7

Selected Proofs on Sphere Packings

7.1 Proof of Theorem 1.3.5

7.1.1 A proof by estimating the surface area of unions of balls

Let $\mathbf{B}$ denote the unit ball centered at the origin $\mathbf{o}$ of $\mathbb{E}^3$ and let $\mathcal{P} := \{\mathbf{c}_1 + \mathbf{B}, \mathbf{c}_2 + \mathbf{B}, \dots, \mathbf{c}_n + \mathbf{B}\}$ denote the packing of n unit balls with centers $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ in $\mathbb{E}^3$ having the largest number $C(n)$ of touching pairs among all packings of n unit balls in $\mathbb{E}^3$. ($\mathcal{P}$ might not be uniquely determined up to congruence in which case $\mathcal{P}$ stands for any of those extremal packings.) First, observe that Theorem 1.4.1 and Theorem 2.4.3 imply the following inequality in a straightforward way.

Lemma 7.1.1

$$\frac{n\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{B})} \leq \delta(\mathbf{B}) = \frac{\pi}{\sqrt{18}}.$$

Second, the well-known isoperimetric inequality [97] yields the following.

Lemma 7.1.2

$$36\pi\mathrm{vol}_3^2\left(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{B}\right) \leq \mathrm{svol}_2^3\left(\mathrm{bd}\left(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{B}\right)\right).$$

Thus, Lemma 7.1.1 and Lemma 7.1.2 generate the following inequality.

Corollary 7.1.3

$$4(18\pi)^{\frac{1}{3}} n^{\frac{2}{3}} \leq \mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{B}\right)\right).$$

Now, assume that $\mathbf{c}_i + \mathbf{B} \in \mathcal{P}$ is tangent to $\mathbf{c}_j + \mathbf{B} \in \mathcal{P}$ for all $j \in T_i$, where $T_i \subset \{1, 2, \dots, n\}$ stands for the family of indices $1 \leq j \leq n$ for which $\|\mathbf{c}_i - \mathbf{c}_j\| = 2$. Then let $S_i := \mathrm{bd}(\mathbf{c}_i + 2\mathbf{B})$ and let $C_{S_i}(\mathbf{c}_j, \frac{\pi}{6})$ denote the open

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spherical cap of S_i centered at $\mathbf{c}_j \in S_i$ having angular radius $\frac{\pi}{6}$. Clearly, the family $\{C_{S_i}(\mathbf{c}_j, \frac{\pi}{6}), j \in T_i\}$ consists of pairwise disjoint open spherical caps of S_i; moreover,

$$\frac{\sum_{j \in T_i} \mathrm{svol}_2\left(C_{S_i}(\mathbf{c}_j, \frac{\pi}{6})\right)}{\mathrm{svol}_2\left(\cup_{j \in T_i} C_{S_i}(\mathbf{c}_j, \frac{\pi}{3})\right)} = \frac{\sum_{j \in T_i} \mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j \in T_i} C(\mathbf{u}_{ij}, \frac{\pi}{3})\right)}, \tag{7.1}$$

where $\mathbf{u}_{ij} := \frac{1}{2}(\mathbf{c}_j - \mathbf{c}_i) \in \mathbb{S}^2$ and $C(\mathbf{u}_{ij}, \frac{\pi}{6}) \subset \mathbb{S}^2$ (resp., $C(\mathbf{u}_{ij}, \frac{\pi}{3}) \subset \mathbb{S}^2$) denotes the open spherical cap of $\mathbb{S}^2$ centered at $\mathbf{u}_{ij}$ having angular radius $\frac{\pi}{6}$ (resp., $\frac{\pi}{3}$) and where $\mathrm{svol}_2(\cdot)$ (resp., $\mathrm{Sarea}(\cdot)$) denotes the 2-dimensional surface volume measure in $\mathbb{E}^3$ (resp., spherical area measure on $\mathbb{S}^2$) of the corresponding set. Now, Molnár's density bound (see Satz 1 in [200]) implies that

$$\frac{\sum_{j \in T_i} \mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j \in T_i} C(\mathbf{u}_{ij}, \frac{\pi}{3})\right)} < 0.89332\ . \tag{7.2}$$

In order to estimate $\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n} \mathbf{c}_i + 2\mathbf{B}\right)\right)$ from above let us assume that m members of $\mathcal{P}$ have 12 touching neighbours in $\mathcal{P}$ and k members of $\mathcal{P}$ have at most 9 touching neighbours in $\mathcal{P}$. Thus, $n - m - k$ members of $\mathcal{P}$ have either 10 or 11 touching neighbours in $\mathcal{P}$. Without loss of generality we may assume that $4 \leq k \leq n - m$. Based on the notation just introduced, it is rather easy to see, that (7.1) and (7.2) together with the well-known fact that the kissing number of $\mathbf{B}$ is 12, imply the following estimate.

Corollary 7.1.4

$$\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n} \mathbf{c}_i + 2\mathbf{B}\right)\right) < 12.573(n - m - k) + 38.9578k$$

$$< \frac{38.9578}{3}(n - m - k) + 38.9578k\ .$$

Hence, Corollary 7.1.3 and Corollary 7.1.4 yield in a straightforward way that

$$1.1822n^{\frac{2}{3}} - 3k < n - m - k\ . \tag{7.3}$$

Finally, as the number $C(n)$ of touching pairs in $\mathcal{P}$ is obviously at most

$$\frac{1}{2}\left(12n - (n - m - k) - 3k\right)\ ,$$

therefore (7.3) implies that

$$C(n) \leq \frac{1}{2}\left(12n - (n - m - k) - 3k\right) < 6n - 0.5911n^{\frac{2}{3}} < 6n - 0.59n^{\frac{2}{3}},$$

finishing the proof of Theorem 1.3.5.

7.1.2 On the densest packing of congruent spherical caps of special radius

We feel that it is worth making the following comment: it is likely that (7.2) can be replaced by the following sharper estimate.

Conjecture 7.1.5

$$\frac{\sum_{j\in T_i} \mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j\in T_i} C(\mathbf{u}_{ij}, \frac{\pi}{3})\right)} \le 6\left(1-\frac{\sqrt{3}}{2}\right) = 0.8038\ldots \ ,$$

with equality when 12 *spherical caps of angular radius* $\frac{\pi}{6}$ *are packed on* $\mathbb{S}^2$.

If so, then one can improve Theorem 1.3.5 as follows.

Proposition 7.1.6 *Conjecture 7.1.5 implies that*

$$C(n) \le 6n - \frac{3(18\pi)^{\frac{1}{3}}}{2\pi} n^{\frac{2}{3}} = 6n - 1.8326\ldots n^{\frac{2}{3}} \ .$$

Proof: Indeed, Conjecture 7.1.5 implies in a straightforward way that

$$\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n} \mathbf{c}_i + 2\mathbf{B}\right)\right)$$

$$\le 16\pi n - \frac{1}{6\left(1-\frac{\sqrt{3}}{2}\right)} 16\pi \left(1-\frac{\sqrt{3}}{2}\right) C(n) = 16\pi n - \frac{8\pi}{3} C(n) \ .$$

The above inequality combined with Corollary 7.1.3 yields

$$4(18\pi)^{\frac{1}{3}} n^{\frac{2}{3}} \le 16\pi n - \frac{8\pi}{3} C(n) \ ,$$

from which the inequality of Proposition 7.1.6 follows. □

7.2 Proof of Theorem 1.4.7

7.2.1 The Voronoi star of a Voronoi cell in unit ball packings

Without loss of generality we may assume that the d-dimensional unit ball $\mathbf{B} \subset \mathbb{E}^d$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of the given unit ball packing in $\mathbb{E}^d, d \ge 2$. Let $\mathbf{V}$ be the Voronoi cell assigned to $\mathbf{B}$. We may assume that $\mathbf{V}$ is bounded; that is, $\mathbf{V}$ is a d-dimensional convex polytope in $\mathbb{E}^d$.

First, following [218], we dissect $\mathbf{V}$ into finitely many d-dimensional simplices as follows. Let F_i denote an arbitrary i-dimensional face of $\mathbf{V}$,

$0 \le i \le d-1$. Let the chain $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ be called a *flag* of $\mathbf{V}$, and let $\mathcal{F}$ be the family of all flags of $\mathbf{V}$. Now, let $f \in \mathcal{F}$ be an arbitrary flag of $\mathbf{V}$ with the associated chain $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$. Then let $\mathbf{v}_i \in F_{d-i}$ be the point of F_{d-i} closest to $\mathbf{o}$, $1 \le i \le d$. Finally, let $\mathbf{V}_f := \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \dots, \mathbf{v}_d\}$, where $\mathrm{conv}(\cdot)$ stands for the convex hull of the given set. It is easy to see that the family $\mathcal{V} := \{\mathbf{V}_f \mid f \in \mathcal{F} \text{ and } \dim(\mathbf{V}_f) = d\}$ of d-dimensional simplices forms a tiling of $\mathbf{V}$ (i.e., $\cup_{\mathbf{V}_f \in \mathcal{V}} \mathbf{V}_f = \mathbf{V}$ and no two simplices of $\mathcal{V}$ have an interior point in common). This tiling is a rather special one, namely the d-dimensional simplices of $\mathcal{V}$ have $\mathbf{o}$ as a common vertex; moreover the union of their facets opposite to $\mathbf{o}$ is the boundary $\mathrm{bd}\mathbf{V}$ of $\mathbf{V}$. Finally, as shown in [218], for any $\mathbf{V}_f \in \mathcal{V}$ with $\mathbf{V}_f = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \dots, \mathbf{v}_d\}$ we have that

$$\sqrt{\frac{2i}{i+1}} \le \|\mathbf{v}_i\| = \mathrm{dist}\left(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_d\}\right), 1 \le i \le d, \tag{7.4}$$

where $\mathrm{dist}(\cdot,\cdot)$ (resp., $\|\cdot\|$) stands for the Euclidean distance function (resp., norm) in $\mathbb{E}^d$.

