Chapter 8
Equipartitioning Operations for Connected Numbers: Their Use and Interiorization

Leslie P. Steffe and Catherine Ulrich

During her fourth grade, Melissa had been paired with another child who had constructed only the tacitly nested number sequence. The teacher of the two children geared her activities to the other child and Melissa essentially served as the other child's interlocutor. The other child did not construct any fraction schemes during her fourth grade and, Melissa, whose role was to interact with the other child at her level, constructed at most the partitive fraction scheme. Melissa had constructed the explicitly nested number sequence, so we paired her with Joe in their fifth grade in order to investigate her constructive itinerary. Our purpose in the upcoming analysis is to investigate her construction of the iterative fraction scheme as well as the unit fraction composite scheme and compare her progress with that of Laura while she was in the fifth grade. Although Joe served as Melissa's interlocutor, we investigate whether the schemes that he established during his fourth grade were permanent as well as any accommodations he might make in them. In what follows, we analyze 17 protocols that were selected from the teaching episodes that started on the 20th of October and proceeded on through the 4th of May of the children's fifth grade.

Melissa's Initial Fraction Schemes

As a result of the first two teaching episodes held on the 20th and the 27th of October, it was clear that Melissa had already constructed the partitive fraction scheme. In the third teaching episode held on the 3rd of November, Melissa's use of her units-coordinating scheme in the context of the children partitioning already-partitioned sticks was analyzed to investigate whether she had constructed recursive partitioning. The first task involved the children making fraction sticks, beginning with a 4/4-stick.

Protocol I. Using a 4/4-stick to make new fraction sticks.

T: (Points to a 4/4-stick on the screen.) Each one of those pieces is how much of the whole stick?

J: One-fourth.

T: I want you to use this fraction stick to make a new fraction stick. What could you do to make this fraction stick into a new fraction stick?

L.P. Steffe and J. Olive, *Children's Fractional Knowledge*,
DOI 10.1007/978-1-4419-0591-8_8, © Springer Science+Business Media, LLC 2010

M: You could use PARTS.
T: What would be the next fraction stick you could make?
J: One-half.
M: (Dials Parts to "10" and clicks on the right-most part of the 4/4-stick. PARTS partitioned just the selected part into ten parts.)
T: OK. Before clicking any more, Melissa, how many pieces would be in the whole stick if you kept that up?
M: Forty.
T: (To Joe.) How much of the whole stick would each tiny piece be?
J: One-fortieth.
M: (Completes partitioning the 4/4-stick into forty parts and pulls out one-fortieth of the stick upon being directed by the teacher. She then erases all marks and repartitions the stick into a 4/4-stick again upon being directed by the teacher.)
T: What would be the next fraction stick that you could make doing the same thing? What would be the very next one you could make?
J: Two!
T: OK, go ahead.
J: (Partitions the first of the four parts of a copy of the 4/4-stick into two parts.)
T: What is each piece of that one going to be?
M: (Immediately.) It would be eighths!
T: (After Joe finishes partitioning the four parts of the 4/4-stick.) You make the next one. (To Melissa). What is it going to be?
M: Umm…, eleven…no, umm, five!
T: (Looks at Melissa sharply.) You used two, right? How many pieces would be in the next one?
M: It would be twelve.
T: OK, go ahead. Copy the stick first.
M: (Partitions each part of the 4/4-stick into three parts.)

Melissa's choice to use PARTS was made independently of the teacher's language and actions after the teacher's question, "What could you do to make this fraction stick into a new fraction stick?" Further, after the teacher asked Melissa how many pieces would be in the whole stick if she kept up partitioning each fourth into ten parts, she immediately answered, "Forty." Independently choosing to use PARTS and answering "forty" together constitute solid indication of a generalizing assimilation of her units-coordinating scheme in the context of connected numbers. Assimilation follows from her independently choosing to use Parts and its generalizing nature follows from her using her number concept, ten, to partition a part of the 4/4-stick into ten parts.[1] Further, she at least anticipated partitioning each of the other parts of the 4/4-stick into ten parts, which is the basis for inferring that she coordinated her two number concepts, ten and four, by using ten to partition each of the four parts.

An anticipatory units-coordinating scheme is certainly basic to recursive partitioning because, to find how much one-tenth of one-fourth is of the whole stick, the child mentally partitions each fourth into ten parts to produce forty, exactly as

[1] In the discrete case, a units-coordination would consist of Melissa inserting the unit of ten into each unit of the four units of four.

Melissa demonstrated. Melissa's answer, "It would be eighths!" to the teacher's question, "What is each piece going to be?" is indicative of recursive partitioning, but her answer followed on from Joe partitioning one-fourth of the unit stick into halves and previously saying "one-fortieth" as the fraction of the unit stick constituted by one of the forty parts that Melissa would make had she completed partitioning the 4/4-stick. Consequently, it is uncertain whether Melissa could have independently answered in that way. On the other hand, Joe's answer of "one-fortieth" corroborates that, for him, recursive partitioning was a permanent operation (cf. Protocol X of Chap. 7).

Contraindication of Recursive Partitioning in Melissa

In further exploration of whether Melissa had constructed recursive partitioning, the teacher began by asking Joe to make the next one.

Protocol I. (First Cont)

T: Joe, you make the next one.
J: Five!
T: Five?!
J: You have already got four. (Pointing the cursor arrow at the 4/4-stick.)
T: Ooooh. You already have four there, but did you use four?
J: No.
T: All right, make a copy.
J: (After some discussion, the teacher asked Joe if he wanted to do his fraction stick.) Oh, I forgot! (Partitions each part of a copy of the 4/4-stick into four parts.)
T: OK, Melissa, its your turn now. What will the next one be?
M: Twenty. (Makes a 20/20-sick using the 4/4-stick.)
T: Each piece will be how much of the whole stick?
M: (Hesitates.)
J: One-twentieth.
T: OK, Joe, you make one more. What is this one going to be?
J: Twenty-fourths. (Makes a 24/24-stick using the 4/4-stick.)

Melissa's hesitation after the teacher asked the additional question, "Each piece will be how much of the whole stick?" indicates an uncertainty on her part about what the answer should be. It also indicates that she was still in the process of linking her units-coordinating scheme and her partitive fraction scheme in order to establish a way of operating that resembles a unit fraction composition scheme. If she established a link between the two schemes, it was after she answered "twenty." Before she answered, "twenty," Melissa's goal seemed to be to produce the next partition of her partitioned 4/4-stick. After she answered "twenty" and partitioned each part of the 4/4-stick into five parts, she closed off using her units-coordinating scheme because she had reached her goal. So, when the teacher asked her how much each piece would be of the whole stick, the task seemed to appear to her as a new task. If the question evoked her partitive fraction scheme she would use it to assimilate the results of using her units-coordinating scheme, and in this way establish a link between the two schemes. So, she may have been in the process of establishing using the results of her units-coordinating scheme as a situation of her partitive fraction scheme. Even if Melissa did establish a link between the two schemes, she would need to demonstrate recursive partitioning in order for us to judge the linked schemes as a unit fraction composition scheme.[2]

Reversibility of Joe's Unit Fraction Composition Scheme

Joe's answer, "One-twentieth". again corroborates the inference that he had constructed a unit fraction composition scheme while he was in his fourth grade (cf. Protocol X of Chap. VII). The teacher continued on and changed the question in such a way that answering it would seem to necessarily involve reversibility of the unit fraction composition scheme. What resulted was a chance to test if Joe's scheme was reversible and a new chance to test whether Melissa had constructed a unit fraction composition scheme and, hence, recursive partitioning.

Protocol I. (Second Cont).

T: Make one-fourth using that stick. (Points to the 4/4-stick.)
M: Pulls one part out of the 4/4-stick.
T: (Points to the 1/4-stick.) I want you to cut that up so each little piece is one-twentieth of the whole stick.
J: (Immediately partitions the 1/4-stick into five parts.)
T: (Asks Melissa to pull another 1/4-stick out of the 4/4-stick.) I want you to cut that up so each little piece is one-thirty-second of the whole stick.
M: (Immediately partitions the 1/4-stick into thirty-two parts.)
T: Would that be right?
J: (Shakes his head.) No, because thirty-two times four is not thirty-two!
T: OK. You can erase marks if you want to. (To Melissa.) (After Melissa erases marks.) What would you have to do, Melissa?
M: Cut it into eight pieces! (Partitions the 1/4-stick into eight parts.)
T: Why wouldn't each one of those little pieces be one thirty-second of the whole stick? (Referring to her partitioning the 1/4-stick into thirty-two parts.)
M: Because I thought you meant thirty-two pieces in all the parts.
T: Would you like to ask Melissa a question, Joe?
J: (After a long pause when Melissa was rearranging the screen.) Each little piece would be one-sixteenth.
M: (Partitions a 1/4-stick into four parts.)
T: How could you prove that each part is one-sixteenth?

[2] It would also require that she could solve tasks like, "Find how much one-fifth of one-fourth of a stick is of the whole stick."

M: Because four times four is sixteen.

T: You give Joe one.

M: (Puts up fingers apparently counting by seven four times. But the monitor partially hid her hands, so it was not possible to see all of what she did.) Each little part would be one-twenty-eighth!

J: (Immediately dials PARTS to "7" and clicked on a copy of the 1/4-stick.)

After a long pause, Joe then told Melissa that each part would be one-thirty-sixth in posing a problem to her. She immediately dialed PARTS to "9" and clicked on a copy of the 1/4-stick. She seemed to have abstracted how to use her anticipatory units-coordinating scheme to solve the situations that Joe presented to her. However, to infer that she could engage in recursive partitioning even though she performed flawlessly after Joe's corrective, "No, because thirty-two times four is not thirty-two!" was not plausible. The fact that she initially partitioned a 1/4-stick into thirty-two rather than eight parts after the teacher asked her to cut it up so each little piece would be one-thirty-second of the whole stick is contraindication of recursive partitioning in the context of solving the problem. At that point, "Thirty-two" referred to the result of a single partition rather than a result of a coordination of two partitions. Joe's corrective contained explicit multiplicative language and that would be sufficient to provoke her units-coordination scheme. It is reasonable to infer that her units-coordinating scheme was a reversible scheme because of the way she used it, for example, to pose the problem, "Each little part would be one-twenty-eighth!" to Joe.

When Joe presented the problem "Each little piece would be one-sixteenth." to Melissa, she partitioned a 1/4-stick into four parts and explained why she did it by saying, "Because four times four is sixteen." "One-sixteenth," then, apparently referred to partitioning the whole stick into sixteen parts, i.e., her partitive fraction scheme was a reversible scheme. Rather than attribute recursive partitioning to Melissa, the conjecture is that her reversible partitive fraction scheme became linked to her reversible units-coordinating scheme in that a result of the former was used as a situation of the latter. For example, when she posed the problem "Each little part would be one-twenty-eighth!" involved using her reversible partitive fraction scheme because one-twenty-eighth referred to partitioning the whole stick into twenty-eight parts. She could then reason, "Four times what is twenty-eight?" which is an indication of a reversible units-coordinating scheme. The linkage between the two schemes might seem sufficient to infer that she had constructed a reversible unit fraction composition scheme. However, the unit fraction composition scheme contains recursive partitioning as a subscheme whereas the linkage of the two schemes as we have explained it does not involve recursive partitioning.

The indication is solid that Joe had indeed constructed his unit fraction composition scheme as a reversible scheme when he partitioned the 1/4-stick into five parts after the teacher's request, "I want you to cut that up so each little piece is one-twentieth of the whole stick." His goal was to make a piece that is one-twentieth of the whole stick, whereas the most that can be inferred is that Melissa's goal was to make so many pieces of the whole stick using the 4/4-stick. These differences in the schemes the two children used are subtle. If the differences are consequential, they should be manifest in the adaptability of the two schemes.

After the tasks of Protocol I, the teacher proceeded to ask the children a more complex question that we regard as a test of the adaptability of the two children's schemes.

Protocol II. A test of adaptability.

T: Joe, make three-fourths of this stick.
J: (Pulls a 3/4-stick out of the 4/4-stick.)
T: I want you to cut this up so each little piece is one-eighth.
M: (Dials PARTS to "8" and clicks on each of the parts of the 3/4-stick.)
T: Is each little piece one-eighth of the big stick?
M: Uh, huh. (Yes.)
T: Well, pull a piece out and see.
M: (Pulls out one of the twenty-four parts she made by clicking on the 3/4-stick three times. She then activates MEASURE[3] but hesitates before clicking on the 1/32-stick she pulled out.)
J: (With urgency.) Click on it and see!!
M: (Clicks on the 1/32-stick and "1/32" appears in the number box.)
J: (Smiles knowingly.)
T: I want you to cut three-fourths up so each little piece is one-eighth of the whole stick.
J: (Takes the mouse and starts to dial PARTS. The teacher asks him to take another three-fourths out first, so he pulls a 3/4-stick from the 4/4-stick and, after the teacher reposes the question, he dials PARTS to "2" and clicks on each part of the 3/4-stick.)
T: How did you do that, Joe?!! Is that right?
M: (Counts the six parts on the now 6/8-stick.)
J: (Pulls a 1/8-stick out from the 6/8-stick and measures it. "1/8" appears in the number box.)
T: Can you tell us how you thought that out?
J: (Smiling.) I put it on two parts, and two right here. (The 3/4-stick.) And six and eight in this one. (The 4/4-stick.)
T: (To Melissa.) Did you hear what he said?
M: (Nods.)
T: (Asks Melissa to pull out another 3/4-stick and cut it up into pieces so that each little piece would be one-twelfth of the whole stick.)
M: (Dials PARTS to "4" and clicks on each of the three parts of the 3/4-stick.)
T: What do you think, Joe?
J: That would be one-sixteenth!

Having three rather than one part of the 4/4-stick to partition to make a stick that was one-eighth of the 4/4-stick introduced a constraint that Melissa did not eliminate by making an adjustment in her way of operating in the second continuation of

[3] MEASURE works by designating the 4/4-stick as a unit stick and then using the mouse to click on the stick to be measured relative to the unit stick.

