I

Introduction to Formal Languages

Gelegentlich ergreifen wir die Feder
Und schreiben Zeichen auf ein weisses Blatt,
Die sagen dies und das, es kennt sie jeder,
Es ist ein Spiel, das seine Regeln hat.
H. Hesse, "Buchstaben"

We now and then take pen in hand
And make some marks on empty paper.
Just what they say, all understand.
It is a game with rules that matter.
H. Hesse, "Alphabet"
(translated by Prof. Richard S. Ellis)

1 General Information

1.1. Let A be any abstract set. We call A an *alphabet*. Finite sequences of elements of A are called *expressions* in A. Finite sequences of expressions are called *texts*.

We shall speak of a *language with alphabet* A if certain expressions and texts are distinguished (as being "correctly composed," "meaningful," etc.). Thus, in the Latin alphabet A we may distinguish English word forms and grammatically correct English sentences. The resulting set of expressions and texts is a working approximation to the intuitive notion of the "English language."

The language Algol 60 consists of distinguished expressions and texts in the alphabet $\{\text{Latin letters}\} \cup \{\text{digits}\} \cup \{\text{logical signs}\} \cup \{\text{separators}\}$. *Programs* are among the most important distinguished texts.

In natural languages the set of distinguished expressions and texts usually has unsteady boundaries. The more formal the language, the more rigid these boundaries are.

The rules for forming distinguished expressions and texts make up the *syntax* of the language. The rules that tell how they correspond with reality make

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up the *semantics* of the language. Syntax and semantics are described in a *metalanguage.*

1.2. “Reality” for the languages of mathematics consists of certain classes of (mathematical) arguments or certain computational processes using (abstract) automata. Corresponding to these designations, the languages are divided into formal and algorithmic languages. (Compare: in natural languages, the declarative versus imperative moods, or—on the level of texts—statement versus command.)

Different formal languages differ from one another, in the first place, by the scope of the formalizable types of arguments—their expressiveness; in the second place, by their orientation toward concrete mathematical theories; and in the third place, by their choice of elementary modes of expression (from which all others are then synthesized) and written forms for them.

In the first part of this book a certain class of formal languages is examined systematically. Algorithmic languages are brought in episodically.

The “language–parole” dichotomy, which goes back to Humboldt and Saussure, is as relevant to formal languages as to natural languages. In §3 of this chapter we give models of “speech” in two concrete languages, based on set theory and arithmetic, respectively, because, as many believe, habits of speech must precede the study of grammar.

The language of set theory is among the richest in expressive means, despite its extreme economy. In principle, a formal text can be written in this language corresponding to almost any segment of modern mathematics—topology, functional analysis, algebra, or logic.

The language of arithmetic is one of the poorest, but its expressive possibilities are sufficient for describing all of elementary arithmetic, and also for demonstrating the effects of self-reference à la Gödel and Tarski.

1.3. As a means of communication, discovery, and codification, no formal language can compete with the mixture of mathematical argot and formulas that is common to every working mathematician.

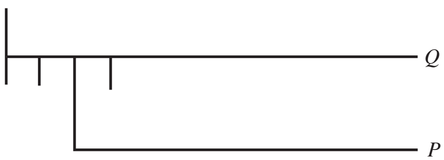
However, because they are so rigidly normalized, formal texts can themselves serve as an object for mathematical investigation. The results of this investigation are themselves *theorems of mathematics.* They arouse great interest (and strong emotions) because they can be interpreted as *theorems about mathematics.* But it is precisely the possibility of these and still broader interpretations that determines the general philosophical and human value of mathematical logic.

1.4. We have agreed that the expressions and texts of a language are elements of certain abstract sets. In order to work with these elements, we must somehow fix them materially. In the modern European tradition (as opposed to the ancient Babylonian tradition, or the latest American tradition, using computer memory), the following notation is customary. The elements of the alphabet are indicated by certain symbols on paper (letters of different kinds of type, digits, additional signs, and also combinations of these). An expression in an alphabet A is written in the form of a sequence of symbols, read from left to right, with hyphens when necessary. A text is written as a sequence of written expressions, with spaces or punctuation marks between them.

1.5. If written down, most of the interesting expressions and texts in a formal language either would be physically extremely long, or else would be psychologically difficult to decipher and learn in an acceptable amount of time, or both.

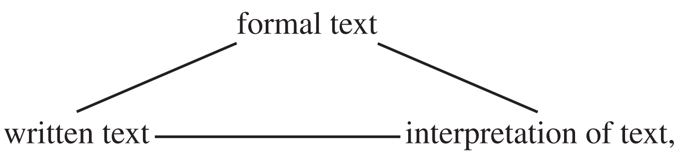
They are therefore replaced by "abbreviated notation" (which can sometimes turn out to be physically longer). The expression "$xxxxxx$" can be briefly written "$x \dots x$ (six times)" or "x^6." The expression "$\forall z(z \in x \Leftrightarrow z \in y)$" can be briefly written "$x = y$." Abbreviated notation can also be a way of denoting any expression of a definite type, not only a single such expression (any expression $101010 \dots 10$ can be briefly written "the sequence of length $2n$ with ones in odd places and zeros in even places" or "the binary expansion of $\frac{2}{3}(4^n - 1)$").