Second, we define the *Voronoi star* $\mathbf{V}^* \subset \mathbf{V}$ assigned to the Voronoi cell $\mathbf{V}$ as follows. Let $\mathbf{V}_f \in \mathcal{V}$ with $\mathbf{V}_f = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \dots, \mathbf{v}_d\}$. Then let $\mathbf{v}_1^* := H \cap \mathrm{lin}\{\mathbf{v}_1\}$, where H denotes the hyperplane parallel to the hyperplane $\mathrm{aff}\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ and tangent to $\mathbf{B}$ such that it separates $\mathbf{o}$ from $\mathrm{aff}\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ (with $\mathrm{lin}(\cdot)$ and $\mathrm{aff}(\cdot)$ standing for the linear and affine hulls of the given sets in $\mathbb{E}^d$). Finally, let $\mathbf{V}_f^* := \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1^*, \mathbf{v}_2, \dots, \mathbf{v}_d\}$ and let the Voronoi star $\mathbf{V}^*$ of $\mathbf{V}$ be defined as $\mathbf{V}^* := \cup_{\mathbf{V}_f \in \mathcal{V}} \mathbf{V}_f^*$. It follows from the definition of the Voronoi star and from (7.4) that the following inequalities and (surface) volume formula hold:

$$1 \le \|\mathbf{v}_1^*\| = \mathrm{dist}\left(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_1^*, \mathbf{v}_2, \dots, \mathbf{v}_d\}\right) \le \|\mathbf{v}_1\|, \tag{7.5}$$

$$\sqrt{\frac{2i}{i+1}} \le \|\mathbf{v}_i\| = \mathrm{dist}\left(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_d\}\right), 2 \le i \le d, \text{ and} \tag{7.6}$$

$$\mathrm{vol}_d(\mathbf{V}^*) = \frac{1}{d}\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{V}), \tag{7.7}$$

where $\mathrm{vol}_d(\cdot)$ (resp., $\mathrm{svol}_{d-1}(\cdot)$) refers to the d-dimensional (resp., $(d-1)$-dimensional) volume (resp., surface volume) measure.

7.2.2 Estimating the volume of a Voronoi star from below

As an obvious corollary of (7.7), we find that Theorem 1.4.7 follows from the following theorem.

Theorem 7.2.1 $\mathrm{vol}_d(\mathbf{V}^*) \ge \frac{\omega_d}{\sigma_d}$.

Proof: The main tool of our proof is the following lemma of Rogers. (See [218] and [219] for the original version of the lemma, which is somewhat different from the equivalent version below. Also, for a strengthening we refer the interested reader to Lemma 7.3.11.)

Lemma 7.2.2 *Let* $\mathbf{W} := \mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$ *be a d-dimensional simplex of* $\mathbb{E}^d$ *having the property that* $\mathrm{lin}\{\mathbf{w}_j - \mathbf{w}_i \mid i < j \leq d\}$ *is orthogonal to the vector* $\mathbf{w}_i$ *in* $\mathbb{E}^d$ *for all* $1 \leq i \leq d-1$ *(i.e., let* $\mathbf{W}$ *be a d-dimensional orthoscheme in* $\mathbb{E}^d$*). Moreover, let* $\mathbf{U} := \mathrm{conv}\{\mathbf{o}, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *be a d-dimensional simplex of* $\mathbb{E}^d$ *such that* $\|\mathbf{u}_i\| = \mathrm{dist}\,(\mathbf{o}, \mathrm{conv}\{\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_d\})$ *for all* $1 \leq i \leq d$*. If* $\|\mathbf{w}_i\| \leq \|\mathbf{u}_i\|$ *holds for all* $1 \leq i \leq d$*, then*

$$\frac{\mathrm{vol}_d(\mathbf{W})}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{W})} \leq \frac{\mathrm{vol}_d(\mathbf{U})}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{U})},$$

where $\mathbf{B}$ *stands for the d-dimensional unit ball centered at the origin* $\mathbf{o}$ *of* $\mathbb{E}^d$*.*

Now, let $\mathbf{W}$ be the orthoscheme of Lemma 7.2.2 with the additional property that $\|\mathbf{w}_i\| = \sqrt{\frac{2i}{i+1}}$ for all $1 \leq i \leq d$. Notice that a regular d-dimensional simplex of edge length 2 in $\mathbb{E}^d$ can be dissected into $(d+1)!$ d-dimensional simplices, each congruent to $\mathbf{W}$. This implies that

$$\sigma_d = \frac{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{W})}{\mathrm{vol}_d(\mathbf{W})}. \tag{7.8}$$

Finally, let $\mathbf{U} := \mathbf{V}_f^* = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1^*, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ for $\mathbf{V}_f \in \mathcal{V}$. Clearly, (7.5) and (7.6) show that $\mathbf{W}$ and $\mathbf{U}$, just introduced, satisfy the assumptions of Lemma 7.2.2. Thus, Lemma 7.2.2 and (7.8) imply that

$$\frac{1}{\sigma_d} \leq \frac{\mathrm{vol}_d(\mathbf{V}_f^*)}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*)}. \tag{7.9}$$

Hence, (7.9) yields that

$$\frac{\omega_d}{\sigma_d} \leq \sum_{\mathbf{V}_f \in \mathcal{V}} \mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*) \frac{\mathrm{vol}_d(\mathbf{V}_f^*)}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*)} = \sum_{\mathbf{V}_f \in \mathcal{V}} \mathrm{vol}_d(\mathbf{V}_f^*) = \mathrm{vol}_d(\mathbf{V}^*),$$

finishing the proof of Theorem 7.2.1. □

7.3 Proof of Theorem 1.4.8

7.3.1 Basic metric properties of Voronoi cells in unit ball packings

Let $\mathbf{P}$ be a bounded Voronoi cell, that is, a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbb{E}^d$. Without loss of generality we may assume that the unit ball $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d | \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq 1\}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of $\mathcal{P}$ with $\mathbf{P}$ as its Voronoi cell. Then $\mathbf{P}$ is the intersection of finitely many closed halfspaces of $\mathbb{E}^d$ each of which is bounded by a hyperplane that is the perpendicular bisector of a line segment $\mathbf{ox}$ with $\mathbf{x}$ being the center of some unit ball of $\mathcal{P}$. Now, let F_{d-i} be an arbitrary $(d-i)$-dimensional face of $\mathbf{P}$, $1 \leq i \leq d$. Then clearly there are at least $i+1$ Voronoi cells of $\mathcal{P}$ which meet along the face F_{d-i}, that is, contain F_{d-i} (one of which is, of course, $\mathbf{P}$). Also, it is clear from the construction that the affine hull of centers of the unit balls sitting in all of these Voronoi cells is orthogonal to $\operatorname{aff} F_{d-i}$. Thus, there are unit balls of these Voronoi cells with centers $\{\mathbf{o}, \mathbf{x}_1, \ldots, \mathbf{x}_i\}$ such that $X = \operatorname{conv}\{\mathbf{o}, \mathbf{x}_1, \ldots, \mathbf{x}_i\}$ is an i-dimensional simplex and of course, $\operatorname{aff} X$ is orthogonal to $\operatorname{aff} F_{d-i}$. Hence, if $R(F_{d-i})$ denotes the radius of the $(i-1)$-dimensional sphere that passes through the vertices of X, then

$$R(F_{d-i}) = \operatorname{dist}(\mathbf{o}, \operatorname{aff} F_{d-i}), \text{ where } 1 \leq i \leq d.$$

As the following statements are well known and their proofs are relatively straightforward, we refer the interested reader to the relevant section in [56] for the details of those proofs.

Lemma 7.3.1 *If $F_{d-i-1} \subset F_{d-i}$ and $R(F_{d-i}) = R < \sqrt{2}$ for some $i, 1 \leq i \leq d-1$, then*

$$\frac{2}{\sqrt{4 - R^2}} \leq R(F_{d-i-1}).$$

Corollary 7.3.2 $\sqrt{\frac{2i}{i+1}} \leq R(F_{d-i})$ *for all* $1 \leq i \leq d$.