Protocol I. From the observer's perspective, Melissa regressed to the way she operated at the beginning of the second continuation of Protocol I. Her way of operating in Protocol II proved to be nonadaptive, and when contrasted with Joe's insightful comments, this nonadaptability serves as contraindication that she had constructed a reversible unit fraction composition scheme. Joe, on the other hand, operated in a way that was compatible with the hypothesis that he had in fact constructed his unit fraction composition scheme as reversible.

Based on Joe's partitioning each part of the 3/4-stick into two parts and on his explanation, Melissa did modify her initial partitioning activity where she partitioned each of the three parts into eight parts. Rather than partition each of the three parts into twelve parts upon being asked to cut the three parts into pieces so that each piece would be one-twelfth of the whole stick, she partitioned each of the three parts into four parts to produce twelve parts. However, partitioning each of the three parts into four parts to produce twelve parts did not constitute the insight that stems from using a reversible unit fraction composition scheme.

A Reorganization in Melissa's Units-Coordinating Scheme

The goal in the analysis of the teaching episode held on the 1st of December is to explain the schemes the children used to transform a unit fraction into a commensurate fraction. To begin the teaching episode, Melissa made a long and rather narrow bar. The teacher asked Joe to partition it into five parts and to pull out one-fifth of the bar. Joe then made unmarked copies of the bar at the teacher's request and, after the teacher explained that he was to partition the first copy into a different number of parts so that one-fifth could be still pulled out, Joe immediately partitioned it into ten parts and pulled out two after the teacher asked him to pull out one-fifth of the bar. He also said "two-tenths" after the teacher asked him for another name for that fraction.

As Joe spoke, Melissa was counting the number of parts in the 10/10-bar by twos. After she was done, she agreed with Joe that another name would be two-tenths. This seemed to evoke Melissa's units-coordinating scheme because she immediately dialed PARTS to "15" and partitioned the next copy into fifteen parts and pulled out three of the parts after the teacher asked her to pull out one-fifth. She also immediately said that another fraction name for the part she pulled out would be three-fifteenths. Protocol III starts with Joe's next turn.

Protocol III. Joe and Melissa produce fractions commensurate with one-fifth.

T: OK, it's your turn Joe. Make it go as far as you can. (Three unmarked copies of the 5/5-bar remained.)

J: (Dials Parts to "25" and clicks on the next copy. He then drags a rectangle over the first five parts, one at a time, while intently looking at what he was doing. He did not appear to look at the previous one-fifth fractional parts. Melissa looked intently at the computer screen while Joe was working silently.)

M: (Dials Parts to "30" and clicks on the next copy. She then drags a rectangle over the first six parts, one at a time, while intently looking at what she was doing. She did not appear to look at the previous one-fifth fractional parts.)

T: (To Joe.) What was the name for your fraction?

J: Five-twenty-fifths.

T: (To Melissa.) What was the name for your fraction, Melissa?

M: (Immediately.) Six-thirtieths.

Melissa knew that she was going to pull out the first six parts of the 30/30-bar that she made before she began dragging the rectangle over the parts. Even though it is quite likely that she knew that Joe pulled out five parts because she watched him intently as he was doing so, she did proceed with confidence when she pulled out six parts. Moreover, she knew immediately that she had made six-thirtieths of the bar. For these reasons, it is plausible that she understood that she needed to pull out six parts because five times six is thirty. My hypothesis is that after Joe partitioned a copy of the unit stick into twenty-five parts and pulled out five, she conceived of the stick as partitioned into the original five parts where each part was partitioned into five parts using her units-coordinating scheme. She then proceeded to use her units-coordinating scheme to mentally partition each of these five parts into six parts each to produce thirty parts and then pulled out six of the thirty parts.

To illustrate the children's reliance on the counting-by-five patterns involved in producing the sequence of fractions commensurate with one-fifth, the teacher asked Melissa what the next one would be. She said thirty-five and seven, and then seven-thirty-fifths. Joe then said the next one would be eight-fortieths, and Melissa said the next one after that would be nine-forty-fifths. The two children continued without hesitation, taking turns, until they reached fifteen-seventy-fifths. The children did not know that, say, five times fifteen is seventy-five nor that five times fourteen is seventy without computing, but they had a way of producing a sequence of fractions, each commensurate with one-fifth, that avoided multiplicative computation. Nevertheless, we infer that each fraction of the sequence symbolized an invariant conceptual structure. For example, "fifteen-seventy-fifths" symbolized the 5/5-unit stick where each of the five parts was partitioned into fifteen parts as indicated by Melissa saying "one-fifth" when asked for another fraction name by the teacher. The children knew that fifteen-seventy-fifths was one-fifth, not because they could engage in the operations involved in reducing fifteen-seventy-fifths to one-fifth,[4] but because they maintained the original partition of the unit stick into five parts across their partitioning activity.

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[4]These operations involve partitioning two composite units, one of numerosity seventy-five and the other of numerosity fifteen, into subunits of equal numerosity. It also involves an awareness of both unit structures produced in each case (e.g., a unit of five units of three and unit of twenty-five units of three) as well as awareness that five out of the twenty-five units of three determines fifteen out of the seventy-five units of one.

Given Melissa's strong performance in Protocol III, the teacher decided to change the task to one similar to the second continuation of Protocol I. He asked them how many parts the 1/5-stick would need to be cut into so that each part would be one-one-hundredth of the whole stick. In solving the task, it would have been possible for the children to continue coordinating the two number sequences for one and five beyond "fifteen and seventy-five." However, introducing reversibility led the children to abandon this way of proceeding and to using their units-coordinating schemes to make conjectures when trying to solve the task.

Protocol IV. Melissa's conjecture.

T: (Points to the 1/5-bar with the cursor.) This is the part you are going to break – break this piece up so that each little part of this is one-one-hundredth of the unit bar. (Running the cursor over the 5/5-unit bar.)

M: You would put one hundred in this bar. (The 5/5-unit bar.)

T: But what would you have to put in this one? (Pointing to the 1/5-bar.)

M: You would have to pull one out of this one? (The 5/5-unit bar.) You would have to pull ten out of this bar.

T: So, there would be ten in this? (The 1/5-bar.)

M: Yeah. There would be ten in this. (The 1/5-bar.)

T: And that would give you a hundred in the unit bar?

M: Uh-huh. (Yes.)

It is quite significant that Melissa's partitive fraction scheme was reversible when there was no bar in her visual field that was one-one-hundredth of the 5/5-bar: She knew if each little part of the 1/5-bar was one-one-hundredth, then the 5/5-unit bar would be partitioned into one hundred parts. She also seemed to understand that the unit bar, when partitioned into one hundred parts, was one hundred times one of its parts. "One-one-hundredth" referred to a partitive unit fraction and her comment, "You would put one hundred in this bar," seemed to refer to *splitting* the bar into one hundred parts. We emphasize "splitting" because to split a bar into one hundred parts means that the bar is conceived of as partitioned into one hundred parts, and of one hundred times any one of its parts. Given that one-one-hundredth was posited as a part of the 1/5-bar, Melissa would need to inject that 1/5-bar along with its part (one-one-hundredth) into the 5/5-bar and then mentally split the 5/5-bar into one hundred parts. When she said that "You would have to pull one out of this one (the 5/5-bar)?" and then "You would have to pull ten out of this bar." there was no bar in her visual field partitioned into one hundred parts. She was speaking hypothetically, and the ten parts she referred to were apparently conceived of as ten parts of one of the five parts of the 5/5-bar ["Yeah, there would be ten in this. (The 1/5-bar.)"]. She seemed to establish a connected number, one hundred, where the one hundred units were partitioned into five parts, which constitutes the structure of a unit of units of units in the context of connected numbers. Melissa performed these operations without considering how many parts each one of the five parts would need to be partitioned into. For this reason, and because it was legitimate to infer that she engaged in mentally splitting the 5/5-bar into one hundred parts, we infer that she engaged in reciprocal reasoning in the case of one-one-hundredth and one hundred.[5]

Her saying that there would be ten in the 1/5-bar indicates that she did not establish how many of the one hundred parts would be in each one of the five parts. Melissa was very precise in her calculations, and she would not have made such a blatant mistake had she actually counted by ten five times in a test to find if ten worked. So, the teacher decided to ask Melissa to check her estimate of ten.

Protocol IV. (Cont)

T: (To Melissa.) Make a copy of this. (The 5/5-bar.)
M: (Breaks the 5/5-bar into its parts.)
T: Do you think ten in each part is going to give you a hundred parts?
M: (Nods.)
J: I know how to do it!
M: Twenty!
T: OK! Go ahead and do it.
J: There is another way, too!
M: (Partitions each of the five parts into twenty parts and then joins the five parts together into a 100/100-bar.)
T: Is there a hundred parts in that?
J: (Uses the menu PARTS IN A BAR to verify that there are one hundred parts in the bar and "100" appears in the NUMBER BOX.)
T: Can you pull one out, Joe?
J: (Pulls one of the one hundred parts out from the 100/100-bar.)
T: Try measuring that.
J: (Measures the part he pulled out and "1/100" appears in the NUMBER BOX. Both children express pleasure at being able to measure such a small part of the bar.)
T: (Points to the original 1/5-bar.) So how many parts do you have to break this into to get one-one-hundredth?
M: Twenty.

It was not until the teacher asked, "Do you think ten in each part is going to give you a hundred parts?" and Joe said, "I know how to do it!" that Melissa doubted her answer of ten and made a corrective. In that she quickly said, "twenty," it is quite possible that her original answer of ten was based on her associations among the numbers five, ten, and one hundred. In fact, Joe's "another way to do it" was to partition the bar into ten parts and then partition each part again into ten parts as he demonstrated after the continuation of Protocol IV. By their fifth grade in school, children have learned well that "ten times ten is one hundred," so it is very plausible that Melissa relied on this knowledge to produce ten. In any event, this is the first time that we were able to infer that Melissa had constructed the opera-

[5] From this, I do not infer that she had constructed the concept of the reciprocal of a fraction. Nor do I infer that she had constructed more general reciprocal reasoning involving proper and/or improper fractions.

tions that produce a unit of units of units in the context of connected numbers and the splitting operation. It is also the first time that we were able to infer that she had constructed her partitive unit fraction scheme and her units-coordinating scheme as reversible schemes. These inferences are corroborated in Protocol V when the teacher changed the number of parts from one hundred to eighty.

Protocol V. Inferring equipartitioning operations for connected numbers.

T: How many parts would you have to break this up into, the one-fifth, if each little part was one-eightieth of the unit bar?

M: (After approximately 25 seconds) Twenty-five. Cause, um, three times twenty is sixty and you add two more tens to it and you would get eighty. (Three twenties and two tens!)

T: That would give you eighty parts, but would they all be the same size?

M: Uh-uh. (No.)

T: So that wouldn't work, would it?

M: No.

J: I know how many sticks. Put twenty in one and then thirty in another, and then twenty in another.

T: Would all those parts be the same size?

J: (Shakes his head.) I don't know.

M: (While Joe and the teacher are talking, Melissa intently enacts a computational algorithm on the table by tracing numerals with her right forefinger. She then sits back in her chair and appears to be in deep concentration.)

J: (Continues on working while Melissa is attempting to find a result. The children do not communicate verbally. Joe proposes a possible answer, and the teacher casts doubt on it, so he keeps working.)

T: Maybe I have forgotten the problem. What is it that we are trying to do?

M: How many eightieths can you get to fit into one-fifth!

T: That's right! Very good! How many eightieths can you get to fit into one-fifth?

M: (Asks for a paper and pencil and leaves her seat to retrieve her own supply of paper and pencil.)

J: (Makes two copies of the 1/5-bar and partitions the original 1/5-bar and the two copies into thirty parts each. He then indicates to the teacher that he is trying to make ninetieths!)

T: Melissa, what are you looking for? We have a calculator on the computer that you can use. (Joe activates the calculator and Melissa retrieves a pencil and pad and returns to her seat. The teacher re-poses the problem to each of the two children. Each child works separately, Melissa using her pencil and paper and Joe mentally without using the calculator. She tries five times twelve, and then others. She then says that she got it. Joe then says, "sixteen.") Sixteen!! How did you get that? (Checks Melissa's answer before he allows Joe to explain.) What did you get?

M: I got fourteen! (Explains that she divided five into eighty. She then checks Joe's answer by computing five times sixteen.)

T: How did you get sixteen, Joe?

J: I know five times twelve, so I did thirteen would be sixty-five, fourteen would be seventy, fifteen would be seventy-five, and sixteen would be eighty!

T: Fantastic! (Guides the children to partition the 1/5-bar into sixteen parts, pull out one part and measure it to check.)

Due to the difficulty of finding sixteen mainly, both Melissa and Joe tried to fit eighty-eightieths into five-fifths without maintaining an equal number of the eightieths in each fifth. When the teacher asked the children what they were trying to do, Melissa answered, “How many eightieths can you get to fit into one-fifth!” In that Melissa used fraction language in her answer, it is clear that she was aware that each of the eighty parts was one-eightieth of the unit stick, and that it was implicit in her answer that finding how many eightieths would fit into one-fifth would inform her about how many eightieths would fit into each of the one-fifths. Her clearly stated problem is solid indication of equipartitioning operations for connected numbers with the proviso that the five-part stick was one of her givens. In that she computed to check whether five times sixteen is eighty, there is indication that she understood that it was a matter of necessity that the number of eightieths in each fifth could be iterated five times to produce eighty. Melissa still focused on portioning in her language. Nevertheless, her equipartitioning operations permitted her to work at the level of re-presentation and there was no necessity for there to be material in her visual field to stand in for the eightieths. Moreover, she worked symbolically using her division algorithm to achieve her goal.

