IV

The Continuum Problem and Constructible Sets

1 Gödel's Constructible Universe

1.1. In this section we introduce the subclass $L \subset V$—"Gödel's constructible universe"—and establish its fundamental properties. Perhaps the shortest description of L is that it is the smallest transitive model of the axioms of $\mathrm{L_1Set}$ that contains all the ordinals. But the working definition of L, from which the name "constructible universe" is derived, is rather different.

We consider the following operations $F_1, \dots, F_8$ on sets:

$$\begin{aligned}
F_1(X,Y) &= \{X,Y\},\\
F_2(X,Y) &= X\backslash Y,\\
F_3(X,Y) &= X \times Y,\\
F_4(X) &= \{U \mid \exists W(\langle U,W\rangle \in X)\} = \mathrm{dom}\ X,\\
F_5(X) &= \{\langle U,W\rangle \mid U,W \in X;\ U \in W\},\\
F_6(X) &= \{\langle U_1,U_2,U_3\rangle \mid \langle U_2,U_3,U_1\rangle \in X\},\\
F_7(X) &= \{\langle U_1,U_2,U_3\rangle \mid \langle U_3,U_2,U_1\rangle \in X\},\\
F_8(X) &= \{\langle U_1,U_2,U_3\rangle \mid \langle U_1,U_3,U_2\rangle \in X\}.
\end{aligned}$$

We say that a set (or class) Y is closed with respect to an operation F of degree r if we have $F(Z_1, \dots, Z_r) \in Y$ for all $Z_1, \dots, Z_r \in Y$ such that $F(Z_1, \dots, Z_r)$ is defined. For every $X \in V$ we let $\mathcal{J}(X)$ denote the smallest set $Y \supset X$ that is closed with respect to the operations $F_1, \dots, F_8$. It will later be shown (Section 1.4) that $\mathcal{J}(X)$ actually is a set. The following construction is analogous to the definition of V.

1.2. **Definition.**

$$\begin{aligned}
L_0 &= \varnothing;\\
L_{\alpha+1} &= \mathcal{P}(L_\alpha) \cap \mathcal{J}(L_\alpha \cup \{L_\alpha\});\\
L_\alpha &= \bigcup_{\beta<\alpha} L_\beta,\ \textit{if } \alpha \textit{ is a limit ordinal};\\
L &= \cup\, L_\alpha.
\end{aligned}$$

The elements of L are called *constructible sets*.

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The operations $F_1, \ldots, F_8$ and simple combinations of them, together with the transfinite recursion in the definition of L, exhaust the arsenal of primitive set-theoretic constructions used in mathematics. This can be seen by looking at Bourbaki's "compendium of the results of set theory," upon which all subsequent material in their voluminous treatise on the foundations of mathematics is based. The only way we could possibly (but not necessarily) leave L would be to apply the axiom of choice. This could happen provided that L is strictly less than V; but, as mentioned before, this question is undecidable in the Zermelo–Fraenkel axiom system (see also 5.16 below). Gödel was of the opinion that L does not exhaust V, as are most specialists who accept the semantics of $\mathrm{L_1Set}$.

Of course, the constructibility of the elements of L should not be understood in a finitistic sense. The sets we construct at the $(\alpha+1)$th stage are only the subsets of L_α that are obtained from the elements of the sets L_α and $\{L_\alpha\}$ using the explicit constructions F_i. But when we consider all the ordinals indexing the stages, we see that L is hopelessly infinite. Nevertheless, in many respects the construction of L is simpler than that of V, and L seems to provide a convenient framework for mathematics.

We now list some properties of L that follow easily from the definitions. The specific nature of the operations F_i plays a very secondary role in these properties.

1.3. $L_n = V_n$ *for all* $n \leqslant \omega_0$. This is true for L_0. Suppose it is true for L_n. It is clear from the definition that $L_n \in L_{n+1}$ and $\{X\} \in L_{n+1}$ for all $X \in L_n$. Moreover, any subset of L_n can be represented as a finite difference $(\cdots(L_n\backslash\{X_1\})\backslash\{X_2\})\backslash\cdots\backslash\{X_k\}$, where the $X_i \in L_n$ are the elements not in the given subset.

1.4. card $L_\alpha =$ card α *for all infinite ordinals* α. In fact, for $X \in V$ let

$$\Phi(X) = X \cup \bigcup_{i=1}^{3} F_i^{''}(X \times X) \cup \bigcup_{j=4}^{8} F_i^{''}(X),$$

where $F^{''}(X) = \{F(Y) | Y \in X\}$ is the image of F restricted to the elements of X. Then $\mathcal{J}(X) = \bigcup_{n=0}^{\infty} \Phi^n(X)$. It is hence clear that card $\mathcal{J}(X) =$ card X if X is infinite. We now prove the assertion 1.4 by induction on α.

Obviously card $L_\alpha \geqslant$ card α. Suppose that α is the least infinite ordinal for which card $L_\alpha >$ card α. By 1.3, we have $\alpha > \omega_0$. α cannot be a limit ordinal, or we would have card $L_\alpha = \Sigma_{\beta<\alpha}$ card $\beta =$ card α. But the case $\alpha = \beta + 1$ is also impossible, since in that case card $L_\alpha \leqslant$ card $\mathcal{J}(L_\beta \cup \{L_\beta\}) =$ card $(L_\beta \cup \{L_\beta\}) =$ card $\beta =$ card α. □

In particular, the result 1.4 shows that beginning with $w_0 + 1$, the inclusion $L_\alpha \subset V_\alpha$ becomes a strict inequality, since card $V_{\omega_0+1} = 2^{\omega_0}$. Of course, this does not in principle exclude the possibility that $\forall_\alpha\ \exists\beta > \alpha,\ L_\beta \supset V_\alpha$, but it seems that there is no such β even for $\alpha = \omega_0 + 1$.

1.5. *L is transitive:* $Y \in X \in L_\alpha \Rightarrow Y \in L_\alpha,$ *i.e.,* $L_\alpha \subset L_{\alpha+1}$. See Section 13 of the appendix to Chapter II; the proof is no different for L.

1.6. *L is a big class*: by definition, this means that *for any $X \in V$ with $X \subset L$ there exists a $Y \in L$ such that $X \subset Y$.*

On L we consider the function $\phi(x)$ that is equal to the least α for which $x \in L_\alpha$. Let $X \in V,\ X \subset L$. We consider the map ϕ restricted to X. By the replacement axiom, the values of ϕ form some set Y. The elements of Y are ordinals. Let $\beta = \cup Y$. Then for each $x \in X$ we have $\beta \geqslant \phi(x)$, so that $X \subset L_\beta$.

Effective numbering of L by ordinals.

We order pairs of ordinals $\langle\alpha,\ \beta\rangle$ by the relation

$$\begin{aligned}\langle\alpha_1,\ \beta_1\rangle < \langle\alpha_2,\ \beta_2\rangle \Leftrightarrow\ &\text{either}\max(\alpha_1,\ \beta_1) < \max(\alpha_2,\ \beta_2),\\ &\text{or else these maxima are equal and } \alpha_1 < \alpha_2,\\ &\text{or else these maxima are equal and } \alpha_1 = \alpha_2\\ &\text{and } \beta_1 < \beta_2.\end{aligned}$$

Further, we order triples $\langle i, \alpha, \beta\rangle$, where $i = 0, \dots, 8$, by the relation

$$\begin{aligned}\langle i_1,\ \alpha_1,\ \beta_1\rangle < \langle i_2,\ \alpha_2,\ \beta_2\rangle \Leftrightarrow\ &\text{either } \langle\alpha_1, \beta_1\rangle < \langle\alpha_2, \beta_2\rangle,\\ &\text{or else } \langle\alpha_1, \beta_1\rangle = \langle\alpha_2, \beta_2\rangle \text{ and } i_1 < i_2.\end{aligned}$$

We call these triples *important.*

1.7. **Lemma.** *The class of important triples is well-ordered by the relation $<$. In addition, the following assertions hold:*

(a) *The next triple after $\langle i,\ \alpha,\ \beta\rangle$ has the form*

$$\begin{aligned}&\langle i+1,\ \alpha,\ \beta\rangle, && \textit{if } i \leqslant 7;\\ &\langle 0,\ \alpha+1,\ \beta\rangle, && \textit{if } i = 8 \textit{ and } \alpha+1 < \beta;\\ &\langle 0,\ \alpha+1,\ 0\rangle, && \textit{if } i = 8 \textit{ and } \alpha+1 = \beta;\\ &\langle 0,\ \alpha,\ \beta+1\rangle, && \textit{if } i = 8 \textit{ and } \alpha > \beta;\\ &\langle 0,\ 0,\ \beta+1\rangle, && \textit{if } i = 8 \textit{ and } \alpha = \beta.\end{aligned}$$

(b) *Limit triples have the form*

$\langle 0,\ \alpha,\ \beta\rangle$, *if $\alpha+1 \leqslant \beta$ and α is a limit ordinal*: *this is the limit of $\langle i,\ \gamma,\ \beta\rangle, \gamma < \alpha$;*

$\langle 0,\ \alpha,\ 0\rangle$, *if a is a limit ordinal*: *this is the limit of $\langle i,\ \gamma,\ \alpha\rangle, \gamma < \alpha$;*

$\langle 0,\ \alpha,\ \beta\rangle$, *if $\alpha \geqslant \beta$ and β is a limit ordinal*: *this is the limit of $\langle i, \alpha, \gamma\rangle, \gamma < \beta$;*

$\langle 0,\ 0,\ \beta\rangle$, *if β is a limit ordinal*: *this is the limit of $\langle i,\ \alpha,\ \gamma\rangle$, $\alpha < \beta$, $\gamma < \beta$.*

Proof. The proof follows immediately from the definitions. We shall illustrate this by showing explicitly how to find the least triple in any nonempty class C of triples. We set

$$\gamma = \min\{\max(\alpha,\ \beta) | \langle i,\ \alpha,\ \beta\rangle \in C\};$$
$$C_\gamma = \{\langle i,\ \alpha,\ \beta\rangle \in C | \max(\alpha,\ \beta) = \gamma\}.$$

If C_γ does not contain any triples of the form $\langle i,\ \alpha,\ \gamma\rangle$, then let β_0 be the minimum of the third coordinates of triples in C_γ, and let i_0 be the least i such that $\langle i,\ \gamma,\ \beta_0\rangle \in C_\gamma$. Then $\langle i_0,\ \gamma,\ \beta_0\rangle$ is the least triple in C. Otherwise, let C'_γ consist of triples of the form $\langle i,\ \alpha,\ \gamma\rangle \in C_\gamma$, let α_0 be the minimum of the second coordinates in C'_γ, and let i_0 be the least i such that $\langle i,\ \alpha_0,\ \gamma\rangle \in C_\gamma$. Then $\langle i_0,\ \alpha_0,\ \gamma\rangle$ is the least triple in C.

