4

Coverings by Planks and Cylinders

4.1 Plank Theorems

As usual, a *convex body* of the Euclidean space $\mathbb{E}^d$ is a compact convex set with non-empty interior. Let $\mathbf{C} \subset \mathbb{E}^d$ be a convex body, and let $H \subset \mathbb{E}^d$ be a hyperplane. Then the distance $w(\mathbf{C}, H)$ between the two supporting hyperplanes of $\mathbf{C}$ parallel to H is called the *width of* $\mathbf{C}$ *parallel to* H. Moreover, the smallest width of $\mathbf{C}$ parallel to hyperplanes of $\mathbb{E}^d$ is called the *minimal width* of $\mathbf{C}$ and is denoted by $w(\mathbf{C})$.

Recall that in the 1930's, Tarski posed what came to be known as the plank problem. A *plank* $\mathbf{P}$ in $\mathbb{E}^d$ is the (closed) set of points between two distinct parallel hyperplanes. The *width* $w(\mathbf{P})$ of $\mathbf{P}$ is simply the distance between the two boundary hyperplanes of $\mathbf{P}$. Tarski conjectured that if a convex body of minimal width w is covered by a collection of planks in $\mathbb{E}^d$, then the sum of the widths of these planks is at least w. This conjecture was proved by Bang in his memorable paper [18]. (In fact, the proof presented in that paper is a simplification and generalization of the proof published by Bang somewhat earlier in [17].) Thus, we call the following statement Bang's plank theorem.

Theorem 4.1.1 *If the convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$ *(i.e.,* $\mathbf{C} \subset \mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n \subset \mathbb{E}^d$*), then*

$$\sum_{i=1}^{n} w(\mathbf{P}_i) \geq w(\mathbf{C}).$$

In [18], Bang raised the following stronger version of Tarski's plank problem called the affine plank problem. We phrase it via the following definition. Let $\mathbf{C}$ be a convex body and let $\mathbf{P}$ be a plank with boundary hyperplanes parallel to the hyperplane H in $\mathbb{E}^d$. We define the $\mathbf{C}$-*width* of the plank $\mathbf{P}$ as $\frac{w(\mathbf{P})}{w(\mathbf{C},H)}$ and label it $w_{\mathbf{C}}(\mathbf{P})$. (This notion was introduced by Bang [18] under the name "relative width".)

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Conjecture 4.1.2 *If the convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$, *then*

$$\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq 1.$$

The special case of Conjecture 4.1.2, when the convex body to be covered is centrally symmetric, has been proved by Ball in [12]. Thus, the following is Ball's plank theorem.

Theorem 4.1.3 *If the centrally symmetric convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$, *then*

$$\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq 1.$$

From the point of view of discrete geometry it seems natural to mention that after proving Theorem 4.1.3 Ball [13] used Bang's proof of Theorem 4.1.1 to derive a new argument for an improvement of the Davenport–Rogers lower bound on the density of economical sphere lattice packings.

It was Alexander [3] who noticed that Conjecture 4.1.2 is equivalent to the following generalization of a problem of Davenport.

Conjecture 4.1.4 *If a convex body* $\mathbf{C}$ *in* $\mathbb{E}^d, d \geq 2$ *is sliced by* $n-1$ *hyperplane cuts, then there exists a piece that covers a translate of* $\frac{1}{n}\mathbf{C}$.

We note that the paper [33] of A. Bezdek and the author proves Conjecture 4.1.4 for successive hyperplane cuts (i.e., for hyperplane cuts when each cut divides one piece). Also, the same paper ([33]) introduced two additional equivalent versions of Conjecture 4.1.2. As they seem to be of independent interest we recall them following the terminology used in [33].

