# 5

# On the Volume of Finite Arrangements of Spheres

## 5.1 The Conjecture of Kneser and Poulsen

Recall that $\|\ldots\|$ denotes the standard Euclidean norm of the $d$-dimensional Euclidean space $\mathbb{E}^d$. So, if $\mathbf{p}_i, \mathbf{p}_j$ are two points in $\mathbb{E}^d$, then $\|\mathbf{p}_i - \mathbf{p}_j\|$ denotes the Euclidean distance between them. It is convenient to denote the (finite) point configuration consisting of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ in $\mathbb{E}^d$ by $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$. Now, if $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ are two configurations of $N$ points in $\mathbb{E}^d$ such that for all $1 \leq i < j \leq N$ the inequality $\|\mathbf{q}_i - \mathbf{q}_j\| \leq \|\mathbf{p}_i - \mathbf{p}_j\|$ holds, then we say that $\mathbf{q}$ is a *contraction* of $\mathbf{p}$. If $\mathbf{q}$ is a contraction of $\mathbf{p}$, then there may or may not be a continuous motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbb{E}^d$ for all $0 \leq t \leq 1$ and $1 \leq i \leq N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is monotone decreasing for all $1 \leq i < j \leq N$. When there is such a motion, we say that $\mathbf{q}$ is a *continuous contraction* of $\mathbf{p}$. Finally, let $\mathbf{B}^d[\mathbf{p}_i, r_i]$ denote the (closed) $d$-dimensional ball centered at $\mathbf{p}_i$ with radius $r_i$ in $\mathbb{E}^d$ and let $\mathrm{vol}_d(\ldots)$ represent the $d$-dimensional volume (Lebesgue measure) in $\mathbb{E}^d$. In 1954 Poulsen [216] and in 1955 Kneser [183] independently conjectured the following for the case when $r_1 = \cdots = r_N$.

**Conjecture 5.1.1** *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$, *then*

$$\mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]\right) \geq \mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r_i]\right).$$

**Conjecture 5.1.2** *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$, *then*

$$\mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]\right) \leq \mathrm{vol}_d\left(\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r_i]\right).$$

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Actually, Kneser seems to be the one who has generated a great deal of interest in the above conjectures also via private letters written to a number of mathematicians. For more details on this see, for example, [181].

## 5.2 The Kneser–Poulsen Conjecture for Continuous Contractions

For a given point configuration $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ in $\mathbb{E}^d$ and radii $r_1, r_2, \dots, r_N$ consider the following sets,

$$\mathbf{V}_i = \{\mathbf{x} \in \mathbb{E}^d \mid \text{for all } j,\ \|\mathbf{x} - \mathbf{p}_i\|^2 - r_i^2 \leq \|\mathbf{x} - \mathbf{p}_j\|^2 - r_j^2\},$$

$$\mathbf{V}^i = \{\mathbf{x} \in \mathbb{E}^d \mid \text{for all } j,\ \|\mathbf{x} - \mathbf{p}_i\|^2 - r_i^2 \geq \|\mathbf{x} - \mathbf{p}_j\|^2 - r_j^2\}.$$

The set $\mathbf{V}_i$ (resp., $\mathbf{V}^i$) is called the *nearest (resp., farthest) point Voronoi cell* of the point $\mathbf{p}_i$. (For a detailed discussion on nearest as well as farthest point Voronoi cells we refer the interested reader to [124] and [230].) We now restrict each of these sets as follows.

$$\mathbf{V}_i(r_i) = \mathbf{V}_i \cap \mathbf{B}^d[\mathbf{p}_i, r_i],$$

$$\mathbf{V}^i(r_i) = \mathbf{V}^i \cap \mathbf{B}^d[\mathbf{p}_i, r_i].$$

We call the set $\mathbf{V}_i(r_i)$ (resp., $\mathbf{V}^i(r_i)$) the *nearest (resp., farthest) point truncated Voronoi cell* of the point $\mathbf{p}_i$. For each $i \neq j$ let $W_{ij} = \mathbf{V}_i \cap \mathbf{V}_j$ and $W^{ij} = \mathbf{V}^i \cap \mathbf{V}^j$. The sets $W_{ij}$ and $W^{ij}$ are the *walls* between the nearest and farthest point Voronoi cells. Finally, it is natural to define the relevant *truncated walls* as follows.