Ever since our tradition started, with Viète, Descartes, and Leibniz, abbreviated notation has served as an inexhaustible source of inspiration and errors. There is no sense in, or possibility of, trying to systematize its devices; they bear the indelible imprint of the fashion and spirit of the times, the artistry and pedantry of the authors. The symbols Σ, $\int$, $\in$ are classical models worthy of imitation. Frege's notation, now forgotten, for "P and Q" (actually "not [if P, then not Q]" whence the asymmetry):



shows what should be avoided. In any case, abbreviated notation permeates mathematics.

The reader should become used to the trinity



which replaces the unconscious identification of a statement with its form and its sense, as one of the first priorities in his study of logic.

2 First-Order Languages

In this section we describe the most important class of formal languages $\mathfrak{L}_1$—the first-order languages—and give two concrete representatives of this class: the Zermelo–Fraenkel language of set theory $\mathrm{L_1Set}$, and the Peano language of arithmetic $\mathrm{L_1Ar}$. Another name for $\mathfrak{L}_1$ is *predicate languages.*

2.1. The alphabet of any language in the class $\mathfrak{L}_1$ is divided into six disjoint subsets. The following table lists the generic name for the elements in each subset, the standard notation for these elements in the general case, the special notation used in this book for the languages $\mathrm{L_1Set}$ and $\mathrm{L_1Ar}$. We then describe the rules for forming distinguished expressions and briefly discuss semantics.

The distinguished expressions of any language L in the class $\mathfrak{L}_1$ are divided into two types: *terms* and *formulas.* Both types are defined recursively.

2.2. **Definition.** *Terms* are the elements of the least subset of the expressions of the language that satisfies the following two conditions:

(a) Variables and constants are (atomic) terms.
(b) If f is an operation of degree r and $t_1, \ldots, t_r$ are terms, then $f(t_1, \ldots, t_r)$ is a term.

In (a) we identify an element with a sequence of length one. The alphabet does not include commas, which are part of our abbreviated notation: $f(t_1,\ t_2,\ t_3)$ means the same as $f(t_1 t_2 t_3)$. In §1 of Chapter II we explain how a sequence of terms can be uniquely deciphered despite the absence of commas.

If two sets of expressions in the language satisfy conditions (a) and (b), then the intersection of the two sets also satisfies these conditions. Therefore the definition of the set of terms is correct.

Language Alphabets

Subsets of the Alphabet	Names and Notation		
	General	in $\mathrm{L_1Set}$	in $\mathrm{L_1Ar}$
connectives and quantifiers	$\Leftrightarrow$(equivalent); $\Rightarrow$(implies); $\vee$(inclusive or); $\wedge$ (and); $\neg$(not); $\forall$ (universal quantifier); $\exists$ (existential quantifier)		
variables	$x,\ y,\ z,\ u,\ v, \ldots$ with indices		
constants	c $\cdots$ with indices	$\emptyset$ (empty set)	$\bar{0}$ (zero); $\bar{1}$ (one)
operations of degree 1, 2, 3, …	$f,\ g,\ \ldots$ with indices	none	$+$ (addition, degree 2); $\cdot$(multiplication, degree 2)
relations (predicates) of degree 1, 2, 3, …	$p,\ q,\ \ldots$ with indices	$\in$ (is an element of, degree 2); $=$ (equals, degree 2)	$=$ (equality, degree 2)
parentheses	((left parenthesis);)(right parenthesis)		

2.3. **Definition.** *Formulas* are the elements of the least subset of the expressions of the language that satisfies the following two conditions:

(a) If p is a relation of degree r and $t_1, \ldots, t_r$ are terms, then $p(t_1, \ldots, t_r)$ is an (atomic) formula.

(b) If P and Q are formulas (abbreviated notation!), and x is a variable, then the expressions

$$(P) \Leftrightarrow (Q),\ (P) \Rightarrow (Q),\ (P) \vee (Q),\ (P) \wedge (Q),$$
$$\neg(P),\ \forall x(P),\ \exists x(P)$$

are formulas.

It is clear from the definitions that any term is obtained from atomic terms in a finite number of steps, each of which consists in "applying an operation symbol" to the earlier terms. The same is true for formulas. In Chapter II, §1 we make this remark more precise.

The following initial interpretations of terms and formulas are given for the purpose of orientation and belong to the so-called "standard models" (see Chapter II, §2 for the precise definitions).

