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Selected Proofs on Ball-Polyhedra

12.1 Proof of Theorem 6.2.1

12.1.1 Finite sets that cannot be translated into the interior of a convex body

We start with the following rather natural statement that can be proved easily with the help of Helly's theorem [85].

Lemma 12.1.1 *Let* $\mathbf{F}$ *be a finite set of at least* $d+1$ *points and* $\mathbf{C}$ *be a convex set in* $\mathbb{E}^d, d \geq 2$. *Then* $\mathbf{C}$ *has a translate that covers* $\mathbf{F}$ *if and only if every* $d+1$ *points of* $\mathbf{F}$ *can be covered by a translate of* $\mathbf{C}$.

Proof: For each point $\mathbf{p} \in \mathbf{F}$ let $\mathbf{C_p}$ denote the set of all translation vectors in $\mathbb{E}^d$ with which one can translate $\mathbf{C}$ such that it contains $\mathbf{p}$; that is, let $\mathbf{C_p} := \{\mathbf{t} \in \mathbb{E}^d \mid \mathbf{p} \in \mathbf{t} + \mathbf{C}\}$. Now, it is easy to see that $\mathbf{C_p}$ is a convex set of $\mathbb{E}^d$ for all $\mathbf{p} \in \mathbf{F}$ moreover, $\mathbf{F} \subset \mathbf{t}+\mathbf{C}$ if and only if $\mathbf{t} \in \cap_{\mathbf{p}\in\mathbf{F}}\mathbf{C_p}$. Thus, Helly's theorem [85] applied to the convex sets $\{\mathbf{C_p} \mid \mathbf{p} \in \mathbf{F}\}$ implies that $\mathbf{F} \subset \mathbf{t}+\mathbf{C}$ if and only if $\mathbf{C}_{\mathbf{p}_1} \cap \mathbf{C}_{\mathbf{p}_2} \cap \cdots \cap \mathbf{C}_{\mathbf{p}_{d+1}} \neq \emptyset$ holds for any $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1} \in \mathbf{F}$, i.e. if and only if any $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{d+1} \in \mathbf{F}$ can be covered by a translate of $\mathbf{C}$, finishing the proof of Lemma 12.1.1. $\square$

Also the following statement plays a central role in our investigations. This is a generalization of the analogue 2-dimensional statement proved in [42].

Lemma 12.1.2 *Let* $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ *be a finite set of points and* $\mathbf{C}$ *be a convex body in* $\mathbb{E}^d, d \geq 2$. *Then* $\mathbf{F}$ *cannot be translated into the interior of* $\mathbf{C}$ *if and only if the following two conditions hold. There are closed supporting halfspaces* $H^+_{i_1}, H^+_{i_2}, \ldots, H^+_{i_s}$ *of* $\mathbf{C}$ *assigned to some points of* $\mathbf{F}$ *say, to* $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_s}$ *with* $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ *and a translation vector* $\mathbf{t} \in \mathbb{E}^d$ *such that*
(i) the translated point $\mathbf{t}+\mathbf{f}_{i_j}$ *belongs to the closed halfspace* $H^-_{i_j}$ *for all* $1 \leq j \leq s$, *where the interior of* $H^-_{i_j}$ *is disjoint from the interior of* $H^+_{i_j}$ *and its*

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boundary hyperplane is identical to the boundary hyperplane of $H_{i_j}^+$ *(which is in fact, a supporting hyperplane of* $\mathbf{C}$*);*
(ii) the intersection $\cap_{j=1}^s H_{i_j}^+$ *is nearly bounded, meaning that it lies between two parallel hyperplanes of* $\mathbb{E}^d$*.*

Proof: First, we assume that there are closed supporting halfspaces $H_{i_1}^+$, $H_{i_2}^+, \ldots, H_{i_s}^+$ of $\mathbf{C}$ assigned to some points of $\mathbf{F}$ say, to $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_s}$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and a translation vector $\mathbf{t} \in \mathbb{E}^d$ satisfying (i) as well as (ii). Based on this our goal is to show that $\mathbf{F}$ cannot be translated into the interior of $\mathbf{C}$ or equivalently that $\mathbf{F}$ cannot be covered by a translate of the interior $\mathrm{int}\mathbf{C}$ of $\mathbf{C}$. We prove this in an indirect way: we assume that $\mathbf{F}$ can be covered by a translate of $\mathrm{int}\mathbf{C}$ and look for a contradiction. Indeed, if $\mathbf{F}$ can be covered by a translate of $\mathrm{int}\mathbf{C}$, then $\mathbf{t}+\mathbf{F}$ can be covered by a translate of $\mathrm{int}\mathbf{C}$; that is, there is a translation vector $\mathbf{t}^* \in \mathbb{E}^d$ such that $\mathbf{t}+\mathbf{F} \subset \mathbf{t}^* + \mathrm{int}\mathbf{C}$. In particular, if $\mathbf{F}^* := \{\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_s}\}$, then $\mathbf{t}+\mathbf{F}^* \subset \mathbf{t}^* + \mathrm{int}\mathbf{C}$. Clearly, this implies that $\cap_{j=1}^s H_{i_j}^+ \subset \mathrm{int}\big(\cap_{j=1}^s \mathbf{t}^* + H_{i_j}^+\big)$, a contradiction to (ii).

Second, we assume that $\mathbf{F}$ cannot be translated into the interior of $\mathbf{C}$ and look for closed supporting halfspaces $H_{i_1}^+, H_{i_2}^+, \ldots, H_{i_s}^+$ of $\mathbf{C}$ assigned to some points of $\mathbf{F}$ say, to $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_s}$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and a translation vector $\mathbf{t} \in \mathbb{E}^d$ satisfying (i) as well as (ii). In order to simplify matters let us start to investigate the case when $\mathbf{C}$ is a smooth convex body in $\mathbb{E}^d$, that is, when through each boundary point of $\mathbf{C}$ there exists precisely one supporting hyperplane of $\mathbf{C}$. (Also, without loss of generality we assume that the origin $\mathbf{o}$ of $\mathbb{E}^d$ is an interior point of $\mathbf{C}$.) As $\mathbf{F}$ cannot be translated into $\mathrm{int}\mathbf{C}$ therefore Lemma 12.1.1 implies that there are $m \leq d+1$ points of $\mathbf{F}$ say, $\mathbf{F}_m := \{\mathbf{f}_{j_1}, \mathbf{f}_{j_2}, \ldots, \mathbf{f}_{j_m}\}$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ such that $\mathbf{F}_m$ cannot be translated into $\mathrm{int}\mathbf{C}$. Now, let $\lambda_0 := \inf\{\lambda > 0 \mid \lambda\mathbf{F}_m$ cannot be translated into $\mathrm{int}\mathbf{C}\}$. Clearly, $\lambda_0 \leq 1$ and $\lambda_0\mathbf{F}_m$ cannot be translated into $\mathrm{int}\mathbf{C}$; moreover, as $\lambda_0 = \sup\{\delta > 0 \mid \delta\mathbf{F}_m$ can be translated into $\mathbf{C}\}$, therefore there exists a translation vector $\mathbf{t} \in \mathbb{E}^d$ such that $\mathbf{t}+\lambda_0\mathbf{F}_m \subset \mathbf{C}$. Let $\mathbf{t}+\lambda_0\mathbf{f}_{i_1}, \mathbf{t}+\lambda_0\mathbf{f}_{i_2}, \ldots, \mathbf{t}+\lambda_0\mathbf{f}_{i_s}$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n, 2 \leq s \leq m \leq d+1$ denote the points of $\mathbf{t}+\lambda_0\mathbf{F}_m$ that are boundary points of $\mathbf{C}$ and let $H_{i_1}^+, H_{i_2}^+, \ldots, H_{i_s}^+$ be the corresponding closed supporting halfspaces of $\mathbf{C}$. We claim that $\mathbf{H}^+ := \cap_{k=1}^s H_{i_k}^+$ is nearly bounded. Indeed, if $\mathbf{H}^+$ were not nearly bounded, then there would be a translation vector $\mathbf{t}' \in \mathbb{E}^d$ with $\mathbf{H}^+ \subset \mathbf{t}' + \mathrm{int}\mathbf{H}^+$. As $\mathbf{C}$ is a smooth convex body therefore this would imply the existence of a sufficiently small $\mu > 0$ with the property that $\{\mathbf{t}+\lambda_0\mathbf{f}_{i_1}, \mathbf{t}+\lambda_0\mathbf{f}_{i_2}, \ldots, \mathbf{t}+\lambda_0\mathbf{f}_{i_s}\} \subset \mu\mathbf{t}' + \mathrm{int}\mathbf{C}$, a contradiction. Thus, as $\mathbf{o} \in \mathrm{int}\mathbf{C}$ therefore the points $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_s}$ and the closed supporting halfspaces $H_{i_1}^+, H_{i_2}^+, \ldots, H_{i_s}^+$ and the translation vector $\mathbf{t} \in \mathbb{E}^d$ satisfy (i) as well as (ii). We are left with the case when $\mathbf{C}$ is not necessarily a smooth convex body in $\mathbb{E}^d$. In this case let $\mathbf{C}_N, N = 1, 2, \ldots$ be a sequence of smooth convex bodies lying in $\mathrm{int}\mathbf{C}$ with $\lim_{N\to+\infty} \mathbf{C}_N = \mathbf{C}$. As $\mathbf{F}$ cannot be translated into the interior of $\mathbf{C}_N$ for all $N = 1, 2, \ldots$ therefore applying the method described above to each $\mathbf{C}_N$ and taking proper subsequences if necessary we end up with some points of $\mathbf{F}$ say, $\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \dots, \mathbf{f}_{i_s}$ with $1 \le i_1 < i_2 < \cdots < i_s \le n$ and with s convergent sequences of closed supporting halfspaces $H^+_{N,i_1}, H^+_{N,i_2}, \dots, H^+_{N,i_s}$ of $\mathbf{C}_N$ and a convergent sequence of translation vectors $\mathbf{t}_N$ that satisfy (i) and (ii) for each N. By taking the limits $H^+_{i_1} := \lim_{N\to+\infty} H^+_{N,i_1}, H^+_{i_2} := \lim_{N\to+\infty} H^+_{N,i_2}, \dots, H^+_{i_s} := \lim_{N\to+\infty} H^+_{N,i_s}$, and $\mathbf{t} := \lim_{N\to+\infty} \mathbf{t}_N$ we get the desired nearly bounded family of closed supporting halfspaces of $\mathbf{C}$ and the translation vector $\mathbf{t} \in \mathbb{E}^d$ satisfying (i) as well as (ii). This completes the proof of Lemma 12.1.2. $\square$

