\dot{q}_k\right) \end{aligned} \qquad (12.400)$$

The integral in Equation (12.400) is the pseudo inertia matrix (11.143) for the link (i)

$${}^i\bar{I}_i = \int_{B_i}{}^i\mathbf{r}_P\,{}^i\mathbf{r}_P^T\,dm. \qquad (12.401)$$

Hence, the kinetic energy of link (i) becomes

$$K_i = \frac{1}{2}\operatorname{tr}\left(\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial\,{}^0T_i}{\partial q_j}\,{}^i\bar{I}_i\,\frac{\partial\,{}^0T_i}{\partial q_k}^T\dot{q}_j\,\dot{q}_k\right). \qquad (12.402)$$

The kinetic energy of a robot having n links is a summation of the kinetic energies of each link.

$$K = \sum_{i=1}^{n}K_i = \frac{1}{2}\operatorname{tr}\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial\,{}^0T_i}{\partial q_j}\,{}^i\bar{I}_i\,\frac{\partial\,{}^0T_i}{\partial q_k}^T\dot{q}_j\,\dot{q}_k\right) \qquad (12.403)$$

We may also add the kinetic energy due to the actuating motors K_a that are installed at the joints of the robot

$$K_a = \begin{cases} \sum_{i=1}^{n}\frac{1}{2}I_i\dot{q}_i^2 & \text{if joint } i \text{ is R} \\ \sum_{i=1}^{n}\frac{1}{2}m_i\dot{q}_i^2 & \text{if joint } i \text{ is P} \end{cases} \qquad (12.404)$$

where, I_i is the moment of inertia of the rotary actuator at joint i, and m_i is the mass of the translatory actuator. However, we may assume that the motors are concentrated masses at joints and add the mass of the motor at joint i to the mass of the link $(i-1)$ and adjust the inertial parameters of the link. The motor at joint i will drive the link (i).

For the potential energy we assume the gravity is the only source of potential energy. Therefore, the potential energy of the link (i) with respect to the base coordinate frame is

$$V_i = -m_i\,{}^0\mathbf{g}\cdot{}^0\mathbf{r}_i = -m_i\,{}^0\mathbf{g}^T\;{}^0T_i\,{}^i\mathbf{r}_i \qquad (12.405)$$

where ${}^0\mathbf{g} = \begin{bmatrix} g_x & g_y & g_z & 0 \end{bmatrix}^T$ is the gravitational acceleration usually in the direction $-z_0$, and ${}^0\mathbf{r}_i$ is the position vector of C of link (i) in the base frame. The potential energy of the whole robot is then equal to

$$V = \sum_{i=1}^{n}V_i = -\sum_{i=1}^{n}m_i\mathbf{g}^T\;{}^0T_i\,{}^i\mathbf{r}_i. \qquad (12.406)$$

The Lagrangean of a robot is found by substituting (12.403) and (12.406) in the Lagrange equation (12.214).

$$\mathcal{L} = K - V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_j}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_k}^T\right) \dot{q}_j\, \dot{q}_k + \sum_{i=1}^{n} m_i\, {}^{0}\mathbf{g}^T\ {}^{0}T_i\, {}^{i}\mathbf{r}_i. \tag{12.407}$$

The dynamic equations of motion of a robot can now be found by applying the Lagrange equations (12.215) to Equation (12.407). We develop the equations of motion term by term. Differentiating the $\mathcal{L}$ with respect to $\dot{q}_r$ is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{q}_r} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_r}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_k}^T\right) \dot{q}_k \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_j}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_r}^T\right) \dot{q}_j \\ &= \sum_{i=r}^{n}\sum_{j=1}^{i} \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_j}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_r}^T\right) \dot{q}_j \end{aligned} \tag{12.408}$$

because

$$\frac{\partial\, {}^{0}T_i}{\partial q_r} = 0 \qquad \text{for } \ r > i \tag{12.409}$$

and

$$\operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_j}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_k}^T\right) = \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_k}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_j}^T\right). \tag{12.410}$$

Time derivative of $\partial\mathcal{L}/\partial\dot{q}_r$ is:

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_r} &= \sum_{i=r}^{n}\sum_{j=1}^{i} \operatorname{tr}\left(\frac{\partial\, {}^{0}T_i}{\partial q_j}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_r}^T\right) \ddot{q}_j \\ &\quad + \sum_{i=r}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \operatorname{tr}\left(\frac{\partial^2\, {}^{0}T_i}{\partial q_j \partial q_k}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_r}^T\right) \dot{q}_j\dot{q}_k \\ &\quad + \sum_{i=r}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \operatorname{tr}\left(\frac{\partial^2\, {}^{0}T_i}{\partial q_r \partial q_k}\, {}^{i}\bar{I}_i\, \frac{\partial\, {}^{0}T_i}{\partial q_j}^T\right) \dot{q}_j\dot{q}_k \end{aligned} \tag{12.411}$$

The last term of the Lagrange equation is:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_r} &= \frac{1}{2}\sum_{i=r}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \mathrm{tr}\left(\frac{\partial^2 \, {}^0T_i}{\partial q_j \partial q_r} \, {}^i\bar{I}_i \, \frac{\partial \, {}^0T_i}{\partial q_k}^T\right) \dot{q}_j \, \dot{q}_k \\ &\quad + \frac{1}{2}\sum_{i=r}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \mathrm{tr}\left(\frac{\partial^2 \, {}^0T_i}{\partial q_k \partial q_r} \, {}^i\bar{I}_i \, \frac{\partial \, {}^0T_i}{\partial q_j}^T\right) \dot{q}_j \, \dot{q}_k \\ &\quad + \sum_{i=r}^{n} m_i \mathbf{g}^T \, \frac{\partial \, {}^0T_i}{\partial q_r} \, {}^i\mathbf{r}_i \end{aligned} \tag{12.412}$$

which can be simplified to

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_r} &= \sum_{i=r}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \mathrm{tr}\left(\frac{\partial^2 \, {}^0T_i}{\partial q_r \partial q_j} \, {}^i\bar{I}_i \, \frac{\partial \, {}^0T_i}{\partial q_k}^T\right) \dot{q}_j \, \dot{q}_k \\ &\quad + \sum_{i=r}^{n} m_i \mathbf{g}^T \, \frac{\partial \, {}^0T_i}{\partial q_r} \, {}^i\mathbf{r}_i. \end{aligned} \tag{12.413}$$

Interestingly, the third term in Equation (12.411) is equal to the first term in (12.413). So, substituting these equations in the Lagrange equation can be simplified to

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} &= \sum_{j=i}^{n}\sum_{k=1}^{j} \mathrm{tr}\left(\frac{\partial \, {}^0T_j}{\partial q_k} \, {}^j\bar{I}_j \, \frac{\partial \, {}^0T_j}{\partial q_i}^T\right) \ddot{q}_k \\ &\quad + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \mathrm{tr}\left(\frac{\partial^2 \, {}^0T_j}{\partial q_k \partial q_m} \, {}^j\bar{I}_j \, \frac{\partial \, {}^0T_j}{\partial q_i}^T\right) \dot{q}_k \dot{q}_m \\ &\quad - \sum_{j=i}^{n} m_j \mathbf{g}^T \, \frac{\partial \, {}^0T_j}{\partial q_i} \, {}^j\mathbf{r}_j. \end{aligned} \tag{12.414}$$

Finally, the equations of motion for an n link robot are

$$\begin{aligned}Q_i &= \sum_{i=i}^{n}\sum_{k=1}^{j} \mathrm{tr}\left(\frac{\partial \, {}^0T_j}{\partial q_k} \, {}^j\bar{I}_j \, \frac{\partial \, {}^0T_j}{\partial q_i}^T\right) \ddot{q}_k \\ &\quad + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \mathrm{tr}\left(\frac{\partial^2 \, {}^0T_j}{\partial q_k \partial q_m} \, {}^j\bar{I}_j \, \frac{\partial \, {}^0T_j}{\partial q_i}^T\right) \dot{q}_k \dot{q}_m \\ &\quad - \sum_{j=i}^{n} m_j \mathbf{g}^T \, \frac{\partial \, {}^0T_j}{\partial q_i} \, {}^j\mathbf{r}_j. \end{aligned} \tag{12.415}$$

The equations of motion can be written in a more concise form

$$Q_i = \sum_{j=1}^{n} D_{ij} \ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} H_{ijk} \dot{q}_j \, \dot{q}_k + G_i \tag{12.416}$$

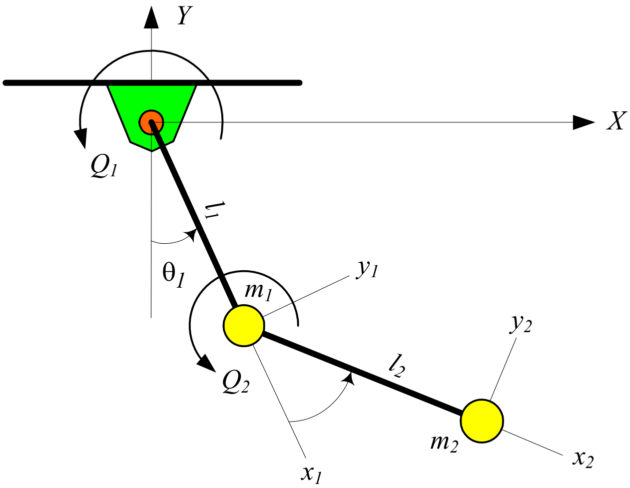

FIGURE 12.19. $2R$ manipulator mounted on a ceiling.

where

$$D_{ij} = \sum_{r=\max i,j}^{n} \operatorname{tr}\left(\frac{\partial\, {}^0T_r}{\partial q_j}\, {}^r\bar{I}_r\, \frac{\partial\, {}^0T_r}{\partial q_i}^T\right) \tag{12.417}$$

$$H_{ijk} = \sum_{r=\max i,j,k}^{n} \operatorname{tr}\left(\frac{\partial^2\, {}^0T_r}{\partial q_j \partial q_k}\, {}^r\bar{I}_r\, \frac{\partial\, {}^0T_r}{\partial q_i}^T\right) \tag{12.418}$$

$$G_i = -\sum_{r=i}^{n} m_r \mathbf{g}^T \frac{\partial\, {}^0T_r}{\partial q_i}\, {}^r\mathbf{r}_r. \tag{12.419}$$

■

Example 343 *$2R$ manipulator mounted on ceiling.*

Figure 12.19 depicts an ideal $2R$ planar manipulator mounted on a ceiling. Ceiling mounting is an applied method in some robotic operated assembly lines.