The exact form of assertions (a) and (b) will be needed only in §5. The lemma implies that there exists a unique order-preserving isomorphism

$$K : \{\text{ordinals}\} \ \Rightarrow\ \{\text{important triples}\}.$$

Using this isomorphism, we recursively define a numbering mapping

$$N : \{\text{ordinals}\} \ \Rightarrow L.$$

Since we have $\alpha < \gamma$ and $\beta < \gamma$ if $\gamma > 0,\ i > 0$, and $K(\gamma) = \langle i,\ \alpha,\ \beta\rangle$, we may set

$$N(\gamma) = \begin{cases} L_\alpha, & \text{for } i = 0; \\ F_i(N(\alpha),\ N(\beta)), & \text{for } i = 1, 2, 3; \\ F_i(N(\alpha)), & \text{for } i = 4, 5, 6, 7, 8. \end{cases}$$

1.8. **Lemma.**

(a) *The mapping N is correctly defined.*
(b) *The image of N coincides with all of L.*

Proof.

(a) To verify correctness, it suffices to show that $\{L_\alpha\} \in L$ and that the class L is closed with respect to the operations F_i. In fact, then induction on γ shows that $N(\gamma) \in L$ if $N(\alpha) \in L$ for all $\alpha < \gamma$.

Let $X, Y \in L_\alpha$. Since L is transitive (see 1.5), we easily find that $F_1(X, Y)$, $F_2(X, Y)$, and $F_4(X)$ belong to $\mathcal{P}(L_\alpha)$, and hence to $L_{\alpha+1}$. For example,

$$U \in F_4(X) \Rightarrow \exists W \langle U, W\rangle \in L_\alpha \Rightarrow \{U\} \in L_\alpha \Rightarrow U \in L_\alpha.$$

Further, $X \times Y$ is a subset of the ordered pairs of elements in L_α. We showed that the unordered pairs lie in $L_{\alpha+1}$, so that the ordered pairs lie in $L_{\alpha+2}$, and finally $X \times Y \in L_{\alpha+3}$ and $F_5(X) \in L_{\alpha+4}$. Analogously, the elements of $F_i(X)$ for $i = 6, 7, 8$ are ordered triples of elements in L_α, so that $F_i(X) \in L_{\alpha+6}$.

(b) Let Z be the image of N. We show by induction on α that $L_\alpha \subset Z$. If α is a limit ordinal and $L_\gamma \subset Z$ for each $\gamma < \alpha$ then also $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma \subset Z$. Suppose $\alpha = \beta + 1$ and $L_\beta \subset Z$, and let $X \in L_\alpha$. Then $X \in \Phi^n(L_\beta \cup \{L_\beta\})$ and we show that $X \in Z$ by induction on n.

(b_1) $n = 0$. Then either $X \in L_\beta$ so $X \in Z$ by the induction hypothesis, or else $X = L_\beta$, in which case $X = N(\gamma)$ for γ such that $K(\gamma) = \langle 0,\ \beta,\ 0\rangle$.

(b_2) $n > 0$. Let $X = F_i(Y, Z),\ i = 1, 2, 3;\ Y, Z \in \Phi^{n-1}(L_\beta \cup \{L_\beta\})$. By the induction hypothesis, $Y = N(\gamma_1)$ and $Z = N(\gamma_2)$ for some ordinals γ_1, γ_2. Therefore $X = N(\gamma)$, where $K(\gamma) = \langle i, \gamma_1, \gamma_2\rangle$.

Let $X = F_i(Y),\ i = 4, \ldots, 8;\ Y \in \Phi^{n-1}(L_\beta \cup \{L_\beta\})$. The verification is analogous.

The lemma is proved. □

In §3 the numbering N will allow us to prove that a strong form of the axiom of choice is L-true. The fundamental step in the proof is to choose the element with the least N-number in each constructible set.

2 Definability and Absoluteness

2.1. Let $M \subset V$ be a nonempty class, and let P be a formula in $\mathrm{L_1Set}$. As in §7 of Chapter II, we shall consider the truth values $|P|_M(\xi)$ for $\xi \in \overline{M}$, where we take the standard interpetation of $\mathrm{L_1Set}$ in V restricted to M. We then say that the formula P is M-true if $|P|_M = 1$ for all ξ.

We shall also consider formulas "with constants in M," where we assume that the language $\mathrm{L_1Set}$ has been extended so that its alphabet includes names for all the elements of M. We shall designate these elements by the same letters as in the metalanguage ($X, Y, \ldots$ for sets; $\alpha, \beta, \ldots$ for ordinals, etc.), which we hope will not lead to confusion. We extend the definition of $|P|_M(\xi)$ to formulas with constants in M in the obvious way: we take $X^\xi = X$ for any constant X and any point ξ.

2.2. **Definition.** Let $X_i \in M,\ i = 1, \ldots, n$. Sets of the form

$$\{\langle y_1^\xi, \ldots, y_n^\xi\rangle | \xi \in \overline{M},\ y_i^\xi \in X_i \text{ for } i = 1, \ldots, n;\ |P|_M(\xi) = 1\} \\ \subset X_1 \times \cdots \times X_n$$

are called M-definable sets. Here P runs through all formulas with constants in M and free variables in the set $\{y_1, \ldots, y_n\}$.

If $P(y_1, \ldots, y_n,\ Z_1, \ldots, Z_m)$ is such a formula (where the notation shows the constants and free variables) and if $y_i^\xi = Y_i$, we shall often write "$P(Y_1, \ldots, Y_n, Z_1, \ldots, Z_m)$ is M-true" instead of $|P|_M(\xi) = 1$.

The next proposition, which, in particular, is applicable to L, is a basic instrument for proving many assertions about L.

2.3. **Proposition.** *Let $M \subset V$ be a transitive big class (see 1.6) that is closed with respect to the operations $F_1, \ldots, F_8$. Then all M-definable sets are elements of M.*

PROOF. The proof is by induction on the number of connectives and quantifiers in the defining formula P.

(a) $P(y_1, \ldots, y_n;\ Z_1, \ldots, Z_m)$ is an atomic formula. It can have one of eight possible forms: the predicate can be either $\in$ or $=$, and on each side of $\in$ or $=$ we can have either a constant or a variable. But all of these cases reduce to two: $y_i \in y_j$ and $y_i \in Z_j$, if we are willing to make the formula a little more complicated. For example, since M is transitive, we have

$$\text{“}y = Z\text{”} \quad \text{defines the same set as } \forall z(z \in Z \Leftrightarrow z \in y),$$
$$\text{“}Z \in y\text{”} \quad \text{defines the same set as } \exists z(z = Z \wedge z \in y),$$

and so on. We therefore analyze these two basic cases.

(a$_1$) $y_i \in Z$. We have $Z \cap X_i = Z\backslash(Z\backslash X_i) \in M$, since Z and $X_i \in M$, and M is F_2-closed; and we have $X_1 \times \cdots \times X_{i-1} \times Z \cap X_i \times \cdots \times X_n \in M$, since M is F_3-closed. This last set is M-definable by the formula $y_i \in Z$, because M is transitive.

(a$_2$) $y_i \in y_j$. We use induction on $n \geqslant 3$. Let

$$Y = \{\langle Y_1, \ldots, Y_n\rangle | Y_k \in X_k \quad \text{for } k = 1, \ldots, n;\ Y_i \in Y_j\}.$$

The case $\langle i, j\rangle = \langle n-1, n\rangle$. Let $X_{n-1} \cup X_n \subset X \in M$. Then

$$Y = \\ \times F_6(F_5(X) \times (X_1 \times \cdots \times X_{n-2}) \cap (X_{n-1} \times X_n) \times (X_1 \times \cdots \times X_{n-2})).$$

The case $\langle i, j\rangle = \langle n, n-1\rangle$. Again let $X_{n-1} \cup X_n \subset X \in M$. Then

$$Y = \\ \times F_7(F_5(X) \times (X_1 \times \cdots \times X_{n-2}) \cap (X_{n-1} \times X_n) \times (X_1 \times \cdots \times X_{n-2})).$$

The case $n \notin \{i, j\}$. By the induction assumption, the set $Y^{'}$, which is M-defined by the formula $y_i \in y_j$ in $X_1 \times \cdots \times X_{n-1}$, lies in M. But $Y = Y^{'} \times X_n$.

The case $n - 1 \notin \{i, j\}$. Let $Y^{'}$ be M-defined by the formula $y_i \in y_j$ in $X_1 \times \cdots \times X_{n-2} \times X_n$. Then $Y = F_8(Y^{'} \times X_{n-1})$.

The case $n = 2$ reduces to the case $n = 3$ by taking the direct product with $\{\varnothing\}$ and projecting. The projection of $X_1 \times \cdots \times X_n$ onto X_1 is $F_4 \circ \cdots \circ F_4$ $(n-1$ times$)$.

(b) *Connectives.* $\wedge$ corresponds to intersection, and $\neg$ corresponds to taking the complement (relative to $X_1 \times \cdots \times X_n$). M is closed with respect to these operations, and the other connectives can be expressed in terms of these two.