12.1.2 From generalized billiard trajectories to shortest ones

Lemma 12.1.3 *Let $\mathbf{C}$ be a convex body in $\mathbb{E}^d, d \ge 2$. If $\mathbf{P}$ is a generalized billiard trajectory in $\mathbf{C}$, then $\mathbf{P}$ cannot be translated into the interior of $\mathbf{C}$.*

Proof: Let $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ be the vertices of $\mathbf{P}$ and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be the points of the unit sphere $\mathbb{S}^{d-1}$ centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$ whose position vectors are parallel to the inner angle bisectors (halflines) of $\mathbf{P}$ at the vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ of $\mathbf{P}$. Moreover, let $H_1^+, H_2^+, \dots, H_n^+$ denote the closed supporting halfspaces of $\mathbf{C}$ whose boundary hyperplanes are perpendicular to the inner angle bisectors of $\mathbf{P}$ at the vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$. Based on Lemma 12.1.2 in order to prove that $\mathbf{P}$ cannot be translated into the interior of $\mathbf{C}$ it is sufficient to show that $\cap_{i=1}^n H_i^+$ is nearly bounded or equivalently that $\mathbf{o} \in \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\})$, where $\mathrm{conv}(.)$ denotes the convex hull of the corresponding set in $\mathbb{E}^d$. It is easy to see that $\mathbf{o} \in \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\})$ if and only if for any hyperplane H of $\mathbb{E}^d$ passing through $\mathbf{o}$ and for any of the two closed halfspaces bounded by H say, for H^+, we have that $H^+ \cap \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}) \neq \emptyset$. Indeed, for a given H^+ let $\mathbf{t} \in \mathbb{E}^d$ be chosen so that $\mathbf{t} + H^+$ is a supporting halfspace of $\mathrm{conv}(\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\})$. Clearly, at least one vertex of $\mathbf{P}$ say, $\mathbf{p}_{i_0}$ must belong to the boundary of $\mathbf{t} + H^+$ and therefore $\mathbf{v}_{i_0} \in H^+ \cap \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\})$, finishing the proof of Lemma 12.1.3. $\square$

For the purpose of the following statement it seems natural to introduce *generalized $(d+1)$-gons* in $\mathbb{E}^d$ as closed polygonal paths (possibly with self-intersections) having at most $d+1$ sides.

Theorem 12.1.4 *Let $\mathbf{C}$ be a convex body in $\mathbb{E}^d, d \ge 2$ and let $\mathcal{F}_{d+1}(\mathbf{C})$ denote the family of all generalized $(d+1)$-gons of $\mathbb{E}^d$ that cannot be translated into the interior of $\mathbf{C}$. Then $\mathcal{F}_{d+1}(\mathbf{C})$ possesses a minimal length member; moreover, the shortest perimeter members of $\mathcal{F}_{d+1}(\mathbf{C})$ are identical (up to translations) with the shortest generalized billiard trajectories of $\mathbf{C}$.*

Proof: If $\mathbf{P}$ is an arbitrary generalized billiard trajectory of the convex body $\mathbf{C}$ in $\mathbb{E}^d$ with vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, then according to Lemma 12.1.3 $\mathbf{P}$ cannot be translated into the interior of $\mathbf{C}$. Thus, by Lemma 12.1.1 $\mathbf{P}$ possesses at most $d+1$ vertices say, $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \dots, \mathbf{p}_{i_{d+1}}$ with $1 \le i_1 \le i_2 \le \cdots \le i_{d+1} \le n$

such that $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \ldots, \mathbf{p}_{i_{d+1}}$ cannot be translated into the interior of $\mathbf{C}$. This implies that by connecting the consecutive points of $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \ldots, \mathbf{p}_{i_{d+1}}$ by line segments according to their cyclic ordering the generalized $(d+1)$-gon $\mathbf{P}_{d+1}$ obtained, has length $l(\mathbf{P}_{d+1})$ at most as large as the length $l(\mathbf{P})$ of $\mathbf{P}$; moreover, $\mathbf{P}_{d+1}$ cannot be covered by a translate of int$\mathbf{C}$ (i.e., $\mathbf{P}_{d+1} \in \mathcal{F}_{d+1}(\mathbf{C})$). Now, by looking at only those members of $\mathcal{F}_{d+1}(\mathbf{C})$ that lie in a d-dimensional ball of sufficiently large radius in $\mathbb{E}^d$ we get via a standard compactness argument and Lemma 12.1.2 that $\mathcal{F}_{d+1}(\mathbf{C})$ possesses a member of minimal length say, $\Delta_{d+1}(\mathbf{C})$. As the inequalities $l(\Delta_{d+1}(\mathbf{C})) \leq l(\mathbf{P}_{d+1}) \leq l(\mathbf{P})$ hold for any generalized billiard trajectory $\mathbf{P}$ of $\mathbf{C}$, therefore in order to finish our proof it is sufficient to show that $\Delta_{d+1}(\mathbf{C})$ is a generalized billiard trajectory of $\mathbf{C}$. Indeed, as $\Delta_{d+1}(\mathbf{C}) \in \mathcal{F}_{d+1}(\mathbf{C})$ therefore $\Delta_{d+1}(\mathbf{C})$ cannot be translated into int$\mathbf{C}$. Thus, the minimality of $\Delta_{d+1}(\mathbf{C})$ and Lemma 12.1.2 imply that if $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m$ denote the vertices of $\Delta_{d+1}(\mathbf{C})$ with $m \leq d+1$, then there are closed supporting halfspaces $H_1^+, H_2^+, \ldots, H_m^+$ of $\mathbf{C}$ whose boundary hyperplanes $H_1, H_2, \ldots, H_m$ pass through the points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m$ (each being a boundary point of $\mathbf{C}$) and have the property that $\cap_{i=1}^m H_i^+$ is nearly bounded in $\mathbb{E}^d$. If the inner angle bisector at a vertex of $\Delta_{d+1}(\mathbf{C})$ say, at $\mathbf{q}_i$ were not perpendicular to H_i, then it is easy to see via Lemma 12.1.2 that one could slightly move $\mathbf{q}_i$ along H_i to a new position $\mathbf{q}_i'$ (which is typically an exterior point of $\mathbf{C}$ on H_i) such that the new generalized $(d+1)$-gon $\Delta'_{d+1}(\mathbf{C}) \in \mathcal{F}_{d+1}(\mathbf{C})$ would have a shorter length, a contradiction. This completes the proof of Lemma 12.1.4. $\square$

Finally, notice that Theorem 6.2.1 follows from Theorem 12.1.4 in a straightforward way.