The Lagrangean of the manipulator is

$$\begin{aligned}\mathcal{L} &= K - V = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 \\ &\quad + \frac{1}{2} m_2 \left(l_1^2 \dot\theta_1^2 + l_2^2 \left(\dot\theta_1 + \dot\theta_2\right)^2 + 2 l_1 l_2 \dot\theta_1 \left(\dot\theta_1 + \dot\theta_2\right)\cos\theta_2\right) \\ &\quad + m_1 g l_1 \cos\theta_1 + m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\left(\theta_1 + \theta_2\right)\right)\end{aligned} \tag{12.420}$$

which leads to the following equations of motion:

$$\begin{aligned} Q_1 &= \left((m_1+m_2)\,l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2\right)\ddot{\theta}_1 \\ &\quad + m_2 l_2 \left(l_2 + l_1 \cos\theta_2\right)\ddot{\theta}_2 \\ &\quad - 2m_2 l_1 l_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 \sin\theta_2 \dot{\theta}_2^2 \\ &\quad + (m_1+m_2)\, g l_1 \sin\theta_1 + m_2 g l_2 \sin(\theta_1+\theta_2) \end{aligned} \tag{12.421}$$

$$\begin{aligned} Q_2 &= m_2 l_2 \left(l_2 + l_1\cos\theta_2\right)\ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 \\ &\quad - 2m_2 l_1 l_2 \sin\theta_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2\right) - m_2 g l_2 \sin(\theta_1+\theta_2). \end{aligned} \tag{12.422}$$

The equations of motion can be rearranged to

$$\begin{aligned} Q_1 &= D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 + H_{111}\dot{\theta}_1^2 + H_{122}\dot{\theta}_2^2 \\ &\quad + D_{112}\dot{\theta}_1\dot{\theta}_2 + D_{121}\dot{\theta}_2\dot{\theta}_1 + G_1 \end{aligned} \tag{12.423}$$

$$\begin{aligned} Q_2 &= D_{12}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + H_{211}\dot{\theta}_1^2 + H_{222}\dot{\theta}_2^2 \\ &\quad + D_{212}\dot{\theta}_1\dot{\theta}_2 + D_{221}\dot{\theta}_2\dot{\theta}_1 + G_2 \end{aligned} \tag{12.424}$$

where,

$$D_{11} = (m_1+m_2)\,l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2 \tag{12.425}$$

$$D_{12} = m_2 l_2 \left(l_2 + l_1\cos\theta_2\right) \tag{12.426}$$

$$D_{21} = D_{12} = m_2 l_2 \left(l_2 + l_1\cos\theta_2\right) \tag{12.427}$$

$$D_{22} = m_2 l_2^2 \tag{12.428}$$

$$H_{111} = 0 \tag{12.429}$$

$$H_{122} = -m_2 l_1 l_2 \sin\theta_2 \tag{12.430}$$

$$H_{211} = -m_2 l_1 l_2 \sin\theta_2 \tag{12.431}$$

$$H_{222} = 0 \tag{12.432}$$

$$H_{112} = H_{121} = -m_2 l_1 l_2 \sin\theta_2 \tag{12.433}$$

$$H_{212} = H_{221} = -m_2 l_1 l_2 \sin\theta_2 \tag{12.434}$$

$$G_1 = (m_1+m_2)\, g l_1 \sin\theta_1 + m_2 g l_2 \sin(\theta_1+\theta_2) \tag{12.435}$$

$$G_2 = m_2 g l_2 \sin(\theta_1+\theta_2). \tag{12.436}$$

Example 344 *$2R$ manipulator with massive links.*

A $2R$ planar manipulator with massive links is shown in Figure 12.20. We assume the mass center C of each link is in the middle of the link and

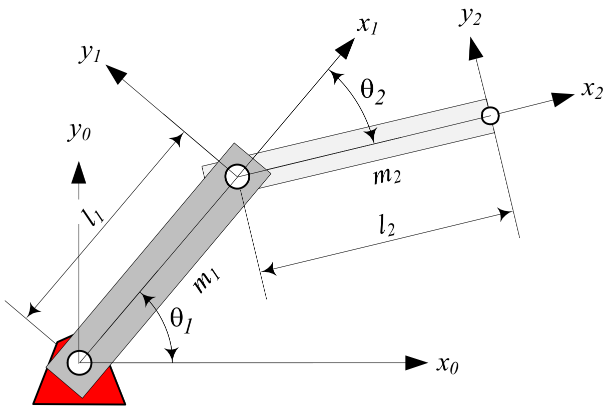

FIGURE 12.20. A $2R$ planar manipulator with massive links.

the motors at each joint is massless. The links' transformation matrices are

$$
{}^0T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.437}
$$

$$
{}^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.438}
$$

$$
\begin{aligned}
{}^0T_2 &= {}^0T_1\,{}^1T_2 \qquad\qquad (12.439) \\
&= \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1c\theta_1 + l_2c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & l_1s\theta_1 + l_2s(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Employing the velocity coefficient matrix Δ_R for revolute joints, we can *write*

$$\begin{aligned}\frac{\partial\,{}^0T_1}{\partial\theta_1} &= \Delta_R\,{}^0T_1 \qquad (12.440)\\ &= \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1\\ \sin\theta_1 & \cos\theta_1 & 0 & l_1\sin\theta_1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}-\sin\theta_1 & -\cos\theta_1 & 0 & -l_1\sin\theta_1\\ \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial\,{}^0T_2}{\partial\theta_1} &= \Delta_R\,{}^0T_2 \qquad (12.441)\\ &= \begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}c\theta_{12} & -s\theta_{12} & 0 & l_1c\theta_1 + l_2c\theta_{12}\\ s\theta_{12} & c\theta_{12} & 0 & l_1s\theta_1 + l_2s\theta_{12}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}-s(\theta_1+\theta_2) & -c(\theta_1+\theta_2) & 0 & -l_1s\theta_1 - l_2s(\theta_1+\theta_2)\\ c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_1c\theta_1 + l_2c(\theta_1+\theta_2)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\frac{\partial\,{}^0T_2}{\partial\theta_2} &= {}^0T_1\,\Delta_R\,{}^1T_2 \qquad (12.442)\\ &= \begin{bmatrix}-s(\theta_1+\theta_2) & -c(\theta_1+\theta_2) & 0 & -l_2s(\theta_1+\theta_2)\\ c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & l_2c(\theta_1+\theta_2)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.\end{aligned}$$

Assuming all the product of inertias are zero, we find

$${}^1\bar{I}_1 = \begin{bmatrix}\frac{1}{3}m_1l_1^2 & 0 & 0 & -\frac{1}{2}m_1l_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{1}{2}m_1l_1 & 0 & 0 & m_1\end{bmatrix} \qquad (12.443)$$

$${}^2\bar{I}_2 = \begin{bmatrix}\frac{1}{3}m_2l_2^2 & 0 & 0 & -\frac{1}{2}m_2l_2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{1}{2}m_2l_2 & 0 & 0 & m_2\end{bmatrix}. \qquad (12.444)$$

Using inertia and derivative of transformation matrices we can calculate the inertial-type symmetric matrix $\mathbf{D}(\mathbf{q})$.

$$\begin{aligned}
D_{11} &= \operatorname{tr}\left(\frac{\partial\,{}^0T_1}{\partial q_1}\,{}^1\bar{I}_1\,\frac{\partial\,{}^0T_1}{\partial q_1}^T\right)+\operatorname{tr}\left(\frac{\partial\,{}^0T_2}{\partial q_1}\,{}^2\bar{I}_2\,\frac{\partial\,{}^0T_2}{\partial q_1}^T\right) \\
&= \frac{1}{3}m_1l_1^2+m_2\left(l_1^2+\frac{1}{3}l_2^2\right)+m_2l_1l_2\cos\theta_2 \qquad (12.445)
\end{aligned}$$

$$\begin{aligned}
D_{12} &= D_{21}=\operatorname{tr}\left(\frac{\partial\,{}^0T_2}{\partial q_1}\,{}^2\bar{I}_2\,\frac{\partial\,{}^0T_2}{\partial q_1}^T\right) \\
&= \frac{1}{3}m_2l_2^2+m_2l_1^2+m_2l_1l_2\cos\theta_2 \qquad (12.446)
\end{aligned}$$

$$D_{22}=\operatorname{tr}\left(\frac{\partial\,{}^0T_2}{\partial q_2}\,{}^2\bar{I}_2\,\frac{\partial\,{}^0T_2}{\partial q_2}^T\right)=\frac{1}{3}l_2^2m_2 \qquad (12.447)$$

The coupling terms $\mathbf{H}(\mathbf{q},\dot{\mathbf{q}})$ *are calculated as below*

$$\begin{aligned}
H_1 &= \sum_{k=1}^{2}\sum_{m=1}^{2}H_{1km}\dot{q}_k\dot{q}_m \\
&= H_{111}\dot{q}_1\dot{q}_1+H_{112}\dot{q}_1\dot{q}_2+H_{121}\dot{q}_2\dot{q}_1+H_{122}\dot{q}_2\dot{q}_2 \qquad (12.448)
\end{aligned}$$

$$\begin{aligned}
H_2 &= \sum_{k=1}^{2}\sum_{m=1}^{2}H_{2km}\dot{q}_k\dot{q}_m \\
&= H_{211}\dot{q}_1\dot{q}_1+H_{212}\dot{q}_1\dot{q}_2+H_{221}\dot{q}_2\dot{q}_1+H_{222}\dot{q}_2\dot{q}_2 \qquad (12.449)
\end{aligned}$$

where

$$H_{ijk}=\sum_{r=\max i,j,k}^{n}\operatorname{tr}\left(\frac{\partial^2\,{}^0T_r}{\partial q_j\partial q_k}\,{}^r\bar{I}_r\,\frac{\partial\,{}^0T_r}{\partial q_i}^T\right). \qquad (12.450)$$

These calculations lead to

$$\mathbf{H}=\begin{bmatrix} -\frac{1}{2}m_2l_1l_2\dot{\theta}_2^2\sin\theta_2-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2 \\ \frac{1}{2}m_2l_1l_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix}. \qquad (12.451)$$