Let $\mathbf{C}$ and $\mathbf{K}$ be convex bodies in $\mathbb{E}^d$ and let H be a hyperplane of $\mathbb{E}^d$. The $\mathbf{C}$-*width of* $\mathbf{K}$ *parallel to* H is denoted by $w_{\mathbf{C}}(\mathbf{K}, H)$ and is defined as $\frac{w(\mathbf{K},H)}{w(\mathbf{C},H)}$. The *minimal* $\mathbf{C}$-*width of* $\mathbf{K}$ is denoted by $w_{\mathbf{C}}(\mathbf{K})$ and is defined as the minimum of $w_{\mathbf{C}}(\mathbf{K}, H)$, where the minimum is taken over all possible hyperplanes H of $\mathbb{E}^d$. Recall that the inradius of $\mathbf{K}$ is the radius of the largest ball contained in $\mathbf{K}$. It is quite natural then to introduce the $\mathbf{C}$-*inradius of* $\mathbf{K}$ as the factor of the largest (positively) homothetic copy of $\mathbf{C}$, a translate of which is contained in $\mathbf{K}$. We need to do one more step to introduce the so-called successive $\mathbf{C}$-inradii of $\mathbf{K}$ as follows. Let r be the $\mathbf{C}$-inradius of $\mathbf{K}$. For any $0 < \rho \leq r$ let the $\rho\mathbf{C}$-*rounded body of* $\mathbf{K}$ be denoted by $\mathbf{K}^{\rho\mathbf{C}}$ and be defined as the union of all translates of $\rho\mathbf{C}$ that are covered by $\mathbf{K}$. Now, take a fixed integer $n \geq 1$. On the one hand, if $\rho > 0$ is sufficiently small, then $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) > n\rho$. On the other hand, $w_{\mathbf{C}}(\mathbf{K}^{r\mathbf{C}}) = r \leq nr$. As $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}})$ is a decreasing continuous function of $\rho > 0$ and $n\rho$ is a strictly increasing continuous function of ρ, there exists a uniquely determined $\rho > 0$ such that

$$w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) = n\rho.$$

This uniquely determined ρ is called the *nth successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$ and is denoted by $r_{\mathbf{C}}(\mathbf{K}, n)$. Notice that $r_{\mathbf{C}}(\mathbf{K}, 1) = r$. Now, the two equivalent versions of Conjecture 4.1.2 and Conjecture 4.1.4 introduced in [33] can be phrased as follows.

Conjecture 4.1.5 *If a convex body* $\mathbf{K}$ *in* $\mathbb{E}^d, d \geq 2$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$*, then* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq w_{\mathbf{C}}(\mathbf{K})$ *for any convex body* $\mathbf{C}$ *in* $\mathbb{E}^d$*.*

Conjecture 4.1.6 *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$*. If* $\mathbf{K}$ *is sliced by* $n-1$ *hyperplanes, then the minimum of the greatest* $\mathbf{C}$*-inradius of the pieces is equal to the nth successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$*; that is, it is* $r_{\mathbf{C}}(\mathbf{K}, n)$*.*

A. Bezdek and the author [33] proved the following theorem that (under the condition that $\mathbf{C}$ is a ball) answers a question raised by Conway ([32]) as well as proves Conjecture 4.1.6 for successive hyperplane cuts.

Theorem 4.1.7 *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d$*,* $d \geq 2$*. If* $\mathbf{K}$ *is sliced into n pieces by* $n-1$ *successive hyperplane cuts (i.e., when each cut divides one piece), then the minimum of the greatest* $\mathbf{C}$*-inradius of the pieces is the nth successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$ *(i.e.,* $r_{\mathbf{C}}(\mathbf{K}, n)$*). An optimal partition is achieved by* $n-1$ *parallel hyperplane cuts equally spaced along the minimal* $\mathbf{C}$*-width of the* $r_{\mathbf{C}}(\mathbf{K}, n)\mathbf{C}$*-rounded body of* $\mathbf{K}$*.*

4.2 Covering Convex Bodies by Cylinders

In his paper [18], Bang, by describing a concrete example and writing that it may be extremal, proposes investigating a quite challenging question that can be phrased as follows.

Problem 4.2.1 *Prove or disprove that the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is at least half of the minimum area 2-dimensional projection of the body.*

If true, then the estimate of Problem 4.2.1 is a sharp one due to a covering of a regular tetrahedron by two cylinders described in [18]. A very recent paper of the author and Litvak ([71]) investigates Problem 4.2.1 as well as its higher-dimensional analogue. Their main result can be summarized as follows.

Given $0 < k < d$ define a k-codimensional cylinder $\mathbf{C}$ in $\mathbb{E}^d$ as a set which can be presented in the form $\mathbf{C} = H + B$, where H is a k-dimensional linear subspace of $\mathbb{E}^d$ and B is a measurable set (called the base) in the orthogonal complement $H^{\perp}$ of H. For a given convex body $\mathbf{K}$ and a k-codimensional cylinder $\mathbf{C} = H + B$ we define the cross-sectional volume $\mathrm{crv}_{\mathbf{K}}(\mathbf{C})$ of $\mathbf{C}$ with respect to $\mathbf{K}$ as follows,

$$\mathrm{crv}_{\mathbf{K}}(\mathbf{C}) := \frac{\mathrm{vol}_{d-k}(\mathbf{C}\cap H^{\perp})}{\mathrm{vol}_{d-k}(P_{H^{\perp}}\mathbf{K})} = \frac{\mathrm{vol}_{d-k}(P_{H^{\perp}}\mathbf{C})}{\mathrm{vol}_{d-k}(P_{H^{\perp}}\mathbf{K})} = \frac{\mathrm{vol}_{d-k}(B)}{\mathrm{vol}_{d-k}(P_{H^{\perp}}\mathbf{K})},$$