Joe’s strategic reasoning, starting with five times twelve is sixty and coordinating incrementing twelve by one and sixty by five until he reached eighty, is also an indication of equipartitioning operations for connected numbers. The coordination corroborates that he regarded it as necessary for five iterations of the numerosity of the items placed into one of the fifths to yield eighty-eightieths. His reasoning warrants the inference that Joe, like Melissa, assimilated the situation using three levels of units, i.e., he constructed the situation as a unit containing five connected units each of which contained an unknown numerosity of units, where his goal was to find how many parts he should partition each of the five connected units into so that, together, there would be eighty one-eightieths in the 5/5-stick. This situation and goal was a situation and goal of his *reversible* units-coordinating scheme that he used strategically. Using his reversible units-coordinating scheme in this way can be thought of as a reversible unit commensurate fraction scheme if the goal is to find how many eightieths is equal to one-fifth, which was Melissa’s explicitly stated goal. This goal, along with the equipartitioning operations for connected numbers that induced a modification in the units-coordinating scheme, is what differentiates a reversible unit commensurate fraction scheme from a reversible units-coordinating scheme. It is quite significant to realize that a reversible unit commensurate fraction scheme is a *multiplicative* scheme. But it is yet to be constructed as an *equivalence* scheme.

Melissa’s Construction of a Fractional Connected Number Sequence

Joe already constructed a fractional connected number sequence in the teaching episode that was held on the 28th of April of his fourth grade as a functional accommodation of his iterative fraction scheme. So, the primary interest was in analyzing

Melissa's language and activities in Protocol VI, which involved the production of improper fractions. It was important to know whether the operations that she used to produce an equipartitioning of the composite unit eighty-eighths into five connected composite units were permanent operations or whether they were specific to the situation of Protocol V. In Joe's case, whether his iterative fraction scheme was permanent and ready-at-hand for him to use, or whether he had to reconstruct it using the operations that were available to him was investigated.

Protocol VI was extracted from the teaching episode held on the 12th of January. The children had made an 11/11-bar and pulled out several fractional parts, one of which was a 6/11-bar.

Protocol VI. Melissa's construction of the iterative fraction scheme.

T: Tell me, is that (The 6/11-bar.) more or less than a half?

M: More.

T: (To Joe.) Can you make one that is a little bit less than that, that would be less than a half? What would be left of the candy bar if you took six-elevenths out? (Pretending that the bar was a candy bar.)

M: Five.

T: Five what?

M: Five-elevenths.

T: OK. Can you do that one, Joe?

J: Five-elevenths?

T: Yeah.

J: (Pulls a 5/11-bar from the 11/11-bar in such a way that it is the complement of the 6/11-bar that Melissa pulled out.)

T: If you had two pieces like that, how much of the bar would you have?

M: Ten-elevenths! (Joe also answered "ten-elevenths.")

T: Would that be the whole bar?

J&M: Less than the whole bar.

T: How much less?

J&M: One-eleventh.

T: OK, now put out the six-elevenths.

M: (Drags the 6/11-bar from the side of the screen and places it and the 5/11-bar end-to-end.)

T: If each of you have a bar like that… (The 6/11-bar.)

M: It wouldn't be any more of the bar. (The 6/11-bar and the 5/11-bar together made a bar commensurate with the 11/11-bar.)

T: What if you both had six-elevenths?

M: It would be one more than the bar.

J: Twelve-elevenths.

T: Twelve-elevenths! How much more than a bar would that be?

J&M: One-eleventh.

T: If you took eight of these two-elevenths… (Melissa dragged a 2/11-bar under the 11/11-bar.)

M: Sixteen-elevenths!

T: And how much of the bar would that be?

M: That would be…

J: Five-elevenths.

M: Five-elevenths more…

Protocol VI (continued)

T: Five-elevenths more than…

J&M: The whole bar!!

T: Referring to a 4/11-bar.) If you took eleven pieces like this, can you tell me how much you would have?

M: Forty-four!

T: Forty-four what?

M: Forty-four elevenths!

T: How many whole bars would that be?

J: Four!

T: Why would forty-four elevenths be four bars?

M: Because four times eleven is forty-four!

In this segment of the teaching episode, Joe and Melissa did not communicate directly with each other, but rather with the teacher. Nevertheless, the thinking of the two children seemed to be in harmony. Melissa's comment that the 6/11-bar was more than one-half of the 11/11-bar led the teacher to ask the question, "What would be left of the candy bar if you took six-elevenths out"? Melissa's answer of five-elevenths serves as confirmation that, for her, the six-elevenths when joined to five-elevenths constituted the whole of the 11/11-bar. This inference is corroborated by her subsequent comment that the 6/11-bar together with the 5/11-bar would not be any more of the bar, meaning that together, they would not be more than the bar. Not only did she enact pulling the 6/11-bar from the 11/11-bar using PULL PARTS, but also she mentally disembedded five-elevenths from eleven-elevenths, six-elevenths from eleven-elevenths, and integrated them together to reconstitute eleven-elevenths.

Both children convincingly demonstrated that they understood that a 12/11-bar contains the 11/11-bar and that it is one-eleventh more than the 11/11-bar. That is, they demonstrated a reversal in relation between the fractional part and the fractional whole in that what was before a fractional part now contained the fractional whole as a part. Because the children knew that the composite unit of numerosity twelve is one more than the composite unit of numerosity eleven, it would seem that it would be rather straightforward for them to understand that the fractional connected number twelve-elevenths is one-eleventh more than the fractional connected number eleven-elevenths. However, this relation between twelve and eleven has to be constructed anew in the case of fractions. The children's operations opened the way for their construction of the reversal in relation between the part and the whole and, as demonstrated in Protocol VII, for the construction of the relation between twelve-elevenths and eleven-elevenths as well as a more general iterative fraction scheme for composite fractions.

Both children knew that if they each had a 6/11-bar, then together they would have a 12/11-bar and that that bar would be 1/11 more than the whole bar. This was especially remarkable because they seemed explicitly aware of their reasoning and of the elements on which they operated: Melissa said, "It would be one more than the bar." In reply to the teacher's question, "What if you both had six-elevenths?"

and Joe knew that it would be twelve-elevenths as well. The children seemed aware of integrating the two composite units together, each containing six 1/11-bars.[6] In that one of these composite units was not in the children's visual field, the children worked in re-presentation when performing the integration operation. That is, the children operated on elements in visualized imagination, which also supports the inference of awareness.

Another indication of their explicit awareness of how they operated is that they mentally produced a 1/11-bar that was a part of the twelve-elevenths bar they mentally established but was not contained in the 11/11-unit bar. Mentally producing this 1/11-bar was a major achievement for the children and saying that the 12/11-bar was one-eleventh more than the 11/11-bar stood in for actually producing this 1/11-bar. My judgment is that the children's comments symbolized producing the 1/11-bar, which would entail an awareness of the operations involved.

The inference that the children were explicitly aware of the numerical whole-to-part relation between twelve-elevenths and eleven-elevenths as well as of the status of each 1/11-bar contained in the 12/11-bar as a unit fractional part of the 11/11-bar is corroborated by their knowing that sixteen-elevenths is five-elevenths more than the bar (the 11/11-bar). Producing five-elevenths further indicates that one-eleventh had become an iterable unit for the children that was on a par with their iterable unit of one. The children had constructed a fractional connected number sequence of which one-eleventh was the basic unit element, a sequence analogous to their explicitly nested number sequence where the basic unit element was one.

Further, the children seemed to operate with their fractional connected number sequence for one-eleventh in a way that was analogous to how they operated with their explicitly nested number sequence for one. This claim also finds corroboration in how the children operated after the teacher asked them the incomplete question, "If you took eight of these two-elevenths…?" Melissa almost immediately replied "Sixteen-elevenths!" and both children knew that this result was five-elevenths more than the unit bar. The children's way of operating indicates not only that two-elevenths was an iterable composite fractional unit for the children, it also indicates that the children engaged in a generalizing assimilation[7] of their units-coordinating scheme for composite units. They could now iterate two-elevenths eight times and produce sixteen-elevenths as the result.

It is quite impressive that the children also knew that eleven 4/11-bars would yield forty-four-elevenths. This knowledge is another corroboration of the inference

[6] To integrate the two 6/11-bars together means to unite them into a composite unit containing the two bars as component units and, further, to disunite the two bars into their elements and then unite these elements into a composite unit while maintaining an awareness of the two component units.

[7] Recall that an assimilation is generalizing if, from the point of view of the observer, the scheme involved is used in situations that contain elements that are novel for the scheme and if there is an adjustment in the scheme without the activity of the scheme being implemented.

that the children had engaged in generalizing assimilation of their units coordinating scheme for composite units. It is especially impressive that both Melissa and Joe knew that there were four unit bars in a 44/11-bar, "Because four times eleven is forty-four!" They were aware that if they iterated a 4/11-bar eleven times, they would produce a 44/11-bar. They were also aware that they could produce this 44/11-bar by iterating the 11/11-bar four times, "Because four times eleven is forty-four!" Although this knowledge may have been based on a functional interchange of the number of iterations and the number of elements in the composite unit being iterated rather than on operations that produce an awareness of commutativity, it still indicates a generalization of their units-coordinating scheme for composite units. Not only was the 44/11-bar eleven times the 4/11-bar, it was also forty-four times the 1/11-bar. That is, the fraction 44/11 now stood in multiplicative relation to one-eleventh in that it was forty-four times one-eleventh.

It could be said that both Joe and Melissa had constructed *fractional numbers* because, after making the 12/11-bar using the 6/11-bar, they knew that the 12/11-bar was one-eleventh more than the 11/11-bar. They did not resort to constituting one part of the 12/11-bar as one-twelfth because the 12/11-bar had twelve parts. This indicates that the children regarded a 1/11-bar as contained in the 11/11-bar but also as a unit fractional number that could be iterated enough times to produce improper fractions. Maintaining an awareness of both of these aspects of the 1/11-bar was essential for the children to designate each part of the 12-part 12/11-bar as one-eleventh. But it is not sufficient. The children also regarded the 11/11-bar as both a composite unit (i.e., a single entity) and as consisting of eleven times one of its unit parts (Melissa said that the 12/11-bar would be one more than the bar). Speaking of "the bar" indicates that she regarded the 11/11-bar as an entity, and by saying that the 12/11-bar was one more than the bar, she indicated an awareness of "the bar" as being eleven times one of its parts. Moreover, both children said that a 16/11-bar would be five-elevenths more than the whole bar. Implicit in this statement is the understanding that the whole bar is also an 11/11-bar, i.e., a bar that is eleven times one of its parts. A fraction was no longer simply a part of a fractional whole for the children, because the relationship between the fractional numbers twelve-elevenths, sixteen-elevenths, and forty-four elevenths and the fractional whole had changed in that the fractional whole was now contained in the fractional numbers. This is a defining characteristic of fractional numbers and clearly differentiates the iterative fraction scheme from the partitive fraction scheme. The splitting operation is lacking in the partitive fraction scheme, and it is this operation along with the operations that produce three levels of units that permits a child to reorganize the partitive fraction scheme into an iterative fraction scheme. These operations are also the operations that permit the construction of the unit fraction composition scheme.

The question concerning whether Joe's iterative fraction scheme was permanent and ready-at-hand for him to use is answered in the affirmative. Further, establishing the iterative fraction scheme and a fractional connected number sequence for elevenths in the context of Protocol VII is indication that Melissa's equipartitioning operations for connected numbers, which were used to find how many eightieths you can fit into one-fifth, were permanent operations[8] that she could at least use in activity. It was indeed surprising that Melissa could operate so powerfully the first time that we presented improper fraction tasks to her and it is testimony to the adaptability of children who can use three levels of units in assimilating improper fraction situations.

Testing the Hypothesis that Melissa Could Construct a Commensurate Fraction Scheme

Given that Melissa had constructed equipartitioning operations in Protocol V and the iterative fraction scheme in Protocol VI, the hypothesis that she could construct a commensurate fraction scheme seemed plausible. The test of this hypothesis turns on whether she could transform a proper fraction into a commensurate fraction. In Protocol VII, which served in testing the hypothesis, the unit bar was still the 11/11-bar and the teacher began by asking the children to partition the bar so they could still pull out elevenths.

Protocol VII. Making six-elevenths using a 22/22-bar.

T: Joe, and Melissa, both of you, can you think of another number of parts you could put in this bar (A copy of the 11/11-bar.) and still pull your elevenths out of it?
M: (Partitions the bar into twenty-two parts using PARTS.)
T: Pull three-elevenths out for me.
M: (Pulls a 3/22-bar from the 22/22-bar.)
J: (Watches Melissa intently.) That's three twenty-two.
T: (To Joe.) Could you pull out three-elevenths?
J: (After Melissa erases the 3/22-bar, mistakenly pulls out a 5/22-bar, but tries again this time pulling out a 6/22-bar.)
T: OK, now Melissa, why is that three-elevenths?
M: (Drags the 6/22-bar to the end of the 5/22-bar, and doesn't respond to the teacher's inquiry.)
T: (After asking Melissa to explain four more times.) OK, explain to Melissa why that's three-elevenths.
J: Because…I know if you pull out two twenty-twos it will be one-eleventh. So, if you pull out six of those it will be three-elevenths!
T: (To Melissa.) Does that make sense to you?
M: (Nods.)

Because it was Melissa who independently made the decision to partition the unmarked copy of the 11/11-bar into twenty-two parts, it was a surprise that she pulled out a 3/22-bar instead of a 6/22-bar after the teacher asked her to pull out three-elevenths. It was especially surprising in view of her clearly stating in

[8]These operations are the same operations that produce three levels of units that both children used to produce the iterative fractional scheme.

Protocol V, "How many eightieths can you get to fit into one-fifth!" when it was her goal to find how many parts she would need to break a 1/5-bar into so that each part was one-eightieth of the unit bar. In contrast to Joe, she seemed to operate as she did in Protocols III and IV; she produced the 22-part bar without establishing, prior to activity, the 11/11-bar as a unit containing eleven connected units each of which could be partitioned into two connected units using her equipartitioning operations. That she could not offer an explanation for why the six-part bar Joe pulled out was three-elevenths of the bar contraindicates that she used her equipartitioning operations to produce the 22-part bar at the very beginning of the protocol.

Joe, on the other hand, not only pulled out a 6/22-bar when asked to pull out three-elevenths of the bar, but also explained that, "I know if you pull out two twenty-twos it will be one-eleventh. So, if you pull out six of those it will be three-elevenths!" This explanation indicates that he iterated a 2/22-stick three times where the 2/22-stick and the 1/11-stick were identical, which corroborates Joe's use of equipartitioning operations applied to connected numbers as the key operations that permitted Joe to act so powerfully. So, equipartitioning operations were assimilating operations of Joe's, but not Melissa's, units-coordinating scheme for connected numbers.