The last terms are the gravitational vector $\mathbf{G}(\mathbf{q})$

$$G_1 = -m_1 \mathbf{g}^T \frac{\partial\, {}^0T_1}{\partial q_1}\, {}^1\mathbf{r}_1 - m_2 \mathbf{g}^T \frac{\partial\, {}^0T_2}{\partial q_1}\, {}^2\mathbf{r}_2$$

$$\begin{aligned}
= \quad & -m_1 \begin{bmatrix} 0 \\ -g \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} -\sin\theta_1 & -\cos\theta_1 & 0 & -l_1\sin\theta_1 \\ \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cos\theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l_1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
& -m_2 \begin{bmatrix} 0 \\ -g \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} -s\theta_{12} & -c\theta_{12} & 0 & -l_1 s\theta_1 - l_2 s\theta_{12} \\ c\theta_{12} & -s\theta_{12} & 0 & l_1 c\theta_1 + l_2 c\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l_1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
= \quad & \frac{1}{2} m_1 g l_1 \cos\theta_1 + \frac{1}{2} m_2 g l_1 \cos(\theta_1 + \theta_2) + m_2 g l_1 \cos\theta_1 \qquad (12.452)
\end{aligned}$$

$$\begin{aligned}
G_2 &= -m_2 \mathbf{g}^T \frac{\partial\, {}^0T_2}{\partial q_2}\, {}^2\mathbf{r}_2 \\
&= -m_2 \begin{bmatrix} 0 \\ -g \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} -s\theta_{12} & -c\theta_{12} & 0 & -l_2 s\theta_{12} \\ c\theta_{12} & -s\theta_{12} & 0 & l_2 c\theta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l_1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{2} m_2 g l_2 \cos(\theta_1 + \theta_2)\,. \qquad (12.453)
\end{aligned}$$

Finally the equations of motion for the $2R$ *planar manipulator are*

$$\begin{aligned}
& \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \\
& \begin{bmatrix} \frac{1}{3} m_1 l_1^2 + m_2 \left(l_1^2 + \frac{1}{3} l_2^2 + l_1 l_2 c\theta_2\right) & m_2 \left(l_1^2 + \frac{1}{3} l_2^2 + l_1 l_2 c\theta_2\right) \\ \left(l_1^2 + \frac{1}{3} l_2^2\right) m_2 + m_2 l_1 l_2 c\theta_2 & \frac{1}{3} l_2^2 m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\
& + \begin{bmatrix} -\frac{1}{2} m_2 l_1 l_2 \dot{\theta}_2^2 \sin\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 \\ \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \\
& + \begin{bmatrix} \frac{1}{2} m_1 g l_1 \cos\theta_1 + \frac{1}{2} m_2 g l_1 \cos(\theta_1 + \theta_2) + m_2 g l_1 \cos\theta_1 \\ \frac{1}{2} m_2 g l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \qquad (12.454)
\end{aligned}$$

12.5 Robot Statics

At the beginning and at the end of a rest-to-rest mission, a robot must keep the specified configurations. To hold the position and orientation, the actuators must apply some required forces to balance the external loads

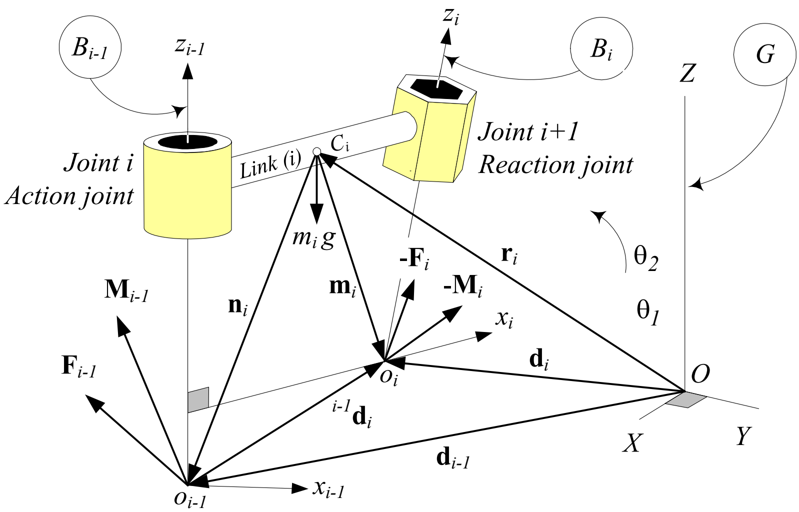

FIGURE 12.21. Position vectors and force system on link (i).

applied to the robot. Calculating the required actuators' force to hold a robot in a specific configuration is called *robot statics analysis*.

In a static condition, the globally expressed Newton-Euler equations for the link (i) can be written in a recursive form

$$ {}^0\mathbf{F}_{i-1} = {}^0\mathbf{F}_i - \sum {}^0\mathbf{F}_{e_i} \tag{12.455} $$

$$ {}^0\mathbf{M}_{i-1} = {}^0\mathbf{M}_i - \sum {}^0\mathbf{M}_{e_i} + {}_{i-1}^{\;\;0}\mathbf{d}_i \times {}^0\mathbf{F}_i \tag{12.456} $$

where

$$ {}_{i-1}^{\;\;0}\mathbf{d}_i = {}^0\mathbf{d}_i - {}^0\mathbf{d}_{i-1}. \tag{12.457} $$

Therefore, we are able to calculate the action force system $(\mathbf{F}_{i-1}\ ,\ \mathbf{M}_{i-1})$ when the reaction force system $(-\mathbf{F}_i\ ,\ -\mathbf{M}_i)$ is given. The position vectors and force systems on link (i) are shown in Figure 12.21.

Proof. In a static condition, the Newton-Euler equations of motion (12.1) and (12.2) for the link (i) reduce to force and moment balance equations.

$$ {}^0\mathbf{F}_{i-1} - {}^0\mathbf{F}_i + \sum {}^0\mathbf{F}_{e_i} = 0 \tag{12.458} $$

$$ {}^0\mathbf{M}_{i-1} - {}^0\mathbf{M}_i + \sum {}^0\mathbf{M}_{e_i} + {}^0\mathbf{n}_i \times {}^0\mathbf{F}_{i-1} - {}^0\mathbf{m}_i \times {}^0\mathbf{F}_i = 0 \tag{12.459} $$

These equations can be rearranged into a *backward recursive* form.

$$ {}^0\mathbf{F}_{i-1} = {}^0\mathbf{F}_i - \sum {}^0\mathbf{F}_{e_i} \tag{12.460} $$

$$ {}^0\mathbf{M}_{i-1} = {}^0\mathbf{M}_i - \sum {}^0\mathbf{M}_{e_i} - {}^0\mathbf{n}_i \times {}^0\mathbf{F}_{i-1} + {}^0\mathbf{m}_i \times {}^0\mathbf{F}_i \tag{12.461} $$

However, we may transform the Euler equation from C_i to O_{i-1} and find the Equation (12.456).

Practically, we measure the position of mass center $\mathbf{r}_i$ and the relative position of B_i and B_{i-1} in the coordinate frame B_i attached to the link (i). Hence, we must transform ${}^i\mathbf{r}_i$ and ${}_{i-1}^{i}\mathbf{d}_i$ to the base frame.

$$\begin{aligned} {}^0\mathbf{r}_i &= {}^0T_i \, {}^i\mathbf{r}_i && (12.462) \\ {}_{i-1}^{0}\mathbf{d}_i &= {}^0T_i \, {}_{i-1}^{i}\mathbf{d}_i && (12.463) \end{aligned}$$

The external load is usually the gravitational force $m_i\mathbf{g}$ and hence,

$$\begin{aligned} \sum {}^0\mathbf{F}_{e_i} &= m_i \, {}^0\mathbf{g} && (12.464) \\ \sum {}^0\mathbf{M}_{e_i} &= {}^0\mathbf{r}_i \times m_i \, {}^0\mathbf{g}. && (12.465) \end{aligned}$$

Using the DH parameters, we may express the relative position vector ${}_{i-1}^{i}\mathbf{d}_i$ by

$$ {}_{i-1}^{i}\mathbf{d}_i = \begin{bmatrix} a_i \\ d_i \sin\alpha_i \\ d_i \cos\alpha_i \\ 1 \end{bmatrix}. \qquad (12.466)$$

The backward recursive equations (12.455) and (12.456) allow us to start with a known force system $(\mathbf{F}_n \,, \mathbf{M}_n)$ at B_n, applied from the end-effector to the environment, and calculate the force system at B_{n-1}.

$$\begin{aligned} {}^0\mathbf{F}_{n-1} &= {}^0\mathbf{F}_n - \sum {}^0\mathbf{F}_{e_n} && (12.467) \\ {}^0\mathbf{M}_{n-1} &= {}^0\mathbf{M}_n - \sum {}^0\mathbf{M}_{e_n} + {}_{n-1}^{0}\mathbf{d}_n \times {}^0\mathbf{F}_n && (12.468) \end{aligned}$$

Following the same procedure and calculating force system at proximal end by having the force system at distal end of each link, ends up to the force system at the base. In this procedure, the force system applied by the end-effector to the environment is assumed to be known.