$$\begin{aligned} W_{ij}(\mathbf{p}_i, r_i) &= W_{ij} \cap \mathbf{B}^d[\mathbf{p}_i, r_i] \\ &= W_{ij}(\mathbf{p}_j, r_j) = W_{ij} \cap \mathbf{B}^d[\mathbf{p}_j, r_j], \end{aligned}$$

$$\begin{aligned} W^{ij}(\mathbf{p}_i, r_i) &= W^{ij} \cap \mathbf{B}^d[\mathbf{p}_i, r_i] \\ &= W^{ij}(\mathbf{p}_j, r_j) = W^{ij} \cap \mathbf{B}^d[\mathbf{p}_j, r_j]. \end{aligned}$$

The following formula discovered by Csikós [113] proves Conjecture 5.1.1 as well as Conjecture 5.1.2 for continuous contractions in a straighforward way in any dimension. (Actually, the planar case of the Kneser–Poulsen conjecture under continuous contractions has been proved independently in [77], [112], [99], and [26].)

**Theorem 5.2.1** *Let $d \geq 2$ and let $\mathbf{p}(t), 0 \leq t \leq 1$ be a smooth motion of a point configuration in $\mathbb{E}^d$ such that for each $t$, the points of the configuration are pairwise distinct. Then*

$$\frac{d}{dt}\mathrm{vol}_d\left(\bigcup_{i=1}^{N}\mathbf{B}^d[\mathbf{p}_i(t), r_i]\right)$$

$$= \sum_{1\leq i<j\leq N}\left(\frac{d}{dt}d_{ij}(t)\right)\cdot \mathrm{vol}_{d-1}\left(W_{ij}(\mathbf{p}_i(t), r_i)\right),$$

$$\frac{d}{dt}\mathrm{vol}_d\left(\bigcap_{i=1}^{N}\mathbf{B}^d[\mathbf{p}_i(t), r_i]\right)$$

$$= \sum_{1\leq i<j\leq N} -\left(\frac{d}{dt}d_{ij}(t)\right)\cdot \mathrm{vol}_{d-1}\left(W^{ij}(\mathbf{p}_i(t), r_i)\right),$$

*where $d_{ij}(t) = \|\mathbf{p}_i(t) - \mathbf{p}_i(t)\|$.*

On the one hand, Csikós [114] managed to generalize his formula to configurations of balls called flowers which are sets obtained from balls with the help of operations $\cap$ and $\cup$. This work extends to hyperbolic as well as spherical space. On the other hand, Csikós [115] has succeeded in proving a Schläfli-type formula for polytopes with curved faces lying in pseudo-Riemannian Einstein manifolds, which can be used to provide another proof of Conjecture 5.1.1 as well as Conjecture 5.1.2 for continuous contractions (for more details see [115]).

## 5.3 The Kneser–Poulsen Conjecture in the Plane

In the recent paper [58] the author and Connelly proved Conjecture 5.1.1 as well as Conjecture 5.1.2 in the Euclidean plane. Thus, we have the following theorem.

**Theorem 5.3.1** *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^2$, then*

$$\mathrm{vol}_2\left(\bigcup_{i=1}^{N}\mathbf{B}^2[\mathbf{p}_i, r_i]\right) \geq \mathrm{vol}_2\left(\bigcup_{i=1}^{N}\mathbf{B}^2[\mathbf{q}_i, r_i]\right);$$

*moreover,*

$$\mathrm{vol}_2\left(\bigcap_{i=1}^{N}\mathbf{B}^2[\mathbf{p}_i, r_i]\right) \leq \mathrm{vol}_2\left(\bigcap_{i=1}^{N}\mathbf{B}^2[\mathbf{q}_i, r_i]\right).$$

In fact, the paper [58] contains a proof of an extension of the above theorem to flowers as well. In what follows we give an outline of the three-step proof published in [58] by phrasing it through a sequence of theorems each being higher-dimensional. Voronoi cells play an essential role in our proofs of Theorems 5.3.2 and 5.3.3.