Lemma 7.3.3 *If $R(F_{d-i}) < \sqrt{2}$ for some $i, 1 \leq i \leq d$, then the orthogonal projection of $\mathbf{o}$ onto $\operatorname{aff} F_{d-i}$ belongs to $\operatorname{relint} F_{d-i}$ and so $R(F_{d-i}) = \operatorname{dist}(\mathbf{o}, F_{d-i})$.*

7.3.2 Wedges of types I, II, and III, and truncated wedges of types I, and II

Let $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ be an arbitrary flag of the Voronoi polytope $\mathbf{P}$. Then let $\mathbf{r}_i \in F_{d-i}$ be the uniquely determined point of the $(d-i)$-dimensional face F_{d-i} of $\mathbf{P}$ that is closest to the center point $\mathbf{o}$ of $\mathbf{P}$; that is, let

$$\mathbf{r}_i \in F_{d-i} \text{ such that } \|\mathbf{r}_i\| = \min\{\|\mathbf{x}\| \mid \mathbf{x} \in F_{d-i}\}, \text{ where } 1 \leq i \leq d.$$

Definition 7.3.4 *If the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_i$ are linearly independent in $\mathbb{E}^d$, then we call $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ the i-dimensional Rogers simplex assigned to the subflag $F_{d-i} \subset \cdots \subset F_{d-1}$ of the Voronoi polytope $\mathbf{P}$, where $1 \leq i \leq d$. If $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\} \subset \mathbb{E}^d$ is the d-dimensional Rogers simplex assigned to the flag $F_0 \subset \cdots \subset F_{d-1}$ of $\mathbf{P}$, then $\operatorname{conv}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}$ is* *called the i-dimensional base of the given d-dimensional Rogers simplex and* $\mathrm{dist}(\mathbf{o}, \mathrm{aff}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}) = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_i) = R(F_i)$ *is called the height assigned to the i-dimensional base, where* $1 \leq i \leq d$.

Definition 7.3.5 *The i-dimensional simplex* $Y = \mathrm{conv}\{\mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i\} \subset \mathbb{E}^d$ *with vertices* $\mathbf{y}_0 = \mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i$ *is called an i-dimensional orthoscheme if for each* $j, 0 \leq j \leq i-1$ *the vector* $\mathbf{y}_j$ *is orthogonal to the linear hull* $\mathrm{lin}\{\mathbf{y}_k - \mathbf{y}_j \mid j+1 \leq k \leq i\}$, *where* $1 \leq i \leq d$.

It is shown in [218] that the union of the d-dimensional Rogers simplices of the Voronoi polytope $\mathbf{P}$ is the polytope $\mathbf{P}$ itself and their interiors are pairwise disjoint. This fact together with Corollary 7.3.2 and Lemma 7.3.3 imply the following metric properties of Rogers simplices in a straightforward way.

Lemma 7.3.6
(1) If $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an i-dimensional Rogers simplex assigned to the subflag* $F_{d-i} \subset \cdots \subset F_{d-1}$ *of the Voronoi polytope* $\mathbf{P}$, *then* $\sqrt{\frac{2j}{j+1}} \leq \|\mathbf{r}_j\|$ *for all* $1 \leq j \leq i$, *where* $1 \leq i \leq d$.
(2) If $F_{d-i} \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* $\mathbf{P}$ *with* $R(F_{d-i}) < \sqrt{2}$, *then* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an i-dimensional Rogers simplex which is, in fact, an i-dimensional orthoscheme (in short, an i-dimensional Rogers orthoscheme) with the property that each* $\mathbf{r}_j \in \mathrm{relint} F_{d-j}, 1 \leq j \leq i$ *is the orthogonal projection of* $\mathbf{o}$ *onto* $\mathrm{aff} F_{d-j}$, *where* $1 \leq i \leq d$.
(3) If $F_2 \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, 3 \leq d$ *with* $R(F_2) < \sqrt{2}$, *then the union of the 2-dimensional bases of the d-dimensional Rogers simplices that contain the orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *is the (uniquely determined) 2-dimensional face* F_2 *of the Voronoi polytope* $\mathbf{P}$ *that is totally orthogonal to* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *at the point* $\mathbf{r}_{d-2}$ *and so,* $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_2)$ *with* $\mathbf{r}_{d-2} \in \mathrm{relint} F_2$.

Now we are ready for the definitions of wedges and truncated wedges. Recall that for any 2-dimensional face F_2 of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ we have that $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2)$.

Definition 7.3.7
(1) Let F_2 *be a 2−dimensional face of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ *with* $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2) < \sqrt{\frac{2(d-1)}{d}}$ *and let* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *be any* $(d-2)$*-dimensional Rogers simplex with* $\mathbf{r}_{d-2} \in \mathrm{relint} F_2$. *Then the union* $\mathbf{W}_I$ *of the d-dimensional Rogers simplices of* $\mathbf{P}$ *that contain the orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *is called a wedge of type I (generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$*).* F_2 *is called the 2-dimensional base of* $\mathbf{W}_I$, *and* $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_2)$ *is the height of* $\mathbf{W}_I$ *assigned to the base* F_2.

(2) Let F_2 *be a* 2*-dimensional face of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ *with* $\sqrt{\frac{2(d-1)}{d}} \leq R(F_2) < \sqrt{\frac{2d}{d+1}}$ *and let* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *be any* $(d-2)$*-dimensional Rogers simplex with* $\mathbf{r}_{d-2} \in \mathrm{relint} F_2$*. Then the union* $\mathbf{W}_{II}$ *of the* d*-dimensional Rogers simplices of* $\mathbf{P}$ *that contain the orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *is called a wedge of type II (generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$*).* F_2 *is called the* 2*-dimensional base of* $\mathbf{W}_{II}$*, and* $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_2)$ *is the height of* $\mathbf{W}_{II}$ *assigned to the base* F_2*.*
(3) Let $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ *be the* d*-dimensional Rogers simplex assigned to the flag* $F_0 \subset F_1 \cdots \subset F_{d-1}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ *with* $\sqrt{\frac{2d}{d+1}} \leq R(F_2)$*. Then* $\mathbf{W}_{III} = \mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ *is called a wedge of type III.*

At this point, it useful to recall, that for any vertex F_0 of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$ we have that $\sqrt{\frac{2d}{d+1}} \leq R(F_0)$.

Definition 7.3.8 *Let* $\overline{\mathbf{B}} = \left\{\mathbf{x} \in \mathbb{E}^d |\ \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq \sqrt{\frac{2d}{d+1}}\right\}$.
(1) If $\mathbf{W}_I$ *is a wedge of type I with the* 2*-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$*, then*

$$\overline{\mathbf{W}}_I = \mathrm{conv}\left((\overline{\mathbf{B}} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \dots, \mathbf{r}_{d-3}\}\right)$$

is called the truncated wedge of type I with the 2*-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *generated by the* $(d-2)$*-dimensional Rogers orthoscheme*

$$\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}.$$

(2) If $\mathbf{W}_{II}$ *is a wedge of type II with the* 2*-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$*, then*

$$\overline{\mathbf{W}}_{II} = \mathrm{conv}\left((\overline{\mathbf{B}} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \dots, \mathbf{r}_{d-3}\}\right)$$

is called the truncated wedge of type II with the 2*-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *generated by the* $(d-2)$*-dimensional Rogers orthoscheme*

$$\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}.$$

As the following claim can be proved by Lemma 7.3.6 in a straightforward way, we leave the relevant details to the reader.

Lemma 7.3.9

(1) Let $\mathbf{W}_I$ *(resp.,* $\mathbf{W}_{II}$*) denote the wedge of type I (resp., of type II) with the* 2*-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$*. If the points* $\mathbf{x}, \mathbf{y} \in \text{aff}F_2$ *are chosen so that the triangle* $\triangle \mathbf{r}_{d-2}\mathbf{x}\mathbf{y}$ *has a right angle at the vertex* $\mathbf{x}$*, then* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}, \mathbf{x}, \mathbf{y}\}$ *is a d-dimensional orthoscheme. Moreover, if* $\mathbf{z} \in \text{aff}F_2$ *is an arbitrary point, then* $\text{conv}\{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}, \mathbf{z}\}$ *is a* $(d-2)$*-dimensional orthoscheme.*
(2) Let $\mathbf{W}_I$ *denote the wedge of type I with the* 2*-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$*. Let* $Q_2 \subset \text{aff}F_2$ *and* $Q_2^* \subset \text{aff}F_2$ *be compact convex sets with* $\text{relint}Q_2 \cap \text{relint}Q_2^* = \emptyset$*. If* $K_2 = Q_2$ *(resp.,* $K_2^* = Q_2^*$*) and* $K_j = \text{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ *(resp.,* $K_j^* = \text{conv}(K_{j-1}^* \cup \{\mathbf{r}_{d-j}\})$*) for* $j = 3, \ldots, d$*, then* $K_d = \text{conv}(Q_2 \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$ *(resp.,* $K_d^* = \text{conv}(Q_2^* \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$*), moreover* $\text{relint}K_d \cap \text{relint}K_d^* = \emptyset$*. A similar statement holds for* $\mathbf{W}_{II}$*.*
(3) Let $\mathbf{W}_I$ *(resp.,* $\overline{\mathbf{W}}_I$*) denote the wedge of type I (resp., truncated wedge of type I) with the* 2*-dimensional base* F_2 *(resp.,* $\overline{\mathbf{B}} \cap F_2$*) which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$*. If* $K_2 = F_2$ *(resp.,* $K_2 = \overline{\mathbf{B}} \cap F_2$*) and* $K_j = \text{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ *for* $j = 3, \ldots, d$*, then* $K_d = \mathbf{W}_I$ *(resp.,* $K_d = \overline{\mathbf{W}}_I$*). Similar statements hold for* $\mathbf{W}_{II}$ *and* $\overline{\mathbf{W}}_{II}$*.*

We close this section with the following important observation published in [56], and refer the interested reader to [56] for the details of the seven-page proof, which is based on Corollary 7.3.2 and Lemma 7.3.3.