The question now is whether Melissa's equipartitioning operations that she used in Protocols V and VI could be activated as assimilating operations in the context of finding fractions commensurate to proper fractions. The continuation of Protocol VII provides an occasion to further analyze Melissa's assimilating operations.

Protocol VII. (Cont)

T: (To Melissa.) You give Joe an elevenths fraction to pull out of your bar. (The 22/22-bar)

M: Umm, let me see. Ten-elevenths.

T: Joe, can you pull out ten-elevenths?

J: (Using PULL PARTS, slowly drags a rectangle around twenty parts and pulls them out of the 22/22-bar. He counts by twos in the process.)

M: (Counts the twenty parts as Joe counted them.)

T: (To Melissa.) How many parts is that?

M: (Immediately.) Twenty.

T: (After Joe drags the 20/22-bar directly underneath the 11/11-bar with left endpoints coinciding.) Can you check it, Melissa?

M: (Counts the ten parts of the 11/11-bar that are spanned by the 20/22-bar.)

T: Joe, you give Melissa one.

J: Six-elevenths.

M: (Counts off six of the eleven parts of the 11/11-bar starting with the left-most part. She then drags the 22/22-bar directly beneath the 11/11-bar.)

T: (Laughing.) How many parts is that, Melissa?

M: Twelve.

Melissa's decision to ask Joe to pull out ten-elevenths seemed to be based on the 11/11-bar rather than on the 22/22-bar. To justify why Joe's answer of twenty was related to 10/11-bar, she counted the ten parts of the 11/11-bar that corresponded to the 20/22-bar. She did not give an explanation based on the logical necessity that two one-twenty-seconds in each one-eleventh implies that twenty-twenty-seconds would be ten-elevenths because two times ten is twenty. Rather, she made a direct reading using the bars to justify Joe's answer and proceeded in a very similar way when Joe asked her to find six-elevenths of the 22/22-bar. For these reasons, we infer that when she produced the 22/22-bar in Protocol VII, she simply added eleven and eleven, which is to say that she counted by eleven. By counting this way, she proceeded in a way that was similar to how she produced the sequences for five in the teaching episode held on the 1st of December (cf. Protocol III). In order to continue probing Melissa's assimilating operations, the teacher gave Joe the problem of making another bar using the unit bar with a different number of parts so that he could still pull out elevenths.

Protocol VIII. Melissa produces a 33/33-bar by inserting three into each part of a 11/11-bar.

T: (To Joe.) Go ahead, Joe. Copy the unit bar and make a different one now.
J: (Makes a copy of the unit bar and wipes the marks off from it using WIPE BAR. He then partitions it into twenty-two parts.)
T: That is the same as Melissa's bar!
J: Oh, I know. (Wipes the bar and tries to dial Parts to "33," but the greatest numeral on the dial is "32.")
T: You are trying to get to what number?
J: Thirty-three.
T: Is there a way to use the unit bar to get a bar with thirty-three parts in it?
M: Make it eleven parts!
T: So you can use a bar with eleven in it. OK.
J: (Dials PARTS to "11" and clicks on the copied, but blank, bar.)
M: Then put three in each one. (Pointing to the first three parts of the 11/11-bar that Joe made.)

Melissa's decision to "make it be eleven parts" and her comment, "then put three in each one," are the first indications that she made a units-coordination to produce a different number of parts in the unit bar so she could still pull out elevenths. The teacher in the continuation of Protocol VIII explored the consequences of her perhaps momentary and contextual insight.

Protocol VIII. (Cont)

J: (Joe breaks the 11/11-bar and partitions each part into three parts. He then joins the eleven 3/33-bars together to form a 33/33-bar and indicates to the teacher that he wants to pull out six-elevenths. He then slowly drags a rectangle over a few pieces but stops after he realizes that the rectangle did not contain the first part he wanted to pull out.)
T: Is there a way you could pull one-eleventh out? How many times would you have to repeat it?
J: Six.
T: OK. Do that.
J: (Pulls a 3/33-bar out.)
T: How many of them will you need – how many thirty-thirds?
J: Eighteen.
T: Do you think he is right, Melissa?

Protocol VIII (continued)

M: Yep.
T: Why?
M: No! He needs twelve!
J: (Makes the 18/33-bar and drags it directly underneath the 11/11-bar with left endpoints coinciding.)
M: (Counts the parts in the 18/33-bar and agrees that there are eighteen.)
T: How did you get twelve?
M: I was using the twenty-seconds!

Even though Melissa conflated using twenty-seconds and thirty-thirds, the fact that she said that Joe would need twelve of what she considered to be twenty-seconds indicates that she regarded each 1/11-bar as containing a 2/22-bar. There was at least a temporary modification in Melissa's units-coordinating scheme in that equipartitioning operations were used in constituting the situation. This local modification opens the possibility that Melissa could construct a commensurate fraction scheme, but it does not establish it at this point in the teaching experiment.

Whether Joe could produce a plurality of fractions commensurate to one-eleventh[9] is doubtful. He did seem to be aware that, for any number of elevenths up to and including eleven-elevenths, he could produce a commensurate fraction using the 33/33-bar. But he seemed unaware that he could simply use his number sequence in systematically generating fractions commensurate to one-eleventh. There is no reason to believe that either child could produce a plurality of fractions commensurate to one-eleventh in the sense that they produced a fractional connected number sequence (cf. Protocol VI).

To test the hypothesis that Melissa's use of equipartitioning operations as assimilating operations in the first part of her units-coordinating scheme in Protocols VIII and its continuation was an accommodation, Protocol IX is selected from the teaching episode held on the 2nd of February. The teacher asked the children to use one-ninth, one-eighteenth, and one-twenty-sevenths to produce a fraction commensurate with a multiple of one of the other fractions.[10] The solution of these tasks would involve a coordination of the iterative fraction scheme and commensurate fraction scheme. These tasks were chosen to test the hypothesis that Melissa's equipartitioning operations for connected numbers were assimilating operations for her units-coordinating scheme because, in the case of her iterative fraction scheme, Melissa apparently took the results of her equipartitioning operations

[9] By a plurality of commensurate fractions, I mean an unbounded sequence of fractions. Such a sequence could be realized as an unbounded sequence where the next fraction of the sequence entails partitioning the 11/11-stick, say, using the next number in the number sequence for one. The children "produce" this sequence only in the sense that they are aware that they could continue on making fractions of the sequence an indefinite number of times.

[10] For example, use one-eighteenth to produce four-ninths.

as material in further operating when she justified why forty-four-elevenths (which she produced as eleven times four-elevenths) was four whole bars (cf. Protocol VI). Hence, if her construction of equipartitioning operations in Protocol VIII was on a par with her construction and use of operations that produce three levels of units to assimilate fraction situations in Protocol V, then she should be able to use her equipartitioning operations to establish how many of one unit fraction are in another, which would open the way for her to coordinate her scheme to find a particular commensurate unit fraction with her iterative fraction scheme in order to solve this novel task. Joe had already established equipartitioning operations as permanent operations in the first part of his units-coordinating scheme and the operations that produce three levels of units as the first part of his iterative fraction scheme, so a comparison of how the two children operated in the coordination tasks can be used in testing the hypothesis. There were three congruent bars in the children's visual field prior to Protocol IX: a 9/9-bar, an 18/18-bar, and a 27/27-bar. There was also a 1/9-bar, an 1/18-bar, and a 1/27-bar available.

Protocol IX. Melissa attempts to use the 1/18-bar to make a fraction that is not made up of eighteenths.

T: You have to make a different kind of fraction from the one you are using. If you make ninths, you can't use ninths, and if you make twenty-sevenths, you can't use twenty-sevenths. (To Melissa.) You pose a problem for Joe, now. Choose which units to use. (Pointing to the 1/9-bar, the 1/18-bar, and the 1/27-bar.)

M: (Chooses the 1/18-bar. She drags the 1/18-bar to a position beneath the three partitioned whole bars on the screen; the 9/9-bar, the 18/18-bar, and the 27/27-bar.) I want you to make…. (Sits looking intently at the screen for approximately 23 seconds) I know what I want to do, but I don't know how to say it!

T: What kind of a fraction do you want him to make? You have the eighteenth, and you want him to make what?

M: (Looks intently at the screen for approximately 20 seconds) Um, one-fourteenth.

T: (Clasps his hands to the side of his head.) Make one-fourteenth? Of the unit bar?

M: Use this (The 1/18-bar.) to make one-fourteenth of that. (Runs her hands along the 27/27-bar.)

T: You mean make fourteen parts out of this bar? (Pointing to the 27/27-bar.) Show me, I am not sure I understand.

M: (Repeats the 1/18-bar into an 8/18-bar and drags it directly underneath the 27/27-bar with left endpoints coinciding. The 8/18-bar is adjacent to twelve parts of the 27/27-bar. Shakes her head.)

J: Two-eighteenths makes three-twenty-sevenths!

T: (To Melissa.) Did you hear what he said?

M: Two-eighteenths makes twenty-sevenths.

T: (Asks Joe to repeat what he said.)

J: Two-eighteenths makes *three*-twenty-sevenths!

M: Oh.

T: Can you show us that, Joe?

J: (Fills three parts of the 27/27-bar directly beneath two parts of the 18/18-bar at the rightmost end.)

M: (After the teacher asks her if she sees that, nods.) Mm-hmm.

Melissa sitting and looking intently at the screen for approximately 23 seconds coupled with the comment, "I know what I want to do, but I don't know how to say it!" indicates that she experienced a perturbation that she could not resolve. She could iterate the 1/18-bar as many times as she wanted, so iteration or lack thereof was not a source of her experienced perturbation. To make a 1/9-bar using her 1/18-bar certainly would entail establishing two-eighteenths as commensurate to one-ninth prior to activity. So, her comment indicates that she could not establish this relation as she sat intently looking at the screen, although she had used the operations that produce three levels of units to produce a fractional connected number sequence (Protocol VI). Her choice of one-fourteenth as the bar that she wanted Joe to make is corroboration that she could not establish the two fractions as commensurate. She certainly did not coordinate her iterative fraction scheme with making commensurate fractions. It is important to note that if such a coordination had occurred, it would have also occurred prior to activity as it did for Joe when he said that two-eighteenths makes three-twenty-sevenths. That is, an inference that Melissa made a coordination of the two schemes would need to be based on her choice of an appropriate fraction that she wanted Joe to make, prior to observable activity.

Her use of the operations that produce three levels of units as constitutive operations of her iterative fraction scheme in Protocol VI apparently was not accompanied by an explicit awareness of herself as an operating agent, which means, in other terms, that she did not willfully execute the operations. Although she did know that sixteen-elevenths was five-elevenths more than the whole bar, this reasoning can be carried out without being aware of the operations involved in producing the unit of units of units on which it is based. Further, after Melissa produced forty-four-elevenths as eleven times four-elevenths and after Joe said that there were four unit bars in the forty-four-elevenths, she explained that there were four because four times eleven is forty-four. Although her explanation seems to imply that she was aware of the involved equipartitioning operations on which transforming eleven times four-elevenths into four times eleven-elevenths are based, her adding and multiplying schemes for whole numbers were evoked and used throughout the protocol. My interpretation is that the reason these whole number schemes were evoked was because she used her equipartitioning operations for connected numbers in assimilating the tasks and these operations evoked her adding and multiplying schemes for whole numbers. Such massive transfer, although it is solid corroboration of the reorganization hypothesis, does render opaque the issue of whether she was aware of the operations to which her explanation pointed. These operations entail reassembling a composite unit that contains eleven units each of which contains four connected 1/11-units into a composite unit containing four units each of which contains eleven connected 1/11-units. Joe, on the other hand, did willfully use equipartitioning operations in Protocol IX, which is compatible with his explicit awareness of the operation of recursive partitioning and its inverse (cf. the continuation of Protocol I) because equipartitioning operations contain the operation of recursive partitioning.

Melissa's Use of the Operations that Produce Three Levels of Units in Re-presentation

How Melissa might willfully use her operations that produce three levels of units in re-presentation was of primary interest in the analysis. In Protocol V, Melissa had constructed two composite units, a composite unit of eighty-eightieths and a composite unit of five connected units comprising five-fifths, and asked how many eightieths would go into each fifth. She produced an equipartitioning of eighty-eightieths and engaged in finding if fourteen of the eightieths would fit into each fifth. Although she did not explicitly iterate fourteen five times, she did use her multiplying algorithm to test whether five times fourteen is eighty. Her complete activity, when considered together, implies that she formed an image of eighty-eightieths, whatever it might have been, and was aware of partitioning re-presentations of a part of the 5/5-bar in order to produce eighty-eightieths. So, it was a puzzle to us why recursive partitioning was not ready-at-hand for her in those cases where there was no material in her visual field on which she could operate because recursive partitioning operations are operations that produce three levels of units. In the next section, we analyze protocols selected from teaching episodes where the children were asked to repeatedly make fractions of fractional parts of bars. The children's engagement in the tasks presented an opportunity to analyze Melissa's interiorization of recursive partitioning operations and Joe's construction of a scheme of recursive partitioning operations.

Repeatedly Making Fractions of Fractional Parts of a Rectangular Bar

In the teaching episode held on the 9th of February, the children made repeated partitions of a bar. At each step, the children were to find the fractional part of the whole bar that one of the parts comprised.

Protocol X. Joe and Melissa find a half of a half of a half of a….

T: (The children had made a rather large rectangular bar on the screen.) Melissa, you go first. You cut it into halves. Make it into two parts. Then Joe, you take one of those pieces and cut it into halves.

M: And then cut it and cut it and cut it….

T: And then take one of those parts and cut it again, and then keep on. At each one, I want you to tell me how much of the unit bar it is and why did you say that.

M: (Uses VERTICAL PARTS to partition the rectangular bar into two parts, breaks it upon the request of the teacher, and fills one of the parts a different color. She then says the part she filled is one-half.)

J: (Partitions the filled part into two parts using HORIZONTAL PARTS, breaks the two parts and fills one of them.) One-fourth. (Explains that it takes four of them to cover the whole piece.)