**Theorem 5.3.2** *Consider $N$ moving closed $d$-dimensional balls $\mathbf{B}^d[\mathbf{p}_i(t), r_i]$ with $1 \leq i \leq N, 0 \leq t \leq 1$ in $\mathbb{E}^d, d \geq 2$. If $F_i(t)$ is the contribution of the ith ball to the boundary of the union $\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i(t), r_i]$ (resp., of the intersection $\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i(t), r_i]$), then*

$$\sum_{1 \leq i \leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i(t))$$

*decreases (resp., increases) in $t$ under any analytic contraction $\mathbf{p}(t)$ of the center points, where $0 \leq t \leq 1$ and $\operatorname{svol}_{d-1}(\dots)$ refers to the relevant $(d-1)$-dimensional surface volume.*

**Theorem 5.3.3** *Let the centers of the closed $d$-dimensional balls $\mathbf{B}^d[\mathbf{p}_i, r_i]$, $1 \leq i \leq N$ lie in the $(d-2)$-dimensional affine subspace $L$ of $\mathbb{E}^d, d \geq 3$. If $F_i$ stands for the contribution of the ith ball to the boundary of the union $\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]$ (resp., of the intersection $\bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r_i]$), then*

$$\operatorname{vol}_{d-2}\left(\bigcup_{i=1}^{N} \mathbf{B}^{d-2}[\mathbf{p}_i, r_i]\right) = \frac{1}{2\pi} \sum_{1 \leq i \leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i)$$

$$\left(\text{resp., } \operatorname{vol}_{d-2}\left(\bigcap_{i=1}^{N} \mathbf{B}^{d-2}[\mathbf{p}_i, r_i]\right) = \frac{1}{2\pi} \sum_{1 \leq i \leq N} \frac{1}{r_i} \operatorname{svol}_{d-1}(F_i)\right),$$

*where $\mathbf{B}^{d-2}[\mathbf{p}_i, r_i] = \mathbf{B}^d[\mathbf{p}_i, r_i] \cap L, 1 \leq i \leq N$.*

**Theorem 5.3.4** *If $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ is a contraction of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ in $\mathbb{E}^d, d \geq 1$, then there is an analytic contraction $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_N(t)), 0 \leq t \leq 1$ in $\mathbb{E}^{2d}$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$.*

Note that Theorems 5.3.2, 5.3.3, and 5.3.4 imply Theorem 5.3.1 in a straighforward way.

Also, we note that Theorem 5.3.4 (called the Leapfrog Lemma) cannot be improved; namely, it has been shown in [24] that there exist point configurations $\mathbf{q}$ and $\mathbf{p}$ in $\mathbb{E}^d$, actually constructed in the way suggested in [58], such that $\mathbf{q}$ is a contraction of $\mathbf{p}$ in $\mathbb{E}^d$ and there is no continuous contraction from $\mathbf{p}$ to $\mathbf{q}$ in $\mathbb{E}^{2d-1}$.

In order to describe a more complete picture of the status of the Kneser–Poulsen conjecture we mention two additional corollaries obtained from the proof published in [58] and just outlined above. (For more details see [58].)

**Theorem 5.3.5** *Let* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *be two point configurations in* $\mathbb{E}^d$ *such that* $\mathbf{q}$ *is a piecewise-analytic contraction of* $\mathbf{p}$ *in* $\mathbb{E}^{d+2}$*. Then the conclusions of Conjecture 5.1.1 as well as Conjecture 5.1.2 hold in* $\mathbb{E}^d$*.*

The following generalizes a result of Gromov in [150], who proved it in the case $N \leq n + 1$.

**Theorem 5.3.6** *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)$ *is an arbitrary contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$ *and* $N \leq n + 3$*, then both Conjecture 5.1.1 and Conjecture 5.1.2 hold.*

As a next step it would be natural to investigate the case $N = n + 4$.

## 5.4 Non-Euclidean Kneser–Poulsen-Type Results

It is somewhat surprising that in spherical space for the specific radius of balls (i.e., spherical caps) one can find a proof of both Conjecture 5.1.1 and Conjecture 5.1.2 in all dimensions. The magic radius is $\frac{\pi}{2}$ and the following theorem describes the desired result in details.