(c) *Quantifiers.* It suffices to verify $\exists$. This corresponds to projecting, because M is a big class. More precisely, let Y be M-defined by the formula $\exists y_{n+1} P(y_1, \ldots, y_n, y_{n+1})$ in $X_1 \times \cdots \times X_n$. We have

$$\langle Y_1, \ldots, Y_n\rangle \in Y \Leftrightarrow$$
$$\text{there exists a } Y_{n+1} \in M \text{ such that } P(Y_1, \ldots, Y_{n+1}) \text{ is } M\text{-true}.$$

To each $\langle Y_1, \ldots, Y_n\rangle \in X_1 \times \cdots \times X_n$ we associate the least ordinal a for which there exists $Y_{n+1} \in M \cap V_\alpha$ such that $P(Y_1, \ldots, Y_{n+1})$ is M-true, if there is such a Y_{n+1}. This gives rise to a function on $Y \subset X_1 \times \cdots \times X_n$. Let A be the set of its values, and let $\beta = \cup A$. Then $X = M \cap V_\beta$ is a set, and $X \subset M$. Since M is a big class, there exists $X_{n+1} \in M$ such that $X \subset X_{n+1}$. By the induction assumption, the M-definable subset $Y^{'} \subset X_1 \times \cdots \times X_n \times X_{n+1}$ consisting of those points $\langle Y_1, \ldots, Y_{n+1}\rangle$ for which $P(Y_1, \ldots, Y_{n+1})$ is M-true belongs to M. But $Y = F_4(Y^{'})$, and M is closed under F_4.

The proposition is proved. □

In order to be able to use Proposition 2.3, we need criteria for verifying M-truth. As remarked in §7 of Chapter II, the basic technical tool for this is the notion of *absoluteness*. A formula P is called M-absolute ((M, V)-absolute in the terminology of Chapter II) if $|P|_M(\xi) = |P|_V(\xi)$ for all $\xi \in \overline{M} \subset \overline{V}$. The standard method of proving that a formula is M-true is to prove that it is V-true and M-absolute.

The following lemma provides us with a large class of M-absolute formulas.

2.4. **Lemma.**

(a) *Atomic formulas are M-absolute for all M.*

(b) *If the formulas P, P_1, and P_2 are M-absolute, then so are the formulas $\neg P$ and $P_1 * P_2$ (where $*$ is any connective).*

(c) *Suppose that the class M is transitive, and is closed with respect to an operation f of degree r. If the formula P is M-absolute, then the "restricted quantifier" formulas*

$$\forall x(x \in f(y_1, \ldots, y_r) \Rightarrow P),$$
$$\exists x(x \in f(y_1, \ldots, y_r) \wedge P)$$

are also M-absolute.

PROOF. Part (c) is the only assertion that might not be completely obvious. Before proving it, we make one remark. The formula $x \in f(y_1, \ldots, y_r)$ is written in a suitable extension of $\mathrm{L_1Set}$, and may be assumed to be V-equivalent to some formula $P(x, y_1, \ldots, y_r)$ in $\mathrm{L_1Set}$ (with constants in M) for which $\forall y_1, \ldots, \forall y_r\ \exists! x\, P$ or a restricted version of this formula is deducible from the Zermelo–Fraenkel axioms. This P determines the operation f. We also allow the case $r = 0$; then f is simply a constant in M. We shall identify f with its standard interpretation, i.e., we shall denote terms by $f(Y_1, \ldots, Y_r) \in M$ for $Y_1, \ldots, Y_r \in M$.

Now let $\xi \in \overline{M}$, $y_i^\xi = Y_i \in M$, $Q = \exists x(x \in f(y_1, \ldots, y_r) \wedge P)$, $Y = f(Y_1, \ldots, Y_r) \in M$. Then

$$|Q|_M(\xi) = \sup_{X \in M} (|X \in Y|_M \cdot |P|_M(\xi^{'})),$$

where the $\xi^{'} \in \overline{M}$ are variations of ξ along x such that $x^{\xi^{'}} = X$. Since P is absolute, it follows that $|P|_M(\xi^{'}) = |P|_V(\xi^{'})$, and since M is transitive, it follows that if $X \notin M$, then $|X \in Y|_M = |X \in Y|_V = 0$. Hence, on the right we can write V everywhere in place of M and can let ξ' run through all variations of ξ along x in V with $x^{\xi'} = X$. The resulting expression equals $|Q|_V(\xi)$.

The quantifier $\forall$ can be handled analogously, or else can be reduced to $\exists$. The lemma is proved. □

We shall abbreviate the restricted quantifier formulas in 2.4(c) as

$$(\forall x \in f(y_1, \ldots, y_r))P, \qquad (\exists x \in f(y_1, \ldots, y_r))P,$$

respectively.

If all the quantifiers in a formula Q are restricted in this way, we say that Q is a Σ_0-formula.

As a first application of the results in 2.3 and 2.4, we prove the following fact.

2.5. **Proposition**. *All ordinals are constructible.*

Proof. Suppose that this is not the case, and that β is the least nonconstructible ordinal. All of the elements in β are contained in L_α. Since L is transitive, it follows that all $\gamma \geqslant \beta$ are nonconstructible. Hence,

$$\beta = \{x | (x \text{ is an ordinal} \wedge x \in L_\alpha) \text{ is } V\text{-true}\}.$$

If we show that "V-true" may be replaced by "L-true" here, we immediately have a contradiction, since then $\beta \in L$ by Proposition 2.3.

To do this, it suffices to verify that the formula "x is an ordinal" is L-absolute. Using the regularity axiom, from which $\neg(y \in y)$ is deducible, we can write this formula in the following Σ_0-form:

$$(\forall y \in x)(\forall z \in y)(z \in x) \wedge (\forall y_1 \in x)(\forall y_2 \in x)(y_1 \in y_2 \vee y_2 \in y_1 \vee y_1 = y_2)$$

and then apply Lemma 2.4. □

3 The Constructible Universe as a Model for Set Theory

3.1. **Theorem.** *The Zermelo–Fraenkel axioms are L-true.*

Proof. The general principle for verifying the axioms is to note that every set whose existence is stipulated in a given axiom can be represented as a set defined by a Σ_0-formula with constants in L. We only occasionally have to perform a direct verification that a subformula is L-absolute.

(a) *Empty set.* This axiom is equivalent to the Σ_0-formula $\neg\exists x(x \in \varnothing)$, which is V-true.

(b) *Extensionality.* This axiom can be represented in Σ_0-form. In addition, in Section 4.8 of Chapter II we verified this axiom for any transitive class.

(c) *Pairing.* A direct computation of the L-truth function gives 1, since L is closed with respect to forming pairs.

(d) *Regularity.* This follows by a direct computation using the transitivity of L.

(e) *Union.* Here it is somewhat more complicated to reduce the axiom to a Σ_0-formula. The axiom is written in the form

$$\forall x\ \exists y\ \forall u(\exists z(u \in z \wedge z \in x) \Leftrightarrow u \in y).$$

Let $\xi \in \overline{L}$, let $\xi^{'}$ be any variation of ξ along x, and let $X = x^{\xi^{'}} \in L$. We must show that

$$|\exists y\ \forall u(\exists z(u \in z \wedge z \in X) \Leftrightarrow u \in y)|_L(\xi^{'}) = 1.$$

It suffices to find a $Y \in L$ such that

$$|\forall u(\exists z(u \in z \wedge z \in X) \Leftrightarrow u \in Y)|_L = 1,$$

i.e., such that for all $U \in L$,

$$|(\exists z \in X)(U \in z)|_L = |U \in Y|_L.$$

We can clearly take $Y = \bigcup_{z \in X} Z$ if we show that Y is constructible. Since L is transitive, we know that all the elements of Y are constructible. Hence, there exists a constructible set $Y^{'}$ such that $Y^{'} \supset Y$. Then Y can be represented as follows (where we replace V-truth by L-truth using Lemma 2.4):

$$Y = \{U | U \in Y^{'};\ (\exists z \in X)(U \in z) \text{ is } L\text{-true}\}.$$

Now the required assertion follows by Proposition 2.3.

In what follows we shall usually omit explicit mention of the points $\xi \in \overline{L}$.

(f) *Power set axiom* $\forall x\ \exists y\ \forall z(z \subset x \Leftrightarrow z \in y)$. We fix $X \in L$, form the set $Y = \mathcal{P}(X) \cap L$ of constructible subsets of X, and show that Y is constructible. In fact, let $Y^{'} \supset Y$, where $Y^{'}$ is constructible. Then by Lemma 2.4,

$$Y = \{Z | Z \in Y^{'};\ (Z \subset X) \text{ is } L\text{-true}\},$$

because $Z \subset X$ has the Σ_0-form $(\forall z \in Z)(z \in X)$. Now a direct computation gives

$$|\forall z(z \subset X \Leftrightarrow z \in Y)|_L = 1.$$

(g) *Infinity.* This axiom is L-true because of the constructibility of the set $\big\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \dots\big\}$, which can be represented in the form

$$\big\{Y | Y \in L_{\omega_0};\ \big[Y = \varnothing \vee (\exists y \in L_{\omega_0})(Y = \{y\})\big] \text{ is } L\text{-true}\big\}.$$

(h) *Replacement.* Let $\overline{z} = \langle z_1, \dots, z_n \rangle$. This axiom is written in the form

$$\forall \overline{z}\ \forall u\Big(\forall x\big(x \in u \Rightarrow \exists! y P(x,\ y,\ \overline{z})\big) \Rightarrow \exists w\ \forall y(y \in w \Leftrightarrow \exists x(x \in u \wedge P(x, y, \overline{z})))\Big).$$

We fix $Z_1, \dots, Z_n \in L, \overline{Z} = \langle Z_1, \dots, Z_n \rangle$, and $U \in L$. It is sufficient to consider the case that the premise is L-true, i.e., for all $X \in L$,

$$|X \in U \Rightarrow \exists! y P(X, y, \overline{Z})|_L = 1.$$

We must find a value $W \in L$ of w for which the conclusion is L-true. We set $W^{'}$ = a constructive set containing as elements all constructive Y for which

$$(\exists x \in U) P(x, Y, \overline{Z}) \text{ is } L\text{-true}.$$

This set exists because since the premise of the axiom is L-true, it follows that each $X \in U$ corresponds to at most one constructible Y. We then set

$$W = \Big\{ Y | Y \in W^{'}; (\exists x \in U) \Big(P(x, Y, \overline{Z}) \Big) \text{ is } L\text{-true} \Big\}.$$

This set is constructible by Proposition 2.3, and it follows from the way it is defined that

$$\Big| \forall y \Big(y \in W \Leftrightarrow \exists x \big(x \in U \wedge P(x, y, \overline{Z}) \big) \Big) \Big|_L = 1.$$

(i) *Axiom of choice.* The main intuitive point in the verification is the numbering N of the universe L that was constructed in 1.8. But the formal verification is much more complicated here than in the previous cases. A fair amount of work is needed to give a formalization of the construction in 1.7–1.8 that is sufficiently detailed to prove the following fact:

3.2. **Proposition.** *There exists a formula $N(x, y)$ in L with two free variables such that*

(a) *For any $X, Y \in V$, the formula $N(X, Y)$ is V-true if and only if X is an ordinal and $Y = N(X)$.*
(b) *$N(x, y)$ is L-absolute.*

We shall postpone the proof until §5, and shall make use of this proposition to verify the axiom of choice. We divide this verification into two steps.