It is also possible to rearrange the static Equations (12.458) and (12.459) into a *forward recursive* form.

$$\begin{aligned} {}^0\mathbf{F}_i &= {}^0\mathbf{F}_{i-1} + \sum {}^0\mathbf{F}_{e_i} && (12.469) \\ {}^0\mathbf{M}_i &= {}^0\mathbf{M}_{i-1} + \sum {}^0\mathbf{M}_{e_i} + {}^0\mathbf{n}_i \times {}^0\mathbf{F}_{i-1} - {}^0\mathbf{m}_i \times {}^0\mathbf{F}_i && (12.470) \end{aligned}$$

Transforming the Euler equation from C_i to O_i simplifies the forward recursive equations into the more practical equations.

$$\begin{aligned} {}^0\mathbf{F}_i &= {}^0\mathbf{F}_{i-1} + \sum {}^0\mathbf{F}_{e_i} && (12.471) \\ {}^0\mathbf{M}_i &= {}^0\mathbf{M}_{i-1} + \sum {}^0\mathbf{M}_{e_i} - {}_{i-1}^{0}\mathbf{d}_i \times {}^0\mathbf{F}_{i-1} && (12.472) \end{aligned}$$

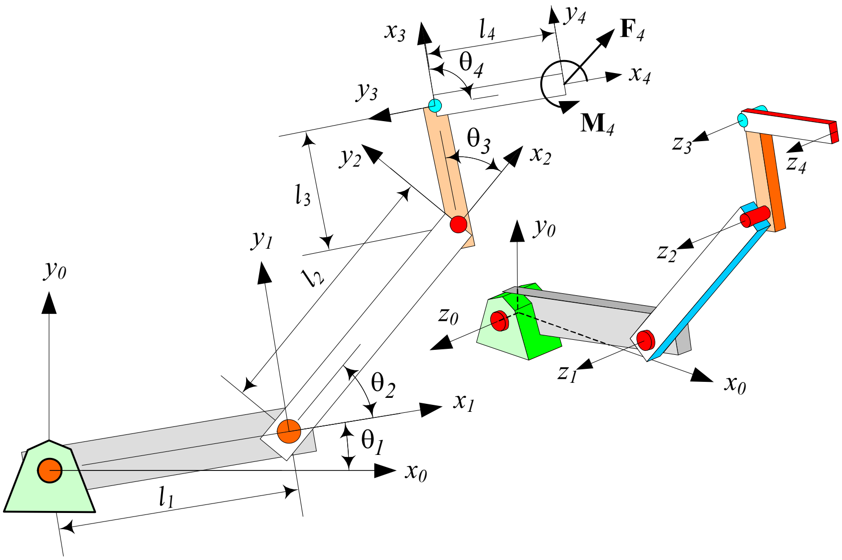

FIGURE 12.22. A 4R planar manipulator.

Using the forward recursive Equations (12.471) and (12.472) we can start with a known force system ($\mathbf{F}_0$, $\mathbf{M}_0$) at B_0, applied from the base to the link (1), and calculate the force system at B_1.

$$ {}^0\mathbf{F}_1 = {}^0\mathbf{F}_0 + \sum {}^0\mathbf{F}_{e_1} \tag{12.473} $$

$$ {}^0\mathbf{M}_1 = {}^0\mathbf{M}_0 + \sum {}^0\mathbf{M}_{e_1} - {}^0\mathbf{d}_1 \times {}^0\mathbf{F}_0 \tag{12.474} $$

Following this procedure and calculating force system at the distal end by having the force system at the proximal end of each link, ends up at the force system applied to the environment by the end-effector. In this procedure, the force system applied by the base actuators to the first link is assumed to be known. ■

Example 345 *Statics of a 4R planar manipulator.*

Figure 12.22 illustrates a 4R *planar manipulator with the DH coordinate frames set up for each link. Assume the end-effector force system applied to the environment is measured as*

$$ {}^4\mathbf{F}_4 = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} \qquad {}^4\mathbf{M}_4 = \begin{bmatrix} 0 \\ 0 \\ M_z \end{bmatrix}. \tag{12.475} $$

In addition, we assume that the links are uniform such that their C *are located at the midpoint of each link, and the gravitational acceleration is:*

$$ \mathbf{g} = -g\,\hat{\jmath}_0 \tag{12.476} $$

The manipulator consists of four $R\|R(0)$ links, therefore their transformation matrices ${}^{i-1}T_i$ are of class (5.32) that because $d_i = 0$ and $a_i = l_i$, simplifies to

$$
{}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & l_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & 0 & l_i\sin\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12.477}
$$

The C position vectors ${}^i\mathbf{r}_i$ and the relative position vectors ${}_{i-1}^{\ 0}\mathbf{d}_i$ are:

$$
{}^i\mathbf{r}_i = \begin{bmatrix} l_i/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}_{i-1}^{\ i}\mathbf{d}_i = \begin{bmatrix} l_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12.478}
$$

and therefore,

$$
{}^0\mathbf{r}_i = {}^0T_i\,{}^i\mathbf{r}_i \tag{12.479}
$$

$$
{}_{i-1}^{\ 0}\mathbf{d}_i = {}^0T_i\,{}_{i-1}^{\ i}\mathbf{d}_i \tag{12.480}
$$

where

$$
{}^0T_i = {}^0T_1 \cdots {}^{i-1}T_i. \tag{12.481}
$$

The static force at joints 3, 2, and 1 are

$$
\begin{aligned}
{}^0\mathbf{F}_3 &= {}^0\mathbf{F}_4 - \sum {}^0\mathbf{F}_{e_4} = {}^0\mathbf{F}_4 + m_4 g\,{}^0\hat{\jmath}_0 \qquad (12.482) \\
&= \begin{bmatrix} F_x \\ F_y \\ 0 \\ 0 \end{bmatrix} + m_4 g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y + m_4 g \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^0\mathbf{F}_2 &= {}^0\mathbf{F}_3 - m_3 g\,{}^0\hat{\jmath}_0 \qquad (12.483) \\
&= \begin{bmatrix} F_x \\ F_y + m_4 g \\ 0 \\ 0 \end{bmatrix} + m_3 g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y + (m_3 + m_4)\,g \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^0\mathbf{F}_1 &= {}^0\mathbf{F}_2 - m_2 g\,{}^0\hat{\jmath}_0 \\
&= \begin{bmatrix} F_x \\ F_y + (m_3 + m_4)\,g \\ 0 \\ 0 \end{bmatrix} + m_2 g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} F_x \\ F_y + g\,(m_2 + m_3 + m_4) \\ 0 \\ 0 \end{bmatrix} \qquad (12.484)
\end{aligned}
$$

$$\begin{aligned}
{}^0\mathbf{F}_0 &= {}^0\mathbf{F}_1 - m_1 g\, {}^0\hat{\jmath}_0 \\
&= \begin{bmatrix} F_x \\ F_y + g\left(m_2 + m_3 + m_4\right) \\ 0 \\ 0 \end{bmatrix} + m_1 g \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} F_x \\ F_y + g\left(m_1 + m_2 + m_3 + m_4\right) \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{12.485}$$

The static moment at joints 3, 2, *and* 1 *are*

$$\begin{aligned}
{}^0\mathbf{M}_3 &= {}^0\mathbf{M}_4 - \sum {}^0\mathbf{M}_{e_i} + {}^0_3\mathbf{d}_4 \times {}^0\mathbf{F}_4 \\
&= {}^0\mathbf{M}_4 + {}^0\mathbf{r}_4 \times m_4 g\, {}^0\hat{\jmath}_0 + {}^0_3\mathbf{d}_4 \times {}^0\mathbf{F}_4 \\
&= {}^0\mathbf{M}_4 + {}^0\mathbf{r}_4 \times m_4 g\, {}^0\hat{\jmath}_0 + {}^0_3\mathbf{d}_4 \times {}^0\mathbf{F}_4 \\
&= {}^0\mathbf{M}_4 + m_4 g\, \left({}^0T_4\, {}^4\mathbf{r}_4 \times {}^0\hat{\jmath}_0\right) + \left({}^0T_4\, {}^4_3\mathbf{d}_4\right) \times {}^0\mathbf{F}_4 \\
&= \begin{bmatrix} 0 \\ 0 \\ M_{3z} \\ 0 \end{bmatrix}
\end{aligned} \tag{12.486}$$

$$M_{3z} = M_z + l_4 F_y \cos\theta_{1234} - l_4 F_x \sin\theta_{1234} + \frac{1}{2} g l_4 m_4 \cos\theta_{1234} \tag{12.487}$$

$$\begin{aligned}
{}^0\mathbf{M}_2 &= {}^0\mathbf{M}_3 + m_3 g\, \left({}^0T_3\, {}^3\mathbf{r}_3 \times {}^0\hat{\jmath}_0\right) + \left({}^0T_3\, {}^3_2\mathbf{d}_3\right) \times {}^0\mathbf{F}_3 \\
&= \begin{bmatrix} 0 \\ 0 \\ M_{2z} \\ 0 \end{bmatrix}
\end{aligned} \tag{12.488}$$

$$\begin{aligned}
M_{2z} &= M_z + l_4 F_y \cos\theta_{1234} - l_4 F_x \sin\theta_{1234} + \frac{1}{2} g l_4 m_4 \cos\theta_{1234} \\
&\quad + \frac{1}{2} g l_3 m_3 \cos\theta_{123} - l_3 F_x \sin\theta_{123} \\
&\quad + l_3 \left(F_y + g m_4\right) \cos\theta_{123}
\end{aligned} \tag{12.489}$$

$$\begin{aligned}
{}^0\mathbf{M}_1 &= {}^0\mathbf{M}_2 + m_2 g\, \left({}^0T_2\, {}^2\mathbf{r}_2 \times {}^0\hat{\jmath}_0\right) + \left({}^0T_2\, {}^2_1\mathbf{d}_2\right) \times {}^0\mathbf{F}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ M_{1z} \\ 0 \end{bmatrix}
\end{aligned} \tag{12.490}$$

$$\begin{aligned} M_{1z} &= M_z + l_4 F_y \cos\theta_{1234} - l_4 F_x \sin\theta_{1234} + \frac{1}{2} g l_4 m_4 \cos\theta_{1234} \\ &\quad + \frac{1}{2} g l_3 m_3 \cos\theta_{123} - l_3 F_x \sin\theta_{123} + l_3 \cos\theta_{123} \left(F_y + g m_4\right) \\ &\quad + \frac{1}{2} g l_2 m_2 \cos\theta_{12} - l_2 F_x \sin\theta_{12} \\ &\quad + l_2 \cos\theta_{12} \left(F_y + g \left(m_3 + m_4\right)\right) \end{aligned} \tag{12.491}$$

$$\begin{aligned} {}^0\mathbf{M}_0 &= {}^0\mathbf{M}_1 + m_1 g \ \left({}^0T_1 \, {}^1\mathbf{r}_1 \times {}^0\hat{\jmath}_0\right) + \left({}^0T_1 \, {}_0^1\mathbf{d}_1\right) \times {}^0\mathbf{F}_1 \\ &= \begin{bmatrix} 0 \\ 0 \\ M_{0z} \\ 0 \end{bmatrix} \end{aligned} \tag{12.492}$$