**Theorem 5.4.1** *If a finite set of closed d-dimensional balls of radius* $\frac{\pi}{2}$ *(i.e., of closed hemispheres) in the d-dimensional spherical space* $\mathbb{S}^d, d \geq 2$ *is rearranged so that the (spherical) distance between each pair of centers does not increase, then the (spherical) d-dimensional volume of the intersection does not decrease and the (spherical) d-dimensional volume of the union does not increase.*

The method of the proof published by the author and Connelly in [61] can be described as follows. First, one can use a leapfrog lemma to move one configuration to the other in an analytic and monotone way, but only in higher dimensions. Then the higher-dimensional balls have their combined volume (their intersections or unions) change monotonically, a fact that one can prove using Schläfli's differential formula. Then one can apply an integral formula to relate the volume of the higher-dimensional object to the volume of the lower-dimensional object, obtaining the volume inequality for the more general discrete motions.

The following statement is a corollary of Theorem 5.4.1, the Euclidean part of which has been proved independently by Alexander [4], Capoyleas, and Pach [98] and Sudakov [234]. For the sake of completeness in what follows, we recall the notion of spherical mean width, which is most likely less known than its widely used Euclidean counterpart. Let $\mathbb{S}^d$ be the $d$-dimensional unit sphere centered at the origin in $\mathbb{E}^{d+1}$. A spherically convex body is a closed, spherically convex subset of $\mathbb{S}^d$ with interior points and lying in some closed hemisphere, thus, the intersection of $\mathbb{S}^d$ with a $(d+1)$-dimensional closed convex cone of $\mathbb{E}^{d+1}$ different from $\mathbb{E}^{d+1}$. Recall that $\mathrm{Svol}_d(\dots)$ denotes the spherical Lebesgue measure on $\mathbb{S}^d$, and recall that $(d+1)\omega_{d+1} = \mathrm{Svol}_d(\mathbb{S}^d)$. Moreover, as usual we denote the standard inner product of $\mathbb{E}^{d+1}$ by $\langle\cdot,\cdot\rangle$, and for $\mathbf{u} \in \mathbb{S}^d$ we write $\mathbf{u}^\perp := \{\mathbf{x} \in \mathbb{E}^{d+1} : \langle \mathbf{u}, \mathbf{x}\rangle = 0\}$ for the orthogonal complement of $\mathrm{lin}\{\mathbf{u}\}$. For a spherically convex body $\mathbf{K}$, the polar body is defined by

$$\mathbf{K}^* := \{\mathbf{u} \in \mathbb{S}^d : \langle \mathbf{u}, \mathbf{v}\rangle \le 0 \text{ for all } \mathbf{v} \in \mathbf{K}\}.$$

It is also spherically convex, but need not have interior points. The number

$$U(\mathbf{K}) := \frac{1}{2}\mathrm{Svol}_d(\{\mathbf{u} \in \mathbb{S}^d : \mathbf{u}^\perp \cap \mathbf{K} \neq \emptyset\})$$

can be considered as the *spherical mean width* of $\mathbf{K}$. Obviously, a vector $\mathbf{u} \in \mathbb{S}^d$ satisfies $\mathbf{u} \in \mathbf{K}^* \cup (-\mathbf{K}^*)$ if and only if $\mathbf{u}^\perp$ does not meet the interior of $\mathbf{K}$, hence

$$(d+1)\omega_{d+1} - 2\mathrm{Svol}_d(\mathbf{K}^*) = 2U(\mathbf{K}). \tag{5.1}$$

Now, (5.1) and Theorem 5.4.1 imply the following theorem in a rather straighforward way.

**Theorem 5.4.2** *Let* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *be* $N$ *points on a closed hemisphere of* $\mathbb{S}^d, d \ge 2$ *(resp., points in* $\mathbb{E}^d, d \ge 2$*), and let* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *be a contraction of* $\mathbf{p}$ *in* $\mathbb{S}^d$ *(resp., in* $\mathbb{E}^d$*). Then the spherical mean width (resp., mean width) of the spherical convex hull (resp., convex hull) of* $\mathbf{q}$ *is less than or equal to the spherical mean width (resp., mean width) of the spherical convex hull (resp., convex hull) of* $\mathbf{p}$.