12.2 Proofs of Theorems 6.6.1, 6.6.3, and 6.6.4

12.2.1 Strict separation by spheres of radii at most one

For the proof of Theorem 6.6.1 we need the following weaker version of it due to Houle [169] as well as the following lemma proved in [69].

Theorem 12.2.1 *Let $A, B \subset \mathbb{E}^d$ be finite sets. Then A and B can be strictly separated by a sphere $S^{d-1}(\mathbf{c}, r)$ such that $A \subset \mathbf{B}^d(\mathbf{c}, r)$ if and only if for every $T \subset A \cup B$ with* card$T \leq d+2$, *$T \cap A$ and $T \cap B$ can be strictly separated by a sphere $S^{d-1}(\mathbf{c}_T, r_T)$ such that $T \cap A \subset \mathbf{B}^d(\mathbf{c}_T, r_T)$.*

Lemma 12.2.2 *Let $A, B \subset \mathbb{E}^d$ be finite sets and suppose that $S^{d-1}(\mathbf{o}, 1)$ is the smallest sphere that separates A from B such that $A \subset \mathbf{B}^d[\mathbf{o}, 1]$. Then there is a set $T \subset A \cup B$ with* card$T \leq d+1$ *such that $S^{d-1}(\mathbf{o}, 1)$ is the smallest sphere $S^{d-1}(\mathbf{c}, r)$ that separates $T \cap A$ from $T \cap B$ and satisfies $T \cap A \subset \mathbf{B}^d[\mathbf{c}, r]$.*

We prove the "if" part of Theorem 6.6.1; the opposite direction is trivial. Theorem 12.2.1 guarantees the existence of the smallest sphere $S^{d-1}(\mathbf{c}', r')$

that separates A and B such that $A \subset \mathbf{B}^d[\mathbf{c}', r']$. According to Lemma 12.2.2, there is a set $T \subset A \cup B$ with card$T \leq d+1$ such that $S^{d-1}(\mathbf{c}', r')$ is the smallest sphere that separates $T \cap A$ from $T \cap B$ and whose convex hull contains $T \cap A$. By the assumption, we have $r' < r_T \leq 1$. Note that Theorem 12.2.1 guarantees the existence of a sphere $S^{d-1}(\mathbf{c}^*, r^*)$ that strictly separates A from B and satisfies $A \subset \mathbf{B}^d(\mathbf{c}^*, r^*)$. Because $r' < 1$, there is a sphere $S^{d-1}(\mathbf{c}, r)$ with $r \leq 1$ such that $\mathbf{B}^d[\mathbf{c}', r'] \cap \mathbf{B}^d(\mathbf{c}^*, r^*) \subset \mathbf{B}^d(\mathbf{c}, r) \subset \mathbb{E}^d \setminus \left(\mathbf{B}^d(\mathbf{c}', r') \cup \mathbf{B}^d[\mathbf{c}^*, r^*]\right)$. This sphere clearly satisfies the conditions in Theorem 6.6.1 and so, the proof of Theorem 6.6.1 is complete.

12.2.2 Characterizing spindle convex sets

Our proof of Theorem 6.6.3 is based on the following statement.

Lemma 12.2.3 *Let a spindle convex set $\mathbf{C} \subset \mathbb{E}^d$ be supported by the hyperplane H in $\mathbb{E}^d$ at $\mathbf{x} \in \mathrm{bd}\mathbf{C}$. Then the closed unit ball supported by H at $\mathbf{x}$ and lying in the same side as $\mathbf{C}$ contains $\mathbf{C}$.*

Proof: Let $\mathbf{B}^d[\mathbf{c}, 1]$ be the closed unit ball that is supported by H at $\mathbf{x}$ and is in the same closed half-space bounded by H as $\mathbf{C}$. We show that $\mathbf{B}^d[\mathbf{c}, 1]$ is the desired unit ball.

Assume that $\mathbf{C}$ is not contained in $\mathbf{B}^d[\mathbf{c}, 1]$. So, there is a point $\mathbf{y} \in \mathbf{C}$, $y \notin \mathbf{B}^d[\mathbf{c}, 1]$. Then, by taking the intersection of the configuration with the plane that contains $\mathbf{x}, \mathbf{y}$, and $\mathbf{c}$, we see that there is a shorter unit circular arc connecting $\mathbf{x}$ and $\mathbf{y}$ that does not intersect $\mathbf{B}^d(\mathbf{c}, 1)$. Hence, H cannot be a supporting hyperplane of $\mathbf{C}$ at $\mathbf{x}$, a contradiction. $\square$

Indeed, it is easy to see that Lemma 12.2.3 implies Theorem 6.6.3 in a rather straightforward way.

12.2.3 Separating spindle convex sets

Finally, we prove Theorem 6.6.4 as follows. Since $\mathbf{C}$ and $\mathbf{D}$ are spindle convex, they are convex bounded sets with disjoint relative interiors. So, their closures are convex compact sets with disjoint relative interiors. Hence, they can be separated by a hyperplane H that supports $\mathbf{C}$ at a point, say $\mathbf{x}$. The closed unit ball $\mathbf{B}^d[\mathbf{c}, 1]$ of Lemma 12.2.3 satisfies the conditions of the first statement of Theorem 6.6.4. For the second statement of Theorem 6.6.4, we assume that $\mathbf{C}$ and $\mathbf{D}$ have disjoint closures, so $\mathbf{B}^d[\mathbf{c}, 1]$ is disjoint from the closure of $\mathbf{D}$ and remains so even after a sufficiently small translation. Furthermore, $\mathbf{C}$ is a spindle convex set that is different from a unit ball, so $\mathbf{c} \notin \mathrm{conv}(\mathbf{C} \cap S^{d-1}(\mathbf{c}, 1))$. Hence, there is a sufficiently small translation of $\mathbf{B}^d[\mathbf{c}, 1]$ that satisfies the second statement of Theorem 6.6.4, finishing the proof of Theorem 6.6.4.

12.3 Proof of Theorem 6.7.1

12.3.1 On the boundary of spindle convex hulls in terms of supporting spheres

Let $S^k(\mathbf{c}, r) \subset \mathbb{E}^d$ be a k-dimensional sphere centered at $\mathbf{c}$ and having radius r with $0 \leq k \leq d-1$. Recall the following strong version of spherical convexity. A set $F \subset S^k(\mathbf{c}, r)$ is *spherically convex* if it is contained in an open hemisphere of $S^k(\mathbf{c}, r)$ and for every $\mathbf{x}, \mathbf{y} \in F$ the shorter great-circular arc of $S^k(\mathbf{c}, r)$ connecting $\mathbf{x}$ with $\mathbf{y}$ is in F. The *spherical convex hull* of a set $X \subset S^k(\mathbf{c}, r)$ is defined in the natural way and it exists if and only if X is in an open hemisphere of $S^k(\mathbf{c}, r)$. We denote it by $\mathrm{Sconv}(X, S^k(\mathbf{c}, r))$. Carathéodory's theorem can be stated for the sphere in the following way. If $X \subset S^k(\mathbf{c}, r)$ is a set in an open hemisphere of $S^k(\mathbf{c}, r)$, then $\mathrm{Sconv}(X, S^k(\mathbf{c}, r))$ is the union of spherical simplices with vertices in X. The proof of this spherical equivalent of the original Carathéodory's theorem uses the central projection of the open hemisphere of $S^k(\mathbf{c}, r)$ to $\mathbb{E}^k$.

Recall that the circumradius $\mathrm{cr}(X)$ of a bounded set $X \subset \mathbb{E}^d$ is defined as the radius of the unique smallest d-dimensional closed ball that contains X (also known as the circumball of X). Now, it is easy to see that if $C \subset \mathbb{E}^d$ is a spindle convex set such that $C \subset \mathbf{B}^d[\mathbf{q}, 1]$ and $\mathrm{cr}(C) < 1$, then $C \cap S^{d-1}(\mathbf{q}, 1)$ is spherically convex on $S^{d-1}(\mathbf{q}, 1)$.

The following lemma describes the surface of a spindle convex hull.

Lemma 12.3.1 *Let $X \subset \mathbb{E}^d$ be a closed set such that $\mathrm{cr}(X) < 1$ and let $\mathbf{B}^d[\mathbf{q}, 1]$ be a closed unit ball containing X. Then*
(i) $X \cap S^{d-1}(\mathbf{q}, 1)$ is contained in an open hemisphere of $S^{d-1}(\mathbf{q}, 1)$,
(ii) $\mathrm{conv}_s(X) \cap S^{d-1}(\mathbf{q}, 1) = \mathrm{Sconv}(X \cap S^{d-1}(\mathbf{q}, 1), S^{d-1}(\mathbf{q}, 1))$.