3.3. Universal choice function. Let $X \in L$ be a nonempty set. We construct the function Y that for every nonempty $Z \in X$ chooses the element U in Z with the least N-number (see 1.8):

$$Y = \Big\{ \langle Z, U \rangle | Z \in X,\ U \in \bigcup_{X' \in X} X^{'};\ U \in Z \wedge \exists w \Big(N(w, U) \wedge \forall z \big(z \in Z \\ \Rightarrow (z = U \ \vee \ \forall w^{'} (N(w^{'}, z) \Rightarrow w \in w^{'}))\big) \Big) \text{ is } V\text{-true} \Big\}.$$

We want to prove that $Y \in L$. By Proposition 2.3, this holds if we can define Y by means of the L-truth of a formula. We are not allowed mechanically to replace V by L, since it is not immediately obvious from its external form that this formula is L-absolute. We proceed as follows: taking into account the constructibility of the ordinals, we take all ordinals that occur as the least

N-numbers of the elements of the constructible set $\cup_{X' \in X} X' = \cup(X)$, and we find a constructible set W that contains these ordinals. Then we replace $\exists w$ by $\exists w \in W$ and $\forall w'$ by $\forall w' \in W$ in the formula. The set Y does not change, and now V-truth may be replaced by L-truth, as can be seen using Proposition 3.2 and Lemma 2.4.

3.4. We now compute the L-truth value of the axiom of choice:

$$\forall x\big(x \neq \varnothing \Rightarrow \exists y(y \text{ is a function} \wedge \operatorname{dom} y = x \\ \wedge (\forall z \in x)(z \neq \varnothing \Rightarrow y(z) \in z))\big).$$

It suffices to show that if we take a nonempty $X \in L$ and the constructible choice function $Y \in L$ in 3.3, then

$$|Y \text{ is a function}|_L = |\operatorname{dom} Y = X|_L = |(\forall z \in X)(z \neq \varnothing \Rightarrow Y(z) \in z)|_L = 1.$$

The third formula here is V-true, and is written in Σ_0-form except for the subformula $Y(z) \in z$, which can be replaced by $(\forall u \in U(Y))(\langle z, u\rangle \in Y \Rightarrow u \in z)$. Thus, the third formula is L-absolute and hence L-true.

We verify that the first two formulas are absolute in §5. They are V-true by construction. This completes the proof of Proposition 3.1. □

We note that the same argument shows the following: *all the axioms*, with the possible exception of the axiom of choice, *are M-true for any transitive big class M that is closed with respect to the operations* $F_1, \ldots, F_8$.

4 The Generalized Continuum Hypothesis Is *L*-True

4.1. We wish to show that the assertion "card $\mathcal{P}(\omega_\alpha) = \omega_{\alpha+1}$" is L-true. A certain amount of caution is essential here, because cardinality is not an L-absolute notion. If Y is a constructible set, let $\operatorname{card}_L(Y)$ be the least ordinal β for which there exists in L a one-to-one onto function $f : Y \to \beta$. Hence "card $(Y) =$ card (Z)" is L-true iff $\operatorname{card}_L(Y) = \operatorname{card}_L(Z)$. Note that although $\operatorname{card}_L(Y) \geqslant \operatorname{card}(Y)$, equality fails if there are one-to-one onto functions $Y \to \beta$ in V, but no such function lies in L. The cardinal ω_α in L is the αth ordinal $\beta > \omega_0$ such that $\operatorname{card}_L(\beta) = \beta$. Thus ω_α in L may not coincide with the "real" ω_α, that is, with ω_α in V.

We shall show that for each ordinal β and each constructible $X \subset \beta$ there is an ordinal γ with $X \in L_\gamma$ and $\operatorname{card}_L(\gamma) = \operatorname{card}_L(\beta)$. Hence $\mathcal{P}(\beta) \cap L \subset L_{\beta^+}$, where β^+ is the least ordinal greater than β such that $\operatorname{card}_L(\beta^+) \neq \operatorname{card}_L(\beta)$. The L-truth of the generalized continuum hypothesis will then follow if we show the L-truth of "card $(\beta^+) = \beta^+$."

Our proof exploits throughout a proposition that requires a good deal of work formalizing the construction of L within $\mathrm{L_1Set}$.

4.2. **Proposition.** *There exists a formula* $L(x, y)$ *of* $\mathrm{L_1Set}$ *with two independent variables* *such that*

(a) *for any X and Y in V, $L(X,Y)$ is V-true $\Leftrightarrow$ Y is an ordinal and $X \in L_\gamma$;*
(b) *for any transitive model $M \subset V$ of the axioms (without the axiom of choice), the formula $L(x,y)$ is M-absolute. In particular, it is L-absolute.*

We again postpone the proof until §5.

4.3. **Lemma.** *Let $X \subseteq \beta$ be constructible. Then $X \in L_\gamma$ for some ordinal γ such that* card $_L(\gamma) = \mathrm{card}_L(\beta)$.

Proof. In this deduction, in addition to Proposition 4.2 we use versions of Propositions 7.3 and 7.6 of Chapter II that apply to the constructible universe. They are formulated precisely and proved below, in Sections 4.5 and 4.6.

Suppose that $X \subset \beta$ is constructible. Let δ be an ordinal such that $X \in L_\delta$. We enlarge the alphabet of $\mathrm{L_1Set}$ by adding names $\bar{\delta}$ and $\overline{X}$ for δ and X. Let $\mathcal{E}$ be the set of formulas

$$\{\text{axioms of } \mathrm{L_1Set}\} \cup \left\{L(\overline{X}, \bar{\delta})\right\}.$$

Let $N_0 \subset L$ be the set $\beta \cup \{X\} \cap \{\delta\}$. By Proposition 4.5 there is a constructible set N such that $N_0 \subset N$, all formulas in $\mathcal{E}$ are (N, L)-absolute, and $\mathrm{card}_L(N) = \mathrm{card}_L(\beta)$. Thus $(N, \in)$ is a model for the axioms and, by Proposition 4.2 (a), for $L(\overline{X}, \bar{\delta})$. Now N might not be transitive, but then by Proposition 4.6 there are a transitive axiom model (M, ε) and a constructible isomorphism $f : (N, \in) \xrightarrow{\sim} (M, \varepsilon)$. Hence $L(\overline{X}, \bar{\delta})$ is M-true and $\mathrm{card}_L(M) = \mathrm{card}_L(N)$. What are the interpretations of the constants $\overline{X}$ and $\bar{\delta}$ in M?

Since the set $\beta \subset N$ is transitive, it goes to itself under the isomorphism f; hence so does the set $X \subset \beta$. Let δ_M be the image of δ under f. Since by Proposition 4.2(b) the formula $L(x,y)$ is M-absolute, and $L(\overline{X}, \bar{\delta})$ is M-true, it follows that $L(X, \delta_M)$ is V-true, so that δ_M is an actual ordinal and $X \in L_{\delta_M}$. Moreover, since $\delta_M \in M$ and M is transitive, $\delta_M \subset M$; hence $\mathrm{card}_L(\delta_M) \leqslant \mathrm{card}_L(M)$. Letting γ be the larger of δ_M and β, we have $\mathrm{card}_L(\gamma) = \mathrm{card}_L(\beta)$ and $X \in L_\gamma$. The lemma is proved. $\square$

4.4. Deduction that the GCH is L-true from the lemma. Let β^+ be the smallest ordinal greater than β such that $\mathrm{card}_L(\beta^+) \neq \mathrm{card}_L(\beta)$. Then Lemma 4.3 implies the V-truth of the formula

$$\forall z (z \in L \Rightarrow (z \subset \beta \Rightarrow z \in L_{\beta+})).$$

Since "$z \in L_{\beta+}$" (i.e., the formula $L(z, \beta^+)$) is L-absolute, it follows that

$$\forall z (z \subset \beta \Rightarrow z \in L_{\beta+})$$

is L-true. Now if β is the cardinal ω_α in L then β^+ is the cardinal $\omega_{\alpha+1}$ in L. Hence for each α we have shown the L-truth of

$$\mathcal{P}(\omega_\alpha) \subset L_{\omega_{\alpha+1}}.$$

We claim that the following formula is also L-true:

$$\mathrm{card}(L_{\omega_{\alpha+1}}) = \omega_{\alpha+1}.$$

Since "$\mathrm{card}(\mathcal{P}\omega_\alpha)) \leqslant \omega_{\alpha+1}$" is formally deducible in $\mathrm{L_1Set}$ from the preceding two formulas, and since all the axioms are L-true, this will show that the GCH is L-true.

Our claim is verified thus: In Section 1.4 we proved that $\mathrm{card}(L_\gamma) = \mathrm{card}(\gamma)$ for each ordinal γ. Indeed, that proof can be formalized in $\mathrm{L_1Set}$, using the formula $L(x, y)$ of Proposition 4.2. That is, the assertion "$\forall\gamma(\mathrm{card}(L_\gamma) = \mathrm{card}(\gamma))$" is deducible from the axioms (see 5.17). Since the axioms are L-true, this assertion is then L-true. But since "$\mathrm{card}(w_{\alpha+1}) = w_{\alpha+1}$" is trivially L-true, the claim follows. This completes the proof. $\square$

4.5. **Proposition.** *Let $\mathcal{E}$ be a constructible countable set of L-true formulas in the language* $\mathrm{L_1Set}$*, and let M_0 be a constructible set. Then there exists a constructible set $M \supset M_0, \mathrm{card}_L(M) \leqslant \mathrm{card}_L(M_0) + \omega_0$, such that all of the formulas in $\mathcal{E}$ are (M, L)-absolute.*

Proof. The general scheme is the same as in Section 7.3 of Chapter II, but some additional precautions are required. The main point is to prove that if $P(x, \overline{y}), \overline{y} = (y_1, \ldots, y_n)$, is a formula in $\mathcal{E}$, then there exists a constructible set $M \supset M_0$ with $\mathrm{card}_L(M) \leqslant \mathrm{card}_L(M_0) + \omega_0$ that can be constructed constructibly from P and has the property that $\exists x(P(x, \overline{y}))$ is (M, L)-absolute. After this we must verify constructible closure over all $P \in \mathcal{E}$.