2.4. Examples and interpretations

(a) The terms stand for (are notation for) the objects of the theory. Atomic terms stand for indeterminate objects (variables) or concrete objects (constants). The term $f(t_1, \dots, t_r)$ is the notation for the object obtained by applying the operation denoted by f to the objects denoted by $t_1, \dots, t_r$. Here are some examples from $\mathrm{L_1Ar}$:

$$\begin{aligned} \bar{0} &\quad \text{denotes zero;} \\ \bar{1} &\quad \text{denotes one;} \\ +(\bar{1}, \bar{1}) &\quad \text{denotes two } (1+1=2 \text{ in the usual notation);} \\ +\Big(\bar{1} + (\bar{1}, \bar{1})\Big) &\quad \text{denotes three;} \\ \cdot\Big(+(\bar{1}, \bar{1}) + (\bar{1}, \bar{1})\Big) &\quad \text{denotes four } (2 \times 2 = 4). \end{aligned}$$

Since this normalized notation is different from what we are used to in arithmetic, in $\mathrm{L_1Ar}$ we shall usually write simply $t_1 + t_2$ instead of $+(t_1, t_2)$ and $t_1 \cdot t_2$ instead of $\cdot(t_1, t_2)$. This convention may be considered as another use of abbreviated notation:

$$x \text{ stands for an indeterminate integer;}$$
$$x + \bar{1} \text{ (or } +(x, \bar{1})\text{) stands for the next integer.}$$

In the language $\mathrm{L_1Set}$ all terms are atomic:

$$x \text{ stands for an indeterminate set;}$$
$$\varnothing \text{ stands for the empty set.}$$

(b) The formulas stand for statements (arguments, propositions, ...) of the theory. When translated into formal language, a statement may be either true, false, or indeterminate (if it concerns indeterminate objects); see Chapter II for the precise definitions. In the general case the atomic formula $p\,(t_1, \ldots, t_r)$ has roughly the following meaning: "The ordered r-tuple of objects denoted by $t_1, \ldots, t_r$ has the property denoted by p." Here are some examples of atomic formulas in $\mathrm{L}_1\mathrm{Ar}$. Their general structure is $= (t_1, t_2)$, or, in nonnormalized notation, $t_1 = t_2$:

$$\bar{0} = \bar{1}, \qquad x + \bar{1} = y.$$

Here are some examples of formulas which are not atomic:

$$\begin{aligned}
&\neg(\bar{0} = \bar{1}),\\
&(x = \bar{0}) \Leftrightarrow (x + \bar{1} = \bar{1}),\\
&\forall\, x\Big((x = \bar{0}) \vee \big(\neg(x \cdot x = \bar{0})\big)\Big).
\end{aligned}$$

Some atomic formulas in L_1 Set

$$y \in x \qquad (y \text{ is an element of } x),$$

and also $\varnothing \in y,\ x \in \varnothing$, etc. Of course, normalized notation must have the form $\in (xy)$, and so on.

Some nonatomic formulas:

$$\exists\, x\big(\forall y(\neg(y \in x))\big): \qquad \text{there exists an } x \text{ of which no } y \text{ is an element.}$$

Informally this means: "The empty set exists." We once again recall that an informal interpretation presupposes some standard interpretive system, which will be introduced explicitly in Chapter II.

$$\forall\, y(y \in z \Rightarrow y \in x): \qquad z \text{ is a subset of } x.$$

This is an example of a very useful type of abbreviated notation: four parentheses are omitted in the formula on the left. We shall not specify precisely when parentheses may be omitted; in any case, it must be possible to reinsert them in a way that is unique or is clear from the context without any special effort.

We again emphasize: the abbreviated notation for formulas are only material designations. Abbreviated notation is chosen for the most part with psychological goals in mind: speed of reading (possibly with a loss in formal uniqueness), tendency to encourage useful associations and discourage harmful ones, suitability to the habits of the author and reader, and so on. The mathematical objects in the theory of formal languages are the formulas themselves, and not any particular designations.

Digression: Names

On several occasions we have said that a certain object (a sign on paper, an element of an alphabet as an abstract set, etc.) is a notation for, or denotes, another element. A convenient general term for this relationship is naming.

The letter x is the name of an element of the alphabet; when it appears in a formula, it becomes the name of a set or a number; the notation $x \in y$ is the name of an expression in the alphabet A, and this expression, in turn, is the name of an assertion about indeterminate sets; and so on.

When we form words, we often identify the names of objects with the objects themselves: we say "the variable x," "the formula P," "the set z." This can sometimes be dangerous. The following passage from Rosser's book *Logic for Mathematicians* points up certain hidden pitfalls:

> The gist of the matter is that, if we have a statement such as "3 is greater than $\frac{9}{12}$" about the rational number $\frac{9}{12}$ and containing a name "$\frac{9}{12}$" of this rational number, one can replace this name by any other name of the same rational number, for instance, "$\frac{3}{4}$." If we have a statement such as "3 divides the denominator of '$\frac{9}{12}$' " about a name of a rational number and containing a name of this name, one can replace this name of the name by some other name of the same name, but not in general by the name of some other name, if it is a name of some other name of the same rational number.

Rosser adds that "failure to observe such distinctions carefully can seldom lead to confusion in logic and still more seldom in mathematics." However, these distinctions play a significant role in philosophy and in mathematical practice.

"A rose by any other name would smell as sweet"—this is true because roses exist outside of us and smell in and of themselves. But, for example, it seems that Hilbert spaces "exist" only insofar as we talk about them, and the choice of terminology here makes a difference. The word "space" for the set of equivalence classes of square integrable functions was at the same time a codeword for an entire circle of intuitive ideas concerning "real" spaces. This word helped organize the concept and led it in the right direction.

