2

Finite Packings by Translates of Convex Bodies

2.1 Hadwiger Numbers of Convex Bodies

Let $\mathbf{K}$ be a convex body (i.e., a compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. Then the *Hadwiger number* $H(\mathbf{K})$ of $\mathbf{K}$ is the largest number of non-overlapping translates of $\mathbf{K}$ that can all touch $\mathbf{K}$. An elegant observation of Hadwiger [154] is the following.

Theorem 2.1.1 *For every d-dimensional convex body* $\mathbf{K}$*,*

$$H(\mathbf{K}) \leq 3^d - 1,$$

where equality holds if and only if $\mathbf{K}$ *is an affine d-cube.*

On the other hand, in another elegant paper Swinnerton–Dyer [236] proved the following lower bound for Hadwiger numbers of convex bodies.

Theorem 2.1.2 *For every d-dimensional* ($d \geq 2$)*convex body* $\mathbf{K}$*,*

$$d^2 + d \leq H(\mathbf{K}).$$

Actually, finding a better lower bound for Hadwiger numbers of d-dimensional convex bodies is a highly challenging open problem for all $d \geq 4$. (It is not hard to see that the above theorem of Swinnerton–Dyer is sharp for dimensions 2 and 3.) The best lower bound known in dimensions $d \geq 4$ is due to Talata [239], who by applying Dvoretzky's theorem on spherical sections of centrally symmetric convex bodies succeeded in showing the following inequality.

Theorem 2.1.3 *There exists an absolute constant* $c > 0$ *such that*

$$2^{cd} \leq H(\mathbf{K})$$

holds for every positive integer d and for every d-dimensional convex body $\mathbf{K}$*.*

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Now, if we look at convex bodies different from a Euclidean ball in dimensions larger than two, then our understanding of their Hadwiger numbers is very limited. Namely, we know the Hadwiger numbers of the following convex bodies different from a ball. The result for tetrahedra is due to Talata [241] and the rest was proved by Larman and Zong [187].

Theorem 2.1.4 *The Hadwiger numbers of tetrahedra, octahedra, and rhombic dodecahedra are all equal to 18.*

In order to gain some more insight on Hadwiger numbers it is natural to pose the following question.

Problem 2.1.5 *For what integers k with $12 \leq k \leq 26$ does there exist a 3-dimensional convex body with Hadwiger number k? What is the Hadwiger number of a d-dimensional simplex (resp., crosspolytope) for $d \geq 4$?*

2.2 One-Sided Hadwiger Numbers of Convex Bodies

The author and Brass [60] assigned to each convex body $\mathbf{K}$ in $\mathbb{E}^d$ a specific positive integer called the *one-sided Hadwiger number* $h(\mathbf{K})$ as follows: $h(\mathbf{K})$ is the largest number of non-overlapping translates of $\mathbf{K}$ that touch $\mathbf{K}$ and that all lie in a closed supporting halfspace of $\mathbf{K}$. In [60], using the Brunn–Minkowski inequality, the author and Brass proved the following sharp upper bound for the one-sided Hadwiger numbers of convex bodies.

Theorem 2.2.1 *If $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d, d \geq 2$, then*

$$h(\mathbf{K}) \leq 2 \cdot 3^{d-1} - 1.$$

Moreover, equality is attained if and only if $\mathbf{K}$ is a d-dimensional affine cube.

The following is an open problem raised in [60].

Problem 2.2.2 *Find the smallest positive integer $n(d)$ with the property that if $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$, then the maximum number of non-overlapping translates of $\mathbf{K}$ that can touch $\mathbf{K}$ and can lie in an open supporting halfspace of $\mathbf{K}$ is at most $n(d)$.*

The notion of one-sided Hadwiger numbers was introduced to study the (discrete) geometry of the so-called k^+-neighbour packings, which are packings of translates of a given convex body in $\mathbb{E}^d$ with the property that each packing element is touched by at least k others from the packing, where k is a given positive integer. As this area of discrete geometry has a rather large literature we refer the interested reader to [60] for a brief survey on the relevant results. Here, we emphasize the following corollary of the previous theorem proved in [60].

Theorem 2.2.3 *If* $\mathbf{K}$ *is an arbitrary convex body in* $\mathbb{E}^d$*, then any* k^+*-neighbour packing by translates of* $\mathbf{K}$ *with* $k \geq 2 \cdot 3^{d-1}$ *must have a positive density in* $\mathbb{E}^d$*. Moreover, there is a* $(2 \cdot 3^{d-1} - 1)^+$*-neighbour packing by translates of a* d*-dimensional affine cube with density* 0 *in* $\mathbb{E}^d$*.*

2.3 Touching Numbers of Convex Bodies

The *touching number* $t(\mathbf{K})$ of a convex body $\mathbf{K}$ in d-dimensional Euclidean space $\mathbb{E}^d$ is the largest possible number of mutually touching translates of $\mathbf{K}$ lying in $\mathbb{E}^d$. The elegant paper [116] of Danzer and Grünbaum gives a proof of the following fundamental inequality. In fact, this inequality was phrased by Petty [213] as well as by P. Soltan [231] in another equivalent form saying that the cardinality of an equilateral set in any d-dimensional normed space is at most 2^d.