$$\begin{aligned} M_{0z} &= M_z + l_4 F_y \cos\theta_{1234} - l_4 F_x \sin\theta_{1234} + \frac{1}{2} g l_4 m_4 \cos\theta_{1234} \\ &\quad + \frac{1}{2} g l_3 m_3 \cos\theta_{123} - l_3 F_x \sin\theta_{123} + l_3 \cos\theta_{123} \left(F_y + g m_4\right) \\ &\quad - l_2 F_x \sin\theta_{12} + \frac{1}{2} g l_2 m_2 \cos\theta_{12} + l_2 \cos\theta_{12} \left(F_y + g \left(m_3 + m_4\right)\right) \\ &\quad + \frac{1}{2} g l_1 m_1 \cos\theta_1 - l_1 F_x \sin\theta_1 \\ &\quad + l_1 \cos\theta_1 \left(F_y + g \left(m_2 + m_3 + m_4\right)\right) \end{aligned} \tag{12.493}$$

where

$$\theta_{1234} = \theta_1 + \theta_2 + \theta_3 + \theta_4 \tag{12.494}$$

$$\theta_{123} = \theta_1 + \theta_2 + \theta_3 \tag{12.495}$$

$$\theta_{12} = \theta_1 + \theta_2. \tag{12.496}$$

Example 346 *Recursive force equation in link's frame.*

Practically, it is easier to measure and calculate the force systems in the kink's frame. Therefore, we may write the backward recursive Equations (12.455) and (12.456) in the following form:

$${}^i\mathbf{F}_{i-1} = {}^i\mathbf{F}_i - \sum {}^i\mathbf{F}_{e_i} \tag{12.497}$$

$${}^i\mathbf{M}_{i-1} = {}^i\mathbf{M}_i - \sum {}^i\mathbf{M}_{e_i} + {}_{i-1}^{\;\;\;i}\mathbf{d}_i \times {}^i\mathbf{F}_i \tag{12.498}$$

and calculate the proximal force system from the distal force system in the link's frame. The calculated force system, then, may be transformed to the previous link's coordinate frame by a transformation

$${}^{i-1}\mathbf{F}_{i-1} = {}^{i-1}T_i \, {}^i\mathbf{F}_{i-1} \tag{12.499}$$

$${}^{i-1}\mathbf{M}_{i-1} = {}^{i-1}T_i \, {}^i\mathbf{M}_{i-1} \tag{12.500}$$

or they may be transformed to any other coordinate frame including the base frame.

$$
\begin{aligned}
{}^{0}\mathbf{F}_{i-1} &= {}^{0}T_{i}\,{}^{i}\mathbf{F}_{i-1} && (12.501) \\
{}^{0}\mathbf{M}_{i-1} &= {}^{0}T_{i}\,{}^{i}\mathbf{M}_{i-1} && (12.502)
\end{aligned}
$$

Example 347 *Actuator's force and torque.*

Applying a backward or forward recursive static force analysis ends up with a set of known force systems at joints. Each joint is driven by a motor or generally an actuator that applies a force in a P joint, or a torque in an R joint. When the joint i is prismatic, the actuator force is applied along the axis of the joint i. Therefore, the force of the driving motor is along the z_{i-1}-axis

$$F_m = {}^{0}\hat{k}_{i-1}^{T}\,{}^{0}\mathbf{F}_i \tag{12.503}$$

showing that the $\hat{k}_{i-1}$ component of the joint force $\mathbf{F}_i$ is supported by the actuator, while the $\hat{\imath}_{i-1}$ and $\hat{\jmath}_{i-1}$ components of $\mathbf{F}_i$ must be supported by the bearings of the joint.

Similarly, when joint i is revolute, the actuator torque is applied about the axis of joint i. Therefore, the torque of the driving motor is along the z_{i-1}-axis

$$M_m = {}^{0}\hat{k}_{i-1}^{T}\,{}^{0}\mathbf{M}_i \tag{12.504}$$

showing that the $\hat{k}_{i-1}$ component of the joint torque $\mathbf{M}_i$ is supported by the actuator, while the $\hat{\imath}_{i-1}$ and $\hat{\jmath}_{i-1}$ components of $\mathbf{M}_i$ must be supported by the bearings of the joint.

12.6 Summary

Dynamics equations of motion for a robot can be found by both Newton-Euler and Lagrange methods. In the Newton-Euler method, each link (i) is a rigid body and therefore, its translational and rotational equations of motion in the base coordinate frame are:

$$
\begin{aligned}
m_i\, {}^0\mathbf{a}_i &= {}^0\mathbf{F}_{i-1} - {}^0\mathbf{F}_i + \sum {}^0\mathbf{F}_{e_i} && (12.505) \\
{}^0I_i\, {}_0\boldsymbol{\alpha}_i &= {}^0\mathbf{M}_{i-1} - {}^0\mathbf{M}_i + \sum {}^0\mathbf{M}_{e_i} \\
&\quad + \left({}^0\mathbf{d}_{i-1} - {}^0\mathbf{r}_i\right) \times {}^0\mathbf{F}_{i-1} - \left({}^0\mathbf{d}_i - {}^0\mathbf{r}_i\right) \times {}^0\mathbf{F}_i && (12.506)
\end{aligned}
$$

The force $\mathbf{F}_{i-1}$ and moment $\mathbf{M}_{i-1}$ are the resultant force and moment that link $(i-1)$ applies to link (i) at joint i. Similarly, $\mathbf{F}_i$ and $\mathbf{M}_i$ are the resultant force and moment that link (i) applies to link $(i+1)$ at joint $i+1$. We measure the force systems $(\mathbf{F}_{i-1}\ ,\ \mathbf{M}_{i-1})$ and $(\mathbf{F}_i\ ,\ \mathbf{M}_i)$ at the origin of the coordinate frames B_{i-1} and B_i respectively. The sum of the external loads acting on the link (i) are $\sum \mathbf{F}_{e_i}$ and $\sum \mathbf{M}_{e_i}$. The vector ${}^0\mathbf{r}_i$ is the global position vector of C_i and ${}^0\mathbf{d}_i$ is the global position vector of the origin of B_i. The vector ${}^0\boldsymbol{\alpha}_i$ is the angular acceleration and ${}^0\mathbf{a}_i$ is the translational acceleration of the link (i) measured at the mass center C_i.

$$
{}^0\mathbf{a}_i = {}^0\ddot{\mathbf{d}}_i + {}_0\boldsymbol{\alpha}_i \times \left({}^0\mathbf{r}_i - {}^0\mathbf{d}_i\right) + {}_0\boldsymbol{\omega}_i \times \left({}_0\boldsymbol{\omega}_i \times \left({}^0\mathbf{r}_i - {}^0\mathbf{d}_i\right)\right) \tag{12.507}
$$

$$
{}_0\boldsymbol{\alpha}_i = \begin{cases} {}_0\boldsymbol{\alpha}_{i-1} + \ddot{\theta}_i\, {}^0\hat{k}_{i-1} + {}_0\boldsymbol{\omega}_{i-1} \times \dot{\theta}_i\, {}^0\hat{k}_{i-1} & \text{if joint } i \text{ is R} \\ {}_0\boldsymbol{\alpha}_{i-1} & \text{if joint } i \text{ is P} \end{cases} \tag{12.508}
$$

Weight is usually the only external load on middle links of a robot, and reactions from the environment are extra external force systems on the base and end-effector links. The force and moment that the base actuator applies to the first link are $\mathbf{F}_0$ and $\mathbf{M}_0$, and the force and moment that the end-effector applies to the environment are $\mathbf{F}_n$ and $\mathbf{M}_n$. If weight is the only external load on link (i) and it is in $-{}^0\hat{k}_0$ direction, then we have

$$
\begin{aligned}
\sum {}^0\mathbf{F}_{e_i} &= m_i\, {}^0\mathbf{g} = -m_i\, g\, {}^0\hat{k}_0 && (12.509) \\
\sum {}^0\mathbf{M}_{e_i} &= {}^0\mathbf{r}_i \times m_i\, {}^0\mathbf{g} = -\,{}^0\mathbf{r}_i \times m_i\, g\, {}^0\hat{k}_0 && (12.510)
\end{aligned}
$$

where $\mathbf{g}$ is the gravitational acceleration vector.

The Newton-Euler equation of motion can also be written in link's coordinate frame in a forward or backward method. The backward Newton-Euler equations of motion for link (i) in the the local coordinate frame B_i are

$$
{}^i\mathbf{F}_{i-1} = {}^i\mathbf{F}_i - \sum {}^i\mathbf{F}_{e_i} + m_i\, {}^i_0\mathbf{a}_i \tag{12.511}
$$

$$
\begin{aligned}
{}^i\mathbf{M}_{i-1} &= {}^i\mathbf{M}_i - \sum {}^i\mathbf{M}_{e_i} - \left({}^i\mathbf{d}_{i-1} - {}^i\mathbf{r}_i\right) \times {}^i\mathbf{F}_{i-1} \\
&\quad + \left({}^i\mathbf{d}_i - {}^i\mathbf{r}_i\right) \times {}^i\mathbf{F}_i + {}^iI_i\, {}^i_0\boldsymbol{\alpha}_i + {}^i_0\boldsymbol{\omega}_i \times {}^iI_i\, {}^i_0\boldsymbol{\omega}_i && (12.512)
\end{aligned}
$$

where

$$\begin{aligned} {}^{i}\mathbf{n}_i &= {}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i & (12.513) \\ {}^{i}\mathbf{m}_i &= {}^{i}\mathbf{d}_i - {}^{i}\mathbf{r}_i. & (12.514) \end{aligned}$$

and

$$ {}^{i}_{0}\mathbf{a}_i = {}^{i}\ddot{\mathbf{d}}_i + {}^{i}_{0}\boldsymbol{\alpha}_i \times \left({}^{i}\mathbf{r}_i - {}^{i}\mathbf{d}_i\right) + {}^{i}_{0}\boldsymbol{\omega}_i \times \left({}^{i}_{0}\boldsymbol{\omega}_i \times \left({}^{i}\mathbf{r}_i - {}^{i}\mathbf{d}_i\right)\right) \quad (12.515)$$

$$ {}^{i}_{0}\boldsymbol{\alpha}_i = \begin{cases} {}^{i}T_{i-1}\left({}^{i-1}_{0}\boldsymbol{\alpha}_{i-1} + \ddot{\theta}_i\, {}^{i-1}\hat{k}_{i-1}\right) & \\ \quad + {}^{i}T_{i-1}\left({}^{i-1}_{0}\boldsymbol{\omega}_{i-1} \times \dot{\theta}_i\, {}^{i-1}\hat{k}_{i-1}\right) & \text{if joint } i \text{ is R} \\ {}^{i}T_{i-1}\, {}^{i-1}_{0}\boldsymbol{\alpha}_{i-1} & \text{if joint } i \text{ is P.} \end{cases} \quad (12.516)$$

In this method, we search for the driving force system $({}^{i}\mathbf{F}_{i-1}, {}^{i}\mathbf{M}_{i-1})$ by having the driven force system $({}^{i}\mathbf{F}_i, {}^{i}\mathbf{M}_i)$ and the resultant external force system $({}^{i}\mathbf{F}_{e_i}, {}^{i}\mathbf{M}_{e_i})$. When the driving force system $({}^{i}\mathbf{F}_{i-1}, {}^{i}\mathbf{M}_{i-1})$ is found in frame B_i, we can transform them to the frame B_{i-1} and apply the Newton-Euler equation for link $(i-1)$.

$$\begin{aligned} {}^{i-1}\mathbf{F}_{i-1} &= {}^{i-1}T_i\, {}^{i}\mathbf{F}_{i-1} & (12.517) \\ {}^{i-1}\mathbf{M}_{i-1} &= {}^{i-1}T_i\, {}^{i}\mathbf{M}_{i-1} & (12.518) \end{aligned}$$

The negative of the converted force system acts as the driven force system $(-\,{}^{i-1}\mathbf{F}_{i-1}, -\,{}^{i-1}\mathbf{M}_{i-1})$ for the link $(i-1)$.

