8

Selected Proofs on Finite Packings of Translates of Convex Bodies

8.1 Proof of Theorem 2.2.1

8.1.1 Monotonicity of a special integral function

Lemma 8.1.1 *Let* $f : [0,1] \to \mathbb{R}$ *be a function such that* f *is positive and monotone increasing on* $(0,1]$*; moreover,* $f(x) = (g(x))^k$ *for some concave function* $g : [0,1] \to \mathbb{R}$*, where* k *is a positive integer. Then*

$$F(y) := \frac{1}{f(y)} \int_0^y f(x)dx$$

is strictly monotone increasing on $(0,1]$.

Proof: Without loss of generality we may assume that f is differentiable. So, to prove that $F(y) := \frac{1}{f(y)} \int_0^y f(x)dx$ is strictly monotone increasing, it is sufficient to show that $\frac{d}{dy}F > 0$ or equivalently that $\int_0^y f(x)dx < \frac{(f(y))^2}{f'(y)}$. From now on, let $0 < y < 1$ be fixed (with $f'(y) > 0$).

As $f = g^k$ for some concave g therefore the linear function $l(x) = b_1 + b_2(x-y)$ with $b_1 = (f(y))^{\frac{1}{k}}$ and $b_2 = \frac{f'(y)}{k(f(y))^{\frac{k-1}{k}}}$ satisfies the inequality $g(x) \leq l(x)$ for all $0 \leq x \leq 1$, and so we have that $f(x) \leq (l(x))^k$ holds for all $0 \leq x \leq 1$. Thus, for all $0 \leq x \leq 1$ we have

$$f(x) \leq \left((f(y))^{\frac{1}{k}} + \frac{f'(y)}{k\,(f(y))^{\frac{k-1}{k}}}(x-y) \right)^k = f(y) \left(1 + \frac{f'(y)}{kf(y)}(x-y) \right)^k . \tag{8.1}$$

By integration we get

$$\int_0^y f(x)dx \leq \int_0^y f(y) \left(1 + \frac{f'(y)}{kf(y)}(x-y) \right)^k dx$$

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$$= \frac{k}{k+1}\frac{(f(y))^2}{f'(y)}\left(1-\left(1-\frac{yf'(y)}{kf(y)}\right)^{k+1}\right). \tag{8.2}$$

Now, because the first factor of (8.2) is strictly between 0 and 1, it is sufficient to show that the last factor is at most 1; that is, we are left to show the inequality

$$0 \leq \left(1-\frac{yf'(y)}{kf(y)}\right)^{k+1} \tag{8.3}$$

Suppose that (8.3) is not true; then $\left(1-\frac{yf'(y)}{kf(y)}\right)^{k+1} < 0$. Let $G(x) := \left(1+\frac{f'(y)}{kf(y)}(x-y)\right)^{k+1}$. As $G(y)=1$ and by assumption $G(0)<0$, therefore there must be an $0<x_0<y$ such that $G(x_0)=\left(1+\frac{f'(y)}{kf(y)}(x_0-y)\right)^{k+1}=0$. But then this and (8.1) imply in a straightforward way that $f(x_0) \leq f(y)\left(1+\frac{f'(y)}{kf(y)}(x_0-y)\right)^{k}=0$. However, by the assumptions of Lemma 8.1.1 we have that $f(x_0)>0$, a contradiction. This completes our proof of Lemma 8.1.1. □