Before we continue our non-Euclidean discussions it seems natural to mention a Euclidean Kneser–Poulsen-type result supported by Theorem 5.4.2. For that purpose, let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ be $N$ points in $\mathbb{E}^d, d \ge 2$, and let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ be an arbitrary contraction of $\mathbf{p}$ in $\mathbb{E}^d$. Now, if $r > 0$ is sufficiently large, then the union of the balls of radius $r$ centered at the points of $\mathbf{q}$ (resp., $\mathbf{p}$) is eventually the same as the outer parallel domain of radius $r$ of the convex hull of $\mathbf{q}$ (resp., $\mathbf{p}$). Then writing out Steiner's formula for the volumes of the outer parallel domains just mentioned with coefficients equal to the proper intrinsic volumes and noting that the first intrinsic volume is equal to the mean width (up to some constant), Theorem 5.4.2 implies that Conjecture 5.1.1 holds for sufficiently large equal radii (provided of course, that the mean width in question is non-zero). A similar argument supports the inequality of Conjecture 5.1.2 to hold for sufficiently large equal radii. Thus we have arrived at the following theorem that was proved regorously by Gorbovickis in [149] (using a different approach).

**Theorem 5.4.3** *If* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *is a contraction of* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$*, then there exists* $r_0 > 0$ *such that for any* $r \ge r_0$*,*

$$\mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r]\right) \ge \mathrm{vol}_d\left(\bigcup_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r]\right)$$

$$\left( \textit{resp.}, \ \mathrm{vol}_d \left( \bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{p}_i, r] \right) \leq \mathrm{vol}_d \left( \bigcap_{i=1}^{N} \mathbf{B}^d[\mathbf{q}_i, r] \right) \right).$$

We note that Theorem 5.4.1 extends to flowers as well; moreover, a positive answer to the following problem would imply that both Conjecture 5.1.1 and Conjecture 5.1.2 hold for circles in $\mathbb{S}^2$ (for more details on this see [61]).

**Problem 5.4.4** *Suppose that* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *and* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *are two point configurations in* $\mathbb{S}^2$. *Then prove or disprove that there is a monotone piecewise-analytic motion from* $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$ *to* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ *in* $\mathbb{S}^4$.

Note that in fact, Theorem 5.4.1 states a volume inequality between two spherically convex polytopes satisfying some metric conditions. The following problem searches for a natural analogue of that in the hyperbolic 3-space $\mathbb{H}^3$. In order to state it properly we recall the following. Let $A$ and $B$ be two planes in $\mathbb{H}^3$ and let $A^+$ (resp., $B^+$) denote one of the two closed halfspaces bounded by $A$ (resp., $B$) such that the set $A^+ \cap B^+$ is nonempty. Recall that either $A$ and $B$ intersect or $A$ is parallel to $B$ or $A$ and $B$ have a line perpendicular to both of them. Now, "the dihedral angle $A^+ \cap B^+$" means not only the set in question, but also refers to the standard angular measure of the corresponding angle between $A$ and $B$ in the first case, it refers to 0 in the second case, and finally, in the third case it refers to the negative of the hyperbolic distance between $A$ and $B$.

**Problem 5.4.5** *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be compact convex polyhedra of* $\mathbb{H}^3$ *with* $\mathbf{P}$ *(resp.,* $\mathbf{Q}$*) being the intersection of the closed halfspaces* $H^+_{P,1}, H^+_{P,2}, \ldots, H^+_{P,N}$ *(resp.,* $H^+_{Q,1}, H^+_{Q,2}, \ldots, H^+_{Q,N}$*). Assume that the dihedral angle* $H^+_{Q,i} \cap H^+_{Q,j}$ *(containing* $\mathbf{Q}$*) is at least as large as the corresponding dihedral angle* $H^+_{P,i} \cap H^+_{P,j}$ *(containing* $\mathbf{P}$*) for all* $1 \leq i < j \leq N$. *Then prove or disprove that the volume of* $\mathbf{P}$ *is at least as large as the volume of* $\mathbf{Q}$.

Using Andreev's version [6], [7] of the Koebe–Andreev–Thurston theorem and Schläfli's differential formula the author [64] proved the following partial analogue of Theorem 5.4.1 in $\mathbb{H}^3$.

**Theorem 5.4.6** *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in* $\mathbb{H}^3$. *If each inner dihedral angle of* $\mathbf{Q}$ *is at least as large as the corresponding inner dihedral angle of* $\mathbf{P}$, *then the volume of* $\mathbf{P}$ *is at least as large as the volume of* $\mathbf{Q}$.