The forward Newton-Euler equations of motion for link (i) in the the local coordinate frame B_i are

$$ {}^{i}\mathbf{F}_i = {}^{i}\mathbf{F}_{i-1} + \sum {}^{i}\mathbf{F}_{e_i} - m_i\, {}^{i}_{0}\mathbf{a}_i \quad (12.519)$$

$$\begin{aligned} {}^{i}\mathbf{M}_i = {} & {}^{i}\mathbf{M}_{i-1} + \sum {}^{i}\mathbf{M}_{e_i} + \left({}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_{i-1} \\ & - \left({}^{i}\mathbf{d}_i - {}^{i}\mathbf{r}_i\right) \times {}^{i}\mathbf{F}_i - {}^{i}I_i\, {}^{i}_{0}\boldsymbol{\alpha}_i - {}^{i}_{0}\boldsymbol{\omega}_i \times {}^{i}I_i\, {}^{i}_{0}\boldsymbol{\omega}_i. \quad (12.520) \end{aligned}$$

$$\begin{aligned} {}^{i}\mathbf{n}_i &= {}^{i}\mathbf{d}_{i-1} - {}^{i}\mathbf{r}_i & (12.521) \\ {}^{i}\mathbf{m}_i &= {}^{i}\mathbf{d}_i - {}^{i}\mathbf{r}_i & (12.522) \end{aligned}$$

Using the forward Newton-Euler equations of motion, we can calculate the reaction force system $({}^{i}\mathbf{F}_i, {}^{i}\mathbf{M}_i)$ by having the action force system $({}^{i}\mathbf{F}_{i-1}, {}^{i}\mathbf{M}_{i-1})$. When the reaction force system $({}^{i}\mathbf{F}_i, {}^{i}\mathbf{M}_i)$ is found in frame B_i, we can transform them to the frame B_{i+1}

$$\begin{aligned} {}^{i+1}\mathbf{F}_i &= {}^{i}T_{i+1}^{-1}\, {}^{i}\mathbf{F}_i & (12.523) \\ {}^{i+1}\mathbf{M}_i &= {}^{i}T_{i+1}^{-1}\, {}^{i}\mathbf{M}_i. & (12.524) \end{aligned}$$

The negative of the converted force system acts as the action force system $(-\,{}^{i+1}\mathbf{F}_i,\,-\,{}^{i+1}\mathbf{M}_i)$ for the link $(i+1)$ and we can apply the Newton-Euler equation to the link $(i+1)$. The forward Newton-Euler equations of motion allows us to start from a known action force system $({}^{1}\mathbf{F}_0,\,{}^{1}\mathbf{M}_0)$, that the base link applies to the link (1), and calculate the action force of the next link. Therefore, analyzing the links of a robot, one by one, we end up with the force system that the end-effector applies to the environment.

The Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)-\frac{\partial\mathcal{L}}{\partial q_i} = Q_i \qquad i=1,2,\cdots n \tag{12.525}$$

$$\mathcal{L} = K-V \tag{12.526}$$

provides a systematic approach to obtain the dynamics equations for robots. The variables q_i are the coordinates by which the energies are expressed and the Q_i is the corresponding generalized nonpotential force.

The equations of motion for an n link serial manipulator, based on Newton-Euler or Lagrangian, can always be set in a matrix form

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}}+\mathbf{H}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{G}(\mathbf{q})=\mathbf{Q} \tag{12.527}$$

or

$$\mathbf{D}(\mathbf{q})\,\ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+\mathbf{G}(\mathbf{q})=\mathbf{Q} \tag{12.528}$$

or in a summation form

$$\sum_{j=1}^{n}D_{ij}(q)\,\ddot{q}_j+\sum_{k=1}^{n}\sum_{m=1}^{n}H_{ikm}\dot{q}_k\dot{q}_m+G_i=Q_i \tag{12.529}$$

where, $\mathbf{D}(\mathbf{q})$ is an $n\times n$ inertial-type symmetric matrix

$$D=\sum_{i=1}^{n}\left(\mathbf{J}_{Di}^{T}\,m_i\,\mathbf{J}_{Di}+\frac{1}{2}\mathbf{J}_{Ri}^{T}\,{}^{0}I_i\,\mathbf{J}_{Ri}\right) \tag{12.530}$$

H_{ikm} is the velocity coupling vector

$$H_{ijk}=\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial D_{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial D_{jk}}{\partial q_i}\right) \tag{12.531}$$

G_i is the gravitational vector

$$G_i=\sum_{j=1}^{n}m_j\mathbf{g}^{T}\,\mathbf{J}_{Dj}^{(i)} \tag{12.532}$$

and $\mathbf{J}_i$ is the Jacobian matrix of the robot

$$\dot{\mathbf{X}}_i=\begin{bmatrix}{}^{0}\mathbf{v}_i\\ {}^{0}\boldsymbol{\omega}_i\end{bmatrix}=\begin{bmatrix}\mathbf{J}_{Di}\\ \mathbf{J}_{Ri}\end{bmatrix}\dot{\mathbf{q}}=\mathbf{J}_i\,\dot{\mathbf{q}}. \tag{12.533}$$

To hold a robot in a stationary configuration, the actuators must apply some required forces to balance the external loads applied to the robot. In the static condition, the globally expressed Newton-Euler equations for the link (i), can be written in a recursive form

$$
\begin{aligned}
{}^{0}\mathbf{F}_{i-1} &= {}^{0}\mathbf{F}_{i} - \sum {}^{0}\mathbf{F}_{e_i} &\quad (12.534)\\
{}^{0}\mathbf{M}_{i-1} &= {}^{0}\mathbf{M}_{i} - \sum {}^{0}\mathbf{M}_{e_i} + {}_{i-1}^{\;\;\,0}\mathbf{d}_{i} \times {}^{0}\mathbf{F}_{i}. &\quad (12.535)
\end{aligned}
$$

Now we are able to calculate the action force system $(\mathbf{F}_{i-1}\ ,\ \mathbf{M}_{i-1})$ when the reaction force system $(-\mathbf{F}_{i}\ ,\ -\mathbf{M}_{i})$ is given.

12.7 Key Symbols

a	kinematic link length,
$\mathbf{a}$	acceleration vector
$[A]$	coefficient matrix of a set of linear equations
$\mathbf{b}$	vector of known values in a set of linear equations
B	body coordinate frame
c	cos
c_i	position of the mass center of link (i) in B_i
$\mathbf{c}$	Jacobian generating vector
C	mass center
$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$	damping-type matrix of equation of motion
d_x, d_y, d_z	elements of $\mathbf{d}$
$\mathbf{d}$	translation vector, joint position vector
$\mathbf{d}_i$	position vector of the origin of B_i
D	displacement transformation matrix
$\mathbf{D}(\mathbf{q})$	inertial-type matrix of equation of motion
$\mathbf{F}_{e_i}$	external force acting on the link (i)
$\mathbf{F}_i$	the force that link (i) applies to $(i+1)$ at joint $i+1$
$\mathbf{F}_{i-1}$	the force that link $(i-1)$ applies to link (i) at joint i
$\mathbf{F}_s$	shaking force
$\mathbf{g}$	gravitational acceleration vector
G, B_0	global coordinate frame, Base coordinate frame
$\mathbf{G}(\mathbf{q})$	gravitational vector of equation of motion
$\mathbf{H}(\mathbf{q}, \dot{\mathbf{q}})$	velocity coupling vector of equation of motion
$\hat{\imath}, \hat{\jmath}, \hat{k}$	local coordinate axes unit vectors
$\hat{I}, \hat{J}, \hat{K}$	global coordinate axes unit vectors
$I = [I]$	mass moment matrix
$\bar{I} = [\bar{I}]$	pseudo inertia matrix
$\mathbf{I} = [\mathbf{I}]$	identity matrix
$\mathbf{J}$	Jacobian
K	kinetic energy
l	length
$\mathbf{L}$	angular moment vector, moment of moment
$\mathcal{L}$	Lagrangean
m	mass
$\mathbf{m}_i$	position vector of o_i from C_i
$\mathbf{n}_i$	position vector of o_{i-1} from C_i
$\mathbf{M}_{e_i}$	external moment acting on the link (i)
$\mathbf{M}_i$	the moment that link (i) applies to $(i+1)$ at joint $i+1$
$\mathbf{M}_{i-1}$	the moment that link $(i-1)$ applies to link (i) at joint i
q	generalized coordinate
Q	torque of an actuator, generalized nonpotential force
$\mathbf{Q}$	moment vector at a joint

$\mathbf{r}$	position vectors, homogeneous position vector, global position of the mass center of a link
r_i	the element i of $\mathbf{r}$
r_{ij}	the element of row i and column j of a matrix
R	rotation transformation matrix
s	sin
T	homogeneous transformation matrix
$\mathbf{v}$	translational velocity vector
V	potential energy
x, y, z	local coordinate axes
$\mathbf{x}$	vector of unknown values in a set of linear equations
X, Y, Z	global coordinate axes
Z_i	short notation of an equation