where $P_{H^{\perp}} : \mathbb{E}^d \to H^{\perp}$ denotes the orthogonal projection of $\mathbb{E}^d$ onto $H^{\perp}$. Notice that for every invertible affine map $T : \mathbb{E}^d \to \mathbb{E}^d$ one has $\mathrm{crv}_{\mathbf{K}}(\mathbf{C}) = \mathrm{crv}_{T\mathbf{K}}(T\mathbf{C})$. The following theorem is proved in [71].

Theorem 4.2.2 *Let* $\mathbf{K}$ *be a convex body in* $\mathbb{E}^d$*. Let* $\mathbf{C}_1, \ldots, \mathbf{C}_N$ *be* k*-codimensional cylinders in* $\mathbb{E}^d, 0 < k < d$ *such that* $\mathbf{K} \subset \bigcup_{i=1}^N \mathbf{C}_i$*. Then*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(\mathbf{C}_i) \geq \frac{1}{\binom{d}{k}}.$$

Moreover, if $\mathbf{K}$ *is an ellipsoid and* $\mathbf{C}_1, \ldots, \mathbf{C}_N$ *are* 1*-codimensional cylinders in* $\mathbb{E}^d$ *such that* $\mathbf{K} \subset \bigcup_{i=1}^N \mathbf{C}_i$*, then*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(\mathbf{C}_i) \geq 1.$$

The case $k = d-1$ of Theorem 4.2.2 corresponds to Conjecture 4.1.2, that is, to the affine plank problem. Theorem 4.2.2 for $k = d-1$ implies the lower bound $1/d$ that can be somewhat further improved (for more details see [71]).

As an immediate corollary of Theorem 4.2.2 we get the following estimate for Problem 4.2.1.

Corollary 4.2.3 *The sum of the base areas of finitely many (*1*-codimensional) cylinders covering a* 3*-dimensional convex body is always at least one third of the minimum area* 2*-dimensional projection of the body.*

Also, note that the inequality of Theorem 4.2.2 on covering ellipsoids by 1-codimensional cylinders is best possible. By looking at this result from the point of view of k-codimensional cylinders we are led to ask the following quite natural question. Unfortunately, despite its elementary character it is still open.

Problem 4.2.4 *Let* $0 < c(d,k) \leq 1$ *denote the largest real number with the property that if* $\mathbf{K}$ *is an ellipsoid and* $\mathbf{C}_1, \ldots, \mathbf{C}_N$ *are* k*-codimensional cylinders in* $\mathbb{E}^d, 1 \leq k \leq d-1$ *such that* $\mathbf{K} \subset \bigcup_{i=1}^N \mathbf{C}_i$*, then* $\sum_{i=1}^N \mathrm{crv}_{\mathbf{K}}(\mathbf{C}_i) \geq c(d,k)$*. Determine* $c(d,k)$ *for given* d *and* k*.*

On the one hand, Theorems 4.1.1 and 4.2.2 imply that $c(d, d-1) = 1$ and $c(d,1) = 1$; moreover, $c(d,k) \geq \frac{1}{\binom{d}{k}}$. On the other hand, a clever construction due to Kadets [174] shows that if $d - k \geq 3$ is a fixed integer, then $\lim_{d\to\infty} c(d,k) = 0$. Thus the following as a subquestion of Problem 4.2.4 seems to be open as well.

Problem 4.2.5 *Prove or disprove the existence of a universal constant $c > 0$ (independent of d) with the property that if $\mathbf{B}^d$ denotes the unit ball centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$ and $\mathbf{C}_1, \dots, \mathbf{C}_N$ are $(d-2)$-codimensional cylinders in $\mathbb{E}^d$ such that $\mathbf{B}^d \subset \bigcup_{i=1}^N \mathbf{C}_i$, then the sum of the 2-dimensional base areas of $\mathbf{C}_1, \dots, \mathbf{C}_N$ is at least c.*

4.3 Covering Lattice Points by Hyperplanes

In their paper [51], the author and Hausel established the following discrete version of Tarski's plank problem.