M: (Repeats the action using VERTICAL PARTS on one of Joe's parts.) One-sixth – (Shakes her head "no.") one-eighth! If you would do it to all of them, you would have eight. (With Joe, counts the parts that would have been made if a complete partitioning had been made at each step in verification after the teacher asked Joe if he thought it would be one-sixth or one-eighth.) You have four sets of two! (Fig. 8.1)

J: (Repeats the actions on the lower right rectangle using VERTICAL PARTS.) One-sixteenth.

T: Why do you say its one-sixteenth?

J: Cause that's the way we were doing it.

M: Cause that's the way we were doing it by twos. Two, four, six, eight, ten, twelve, fourteen, sixteen. (Pointing to eight places on the original rectangle where pairs of rectangles would have been made if a complete partitioning was made at each step.)

T: (Asks Joe to measure the part he made, and "1/16" appears in the NUMBER BOX.)

M: (Repeats the action on the right most one-sixteenth of the rectangle using HORIZONTAL PARTS.) One-eighteenth. (Fig. 8.2)

T: How much do you think it is, Joe?

J: (Counts from the bottom to the top of eight imagined columns of four rectangles starting from his right to proceeding to his left.) One-thirty-two.

T: (To Melissa.) How did you get one-eighteenth?

M: Well, I used doubles. (After the teacher asks her how she used doubles, she counts the imagined parts as did Joe and agrees with Joe's answer.)

J: (Repeats the action on one-thirty-second.) One-sixty-fourth. (Immediately.)

T: Wow! How did you get one-sixty-fourth?

J: Thirty-two and thirty-two. (A witness asked him how he knew to double thirty-two.) Because the first time we did it we got two. Then we doubled it and got four. Then we doubled that and got eight, and then doubled that, sixteen, and then kept on going!

T: Do you know why those numbers double like that?

J: Uh-uh. (No.)

M: One one-hundred twenty-fourths. (In anticipation of the fractional part of the rectangle she will make next.) (Repeats the actions on the lower right most rectangle.) (Fig. 8.3)

T: One one-hundred twenty-fourth, and you say? (Pointing to Joe.)

J: One one-hundred and twenty-eighth.

M: Oh, I guessed! (Repeats Joe's answer.)

J: (In explanation.) I took one four off from sixty-four. And then I added sixty plus sixty is a hundred twenty and I added the fours.

T: (To Melissa.) Do you know why we are going from sixty-four to one hundred twenty-eight?

M: Because we are counting by…. (Stops and looks confused.)

J: Double it.

M: We are doubling the numbers we get.

T: OK. Let's keep going.

J: (Repeats the action on the lower rightmost one one-hundred twenty-eighth. Looks into space.) I can do it! Forty (After a pause.) two fifty-six! One two hundred-fifty-sixths.

T: Wow!

J: I added the hundreds first, then I added the twenties and got forty, and then added the eights.

M: (Requests paper and pencil, and uses a computational algorithm.)

J: (Measures the part and "1/256" appears in the number box. Both children express pleasure that the computer confirmed their answers.)

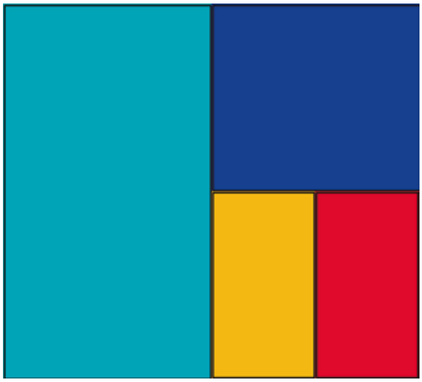

Fig. 8.1. Making one-eighth by partitioning one-fourth.

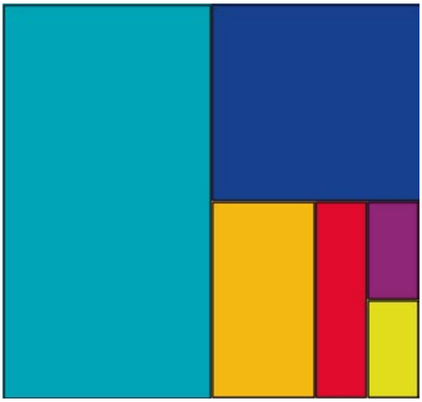

Fig. 8.2. Making one-thirty-second by partitioning one-sixteenth.

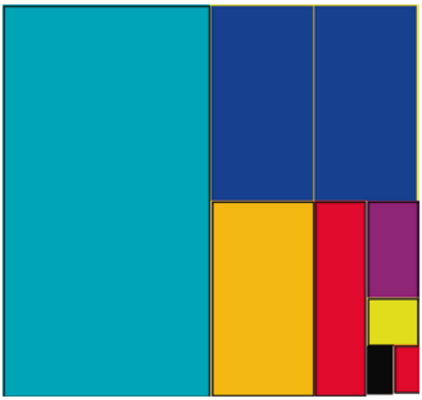

Fig. 8.3. Making one-one-hundred twenty-eighth by partitioning one-sixty-fourths.

Melissa's initial comment, "And then you cut it and cut it and cut it…" indicates that she imagined the steps of cutting into one-half. This is crucial because when engaging in recursive partitioning in re-presentation, the child regenerates the results of an immediately prior partitioning as input for further partitioning. However, Melissa's anticipation of cutting into one-half proceeded forward in sequence rather than recursively – "And then cut it and cut it and cut it…" Nevertheless, the fact that she could anticipate sequential partitioning acts is encouraging because it is fundamental in constructing a scheme of recursive partitioning operations.

After Melissa cut one-fourth of the rectangular bar into two pieces, she momentarily said that one of the parts was one-sixth of the rectangular bar. This same way of proceeding occurred after she cut one-sixteenth of the rectangular bar into two parts and said, "one-eighteenth." In both cases, she operated sequentially and added the number of parts she made to the preceding number of parts. In the first case, she made a self-correction and proceeded to count the parts that would have been made if a complete partitioning had been made at each step. Although we would not refer to these counting actions as partitioning prior partitions in re-presentation, they certainly constituted a modification in her way of judging how much a part of a current rectangle was of the whole rectangular bar. She definitely was aware of the history of the partitioning actions in that she organized their potential results as "four sets of two." In doing so, she projected partitioning into two parts into the unpartitioned parts that were in her visual field. She also projected partitioning into two parts into the unpartitioned parts in explaining why Joe said, "one-sixteenth" – "Cause that's the way we were doing it by twos. Two, four, six, eight, ten, twelve, fourteen, sixteen (pointing to eight places on the original rectangular bar where pairs of rectangles would have been made if a complete partitioning had been made at each step)." These were definitely enactments of partitioning the unpartitioned rectangles into two parts.

Joe's recursive partitioning operations enabled him to independently produce the parts of the rectangular bar that would be produced if partitioning into two parts had been completed at each step. His abstraction of the sequence of doubling acts – "Because the first time we did it we got two. Then we doubled it and got four. Then we doubled that and got eight, and then doubled that, sixteen, and then kept on going!" – indicates that he was aware of his acts of partitioning. Repeatedly engaging in recursive partitioning was not simply an operation that Joe carried out without an awareness of how he proceeded. Rather, he seemed aware of recursive partitioning in activity. If he took the results of recursive partitioning operations at each step as a unity and set those results at a distance, this would enable him to abstract that partitioning each part into one-half doubles the number of parts. This is a major step in the construction of a *scheme of recursive partitioning operations* because, in such a scheme, the child does not need the results of the preceding partitioning operations in his or her visual field to produce the next partition. But Joe would need to generalize it to other partitionings before we would judge that he had indeed constructed such a scheme. In addition, when the teacher asked him "Do you know why those numbers double like that?"

Joe did not know. This is a contraindication that he had explicitly correlated numerical doubling with doubling the number of parts made in the rectangular bar.

Melissa assimilated Joe's way of doubling the preceding numerical result to produce the next partition and guessed what the double of sixty-four would be before she partitioned the rectangle that was one-sixty-fourth of the original rectangular bar into two pieces. Her guess, "One one-hundred-twenty-fourths," was close, but she did not engage in strategic reasoning of the kind in which Joe engaged to produce "One one-hundred and twenty-eighths" and "One two hundred-fifth-sixths". Joe's strategic reasoning is a corroboration of his ability to take partitioning operations as a unity[11] and set their results at a distance and operate on the material he set at a distance using powerful numerical operations.[12] We take Melissa's lack of strategic reasoning as an indication of her more or less general way of organizing her experience into definite and knowable structures. Whether this orientation to mathematical activity constrained her progress in the creative construction of recursively partitioning the re-presented results of a prior partitioning will be addressed in the analysis of the remaining teaching episodes.

Melissa Enacting a Prior Partitioning by Making a Drawing

The modifications Melissa made in her partitioning operations in the teaching episode held on the 9th of February, where she repeatedly projected units of two into the partially partitioned portions of the rectangular bar, reemerged as a drawing in a similar task in TIMA: Sticks in the teaching episode held on the 16th of February. But prior to making the drawing, she enacted partitioning the parts of a 16-stick into two parts each by making cutting motions with her hand. This enactment preceded enacting partitioning by making drawings. The teacher used TIMA: Sticks during the teaching episode because of the way in which PARTS was programmed. In TIMA: Sticks, the children did not need to break a stick in order to partition a part of the stick. Rather, if a stick was partitioned into, say, eight parts, each of the eight parts could be partitioned into as many parts as desired up to and including thirty-two parts. At the beginning of Protocol XIII, Joe had partitioned a stick into eight equal parts and said that each part was one-eighth of the stick, and Melissa took her turn without comment from the teacher.

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[11] Taking a partitioning operation as a unity means that the child abstracts the relation between the input and output and can use a current output as input for further partitioning. Abstracting the relation means that the child is on the "outside" of partitioning and is able to generate and analyze relations between intermediate states with the proviso that the relation is reversible and can be regenerated at will.

[12] For this reason, I believe that he was very close to constructing a scheme of recursive partitioning operations and, hence, the operations necessary to construct, for example, the cyclic nature of the Hindu-Arabic numeration system as a result of productive thinking.

Protocol XI. Melissa enacts partitioning the parts of a 32-stick.

M: (Partitions the parts of the eight-part stick into two parts each and then counts the sixteen parts.) One-sixteenth.
J: (Partitions the seventh one-sixteenth part into two parts and fills one of the parts gray.)
T: How much is that little gray piece in there?
J: (After a short pause.) One-thirty-second.
T: Will you tell me how you got one-thirty-second?
J: I had one-sixteenth, and I just cut that in half and added sixteen and sixteen.
M: (Partitions the 1/32-part Joe made into two parts.) One-sixty-fourth. (Immediately.)
T: How did you get one-sixty-fourth?
M: I doubled two times thirty-two.
W: Why did that work, Melissa?
M: Because you would do it by counting by eights, oh not by eights, by sixteens.
T: You did not say anything about sixteen, did you?
J: You would make them in halves.
M: You would make one-half (Making cutting motions with her hand.) in all of them.

Melissa's comment that, "You would make one-half in all of them" while making cutting motions with her hand indicates awareness, however unarticulated, of partitioning each of the thirty-two parts into two parts.[13] Her saying that she doubled thirty-two to produce sixty-four seemed to be based on Joe's explanation of his answer of one-thirty-seconds rather than on the results of using her units-coordinating scheme because she also said that they counted by sixteens, which was a recapitulation of Joe's previous explanation. Still, we can deduce that she was aware that her action of partitioning one of the thirty-two parts also partitioned each of the other parts because she made cutting motions with her hand while saying, "You would make one-half in all of them." This enactment of imagined acts of partitioning into two parts in the presence of the 32-stick, which is indeed a units-coordination of thirty-two and two, will prove to be a crucial step in her ability to recursively partitioning a stick in re-presentation.

Joe saying, "I just cut that in half and added sixteen and sixteen," was another example of strategic reasoning like that in Protocol X. It again corroborates his ability to take partitioning operations as a unity, set their results at a distance, and operate on the material he set at a distance using powerful numerical operations, because the result of cutting each of the sixteen parts into two parts was reorganized into two composite units of sixteen. Such a reorganization involves an awareness of a composite unit of sixteen units of two, splitting each unit of two into two separated parts, and integrating these parts into two composite units of sixteen elements.

Before judging whether Melissa's imagining of partitioning acts and her enacting them by cutting motions with her hand constituted an accommodation in her units-

[13]In the first part of the protocol, the fact that Melissa counted the sixteen parts of the stick she just made by partitioning each of eight parts into two parts before she answered, "one-sixteenth," should not be interpreted as meaning that she had constructed a fractional composition scheme. Retrospectively, I will interpret it as an important step in such a construction.

coordinating scheme, a recurrence of them in the case of a partitioning of the elements of a partition into three or more parts would need to be observed. Fortunately, the teacher turned to a similar task that involved making thirds. We pick up the task just after Joe partitioned each of the three parts of a stick into three parts and said that one of these parts was one-ninth of the whole stick.

Protocol XII. Melissa's drawing of a partition of a partition of a partition**.**

M: (Partitions the first part of the 9/9-stick into three parts and fills one of the parts.) (Fig. 8.4)
T: OK. How big is that?
M: One-eighteenth!
T: What do you think, Joe?
M: Oh, no, no. One-twelfth!
M: (Before Joe replies, she tries to explain, and then starts to make a drawing, but her drawing is hidden by the monitor.)
J: One-twenty-seventh!
T: Melissa is drawing a number line.
M: (Looks up from her drawing.) It would be about one-twenty-seventh!
T: Joe says it is about one-twenty-seventh. Oh, show them how you figured that out.
M: (Holds her drawing over the screen of the monitor and it was structured something like the diagram that follows.) (Fig. 8.5)

Fig. 8.4. Melissa's partition of the first part of a nine-part stick into three parts.

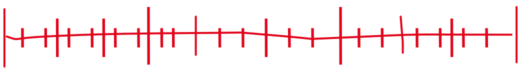


Fig. 8.5. Melissa's drawing of a partition a partition of a partition.