A successfully chosen name is a bridge between scientific knowledge and common sense, between new experience and old habits. The conceptual foundation of any science consists of a complicated network of names of things, names of ideas, and names of names. It evolves itself, and its projection on reality changes.

3 Beginners' Course in Translation

3.1. We recall that the formulas in $\mathrm{L_1Set}$ stand for statements about sets; the formulas in $\mathrm{L_1Ar}$ stand for statements about natural numbers; these formulas contain names of sets and numbers, which may be indeterminate.

In this section we give the first basic examples of two-way translation "argot $\Leftrightarrow$ formal language." One of our purposes will be to indicate the great expressive possibilities in L_1Set and L_1Ar, despite the extremely limited modes of expression.

As in the case of natural languages, this translation cannot be given by rigid rules, is not uniquely determined, and is a creative process. Compare Hesse's quatrain with its translation in the epigraph to this book: the most important aim of translation is to "understand ... just what they say."

Before reading further, the reader should look through the appendix to Chapter II: "The von Neumann Universe." The semantics implicit in L_1Set relates to this universe, and not to arbitrary "Cantor" sets.

A more complete picture of the meaning of the formulas can be obtained from §2 of Chapter II.

Translation from L_1Set *to argot.*

3.2. $\forall\, x(\neg(x \in \varnothing))$: "for all (sets) x it is false that x is an element of (the set) $\varnothing$" (or "$\varnothing$ is the empty set").

The second assertion is equivalent to the first only in the von Neumann universe, where the elements of sets can only be sets, and not real numbers, chairs, or atoms.

3.3. $\forall\, z(z \in x \Leftrightarrow z \in y) \Leftrightarrow x = y$: "if for all z it is true that z is an element of x if and only if z is an element of y, then it is true that x coincides with y; and conversely," or "a set is uniquely determined by its elements."

In the expression 3.3 at least six parentheses have been omitted; and the subformulas $z \in x,\ z \in y,\ x = y$ have not been normalized according to the rules of $\mathfrak{L}_1$.

3.4. $\forall u\ \forall v\ \exists x\ \forall z(z \in x \Leftrightarrow (z = u \vee z = v))$: "for any two sets $u,\ v$ there exists a third set x such that u and v are its only elements."

This is one of the axioms of Zermelo–Fraenkel. The set x is called the "unordered pair of sets $u,\ v$" and is denoted $\{u,\ v\}$ in the appendix.

3.5. $\forall y\ \forall z\big(((z \in y \wedge y \in x) \Rightarrow z \in x) \wedge (y \in x \Rightarrow \neg(y \in y))\big)$: "the set x is partially ordered by the relation $\in$ between its elements."

We mechanically copied the condition $y \in x \Rightarrow \neg(y \in y)$ from the definition of partial ordering. This condition is automatically fulfilled in the von Neumann universe, where no set is an element of itself.

A useful exercise would be to write the following formulas:

"x is totally ordered by the relation $\in$";

"x is linearly ordered by the relation $\in$";

"x is an ordinal."

3.6. $\forall x(y \in z)$: The literal translation "for all x it is true that y is an element of z" sounds a little strange. The formula $\forall x\ \exists x(y \in z)$, which agrees with the rules for constructing formulas, looks even worse. It would be possible to make the rules somewhat more complicated, in order to rule out such formulas, but in general they cause no harm. In Chapter II we shall see that from the point of view of "truth" or "deducibility," such a formula is equivalent to the formula $y \in z$. It is in this way that they must be understood.

Translation from argot to $\mathbf{L_1}$Set.

We choose several basic constructions having general mathematical significance and show how they are realized in the von Neumann universe, which contains only sets obtained from $\varnothing$ by the process of "collecting into a set," and in which all relations must be constructed from $\in$.

3.7. "x is the direct product $y \times z$."

This means that the elements of x are the ordered pairs of elements of y and z, respectively. The definition of an unordered pair is obvious: the formula

$$\forall u\ (u \in x \Leftrightarrow (u = y_1 \vee u = z_1))$$

"means," or may be briefly written in the form, $x = \{y_1, z_1\}$ (compare 3.4). The ordered pair y_1 and z_1 is introduced using a device of Kuratowski and Wiener: this is the set x_1 whose elements are the unordered pairs $\{y_1, y_1\}$ and $\{y_1, z_1\}$.

We thus arrive at the formula

$$\exists y_2\ \exists z_2(\text{“}x_1 = \{y_2, z_2\}\text{”} \wedge \text{“}y_2 = \{y_1, y_1\}\text{”} \wedge \text{“}z_2 = \{y_1, z_1\}\text{”}),$$

which will be abbreviated

$$x_1 = \langle y_1, z_1 \rangle$$

and will be read "x_1 is the ordered pair with first element y_1 and second element z_1." The abbreviated notation for the subformulas is in quotes; we shall later omit the quotation marks.