Recall that the lattice width of a convex body $\mathbf{K}$ in $\mathbb{E}^d$ is defined as

$$w(\mathbf{K}, \mathbb{Z}^d) = \min\left\{\max_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle - \min_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle \ \mid \ \mathbf{y} \in \mathbb{Z}^d, \ \mathbf{y} \neq \mathbf{o}\right\},$$

where $\mathbb{Z}^d$ denotes the integer lattice of $\mathbb{E}^d$. It is well known that if $\mathbf{y} \in \mathbb{Z}^d$, $\mathbf{y} \neq \mathbf{o}$ is chosen such that $\lambda\mathbf{y} \notin \mathbb{Z}^d$ for any $0 < \lambda < 1$ (i.e., $\mathbf{y}$ is a primitive integer point), then

$$\max_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle - \min_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle$$

is equal to the Euclidean width of $\mathbf{K}$ in the direction $\mathbf{y}$ divided by the Euclidean distance between two consecutive lattice hyperplanes of $\mathbb{Z}^d$ that are orthogonal to $\mathbf{y}$. Thus if $\mathbf{K}$ is the convex hull of finitely many points of $\mathbb{Z}^d$, then

$$\max_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle - \min_{\mathbf{x}\in\mathbf{K}}\langle \mathbf{x}, \mathbf{y}\rangle$$

is an integer namely, it is less by one than the number of lattice hyperplanes of $\mathbb{Z}^d$ that intersect $\mathbf{K}$ and are orthogonal to $\mathbf{y}$. Now, we are ready to state the following conjecture of the author and Hausel ([51]).

Conjecture 4.3.1 *Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$. Let $H_1, \dots, H_N$ be hyperplanes in $\mathbb{E}^d$ such that*

$$\mathbf{K} \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} H_i.$$

Then

$$N \geq w(\mathbf{K}, \mathbb{Z}^d) - d.$$

Properly translated copies of cross-polytopes, described in [51], show that if true, then the above inequality is best possible.

The special case, when $N = 0$, is of independent interest. (In particular, this case seems to be "responsible" for the term d in the inequality of Conjecture 4.3.1.) Namely, it seems reasonable to conjecture (see also [16]) that if $\mathbf{K}$ is an integer point free convex body in $\mathbb{E}^d$, then $w(\mathbf{K}, \mathbb{Z}^d) \leq d$. On the one hand, this has been proved by Banaszczyk [15] for ellipsoids. On the other hand, for general convex bodies containing no integer points, Banaszczyk, Litvak, Pajor, and Szarek [16] have proved the inequality $w(\mathbf{K}, \mathbb{Z}^d) \leq C\, d^{\frac{3}{2}}$, where C is an absolute positive constant. This improves an earlier result of Kannan and Lovász [177].

Although Conjecture 4.3.1 is still open we have the following partial results which were recently published. Improving the estimates of [51], Talata [238] has succeeded in deriving a proof of the following inequality.

Theorem 4.3.2 *Let* $\mathbf{K}$ *be a convex body in* $\mathbb{E}^d$*. Let* $H_1, \ldots, H_N$ *be hyperplanes in* $\mathbb{E}^d$ *such that*

$$\mathbf{K} \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} H_i.$$

Then

$$N \geq c\, \frac{w(\mathbf{K}, \mathbb{Z}^d)}{d} - d,$$

where c *is an absolute positive constant.*

In the paper [71], the author and Litvak have shown that the plank theorem of Ball [12] implies a slight improvement on the above inequality for centrally symmetric convex bodies whose lattice width is at most quadratic in dimension. (Actually, this approach is different from Talata's technique and can lead to a somewhat even stronger inequality in terms of the relevant basic measure of the given convex body. For more details on this we refer the interested reader to [71].)

Theorem 4.3.3 *Let* $\mathbf{K}$ *be a centrally symmetric convex body in* $\mathbb{E}^d$*. Let* $H_1, \ldots, H_N$ *be hyperplanes in* $\mathbb{E}^d$ *such that*

$$\mathbf{K} \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} H_i.$$

Then

$$N \geq c\, \frac{w(\mathbf{K}, \mathbb{Z}^d)}{d \ln(d+1)},$$

where c *is an absolute positive constant.*

Motivated by Conjecture 4.3.1 and by a conjecture of Corzatt [109] (according to which if in the plane the integer points of a convex domain can be covered by N lines, then those integer points can also be covered by N lines having at most four different slopes), Brass, Moser, and Pach [96] have raised the following related question.

Problem 4.3.4 *For every positive integer* d *find the smallest constant* $c(d)$ *such that if the integer points of a convex body in* $\mathbb{E}^d$ *can be covered by* N *hyperplanes, then those integer points can also be covered by* $c(d)N$ *parallel hyperplanes.*

Theorem 4.3.2 implies that $c(d) \leq c\,d^2$ for convex bodies in general and for centrally symmetric convex bodies Theorem 4.3.3 yields the somewhat better upper bound $c\,d\ln(d+1)$. As a last note we mention that the problem of finding good estimates for the constants of Theorems 4.3.2 and 4.3.3 is an interesting open question as well.