## 5.5 Alexander's Conjecture

It seems that in the Euclidean plane, for the case of the intersection of congruent disks, one can sharpen the results proved by the author and Connelly [58]. Namely, Alexander [4] conjectures the following.

**Conjecture 5.5.1** *Under arbitrary contraction of the center points of finitely many congruent disks in the Euclidean plane, the perimeter of the intersection of the disks cannot decrease.*

The analogous question for the union of congruent disks has a negative answer, as was observed by Habicht and Kneser long ago (for details see [58]). In [68] some supporting evidence for the above conjecture of Alexander has been collected; in particular, the following theorem was proved.

**Theorem 5.5.2** *Alexander's conjecture holds for continuous contractions of the center points and it holds up to* 4 *congruent disks under arbitrary contractions of the center points.*

We note that Alexander's conjecture does not hold for incongruent disks (even under continuous contractions of their center points) as shown in [68]. Finally we remark that if Alexander's conjecture were true, then it would be a rare instance of an asymmetry between intersections and unions for Kneser–Poulsen-type questions.

## 5.6 Densest Finite Sphere Packings

Let $\mathbf{B}^d$ denote the closed $d$-dimensional unit ball centered at the origin $\mathbf{o}$ of $\mathbb{E}^d, d \geq 2$ and let $\mathcal{P} := \{\mathbf{c}_1 + \mathbf{B}^d, \mathbf{c}_2 + \mathbf{B}^d, \dots, \mathbf{c}_n + \mathbf{B}^d\}$ be a packing of $n$ unit balls with centers $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ in $\mathbb{E}^d$. We say that $\mathcal{P}$ is a *densest packing* among all packings of $n$ unit balls in $\mathbb{E}^d$ if there exists a parameter $r > 1$ with the property that

$$\delta(\mathcal{P}) := \frac{n\mathrm{vol}_d(\mathbf{B}^d)}{\mathrm{vol}_d\left(\bigcup_{i=1}^n \mathbf{c}_i + r\mathbf{B}^d\right)} = \frac{n\omega_d}{\mathrm{vol}_d\left(\bigcup_{i=1}^n \mathbf{c}_i + r\mathbf{B}^d\right)}$$

$$= \max\left\{\frac{n\omega_d}{\mathrm{vol}_d\left(\bigcup_{i=1}^n \mathbf{x}_i + r\mathbf{B}^d\right)} \mid \|\mathbf{x}_j - \mathbf{x}_k\| \geq 2 \text{ for all } 1 \leq j < k \leq n\right\};$$

that is,

$$\mathrm{vol}_d\left(\bigcup_{i=1}^n \mathbf{c}_i + r\mathbf{B}^d\right) = \min_{\|\mathbf{x}_j - \mathbf{x}_k\| \geq 2 \text{ for all } 1 \leq j < k \leq n} \left\{\mathrm{vol}_d\left(\bigcup_{i=1}^n \mathbf{x}_i + r\mathbf{B}^d\right)\right\}. \quad (5.2)$$

The definition (5.2) is rather natural from the point of the Kneser–Poulsen Conjecture and it seems to lead to a new definition of densest finite sphere packings. The closest related notion is the definition of parametric density, introduced by Wills in [247] (see also [28]), where the union of balls is replaced by the convex hull of the union of balls thereby replacing our concave container by a convex one.

First, let us investigate (5.2) in $\mathbb{E}^2$. If (5.2) holds with parameter $r$ satisfying $1 < r \leq \frac{2}{\sqrt{3}} = 1.1547\ldots$, then it is easy to see that $\mathcal{P}$ must be a packing with the largest number of touching pairs among all packings of $n$ unit disks, and therefore according to the well-known result of Harborth [166], $\mathcal{P}$ must be a subset of the densest infinite hexagonal packing of unit disks in $\mathbb{E}^2$. If (5.2) holds with parameter $r$ satisfying $\frac{2}{\sqrt{3}} < r$, then the Hajós Lemma (see, for example, [200]) easily implies that $\delta(\mathcal{P}) < \frac{\pi}{\sqrt{12}}$. This inequality, for any fixed $\frac{2}{\sqrt{3}} < r$, is asymptotically best possible (with respect to $n$). However, the following remains a challenging open question.