Melissa's drawing completed each step of the original partitioning – she first partitioned the stick into three parts, then Joe partitioned each one of the three parts into three parts, and then Melissa partitioned the first of the nine parts of the stick into three parts. Her initial two answers of "one-eighteenth" and "one-twelfth" preceded her drawing so these answers were not based on her drawing. After the teacher cast doubt on her answer of "one-eighteenth" by asking Joe what he thought, Melissa turned inwardly and enacted the complete partitioning of the 9/9-stick by making her drawing. Her enactment of the partitioning solidly indicates that she was in the process of interiorizing the involved operations. It was an indication because, while in the process of drawing, it would be necessary for her to monitor her activity. Should her enactment of partitioning a partition by making a drawing recur, the making of a drawing could be considered as an accommodation in her units-coordinating scheme in the context of fractional connected numbers.

A Test of Accommodation in Melissa's Partitioning Operations

If Melissa's enactment of partitioning activity constituted an accommodation, then it should reoccur in situations similar to the situation of Protocol XII. In the teaching episode held on the 23rd of February, the teacher returned to using TIMA: Bars. This time the teacher asked the children to first partition a bar into three parts, then one of those parts into two parts, then one of those parts into three parts, etc. This change was designed to break the process of always doubling, or always tripling, the last number of parts to produce the current number of parts, a process that had been abstracted by Joe and that Melissa had appropriated without engaging in a similar abstractive process. If the children were to produce the sequence; 1/3, 1/6, 1/18, 1/36, 1/108, 1/216, 1/648, 1/1,296,…, it would be based on at least partitioning the whole of an immediately preceding partitioning. We pick up the conversation where Joe partitioned one-sixth of the bar into three equal parts.

Protocol XIII. Melissa partitions unpartitioned parts of a bar in re-presentation.

J: (Partitions the lower 1/6-bar into three horizontal parts and fills one of them a different color.) (Fig. 8.6)

T: Now, the question is, what's that piece? Figure it out before you say it.

M: (Counts the three parts Joe made from the bottom to the top, and then continues on, pointing to three more places on the 1/6-bar immediately above the three 1/18-bars. She then starts at the top of the adjacent 1/3-bar and points to it six times from the top to the bottom. She then completes her counting episode by starting at the top of the leftmost 1/3-bar and points to it six times from the top to the bottom. Each time she pointed, she subvocally uttered a number word.)

T: Do you know Joe?

J: One-eighteenth.

T: (To Melissa.) What do you say?

M: (Nods.)

T: How did you get that?

M: Because there are three sets of six! (Partitions the bottommost 1/18-bar into two vertical parts, breaks the bar and fills the right most part yellow.) (Fig. 8.7)

T: OK, we've got that little yellow piece out there now. How much is that little yellow piece?

M: (Looks at the screen.) That little yellow one?

J: (Starts to answer, but is hushed by the teacher.) Wait for Melissa to get hers.

M: (Looks downward while she is subvocally uttering number words.) One-thirty-second.

T: One thirty-second. And what did you say, Joe?

J: One-thirty-sixth.

T: (To Melissa.) Show me how you did it.

M: Well, I went by twos. (Points to the two 1/36-bars, then to the 1/18-bar immediately above those two bars, then to the next 1/18-bar, then three times to the 1/6-bar.) Two, four, six, eight, ten, twelve,….

T: How did you do it, Joe?

J: I just doubled it.

T: Why did you double it? Why didn't you triple it, or quadruple it?

J: Because I tried thirds, and that didn't work out, so I went to halves and doubled it.

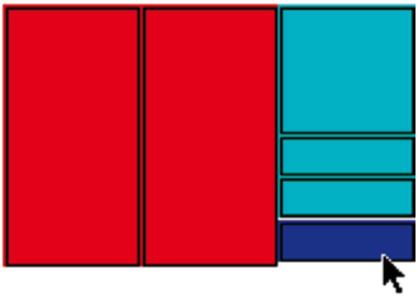


Fig. 8.6. Joe partitioning one-sixth of the bar into three parts.

Fig. 8.7. Melissa's partition of one-eighteenth of a bar into two parts.

In the first step of partitioning in the protocol, Melissa's counting of the parts that would be made had the partitioning been completed at each step and structuring her activity as "three sets of six" constitutes a structuring of the results of completing the prior partitions even though she did not make a drawing as she did in Protocol XII. Melissa was very precise in structuring her activity as three sets of six – as a unit containing three units each containing six units. This comment is a solid indication that she structured her experience as a unit of units of units and that she was aware of her unit structure. The whole of the counting episode constitutes an enactment of recursive partitioning. Further, when Melissa looked downward while she subvocally uttered number words before answering "one-thirty-second," she explained, "Well, I went by twos." Counting in this way would entail making a visualized image of the partially partitioned bar and projecting units of two into parts of the bar as she counted by twos. Partitioning a visualized image of a prior partitioning is precisely the operation that is involved in recursive partitioning at the level of re-presentation given that it was her goal to find how much one-half of one-eighteenth was of the unit bar. We take this to be her goal, and to eliminate the discrepancy between her expectation of establishing a fraction for one-half of one-eighteenth, and not knowing how many parts this partitioning act implied for the unit bar, she completed the partitioning activity of the prior two steps at the level of re-presentation. So, making the drawing in Protocol XII was an accommodation in her recursive partitioning operations.

A Further Accommodation in Melissa's Recursive Partitioning Operations

If she had not monitored "going by twos" in Protocol XIII, Melissa probably would have become lost in her activity. Her monitoring became more explicit in the following protocol that was extracted from the teaching episode held on the 2nd of March. In that teaching episode, Melissa independently introduced a notational system as a correlate of her drawing. Her notational system was a surprise and it seemed to serve two functions for her, as we explain below In Protocol XIV, Joe was to make fourths and Melissa was to make thirds of the resulting bars.

Protocol XIV. Melissa's notational system.

J: Partitions the bar into four horizontal parts and fills one part with a different color. (Both children agree that one-fourth of the bar was filled.)

M: (Partitions the filled part into three vertical parts and fills the middle part with another color.)

T: (Joe starts to say something.) Hold on, I want you both to get the answer. (After both children have the answer, Melissa says she got one-twelfth.) Why did you say its one-twelfth?

M: Because if you put all in those squares, you would get twelve pieces. You would have four sets of three.

J: (As explanation, counts across the uppermost 1/4-bar three times, then across the next 1/4-bar three times, etc.)

T: Joe, I believe it is your turn now.

J: Partitions the middle 1/12-bar into four parts and fills the uppermost part (Fig. 8.8).

M: (Points to three places on the uppermost 1/4-bar as she moves her hand horizontally from the left to the right side. She then points to four places on the uppermost 1/4-bar as she moves her hand downward along the left side. Continuing, she points to four places on the second 1/4-bar, and then to four places on each of the two remaining 1/4-bars as she moves her hand downward.)

J: (Sits silently in deep concentration while Melissa is counting. As Melissa is almost done.) One-forty-eighth!

T: Could be. Wait for Melissa to get her answer, too.

M: (Repeats the four modules of four counting acts downward.)

J: (While Melissa is counting downward the second time, uses MEASURE to measure the 1/48-bar he filled. He tries to hide the answer in the NUMBER BOX using his hand. His activity seems to distract Melissa.)

T: (To Melissa.) What did you get?

M: (Joe removes his hand.) I can see it!

T: What do you think it was before you saw it?

M: I was doing fifteen times four.

T: (Hands Melissa a piece of paper upon her request.) Where did your fifteen times four come from? (Hands Joe a piece of paper.) I will give Joe one so he can use it later.

M: (Writes on her paper which is hidden by the monitor. Looks intently at the screen for approximately 20 seconds. She then returns to writing on her paper.) And the next one was…He did three, so…

T: (After about 15 seconds.) So you're starting to make a sequence of pieces here. (After about 30 seconds.) Now you're drawing a picture of the pieces. (Indicates to Joe that he should show what he did using his paper and pencil.) How did you get your answer, Joe, while she's working on hers?

J: I did twelve times four.

M: (After about 45 seconds.) And the next one was one-twelfth. And the next one was one-forty-eighth.

T: How did you get one-forty-eighth?

M: First, he did his fourths. Then I did my thirds, then he did his fourths.

T: OK, let's hold this up here to see what you have been doing. (Holds Melissa's paper up to the monitor.) Joe started out with a fourth, Melissa thirded it and got a twelfth, Joe fourthed it and got a forty-eighth. Then she drew the picture at the bottom so they could see what's going on (Fig. 8.9).

Fig. 8.8. Joe's partition of one-twelfth of a bar into four parts.

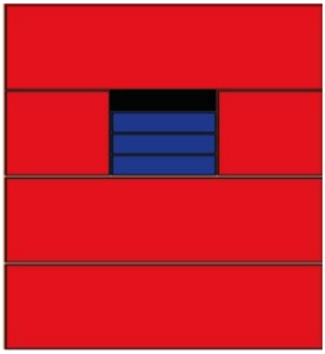
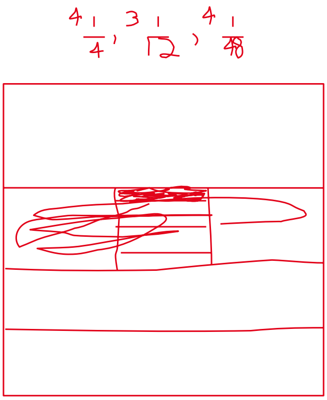


Fig. 8.9. Melissa coordinating her drawing and her notational system.

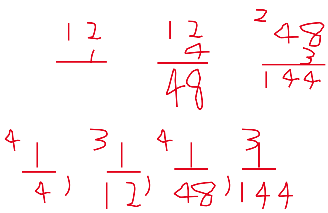


It is quite significant that Melissa recorded both the number of parts into which a particular bar was partitioned and the fraction of the whole bar produced by that partitioning. For example, the "4" in third step in her notational schema [^{4}1/48] referred to partitioning one-twelfth of the whole bar into four parts and the "1/48" referred to both forty-eight parts in the bar and the fraction one of those parts was of the forty-eight parts. Her notational schema not only recorded the history of her partitioning acts, but also indicates that she took the output of each prior partitioning act as input for the next partitioning act. For example, after producing the 1/12-bar in her drawing by partitioning a 1/4-bar into three parts, she partitioned the 1/12-bar into four parts and produced the 1/48-bar. She only completed the first partition in her drawing, so her notation $^{3}1/12$ stood in for partitioning each 1/4-bar into three parts in order to partition the whole bar into twelve parts. That is, she seemed well on the way to constructing her partitioning operations as recursive at the level of re-presentation. To make such an inference, of course, does not require building a notational system because Joe said "one-forty-eighth" after sitting in deep concentration while Melissa was counting over the computer graphics.

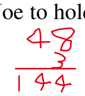
The effect of Melissa's abstractive activity was that, when it was her goal to find, say, how much one-third of one-fourth of a bar was of the bar, she established this situation as a situation of her units-coordinating scheme. It is no exaggeration to say that these partitioning operations had become the operations of her units-coordinating scheme and that the activity of her scheme – finding how many parts was in the partitioning so produced – was recorded in these operations. So, if Melissa produced the goal of finding how much one of the three equal parts of one of the four equal parts was of the unpartitioned bar, this activated the abstracted operations of partitioning into three parts and distributing these operations over the remainder of the four parts while monitoring how many parts were produced as a result of the distribution. The implementation of these operations, along with comparing the part to the partitioned whole, constitutes the activity of a unit fraction composition scheme. The continuation of Protocol XIV contains corroboration that Melissa used her units-coordinating scheme to produce the number of parts of the partition implied by partitioning a 1/48-bar into three parts.

Protocol XIV. (Cont)

T: All right. What's next, the thirds? Who takes the thirds?
M: I do. (Partitions the uppermost 1/48 bar into three vertical parts and fills the middle part.) (Fig. 8.10)
T: Think about it before you actually do it.
M: Count by twelve.
T: Can you tell me what the answer is going to be if you do that? OK, how big is that yellow piece there?
J: One one-hundred forty-fourth.
T: Melissa, what did you get?
M: One one-hundred forty-fourth.
T: OK, let's hold this up. (Melissa's paper.)



T: (Asks Joe to hold up his paper.)
J:



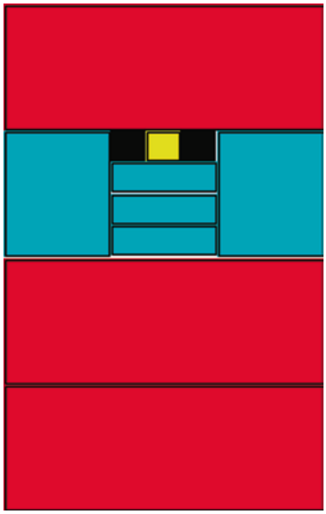


Fig. 8.10. Melissa's partition of a one-forty-eighth bar into three parts.

Melissa extended her sequence of fractions beyond one-forty-eighth by multiplying forty-eight by three, which corroborates that she used her units-coordinating scheme in the production of one hundred forty-four.

Melissa's independent production of her notational system was indeed a surprise and it was not preceded by any intentional action or comment by the teacher. She had already constructed one- and two-digit numerals as symbols in that they referred to at least an iterable unit of one that could be iterated the number of times indicated by a particular numeral to produce a composite unit containing the units produced by iteration, and she could use her numerals to stand in for these operations.[14] Her numerals had also taken on new meaning in that now "4," say, could stand in for partitioning a bar or a stick into four parts. Further, she used her fraction numerals, such as "1/48," to stand in for the operations of her iterative fraction scheme because she did not need to carry out all of those operations to give meaning to the shaded bar in Fig. 8.8. Her drawing that is replicated as Fig. 8.9 solidly indicates that she could give meaning to the numeral prior to making the drawing. So, one source of her creative act of producing the notational system resided in the symbolic nature of her whole number and fraction numerals. Another, of course, was her use of recursive partitioning at the level of re-presentation.