8.1.2 A proof by slicing via the Brunn–Minkowski inequality

Let the convex body $\mathbf{K}$ be positioned in $\mathbb{E}^d$ such that the hyperplane $\{\mathbf{x} \in \mathbb{E}^d \mid \langle \mathbf{x}, \mathbf{v}\rangle = -1\}$ with normal vector $\mathbf{v}$ is a supporting hyperplane for $\mathbf{K}$ and the non-overlapping translates $\mathbf{t}_1+\mathbf{K}, \dots, \mathbf{t}_k+\mathbf{K}$ are all touching $\mathbf{K}$ and (together with $\mathbf{K}$) are all lying in the closed halfspace $\{\mathbf{x} \in \mathbb{E}^d \mid \langle \mathbf{x}, \mathbf{v}\rangle \geq -1\}$. Now, due to the well-known fact that by replacing $\mathbf{K}$ with $\frac{1}{2}(\mathbf{K}+-(\mathbf{K}))$ and performing the same symmetrization for each of the translates $\mathbf{t}_1+\mathbf{K}, \dots, \mathbf{t}_k+\mathbf{K}$ one preserves the packing property, touching pairs, and one-sidedness, we may assume that $\mathbf{K}$ is in fact, a centrally symmetric convex body of $\mathbb{E}^d$ say, it is $\mathbf{o}$-symmetric, where $\mathbf{o}$ stands for the origin of $\mathbb{E}^d$. Moreover, as in the classical proof for the Hadwiger number [151], we use that $\bigcup_{i=0}^{k}(\mathbf{t}_i+\mathbf{K}) \subset 3\mathbf{K}$, where $\mathbf{t}_0=\mathbf{o}$. Furthermore, let the family $\mathbf{t}_0+\mathbf{K}, \mathbf{t}_1+\mathbf{K}, \dots, \mathbf{t}_k+\mathbf{K}$ be scaled so that the normal vector $\mathbf{v}$ is a unit vector (i.e., $\|\mathbf{v}\|=1$). Next, let $H_x := \{\mathbf{p} \in \mathbb{E}^d \mid \langle \mathbf{p}, \mathbf{v}\rangle = x\}$ for $x \in \mathbb{R}$. Then clearly, $\mathbf{K}$ is between the hyperplanes H_{-1} and H_1 touching both, and the translates $\mathbf{t}_1+\mathbf{K}, \dots, \mathbf{t}_k+\mathbf{K}$ (together with $\mathbf{K}=\mathbf{t}_0+\mathbf{K}$) all lie between the hyperplanes H_{-1} and H_3. Obviously, $\int_{-1}^{1}\mathrm{vol}_{d-1}(\mathbf{K}\cap H_x)\,dx = \mathrm{vol}_d(\mathbf{K})$, where $\mathrm{vol}_d(\cdot)$ (resp., $\mathrm{vol}_{d-1}(\cdot)$) denotes the d-dimensional (resp., $d-1$-dimensional) volume measure. Also, it follows from the given setup in a straightforward way that

$$\int_{-1}^{3}\mathrm{vol}_{d-1}\left(H_x \cap \bigcup_{i=0}^{k}(\mathbf{t}_i+\mathbf{K})\right)dx = (k+1)\mathrm{vol}_d(\mathbf{K}). \tag{8.4}$$

Our goal is to write the integral in (8.4) as a sum of two integrals from -1 to 0 and from 0 to 3, and estimate them separately.

First, notice that

$$\int_0^3 \mathrm{vol}_{d-1}\left(H_x \cap \bigcup_{i=0}^{k}(\mathbf{t}_i + \mathbf{K})\right) dx \le \int_0^3 \mathrm{vol}_{d-1}\left(H_x \cap (3\mathbf{K})\right) dx = \frac{3^d}{2}\mathrm{vol}_d(\mathbf{K}). \tag{8.5}$$

Second, notice that

$$\int_{-1}^0 \mathrm{vol}_{d-1}\left(H_x \cap \bigcup_{i=0}^{k}(\mathbf{t}_i + \mathbf{K})\right) dx = \sum_{i=0}^{k}\int_{-1}^0 \mathrm{vol}_{d-1}\left(H_x \cap (\mathbf{t}_i + \mathbf{K})\right) dx \tag{8.6}$$

$$= \sum_{i=0}^{k}\int_0^1 \mathrm{vol}_{d-1}\left(\mathbf{K} \cap (-\mathbf{t}_i + H_{x-1})\right) dx = \sum_{0\le a_i\le 1}\int_0^{1-a_i} f(x)dx, \tag{8.7}$$

where $f(x) := \mathrm{vol}_{d-1}\left(\mathbf{K} \cap H_{x-1}\right), 0 \le x \le 1$ and $a_i := \langle \mathbf{v}, \mathbf{t}_i\rangle, 0 \le i \le k$. We note that $a_i \ge 0$ for all $0 \le i \le k$ (and for some j we have that $a_j \ge 1$). Moreover, f is positive and monotone increasing on $(0, 1]$, and by the Brunn–Minkowski inequality (see, e.g., [85]) the function $f^{\frac{1}{d-1}}$ is concave (for all $d \ge 2$). Thus, Lemma 8.1.1 implies that

$$\sum_{0\le a_i\le 1}\int_0^{1-a_i} f(x)dx \le \sum_{0\le a_i\le 1}\left(\frac{f(1-a_i)}{f(1)}\int_0^1 f(x)dx\right) \tag{8.8}$$

$$= \frac{\int_0^1 f(x)dx}{f(1)}\sum_{0\le a_i\le 1} f(1-a_i) = \frac{\int_0^1 f(x)dx}{f(1)}\sum_{0\le a_i\le 1}\mathrm{vol}_{d-1}\left(\mathbf{K} \cap H_{-a_i}\right) \tag{8.9}$$

$$= \frac{\int_0^1 f(x)dx}{f(1)}\sum_{i=0}^{k}\mathrm{vol}_{d-1}\left((\mathbf{t}_i + \mathbf{K}) \cap H_0\right) \tag{8.10}$$