**Problem 5.6.1** *Assume that $\mathcal{P}$ is a densest packing of $n$ unit disks in $\mathbb{E}^2$ with parameter $\frac{2}{\sqrt{3}} < r$ in (5.2). Prove or disprove that $\mathcal{P}$ is a subset of the densest infinite hexagonal packing of unit disks in $\mathbb{E}^2$.*

Next, let us take a closer look of (5.2) in $\mathbb{E}^3$. If (5.2) holds with parameter $r$ satisfying $2 \leq r$, then Theorem 2.4.3 and Theorem 1.4.1 imply in a straightforward way that $\delta(\mathcal{P}) \leq \frac{\pi}{\sqrt{18}}$. Not surprisingly, this inequality, for any fixed $2 \leq r$, is asymptotically best possible (with respect to $n$). Moreover, if (5.2) holds with parameter $r$ satisfying $\sqrt{\frac{3}{2}} = 1.2247\cdots \leq r < 2$, then Theorem 1.4.6 implies that $\delta(\mathcal{P}) \leq \sigma_3 = 0.7796\ldots$. Last but not least, if (5.2) holds with parameter $r$ satisfying $1 < r < \frac{2}{\sqrt{3}} = 1.1547\ldots$, then it is easy to see that $\mathcal{P}$ must be a packing with the largest number $C(n)$ of touching pairs among all packings of $n$ unit balls in $\mathbb{E}^3$. For some exact values as well as estimates on $C(n)$ see Theorem 1.3.5 and the discussion there. The following problem might generate further progress on the problem at hand. For natural reasons we call it the *Truncated Dodecahedral Conjecture.*

**Conjecture 5.6.2** *Let $\mathcal{F}$ be an arbitrary (finite or infinite) family of non-overlapping unit balls in $\mathbb{E}^3$ with the unit ball $\mathbf{B}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^3$ belonging to $\mathcal{F}$. Let $\mathbf{P}$ stand for the Voronoi cell of the packing $\mathcal{F}$ assigned to $\mathbf{B}$ and let $\mathbf{Q}$ denote a regular dodecahedron circumscribed $\mathbf{B}$ having circumradius $\sqrt{3}\tan\frac{\pi}{5} = 1.2584\ldots$. If $r$ is any parameter with $\frac{2}{\sqrt{3}} < r \leq \sqrt{3}\tan\frac{\pi}{5}$, then*

$$\mathrm{vol}_3(\mathbf{P} \cap r\mathbf{B}) \geq \mathrm{vol}_3(\mathbf{Q} \cap r\mathbf{B}) \ .$$

We note that obviously the inequality of Conjecture 5.6.2 holds for any parameter with $1 < r \leq \frac{2}{\sqrt{3}}$. Moreover, for the sake of completeness we mention that the special case, when $r = \sqrt{3}\tan\frac{\pi}{5}$ in Conjecture 5.6.2, had already been conjectured by L. Fejes Tóth in [135], and it is still open, although the closely related (but weaker) Dodecahedral Conjecture has been recently proved by Hales and McLaughlin [164], [165].

Finally, we take a look at (5.2) in $\mathbb{E}^d, d \geq 4$. On the one hand, if (5.2) holds with parameter $r$ satisfying $2 \leq r$, then Theorem 2.4.3 implies the estimate $\delta(\mathcal{P}) \leq \delta(\mathbf{B}^d)$. On the other hand, if (5.2) holds with parameter $r$

satisfying $\sqrt{\frac{2d}{d+1}} \leq r < 2$, then Theorem 1.4.6 implies that $\delta(\mathcal{P}) \leq \sigma_d$. In fact, Theorem 1.4.8 improves that inequality to $\delta(\mathcal{P}) \leq \widehat{\sigma}_d$ $(< \sigma_d)$ for all $d \geq 8$. Last but not least, if (5.2) holds with parameter $r$ satisfying $1 < r < \frac{2}{\sqrt{3}}$, then it is easy to see that $\mathcal{P}$ must be a packing with the largest number of touching pairs (called the contact number of $\mathcal{P}$) among all packings of $n$ unit balls in $\mathbb{E}^d$. Theorem 2.4.2 gives estimates on the contact number of $\mathcal{P}$.