Finally, the statement "$x = y \times z$" may be written in the form

$$\forall x_1(x_1 \in x \Leftrightarrow \exists y_1\ \exists z_1(y_1 \in y \wedge z_1 \in z \wedge \text{“}x_1 = \langle y_1, z_1 \rangle\text{”})).$$

In order to remind the reader for the last time of the liberties taken in abbreviated notation, we write this same formula adhering to all the canons of $\mathfrak{L}_1$:

$$\forall x_1 \Bigg[(\in (x_1 x))$$

$$\Leftrightarrow \Bigg[\exists y_1 \Bigg(\exists z_1 \Bigg(\Bigg(((\in (y_1 y)) \wedge (\in (z_1 z)) \Bigg) \wedge \Bigg(\exists y_2 \Bigg(\exists z_2 \Bigg(\Bigg(\Bigg(\forall u((\in (u x_1))$$

$$\Leftrightarrow ((= (u y_2) \vee (= (u z_2)) \Bigg) \Bigg) \Bigg) \wedge (\forall u((\in (u y_2))$$

$$\Leftrightarrow (= (u y_1))))) \wedge (\forall u((\in (u z_2) \Leftrightarrow ((= (u y_1)) \vee (= (u z_1)))))) \Bigg) \Bigg) \Bigg) \Bigg) \Bigg] \Bigg]$$

Exercise: Find the open parenthesis corresponding to the fifth closed parenthesis from the end. In §1 of Chapter II we give an algorithm for solving such problems.

3.8. "*f is a mapping from the set u to the set v.*"

First of all, mappings, or functions, are identified with their graphs; otherwise, we would not be able to consider them as elements of the universe. The following formula successively imposes three conditions on f: f is a subset of $u \times v$; the projection of f onto u coincides with all of u; and each element of u corresponds to exactly one element of v:

$$\begin{aligned}&\forall z\big(z \in f \Rightarrow (\exists u_1\ \exists v_1(u_1 \in u \wedge v_1 \in v \wedge \text{``}z = \langle u_1, v_1\rangle\text{''}))\big)\\ &\wedge \forall u_1(u_1 \in u \Rightarrow \exists v_1\ \exists z(v_1 \in v \wedge \text{``}z = \langle u_1, v_1\rangle\text{''} \wedge z \in f))\\ &\wedge \forall u_1\ \forall v_1\ \forall v_2(\exists z_1\ \exists z_1(z_1 \in f \wedge z_2 \in f \wedge \text{``}z_1 = \langle u_1, v_1\rangle\text{''} \wedge \text{``}z_2 = \langle u_1, v_2\rangle\text{''}))\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Rightarrow v_1 = v_2).\end{aligned}$$

Exercise: Write the formula "f is the projection of $y \times z$ onto z."

3.9. "*x is a finite set.*"

Finiteness is far from being a primitive concept. Here is Dedekind's definition: "there does not exist a one-to-one mapping f of the set x onto a proper subset." The formula:

$$\begin{aligned}&\neg\exists f\big(\text{``}f \text{ is a mapping from } x \text{ to } x\text{''} \wedge \forall u_1\ \forall u_2\ \forall v_1\ \forall v_2((\text{``}\langle u_1, v_1\rangle \in f\text{''}\\ &\wedge \text{``}\langle u_2, v_2\rangle \in f\text{''} \wedge \neg(u_1 = u_2)) \Rightarrow \neg(v_1 = v_2) \wedge \exists v_1(v_1 \in x \wedge \neg\exists u_1\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\text{``}\langle u_1, v_1\rangle \in f\text{''}))\big).\end{aligned}$$

The abbreviation "$\langle u_1, v_1\rangle \in f$" means, of course, $\exists y(\text{``}y = \langle u_1, v_1\rangle)\text{''} \wedge y \in f)$.

3.10. "*x is a nonnegative integer.*"

The natural numbers are represented in the von Neumann universe by the finite ordinals, so that the required formula has the form

$$\text{``}x \text{ is totally ordered by the relation } \in\text{''} \wedge \text{``}x \text{ is finite.''}$$

Exercise: Figure out how to write the formulas "$x + y = z$" and "$x \cdot y = z$" where x, y, z are integers $\geqslant 0$.

After this it is possible in the usual way to write the formulas "x is an integer," "x is a rational number," "x is a real number" (following Cantor or Dedekind), etc., and then construct a formal version of analysis. The written statements will have acceptable length only if we periodically extend the language $\mathrm{L_1Set}$ (see §8 of Chapter II). For example, in $\mathrm{L_1Set}$ we are not allowed to write term-names for the numbers 1, 2, 3, ... ($\varnothing$ is the name for 0), although we may construct the formulas "x is the finite ordinal containing 1 element," "x is the finite ordinal containing 2 elements," etc. If we use such roundabout methods of expression, the simplest numerical identities become incredibly long; but of course, in logic we are mainly concerned with the theoretical possibility of writing them.