Greek

α	angular acceleration
$\boldsymbol{\alpha}$	angular acceleration vector
$\alpha_1, \alpha_2, \alpha_3$	components of $\boldsymbol{\alpha}$
θ	rotary joint angle
θ_{ijk}	$\theta_i + \theta_j + \theta_k$
$\omega_1, \omega_2, \omega_3$	components of $\boldsymbol{\omega}$
ϵ	small test number to terminate a procedure
θ	rotary joint angle
θ_{ijk}	$\theta_i + \theta_j + \theta_k$
ω	angular velocity
$\boldsymbol{\omega}$	angular velocity vector
$\tilde{\omega}$	skew symmetric matrix of the vector $\boldsymbol{\omega}$

Symbol

$[\quad]^{-1}$	inverse of the matrix $[\quad]$
$[\quad]^{T}$	transpose of the matrix $[\quad]$
$\equiv$	equivalent
$\vdash$	orthogonal
(i)	link number i
$\parallel$	parallel sign
$\perp$	perpendicular
$\times$	vector cross product
FBD	free body diagram
tr	trace

Exercises

1. Notation and symbols.

   Describe their meaning.

a- $\mathbf{F}_2$	b- ${}^0\mathbf{F}_1$	c- ${}^1\mathbf{F}_1$	d- ${}^2\mathbf{M}_1$	e- ${}^2\mathbf{M}_{e1}$	f- ${}^B\mathbf{M}$
g- $\mathbf{m}_2$	h- ${}^0\mathbf{n}_2$	i- ${}_{i-1}^{0}\mathbf{d}_i$	j- ${}^0\mathbf{d}_i$	k- ${}^0\mathbf{r}_i$	l- ${}^{i-1}\mathbf{d}_i$
m- ${}^0\mathbf{L}_2$	n- ${}^0\mathbf{I}_2$	o- ${}_{i-1}^{0}\mathbf{L}_i$	p- $\mathbf{K}_i$	q- $\mathbf{V}_i$	r- ${}^{i-1}\mathbf{I}_i$

2. ★ Even order recursive translational velocity.

   Find an equation to relate the velocity of link (i) to the velocity of link $(i-2)$, and the velocity of link (i) to the velocity of link $(i+2)$.

3. ★ Even order recursive angular velocity.

   Find an equation to relate the angular velocity of link (i) to the angular velocity of link $(i-2)$, and the angular velocity of link (i) to the angular velocity of link $(i+2)$.

4. ★ Even order recursive translational acceleration.

   Find an equation to relate the acceleration of link (i) to the acceleration of link $(i-2)$, and the acceleration of link (i) to the acceleration of link $(i+2)$.

5. ★ Even order recursive angular acceleration.

   Find an equation to relate the angular acceleration of link (i) to the angular acceleration of link $(i-2)$, and the angular acceleration of link (i) to the angular acceleration of link $(i+2)$.

6. ★ Acceleration in different frames.

   For the $2R$ planar manipulator shown in Figure 12.7, find ${}_1^0\mathbf{a}_2$, ${}_0^1\mathbf{a}_2$, ${}_2^0\mathbf{a}_1$, ${}_0^2\mathbf{a}_1$, ${}_0^2\mathbf{a}_2$, and ${}_1^0\mathbf{a}_1$.

7. Slider-crank mechanism dynamics.

   A planar slider-crank mechanism is shown in Figure 12.23. Set up the link coordinate frames, develop the Newton-Euler equations of motion, and find the driving moment at the base revolute joint.

8. PR manipulator dynamics.

   Find the equations of motion for the planar polar manipulator shown in Figure 5.56. Eliminate the joints' constraint force and moment to derive the equations for the actuators' force or moment.

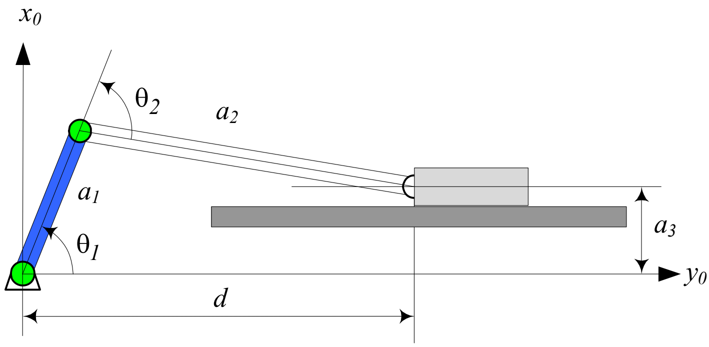

FIGURE 12.23. A planar slider-crank machanism.

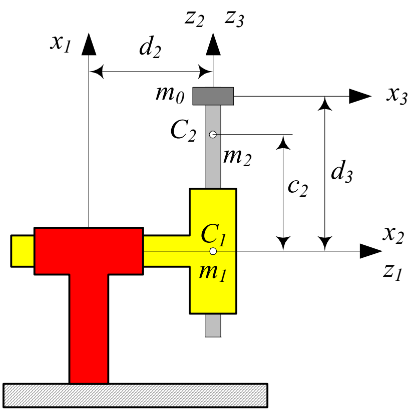

FIGURE 12.24. A 2 *DOF* Cartesian manipulator.

9. A planar Cartesian manipulator.

   Determine the equations of motion of the planar Cartesian manipulator shown in Figure 12.24. *Hint*: The coordinate frames are not based on *DH* rules.

10. ★ Global differential of a link momentum.

    In recursive Newton-Euler equations of motion, why we do not use the following Newton equation?

$$ {}^{i}\mathbf{F} = \frac{{}^{G}d}{dt}\,{}^{i}\mathbf{F} = \frac{{}^{G}d}{dt} m\,{}^{i}\mathbf{v} = m\,{}^{i}\dot{\mathbf{v}} + {}_{0}^{i}\omega_{i} \times m\,{}^{i}\mathbf{v} $$

11. $3R$ planar manipulator dynamics.

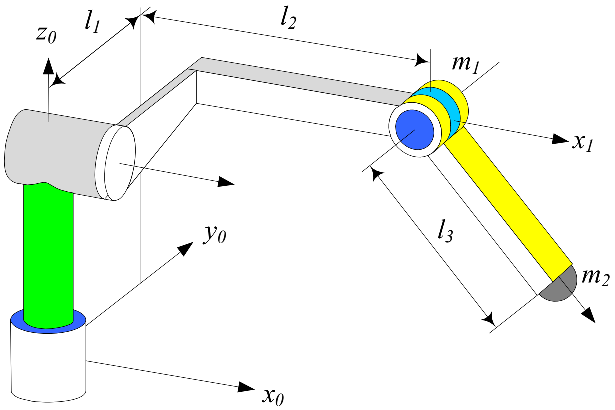

FIGURE 12.25. An articulated manipulator.

A $3R$ planar manipulator is shown in Figure 12.29. The manipulator is attached to a wall and therefore, $\mathbf{g} = g\,{}^0\hat{\imath}_0$.

(a) Find the Newton-Euler equations of motion for the manipulator. Do your calculations in the global frame and derive the dynamic force and moment at each joint.

(b) Reduce the number of equations to three for moments at joints.

(c) Substitute the vectorial quantities and calculate the moments in terms of geometry and angular variables of the manipulator.

12. A planar Cartesian manipulator dynamics.

Determine the Newton-Euler equations of motion for the planar Cartesian manipulator shown in Figure 5.57.

13. Articulated manipulator.

Figure 12.25 illustrates an articulated manipulator with massless arms and two massive points m_1 and m_2.

(a) Follow the DH rules and complete the link coordinate frames.

(b) Determine the DH transformation matrices.

(c) Determine the equations of motion of the manipulator using Lagrange method.

14. Polar planar manipulator dynamics.

A polar planar manipulator with 2 DOF is shown in Figure 5.56.

(a) Determine the Newton-Euler equations of motion for the manipulator.

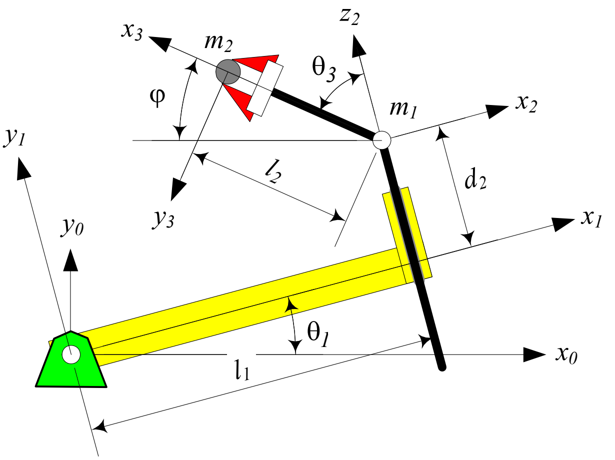

FIGURE 12.26. A planar manipulator.

(b) Reduce the number of equations to two, for moments at the base joint and force at the P joint.

(c) Substitute the vectorial quantities and calculate the action force and moment in terms of geometry and angular variables of the manipulator.

15. ★ Dynamics of a spherical manipulator.

Figure 5.43 illustrates a spherical manipulator attached with a spherical wrist. Analyze the robot and derive the equations of motion for joints action force and moment. Assume $\mathbf{g} = -g\,{}^0\hat{k}_0$ and the end-effector is carrying a mass m.

16. ★ Dynamics of an articulated manipulator.

Figure 5.22 illustrates an articulated manipulator R⊢R‖R. Use $\mathbf{g} = -g\,{}^0\hat{k}_0$ and find the manipulator's equations of motion.

17. A planar manipulator.

Figure 12.26 illustrates a three DOF planar manipulator. Determine the equations of motion of the manipulator if the links are massless and there are two massive points m_1 and m_2.

18. ★ Dynamics of a $SCARA$ robot.

Calculate the dynamic joints' force system for the $SCARA$ robot R‖R‖R‖P shown in Figure 5.23 if $\mathbf{g} = -g\,{}^0\hat{k}_0$.

19. ★ Dynamics of an $SRMS$ manipulator.