4.4 On Some Strengthenings of the Plank Theorems of Ball and Bang

Recall that Ball ([12]) generalized the plank theorem of Bang ([17], [18]) for coverings of balls by planks in Banach spaces (where planks are defined with the help of linear functionals instead of inner product). This theorem was further strengthened by Kadets [175] for real Hilbert spaces as follows. Let $\mathbf{C}$ be a closed convex subset with non-empty interior in the real Hilbert space $\mathbb{H}$ (finite or infinite dimensional). We call $\mathbf{C}$ a *convex body* of $\mathbb{H}$. Then let $\mathrm{r}(\mathbf{C})$ denote the supremum of the radii of the balls contained in $\mathbf{C}$. (One may call $\mathrm{r}(\mathbf{C})$ the *inradius* of $\mathbf{C}$.) Planks and their widths in $\mathbb{H}$ are defined with the help of the inner product of $\mathbb{H}$ in the usual way. Thus, if $\mathbf{C}$ is a convex body in $\mathbb{H}$ and $\mathbf{P}$ is a plank of $\mathbb{H}$, then the width $\mathrm{w}(\mathbf{P})$ of $\mathbf{P}$ is always at least as large as $2\mathrm{r}(\mathbf{C}\cap\mathbf{P})$. Now, the main result of [175] is the following.

Theorem 4.4.1 *Let the ball $\mathbf{B}$ of the real Hilbert space $\mathbb{H}$ be covered by the convex bodies $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_n$ in $\mathbb{H}$. Then*

$$\sum_{i=1}^{n} \mathrm{r}(\mathbf{C}_i \cap \mathbf{B}) \geq \mathrm{r}(\mathbf{B}).$$

We note that an independent proof of the 2-dimensional Euclidean case of Theorem 4.4.1 can be found in [35]. Kadets ([175]) proposes to investigate the analogue of Theorem 4.4.1 in Banach spaces. Thus, an affirmative answer to the following problem would improve the plank theorem of Ball.

Problem 4.4.2 *Let the ball $\mathbf{B}$ be covered by the convex bodies $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_n$ in an arbitrary Banach space. Prove or disprove that*

$$\sum_{i=1}^{n} \mathrm{r}(\mathbf{C}_i \cap \mathbf{B}) \geq \mathrm{r}(\mathbf{B}).$$

In order to complete the picture on plank-type results in spaces other than Euclidean we mention the statement below, proved by Schneider and the author [74]. It is an extension of Theorem 4.4.1 for coverings of large balls in spherical spaces. Recall that $\mathbb{S}^d$ stands for the d-dimensional unit sphere in $(d+1)$-dimensional Euclidean space $\mathbb{E}^{d+1}$, $d \geq 2$. A *spherically convex body* is a closed, spherically convex subset $\mathbf{K}$ of $\mathbb{S}^d$ with interior points and lying in some closed hemisphere, thus, the intersection of $\mathbb{S}^d$ with a $(d+1)$-dimensional closed convex cone of $\mathbb{E}^{d+1}$ different from $\mathbb{E}^{d+1}$. The *inradius* $r(\mathbf{K})$ of $\mathbf{K}$ is the spherical radius of the largest spherical ball contained in $\mathbf{K}$. Also, recall that a *lune* in $\mathbb{S}^d$ is the d-dimensional intersection of $\mathbb{S}^d$ with two closed halfspaces of $\mathbb{E}^{d+1}$ with the origin $\mathbf{o}$ in their boundaries. The intersection of the boundaries (or any $(d-1)$-dimensional subspace in that intersection, if the two subspaces are identical) is called the *ridge* of the lune. Evidently, the inradius of a lune is half the interior angle between the two defining hyperplanes.

Theorem 4.4.3 *If the spherically convex bodies $\mathbf{K}_1, \dots, \mathbf{K}_n$ cover the spherical ball $\mathbf{B}$ of radius $r(\mathbf{B}) \geq \frac{\pi}{2}$ in $\mathbb{S}^d, d \geq 2$, then*

$$\sum_{i=1}^{n} r(\mathbf{K}_i) \geq r(\mathbf{B}).$$

For $r(\mathbf{B}) = \frac{\pi}{2}$ the stronger inequality $\sum_{i=1}^{n} r(\mathbf{K}_i \cap \mathbf{B}) \geq r(\mathbf{B})$ holds. Moreover, equality for $r(\mathbf{B}) = \pi$ or $r(\mathbf{B}) = \frac{\pi}{2}$ holds if and only if $\mathbf{K}_1, \dots, \mathbf{K}_n$ are lunes with common ridge which have pairwise no common interior points.