Theorem 2.3.1 *For an arbitrary convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$*,*

$$t(\mathbf{K}) \leq 2^d$$

with equality if and only if $\mathbf{K}$ *is an affine* d*-cube.*

In connection with the above inequality the author and Pach [41] conjecture the following even stronger result.

Conjecture 2.3.2 *For any convex body* $\mathbf{K}$ *in* $\mathbb{E}^d$*,* $d \geq 3$ *the maximum number of pairwise tangent positively homothetic copies of* $\mathbf{K}$ *is not more than* 2^d*.*

Quite surprisingly this problem is still open. In [41] it was noted that $3^d - 1$ is an easy upper bound for the quantity introduced in Conjecture 2.3.2. More recently Naszódi [207] (resp., Naszódi and Lángi [208]) improved this upper bound to 2^{d+1} in the case of a general convex body (resp., to $3 \cdot 2^{d-1}$ in the case of a centrally symmetric convex body).

It is natural to ask for a non-trivial lower bound for $t(\mathbf{K})$. Brass [94], as an application of Dvoretzky's well-known theorem, gave a partial answer for the existence of such a lower bound.

Theorem 2.3.3 *For each* k *there exists a* $d(k)$ *such that for any convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$ *with* $d \geq d(k)$*,*

$$k \leq t(\mathbf{K}).$$

It is remarkable that the natural sounding conjecture of Petty [213] stated next is still open for all $d \geq 4$.

Conjecture 2.3.4 *For each convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$*,* $d \geq 4$*,*

$$d + 1 \leq t(\mathbf{K}).$$

A generalization of the concept of touching numbers was introduced by the author, Naszódi, and Visy [59] as follows. The mth *touching number* (or the mth *Petty number*) $t(m, \mathbf{K})$ of a convex body $\mathbf{K}$ of $\mathbb{E}^d$ is the largest cardinality of (possibly overlapping) translates of $\mathbf{K}$ in $\mathbb{E}^d$ such that among any m translates there are always two touching ones. Note that $t(2, \mathbf{K}) = t(\mathbf{K})$. The following theorem proved by the author, Naszódi, and Visy [59] states some upper bounds for $t(m, \mathbf{K})$.

Theorem 2.3.5 *Let $t(\mathbf{K})$ be an arbitrary convex body in $\mathbb{E}^d$. Then*

$$t(m, \mathbf{K}) \leq \min\left\{4^d(m-1), \binom{2^d + m - 1}{2^d}\right\}$$

holds for all $m \geq 2$, $d \geq 2$. Also, we have the inequalities

$$t(3, \mathbf{K}) \leq 2 \cdot 3^d, \;\; t(m, \mathbf{K}) \leq (m-1)[(m-1)3^d - (m-2)]$$

for all $m \geq 4$, $d \geq 2$. Moreover, if $\mathbf{B^d}$ (resp., $\mathbf{C^d}$) denotes a d-dimensional ball (resp., d-dimensional affine cube) of $\mathbb{E}^d$, then

$$t(2, \mathbf{B^d}) = d + 1, \;\; t(m, \mathbf{B^d}) \leq (m-1)3^d, \;\; t(m, \mathbf{C^d}) = (m-1)2^d$$

hold for all $m \geq 2$, $d \geq 2$.

We cannot resist raising the following question (for more details see [59]).

Problem 2.3.6 *Prove or disprove that if $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$ with $d \geq 2$ and $m > 2$, then*

$$(m-1)(d+1) \leq t(m, \mathbf{K}) \leq (m-1)2^d.$$

2.4 On the Number of Touching Pairs in Finite Packings

Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$. Then the contact graph of an arbitrary finite packing by non-overlapping translates of $\mathbf{K}$ in $\mathbb{E}^d$ is the (simple) graph whose vertices correspond to the packing elements and whose two vertices are connected by an edge if and only if the corresponding two packing elements touch each other. One of the most basic questions on contact graphs is to find out the maximum number of edges that a contact graph of n non-overlapping translates of the given convex body $\mathbf{K}$ can have in $\mathbb{E}^d$. In a very recent paper [95] Brass extended the earlier mentioned result of Harborth [166] to the "unit circular disk packings" of normed planes as follows.