Proof: Because $\mathrm{cr}(X) < 1$, we obtain that X is contained in the intersection of two distinct closed unit balls which proves (i). Note that by (i), the right-hand side $Z := \mathrm{Sconv}(X \cap S^{d-1}(\mathbf{q}, 1), S^{d-1}(\mathbf{q}, 1))$ of (ii) exists. We show that the set on the left-hand side is contained in Z; the other containment follows from the discussion right before Lemma 12.3.1.

Suppose that $\mathbf{y} \in \mathrm{conv}_s(X) \cap S^{d-1}(\mathbf{q}, 1)$ is not contained in Z. We show that there is a hyperplane H through $\mathbf{q}$ that strictly separates Z from $\mathbf{y}$. Consider an open hemisphere of $S^{d-1}(\mathbf{q}, 1)$ that contains Z, call the spherical center of this hemisphere $\mathbf{p}$. If $\mathbf{y}$ is an exterior point of the hemisphere, H exists. If $\mathbf{y}$ is on the boundary of the hemisphere, then, by moving the hemisphere a little, we find another open hemisphere that contains Z, but with respect to which $\mathbf{y}$ is an exterior point.

Assume that $\mathbf{y}$ is contained in the open hemisphere. Let L be a hyperplane tangent to $S^{d-1}(\mathbf{q}, 1)$ at p. We project Z and $\mathbf{y}$ centrally from $\mathbf{q}$ onto L and, by the separation theorem of convex sets in L, we obtain a $(d-2)$-dimensional affine subspace T of L that strictly separates the image of Z from the image of $\mathbf{y}$. Then $H := \mathrm{aff}(T \cup \{\mathbf{q}\})$ is the desired hyperplane.

Hence, $\mathbf{y}$ is contained in one open hemisphere of $S^{d-1}(\mathbf{q},1)$ and Z is in the other. Let $\mathbf{v}$ be the unit normal vector of H pointing towards the hemisphere of $S^{d-1}(\mathbf{q},1)$ that contains Z. Since X is closed, its distance from the closed hemisphere containing $\mathbf{y}$ is positive. Hence, we can move $\mathbf{q}$ a little in the direction $\mathbf{v}$ to obtain the point $\mathbf{q}'$ such that $X \subset \mathbf{B}^d[\mathbf{q},1] \cap \mathbf{B}^d[\mathbf{q}',1]$ and $\mathbf{y} \notin \mathbf{B}^d[\mathbf{q}',1]$. As $\mathbf{B}^d[\mathbf{q}',1]$ separates X from $\mathbf{y}$, the latter is not in $\mathrm{conv}_s(X)$, a contradiction. $\square$

12.3.2 From the spherical Carathéodory theorem to an analogue for spindle convex hulls

Now, we prove Theorem 6.7.1.

Assume that $\mathrm{cr}(X) > 1$. Recall that the intersection of the d-dimensional closed unit balls of $\mathbb{E}^d$ centered at the points of X is denoted by $\mathbf{B}[X]$. Then $\mathbf{B}[X] = \emptyset$; hence, by Helly's theorem, there is a set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_d\} \subset X$ such that $\mathbf{B}[\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_d\}] = \emptyset$. It follows that $\mathrm{conv}_s(\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_d\}) = \mathbb{E}^d$. Thus, (i) and (ii) follow.

Now, we prove (i) for $\mathrm{cr}(X) < 1$. By the spherical Carathéodory theorem, Lemma 12.2.3, and Lemma 12.3.1 we obtain that

$$\mathbf{y} \in \mathrm{Sconv}(\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}, S^{d-1}(\mathbf{q},1))$$

for some $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\} \subset X$ and some $\mathbf{q} \in \mathbb{E}^d$ such that $X \subset \mathbf{B}^d[\mathbf{q},1]$. Hence, $\mathbf{y} \in \mathrm{conv}_s\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$.

We prove (i) for $\mathrm{cr}(X) = 1$ by a limit argument as follows. Without loss of generality, we may assume that $X \subset \mathbf{B}^d[\mathbf{o},1]$. Let $X^k := (1 - \frac{1}{k})X$ for any $k \in \mathbb{Z}^+$. Let $\mathbf{y}^k$ be the point of $\mathrm{bd}\left(\mathrm{conv}_s(X^k)\right)$ closest to $\mathbf{y}$. Thus, $\lim\limits_{k\to\infty} y^k = y$. Clearly, $\mathrm{cr}(X^k) < 1$, hence there is a set $\{\mathbf{x}_1^k, \mathbf{x}_2^k, \dots, \mathbf{x}_d^k\} \subset X^k$ such that $\mathbf{y}^k \in \mathrm{conv}_s\{\mathbf{x}_1^k, \mathbf{x}_2^k, \dots, \mathbf{x}_d^k\}$. By compactness, there is a sequence $0 < i_1 < i_2 < \dots$ of indices such that all the d sequences $\{\mathbf{x}_1^{i_j} : j \in \mathbb{Z}^+\}, \{\mathbf{x}_2^{i_j} : j \in \mathbb{Z}^+\}, \dots, \{\mathbf{x}_d^{i_j} : j \in \mathbb{Z}^+\}$ converge. Let their respective limits be $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d$. Since X is closed, these d points are contained in X. Clearly, $\mathbf{y} \in \mathrm{conv}_s\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$.

To prove (ii) for $\mathrm{cr}(X) \le 1$, suppose that $\mathbf{y} \in \mathrm{int}\,(\mathrm{conv}_s X)$. Then let $\mathbf{x}_0 \in X \cap \mathrm{bd}\,(\mathrm{conv}_s X)$ be arbitrary and let $\mathbf{y}_1$ be the intersection of $\mathrm{bd}\,(\mathrm{conv}_s X)$ with the ray starting from $\mathbf{x}_0$ and passing through $\mathbf{y}$. Now, by (i), $\mathbf{y}_1 \in \mathrm{conv}_s\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$ for some $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\} \subset X$. Then clearly $\mathbf{y} \in \mathrm{int}\,(\mathrm{conv}_s\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_d\})$.

12.4 Proof of Theorem 6.8.3

12.4.1 On the boundary of spindle convex hulls in terms of normal images

Let $X \subset \mathbb{E}^d, d \geq 3$ be a compact set of Euclidean diameter $\mathrm{diam}(X) \leq 1$. Recall that $\mathbf{B}[X] \subset \mathbb{E}^d$ denotes the convex body which is the intersection of the closed unit balls of $\mathbb{E}^d$ centered at the points of X. For the following investigations it is more proper to use the normal images than the Gauss images of the boundary points of $\mathbf{B}[X]$ defined as follows. The *normal image* $N_{\mathbf{B}[X]}(\mathbf{b})$ of the boundary point $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$ of $\mathbf{B}[X]$ is

$$N_{\mathbf{B}[X]}(\mathbf{b}) := -\nu(\{\mathbf{b}\})$$

In other words, $N_{\mathbf{B}[X]}(\mathbf{b}) \subset \mathbb{S}^{d-1}$ is the set of inward unit normal vectors of all hyperplanes that support $\mathbf{B}[X]$ at $\mathbf{b}$. Clearly, $N_{\mathbf{B}[X]}(\mathbf{b})$ is a closed spherically convex subset of $\mathbb{S}^{d-1}$. (Here we refer to the strong version of spherical convexity introduced for Lemma 12.3.1.)

We need to introduce the following notation as follows. For a set $A \subset \mathbb{S}^{d-1}$ let $A^+ = \{\mathbf{x} \in \mathbb{S}^{d-1} \mid \langle \mathbf{x}, \mathbf{y} \rangle > 0 \text{ for all } \mathbf{y} \in A\}$. (Here $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ refer to the canonical Euclidean norm and the canonical inner product on $\mathbb{E}^d$.)

As is well known, illumination can be reformulated as follows: *The direction* $\mathbf{u} \in \mathbb{S}^{d-1}$ *illuminates the boundary point* $\mathbf{b}$ *of the convex body* $\mathbf{B}[X]$ *if and only if* $\mathbf{u} \in N_{\mathbf{B}[X]}(\mathbf{b})^+$. (Because the proof of this claim is straightforward we leave it to the reader. For more insight on illumination we refer the interested reader to [47] and the relevant references listed there.)