Lemma 7.3.10 *Let* $\overline{\mathbf{B}} \cap F_2$ *be the* 2*-dimensional base of the type I truncated wedge* $\overline{\mathbf{W}}_I$ *(resp., type II truncated wedge* $\overline{\mathbf{W}}_{II}$*) in the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d$ *of dimension* $d \geq 8$*. Then the number of line segments of positive length in* $\text{relbd}(\overline{\mathbf{B}} \cap F_2)$ *is at most* 4*.*

7.3.3 The lemma of comparison and a characterization of regular polytopes

Recall that $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d | \ \text{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq 1\}$ and let

$$S = \{\mathbf{x} \in \mathbb{E}^d | \ \text{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| = 1\}.$$

Then let $H \subset \mathbb{E}^d$ be a hyperplane disjoint from the interior of the unit ball $\mathbf{B}$ and let $Q \subset H$ be an arbitrary $(d-1)$-dimensional compact convex set. If $[\mathbf{o}, Q]$ denotes the convex cone $\text{conv}(\{\mathbf{o}\} \cup Q)$ with apex $\mathbf{o}$ and base Q, then the *(volume) density* $\delta([\mathbf{o}, Q], \mathbf{B})$ of the unit ball $\mathbf{B}$ in the cone $[\mathbf{o}, Q]$ is defined as

$$\delta([\mathbf{o}, Q], B) = \frac{\text{vol}_d([\mathbf{o}, Q] \cap \mathbf{B})}{\text{vol}_d([\mathbf{o}, Q])},$$

where $\mathrm{vol}_d(\cdot)$ refers to the corresponding d-dimensional Euclidean volume measure. It is natural to introduce the following very similar notion. The *surface density* $\widehat{\delta}([\mathbf{o}, Q], S)$ of the unit sphere S in the convex cone $[\mathbf{o}, Q]$ with apex $\mathbf{o}$ and base Q is defined by

$$\widehat{\delta}([\mathbf{o}, Q], S) = \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, Q] \cap S)}{\mathrm{vol}_{d-1}(Q)},$$

where $\mathrm{Svol}_{d-1}(\cdot)$ refers to the corresponding $(d-1)$-dimensional spherical volume measure.

If $h = \mathrm{dist}(\mathbf{o}, H)$, then clearly $h \cdot \delta([\mathbf{o}, Q], \mathbf{B}) = \widehat{\delta}([\mathbf{o}, Q], S)$. We need the following statement, the first part of which is due to Rogers [218] and the second part of which has been proved by the author in [55].

Lemma 7.3.11 *Let* $\mathbf{U} = \mathrm{conv}\{\mathbf{o}, \mathbf{u}_1, \dots, \mathbf{u}_d\}$ *be a d-dimensional orthoscheme in* $\mathbb{E}^d$ *and let* $\mathbf{V} = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \dots, \mathbf{v}_d\}$ *be a d-dimensional simplex of* $\mathbb{E}^d$ *such that* $\|\mathbf{v}_i\| = \mathrm{dist}(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_d\})$ *for all* $1 \leq i \leq d-1$. *If* $1 \leq \|\mathbf{u}_i\| \leq \|\mathbf{v}_i\|$ *holds for all* $1 \leq i \leq d$, *then*
(1) $\delta(\mathbf{U}, \mathbf{B}) \geq \delta(\mathbf{V}, \mathbf{B})$ *and*
(2) $\widehat{\delta}(\mathbf{U}, S) \geq \widehat{\delta}(\mathbf{V}, S)$.

For the sake of completeness we mention the following statement that follows from Lemma 7.3.11 using the special decomposition of convex polytopes into Rogers simplices. Actually, the characterization of regular polytopes through the corresponding volume (resp., surface volume) inequality below was first observed by Böröczky and Máthéné Bognár [91] (resp., by the author [55]). (In fact, it is easy to see that the statement on surface volume implies the one on volume.) For more details on related problems we refer the interested reader to [93].

Corollary 7.3.12 *Let* $\mathbf{U}'$ *be a regular convex polytope in* $\mathbb{E}^d$ *with circumcenter* $\mathbf{o}$ *and let* s_i *denote the distance of an i-dimensional face of* $\mathbf{U}'$ *from* $\mathbf{o}$, $0 \leq i \leq d-1$. *If* $\mathbf{V}'$ *is an arbitrary convex polytope in* $\mathbb{E}^d$ *such that* $\mathbf{o} \in \mathrm{int}\mathbf{V}'$ *and the distance of any i-dimensional face of* $\mathbf{V}'$ *from* $\mathbf{o}$ *is at least* s_i *for all* $0 \leq i \leq d-1$, *then* $\mathrm{vol}_d(\mathbf{V}') \geq \mathrm{vol}_d(\mathbf{U}')$ *(resp.,* $\mathrm{svol}_{d-1}(\mathbf{V}') \geq \mathrm{svol}_{d-1}(\mathbf{U}')$*). Moreover, equality holds if and only if* $\mathbf{V}'$ *is congruent to* $\mathbf{U}'$ *and its circumcenter is* $\mathbf{o}$.

7.3.4 Volume formulas for (truncated) wedges

Definition 7.3.13 *Let* $\mathbf{x}_1, \dots, \mathbf{x}_n, n \geq 1$ *be points in* $\mathbb{E}^d, d \geq 1$ *and let* $X \subset \mathbb{E}^d$ *be an arbitrary convex set. If* $X_0 = X$ *and* $X_m = \mathrm{conv}(\{\mathbf{x}_{n-(m-1)}\} \cup X_{m-1})$ *for* $m = 1, \dots, n$, *then we denote the final convex set* X_n *by*

$$[\mathbf{x}_1, \dots, \mathbf{x}_n, X].$$

Definition 7.3.14 *Let $\mathbf{W}_I$ (resp., $\overline{\mathbf{W}}_I$) denote the wedge (resp., truncated wedge) of type I with the 2-dimensional base F_2 (resp., $\overline{\mathbf{B}} \cap F_2$) which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. Then let*

$$Q_I = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, F_2] \left(resp., \overline{Q}_I = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \overline{\mathbf{B}} \cap F_2]\right)$$

be called the $(d-1)$-dimensional base of the type I wedge $\mathbf{W}_I = [\mathbf{o}, Q_I]$ (resp., type I truncated wedge $\overline{\mathbf{W}}_I = [\mathbf{o}, \overline{Q}_I]$). Similarly, we define the $(d-1)$-dimensional bases Q_{II} and $\overline{Q}_{II}$ of $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$. Finally, let

$$h_1 = \|\mathbf{r}_1\|, h_2 = \|\mathbf{r}_2 - \mathbf{r}_1\|, \ldots, h_{d-2} = \|\mathbf{r}_{d-2} - \mathbf{r}_{d-3}\|.$$

Lemma 7.3.15 *Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. Then we have the following volume formulas.*
(1) $\mathrm{vol}_{d-1}(Q_I) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{vol}_2(F_2)$ *and*
(2) $\mathrm{vol}_d(\mathbf{W}_I) = \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{vol}_2(F_2)$.

Similar formulas hold for the corresponding dimensional volumes of $\overline{Q}_I$, $\overline{\mathbf{W}}_I$, Q_{II}, $\mathbf{W}_{II}$, $\overline{Q}_{II}$, and $\overline{\mathbf{W}}_{II}$.

In general, if $K \subset \mathrm{aff} F_2$ is a convex domain, then
(3) $\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{vol}_2(K)$ *and*
(4) $\mathrm{vol}_d([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) = \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{vol}_2(K)$.

Proof: The proof follows from Lemma 7.3.6 and Lemma 7.3.9 in a straightforward way. □

7.3.5 The integral representation of surface density in (truncated) wedges

The central notion of this section is the limiting surface density introduced as follows.