We reproduce the construction in Section 7.3 of Chapter II. We construct the set M_i by induction. Let $\overline{Y} = \langle Y_1, \ldots, Y_n\rangle \in M_i \times \cdots \times M_i$. We let $\hat{M}_i(\overline{Y})$ denote the class $\{X | P(X, Y_1, \ldots, Y_n)$ is L-true$\}$. We let $\tilde{M}_i(\overline{Y})$ denote $\varnothing$ if $\hat{M}_i(\overline{Y})$ is empty, and $\hat{M}_i(\overline{Y}) \cap L_\alpha$ for the least α for which this intersection is nonempty otherwise. Since $L(x, y)$ is absolute (see §5), it is not hard to see that the function $\tilde{M}_i$, $\mathrm{dom}\, \tilde{M}_i = M_i \times \cdots \times M_i$, is constructible. Because the constructible axiom of choice holds in L, we can obtain a constructible function F_i by choosing one element from each nonempty $\tilde{M}_i(\overline{Y})$. Let N_i be the set of values of $\tilde{M}_i$. This set is constructible, since all of our constructions are absolute; and if M_i is infinite, then $\mathrm{card}_L(N_i) = \mathrm{card}_L(M_i)$. We set $M_{i+1} = M_i \cup N_i$ and $M = \cup M_i$. The set M has the required properties; obviously, $\mathrm{card}_L(M) + \omega_0 = \mathrm{card}_L(M_0) + \omega_0$ in L. The formal transition from $\{M_i\}$ to M is realized by considering a function that "closes" M_0, as in Section 5.11 below. $\square$

4.6. **Proposition.** *For every constructible set N such that the extensionality axiom is N-true there exist a unique constructible transitive set M and isomorphism $f : (N, \in) \xrightarrow{\sim} (M, \varepsilon)$.*

Proof. The plan of proof is the same as in Section 7.6 of Chapter II. First let "f is a continuous $(\alpha+1)$-sequence" be the formula "α is an ordinal"$\wedge$"f is a function"$\wedge$ $\mathrm{dom} f = \alpha + 1 \wedge (\forall\beta \in \alpha + 1)(\beta$ a limit ordinal $\Rightarrow f(\beta) = \bigcup_{\gamma\in\beta} f(\gamma))$. This formula is shown to be L-absolute as in Section 5.14 below.

Now consider the L-absolute operation $\phi(Z) = \{X | X \in N \wedge X \cap N \subset Z\}$, and let $\varnothing_N$ be the unique member of N such that $\varnothing_N \cap N = \varnothing$. Finally, let $\psi(x, y)$ be the formula

$$(\exists f)(\text{“}f \text{ is a continuous } (x+1)\text{-sequence”} \wedge f(0) = \varnothing_N \wedge \\ \times (\forall \beta \in x)(f(\beta+1) = \phi(f(\beta))) \wedge y = f(x)).$$

Then ψ is L-absolute, as can be shown as in Sections 5.14 and 5.15 below, and $\psi(x, y)$ is L-true if and only if $y = N_x$ in the sense of Chapter II, Section 7.6.

We now set $\hat{N} = \cup_\alpha N_\alpha = \{z | (\exists \alpha)(\exists y \subset N)(\psi(\alpha, y) \wedge z \in y)\}$. We show that $\hat{N} = N$. Clearly $\hat{N} \subset N$, and if $N \backslash \hat{N} = Y$ were nonempty, it would follow by the regularity axiom, which holds in L, that $\exists Z (Z \in Y \wedge Z \cap Y = \varnothing)$. For this Z we would have $Z \subset \hat{N}$, hence $Z \subset N_\alpha$ for a suitable α, so that $Z \in N_{\alpha+1}$, which is a contradiction.

The implication $Z \subset \hat{N} \Rightarrow \exists \alpha (Z \subset N_\alpha)$, which we have used here, follows because there exists an absolute function on $\hat{N}$ that associates to each X the least α for which $X \in N_\alpha$. The replacement axiom shows that there exists an ordinal α_0, namely, the least upper bound of the values of this function, for which $\hat{N} = N = N_{\alpha_0}$. This ordinal, which is fixed for N, occurs in our subsequent construction, which is verified to be absolute as in §5.

Let “h is a constructing $(\alpha+1)$-sequence for N, M” be the formula “h is a continuous $(\alpha+1)$-sequence” $\wedge$ $h(0) = \{\langle \varnothing_N, \varnothing \rangle\}$ $\wedge$ “$(\forall \beta \in \alpha)(h(\beta+1)$ is a function $\wedge$ dom $h(\beta+1) = N_{\beta+1} \wedge$ the value of $h(\beta+1))$ on any $X \in N_{\beta+1}$ is the set of $h(\beta)$-images of elements of $X \cap N$).” Then for each α there is a unique such h; let M_α be the image of $h(\alpha)$. For $\alpha = \alpha_0$ we obtain a function $h : N \to M = M_{\alpha_0}$, where M is our desired constructible set and h is a constructible $\in$-isomorphism.

The proposition is proved. □

5 Constructibility Formula

5.1. The purpose of this section is to prove Propositions 4.2 and 3.2. Both proofs are extremely straightforward, and simply consist in writing out explicitly the formulas $L(x, y)$ and $N(x, y)$ and verifying that the conditions in Lemma 2.4 apply. But since these formulas are very long, we perform the verifications in a series of “blocks,” in order to improve their appearance and to make the interpretation and verification of the conditions in 2.4 easier. As soon as a block (subformula) is constructed and its absoluteness is verified, we replace it by an abbreviated notation in the next formula.

The material within each subsection is arranged in the following order: first the abbreviated notation for the formula that is being constructed and shown to be absolute in the subsection; then the complete form of the formula; and finally any remarks that may be needed regarding absoluteness. The “complete form” of the formula may contain abbreviated notation for subformulas. If such a subformula has not yet been interpreted in detail and shown to be absolute, this is done right after the complete form.

By absoluteness we mean "M-absoluteness for any transitive model M for the axioms without the axiom of choice."

Sections 5.2–5.15 are devoted to the formula $L(x, y)$, and Sections 5.18–5.20 are devoted to the formula $N(x, y)$. As the material we are dealing with accumulates, we shall allow ourselves to omit more and more details and to rely on the reader's experience.

The formulas

$$z = \begin{cases} F_i(x, y), & i = 1, 2, 3; \\ F_j(y), & j = 4, 5, 6, 7, 8. \end{cases}$$

5.2. $z = \{x, y\}: \ (\forall u \in z)(u = x \vee u = y) \wedge x \in z \wedge y \in z$. This whole formula is clearly absolute by Lemma 2.4. From now on we shall not even comment on such simple cases.

5.3. $z = x \backslash y\colon \quad (\forall u \in z)(u \in x \wedge u \notin y) \wedge (\forall u \in x)(u \notin y \Rightarrow u \in z)$.

5.4. $z = x \times y\colon \quad (\forall u_1 \in x)(\forall u_2 \in y)(\langle u_1, u_2 \rangle \in z)$

$$\wedge\, (\forall u \in z)(\exists u_1 \in x)(\exists u_2 \in y)(u = \langle u_1, u_2 \rangle);$$

$$\langle u_1, u_2 \rangle \in z\colon \quad (\exists v \in z)(v = \langle u_1, u_2 \rangle);$$

$$u = \langle u_1, u_2 \rangle\colon \quad (\forall v \in u)(v = \{u_1\} \vee v = \{u_1, u_2\}) \wedge \{u_1\} \in u \wedge \{u_1, u_2\} \in u;$$

$$\{u_1, u_2\} \in u\colon \quad (\exists v \in u)(v = \{u_1, u_2\}).$$

5.5. $Z = F_4(y) = \operatorname{dom} y$: $\quad (\forall u \in z)(\exists v \in \cup \cup (y))(\langle u, v \rangle \in y)$

$$\wedge (\forall u \in \cup\cup(y))(\forall v \in \cup\cup(y))(\langle u, v \rangle \in y \Rightarrow u \in z).$$

Here $\cup\, \cup$ appears because $\langle u, v \rangle = \{\{u\}, \{u, v\}\} \in y \Rightarrow u, v \in \cup \cup (y)$. This formula is absolute, since a transitive model is closed with respect to the operation $\cup$ (see 3.1(e)). We shall write $\cup^2 = \cup\cup$, and so on.

5.6. $z = F_5(y)$: $(\forall u \in z)(\exists v \in y)(\exists w \in y)(v \in w \wedge u = \langle v, w \rangle) \wedge (\forall v \in y)(\forall w \in y)(v \in w \Rightarrow \langle v, w \rangle \in z)$.

5.7. $z = F_6(y)$: $(\forall u \in z)(\exists u_1 \in \cup^4(y))(\exists u_2 \in \cup^4(y))(\exists u_3 \in \cup^2(y))(\langle u_1, u_2, u_3 \rangle \in y \wedge u = \langle u_3, u_1, u_2 \rangle) \wedge (\forall u_1 \in \cup^4(y))(\forall u_2 \in \cup^4(y))(\forall u_3 \in \cup^2(y))(\langle u_1, u_2, u_3 \rangle \in y \Rightarrow \langle u_3, u_1, u_2 \rangle \in z)$. Here $\cup^4$ appears for the same reason as $\cup^2$ in 5.5. The formulas $\langle u_1, u_2, u_3 \rangle \in y$, etc., are shown to be absolute in the same way as in 5.4.

The operations F_7 and F_8 are treated analogously to F_6.