3.11. "*x is a topological space.*"

In the formula we must give the topology of x explicitly. We define the topology, for example, in terms of the set y of all open subsets of x. We first write that y consists of subsets of x and contains x and the empty set:

$$P_1: \quad \forall z(z \in y \Rightarrow \forall u(u \in z \Rightarrow u \in x)) \wedge x \in y \wedge \varnothing \in y.$$

The intersection w of any two elements u, v in y is open, i.e., belongs to y:

$$P_2: \quad \forall u\ \forall v\ \forall w((u \in y \wedge v \in y \wedge \forall z((z \in u \wedge z \in v) \Leftrightarrow z \in w)) \Rightarrow w \in y).$$

It is harder to write "the union of any set of open subsets is open." We first write

$$P_3: \quad \forall u(u \in z \Leftrightarrow \forall v(v \in u \Rightarrow v \in y)),$$

that is, "z is the set of all subsets of y." Then

$$P_4: \quad \forall u\ \forall w((u \in z \wedge \forall v_1(v_1 \in w \Leftrightarrow \exists v(v \in u \wedge v_1 \in v))) \Rightarrow w \in y).$$

This means (taking into account P_3, which defines z); "If u is any subset of y, i.e., a set of open subsets of x, then the union w of all these subsets belongs to y, i.e., is open." Now the final formula may be written as follows:

$$P_1 \wedge P_2 \wedge \forall z(P_3 \Rightarrow P_4).$$

The following comments on this formula will be reflected in precise definitions in Chapter II, §§1 and 2. The letters x, y have the same meaning in all the P_i, while z plays different roles: in P_1 it is a subset of x, and in P_3 and P_4 it is the set of subsets of x. We are allowed to do this because as soon as we "bind" z by the quantifier $\forall$, say in P_1, z no longer stands for an (indeterminate) individual set, and becomes a temporary designation for "any set." Where the "scope of action" of $\forall$ ended, z can be given a new meaning. In order to "free" z for later use, $\forall z$ was also put before $P_3 \Rightarrow P_4$.

Translation from argot to $\mathrm{L}_{\mathbf{1}}\mathrm{Ar}$.

3.12. "$x < y$": $\exists z(y = (x + z) + \bar{1})$. Recall that the variables are names for nonnegative integers.

3.13. "*x is a divisor of y*": $\exists z(y = x \cdot z)$.

3.14. "*x is a prime number*": "$\bar{1} < x$" $\wedge$ ("*y is a divisor of x*" $\Rightarrow (y = \bar{1} \vee y = x))$.

3.15. "*Fermat's last theorem*": $\forall x_1\ \forall x_2\ \forall x_3\ \forall u($"$\bar{2} < u$" $\wedge$ "$x_1^u + x_2^u = x_3^u$" $\Rightarrow$ "$x_1 x_2 x_3 = \bar{0}$"). It is not clear how to write the formula $x_1^u + x_2^u = x_3^u$

in L_1Ar. Of course, for any *concrete* $u = 1, 2, 3$ there is a corresponding atomic formula in L_1Ar, but how do we make u into a variable? This is not a trivial problem. In the second part of the book we show how to find an atomic formula $p(x, u, y, z_1, \ldots, z_n)$ such that the assertion that $\exists z_1 \cdots \exists z_n p\ (x, u, y, z_1, \ldots, z_n)$ in the domain of natural numbers is equivalent $y = x^u$. Then $x_1^u + x_2^u = x_3^u$ can be translated as follows:

$$\exists y_1\ \exists y_2\ \exists y_3\ (\text{“}x_1^u = y_1\text{”} \wedge \text{“}x_2^u = y_2\text{”} \wedge \text{“}x_3^u = y_3\text{”} \wedge y_1 + y_2 = y_3).$$

The existence of such a p is a nontrivial number-theoretic fact, so that here the very possibility of performing a translation becomes a mathematical problem.

3.16. "*The Riemann hypothesis.*" The Riemann zeta function $\zeta\ (s)$ is defined by the series $\Sigma_{n=1}^{\infty}\ n^{-s}$ in the half-plane Re $s \geq 1$. It can be continued meromorphically onto the entire complex s-plane. The Riemann hypothesis is the assertion that the nontrivial zeros of $\zeta(s)$ lie on the line Re $s = \frac{1}{2}$. Of course, in this form the Riemann hypothesis cannot be translated into L_1Ar. However, there are several purely arithmetic assertions that are demonstrably equivalent to the Riemann hypothesis. Perhaps the simplest of them is the following.

Let $\mu(n)$ be the Möbius function on the set of integers $\geqslant 1$: it equals 0 if n is divisible by a square, and equals $(-1)^r$, where r is the number of prime divisors of n, if n is square-free. We then have

$$\text{Riemann hypothesis } \Leftrightarrow \forall \varepsilon > 0\ \exists x\ \forall y \left[y > x \Rightarrow \left[\left| \sum_{n=1}^{y} \mu(n) \right| < y^{1/2+\varepsilon} \right] \right].$$

Only the exponent is not an integer on the right; but ε need only run through numbers of the form $1/z$, z an integer $\geqslant 1$, and then we can raise the inequality to the $(2z)$th power. The formula

$$\left(\sum_{n=1}^{y} \mu(n) \right)^{2z} < y^{z+2}$$

can then be translated into L_1Ar, although not completely trivially. The necessary techniques will be developed in the second part of the book.

The last two examples were given in order to show the complexity that is possible in problems that can be stated in L_1Ar, despite the apparent simplicity of the modes of expression and the semantics of the language.

We conclude this section with some remarks concerning higher-order languages.