Definition 7.3.16 *Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. Then choose a coordinate system with two perpendicular axes in the plane $\mathrm{aff} F_2$ meeting at the point $\mathbf{r}_{d-2}$. Now, if $\mathbf{x}$ is an arbitrary point of the plane $\mathrm{aff} F_2$, then for a positive integer n let $T_n(\mathbf{x}) \subset \mathrm{aff} F_2$ denote the square centered at $\mathbf{x}$ having sides of length $\frac{1}{n}$ parallel to the fixed coordinate axes. Then the limiting surface density $\widehat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)$ of the $(d-1)$-dimensional unit sphere S in the $(d-2)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}]$ is defined by*

$$\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{x}],S\right)=\lim_{n\to\infty}\widehat{\delta}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x})],S\right).$$

Based on this we are able to give an integral representation of the surface density in a (truncated) wedge.

Lemma 7.3.17 *Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P}\subset\mathbb{E}^d, d\geq 4$.*
(1) If $\mathbf{x}\in\mathrm{aff}F_2$ and $\mathbf{y}\in\mathrm{aff}F_2$ are points such that $\|\mathbf{x}\|\leq\|\mathbf{y}\|$, then

$$\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{x}],S\right)\geq\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{y}],S\right).$$

(2) For the surface densities of the unit sphere S in the wedge $\mathbf{W}_I$ and in the truncated wedge $\overline{\mathbf{W}}_I$ we have the following formulas.

$$\widehat{\delta}(\mathbf{W}_I,S)=\frac{\mathrm{Svol}_{d-1}([\mathbf{o},Q_I]\cap S)}{\mathrm{vol}_{d-1}(Q_I)}$$

$$=\frac{1}{\mathrm{vol}_2(F_2)}\int_{F_2}\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{x}],S\right)\ d\mathbf{x}$$

and

$$\widehat{\delta}(\overline{\mathbf{W}}_I,S)=\frac{\mathrm{Svol}_{d-1}([\mathbf{o},\overline{Q}_I]\cap S)}{\mathrm{vol}_{d-1}(\overline{Q}_I)}$$

$$=\frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}}\cap F_2)}\int_{\overline{\mathbf{B}}\cap F_2}\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{x}],S\right)\ d\mathbf{x},$$

where $d\mathbf{x}$ stands for the Euclidean area element in the plane $\mathrm{aff}F_2$. *Similar formulas hold for $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$.*
(3) In general, if $K\subset\mathrm{aff}F_2$ is a convex domain, then the surface density of the unit sphere S in the d-dimensional convex cone $[\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K]$ with apex $\mathbf{o}$ and $(d-1)$-dimensional base $[\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K]$ can be computed as follows.

$$\widehat{\delta}([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K],S)=\frac{1}{\mathrm{vol}_2(K)}\int_K\widehat{\delta}_{\lim}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},\mathbf{x}],S\right)\ d\mathbf{x}.$$

Proof:

(1) It is sufficient to look at the case $\|\mathbf{x}\|<\|\mathbf{y}\|$. (The case $\|\mathbf{x}\|=\|\mathbf{y}\|$ follows from this by standard limit procedure.) Then recall that

$$\widehat{\delta}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x})],S\right)=h_1\delta\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x})],S\right)$$

and

$$\widehat{\delta}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{y})],S\right)=h_1\delta\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{y})],S\right).$$

Thus, it is sufficient to show that if n is sufficiently large, then

$$\delta\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S\right) \geq \delta\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S\right).$$

This we can get as follows. We can approximate the d-dimensional convex cone $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})]$ (resp., $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{y})]$) arbitrarily close with a finite (but possibly large) number of non-overlapping d-dimensional orthoschemes each containing the $(d-3)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}]$ as a face and each having all the edge lengths of the 3 edges going out from the vertex $\mathbf{o}$ and not lying on the face $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}]$ close to $\|\mathbf{x}\|$ (resp., $\|\mathbf{y}\|$) for n sufficiently large (see also Lemma 7.3.9). Thus, the claim follows from (1) of Lemma 7.3.11 rather easily.

(2),(3) It is sufficient to prove the corresponding formula for K.

A typical term of the Riemann–Lebesgue sum of

$$\frac{1}{\mathrm{vol}_2(K)} \int_K \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right)\, d\mathbf{x}$$

is equal to

$$\frac{1}{\mathrm{vol}_2(K)} \widehat{\delta}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)], S\right) \mathrm{vol}_2(T_n(\mathbf{x}_m)), m \in M.$$

Using Lemma 7.3.15 this turns out to be equal to

$$\frac{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)])}{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])} \widehat{\delta}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)], S\right)$$

$$= \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])}.$$

Finally, as the union of the non-overlapping squares $T_n(\mathbf{x}_m), m \in M$ is a good approximation of the convex domain K in the plane $\mathrm{aff} F_2$ we get that

$$\sum_{m \in M} \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])}$$

$$= \frac{\sum_{m \in M} \mathrm{Svol}_{d-1}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x}_m)] \cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])}$$

is a good approximation of

$$\frac{\mathrm{Svol}_{d-1}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K] \cap S\right)}{\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K])} = \widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K], S).$$

This completes the proof of Lemma 7.3.17. □

7.3.6 Truncation of wedges increases the surface density

Lemma 7.3.18 *Let* $\mathbf{W}_I$ *(resp.,* $\mathbf{W}_{II}$*) denote the wedge of type I (resp., of type II) with the 2-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$*. Then*

$$\widehat{\delta}(\mathbf{W}_I, S) \leq \widehat{\delta}(\overline{\mathbf{W}}_I, S) \ \left(resp.,\ \widehat{\delta}(\mathbf{W}_{II}, S) \leq \widehat{\delta}(\overline{\mathbf{W}}_{II}, S) \right).$$

Proof: Notice that (1) of Lemma 7.3.17 easily implies that if $0 < \mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})$, then for any $\mathbf{x}^* \in F_2$ with $\|\mathbf{x}^*\| = \sqrt{\frac{2d}{d+1}}$ we have that

$$\frac{1}{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})} \int_{F_2 \setminus \overline{\mathbf{B}}} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$\leq \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}^*], S\right)$$

$$\leq \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}.$$

Thus, if $0 < \mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})$, then (2) of Lemma 7.3.17 yields that

$$\widehat{\delta}(\mathbf{W}_I, S) = \frac{1}{\mathrm{vol}_2(F_2)} \int_{F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$= \frac{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$+ \frac{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})} \int_{F_2 \setminus \overline{\mathbf{B}}} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$\leq \frac{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$+ \frac{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x}$$

$$= \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \widehat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S\right) d\mathbf{x} = \widehat{\delta}(\overline{\mathbf{W}}_I, S).$$

As the same method works for $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$ this completes the proof of Lemma 7.3.18. $\square$

7.3.7 Maximum surface density in truncated wedges of type I

Let $\overline{\mathbf{W}}_I$ denote the truncated wedge of type I with the 2-dimensional base $\overline{\mathbf{B}} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 8$. By assumption F_2 is a 2-dimensional face of the Voronoi polytope $\mathbf{P}$ with

$$\sqrt{\frac{2(d-2)}{d-1}} \leq h = R(F_2) < \sqrt{\frac{2(d-1)}{d}}.$$

Let $G_0 \subset \mathrm{aff} F_2$ (resp., $G \subset \mathrm{aff} F_2$) denote the closed circular disk of radius

$$g_0(h) = \sqrt{\frac{2d}{d+1} - h^2} \left(\text{resp., } g(h) = \frac{2-h^2}{\sqrt{4-h^2}}\right)$$

centered at the point $\mathbf{r}_{d-2}$. It is easy to see that $G \subset \mathrm{relint} G_0$ for all $\sqrt{\frac{2(d-2)}{d-1}} \leq h < \sqrt{\frac{2(d-1)}{d}}$. (Moreover $G = G_0$ for $h = \sqrt{\frac{2(d-1)}{d}}$.) Notice that $G_0 = \overline{\mathbf{B}} \cap \mathrm{aff} F_2$, thus Corollary 7.3.2 implies that there is no vertex of the face F_2 belonging to the relative interior of G_0. Moreover, as $h = R(F_2) < \sqrt{2}$ Lemma 7.3.1 yields that $\frac{2}{\sqrt{4-h^2}} \leq R(F_1)$ holds for any side F_1 of the face F_2, hence $G \subset F_2$ and of course, $G \subset \overline{\mathbf{B}} \cap F_2 = G_0 \cap F_2$. Now, let $M \subset \mathrm{aff} F_2$ be a square circumscribed about G. A straightforward computation yields that $\frac{g_0(h)}{g(h)}$ is a strictly decreasing function on the interval $\left[\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$ (i.e., $\frac{d}{dh}\left(\frac{g_0(h)}{g(h)}\right) < 0$ on the interval $\left(\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$) and

$$\frac{g_0\left(\sqrt{\frac{2(d-2)}{d-1}}\right)}{g\left(\sqrt{\frac{2(d-2)}{d-1}}\right)} = \sqrt{\frac{2d}{d+1}} < \sqrt{2}.$$

Thus, the vertices of the square M do not belong to G_0. Finally, as $d \geq 8$ Lemma 7.3.10 implies that there are at most four sides of the face F_2 that intersect the relative interior of G_0.

The following statement is rather natural from the point of view of the local geometry introduced above, however, its three-page proof based on Lemma 7.3.11 and Lemma 7.3.17 published in [56] is a bit technical and so, for that reason we do not prove it here; instead we refer the interested reader to the proper section in [56].