Finally, we need to recall some further notations as well. Let $\mathbf{a}$ and $\mathbf{b}$ be two points in $\mathbb{E}^d$. If $\|\mathbf{a}-\mathbf{b}\| < 2$, then the *(closed) spindle* of $\mathbf{a}$ and $\mathbf{b}$, denoted by $[\mathbf{a}, \mathbf{b}]_s$, is defined as the union of circular arcs with endpoints $\mathbf{a}$ and $\mathbf{b}$ that are of radii at least one and are shorter than a semicircle. If $\|\mathbf{a}-\mathbf{b}\| = 2$, then $[\mathbf{a}, \mathbf{b}]_s := \mathbf{B}^d[\frac{\mathbf{a}+\mathbf{b}}{\mathbf{2}}, 1]$, where $\mathbf{B}^d[\mathbf{p}, r]$ denotes the (closed) d-dimensional ball centered at $\mathbf{p}$ with radius r in $\mathbb{E}^d$. If $\|\mathbf{a}-\mathbf{b}\| > 2$, then we define $[\mathbf{a}, \mathbf{b}]_s$ to be $\mathbb{E}^d$. Next, a set $\mathbf{C} \subset \mathbb{E}^d$ is called *spindle convex* if, for any pair of points $\mathbf{a}, \mathbf{b} \in \mathbf{C}$, we have that $[\mathbf{a}, \mathbf{b}]_s \subset \mathbf{C}$. Finally, let X be a set in $\mathbb{E}^d$. Then the *spindle convex hull* of X is the set defined by $\mathrm{conv}_s X := \bigcap\{C \subset \mathbb{E}^d | X \subset C \text{ and } C \text{ is spindle convex in } \mathbb{E}^d\}$.

Now, we are ready to state Lemma 12.4.1, which is the core part of this section and whose proof is based on Lemma 12.3.1.

Lemma 12.4.1 *Let* $X \subset \mathbb{E}^d, d \geq 3$ *be a compact set of Euclidean diameter* $\mathrm{diam}(X) \leq 1$. *Then the boundary of the spindle convex hull of* X *can be generated as follows:*

$$\mathrm{bd}\,(\mathrm{conv}_s(X)) = \bigcup_{\mathbf{b} \in \mathrm{bd}(\mathbf{B}[X])} \{\mathbf{b} + \mathbf{y} \mid \mathbf{y} \in N_{\mathbf{B}[X]}(\mathbf{b})\}.$$

Proof: Let $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$. Then (ii) of Lemma 12.3.1 implies that

$$\mathbf{b} + N_{\mathbf{B}[X]}(\mathbf{b}) = \mathrm{Sconv}(X \cap S^{d-1}(\mathbf{b},1), S^{d-1}(\mathbf{b},1)) = \mathrm{conv}_s(X) \cap S^{d-1}(\mathbf{b},1).$$

This together with the fact that

$$\bigcup_{\mathbf{b}\in \mathrm{bd}(\mathbf{B}[X])} N_{\mathbf{B}[X]}(\mathbf{b}) = \mathbb{S}^{d-1}$$

finishes the proof of Lemma 12.4.1. □

12.4.2 On the Euclidean diameter of spindle convex hulls and normal images

Lemma 12.4.2

$$\mathrm{diam}\,(\mathrm{conv}_s(X)) \le 1.$$

Proof: By assumption $\mathrm{diam}(X) \le 1$. Recall that Meissner [196] has called a compact set $M \subset \mathbb{E}^d$ *complete* if $\mathrm{diam}(M \cup \{\mathbf{p}\}) > \mathrm{diam}(M)$ for any $\mathbf{p} \in \mathbb{E}^d \setminus M$. He has proved in [196] that any set of diameter 1 is contained in a complete set of diameter 1. Moreover, he has shown in [196] that a compact set of diameter 1 in $\mathbb{E}^d$ is complete if and only if it is of constant width 1. These facts together with the easy observation that any convex body of constant width 1 in $\mathbb{E}^d$ is in fact a spindle convex set, imply that X is contained in a convex body of convex width 1 and any such convex body must necessarily contain $\mathrm{conv}_s(X)$. Thus, indeed $\mathrm{diam}\,(\mathrm{conv}_s(X)) \le 1$. □

For an arbitrary nonempty subset A of $\mathbb{S}^{d-1}$ let

$$U_{\mathbf{B}[X]}(A) = \left(\bigcup_{N_{\mathbf{B}[X]}(\mathbf{b})\cap A \neq \emptyset} N_{\mathbf{B}[X]}(\mathbf{b}) \right) \subset \mathbb{S}^{d-1}.$$

Lemma 12.4.3 *Let $\emptyset \neq A \subset \mathbb{S}^{d-1}$ be given. Then*

$$\mathrm{diam}\left(U_{\mathbf{B}[X]}(A)\right) \le 1 + \mathrm{diam}(A).$$

Proof: Let $\mathbf{y}_1 \in N_{\mathbf{B}[X]}(\mathbf{b}_1)$ and $\mathbf{y}_2 \in N_{\mathbf{B}[X]}(\mathbf{b}_2)$ be two arbitrary points of $U_{\mathbf{B}[X]}(A)$ with $\mathbf{b}_1, \mathbf{b}_2 \in \mathrm{bd}\,(\mathbf{B}[X])$. We need to show that $\|\mathbf{y}_1 - \mathbf{y}_2\| \le 1 + \mathrm{diam}(A)$.

By Lemma 12.4.1 and by Lemma 12.4.2 we get that

$$\|(\mathbf{y}_1 - \mathbf{y}_2) + (\mathbf{b}_1 - \mathbf{b}_2)\| = \|(\mathbf{b}_1 + \mathbf{y}_1) - (\mathbf{b}_2 + \mathbf{y}_2)\| \le 1.$$

Thus, the triangle inequality yields that

$$\|(\mathbf{y}_1 - \mathbf{y}_2)\| \le 1 + \|(\mathbf{b}_2 - \mathbf{b}_1)\|.$$

This means that in order to finish the proof of Lemma 12.4.3 it is sufficient to show that $\|(\mathbf{b}_2 - \mathbf{b}_1)\| \leq \operatorname{diam}(A)$. This can be obtained easily from the assumption that $N_{\mathbf{B}[X]}(\mathbf{b}_1) \cap A \neq \emptyset$, $N_{\mathbf{B}[X]}(\mathbf{b}_2) \cap A \neq \emptyset$ and from the fact that the sets $\mathbf{b}_1 + N_{\mathbf{B}[X]}(\mathbf{b}_1) \subset \operatorname{bd}(\operatorname{conv}_s(X))$ and $\mathbf{b}_2 + N_{\mathbf{B}[X]}(\mathbf{b}_2) \subset \operatorname{bd}(\operatorname{conv}_s(X))$ are separated by the hyperplane H of $\mathbb{E}^d$ that bisects the line segment connecting $\mathbf{b}_1$ to $\mathbf{b}_2$ and is perpendicular to it with $\mathbf{b}_1 + N_{\mathbf{B}[X]}(\mathbf{b}_1)$ (resp., $\mathbf{b}_2 + N_{\mathbf{B}[X]}(\mathbf{b}_2)$) lying on the same side of H as $\mathbf{b}_2$ (resp., $\mathbf{b}_1$). □

12.4.3 An upper bound for the illumination number based on a probabilistic approach

Let μ_{d-1} denote the standard probability measure on $\mathbb{S}^{d-1}$ and define

$$V_{d-1}(t) := \inf\{\mu_{d-1}(A^+) \mid A \subset \mathbb{S}^{d-1}, \operatorname{diam}(A) \leq t\},$$

where just as before $A^+ = \{\mathbf{x} \in \mathbb{S}^{d-1} \mid \langle \mathbf{x}, \mathbf{y} \rangle > 0 \text{ for all } \mathbf{y} \in A\}$. Moreover, let $n_{d-1}(\epsilon)$ denote the minimum number of closed spherical caps of $\mathbb{S}^{d-1}$ having Euclidean diameter ϵ such that they cover $\mathbb{S}^{d-1}$, where $0 < \epsilon \leq 2$.

Lemma 12.4.4

$$I(\mathbf{B}[X]) \leq 1 + \frac{\ln(n_{d-1}(\epsilon))}{-\ln(1 - V_{d-1}(1+\epsilon))}$$

holds for all $0 < \epsilon \leq \sqrt{2} - 1$ *and* $d \geq 3$.