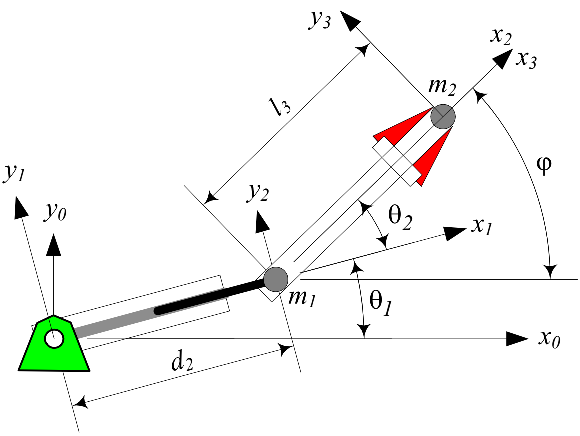

FIGURE 12.27. A RPR planar redundant manipulator.

Figure 5.24 shows a model of the Shuttle remote manipulator system ($SRMS$).

(a) Derive the equations of motion for the $SRMS$ and calculate the joints' force system for $\mathbf{g} = 0$.

(b) Derive the equations of motion for the $SRMS$ and calculate the joints' force system for $\mathbf{g} = -g\,{}^0\hat{k}_0$.

(c) Eliminate the constraint forces and reduce the number of equations equal to the number of action moments.

(d) Assume the links are made of a uniform cylinder with radius $r = .25\,\mathrm{m}$ and $m = 12\,\mathrm{kg/\,m}$. Use the characteristics indicated in Table 5.10 and find the equations of motion when the end-effector is holding a 24 kg mass.

20. $3R$ planar manipulator recursive dynamics.

The manipulator shown in Figure 12.29 is a $3R$ planar manipulator attached to a wall and therefore, $\mathbf{g} = -g\,{}^0\hat{\imath}_0$.

(a) Find the equations of motion for the manipulator utilizing the backward recursive Newton-Euler technique.

(b) ★ Find the equations of motion for the manipulator utilizing the forward recursive Newton-Euler technique.

21. A RPR planar redundant manipulator.

(a) Figure 12.27 illustrates a 3 DOF planar manipulator with joint variables θ_1, d_2, and θ_2. Determine the equations of motion of the manipulator if the links are massless and there are two massive points m_1 and m_2.

22. Polar planar manipulator recursive dynamics.

    Figure 5.56 depicts a polar planar manipulator with 2 DOF.

    (a) Find the equations of motion for the manipulator utilizing the backward recursive Newton-Euler technique.

    (b) ★ Find the equations of motion for the manipulator utilizing the forward recursive Newton-Euler technique.

23. ★ Recursive dynamics of an articulated manipulator.

    Figure 5.22 illustrates an articulated manipulator R⊢R‖R. Use $\mathbf{g} = -g\,{}^0\hat{k}_0$ and find the manipulator's equations of motion

    (a) utilizing the backward recursive Newton-Euler technique.

    (b) utilizing the forward recursive Newton-Euler technique.

24. ★ Recursive dynamics of a $SCARA$ robot.

    A $SCARA$ robot R‖R‖R‖P is shown in Figure 5.23. If $\mathbf{g} = -g\,{}^0\hat{k}_0$ determine the dynamic equations of motion by

    (a) utilizing the backward recursive Newton-Euler technique.

    (b) utilizing the forward recursive Newton-Euler technique.

25. ★ Recursive dynamics of an $SRMS$ manipulator.

    Figure 5.24 shows a model of the Shuttle remote manipulator system ($SRMS$).

    (a) Derive the equations of motion for the $SRMS$ utilizing the backward recursive Newton-Euler technique for $\mathbf{g} = 0$.

    (b) Derive the equations of motion for the $SRMS$ utilizing the forward recursive Newton-Euler technique for $\mathbf{g} = 0$.

26. $3R$ planar manipulator Lagrange dynamics.

    Find the equations of motion for the $3R$ planar manipulator shown in Figure 12.29 utilizing the Lagrange technique. The manipulator is attached to a wall and therefore, $\mathbf{g} = -g\,{}^0\hat{\imath}_0$.

27. Polar planar manipulator Lagrange dynamics.

    Find the equations of motion for the polar planar manipulator, shown in Figure 5.56, utilizing the Lagrange technique.

28. ★ Lagrange dynamics of an articulated manipulator.

Figure 5.22 illustrates an articulated manipulator R⊢R‖R. Use $\mathbf{g} = -g\,{}^0\hat{k}_0$ and find the manipulator's equations of motion utilizing the Lagrange technique.

29. ★ Lagrange dynamics of a *SCARA* robot.

A *SCARA* robot R‖R‖R‖P is shown in Figure 5.23. If $\mathbf{g} = -g\,{}^0\hat{k}_0$ determine the dynamic equations of motion by applying the Lagrange technique.

30. ★ Lagrange dynamics of an *SRMS* manipulator.

Figure 5.24 shows a model of the Shuttle remote manipulator system (*SRMS*). Derive the equations of motion for the *SRMS* utilizing the Lagrange technique for

(a) $\mathbf{g} = 0$

(b) $\mathbf{g} = -g\,{}^0\hat{k}_0$.

31. ★ Work done by actuators.

Consider a $2R$ planar manipulator moving on a given path. Assume that the endpoint of a $2R$ manipulator moves with constant speed $v = 1\,\mathrm{m/sec}$ from P_1 to P_2, on a path made of two semi-circles as shown in Figure 13.35. Calculate the work done by the actuators if $l_1 = l_2 = 1\,\mathrm{m}$ and the manipulator is carrying a 12 kg mass. The center of the circles are at $(0.75\,\mathrm{m}, 0.5\,\mathrm{m})$ and $(-0.75\,\mathrm{m}, 0.5\,\mathrm{m})$.

32. Statics of a $2R$ planar manipulator.

Figure 12.28 illustrates a $2R$ planar manipulator attached to a ceiling. The links are uniform with

$$\begin{aligned} m_1 &= 24\,\mathrm{kg} \qquad m_2 = 18\,\mathrm{kg} \\ l_1 &= 1\,\mathrm{m} \qquad l_2 = 1\,\mathrm{m} \\ \mathbf{g} &= -g\,{}^0\hat{\jmath}_0. \end{aligned}$$

There is a load $\mathbf{F}_e = -14g\,{}^0\hat{\jmath}_0\,\mathrm{N}$ at the endpoint. Calculate the static moments Q_1 and Q_2 for $\theta_1 = 30\,\mathrm{deg}$ and $\theta_2 = 45\,\mathrm{deg}$.

33. Statics of a $2R$ planar manipulator at a different base angle.

In Exercise 32 keep $\theta_2 = 45\,\mathrm{deg}$ and calculate the static moments Q_1 and Q_2 as functions of θ_1. Plot Q_1 and Q_2 versus θ_1 and find the configuration that minimizes Q_1, Q_2, $Q_1 + Q_2$, and the potential energy V.

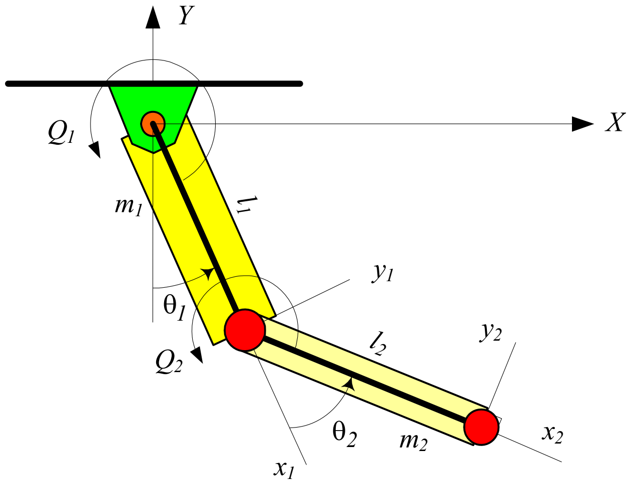

FIGURE 12.28. A 2R planar manipulator attached to a ceiling in static condition.

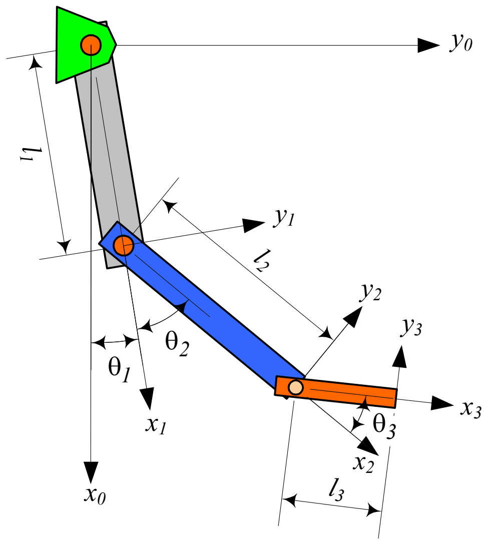

FIGURE 12.29. A 3R planar manipulator attached to a wall.

34. Statics of a $3R$ planar manipulator.

    Figure 12.29 illustrates a $3R$ planar manipulator attached to a wall. Derive the static force and moment at each joint to keep the configuration of the manipulator if $\mathbf{g} = -g\,{}^0\hat{\imath}_0$.

35. ★ Statics of an articulated manipulator.

    An articulated manipulator R⊢R‖R is shown in Figure 5.22. Find the static force and moment at joints for $\mathbf{g} = g\,{}^0\hat{k}_0$. The end-effector is carrying a 20 kg mass. Calculate the maximum base force moment.

36. ★ Statics of a $SCARA$ robot.

    Calculate the static joints' force system for the $SCARA$ robot R‖R‖R‖P shown in Figure 5.23 if $\mathbf{g} = -g\,{}^0\hat{k}_0$ and the end-effector is carrying a 10 kg mass.

37. ★ Statics of a spherical manipulator.

    Figure 5.43 illustrates a spherical manipulator attached with a spherical wrist. Analyze the robot and calculate the static force system in joints for $\mathbf{g} = -g\,{}^0\hat{k}_0$ if the end-effector is carrying a 12 kg mass.

38. ★ Statics of an $SRMS$ manipulator.

    A model of the Shuttle remote manipulator system ($SRMS$) is shown in Figure 5.24. Analyze the static configuration of the $SRMS$ and calculate the joints' force system for $\mathbf{g} = -g\,{}^0\hat{k}_0$.

    Assume the links are made of a uniform cylinder with radius $r = .25$ m and $m = 12$ kg/ m. Use the characteristics indicated in Table 5.10 and find the maximum value of the base force system for a 24 kg mass held by the end-effector. The $SRMS$ is supposed to work in a no-gravity field.