Given that it was Joe who began the teaching experiment in fifth grade having already constructed a reversible unit fractional composition scheme, it was indeed surprising that Melissa seemed to be the stronger of the two students in symbolizing a sequence of recursive partitioning operations using drawings and notation. Joe's abstractive power is well illustrated in the second continuation of Protocol XIII when he partitioned a 1/18-bar into two parts and said that one of the three parts

[14] In this case, I consider Melissa's numerical concepts as multiplicative concepts.

was one-thirty-sixth of the whole bar after sitting silently in deep concentration. Melissa, on the other hand, resorted to doubling and said that the part was one-thirty-second of the whole bar. In this case, Joe used the results of his prior recursive partitioning operations in his current recursive partitioning operations. This opens a possibility that Joe had constructed a scheme of recursive partitioning operations. However, it must be remembered that Joe was not asked in Protocol XIII to explain how he could operate. Had he been able to make such an explanation, this would be a solid indication that he was becoming aware of how he could operate on any but no particular turn,[15] which is essential in inferring a scheme of recursive partitioning operations. The notational system that Melissa generated in Protocol XIV and its continuation is more of an indication of a scheme of recursive partitioning operations than was Joe's ability to produce the next fraction in the sequence of fractions being produced by the children's partitioning actions by multiplying the number of parts in the current partition and the number of parts produced by the preceding partitions. But Melissa's notational system is still not sufficient to infer a scheme of recursive partitioning operations, although it is a strong indication. Making such a notational system is based on an awareness of taking the results of current operation as input for the next operation, so she definitely became aware of how she was operating. This is the basis for my inference that she had finally constructed recursive partitioning. But to infer a scheme of recursive partitioning operations, Melissa would need to use her notational system as input for further operating without resorting to making drawings.

A Child-Generated Fraction Adding Scheme

Both Joe and Melissa had constructed a unit fraction composition scheme that they used recursively in the sense that they used the results of a prior use of the scheme as input for another use of the scheme. During the course of the teaching experiment, we did not emphasize written notation with Joe and Melissa because we were primarily interested in exploring the children's operations and the schemes that they could use to supply meaning for written notation. For example, neither child construed finding how much of the whole bar was constituted by one of the three equal parts of one-forty-eighth as fractional multiplication in the continuation of Protocol XIV, even though their ways of operating were constitutively multiplicative in the fractional sense. Given Melissa's independently generated notational system, however, we became keenly interested in analyzing her construction of a symbolized fraction adding scheme in the context of her recursive use of her unit fraction composition schemes.

In the teaching episode held on the 6th of April, the teacher presented situations like those in which Melissa's drawings appeared to explore the children's creative production of the first few terms of the series $1/2+1/4+1/8+1/16+1/32+\ldots$.

[15] This is akin to operating with a variable.

Protocol XV. Finding partial sums of the series, 1/2 + 1/4 + 1/8 + 1/16 + 1/32.

T: OK, we are going to do some fraction addition today. What we are going to do is take half, then half again, then make half again. I'll show you what I mean. (To Melissa.) Go ahead and make the first one.
M: (Partitions the bar horizontally into two parts, breaks the bar, and fills the lower part.)
T: (To Joe.) OK, you are going to take half of the unfilled part. Go ahead.
J: (Partitions the upper part into two parts vertically, breaks the bar and fills the rightmost part.)
M: Now we've got three-fourths!
T: Now, the question is, how much of the unit bar is filled?
M: Three-fourths.
J: One-half plus one-fourth.
T: Yeah, but what does that work out to?
J: (Picks up his pencil and writes on his paper.) I got three-fourths.
T: OK, let's see how you did it. (Holds Joe's paper up to the monitor)

$$\begin{array}{rcl} \frac{1}{2} & = & \frac{2}{4} \\ \frac{1}{4} & = & \frac{1}{4} \\ \hline & & \frac{3}{4} \end{array}$$

Joe's use of his computational algorithm for adding fractions was a complete surprise. Apparently, the teacher's comment, "OK, we are going to do some fraction addition today." Oriented Joe to use his computational algorithm. Melissa, on the other hand, knew almost immediately that the filled portion of the bar was three-fourths of the bar. Joe replied, "One-half plus one-fourth," but he did not use his recursive partitioning operations to partition the bottom one-half of the bar into two parts and then disembed the three parts from the four parts to produce three out of the four parts. These operations were available to him, but the use of his computational algorithm apparently closed off the activation of relevant operations.

Protocol XV. (First Cont)

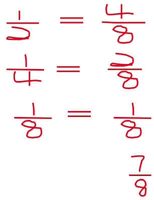
M: (Horizontally partitions the leftmost 1/4-bar into two parts and fills the bottommost part as shown below. The topmost part is unfilled.) (Fig. 8.11)
T: OK!
M: (Points her pencil to the unfilled part and then to the part to its right. She then repeats this pointing action at the filled 1/8-bar and then to the right of that bar. She then points to the 1/2-bar twice and then twice more. She then writes something on her paper.)
J: (While Melissa is pointing.) One-half plus one-fourth plus… (Points to the lower 1/8-bar but does not know the fraction for that bar. He then points in pairs to the bar downward four times as if he is counting 1/8-bars.) One-eighth!
M: (Points to the filled portion of the bar an indefinite number of times while Joe was pointing to the bar eight times. She then again writes something on her paper.)
J: One-half plus one-fourth plus two-eighths!
T: (To a query by a witness.) Two-eighths, he said.
J: One-eighth! (Continues writing on his paper.)

M: (In explaining to the teacher, she points over the bar eight times and then counts over the filled part seven times, where the pointing actions indicate regions of the bar that would be 1/8-bars had the whole bar been partitioned each time.)

J: I got it – I got it.

T: (Places Melissa's paper up to the monitor.)

T: (Places Joe's paper up to the monitor.)



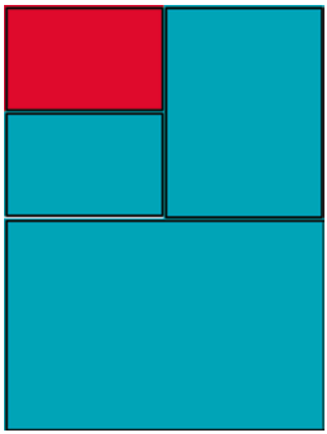


Fig. 8.11. Melissa filling one-eighth of the bar.

When Joe counted over the bar eight times to produce one-eighth, at that point he did not realize that he had essentially solved the problem of finding how much of the whole bar was filled. His goal was to find how much the filled 1/8-bar was of the whole bar. That goal, when coupled with his goal of finding the fractional part each filled bar was of the whole bar so he could use his computational methods, apparently excluded him from asking himself how many eighths could be made from the filled portion of the bar. Because his paper and pencil methods were separated from his recursive partitioning operations that he so powerfully demonstrated in earlier teaching episodes, his goal also apparently excluded finding how many eighths could be made from each of the 1/4-bar and the 1/2-bar using the operation of partitioning the results of prior partitioning, i.e., using reasoning. Melissa, not being in a computational frame of mind, worked insightfully and produced a sequence of partial fractional sums of the bar. However, Melissa had only begun to construct a fraction adding scheme when operating to find seven-eighths. In later tasks, she insightfully produced "15/16" and "31/32," also without explicitly using a fraction adding scheme. It is important to note that in the task where Melissa produced "31/32," when Joe tried to find the sum of whatever fractional meaning he gave to "1/32," "1/16," "1/8," "1/4," and "1/2," he became very despondent and made computational errors in attempting to change each of the last four fractions into thirty-seconds. Apparently, Joe's fraction adding scheme was procedural in nature and it excluded him from reasoning insightfully when he was capable of doing so.

An Attempt to Bring Forth a Unit Fraction Adding Scheme

The current teaching episode held on the 4th of May was conducted for the purpose of investigating the operations involved in finding the sum of two given unit fractions that are not produced as a part of a sequence of recursive operations. The teacher's goal was for the children to find what fractional part a 1/3-bar and a 1/4-bar together is of the whole bar. Because the reversible unit fraction composition scheme is involved in solving this task, the teacher began by presenting a task designed to reinitialize Joe's reversible fraction composition scheme.[16] In the task, the teacher began by asking the children, starting with a 3/3-bar, to partition the bar so that one-ninth could be pulled out.

Protocol XVI. Producing sequences of fractions using the reversible unit fraction composition scheme.

T: (After Joe made a 3/3-bar.) OK, Melissa. Use PARTS so you could pull a ninth out using the thirds.

M: (Dials PARTS to "3" and clicks on the leftmost one-third of the 3/3-bar. She then uses PULL PARTS to pull a 1/9-bar from the unit bar.)

T: (Asks Joe to erase the marks Melissa made to restore the bar to a 3/3-bar.) Joe, you make it so you can pull out a twelfth.

J: (Dials PARTS to "4" and clicks on the leftmost one-third of the 3/3-bar. He accidentally clicks twice so that the second part of the four is again partitioned into four parts. He then erases the extraneous marks and pulls out one of the four parts he made using PULL PARTS.)

T: (To Melissa.) OK, Melissa, it is your turn. What would you like to do now, Melissa?

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[16]In the second continuation of Protocol I, the teacher asked Joe to cut up a 1/4-stick so that each part is one-twentieth of the whole stick and Joe immediately partitioned the stick into five parts. Based on this protocol and Protocol II, where Joe partitioned each part of a 3/4-stick into two parts to make eighths of the whole stick, I inferred that he had constructed his unit fractional composition scheme as a reversible scheme. That is, given a result of the scheme, he could recursively partition a partition to produce the result.

M: One-fifteenth. (After erasing the marks Joe made to pull out twelfths, dials PARTS to "5," clicks on the leftmost one-third of the 3/3-bar, and pulls out one part.)

J: I will pull out an eighteenth!

T: You guys work out all of the ones that you can do.

J: (Dials Parts to "6" without erasing the five parts Melissa just made. He moves the cursor back and forth between the five parts that Melissa made and the middle one-third of the 3/3-bar without clicking. He then clicks on the middle one-third of the 3/3-bar, erases three of the four marks Melissa made, and then uses PULL PARTS to pull out one part of the six parts he made.)

T: (Asks the children to write down on their paper all of the ones they could pull out.)

J: (Writes the sequence of fractions along the top of his paper without difficulty until the teacher interrupts.) 1/3, 1/6, 1/9, 1/12, 1/15, 1/18, 1/21, 1/24, 1/27, 1/30, 1/33, 1/36.

M: (Writes the sequence of fractions across her paper also without difficulty.) 1/3, 1/6, 1/9, 1/12, 1/15, 1/18, 1/21, 1/24, 1/27, 1/30, 1/33, 1/36, 1/39.

T: That's really neat, too. OK, you guys, let's do fourths! Can you do that for fourths?

J: (Nods. He then writes the sequence of fractions without difficulty until the teacher interrupts.) 1/4, 1/8, 1/12, 1/16, 1/20, 1/24, 1/28, 1/32, 1/36, 1/42, 1/48, 1/52. (Holds his paper up to the monitor.)

T: (Reads the first few fractions.) Fourths, eighths, twelfths, sixteenths, twentieths, twenty-fourths, ooooh! (To Melissa.) Hold yours up there.

M: 1/4, 1/8, 1/12, 1/16, 1/20, 1/24, 1/28, 1/32, 1/36, 1/42, 1/48, 1/52, 1/56.

The children could have also produced a sequence of fractions each *commensurate* with one-third or any other reasonable unit fraction had the teacher presented such a task. Because it was the goal of the teacher to investigate whether the children could construct a unit fraction adding scheme involving one-third and one-fourth, after Protocol XVI, however, rather than actually ask the children to produce sequences of commensurate fractions, he asked the children to partition a 3/3-stick so that they could pull out a one-fourth of the whole stick.[17]

Protocol XVII. Attempts to pull a 1/4-stick out from a 3/3-stick.

T: OK, Melissa. I want you to think about this too, Joe. I want you to use PARTS and make it so you can pull out a fourth of this 3/3-stick.

J: (After about 20 seconds.) I know how to do it.

T: OK, I want you to write on your paper what you are going to do. (Both children write on their paper what they plan to do. After the children are done, he asks Melissa to hold her plan up to the monitor.)

M: (Writes: "I need to pull out 1/3 and change it to 1/4.")

J: (Writes: "I will clear the marks and cut it into four parts and pull one out.")

[17] To find how much of a fractional whole $1/3 + 1/4$ comprises involves finding a unit fraction for which both one-third and one-fourth are multiples. To produce this unit fraction, both one-third and one-fourth must be partitioned into a sufficient number of parts to produce such a unit fraction.

T: OK. Let's pretend, Joe, that you can't clear the marks. So, you have to revise your plan. (To Melissa.) Let's pretend that you can't pull a third out. But you can pull a fourth out. You can't pull a third out first. (Encourages the children to write their plans down rather than act using the TIMA: Sticks.) You have to pull one-fourth of the bar out. You have to partition it so you can pull one-fourth of the bar out.

J: (Writes, "I will erase the marks.")

T: (Laughingly reminds Joe that he cannot erase the marks.) I want you to use PARTS.

M: (Writes, "I will make 1/3 into a 1/4 without pulling anything out [she meant without pulling one-third out].")

T: OK, let's see if Melissa can use PARTS first.

M: (Activates PARTS, dials it to "4," and clicks on the leftmost part of the 3/3-stick.)

T: OK, now pull one-fourth out.

M: (Activates PULL PARTS and pulls the leftmost part out of the four parts that she made.)

J: Don't you have to make one-fourth of the whole stick?

M: (Covers the two remaining unmarked parts of the 3/3-stick with her hand.)

T: That is a fourth of what?

J: One-third!

T: That's a fourth of a third, isn't it? I want a fourth of the whole bar.

J: (Erases all of the marks on the stick, including the original two marks that marked the stick into a 3/3-stick.)

T: You've got to leave those third marks in there.

J: (Continues on in spite of the teacher's admonition. He erases all marks and uses MARKS, freehand, to subdivide the stick into four parts. This action circumvented the constraint that he was not to erase the hash marks he made using PARTS.)

T: Oh, I see what you are going to do!

J: (Tries to measure the last part he made, but there was not a unit stick in MEASURE. So, after he uses PULL PARTS to pull the last part out of the four-part stick he made freehand using MARKS, the teacher helps him copy a unit stick into MEASURE. Joe then measures the part he pulled out, and "11/39" appears in the number box.)

T: Is that a fourth?

J: (Hangs his head and laughs.)