Theorem 4.4.3 is a consequence of the following result proved by Schneider and the author in [74]. Recall that $\mathrm{Svol}_d(\dots)$ denotes the spherical Lebesgue measure on $\mathbb{S}^d$, and recall that $(d+1)\omega_{d+1} = \mathrm{Svol}_d(\mathbb{S}^d)$.

Theorem 4.4.4 *If $\mathbf{K}$ is a spherically convex body in $\mathbb{S}^d, d \geq 2$, then*

$$\mathrm{Svol}_d(\mathbf{K}) \leq \frac{(d+1)\omega_{d+1}}{\pi} r(\mathbf{K}).$$

Equality holds if and only if $\mathbf{K}$ is a lune.

Indeed, Theorem 4.4.4 implies Theorem 4.4.3 as follows. If $\mathbf{B} = \mathbb{S}^d$; that is, the spherically convex bodies $\mathbf{K}_1, \dots, \mathbf{K}_n$ cover $\mathbb{S}^d$, then

$$(d+1)\omega_{d+1} \leq \sum_{i=1}^{n} \mathrm{Svol}_d(\mathbf{K}_i) \leq \frac{(d+1)\omega_{d+1}}{\pi} \sum_{i=1}^{n} r(\mathbf{K}_i),$$

and the stated inequality follows. In general, when $\mathbf{B}$ is different from $\mathbb{S}^d$, let $\mathbf{B}' \subset \mathbb{S}^d$ be the spherical ball of radius $\pi - r(\mathbf{B})$ centered at the point antipodal to the center of $\mathbf{B}$. As the spherically convex bodies $\mathbf{B}', \mathbf{K}_1, \dots, \mathbf{K}_n$ cover $\mathbb{S}^d$, the inequality just proved shows that

$$\pi - r(\mathbf{B}) + \sum_{i=1}^{n} r(\mathbf{K}_i) \geq \pi,$$

and the stated inequality follows. If $r(\mathbf{B}) = \frac{\pi}{2}$, then $\mathbf{K}_1 \cap \mathbf{B}, \dots, \mathbf{K}_n \cap \mathbf{B}$ are spherically convex bodies and as $\mathbf{B}', \mathbf{K}_1 \cap \mathbf{B}, \dots, \mathbf{K}_n \cap \mathbf{B}$ cover $\mathbb{S}^d$, the stronger inequality follows. The assertion about the equality sign for the case when $r(\mathbf{B}) = \pi$ or $r(\mathbf{B}) = \frac{\pi}{2}$ follows easily.

4.5 On Partial Coverings by Planks: Bang's Theorem Revisited

The following variant of Tarski's plank problem was introduced very recently by the author in [73]: let $\mathbf{C}$ be a convex body of minimal width $w > 0$ in $\mathbb{E}^d$. Moreover, let $w_1 > 0, w_2 > 0, \dots, w_n > 0$ be given with $w_1 + w_2 + \cdots + w_n < w$. Then find the arrangement of n planks say, of $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$, of width $w_1, w_2, \dots, w_n$ in $\mathbb{E}^d$ such that their union covers the largest volume subset of $\mathbf{C}$, that is, for which $\mathrm{vol}_d((\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n) \cap \mathbf{C})$ is as large as possible. As the following special case is the most striking form of the above problem, we are putting it forward as the main question of this section.

Problem 4.5.1 *Let $\mathbf{B}^d$ denote the unit ball centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$. Moreover, let $w_1, w_2, \dots, w_n$ be positive real numbers satisfying the inequality $w_1 + w_2 + \cdots + w_n < 2$. Then prove or disprove that the union of the planks $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ of width $w_1, w_2, \dots, w_n$ in $\mathbb{E}^d$ covers the largest volume subset of $\mathbf{B}^d$ if and only if $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$ is a plank of width $w_1 + w_2 + \cdots + w_n$ with $\mathbf{o}$ as a center of symmetry.*

Clearly, there is an affirmative answer to Problem 4.5.1 for $n = 1$. Also, we note that it would not come as a surprise to us if it turned out that the answer to Problem 4.5.1 is positive in proper low dimensions and negative in (sufficiently) high dimensions. The following partial results have been obtained in [73].