Lemma 7.3.19 *Let $\overline{\mathbf{W}}_I$ denote the truncated wedge of type I with the 2-dimensional base $\overline{\mathbf{B}} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 8$. Then*

$$\widehat{\delta}(\overline{\mathbf{W}}_I, S) \leq \widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S).$$

It is clear from the construction that we can write $\widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S)$ as a function of $d-2$ variables, namely

$$\widehat{\Delta}(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) = \widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \ldots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 7.3.2 and the assumption on h imply that

$$m_1 = 1 \leq \xi_1, \ldots, m_i = \sqrt{\frac{2i}{i+1}} \leq \xi_i, \ldots, m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \leq \xi_{d-3},$$

$$m_{d-2} = \sqrt{\frac{2(d-2)}{d-1}} \leq \xi_{d-2} = h < \sqrt{\frac{2(d-1)}{d}}.$$

Notice that if $\|\mathbf{r}_i\| = m_i$ for all $1 \leq i \leq d-2$, then $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M]$ can be dissected into four pieces each being congruent to $\mathbf{W}$ and therefore $\widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3},\ G_0 \cap M], S) = \widehat{\sigma}_d$.

Lemma 7.3.20

$$\widehat{\Delta}(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) \leq \widehat{\Delta}(m_1, \ldots, m_{d-3}, m_{d-2}) = \widehat{\sigma}_d.$$

Proof: For any fixed $\xi_{d-2} = h$, (2) of Lemma 7.3.11 easily implies that

$$\widehat{\Delta}(\xi_1, \ldots, \xi_{d-3}, h) \leq \widehat{\Delta}(m_1, \ldots, m_{d-3}, h).$$

Finally, using Lemma 7.3.11 again, it is rather straightforward to show that the function $\widehat{\Delta}(m_1, \ldots, m_{d-3}, h)$ as a function of h is decreasing on the interval $\left(\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$. From this it follows that

$$\widehat{\Delta}(m_1, \ldots, m_{d-3}, h) \leq \widehat{\Delta}(m_1, \ldots, m_{d-3}, m_{d-2}) = \widehat{\sigma}_d,$$

finishing the proof of Lemma 7.3.20. □

Thus, Lemma 7.3.19 and Lemma 7.3.20 yield the following immediate estimate.

Corollary 7.3.21 *Let* $\overline{\mathbf{W}}_I$ *denote the truncated wedge of type I with the* 2*-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 8$*. Then*

$$\widehat{\delta}(\overline{\mathbf{W}}_I, S) \leq \widehat{\sigma}_d.$$

7.3.8 An upper bound for the surface density in truncated wedges of type II

It is sufficient to prove the following statement.

Lemma 7.3.22 *Let* $\overline{\mathbf{W}}_{II}$ *denote the truncated wedge of type II with the 2-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. *Then*

$$\widehat{\delta}(\overline{\mathbf{W}}_{II}, S) \leq \widehat{\sigma}_d.$$

Proof: By assumption F_2 is a 2-dimensional face of the Voronoi polytope $\mathbf{P}$ with

$$\sqrt{\frac{2(d-1)}{d}} \leq h = R(F_2) < \sqrt{\frac{2d}{d+1}}.$$

Let $G_0 \subset \mathrm{aff} F_2$ denote the closed circular disk of radius $g_0(h) = \sqrt{\frac{2d}{d+1} - h^2}$ centered at the point $\mathbf{r}_{d-2}$. As $h = R(F_2) < \sqrt{2}$, therefore Lemma 7.3.1 yields that

$$\sqrt{\frac{2d}{d+1}} \leq \frac{2}{\sqrt{4-h^2}} \leq R(F_1)$$

holds for any side F_1 of the face F_2. Thus,

$$\overline{\mathbf{B}} \cap F_2 = G_0$$

and so

$$\widehat{\delta}(\overline{\mathbf{W}}_{II}, S) = \widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S).$$

It is clear from the construction that we can write $\widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S)$ as a function of $d-2$ variables, namely

$$\widehat{\Delta}^*(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) = \widehat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \ldots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 7.3.2 and the assumption on h imply that

$$m_1 = 1 \leq \xi_1, \ldots, m_i = \sqrt{\frac{2i}{i+1}} \leq \xi_i, \ldots, m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \leq \xi_{d-3},$$

$$m^*_{d-2} = \sqrt{\frac{2(d-1)}{d}} \leq \xi_{d-2} = h < \sqrt{\frac{2d}{d+1}}.$$

For any fixed $\xi_{d-2} = h$, (2) of Lemma 7.3.11 easily implies that

$$\widehat{\Delta}^*(\xi_1, \ldots, \xi_{d-3}, h) \leq \widehat{\Delta}^*(m_1, \ldots, m_{d-3}, h).$$

Finally, again applying (2) of Lemma 7.3.11 we immediately get that

$$\widehat{\Delta}^*(m_1, \ldots, m_{d-3}, h) \leq \widehat{\Delta}^*(m_1, \ldots, m_{d-3}, m^*_{d-2}) \leq \widehat{\sigma}_d.$$

This completes the proof of Lemma 7.3.22. □

7.3.9 The overall estimate of surface density in Voronoi cells

Let $\mathbf{P}$ be a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbb{E}^d, d \geq 8$. Without loss of generality we may assume that the unit ball $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d \mid \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq 1\}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of $\mathcal{P}$ with $\mathbf{P}$ as its Voronoi cell. As before, let S denote the boundary of $\mathbf{B}$.

First, we dissect $\mathbf{P}$ into d-dimensional Rogers simplices. Then let $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ be one of these d-dimensional Rogers simplices assigned to the flag say, $F_0 \subset \cdots \subset F_{d-1}$ of $\mathbf{P}$. As $\mathbf{r}_i \in F_{d-i}, 1 \leq i \leq d$ it is clear that $\mathrm{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\} = \mathrm{aff} F_2$ and so

$$\mathrm{dist}(\mathbf{o}, \mathrm{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\}) = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_2) = R(F_2).$$

Notice that Corollary 7.3.2 implies that $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2)$.

Second, we group the d-dimensional Rogers simplices of $\mathbf{P}$ as follows.

(1): If $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2) < \sqrt{\frac{2(d-1)}{d}}$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to the type I wedge $\mathbf{W}_I$ with the 2-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 8$.

(2): If $\sqrt{\frac{2(d-1)}{d}} \leq R(F_2) < \sqrt{\frac{2d}{d+1}}$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to the type II wedge $\mathbf{W}_{II}$ with the 2-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 8$.

(3): If $\sqrt{\frac{2d}{d+1}} \leq R(F_2)$, then we assign the Rogers simplex $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to itself as the type III wedge $\mathbf{W}_{III}$.

As the wedges of types I, II, and III of the given Voronoi polytope $\mathbf{P}$ sit over the 2-skeleton of $\mathbf{P}$ and form a tiling of $\mathbf{P}$ it is clear that each d-dimensional Rogers simplex of $\mathbf{P}$ belongs to exactly one of them. As a result, in order to show that the surface density $\widehat{\delta}(\mathbf{P}, S) = \frac{\mathrm{Svol}_{d-1}(S)}{\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{P})} = \frac{d\omega_d}{\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{P})}$ of the unit sphere S in the Voronoi polytope $\mathbf{P}$ is bounded from above by $\widehat{\sigma}_d$, it is sufficient to prove the following inequalities.

$\widehat{1}$: $\widehat{\delta}(W_I, S) \leq \widehat{\sigma}_d$,

$\widehat{2}$: $\widehat{\delta}(W_{II}, S) \leq \widehat{\sigma}_d$,

$\widehat{3}$: $\widehat{\delta}(W_{III}, S) \leq \widehat{\sigma}_d$.

This final task is now easy. Namely, Lemma 7.3.18, Corollary 7.3.21, and Lemma 7.3.22 yield $\widehat{1}$ and $\widehat{2}$ in a straightforward way. Finally, $\widehat{3}$ follows with the help of (2) of Lemma 7.3.11 rather easily.

For the details of the proof of $\widehat{\sigma}_d < \sigma_d$, based on the so-called "Lemma of Strict Comparison", we refer the interested reader to the proper section in [56].

This completes the proof of Theorem 1.4.8.

7.4 Proof of Theorem 1.7.3

7.4.1 The signed volume of convex polytopes

Definition 7.4.1 *Let* $\mathbf{P} := \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ *be a d-dimensional convex polytope in* $\mathbb{E}^d, d \geq 2$ *with vertices* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$*. If* $F := \mathrm{conv}\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}\}$ *is an arbitrary face of* $\mathbf{P}$*, then the barycenter of* F *is*

$$\mathbf{c}_F := \frac{1}{k} \sum_{j=1}^{k} \mathbf{p}_{i_j}. \tag{7.10}$$

Let $F_0 \subset F_1 \subset \cdots \subset F_l, 0 \leq l \leq d-1$ denote a sequence of faces, called a (partial) flag of $\mathbf{P}$, where F_0 is a vertex and F_{i-1} is a facet (a face one dimension lower) of F_i for $i = 1, \ldots, l$. Then the simplices of the form $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_l}\}$ constitute a simplicial complex $\mathcal{C}_{\mathbf{P}}$ whose underlying space is the boundary of $\mathbf{P}$.