The formulas

$$y = \begin{cases} F_i''(x \times x), & \text{for } i = 1, 2, 3; \\ F_j''(x), & \text{for } j = 4, 5, 6, 7, 8. \end{cases}$$

5.8. $y = F_i''(x \times x),\ i = 1, 2, 3$:

$$(\forall u \in y)(\exists u_1 \in x)(\exists u_2 \in x)(u = F_i(u_1, u_2)) \\ \wedge(\forall u_1 \in x)(\forall u_2 \in x)(F_i(u_1, u_2) \in y),$$

where $F_i(u_1, u_2) \in y:\ (\exists v \in y)(v = F_i(u_1, u_2))$.

5.9. $y = F_j''(x), j = 4, \ldots, 8$: $(\forall u \in y)(\exists v \in x)(u = F_j''(v)) \wedge (\forall v \in x)$ $(F_j''(v) \in y)$.

5.10. $y = \Phi(x)$ (see 1.4):

$$(\forall z \in y)(z \in x \vee z \in F_1''(x \times x) \vee \cdots \vee z \in F_8''(x)) \wedge (\forall z \in x)(z \in y) \\ \wedge (\forall z \in F_1''(x \times x))(z \in y) \wedge \cdots \wedge (\forall z \in F_8'' y(x))(z \in y).$$

The class L is closed with respect to the operations F_i''. In fact, suppose, for example, that $i \geqslant 4$, and let $X \in L$. Let $U \in L$ be a set containing all $F_i(Y)$ for $Y \in X$. Then

$$F_1''(X) = \{Z | Z \in U,\ (\exists y \in X)(Z = F_i(y)) \text{ is } V\text{-true}\}.$$

Since the formula $Z = F_i(y)$ has been shown to be absolute, we may replace "V-true" by "L-true" here, and then apply Proposition 2.3. Thus, the formula $y = \Phi(x)$ is L-absolute by Lemma 2.4.

If M is an arbitrary transitive model, then the verification that M is closed with respect to F_i'' is somewhat different. Namely, the formula $\forall x\ \exists! y(y = F_i''(x))$ is obviously V-true. The formal deduction of this formula does not use the axiom of choice. Hence, the formula is M-true for any transitive model M. We therefore have $Y \in M$ if $X \in M$, where $Y = F_i''(X)$. We shall use this device many times in what follows.

5.11. "*g closes x*," which is short for "*g is a function on ω_0, and $g(n) = \Phi^n(x)$ for all $n \in \omega_0$*." We write the formula with the constant ω_0 and the free variables g and x:

$$\text{“}g \text{ is a function”} \wedge F_4(g) = \omega_0 \wedge g(0) = x \\ \wedge (\forall n \in \omega_0)(g(n+1) = \Phi(g(n))).$$

Here:

(a) "g is a function":

$$(\forall u \in g)(\exists u_1 \in \cup^2(g))(\exists u_2 \in \cup^2(g))(u = \langle u_1, u_2 \rangle) \\ \wedge (\forall u_1 \in \cup^2(g))(\forall u_2 \in \cup^2(g))(\forall u_3 \in \cup^2(g)) \\ (\langle u_1, u_2 \rangle \in g \wedge \langle u_1, u_3 \rangle \in g \Rightarrow u_2 = u_3).$$

(b) $g(0) = x:\ \langle \varnothing, x \rangle \in g$.
(c) $g(n+1) = \Phi(g(n))$:

$$(\exists y \in \cup^2(g))(\langle n, y \rangle \in g \wedge \langle n \cup \{n\},\ \Phi(y) \rangle \in g),$$

where

$$\langle n \cup \{n\},\ \Phi(y)\rangle \in g: \quad (\exists u \in \cup^2(g))(\exists v \in \cup^2(g)) \\ (u = n \cup \{n\} \wedge v = \Phi(y) \wedge \langle u, v\rangle \in g).$$

Since $\omega_0 \in M$, the formula 5.11 is now easily seen to be absolute by the previous results.

In 5.11 we took the liberty of using g and n for variables of $\mathrm{L_1Set}$ in order to make the formulas intuitively clearer. In what follows we shall also use α, β, K, and N as variables, thereby temporarily ignoring our convention of using only lowercase letters at the end of the Latin alphabet.

5.12. $y \in \mathcal{J}x:\ \exists g($"$g$ closes x"$\wedge(\exists n \in \omega_0)(\langle n, y\rangle \in g))$. Here the quantifier over g is not restricted. Since the formula under the $\exists g$ sign is absolute, we may conclude directly from the definition $\|_L(\xi) = \|_V(\xi), \xi \in \overline{M}$, that $y \in \mathcal{J}x$ is also absolute, provided we show that for any $X \in M$, the function $G \in V$ that closes X lies in M. The formula $\forall x\ \exists!\ g$ ("g closes x") is obviously V-true. If we formalize the verification of this fact, we see that this formula is deducible from the axioms without the axiom of choice. Hence it is M-true. This implies that for any $X \in M$ we have $G \in M$.

5.13. $y \in \mathcal{P}(x) \cap \mathcal{J}(x \cup \{x\}):\ (\forall z \in y)(\forall v \in z)(v \in x) \wedge y \in \mathcal{J}(x \cup \{x\})$.

5.14. "f is the constructing $(\alpha + 1)$-sequence," which is short for "α is an ordinal"$\wedge$ "f is a function"$\wedge$ $\mathrm{dom}\, f = \alpha + 1 \wedge (\forall \beta \in \alpha + 1)(f(\beta) = L_\beta)$.

Here:

(a) $(\forall \beta \in \alpha + 1)(f(\beta) = L_\beta)$:

$$(\forall \beta \in \alpha + 1)((\beta \text{ is a limit ordinal } \Rightarrow f(\beta) = \cup_{\gamma\in\beta} f(\gamma)) \\ \wedge (f(\beta + 1) = \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\}))).$$

(b) "β is a limit ordinal": "β is an ordinal"$\wedge(\forall \alpha \in \beta)(\beta \neq \alpha \cup \{\alpha\})$.

(c) $f(\beta) = \cup_{\gamma\in\beta} f(\gamma):\quad (\exists v \in \cup^2(f))(v = \cup_{\gamma\in\beta} f(\gamma) \wedge \langle \beta, v\rangle \in f)$;

$$v = \cup_{\gamma\in\beta} f(\gamma):\quad (\forall u \in v)(\exists \gamma \in \beta)(u \in f(\gamma)) \\ \wedge (\forall u \in \cup^3(f))(u \in f(\gamma) \Rightarrow u \in v); \\ u \in f(\gamma):\quad (\exists w \in \cup^2(f))(\langle \gamma, w\rangle \in f \wedge u \in w).$$

(d) $f(\beta + 1) = \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\})$:

$$(\exists u \in \cup^2(f))(\langle \beta + 1, u\rangle \in f \wedge (\forall v \in u) \\ (v \in \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\})) \\ \wedge \forall v(v \in \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\}) \Rightarrow v \in u)); \\ v \in \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\}): \\ (\exists u \in \cup^2(f))(\langle \beta, u\rangle \in f \wedge v \in \mathcal{P}(u) \cap \mathcal{J}(u \cup \{u\})).$$

Finally, in order to verify directly that the subformula

$$\forall v(v \in \mathcal{P}(f(\beta)) \cap \mathcal{J}(f(\beta) \cup \{f(\beta)\}) \Rightarrow v \in u)$$

is M-absolute, it suffices to show that M is closed with respect to the operation $X \mapsto \mathcal{P}(X) \cap \mathcal{J}(X \cup \{X\})$. But M is closed with respect to both $\mathcal{J}$ and $X \mapsto \mathcal{P}(X) \cap M$, so the verification is complete.

5.15. $L(x, y)$: "y is an ordinal and $x \in L_y$": "y is an ordinal"$\wedge \exists f$("f is the constructing $(y+1)$-sequence"$\wedge(\exists z \in \cup^2(f))(\langle y, z\rangle \in f \wedge x \in z)$). Since the quantifier $\exists f$ is not bounded, in order to verify this last absoluteness statement we must show that the constructing $(Y+1)$-sequence F is an element of M for any ordinal Y in M. We use the same argument as in 5.12: the formula $\forall y(y$ is an ordinal $\Rightarrow \exists! f(f$ is the constructing $(y+1)$-sequence$))$ not only is V-true, but also is deducible from the axioms without the axiom of choice; therefore it is M-true.

This completes the proof of Proposition 4.2. □

5.16. *Remark.* The formula $\forall x\ \exists y\ L(x, y)$ is often written in the form $V = L$, and is called the *axiom of constructibility.* The absoluteness of $L(x, y)$ implies that the following formula is L-true:

$$|\forall x\ \exists y\ L(x, y)|_L = \inf_{X \in L}\ \sup_{Y \in L} |L(X, Y)|_L = \inf_{X \in L}\ \sup_{Y \in L} |L(X, Y)|_V = 1.$$

Hence, this formula is consistent with the Zermelo–Fraenkel axioms. On the other hand, $V = L$ implies the generalized continuum hypothesis (GCH), and since the negation of the GCH is also consistent with the Zermelo–Fraenkel axioms, it follows that $\neg(V = L)$ is consistent with the axioms.

We now proceed to the proof of Proposition 3.2. This proof follows the same plan as the proof of Proposition 4.2. We return to the conventions and constructions in 1.7–1.8.

5.17. *Remark.* In Section 4.4 we exploited the fact that the assertion "$\alpha \geqslant \omega_0 \Rightarrow \mathrm{card}(L_\alpha) = \mathrm{card}(\alpha)$" is formally deducible from the axioms of $\mathrm{L_1Set}$ (without the axiom of choice). We may now see that such a formal deduction can be obtained by exactly mimicking the proof in Section 1.4. Indeed, from the definition of $L(x, y)$ we have the formal deducibility of "$L_{\alpha+1} = \mathcal{P}(L_\alpha) \cap \mathcal{J}(L_\alpha \cup \{L_\alpha\})$" and "$\beta$ a limit ordinal $\Rightarrow L_\beta = \cup_{\gamma \in \beta} L_\gamma$". Moreover, the following are deducible: "$\mathrm{card}(X) < \omega_0 \Rightarrow \mathrm{card}(X) < \mathrm{card}(\mathcal{P}(X)) < \omega_0$" and "$\mathrm{card}(X) \geqslant \omega_0 \Rightarrow \mathrm{card}(\mathcal{J}(X)) = \mathrm{card}(X)$." As a result, the assertions "$\mathrm{card}(L_{\omega_0}) = \omega_0$," "$\mathrm{card}(L_a) \geqslant \omega_0 \Rightarrow \mathrm{card}(L_{\alpha+1}) = \mathrm{card}(L_\alpha)$," and "$\beta$ a limit ordinal $\Rightarrow \mathrm{card}(L_\beta) = \mathrm{card}(\cup_{\gamma \in \beta} L_\gamma)$" are all deducible. And from these and the axioms of $\mathrm{L_1Set}$ the desired assertion may be deduced (using, in particular, the deducibility of "$\mathrm{card}(\omega_0) = \omega_0$," "$\alpha \geqslant \omega_0 \Rightarrow \mathrm{card}(\alpha + 1) = \mathrm{card}(\alpha)$," "$\beta$ is a limit ordinal $\Rightarrow \beta = \cup_{\gamma \in \beta} \gamma$," and in addition an instance of transfinite induction on the ordinals, which is of course also formally deducible in $\mathrm{L_1Set}$).