$$= \frac{\int_0^1 f(x)dx}{f(1)}\mathrm{vol}_{d-1}\left(H_0 \cap \bigcup_{i=0}^{k}(\mathbf{t}_i + \mathbf{K})\right) \le \frac{\int_0^1 f(x)dx}{f(1)}\mathrm{vol}_{d-1}\left(H_0 \cap (3\mathbf{K})\right) \tag{8.11}$$

$$= \frac{1}{2}\mathrm{vol}_d(\mathbf{K})\frac{1}{\mathrm{vol}_{d-1}(H_0 \cap \mathbf{K})}\mathrm{vol}_{d-1}\left(H_0 \cap (3\mathbf{K})\right) = \frac{3^{d-1}}{2}\mathrm{vol}_d(\mathbf{K}). \tag{8.12}$$

Hence, (8.4), (8.5), (8.6), (8.7), (8.8), (8.9), (8.10), (8.11), and (8.12) yield that

$$(k+1)\mathrm{vol}_d(\mathbf{K}) \le \frac{3^d}{2}\mathrm{vol}_d(\mathbf{K}) + \frac{3^{d-1}}{2}\mathrm{vol}_d(\mathbf{K}),$$

and so, $k \le 2 \cdot 3^{d-1} - 1$ as claimed in Theorem 2.2.1.

To prove that equality can only be reached for d-dimensional affine cubes, notice first that the equality in (8.8) and the strict monotonicity of Lemma 8.1.1 imply that for all a_i with $0 \le a_i < 1$ we have $a_i = 0$ and for all $a_i = 1$ we have $f(1-a_i) = 0$. Taking into account the equality in (8.11), we get that translates of $H_0 \cap \mathbf{K}$ must tile $H_0 \cap (3\mathbf{K})$. Hence, [151] yields that $H_0 \cap \mathbf{K}$ as well as $H_0 \cap (3\mathbf{K})$ are $(d-1)$-dimensional affine cubes. Also, there is only the obvious way to tile $H_0 \cap (3\mathbf{K})$ by 3^{d-1} translates of $H_0 \cap \mathbf{K}$, so the set of the translation vectors $\{\mathbf{t}_i \mid \mathbf{t}_i \in H_0\}$ is $\mathbf{o}$-symmetric. But then the $\left(\left(3^{d-1}-1\right)+1\right) + 2\left(\left(2\cdot 3^{d-1}-1\right) - \left(3^{d-1}-1\right)\right) = 3^d$ translates

$$\{\mathbf{t}_i + \mathbf{K} \mid \mathbf{t}_i \in H_0\} \cup \{\mathbf{t}_i + \mathbf{K} \mid \mathbf{t}_i \notin H_0\} \cup \{-\mathbf{t}_i + \mathbf{K} \mid \mathbf{t}_i \notin H_0\}$$

of $\mathbf{K}$ form a packing in $3\mathbf{K}$. Thus, the Hadwiger number of $\mathbf{K}$ is $3^d - 1$ and so, using [151] we get that $\mathbf{K}$ is indeed a d-dimensional affine cube. This completes the proof of Theorem 2.2.1.

8.2 Proof of Theorem 2.4.3

Let $\mathbf{C}_n := \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and assume that the inequality stated in Theorem 2.4.3 does not hold. Then there is an $\epsilon > 0$ such that

$$\mathrm{vol}_d(\mathbf{C}_n + 2\mathbf{K_o}) = \frac{n\mathrm{vol}_d(\mathbf{K_o})}{\delta(\mathbf{K_o})} - \epsilon. \tag{8.13}$$

Let $\Lambda \subset \mathbb{E}^d$ be a d-dimensional packing lattice of $\mathbf{C}_n + 2\mathbf{K_o}$ such that $\mathbf{C}_n + 2\mathbf{K_o}$ is contained in the fundamental parallelotope $\mathbf{P}$ of Λ. For each $\lambda > 0$ let $\mathbf{Q}_\lambda$ denote the d-dimensional cube of edge length 2λ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ having edges parallel to the corresponding coordinate axes of $\mathbb{E}^d$. Obviously, there is a constant $\mu > 0$ depending on $\mathbf{P}$ only such that for each $\lambda > 0$ there is a subset $L_\lambda \subset \Lambda$ with $\mathbf{Q}_\lambda \subset L_\lambda + \mathbf{P}$ and $L_\lambda + 2\mathbf{P} \subset \mathbf{Q}_{\lambda+\mu}$. Moreover, let $\mathcal{P}_n(\mathbf{K_o})$ be the family of all possible packings of $n > 1$ translates of the $\mathbf{o}$-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d$. The definition of $\delta(\mathbf{K_o})$ implies that for each $\lambda > 0$ there exists a packing in the family $\mathcal{P}_{m(\lambda)}(\mathbf{K_o})$ with centers at the points of $\mathbf{C}_{m(\lambda)}$ such that $\mathbf{C}_{m(\lambda)} + \mathbf{K_o} \subset \mathbf{Q}_\lambda$ and

$$\lim_{\lambda\to\infty} \frac{m(\lambda)\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{Q}_\lambda)} = \delta(\mathbf{K_o}).$$