We regard all points in $\mathbb{E}^d$ as row vectors and use $\mathbf{q}^T$ for the column vector that is the transpose of the row vector $\mathbf{q}$. Moreover, $[\mathbf{q}_1, \ldots, \mathbf{q}_d]$ is the (square) matrix with the ith row $\mathbf{q}_i$.

Choosing a $(d-1)$-dimensional simplex of $\mathcal{C}_{\mathbf{P}}$ to be positively oriented, one can check whether the orientation of an arbitrary $(d-1)$-dimensional simplex $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}$ of $\mathcal{C}_{\mathbf{P}}$ (generated by the given sequence of its vertices), is positive or negative. Let $\mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}\right)$ be equal to 1 (resp., -1) if the orientation of the $(d-1)$-dimensional simplex $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}$ is positive (resp., negative).

Definition 7.4.2 *The signed volume* $V(\mathbf{P})$ *of* $\mathbf{P}$ *is defined as*

$$\frac{1}{d!} \sum_{F_0 \subset F_1 \subset \cdots \subset F_{d-1}} \mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}\right) \det[\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}], \tag{7.11}$$

where the sum is taken over all flags of faces $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ *of* $\mathbf{P}$*, and* $\det[\cdot]$ *is the determinant function.*

The following is clear.

Lemma 7.4.3

$$V(\mathbf{P}) = \frac{1}{d!} \sum_{F_0 \subset \cdots \subset F_{d-1}} \mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}\right) \mathbf{c}_{F_0} \wedge \mathbf{c}_{F_1} \wedge \cdots \wedge \mathbf{c}_{F_{d-1}},$$

where $\wedge$ *stands for the wedge product of vectors. Moreover, one can choose the orientation of the boundary of* $\mathbf{P}$ *such that* $V(\mathbf{P}) = \mathrm{vol}_d(\mathbf{P})$*, where* $\mathrm{vol}_d(\cdot)$ *refers to the d-dimensional volume measure in* $\mathbb{E}^d, d \geq 2$.

7.4.2 The volume force of convex polytopes

We wish to compute the gradient of $V(\mathbf{P})$, where $\mathbf{P} = \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\}$ is regarded as a function of its vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$. To achieve this we consider an arbitrary path $\mathbf{p}(t) = \mathbf{p} + t\mathbf{p}'$ in the space of the configurations $\mathbf{p} := (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$, where $\mathbf{p}' := (\mathbf{p}'_1, \mathbf{p}'_2, \dots, \mathbf{p}'_n)$. Based on Definition 7.4.1, Definition 7.4.2, and Lemma 7.4.3 we introduce $V(\mathbf{P}(t))$ as a function of t (with t being an arbitrary real with sufficiently small absolute value) via

$$\frac{1}{d!} \sum_{F_0 \subset F_1 \subset \cdots \subset F_{d-1}} \mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}(t), \dots, \mathbf{c}_{F_{d-1}}(t)\}\right) \det[\mathbf{c}_{F_0}(t), \dots, \mathbf{c}_{F_{d-1}}(t)]$$

$$= \frac{1}{d!} \sum_{F_0 \subset F_1 \subset \cdots \subset F_{d-1}} \mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}(t), \dots, \mathbf{c}_{F_{d-1}}(t)\}\right) \mathbf{c}_{F_0}(t) \wedge \cdots \wedge \mathbf{c}_{F_{d-1}}(t),$$

where $\mathbf{c}_F(t) := \frac{1}{k}\sum_{j=1}^{k} \mathbf{p}_{i_j}(t)$ for any face $F = \mathrm{conv}\{\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_k}\}$ of $\mathbf{P}$. Clearly, $V(\mathbf{P}(0)) = V(\mathbf{P})$. Moreover, evaluating the derivative $\frac{d}{dt}V(\mathbf{P}(t))$ of $V(\mathbf{P}(t))$ at $t = 0$, collecting terms, and using the anticommutativity of the wedge product we get that

$$\frac{d}{dt}V(\mathbf{P}(t))\,|_{t=0} = \frac{1}{d!}\sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i, \tag{7.12}$$

where each $\mathbf{N}_i$ is some linear combination of wedge products of $d-1$ vectors $\mathbf{p}_j$ with $\mathbf{p}_j$ and $\mathbf{p}_i$ sharing a common face.

Definition 7.4.4 *We call* $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_n)$ *the volume force of the* d*-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d$ *with* n *vertices.*

The following are some simple properties of the volume force. We leave the rather straightforward proofs to the reader.

Lemma 7.4.5 *Let* $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_n)$ *be the volume force of the* d*-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *with vertices* $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$*. Then the following hold.*
(1) Each $\mathbf{N}_i$ *is only a function of the vertices that share a face with* $\mathbf{p}_i$*, but not* $\mathbf{p}_i$ *itself.*
(2) Assume that the origin $\mathbf{o}$ *of* $\mathbb{E}^d$ *is the barycenter of* $\mathbf{P}$*; moreover, let* $T : \mathbb{E}^d \to \mathbb{E}^d$ *be an orthogonal linear map satisfying* $T(\mathbf{P}) = \mathbf{P}$*. If* $T(\mathbf{p}_i) = \mathbf{p}_j$*, then* $T(\mathbf{N}_i) = \mathbf{N}_j$*.*

For more details and examples on volume forces we refer the interested reader to the proper sections in [34].

7.4.3 Critical volume condition

Let $\mathbf{P} := \text{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ be a d-dimensional convex polytope in $\mathbb{E}^d, d \geq 2$ with vertices $\mathbf{p} := (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$. Let G be a graph defined on this vertex set $\mathbf{p}$. Here, G may or may not consist of the edges of $\mathbf{P}$. We think of the edges of G as defining those pairs of vertices of $\mathbf{P}$ that are constrained not to get closer. In the terminology of the geometry of rigid tensegrity frameworks each edge of G is a *strut*. (For more information on rigid tensegrity frameworks and the basic terminology used there we refer the interested reader to [222].)

Let $\mathbf{p}' := (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ be an *infinitesimal flex* of $G(\mathbf{p})$, where $G(\mathbf{p})$ refers to the realization of G over the point configuration $\mathbf{p}$. That is, for each edge (strut) $\{i, j\}$ of G we have

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0, \tag{7.13}$$

where "$\cdot$" denotes the standard inner product (also called the "dot product") in $\mathbb{E}^d$.

Let e denote the number of edges of G. Then the *rigidity matrix* $R(\mathbf{p})$ of $G(\mathbf{p})$ is the $e \times nd$ matrix whose row corresponding to the edge $\{i, j\}$ of G consists of the coordinates of d-dimensional vectors within a sequence of n vectors such that all the coordinates are zero except maybe the ones that correspond to the coordinates of the vectors $\mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{p}_j - \mathbf{p}_i$ listed on the ith and jth position. Another way to introduce $R(\mathbf{p})$ is the following. Let $f : \mathbb{E}^{nd} \to \mathbb{E}^e$ be the map defined by $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \to (\ldots, \|\mathbf{x}_i - \mathbf{x}_j\|^2, \ldots)$. Then it is immediate that $\frac{1}{2}\frac{d}{d\mathbf{x}} f|_{\mathbf{x}=\mathbf{p}} = R(\mathbf{p})$. Now, we can rewrite the inequalities of (7.13) in terms of the rigidity matrix $R(\mathbf{p})$ of $G(\mathbf{p})$ (using the usual matrix multiplication applied to $R(\mathbf{p})$ and the indicated column vector) as follows,

$$R(\mathbf{p})(\mathbf{p}')^T \geq 0, \tag{7.14}$$

where the inequality is meant for each coordinate.

For each edge $\{i, j\}$ of G, let ω_{ij} be a scalar. We collect all such scalars into a single row vector called the *stress* $\omega := (\ldots, \omega_{ij}, \ldots)$ corresponding to the rows of the matrix $R(\mathbf{p})$. Append the volume force $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$ as the last row onto $R(\mathbf{p})$ to get a new matrix $\widehat{R}(\mathbf{p})$, which we call the *augmented rigidity matrix*. So, when performing the matrix multiplication $\widehat{R}(\mathbf{p})(\mathbf{p}')^T$, we find that the result is a column vector of length $e+1$ having $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j)$ on the position corresponding to the edge $\{i, j\}$ of G, and having $\sum_{k=1}^n \mathbf{N}_k \cdot \mathbf{p}'_k$ on the $(e+1)$st position. Also, it is easy to see that

$$(\omega, 1)\widehat{R}(\mathbf{p}) = \left(\ldots, \sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) + \mathbf{N}_i, \ldots \right), \tag{7.15}$$

where each sum is taken over all $\mathbf{p}_j$ adjacent to $\mathbf{p}_i$ in G, and we collect d coordinates at a time.