5.18. *The formula* $H(K,x)$: K is a function $\wedge$ x is an ordinal $\wedge$ dom $K = x+1 \wedge K(0) = \langle 0,0,0\rangle \wedge (\forall y \in x+1)(K(y)$ is an important triple $\wedge K(y+1)$ is the next important triple after $K(y)) \wedge (y$ is a limit ordinal $\Rightarrow K(y) = \lim_{z\in y} K(z))$ *is absolute.*

We shall not analyze the subformulas that have been considered before. The following subformulas remain:

(a) "$K(y)$ is an important triple $\wedge K(y+1)$ is the next important triple after $K(y)$";
(b) $K(y) = \lim_{z\in y} K(z)$.

We shall have to use the absoluteness of the auxiliary formula "$y = x_{(i)}$," which is short for "x is an important triple (i.t.) and y is the ith coordinate of x," where $i = 1, 2$, or 3. That is;

$$\begin{aligned}(\exists u_1 \in \cup^3(x))(\exists u_2 \in \cup^3(x))(\exists u_3 \in \cup(x))& \\ \times (x = \langle u_1,u_2,u_3\rangle \wedge u_1 &\text{ is an ordinal} \wedge u_1 \leqslant 8 \\ \wedge\, u_2 \text{ is an ordinal } \wedge u_3 &\text{ is an ordinal} \wedge y = u_i).\end{aligned}$$

The *complete form of* (a) is

$$\begin{aligned}(\exists u \in \cup(K))(\exists v \in \cup(K))(\langle y,u\rangle \in K \wedge \langle y+1,v\rangle \in K& \\ \wedge\, u \text{ is an i.t.} \wedge v \text{ is the i.t. after } u).&\end{aligned}$$

According to Lemma 1.7(a), "u is an i.t. $\wedge v$ is the i.t. after u" can be written in the form $\bigvee_{i=1}^5 C_i(u,v)$, where $C_i(u,v)$ is the formalization of the ith alternative in 1.7(a). For example,

$$\begin{aligned}C_1:\quad & u \text{ is an i.t.} \wedge v \text{ is an i.t.} \wedge u_{(1)} \leqslant 7 \wedge v_{(1)} = u_{(1)} + 1 \\ &\qquad\qquad \wedge v_{(2)} = u_{(2)} \wedge v_{(3)} = u_{(3)}; \\ C_2:\quad & u \text{ is an i.t.} \wedge v \text{ is an i.t.} \wedge u_{(1)} = 8 \wedge u_{(2)} + 1 < u_{(3)} \\ &\qquad\qquad \wedge v_{(1)} = 0 \wedge v_{(2)} = u_{(2)} + 1 \wedge v_{(3)} = 0.\end{aligned}$$

The other C_i are analogous, and are absolute for the same reasons.

The *complete form of* (b). Here we need to know that the following auxiliary formulas are absolute:

$$\begin{aligned}u = \bigcup_{z\in y} K(z)_{(i)},\ i = 2 \text{ or } 3:\quad & (\forall v \in u)(\exists z \in y)(v = K(z)_{(i)}) \\ &\wedge (\forall z \in y)(\exists v \in u)(v = K(z)_{(i)}); \\ v = K(z)_{(i)}:\quad & (\exists w \in \cup(K))(\langle z,w\rangle \in K \wedge w \text{ is an i.t.} \wedge v = w_{(i)}).\end{aligned}$$

Then, using Lemma 1.7(b), we explain the formula $K(y) = \lim_{z\in y} K(z)$ as follows:

$$K(y)_{(1)} =$$

$$0 \wedge \exists u_2\, \exists u_3 \left(u_2 = \bigcup_{z\in y} K(z)_{(2)} \wedge u_3 = \bigcup_{z\in y} K(z)_{(3)} \wedge \bigvee_{i=1}^{4} D_i(u_2,u_3,y)\right),$$

where the alternatives D_i have the following structure, depending on "how $K(z)$ approaches $K(y)$":

$$\begin{aligned}
D_1: \quad & u_2 \in u_3 \wedge u_2 \text{ is a limit ordinal} \wedge ((\exists z \in y)(K(z)_{(3)} = u_3) \\
& \qquad \rightarrow k(y)_{(2)} = u_2 \wedge K(y)_{(3)} = u_3); \\
D_2: \quad & u_2 = u_3 \wedge u_2 \text{ is a limit ordinal} \wedge ((\exists z \in y)(K(z)_{(3)} = u_3) \\
& \quad \wedge (\forall z \in y)(K(z)_{(2)} \in u_2) \rightarrow k(y)_{(2)} = u_2 \wedge K(y)_{(3)} = 0); \\
D_3: \quad & u_2 \geqslant u_3 \wedge u_3 \text{ is a limit ordinal} \wedge ((\forall z \in y)(K(z)_{(3)} \in u_3) \\
& \qquad \rightarrow K(y)_{(2)} = u_2 \in K(y)_{(3)} = u_3); \\
D_4: \quad & u_2 = u_3 \wedge u_2 \text{ is a limit ordinal} \wedge ((\forall z \in y)(K(z)_{(2)} \in u_2 \\
& \qquad \wedge K(z)_{(3)} \in u_3) \rightarrow K(y)_{(2)} = 0 \wedge K(y)_{(3)} = u_3).
\end{aligned}$$

It is therefore obvious that the D_i are absolute. Even though the quantifiers $\exists u_2$ and $\exists u_3$ are not restricted, there is no problem, since when $K^\xi, y^\xi \in L$, this formula can be V-true only if $u_2^{\xi'}$ and $u_3^{\xi'}$ are uniquely determined ordinals and lie in L, which gives us L-truth.

5.19. *The formula* $S(N, x)$: "x is an ordinal $\wedge$ N is a function $\wedge$ dom $N = x + 1 \wedge (\forall y \leqslant x + 1)(N(y)$ is a constructible set with N-number y)" *is absolute.*

We shall need to know that the following auxiliary formula is absolute:

$$y = (x)_i, \quad i = 1, 2, 3, \quad \text{where } K(x) = \langle (x)_1, (x_2), (x)_3 \rangle$$

(not to be confused with the formula $y = x_{(i)}$ in 5.16, which occurs here as a subformula): x is an ordinal $\wedge \exists K(H(K, x) \wedge (\exists u \in \cup(K))(\langle x, u \rangle \in K \wedge y = u_{(i)}))$. Even though $\exists K$ is not restricted, this does not cause any problem, because for every ordinal $x^\xi \in L$, the value of K^ξ making $H(K^\xi, x^\xi)$ V-true lies in L. In fact, the V-true formula

$$\forall x (x \text{ is an ordinal} \Rightarrow \exists! K(H(K, x)))$$

is deducible from the axioms without the axiom of choice, and hence is L-true.

We now return to $S(N, x)$. We need only show that the subformula "$N(y)$ is a constructible set with N-number y" is absolute. By definition, this subformula can be written as $\bigvee_{i=0}^{8} Q_i(y, N)$, where the alternatives have the form

Q_0: $(y)_1 = 0 \wedge \langle y, L_{(y)_2} \rangle \in N$;
$Q_i,\ 1 \leqslant i \leqslant 3$: $(y)_1 = i \wedge \langle y, F_i(N((y)_2), N((y)_3)) \rangle \in N$;
$Q_i,\ 4 \leqslant i \leqslant 8$: $(y)_1 = i \wedge \langle y, F_i(N((y)_2)) \rangle \in N$.

The absoluteness of the subformulas that have not been analyzed is clear from the following complete forms of these formulas:

(a) $\langle y, L_{(y)_2} \rangle \in N$: $\quad (\exists z \in \cup(N))(\langle y, z \rangle \in N \wedge z \in L_{(y)_2})$;
$z = L_{(y)_2}$: $\quad (\exists u \in y + 1)(u = (y)_2 \wedge z = L_u)$;
$z = L_u$: $\quad (\forall v \in z)(v \in L_u) \wedge \forall v (v \in L_u \Rightarrow v \in z)$.

We can verify directly that the last subformula, with the unrestricted quantifier $\forall v$, *is absolute, since* $L_U \in L$ *for any ordinal* U, *and* L *is transitive.*

(b) $\langle y, F_i(N((y)_2))\rangle \in N, \qquad i = 4, \dots, 8$:
$(\exists u, v, w \in \cup(N))(u = (y)_2 \wedge \langle u, v\rangle \in N \wedge w = F_i(v) \wedge \langle y, w\rangle \in N)$.

(c) $\langle y, F_i(N((y)_2), N((y)_3))\rangle \in N, \qquad i = 1, 2, 3$:
$(\exists u_2, u_3, v_2, v_3, w \in \cup\ (N))(u_2 = (y)_2 \wedge u_3 = (y)_3 \wedge \langle u_2, v_2\rangle \in N$
$\wedge \langle u_3, v_3\rangle \in N \wedge w = F_i(u_2, v_3) \wedge \langle y, w\rangle \in N)$.

5.20. *The formula* $N(x, y)$: "x is an ordinal $\wedge y = N(x)$" *is absolute.*

In fact, this formula is written in the form

$$\exists N(S(N, x + 1) \wedge \langle x, y\rangle \in N).$$

There is no problem with $\exists N$ being unrestricted, since we can apply the same type of argument as we have used many times before: for any ordinal x^ξ there is a unique $N^{\xi'}$ making this formula V-true, and then $N^{\xi'} \in L$, since the formula $\forall x$ (x is an ordinal $\Rightarrow \exists! N(S(N, x + 1)))$ is deducible from the axioms without the axiom of choice, and hence is L-true.