Proof: Let $\emptyset \neq A \subset \mathbb{S}^{d-1}$ be given with Euclidean diameter $\operatorname{diam}(A) \leq 1 + \epsilon \leq \sqrt{2}$. Then the spherical Jung theorem [119] implies that A is contained in a closed spherical cap of $\mathbb{S}^{d-1}$ having angular radius $0 < \arcsin\sqrt{\frac{d-1}{d}} < \frac{\pi}{2}$. Thus, A^+ contains a spherical cap of $\mathbb{S}^{d-1}$ having angular radius $\frac{\pi}{2} - \arcsin\sqrt{\frac{d-1}{d}} > 0$ and of course, A^+ is contained in an open hemisphere of $\mathbb{S}^{d-1}$. Hence, $0 < V_{d-1}(1+\epsilon) < \frac{1}{2}$ and so, the expression on the right in Lemma 12.4.4 is well defined.

Let m be a positive integer satisfying

$$m > \frac{\ln(n_{d-1}(\epsilon))}{-\ln(1 - V_{d-1}(1+\epsilon))}.$$

It is sufficient to show that m directions can illuminate $\mathbf{B}[X]$. Let $n = n_{d-1}(\epsilon)$ and let $A_1, A_2, \ldots, A_n$ be closed spherical caps of $\mathbb{S}^{d-1}$ having Euclidean diameter ϵ and covering $\mathbb{S}^{d-1}$. By Lemma 12.4.3 we have $\operatorname{diam}\left(U_{\mathbf{B}[X]}(A_i)\right) \leq 1 + \epsilon$ for all $1 \leq i \leq n$ and therefore

$$\mu_{d-1}\left(U_{\mathbf{B}[X]}(A_i)^+\right) \geq V_{d-1}(1+\epsilon)$$

for all $1 \leq i \leq n$. Let the directions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be chosen at random, uniformly and independently distributed on $\mathbb{S}^{d-1}$. Thus, the probability that

$\mathbf{u}_j$ lies in $U_{\mathbf{B}[X]}(A_i)^+$ is equal to $\mu_{d-1}\left(U_{\mathbf{B}[X]}(A_i)^+\right) \geq V_{d-1}(1+\epsilon)$. Therefore the probabilty that $U_{\mathbf{B}[X]}(A_i)^+$ contains none of the points $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is at most $(1 - V_{d-1}(1+\epsilon))^m$. Hence, the probability p that at least one $U_{\mathbf{B}[X]}(A_i)^+$ will contain none of the points $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ satisfies

$$p \leq \sum_{i=1}^{n} (1 - V_{d-1}(1+\epsilon))^m < n\,(1 - V_{d-1}(1+\epsilon))^{\frac{\ln(n)}{-\ln\left(1-V_{d-1}(1+\epsilon)\right)}} = 1.$$

This shows that one can choose m directions say, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset \mathbb{S}^{d-1}$ such that each set $U_{\mathbf{B}[X]}(A_i)^+, 1 \leq i \leq n$ contains at least one of them. We claim that the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ illuminate $\mathbf{B}[X]$. Indeed, let $\mathbf{b} \in \mathrm{bd}\,(\mathbf{B}[X])$. We show that at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ illuminates the boundary point $\mathbf{b}$. As the spherical caps $A_1, A_2, \ldots, A_n$ form a covering of $\mathbb{S}^{d-1}$ therefore there exists an A_i with $A_i \cap N_{\mathbf{B}[X]}(\mathbf{b}) \neq \emptyset$. Thus, by definition $N_{\mathbf{B}[X]}(\mathbf{b}) \subset U_{\mathbf{B}[X]}(A_i)$ and therefore

$$N_{\mathbf{B}[X]}(\mathbf{b})^+ \supset U_{\mathbf{B}[X]}(A_i)^+.$$

$U_{\mathbf{B}[X]}(A_i)^+$ contains at least one of the directions $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, say $\mathbf{v}_k$. Hence,

$$\mathbf{v}_k \in U_{\mathbf{B}[X]}(A_i)^+ \subset N_{\mathbf{B}[X]}(\mathbf{b})^+$$

and so, $\mathbf{v}_k$ illuminates the boundary point $\mathbf{b}$ of $\mathbf{B}[X]$, finishing the proof of Lemma 12.4.4. □

12.4.4 Schramm's lower bound for the proper measure of polars of sets of given diameter in spherical space

We need the following notation for the next statement. For $\mathbf{u} \in \mathbb{S}^{d-1}$ let $R_{\mathbf{u}} : \mathbb{E}^d \to \mathbb{E}^d$ denote the reflection about the line passing through the points $\mathbf{u}$ and $-\mathbf{u}$. Clearly, $R_{\mathbf{u}}(\mathbf{x}) = 2\langle \mathbf{x}, \mathbf{u}\rangle \mathbf{u} - \mathbf{x}$ for all $\mathbf{x} \in \mathbb{E}^d$.

Lemma 12.4.5 *Let $A \subset \mathbb{S}^{d-1}$ be a set of Euclidean diameter $0 < \mathrm{diam}(A) \leq t$ contained in the closed spherical cap $C[\mathbf{u}, \arccos a] \subset \mathbb{S}^{d-1}$ centered at $\mathbf{u} \in \mathbb{S}^{d-1}$ having angular radius $0 < \arccos a < \frac{\pi}{2}$ with $0 < a < 1$ and $0 < t \leq 2\sqrt{1-a^2}$. Then*

$$A^+ \cup R_{\mathbf{u}}(A^+) \supset C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right),$$

where $C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right) \subset \mathbb{S}^{d-1}$ denotes the open spherical cap centered at $\mathbf{u}$ having angular radius $0 < \arctan(\frac{2a}{t}) < \frac{\pi}{2}$.

Proof: Suppose that $\mathbf{x} \in \mathbb{S}^{d-1} \setminus (A^+ \cup R_{\mathbf{u}}(A^+))$ and let θ denote the angular distance between $\mathbf{x}$ and $\mathbf{u}$. Clearly $0 < \theta \leq \pi$ and

$$\mathbf{x} = (\cos\theta)\mathbf{u} + (\sin\theta)\mathbf{v}$$

with $\mathbf{v} \in \mathbb{S}^{d-1}$ being perpendicular to $\mathbf{u}$. As $\mathbf{x} \notin A^+$ (resp., $\mathbf{x} \notin R_{\mathbf{u}}(A^+)$ i.e. $R_{\mathbf{u}}(\mathbf{x}) \notin A^+$) therefore there exists a point $\mathbf{y} \in A$ (resp., $\mathbf{z} \in A$) such that

$$0 \geq \langle \mathbf{y}, \mathbf{u} \rangle \cos\theta + \langle \mathbf{y}, \mathbf{v} \rangle \sin\theta \text{ (resp., } 0 \geq \langle \mathbf{z}, \mathbf{u} \rangle \cos\theta - \langle \mathbf{z}, \mathbf{v} \rangle \sin\theta).$$

By adding together the last two inequalities and using the inequalities $\|\mathbf{y} - \mathbf{z}\| \leq t$ and $\sin\theta \geq 0$ we get that

$$0 \geq \langle \mathbf{y} + \mathbf{z}, \mathbf{u} \rangle \cos\theta + \langle \mathbf{y} - \mathbf{z}, \mathbf{v} \rangle \sin\theta \geq \langle \mathbf{y} + \mathbf{z}, \mathbf{u} \rangle \cos\theta - t \sin\theta.$$

As $A \subset C[\mathbf{u}, \arccos a] \subset \mathbb{S}^{d-1}$ therefore if $\cos\theta > 0$, then the last inequality implies that

$$\tan\theta \geq \frac{\langle \mathbf{y} + \mathbf{z}, \mathbf{u} \rangle}{t} = \frac{\langle \mathbf{y}, \mathbf{u} \rangle + \langle \mathbf{z}, \mathbf{u} \rangle}{t} \geq \frac{2a}{t}.$$

Thus, $\theta \geq \arctan\left(\frac{2a}{t}\right)$ follows for all $0 < \theta \leq \pi$, finishing the proof of Lemma 12.4.5. □

Lemma 12.4.6

$$V_{d-1}(t) \geq \frac{1}{\sqrt{8\pi d}} \left(\frac{3}{2} + \frac{\left(2 - \frac{1}{d}\right) t^2 - 2}{4 - \left(2 - \frac{2}{d}\right) t^2} \right)^{-\frac{d-1}{2}}$$

for all $0 < t < \sqrt{\frac{2d}{d-1}}$ *and* $d \geq 3$.