Theorem 2.4.1 *The maximum number of touching pairs in a packing of n translates of a convex domain $\mathbf{K}$ in $\mathbb{E}^2$ is $\lfloor 3n - \sqrt{12n-3} \rfloor$, if $\mathbf{K}$ is not a parallelogram, and $\lfloor 4n - \sqrt{28n-12} \rfloor$, if $\mathbf{K}$ is a parallelogram.*

The main result of this section is an upper bound for the number of touching pairs in an arbitrary finite packing of translates of a convex body, proved by the author in [57]. In order to state the theorem in question in a concise way we need a bit of notation. Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$, $d \geq 3$. Then let $\delta(\mathbf{K})$ denote the density of a densest packing of translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d$, $d \geq 3$. Moreover, let

$$\mathrm{iq}(\mathbf{K}) := \frac{(\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{K}))^d}{(\mathrm{vol}_d(\mathbf{K}))^{d-1}}$$

be the isoperimetric quotient of the convex body $\mathbf{K}$, where $\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{K})$ denotes the $(d-1)$-dimensional surface volume of the boundary $\mathrm{bd}\mathbf{K}$ of $\mathbf{K}$ and $\mathrm{vol}_d(\mathbf{K})$ denotes the d-dimensional volume of $\mathbf{K}$. Moreover, let $\mathbf{B}$ denote the closed d-dimensional ball of radius 1 centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$. Finally, let $\mathbf{K_o} := \frac{1}{2}(\mathbf{K} + (-\mathbf{K}))$ be the normalized (centrally symmetric) difference body assigned to $\mathbf{K}$ with $H(\mathbf{K_o})$ (resp., $h(\mathbf{K_o})$) standing for the Hadwiger number (resp., one-sided Hadwiger number) of $\mathbf{K_o}$.

Theorem 2.4.2 *The number of touching pairs in an arbitrary packing of $n > 1$ translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d$, $d \geq 3$ is at most*

$$\frac{H(\mathbf{K_o})}{2}\,n - \frac{1}{2^d \delta(\mathbf{K_o})^{\frac{d-1}{d}}} \sqrt[d]{\frac{\mathrm{iq}(\mathbf{B})}{\mathrm{iq}(\mathbf{K_o})}}\; n^{\frac{d-1}{d}} - (H(\mathbf{K_o}) - h(\mathbf{K_o}) - 1).$$

In particular, the number of touching pairs in an arbitrary packing of $n > 1$ translates of a convex body in $\mathbb{E}^d$, $d \geq 3$ is at most

$$\frac{3^d - 1}{2}\,n - \frac{\sqrt[d]{\omega_d}}{2^{d+1}}\, n^{\frac{d-1}{d}},$$

where $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of a d-dimensional ball of radius 1 in $\mathbb{E}^d$.

In the proof of Theorem 2.4.2 published by the author [57] the following statement plays an important role that might be of independent interest and so we quote it as follows. For the sake of completeness we wish to point out that Theorem 2.4.3 and Corollary 2.4.4, are actual strengthenings of Theorem 3.1 and Corollary 3.1 of [28] mainly because, in our case the containers of the packings in question are highly non-convex.

Theorem 2.4.3 *Let $\mathbf{K_o}$ be a convex body in $\mathbb{E}^d$, $d \geq 2$ symmetric about the origin $\mathbf{o}$ of $\mathbb{E}^d$ and let $\{\mathbf{c}_1 + \mathbf{K_o}, \mathbf{c}_2 + \mathbf{K_o}, \dots, \mathbf{c}_n + \mathbf{K_o}\}$ be an arbitrary packing of $n > 1$ translates of $\mathbf{K_o}$ in $\mathbb{E}^d$. Then*

$$\frac{n\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{K_o})} \leq \delta(\mathbf{K_o}).$$

We close this section with the following immediate corollary of Theorem 2.4.3.

Corollary 2.4.4 *Let* $\mathcal{P}_n(\mathbf{K_o})$ *be the family of all possible packings of* $n > 1$ *translates of the* $\mathbf{o}$*-symmetric convex body* $\mathbf{K_o}$ *in* $\mathbb{E}^d$, $d \geq 2$. *Moreover, let*

$$\delta(\mathbf{K_o}, n) := \max\left\{ \frac{n\mathrm{vol}_d(\mathbf{K_o})}{\mathrm{vol}_d(\bigcup_{i=1}^n \mathbf{c}_i + 2\mathbf{K_o})} \mid \{\mathbf{c}_1 + \mathbf{K_o}, \dots, \mathbf{c}_n + \mathbf{K_o}\} \in \mathcal{P}_n(\mathbf{K_o}) \right\}.$$

Then

$$\limsup_{n\to\infty} \delta(\mathbf{K_o}, n) = \delta(\mathbf{K_o}).$$