3.17. *Higher-order languages.* Let L be any first-order language. Its modes of expression are limited in principle by one important consideration: we are not allowed to speak of arbitrary properties of objects of the theory, that is, arbitrary subsets of the set of all objects. Syntactically, this is reflected in the prohibition against forming expressions such as $\forall p(p(x))$, where p is a relation of degree 1; relations must stand for fixed rather than variable properties.

Of course, certain properties can be defined using nonatomic formulas. For example, in $\mathrm{L_1Ar}$ instead of "x is even" we may write $\exists y(x = (\bar{1} + \bar{1}) \cdot y)$. However, there is a continuum of subsets of the integers but only a countable set of definable properties (see §2 of Chapter II), so there are automatically properties that cannot be defined by formulas. Thus, it is impossible to replace the forbidden expression $\forall p(p(x))$ by a sequence of expressions $P_1(x),\ P_2(x),\ P_3(x), \ldots$.

Languages in which quantifiers may be applied to properties and/or functions (and also, possibly, to properties of properties, and so on) are called higher-order languages. One such language—$\mathrm{L_2Real}$—will be considered in Chapter III for the purpose of illustrating a simplified version of Cohen forcing.

On the other hand, the same extension of expressive possibilities can be obtained without leaving $\mathfrak{L}_1$. In fact, in the first-order language $\mathrm{L_1Set}$ we may quantify over all subsets of any set, over all subsets of the set of subsets, and so on. Informally this means that we are speaking of all properties, all properties of properties, ... (with transfinite extension). In addition, any higher-order language with a "standard interpretation" in some type of structured sets can be translated into $\mathrm{L_1Set}$ so as to preserve the meanings and truth values in this standard interpretation. (An apparent exception is the languages for describing Gödel–Bernays classes and "large" categories; but it seems, based on our present understanding of paradoxes, that no higher-order languages can be constructed from such a language.)

The attentive reader will notice the contrast between the *possibility* of writing a formula in $\mathrm{L_1Set}$ in which $\forall$ is applied to all subsets (informally, to all properties) of finite ordinals (informally, of integers) and the *impossibility* of writing a formula in $\mathrm{L_1Set}$ that would define any concrete subset in the continuum of undefinable subsets. (There are fewer such subsets in $\mathrm{L_1Set}$ than in $\mathrm{L_1Ar}$, but still a continuum.) We shall examine these problems more closely in Chapter II when we discuss "Skolem's paradox."

Let us summarize. Almost all the basic logical and set-theoretic principles used in the day-to-day work of the mathematician are contained in the first-order languages and, in particular, in $\mathrm{L_1Set}$. Hence, those languages will be the subject of study in the first and third parts of the book. But concrete oriented languages can be formed in other ways, with various degrees of deviation from the rules of $\mathfrak{L}_1$. In addition to $\mathrm{L_2Real}$, examples of such languages examined in Chapter II include SELF (Smullyan's language for self-description) and SAr, which is a language of arithmetic convenient for proving Tarski's theorem on the undefinability of truth.

Digression: Syntax

1. The most important feature that most artificial languages have in common is the ability to encompass a rich spectrum of modes of expression starting with a small finite number of generating principles.

In each concrete case the choice of these principles (including the alphabet and syntax) is based on a compromise between two extremes. Economical use of modes of expression leads to unified notation and simplified mechanical analysis of the text. But then the texts become much longer and farther removed from natural language texts. Enriching the modes of expression brings the artificial texts closer to the natural language texts, but complicates the syntax and the formal analysis. (Compare machine languages with such programming languages as Algol, Fortran, Cobol, etc.)

We now give several examples based on our material.

2. *Dialects of* $\mathfrak{L}_1$

(a) Without changing the logic in $\mathfrak{L}_1$, it is possible to discard parentheses and either of the two quantifiers from the alphabet, and to replace all the connectives by one, namely $\downarrow$ (conjunction of negations). (In addition, constants could be declared to be functions of degree 0, and functions could be interpreted as relations.)

This is accomplished by the following change in the definitions. If $t_1, \ldots, t_r$ are terms, f is an operation of degree r, and p is a relation of degree r, then $ft_1 \ldots t_r$ is a term, and $pt_1 \ldots t_r$ is an atomic formula. If P and Q are formulas, then $\downarrow PQ$ and $\forall xP$ are formulas. The content of $\downarrow PQ$ is "not P and not Q" so that we have the following expressions in this dialect:

$$\begin{aligned} \neg(P): &\quad \downarrow PP, \\ (P) \wedge (Q): &\quad \downarrow\downarrow PP \downarrow QQ, \\ (P) \vee (Q): &\quad \downarrow\downarrow PQ \downarrow PQ. \end{aligned}$$

Clearly, economizing on parentheses and connectives leads to much repetition of the same formula. Nevertheless, it may become simpler to prove theorems about such a language because of the shorter list of syntactic norms.

(b) Bourbaki's language of set theory has an alphabet consisting of the signs $\square$, τ, $\vee$, $\neg$, $=$, $\in$ and the letters. Expressions in this language are not simply sequences of signs in the alphabet, but sequences in which certain elements are paired together by superlinear connectives. For example:

$$\tau \vee \neg \in \square\, A' \in \square\, A''.$$

The main difference between Bourbaki's language and L_1Set is the use of the "Hilbert choice symbol." If, for example, $\in xy$ is the formula "x is an element of y," then

$$\tau \in \square\, y$$

is a term meaning "some element of the set y."