This completes the proof of Proposition 3.2. □

6 Remarks on Formalization

Gödel's theory, to which this chapter is devoted, is usually presented in a more syntactic version. We shall now briefly describe the system of basic ideas and the most important changes in the proofs in this version, in which the least possible appeal is made to the semantics.

6.1. Let $Q(x)$ be a formula in $\mathrm{L_1Set}$ with one free variable x. Let ZF be the set of all the (logical, special, and equality) axioms of $\mathrm{L_1Set}$ except for the axiom of choice. $Q(x)$ is said to be *transitive* if

$$\mathrm{ZF} \vdash (Q(x) \wedge y \in x) \Rightarrow Q(y).$$

6.2. The *relativization* P_Q of a formula P in $\mathrm{L_1Set}$ relative to Q is defined by induction on the number of connectives and quantifiers in P:

$$\begin{aligned}
(x \in y)_Q \quad &\text{is} \quad Q(x) \wedge Q(y) \Rightarrow x \in y;\\
(x = y)_Q \quad &\text{is} \quad Q(x) \wedge Q(y) \Rightarrow x = y;\\
(\neg P)_Q \quad &\text{is} \quad \neg(P_Q);\\
(P_1 * P_2)_Q \quad &\text{is} \quad (P_1)_Q * (P_2)_Q, \quad \text{for any connective } *;\\
(\forall x P)_Q \quad &\text{is} \quad \forall x(Q(x) \Rightarrow P);\\
(\exists x P)_Q \quad &\text{is} \quad \exists x(Q(x) \wedge P).
\end{aligned}$$

6.3. $Q(x)$ is called an (*internal*) *model* of $\mathrm{L_1Set}$ if for any axiom $P \in \mathrm{ZF}$ we have

$$\mathrm{ZF} \vdash P_Q.$$

This model is *transitive* if Q is transitive.

A formula $P(y_1, \ldots, y_n)$ is called *Q-absolute* if

$$\mathrm{ZF} \vdash (Q(y_1) \wedge \cdots \wedge Q(y_n)) \Rightarrow (P \Leftrightarrow P_Q).$$

6.4. The connection between these concepts and our earlier ones is as follows. Every formula $Q(x)$ determines a class $M = \{X \in V | Q(X) \text{ is } V\text{-true}\}$. This class M has the property that

$$|P|_M(\xi) = |P_Q|_V(\xi), \quad \forall \xi \in \overline{M},$$

for any formula P (as can easily be proved by induction on the number of connectives and quantifiers in P). Thus, to give a syntactic reformulation of our proofs we must make the following changes throughout;

(a) We consider only classes M that are defined by formulas Q, and all references to M are replaced by references to Q.
(b) We everywhere replace "P is V-true" by "P is deducible from ZF."
(c) We everywhere replace "P is M-true" by "P_Q is deducible from ZF."
(d) We everywhere replace "P is M-absolute" by "P is Q-absolute."

In order for the new assertions on deducibility from ZF to become sufficiently obvious, we must either do some additional work formalizing the proofs or else give more careful intuitive proofs. In particular, we must find finite subsets of ZF from which the various facts are deducible. The basic results are stated as follows in the new syntactic language:

6.5. $\exists y\ L(x,y)$ "*is*" *a transitive internal model of* $\mathrm{L_1Set}$.

6.6. ZF$\vdash$(axiom of choice)$_{\exists y\ L(x,y)}$.

6.7. ZF$\vdash$(generalized continuum hypothesis)$_{\exists y\ L(x,y)}$.

6.8. Thus, a completely syntactic version of Gödel's theory would consist of all the deductions implicit in 6.5–6.7, without any commentary. Of course, such a treatment has never been written. The formula $\exists y L(x,y)$ alone takes up several pages; without appealing to semantics, it would be impossible either to think up, or to explain, or even to copy down all this without making mistakes. The deductions of all the required relativized formulas $P_{\exists y\ L(x,y)}$ would also be extremely long. This situation gives us an instructive example of what was discussed in "Digression: Proof" in Chapter II.

7 What Is the Cardinality of the Continuum?

After all we have learned about the Zermelo–Fraenkel language and axiom system, it might seem naive to return to this question. But we must do so if we consider mathematical meaning to be our primary concern.

Some specialists in the foundations of mathematics espouse a different point of view. Namely, they answer that the question itself is meaningless. It seems that Paul Cohen himself tends toward this viewpoint, at the same time admitting that "this is a hard decision" (P. Cohen, Comments on the foundations of set theory, *Proc. Symp. Pure Math.*, vol. XIII, part I, American Math. Soc., Providence 1971, p. 12).

From this point of view it is natural to reject almost the entire semantics of $\mathrm{L_1Set}$, including all the V_α starting with $\alpha = \omega_0 + 1$ in the von Neumann universe. No halfway solutions can help matters, especially since questions concerning higher axioms of infinity or the so-called measurable cardinals are in an even worse position than the CH.

It thus becomes necessary to try to find alternative languages and semantics. Here the differences of opinion are wide and irreconcilable. The most clear-cut position is that of the constructivists, although even among them there are different shades of opinion. The constructivists do not recognize infinity as a usable concept, and reject ineffective existence proofs. (It turns out that in practice they often replace these ineffective proofs by a more carefully differentiated word usage—"there cannot not exist," or "there quasi-exists"—which is nearly synonymous with certain linguistic precautions adopted in classical texts.) In our opinion, the shortcoming in their point of view is that constructivism is in no sense "another mathematics." It is, rather, a sophisticated subsystem of classical mathematics, which rejects the extremes in classical mathematics and carefully nourishes its effective computational apparatus.

Unfortunately, it seems that it is these "extremes"—bold extrapolations, abstractions that are infinite and do not lend themselves to a constructivist interpretation—that make classical mathematics effective. One should try to imagine how much help mathematics could have provided twentieth-century quantum physics if for the past hundred years it had developed using only abstractions from "constructive objects." Most likely, the standard calculations with infinite-dimensional representations of Lie groups that today play an important role in understanding the microworld would simply never have occurred to anyone.

It is not impossible that a new (or a completely forgotten old) conception of the continuum, in which the continuum has no "cardinality," could be found in the course of a deep investigation of the external world. The notion of a set consisting of elements may actually be adequate only for finite or countable sets, and "higher infinities" may turn out to be abstractions from objects of a completely different type.

Physics seems to point up a difference in principle between "counting" and the Eudoxus–Dedekind idealization of measurement. The counting procedure applies to regions of attraction—"attractors" (R. Thom)—that are units not having sharp boundaries. The parts of a unit, even if they have physical meaning, are nevertheless attractors of a different sort. But even these ideas apparently stop making sense in the microworld.

If nature has a fundamentally statistical aspect, it might be fruitful to consider mathematical models in which the statistical aspect appears as an undefined concept. The unexpected richness of the nonstandard interpretations of classical mathematics in Boolean-valued models agrees with the suggestion that all the words we say should be understood in a new way.

7.2. We now discuss a less radical point of view on the continuum problem, according to which this question of its cardinality is meaningful. Then the main problem once again becomes how to determine the place of the continuum on the scale of alephs.

Cohen concludes his book with the following opinion: "A point of view which the author feels may eventually come to be accepted is that CH is *obviously* false.... C is greater than $\aleph_n, \aleph_w, \aleph_\alpha$ where $\alpha = \aleph_w$ etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction."

We thus have a conjectural estimate from below for C, and nothing more—not even a conjecture as to whether the cardinal C is regular or singular.

Of course, the real problem consists not only in guessing a plausible conjecture, but in supporting it with sufficiently convincing indirect evidence for it to become widely accepted, even if not proved. What sort of evidence could this be? In discussing new axioms for set theory, Gödel writes:

> there may exist ... other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts.
>
> Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytic number theory frequently allows us to prove number theoretic theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory (K. Gödel, What is Cantor's continuum problem? *Amer. Math. Monthly*, vol. 54, no. 9, 1947).

There is little to add here to this ardently expressed hope. But see §8 of Chapter VII, where it is shown using an idea of Gödel's own that *any* new independent axiom can shorten to an arbitrary extent the proofs of suitable assertions that are provable without the axiom. This result somewhat weakens our confidence in pragmatic criteria for truth.

7.3. More than two decades after the publication of the first edition of this book, Hugh W. Woodin introduced interesting new ideas about the continuum hypothesis.

His constructions enrich both our set-theoretic intuition and its formal language, in an intuitively consistent way.

We will very briefly explain Woodin's approach, following his notes "The continuum hypothesis. I, II," *Notices AMS*, 48 (2001), no. 6, 567–576, and no. 3, 681–690. We will work in the constructible universe of Section IV.1.

Call a set X *transitive* if each element of an element of X belongs to X. The transitive closure of X is the minimal transitive set containing X.

Let k be an infinite cardinal, and $H(k)$ the set of all sets X whose transitive closure is of cardinality $\leq k$. Accepting the axiom of choice, one sees that any constructible set belongs to some $H(k)$. Let $k_0, k_1, k_2, \ldots$ be the increasing sequence of the first infinite cardinals. Woodin easily reinterprets $H(k_0)$ as the semiring of natural numbers $\mathbf{N}$ with addition and multiplication, and, with some effort, $H(k_1)$ as a particular structure on the set of subsets of this semiring. These efforts are justified by providing a list of axioms for these structures that are intuitive and provide a basis for generalization to $H(k_2)$.

Having thus set the stage, Woodin takes up $H(k_2)$ and introduces an extension of first-order logic and a new axiom modestly called $(*)$.

Here the *grand finale* arrives: in this context Woodin can prove that $2^{\aleph_0} = \aleph_2$.

The following quotation from his second paper nicely concludes the discussion of this whole section:

"So, is the continuum hypothesis solvable? Perhaps, I am not completely confident the 'solution' I have sketched *is* the solution, but it is for me a convincing evidence that there is a solution. Thus, I now believe the continuum hypothesis is solvable, which is a fundamental change in my view of set theory."