As $\lim_{\lambda\to\infty} \frac{\mathrm{vol}_d(\mathbf{Q}_{\lambda+\mu})}{\mathrm{vol}_d(\mathbf{Q}_\lambda)} = 1$, therefore there exist $\xi > 0$ and a packing in the family $\mathcal{P}_{m(\xi)}(\mathbf{K_o})$ with centers at the points of $\mathbf{C}_{m(\xi)}$ and with $\mathbf{C}_{m(\xi)} + \mathbf{K_o} \subset \mathbf{Q}_\xi$ such that

$$\frac{\mathrm{vol}_d(\mathbf{P})\delta(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P}) + \epsilon} < \frac{m(\xi)\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})} \tag{8.14}$$

and

$$\frac{n\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})+\epsilon} < \frac{n\mathrm{vol}_d(\mathbf{K_o})\mathrm{card}(L_\xi)}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})}. \tag{8.15}$$

Now, for each $\mathbf{x} \in \mathbf{P}$ we define a packing of $n(\mathbf{x})$ translates of the **o**-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d$ with centers at the points of

$$\mathbf{C}_{n(\mathbf{x})} := \{\mathbf{x} + L_\xi + \mathbf{C}_n\} \cup \{\mathbf{y} \in \mathbf{C}_{m(\xi)} \mid \mathbf{y} \notin \mathbf{x} + L_\xi + \mathbf{C}_n + \mathrm{int}(2\mathbf{K_o})\}.$$

Clearly, $\mathbf{C}_{n(\mathbf{x})} + \mathbf{K_o} \subset \mathbf{Q}_{\xi+\mu}$. As a next step we introduce the (characteristic) function $\chi_{\mathbf{y}} : \mathbf{P} \to \mathbb{R}$ as follows: $\chi_{\mathbf{y}}(\mathbf{x}) := 1$ if $\mathbf{y} \notin \mathbf{x} + L_\xi + \mathbf{C}_n + \mathrm{int}(2\mathbf{K_o})$ and $\chi_{\mathbf{y}}(\mathbf{x}) := 0$ for any other $\mathbf{x} \in \mathbf{P}$. Thus,

$$\int_{\mathbf{x}\in\mathbf{P}} n(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{x}\in\mathbf{P}} \left(n\ \mathrm{card}(L_\xi) + \sum_{\mathbf{y}\in\mathbf{C}_{m(\xi)}} \chi_{\mathbf{y}}(\mathbf{x}) \right) d\mathbf{x}$$

$$= n\mathrm{vol}_d(\mathbf{P})\mathrm{card}(L_\xi) + m(\xi)\left(\mathrm{vol}_d(\mathbf{P}) - \mathrm{vol}_d(\mathbf{C}_n + 2\mathbf{K_o})\right).$$

Hence, there is a point $\mathbf{p} \in \mathbf{P}$ with

$$n(\mathbf{p}) \geq m(\xi)\left(1 - \frac{\mathrm{vol}_d(\mathbf{C}_n + 2\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})}\right) + n\ \mathrm{card}(L_\xi)$$

and so,

$$\frac{n(\mathbf{p})\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})}$$

$$\geq \frac{m(\xi)\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})}\left(1 - \frac{\mathrm{vol}_d(\mathbf{C}_n + 2\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})}\right) + \frac{n\mathrm{vol}_d(\mathbf{K_o})\mathrm{card}(L_\xi)}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})}. \tag{8.16}$$

Thus, (8.16), (8.15), (8.14), and (8.13) imply in a straightforward way that

$$\frac{n(\mathbf{p})\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{Q}_{\xi+\mu})}$$

$$> \frac{\mathrm{vol}_d(\mathbf{P})\delta(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})+\epsilon}\left(1 - \frac{\mathrm{vol}_d(\mathbf{C}_n + 2\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})}\right) + \frac{n\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\mathbf{P})+\epsilon} = \delta(\mathbf{K_o}). \tag{8.17}$$

As $\mathbf{C}_{n(\mathbf{p})} + \mathbf{K_o} \subset \mathbf{Q}_{\xi+\mu}$, therefore (8.17) leads to the existence of a packing by translates of $\mathbf{K_o}$ in $\mathbb{E}^d$ with density strictly larger than $\delta(\mathbf{K_o})$, a contradiction. This finishes the proof of Theorem 2.4.3.