Theorem 4.5.2 *Let $w_1, w_2, \dots, w_n$ be positive real numbers satisfying the inequality $w_1 + w_2 + \cdots + w_n < 2$. Then the union of the planks $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ of width $w_1, w_2, \dots, w_n$ in $\mathbb{E}^3$ covers the largest volume subset of $\mathbf{B}^3$ if and only if $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$ is a plank of width $w_1 + w_2 + \cdots + w_n$ with $\mathbf{o}$ as a center of symmetry.*

Corollary 4.5.3 *If $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ are planks in $\mathbb{E}^d$, $d \geq 3$ of widths w_1, w_2, and w_3 satisfying $0 < w_1 + w_2 + w_3 < 2$, then $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{P}_3$ covers the largest volume subset of $\mathbf{B}^d$ if and only if $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{P}_3$ is a plank of width $w_1 + w_2 + w_3$ having $\mathbf{o}$ as a center of symmetry.*

The following estimate of [73] can be derived from Bang's paper [18]. In order to state it properly we introduce two definitions.

Definition 4.5.4 *Let $\mathbf{C}$ be a convex body in $\mathbb{E}^d$ and let m be a positive integer. Then let $\mathcal{T}^m_{\mathbf{C},d}$ denote the family of all sets in $\mathbb{E}^d$ that can be obtained as the intersection of at most m translates of $\mathbf{C}$ in $\mathbb{E}^d$.*

Definition 4.5.5 *Let $\mathbf{C}$ be a convex body of minimal width $w > 0$ in $\mathbb{E}^d$ and let $0 < x \leq w$ be given. Then for any non-negative integer n let*

$$\mathrm{v}_d(\mathbf{C}, x, n) := \min\{\mathrm{vol}_d(\mathbf{Q}) \mid \mathbf{Q} \in \mathcal{T}^{2^n}_{\mathbf{C},d} \text{ and } w(\mathbf{Q}) \geq x\}.$$

Now, we are ready to state the theorem which although it was not published by Bang in [18], follows from his proof of Tarski's plank conjecture.

Theorem 4.5.6 *Let* $\mathbf{C}$ *be a convex body of minimal width* $w > 0$ *in* $\mathbb{E}^d$*. Moreover, let* $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ *be planks of width* $w_1, w_2, \dots, w_n$ *in* $\mathbb{E}^d$ *with* $w_0 = w_1 + w_2 + \cdots + w_n < w$*. Then*

$$\mathrm{vol}_d(\mathbf{C} \setminus (\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n)) \geq \mathrm{v}_d(\mathbf{C}, w - w_0, n);$$

that is,

$$\mathrm{vol}_d((\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n) \cap \mathbf{C}) \leq \mathrm{vol}_d(\mathbf{C}) - \mathrm{v}_d(\mathbf{C}, w - w_0, n).$$

Clearly, the first inequality above implies (via an indirect argument) that if the planks $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ of width $w_1, w_2, \dots, w_n$ cover the convex body $\mathbf{C}$ in $\mathbb{E}^d$, then $w_1 + w_2 + \cdots + w_n \geq w$. Also, as an additional observation from [73] we mention the following statement, that can be derived from Theorem 4.5.6 in a straightforward way and, on the other hand, represents the only case when the estimate in Theorem 4.5.6 is sharp.

Corollary 4.5.7 *Let* $\mathbf{T}$ *be an arbitrary triangle of minimal width (i.e., of minimal height)* $w > 0$ *in* $\mathbb{E}^2$*. Moreover, let* $w_1, w_2, \dots, w_n$ *be positive real numbers satisfying the inequality* $w_1 + w_2 + \cdots + w_n < w$*. Then the union of the planks* $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ *of width* $w_1, w_2, \dots, w_n$ *in* $\mathbb{E}^2$ *covers the largest area subset of* $\mathbf{T}$ *if* $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$ *is a plank of width* $w_1 + w_2 + \cdots + w_n$ *sitting on the side of* $\mathbf{T}$ *with height* w*.*

It was observed by the author in [73] that there is an implicit connection between problem 4.5.1 and the well-known Blaschke–Lebesgue problem, which is generated by Theorem 4.5.6. The details are as follows.