Bourbaki's language is not very convenient and is not widely used. It became known in the popular literature thanks to an example of a very long abbreviated notation for the term "one," which the authors imprudently introduced:

$$\tau_z\Big((\exists u)(\exists U)(u = (U,\ \{\varnothing\}, Z) \wedge U \subset \{\varnothing\} \times Z \wedge (\forall k)((x \in \{\varnothing\})$$
$$\Rightarrow (\exists y)((x, y) \in U)) \wedge (\forall x)(\forall y)(\forall y^{'})(((x, y \in U \wedge (x, y^{'}) \in U)$$
$$\Rightarrow (y = y^{'})) \wedge (\forall y)((y \in Z) \Rightarrow (\exists x)x((x, y) \in U))))\Big).$$

It would take several tens of thousands of symbols to write out this term completely; this seems a little too much for "one."

(c) A way to greatly extend the expressive possibilities of almost any language in $\mathfrak{L}_1$ is to allow "class terms" of the type $\{x|P(x)\}$, meaning "the class of all objects x having the property P." This idea was used by Morse in his language of set theory and by Smullyan in his language of arithmetic; see §10 of Chapter II.

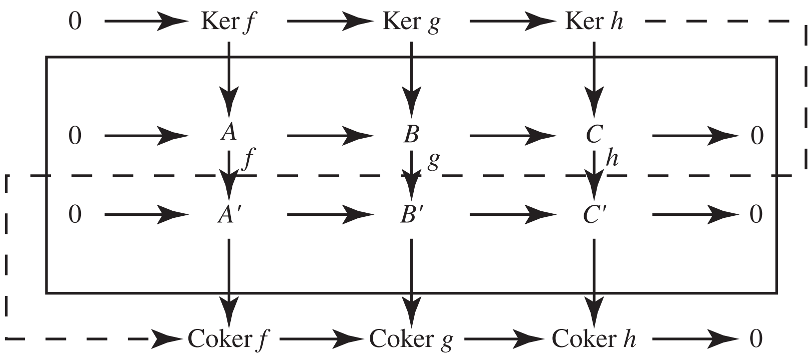
3. *General remarks.* Most natural and artificial languages are characteristically discrete and linear (one-dimensional). On the one hand, our perception of the external world is not felt by us to be either discrete or linear, although these characteristics are observed on the level of physiological mechanisms (coding by impulses in the nervous system). On the other hand, the languages in which we communicate tend to transmit information in a sequence of distinguishable elementary signs. The main reason for this is probably the much greater (theoretically unlimited) uniqueness and reproducibility of information than is possible with other methods of conveyance. Compare with the well-known advantages of digital over analog computers.

The human brain clearly uses both principles. The perception of images as a whole, along with emotions, are more closely connected with nonlinear and nondiscrete processes—perhaps of a wave nature. It is interesting to examine from this point of view the nonlinear fragments in various languages.

In mathematics this includes, first of all, the use of drawings. But this use does not lend itself to formal description, with the exception of the separate and formalized theory of graphs. Graphs are especially popular objects, because they are as close as possible both to their visual image as a whole and to their description using all the rules of set theory. Every time we are able to connect a problem with a graph, it becomes much simpler to discuss it, and large sections of verbal description are replaced by manipulation with pictures.

A less well-known class of examples is the commutative diagrams and spectral sequences of homological algebra. A typical example is the "snake lemma." Here is its precise formulation.

Suppose we are given a commutative diagram of abelian groups and homomorphisms between them (in the box below), in which the rows are exact sequences:



Then the kernels and cokernels of the "vertical" homomorphisms f, g, h form a six-term exact sequence, as shown in the drawing, and the entire diagram of solid arrows is commutative. The "snake" morphism Ker $h \to$ Coker f, which is denoted by the dotted arrow, is the basic object constructed in the lemma.

Of course, it is easy to describe the snake diagram sequentially in a suitable, more or less formal, linear language. However, such a procedure requires an artificial and not uniquely determined breaking up of a clearly two-dimensional picture (as in scanning a television image). Moreover, without having the overall image in mind, it becomes harder to recognize the analogous situation in other contexts and to bring the information together into a single block.

The beginnings of homological algebra saw the enthusiastic recognition of useful classes of diagrams. At first this interest was even exaggerated; see the editor's appendix to the Russian translation of *Homological Algebra* by Cartan and Eilenberg.

There is one striking example of an entire book with an intentional two-dimensional (block) structure: C. H. Lindsey and S. G. van der Meulen, *Informal Introduction to Algol 68* (North-Holland, Amsterdam, 1971). It consists of eight chapters, each of which is divided into seven sections (eight of the 56 sections are empty, to make the system work!). Let (i, j) be the name of the jth section of the ith chapter; then the book can be studied either "row by row" or "column by column" in the (i, j) matrix, depending on the reader's intentions.

As with all great undertakings, this is the fruit of an attempt to solve what is in all likelihood an insoluble problem, since, as the authors remark, Algol 68 "is quite impossible to describe . . . until it has been described."