Proof: Let $\emptyset \neq A \subset \mathbb{S}^{d-1}$ be given with (Euclidean) diameter $\operatorname{diam}(A) \leq t$. The spherical Jung theorem [119] implies that A is contained in the closed spherical cap $C\left[\mathbf{u}, \arcsin\left(\sqrt{\frac{d-1}{2d}} t\right)\right] \subset \mathbb{S}^{d-1}$ centered at the properly chosen $\mathbf{u} \in \mathbb{S}^{d-1}$ having angular radius $0 < \arcsin\left(\sqrt{\frac{d-1}{2d}} t\right) < \frac{\pi}{2}$, where by assumption $0 < t < \sqrt{\frac{2d}{d-1}}$. Thus, Lemma 12.4.5 implies that

$$A^+ \cup R_{\mathbf{u}}(A^+) \supset C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right)$$

with $a = \sqrt{1 - \frac{d-1}{2d} t^2}$. Hence,

$$\mu_{d-1}(A^+) = \frac{1}{2}\left(\mu_{d-1}(A^+) + \mu_{d-1}(R_{\mathbf{u}}(A^+))\right) \geq \frac{1}{2}\mu_{d-1}\left(A^+ \cup R_{\mathbf{u}}(A^+)\right)$$

$$\geq \frac{1}{2}\mu_{d-1}\left(C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right)\right) = \frac{1}{2} \frac{\operatorname{Svol}_{d-1}\left(C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right)\right)}{\operatorname{Svol}_{d-1}(\mathbb{S}^{d-1})}$$

$$= \frac{\mathrm{Svol}_{d-1}\left(C\left(\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right)\right)}{2d\omega_d} = \frac{\mathrm{Svol}_{d-1}\left(C\left[\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right]\right)}{2d\omega_d}.$$

As $\sin\left(\arctan(\frac{2a}{t})\right) = \left(1+\frac{t^2}{4a^2}\right)^{-\frac{1}{2}}$ therefore

$$\mathrm{Svol}_{d-1}\left(C\left[\mathbf{u}, \arctan\left(\frac{2a}{t}\right)\right]\right)$$

$$> \mathrm{vol}_{d-1}\left(\mathbf{B}^{d-1}\left[\cos\left(\arctan\left(\frac{2a}{t}\right)\right)\mathbf{u}, \left(1+\frac{t^2}{4a^2}\right)^{-\frac{1}{2}}\right]\right)$$

$$= \left(1+\frac{t^2}{4a^2}\right)^{-\frac{d-1}{2}} \omega_{d-1} \text{ and so, } \mu_{d-1}(A^+) \geq \frac{\omega_{d-1}}{2d\omega_d}\left(1+\frac{t^2}{4a^2}\right)^{-\frac{d-1}{2}}.$$

Hence, using the well-known estimate (see also [226]) $\frac{\omega_{d-1}}{\omega_d} \geq \sqrt{\frac{d}{2\pi}}$ we get that

$$\mu_{d-1}(A^+) \geq \frac{1}{2d}\sqrt{\frac{d}{2\pi}}\left(1+\frac{t^2}{4a^2}\right)^{-\frac{d-1}{2}}.$$

Finally, substituting $a = \sqrt{1-\frac{d-1}{2d}t^2}$ we are led to the following inequality

$$\mu_{d-1}(A^+) \geq \frac{1}{\sqrt{8\pi d}}\left(1+\frac{t^2}{4-\frac{2(d-1)t^2}{d}}\right)^{-\frac{d-1}{2}}$$

$$= \frac{1}{\sqrt{8\pi d}}\left(\frac{3}{2}+\frac{\left(2-\frac{1}{d}\right)t^2-2}{4-\left(2-\frac{2}{d}\right)t^2}\right)^{-\frac{d-1}{2}}.$$

This finishes the proof of Lemma 12.4.6. □

12.4.5 An upper bound for the number of sets of given diameter that are needed to cover spherical space

Lemma 12.4.7

$$n_{d-1}(\epsilon) < \left(1+\frac{4}{\epsilon}\right)^d$$

for all $0 < \epsilon \leq 2$ *and* $d \geq 3$.

Proof: Let $\{\mathbf{p}_1, \mathbf{p}_2, \dots \mathbf{p}_n\} \subset \mathbb{S}^{d-1}$ be the largest family of points on $\mathbb{S}^{d-1}$ with the property that $\|\mathbf{p}_i - \mathbf{p}_j\| \geq \frac{\epsilon}{2}$ for all $1 \leq i < j \leq n$. Then clearly $\bigcup_{i=1}^n \mathbf{B}^d\left[\mathbf{p}_i, \frac{\epsilon}{2}\right] \supset \mathbb{S}^{d-1}$ and therefore $n \geq n_{d-1}(\epsilon)$. As the balls $\mathbf{B}^d[\mathbf{p}_i, \frac{\epsilon}{4}], 1 \leq i \leq n$ form a packing in $\mathbf{B}^d[\mathbf{o}, 1+\frac{\epsilon}{4}]$ therefore

$$n\left(\frac{\epsilon}{4}\right)^d \omega_d < \left(1+\frac{\epsilon}{4}\right)^d \omega_d,$$

implying that

$$n_{d-1}(\epsilon) \le n < \frac{\left(1+\frac{\epsilon}{4}\right)^d}{\left(\frac{\epsilon}{4}\right)^d} = \left(1+\frac{4}{\epsilon}\right)^d .$$

This completes the proof of Lemma 12.4.7. □

Actually, using [122], one can replace the inequality of Lemma 12.4.7 by the stronger inequality $n_{d-1}(\epsilon) \le (\frac{1}{2} + \mathrm{o}(1)) d \ln d \left(\frac{2}{\epsilon}\right)^d$. As this improves the estimate of Theorem 6.8.3 only in a rather insignificant way, we do not introduce it here.

12.4.6 The final upper bound for the illumination number

Now, we are ready for the proof of Theorem 6.8.3. As $x < -\ln(1-x)$ holds for all $0 < x < 1$, therefore by Lemma 12.4.4 we get that

$$I(\mathbf{B}[X]) \le 1 + \frac{\ln\left(n_{d-1}(\epsilon)\right)}{-\ln\left(1 - V_{d-1}(1+\epsilon)\right)} < 1 + \frac{\ln\left(n_{d-1}(\epsilon)\right)}{V_{d-1}(1+\epsilon)}$$

holds for all $0 < \epsilon \le \sqrt{2} - 1$ and $d \ge 3$. Now, let $\epsilon_0 = \sqrt{\frac{2d}{2d-1}} - 1$. As $0 < \epsilon_0 < \sqrt{2} - 1$ holds for all $d \ge 3$, therefore Lemma 12.4.6 and Lemma 12.4.7 together with the easy inequality $\epsilon_0 > \frac{4}{16d-1}$ yield that

$$I(\mathbf{B}[X]) < 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left(n_{d-1}(\epsilon_0)\right)$$

$$< 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left(\left(1+\frac{4}{\epsilon_0}\right)^d\right) < 1 + \sqrt{8\pi d}\left(\frac{3}{2}\right)^{\frac{d-1}{2}} \ln\left((16d)^d\right)$$

$$= 1 + 4\sqrt{\frac{\pi}{3}} d\sqrt{d}\left(\frac{3}{2}\right)^{\frac{d}{2}} (\ln 16 + \ln d) < 4\left(\frac{\pi}{3}\right)^{\frac{1}{2}} d^{\frac{3}{2}} (3 + \ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}},$$

finishing the proof of Theorem 6.8.3.

12.5 Proof of Theorem 6.9.1

12.5.1 The CW-decomposition of the boundary of a standard ball-polyhedron

Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$ and $\mathbf{b} \in \mathrm{bd}\mathbf{K}$. Then recall that the Gauss image of $\mathbf{b}$ with respect to $\mathbf{K}$ is the set of outward unit normal vectors of hyperplanes that support $\mathbf{K}$ at $\mathbf{b}$. Clearly, it is a spherically convex subset of $S^{d-1}(\mathbf{o}, 1)$ and its dimension is defined in the natural way.

Theorem 12.5.1 *Let* $\mathbf{P}$ *be a standard ball-polyhedron. Then the faces of* $\mathbf{P}$ *form the closed cells of a finite CW-decomposition of the boundary of* $\mathbf{P}$*.*

Proof: Let $\{S^{d-1}(\mathbf{p}_1, 1), \ldots, S^{d-1}(\mathbf{p}_k, 1)\}$ be the reduced family of generating spheres of $\mathbf{P}$. The relative interior (resp., the relative boundary) of an m-dimensional face F of $\mathbf{P}$ is defined as the set of those points of F that are mapped to $\mathbf{B}^m(\mathbf{o}, 1)$ (resp., $S^{m-1}(\mathbf{o}, 1)$) under any homeomorphism between F and $\mathbf{B}^m[\mathbf{o}, 1]$. For every $\mathbf{b} \in \mathrm{bd}\mathbf{P}$ define the following sphere

$$S(\mathbf{b}) := \bigcap \{S^{d-1}(\mathbf{p}_i, 1) : \mathbf{p}_i \in S^{d-1}(\mathbf{b}, 1), i \in \{1, \ldots, k\}\}.$$

Clearly, $S(\mathbf{b})$ is a support sphere of $\mathbf{P}$. Moreover, if $S(\mathbf{b})$ is an m-dimensional sphere, then the face $F := S(\mathbf{b}) \cap \mathbf{P}$ is also m-dimensional as $\mathbf{b}$ has an m-dimensional neighbourhood in $S(\mathbf{b})$ that is contained in F. This also shows that $\mathbf{b}$ belongs to the relative interior of F. Hence, the union of the relative interiors of the faces covers $\mathrm{bd}\mathbf{P}$.