Definition 7.4.6 *Let* $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$ *be the volume force of the d-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *with vertices* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$. *We say that the stress* $\omega = (\ldots, \omega_{ij}, \ldots)$ *resolves* $\mathbf{N}$ *if for each i we have that* $\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) + \mathbf{N}_i = \mathbf{o}$ *or, equivalently,* $(\omega, 1)\widehat{R}(\mathbf{p}) = \mathbf{o}$, *where* $\mathbf{o}$ *denotes the zero vector.*

Definition 7.4.7 *The d-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *and the graph G defined on the vertices of* $\mathbf{P}$ *satisfy the critical volume condition if the volume force* $\mathbf{N}$ *can be resolved by a stress* $\omega = (\ldots, \omega_{ij}, \ldots)$ *such that for each edge* $\{i, j\}$ *of G,* $\omega_{ij} < 0$.

Theorem 7.4.8 *Let the d-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *and the strut graph G, defined on the vertices of* $\mathbf{P}$*, satisfy the critical volume condition. Moreover, let* $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ *be an infinitesimal flex of the strut framework* $G(\mathbf{p})$ *(i.e., let* $\mathbf{p}'$ *satisfy (7.13)). Then*

$$\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} = \frac{1}{d!} \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i \geq 0$$

with equality if and only if $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ *for each edge* $\{i, j\}$ *of G.*

Proof: The assumptions, (7.15), the associativity of matrix multiplication, and (7.12) imply in a straightforward way that

$$0 = \mathbf{o} \cdot \mathbf{p}' = (\omega, 1)\widehat{R}(\mathbf{p})(\mathbf{p}')^T = \sum_{\{i,j\}} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + \sum_{i=1}^{n} \mathbf{N}_i \cdot \mathbf{p}'_i$$

$$= \sum_{\{i,j\}} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i \leq \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i = \frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0},$$

where $\mathbf{N}_i$ is regarded as a d-dimensional vector so that $\mathbf{N}_i \wedge \mathbf{p}'_i$ can be interpreted as the standard inner product $\mathbf{N}_i \cdot \mathbf{p}_i$, with appropriate identification of bases. We clearly get equality if and only if $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ for each edge $\{i, j\}$ of G. □

7.4.4 Strictly locally volume expanding convex polytopes

The following definition recalls standard terminology from the theory of rigid tensegrity frameworks. (See [105] for more information.) Consider now just the *bar graph* $\overline{G}$, which is the graph G with all the struts changed to *bars*, and take its realization $\overline{G}(\mathbf{p})$ sitting over the point configuration $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$. (Here bars mean edges whose lengths are constrained not to change.) We say that the infinitesimal motion $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ is an *infinitesimal flex* of $\overline{G}(\mathbf{p})$ if for each edge (bar) $\{i, j\}$ of $\overline{G}$, we have

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0.$$

This is the same as saying $R(\mathbf{p})(\mathbf{p}')^T = \mathbf{o}$ for the rigidity matrix $R(\mathbf{p})$.

Definition 7.4.9 *We say that* $\mathbf{p}'$ *is a trivial infinitesimal flex if* $\mathbf{p}'$ *is a (directional) derivative of an isometric motion of* $\mathbb{E}^d, d \geq 2$. *We say that* $G(\mathbf{p})$ *(resp.,* $\overline{G}(\mathbf{p})$*) is infinitesimally rigid if* $G(\mathbf{p})$ *(resp.,* $\overline{G}(\mathbf{p})$*) has only trivial infinitesimal flexes.*

Theorem 7.4.10 *Let the d-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *and the strut graph G, defined on the vertices of* $\mathbf{P}$*, satisfy the critical volume condition and assume that the bar framework* $\overline{G}(\mathbf{p})$ *is infinitesimally rigid. Then*

$$\frac{d}{dt} V\left(\mathbf{P}(t)\right)|_{t=0} = \frac{1}{d!} \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i > 0$$

for every non-trivial infinitesimal flex $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \dots, \mathbf{p}'_n)$ *of the strut framework* $G(\mathbf{p})$.

Proof: By Theorem 7.4.8 we have that $\frac{d}{dt} V\left(\mathbf{P}(t)\right)|_{t=0} = \frac{1}{d!} \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i \geq 0$. If $\frac{d}{dt} V\left(\mathbf{P}(t)\right)|_{t=0} = 0$, then applying Theorem 7.4.8 again, $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \dots, \mathbf{p}'_n)$ must be an infinitesimal flex of the bar framework $\overline{G}(\mathbf{p})$. However, then by the infinitesimal rigidity of $\overline{G}(\mathbf{p})$, this would imply that $\mathbf{p}'$ is trivial. Thus, $\frac{d}{dt} V\left(\mathbf{P}(t)\right)|_{t=0} > 0$. □

The following definition leads us to the core part of this section.

Definition 7.4.11 *Let* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *be a d-dimensional convex polytope and let G be a strut graph defined on the vertices* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ *of* $\mathbf{P}$. *We say that* $\mathbf{P}$ *is strictly locally volume expanding over G, if there is an* $\epsilon > 0$ *with the following property. For every* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$ *satisfying*

$$\|\mathbf{p}_i - \mathbf{q}_i\| < \epsilon \text{ for all } i = 1, \dots, n \tag{7.16}$$

and

$$\|\mathbf{p}_i - \mathbf{p}_j\| \leq \|\mathbf{q}_i - \mathbf{q}_j\| \text{ for each edge } \{i, j\} \text{ of } G, \tag{7.17}$$

we have $V(\mathbf{P}) \leq V(\mathbf{Q})$ *(where* $V(\mathbf{Q})$ *is defined via (7.10) and (7.11) substituting* $\mathbf{q}$ *for* $\mathbf{p}$*) with equality only when* $\mathbf{P}$ *is congruent to* $\mathbf{Q}$*, where* $\mathbf{Q}$ *is the polytope generated by the simplices of the barycenters in (7.10) using* $\mathbf{q}$ *instead of* $\mathbf{p}$.

Theorem 7.4.12 *Let the d-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *and the strut graph G, defined on the vertices of* $\mathbf{P}$*, satisfy the critical volume condition and assume that the bar framework* $\overline{G}(\mathbf{p})$ *is infinitesimally rigid. Then* $\mathbf{P}$ *is strictly locally volume expanding over G.*

Proof: The inequalities (7.17) define a semialgebraic set X in the space of all configurations $\{(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) | \mathbf{q}_i \in \mathbb{E}^d, i = 1, \dots, n\}$. Suppose there is no ϵ as in the conclusion. Add $V(\mathbf{P}) \geq V(\mathbf{Q})$ to the constraints defining X. By Wallace [245] (see [105]) there is an *analytic* path $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \dots, \mathbf{p}_n(t)), 0 \leq t < 1$, with $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(t) \in X$, $\mathbf{p}(t)$ not congruent to $\mathbf{p}(0)$ for $0 < t < 1$. So,

$$\|\mathbf{p}_i - \mathbf{p}_j\| \leq \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \text{ for each edge } \{i,j\} \text{ of } G \text{ and} \tag{7.18}$$

$$V(\mathbf{P}) \geq V(\mathbf{P}(t)) \text{ for } 0 \leq t < 1. \tag{7.19}$$

Then after suitably adjusting $\mathbf{p}(t)$ by congruences (as in [105] as well as [107]) we can define

$$\mathbf{p}' := \frac{d^k \mathbf{p}(t)}{dt^k}\Big|_{t=0}$$

for the smallest k that makes $\mathbf{p}'$ a non-trivial infinitesimal flex. (Such k exists by the argument in [105] as well as [107]).

Because (7.18) holds we see that $\mathbf{p}'$ is a non-trivial infinitesimal flex of $G(\mathbf{p})$ and (7.19) implies that

$$\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} \leq 0.$$

But this contradicts Theorem 7.4.10, finishing the proof of Theorem 7.4.12. □

7.4.5 From critical volume condition and infinitesimal rigidity to uniform stability of sphere packings

Here we start with the assumptions of Theorem 1.7.3 and apply Theorem 7.4.12 to each $\mathbf{P}_i$ and $G_{\mathcal{P}}$ restricted to the vertices of $\mathbf{P}_i$, $1 \leq i \leq m$. Then let $\epsilon_0 > 0$ be the smallest $\epsilon > 0$ guaranteed by the strict locally volume expanding property of Theorem 7.4.12. All but a finite number of tiles are fixed. The tiles that are free to move are confined to a region of fixed volume in $\mathbb{E}^d, d \geq 2$. Each $\mathbf{P}_i$ is strictly locally volume expanding, therefore the volume of each of the tiles must be fixed. But the strict condition implies that the motion of each tile must be an isometry. Because the tiling is face-to-face and the vertices are given by $G_{\mathcal{P}}$ we conclude inductively (on the number of tiles) that each vertex of $G_{\mathcal{P}}$ must be fixed. Thus, $\mathcal{P}$ is uniformly stable with respect to ϵ_0 introduced above, finishing the proof of Theorem 1.7.3.