First, recall that the *Blaschke–Lebesgue problem* is about finding the minimum volume convex body of constant width $w > 0$ in $\mathbb{E}^d$. In particular, the Blaschke–Lebesgue theorem states that among all convex domains of constant width w, the Reuleaux triangle of width w has the smallest area, namely $\frac{1}{2}(\pi - \sqrt{3})w^2$. Blaschke [76] and Lebesgue [188] were the first to show this and the succeeding decades have seen other works published on different proofs of that theorem. For a most recent new proof, and for a survey on the state of the art of different proofs of the Blaschke–Lebesgue theorem, see the elegant paper of Harrell [167]. Here we note that the Blaschke–Lebesgue problem is unsolved in three and more dimensions. Even finding the 3-dimensional set of least volume presents formidable difficulties. On the one hand, Chakerian [101] proved that any convex body of constant width 1 in $\mathbb{E}^3$ has volume at least $\frac{\pi(3\sqrt{6}-7)}{3} = 0.365\dots$. On the other hand, it has been conjectured by Bonnesen and Fenchel [85] that Meissner's 3-dimensional generalizations of the Reuleaux triangle of volume $\pi(\frac{2}{3} - \frac{1}{4}\sqrt{3}\arccos(\frac{1}{3})) = 0.420\dots$ are the only extramal sets in $\mathbb{E}^3$.

For our purposes it is useful to introduce the notation $\mathbf{K}_{BL}^{w,d}$ (resp., $\overline{\mathbf{K}}_{BL}^{w,d}$) for a convex body of constant width w in $\mathbb{E}^d$ having minimum volume (resp., surface volume). One may call $\mathbf{K}_{BL}^{w,d}$ (resp., $\overline{\mathbf{K}}_{BL}^{w,d}$) a Blaschke–Lebesgue-type convex body with respect to volume (resp., surface volume). Note that for $d = 2, 3$ one may choose $\mathbf{K}_{BL}^{w,d} = \overline{\mathbf{K}}_{BL}^{w,d}$, however, this is likely not to happen for $d \geq 4$. (For more details on this see [101].) As an important note we mention that Schramm [227] has proved the inequality

$$\mathrm{vol}_d(\mathbf{K}_{BL}^{w,d}) \geq \left(\sqrt{3 + \frac{2}{d+1}} - 1\right)^d \left(\frac{w}{2}\right)^d \mathrm{vol}_d(\mathbf{B}^d),$$

which gives the best lower bound for all $d > 4$. By observing that the orthogonal projection of a convex body of constant width w in $\mathbb{E}^d$ onto any hyperplane of $\mathbb{E}^d$ is a $(d-1)$-dimensional convex body of constant width w one obtains from the previous inequality of Schramm the following one,

$$\mathrm{svol}_{d-1}(\mathrm{bd}(\overline{\mathbf{K}}_{BL}^{w,d})) \geq d\left(\sqrt{3 + \frac{2}{d}} - 1\right)^{d-1} \left(\frac{w}{2}\right)^{d-1} \mathrm{vol}_d(\mathbf{B}^d).$$

Second, let us recall that if X is a finite (point) set lying in the interior of a unit ball in $\mathbb{E}^d$, then the intersection of the (closed) unit balls of $\mathbb{E}^d$ centered at the points of X is called a ball-polyhedron and it is denoted by $\mathbf{B}[X]$. (For an extensive list of properties of ball-polyhedra see the recent paper [69].) Of course, it also makes sense to introduce $\mathbf{B}[X]$ for sets X that are not finite but in those cases we get sets that are typically not ball-polyhedra.

Now, we are ready to state our theorem.

Theorem 4.5.8 *Let $\mathbf{B}[X] \subset \mathbb{E}^d$ be a ball-polyhedron of minimal width x with $1 \leq x < 2$. Then*

$$\mathrm{vol}_d(\mathbf{B}[X]) \geq \mathrm{vol}_d(\mathbf{K}_{BL}^{2-x,d}) + \mathrm{svol}_{d-1}(\mathrm{bd}(\overline{\mathbf{K}}_{BL}^{2-x,d}))(x-1) + \mathrm{vol}_d(\mathbf{B}^d)(x-1)^d.$$

Thus, Theorem 4.5.6 and Theorem 4.5.8 imply the following immediate estimate.

Corollary 4.5.9 *Let $\mathbf{B}^d$ denote the unit ball centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$, $d \geq 2$. Moreover, let $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ be planks of width $w_1, w_2, \ldots, w_n$ in $\mathbb{E}^d$ with $w_0 = w_1 + w_2 + \cdots + w_n \leq 1$. Then*

$$\mathrm{vol}_d((\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n) \cap \mathbf{B}^d) \leq \mathrm{vol}_d(\mathbf{B}^d) - v_d(\mathbf{B}^d, 2 - w_0, n)$$

$$\leq (1 - (1 - w_0)^d)\mathrm{vol}_d(\mathbf{B}^d) - \mathrm{vol}_d(\mathbf{K}_{BL}^{w_0,d}) - \mathrm{svol}_{d-1}(\mathrm{bd}(\overline{\mathbf{K}}_{BL}^{w_0,d}))(1 - w_0).$$