We claim that every face F of $\mathbf{P}$ can be obtained in this way; that is, for any relative interior point $\mathbf{b}$ of F we have $F = S(\mathbf{b}) \cap \mathbf{P}$. Clearly, $F \supset S(\mathbf{b}) \cap \mathbf{P}$, as the support sphere of $\mathbf{P}$ that intersects $\mathbf{P}$ in F contains $S(\mathbf{b})$. It is sufficient to show that F is at most m-dimensional. This is so, because the Gauss image of $\mathbf{b}$ with respect to $\mathbf{P}$ is at least $(d - m - 1)$-dimensional, since the Gauss image of $\mathbf{b}$ with respect to $\bigcap \{\mathbf{B}^d[\mathbf{p}_i, 1] : \mathbf{p}_i \in S^{d-1}(\mathbf{b}, 1), i \in \{1, \ldots, k\}\} \supset \mathbf{P}$ is $(d - m - 1)$-dimensional.

The above argument also shows that no point $\mathbf{b} \in \mathrm{bd}\mathbf{P}$ belongs to the relative interior of more than one face. Moreover, if $\mathbf{b} \in \mathrm{bd}\mathbf{P}$ is on the relative boundary of the face F then $S(\mathbf{b})$ is clearly of smaller dimension than F. Hence, $\mathbf{b}$ belongs to the relative interior of a face of smaller dimension. This concludes the proof of Theorem 12.5.1. □

12.5.2 On the number of generating balls of a standard ball-polyhedron

Corollary 12.5.2 *The generating balls of any standard ball-polyhedron* $\mathbf{P}$ *in* $\mathbb{E}^d$ *consist of at least* $d + 1$ *unit balls.*

Proof: because the faces form a CW-decomposition of the boundary of $\mathbf{P}$, there is a vertex $\mathbf{v}$. The Gauss image of $\mathbf{v}$ is $(d - 1)$-dimensional. So, $\mathbf{v}$ belongs to at least d generating spheres from the family of generating balls. We denote the centers of those spheres by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$. Let $H := \mathrm{aff}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\}$. Then $\mathbf{B}[\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\}]$, which denotes the intersection of the closed d-dimensional unit balls centered at the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$, is symmetric about H. Let σ_H be the reflection of $\mathbb{E}^d$ about H. Then $S := S^{d-1}(\mathbf{x}_1, 1) \cap S^{d-1}(\mathbf{x}_2, 1) \cap \cdots \cap S^{d-1}(\mathbf{x}_d, 1)$ contains the points $\mathbf{v}$ and $\sigma_H(\mathbf{v})$, hence S is a sphere, not a point. Finally, as $\mathbf{P}$ is a standard ball-polyhedron, therefore there is a unit-ball $\mathbf{B}^d[\mathbf{x}_{d+1}, 1]$ in the family of generating balls of $\mathbf{P}$ that does not contain S. □

12.5.3 Basic properties of face lattices of standard ball-polyhedra

Corollary 12.5.3 *Let Λ be the set containing all faces of a standard ball-polyhedron $\mathbf{P} \subset \mathbb{E}^d$ and the empty set and $\mathbf{P}$ itself. Then Λ is a finite bounded lattice with respect to ordering by inclusion. The atoms of Λ are the vertices of $\mathbf{P}$ and Λ is atomic: for every element $F \in \Lambda$ with $F \neq \emptyset$ there is a vertex $\mathbf{x}$ of $\mathbf{P}$ such that $\mathbf{x} \in F$.*

Proof: First, we show that the intersection of two faces F_1 and F_2 is another face (or the empty set). The intersection of the two supporting spheres that intersect $\mathbf{P}$ in F_1 and F_2 is another supporting sphere of $\mathbf{P}$, say $S^l(\mathbf{p}, r)$. Then $S^l(\mathbf{p}, r) \cap \mathbf{P} = F_1 \cap F_2$ is a face of $\mathbf{P}$. From this the existence of a unique maximum common lower bound (i.e., an infimum) for F_1 and F_2 follows.

Moreover, by the finiteness of Λ, the existence of a unique infimum for any two elements of Λ implies the existence of a unique minimum common upper bound (i.e., a supremum) for any two elements of Λ, say C and D, as follows. The supremum of C and D is the infimum of all the (finitely many) elements of Λ that are above C and D.

Vertices of $\mathbf{P}$ are clearly atoms of Λ. Using Theorem 12.5.1 and induction on the dimension of the face it is easy to show that every face is the supremum of its vertices. $\square$

Corollary 12.5.4 *A standard ball-polyhedron $\mathbf{P}$ in $\mathbb{E}^d$ has k-dimensional faces for every $0 \leq k \leq d-1$.*

Proof: We use an inductive argument on k, where we go from $k = d-1$ down to $k = 0$. Clearly, $\mathbf{P}$ has facets. A k-face F of $\mathbf{P}$ is homeomorphic to $\mathbf{B}^k[\mathbf{o}, 1]$, hence its relative boundary is homeomorphic to $S^{k-1}(\mathbf{o}, 1)$, if $k > 0$. Since the $(k-1)$-skeleton of $\mathbf{P}$ covers the relative boundary of F, $\mathbf{P}$ has $(k-1)$-faces. $\square$

Corollary 12.5.5 *Let $d \geq 3$. Any standard ball-polyhedron $\mathbf{P}$ is the spindle convex hull of its $(d-2)$-dimensional faces. Furthermore, no standard ball-polyhedron is the spindle convex hull of its $(d-3)$-dimensional faces.*

Proof: For the first statement, it is sufficient to show that the spindle convex hull of the $(d-2)$-faces contains the facets. Let $\mathbf{p}$ be a point on the facet, $F = \mathbf{P} \cap S^{d-1}(\mathbf{q}, 1)$. Take any great circle C of $S^{d-1}(\mathbf{q}, 1)$ passing through $\mathbf{p}$. Since F is spherically convex on $S^{d-1}(\mathbf{q}, 1)$, $C \cap F$ is a unit circular arc of length less than π. Let $\mathbf{r}, \mathbf{s} \in S^{d-1}(\mathbf{q}, 1)$ be the two endpoints of $C \cap F$. Then $\mathbf{r}$ and $\mathbf{s}$ belong to the relative boundary of F. Hence, by Theorem 12.5.1, $\mathbf{r}$ (resp., $\mathbf{s}$) belongs to a $(d-2)$-face. Clearly, $\mathbf{p} \in \mathrm{conv}_s\{\mathbf{r}, \mathbf{s}\}$.

The proof of the second statement goes as follows. By Corollary 12.5.4 we can choose a relative interior point $\mathbf{p}$ of a $(d-2)$-dimensional face F of $\mathbf{P}$. Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the centers of the generating balls of $\mathbf{P}$ such that $F := S^{d-1}(\mathbf{q}_1, 1) \cap S^{d-1}(\mathbf{q}_2, 1) \cap \mathbf{P}$. Clearly, $\mathbf{p} \notin \mathrm{conv}_s((\mathbf{B}^d[\mathbf{q}_1, 1] \cap \mathbf{B}^d[\mathbf{q}_2, 1]) \setminus \{\mathbf{p}\}) \supset \mathrm{conv}_s(\mathbf{P} \setminus \{\mathbf{p}\})$. $\square$

Corollary 12.5.6 *(Euler–Poincaré Formula) If* $\mathbf{P}$ *is an arbitrary standard* d*-dimensional ball-polyhedron, then*

$$1 + (-1)^{d+1} = \sum_{i=0}^{d-1} (-1)^i f_i(\mathbf{P}),$$

where $f_i(\mathbf{P})$ *denotes the number of* i*-dimensional faces of* $\mathbf{P}$*.*

Proof: It follows from Theorem 12.5.1 and the fact that a ball-polyhedron in $\mathbb{E}^d$ is a convex body, hence its boundary is homeomorphic to $S^{d-1}(\mathbf{o}, 1)$. □