T: Wow. Your plan didn't work (To Joe.) and your plan didn't work. (To Melissa.) Make a new plan. Go back and put in thirds. (Melissa wipes the stick clear of marks and uses PARTS to make another 3/3-stick.) Using PARTS, leaving the thirds there, I want you to use PARTS so that after you are done using PARTS, you can pull one-fourth of the bar out.

M&J: (Sit silently for approximately 60 seconds, so the teacher abandons the situation.)

Melissa did not enact her initial plan, "I need to pull out 1/3 and change it to 1/4" because the teacher introduced another constraint that she was not to pull out a 1/3-stick. In fact, the basic purpose of the teacher in asking the children to write their plans out was to eliminate certain actions which, if executed, might close out the possibility of the children engaging in those actions that would solve the problem. If the teacher had judged that Melissa could have indeed changed a 1/3-stick into a 1/4-stick after she had pulled out the 1/3-stick, there is no question that he would have encouraged her to do so. However, it was the judgment of the teacher that Melissa could not engage in such transformative actions because such actions would imply that Melissa had constructed fractions as rational numbers of arithmetic.[18] In fact, when Melissa executed her second plan, which was, "I will make 1/3 into a 1/4 without pulling anything out," she partitioned the leftmost part of the 3/3-bar into four parts and pulled out one part. This does indicate what she meant by making "1/3 into a 1/4" without pulling anything out. Presumably, had she pulled a 1/3-bar out from the 3/3-bar, she would have made it into a 1/4-bar in a similar way.

Joe eventually enacted his initial plan, which was, "I will clear the marks and cut it into four parts and pull one out." He planned to do this by erasing the marks on the 3/3-bar and using MARKS to mark the bar into four parts. This occurred even after the teacher attempted to induce the constraints of not erasing the marks and using the 3/3-bar in making a 1/4-bar. Rather than being arbitrary, the teacher permitted Joe to use MEASURE in verifying if he had indeed made a 1/4-bar using MARKS. Fortunately, the part Joe pulled out measured "11/39" and not "1/4" so that Joe realized that his way of proceeding did not work. That permitted the teacher to again restate the situation that they were to use PARTS to partition the 3/3-bar without erasing any marks. At this point in the protocol, both children sat silently and made no plans for how they could proceed.

Had the children been successful, we would infer that they had constructed their unit fraction composition scheme as a distributive scheme. That is, to find one-fourth of three-thirds, the children would find one-fourth of each third and unite these three parts together into three-twelfths. So, we close the case study by advancing the hypothesis that the construction of a unit fraction adding scheme entails constructing the unit composition scheme as a distributive scheme.

Discussion of the Case Study

Melissa made rapid progress from the 20th of October to the 1st of December in a way that is reminiscent of the progress that Patricia made during the first part of the teaching experiment when she worked with Joe in her fourth grade (cf. Chap. 7). During this time, however, Melissa seemed to experience internal constraints characteristic of children who have constructed only the partitive fraction scheme when engaging in attempts to solve the tasks of Protocols I and II. In her attempts to solve these tasks, she was dependent on Joe's independent solutions of the tasks in a way that was similar to how Laura was dependent on Jason's independent solutions of similar tasks. In essence, Melissa assimilated Joe's language and actions in Protocol I using her units-coordinating scheme in such a way that we characterized her use of

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[18]Operating on a 1/3-bar to make a 1/4-bar would entail partitioning the 1/3-bar into four parts, pulling out one part and iterating it three times to produce a 3/12-bar or a 1/4-bar. This kind of operating implies that the child has abstracted fractions as an ensemble of operations of which the child is explicitly aware, which is what I mean by the rational numbers of arithmetic.

the scheme as generalizing assimilation. But she could not modify this scheme to remove the constraint that she experienced in the second continuation of Protocol I. In that protocol, the contrast between Joe partitioning each of four parts of a 1/4-stick into five parts so he could pull out one-twentieth of the 4/4-stick and Melissa partitioning a congruent 1/4-stick into thirty-two parts to pull out one-thirty-second of the stick was quite pronounced and served as contraindication that she could modify her units-coordinating scheme to construct recursive partitioning operations at the level of re-presentation at this time in the teaching experiment.

It soon became apparent that there was a distinction between Melissa and Laura in the fall of their fifth grade in that Melissa had constructed the splitting operation. In fact, in the teaching episode held on the 1st of December (cf. Protocol IV), Melissa knew that she would need to put one-hundred parts in a 5/5-bar as a consequence of breaking a 1/5-bar so that each little part would be one-one-hundredth of the 5/5-bar. To know that she needed to put one hundred parts in the 5/5-bar, we inferred that she conceived of the 5/5-bar as partitioned into one hundred parts and that the whole bar was one hundred times any one of its parts. That is, we inferred that she conceptually split the 5/5-bar into one hundred parts and established a unit of five units, each of which contained, mistakenly, ten parts. It was in this same teaching episode that Melissa posed the question, "How many eightieths can you get to fit into one-fifth?" after the teacher changed the task from one hundred parts to eighty. On the basis of her question, we inferred that she was aware of a unit containing the eighty parts that she could partition into five equal parts, each of which contained an unknown but equal number of the eighty parts. Although essential, this inference is not sufficient to infer equipartitioning operations. It must be also possible to infer that each of the five parts is an iterable composite unit. We were able to make this second inference on the basis that she actually tried to find the number of eightieths that would fit into one-fifth using her computational algorithm. So, we inferred that she had constructed equipartitioning operations for connected numbers in the context of operating.

In Joe's case, the second of the two inferences was based on his strategic reasoning to produce sixteen as how many of the eighty parts could be fit into one-fifth. The first of the two inferences hinged on the indicators that he had constructed a reversible unit fraction composition scheme in Protocol II. For this scheme to be reversible means that the results of the scheme are taken as input for further operating. In that these results constitutively involve a unit of units of units, reversibility of the unit fraction composition scheme implies equipartitioning operations for composite units.[19] Joe's equipartitioning operations were more or less permanently constructed and available for him as assimilating operations that he used to constitute and independently solve situations that involved direct or inverse reasoning. However, we soon became aware that had Melissa not been engaged in operating,

[19]In Protocol II, Joe knew that he had to partition each part of the 3/4-stick into two parts in order to pull a 1/8-stick out from each part, so before he acted he had already mentally partitioned each part of the 4/4-stick that contained the 3/4-stick into two parts.

she would not have so clearly posed the question, "How many eightieths can you get to fit into one-fifth?" to herself, nor would she have explicitly formed the goal implied by the question.

The Iterative Fraction Scheme

In retrospect, Melissa's construction of the iterative fraction scheme and a connected number sequence in the teaching episode held on the 12th of January (cf. Protocol VI) might be regarded as an anomaly because, at that point in the teaching experiment, she was yet to use operations that produce three levels of units for connected numbers as assimilating operations. Melissa did, however, mentally split a bar into eighty parts in the context of producing three levels of units in operating in Protocol V. So, in the production of relations among parts and wholes, as indicated by her reasoning that sixteen-elevenths was five-elevenths more than the whole bar in Protocol VI, she could mentally split the whole bar into eleven parts. Therefore, to explain Melissa's construction of the iterative fraction scheme, it was sufficient that she split the fractional whole into eleven parts where each part could be iterated eleven times to produce eleven-elevenths. She could then disembed one-eleventh from the eleven-elevenths and use it as if it were an iterable unit of one. In fact, when she integrated a 6/11-bar with another 6/11-bar, this produced a 12-part bar that she said was *one* more than the 11/11-bar. She then said it was one-eleventh more upon the teacher asking, "How much more than a bar would that be?" Melissa reinterpreted the 12-part bar as containing the 11/11-bar, so each part of the 12-part bar was one-eleventh. She was quite capable of producing three levels of units in operating using discrete quantity, so her interpretation of the iterable unit of one as one-elevenths evoked generalizing those operations in the context of fractional numbers. For example, she construed forty-four-elevenths as forty-four times one-eleventh just as forty-four was forty-four times one. She interpreted forty-four-elevenths as if it were forty-four discrete, rather than connected, segments.

That Melissa was not aware of the operations she used to produce three levels of units[20] when constructing the iterative fraction scheme was corroborated in Protocol IX where she chose to make a fraction that was not made up of eighteenths using a 1/18-bar. To make such a bar entails coordinating the commensurate fraction scheme and the iterative fraction scheme prior to activity, as indicated by the way in which Joe produced three-twenty-sevenths as a fraction that could be made from two-eighteenths. This coordination is based on equipartitioning operations because, when it was Joe's goal to produce a fraction that was not made up of eighteenths using the 1/18-bar, he based his solution on establishing a 9/9-bar in visualized imagination and partitioned each ninth into two or three parts, whichever suited his goal. The operations that produce three levels of units were assimilating operations for Joe's commensurate fraction scheme and his iterative fraction scheme as well as

[20]This is another way saying that she did not use the operations as assimilating operations.

his unit fraction composition scheme. Melissa, on the other hand, produced the operations whose results are a unit of units of units in the context of actually operating. But she was yet to interiorize these operations in such a way that she could produce their results mentally without actually engaging in the operations.

Melissa's Interiorization of Operations that Produce Three Levels of Units

As analysts, we were constrained to the affordances of the teaching episodes when exploring Melissa's interiorization of operations that produce three levels of units and focused on her interiorization of recursive partitioning operations instead. This was justified because the latter operations are constitutively involved in the former. Focusing on recursive partitioning operations led in turn to investigating Melissa's use of her units-coordinating scheme in the context of three levels of units because units-coordinating is the mathematical activity in recursively partitioning a partitioned continuous unit.

Making drawings to complete prior partial partitionings was the key element in Melissa's interiorization of recursive partitioning. Melissa independently contributed her drawings and they were a surprise to the teacher. In explaining the emergence of Melissa's drawings, a reconsideration of how she used her units-coordinating scheme in completing her prior partitions is essential. In Protocol XIV, to explain how she arrived at one-twelfth, Melissa said, "Because if you put all in those squares, you would get twelve pieces. You would have four sets of three." We consider her saying, "because if you put in all those squares," as indicating that she mentally partitioned each one of the four parts into three parts, and her saying, "you would have four sets of three," as indicating that she structured the result into a composite unit containing four units, each of which contained three units. What is left implicit in Melissa's comments is her use of three as a partitioning template that she used to partition each of the four parts into three parts each. That is, the unit of three served as a template to partition the four units of the connected number four when it was her goal to find how much 1/3 of 1/4 of the whole bar was of the whole bar.

So, the change in Melissa's units-coordinating scheme consisted of extending the situation of the scheme from numbers containing discrete, separated units to include numbers containing continuous, connected units. The operations of units-coordination still consisted of two programs of operations, but with an alteration. When activated, the first program of operations produced, say, a connected number, four, and a partitioning unit, three, that could be used to partition each unit of the connected number, four. The second program of operations consisted of partitioning each of the four units into three units, uniting each trio of units together into a composite unit, three, and uniting these units of three into a composite unit containing four units of three, which is an accommodation in the operations of units-coordinating. She still inserted a unit of three into each unit of four, but the meaning of insertion changed from filling each unit of four with a unit of three to partitioning each unit of four into three equal parts.

The activity of the scheme was to progressively integrate the units of three with those preceding and increment the numerosity of the preceding units by three in order to find the numerosity of the parts produced by the partitioning. From an observer's perspective, Melissa's implementation of the activity of the scheme consisted of her counting how many parts would have been made had the partitioning activity been completed at each step (cf. the first continuation of Protocol XIII).

The making of a drawing in Protocol XII introduced a modification in the second program of operations in that, rather than using the stick in her visual field to carry out the units-coordination, she re-enacted units-coordinating activity by making a drawing. Such re-enactment involved sufficiently re-presenting the prior partitioning activity to enable her to make the drawing. However, what she was aware of was the drawing that she was making rather than an image of a stick on which she was operating in re-presentation. One might say that she was *in* the re-presentation and that the re-presentation was found in the activity of making the drawing. There was definitely visualization involved, kinesthetic as well as visual, but the visualization was an activity that was implemented as a drawing.

Had Melissa performed the visualizing activity mentally without actually making a drawing, this would involve monitoring the activity while it was being carried out in a recursive way using her concepts of four and of three. It is this monitoring of the activity which interiorizes the activity. Once interiorized, the child can execute the activity willfully and it can be said to be available to the child without the child actually engaging in the visualizing activity. The visualizing activity is produced by means of operating, so what the child has available is a program of interiorized operations that produce the visualizing activity.

For Melissa, the interiorization of the visualizing activity did not occur in one fell swoop. The interiorization process was also observed in the first continuation of Protocol XIII when she looked downward and away from the computer screen to count all of the parts that would have been produced had partitioning the whole bar been completed after partitioning the whole bar into three parts, then one of these parts into two parts, and then one of these parts into three parts, and then one of these parts into two parts.

After completing this partitioning activity, she partitioned the visible 1/18-bar into two parts and then continued on *in visualized imagination* (cf. Fig. 8.7). So, the material on which she operated was figurative. Metaphorically, she partitioned each visualized unit of her concept of three into two parts, and in this process, recorded this figurative material in operations of partitioning into two parts.[21] If the figurative material is "dropped out," leaving only its records, the figurative material becomes interiorized. So, what became recorded in the units of her concept of three were not

[21] The material on which operations operate become recorded or registered in the operations. The records are interiorized records to the extent that the results of operating – a partitioned bar – can be produced without actually executing the operations. All that is necessary is that the operations be evoked rather than implemented.

simply two unit items. Rather, the operations of partitioning into two parts were recorded into the units of her concept of three by means of the interiorized figurative material.[22]

On the Possible Construction of a Scheme of Recursive Partitioning Operations[23]

Given that it was Joe who began the teaching experiment in fifth grade having already constructed a reversible fraction composition scheme, it was indeed surprising that Melissa seemed to be the stronger of the two students in symbolizing a sequence of recursive partitioning operations using drawings and notation. Joe's abstractive power is well illustrated in the second continuation of Protocol XV when he partitioned a 1/36-bar into three parts and said that one of the three parts was one one-hundred-eighth of the whole bar after sitting silently in deep concentration. Melissa, on the other hand, resorted to doubling 36 and said that the part was 1/72 of the whole bar. In this case, Joe used the results of his prior recursive partitioning operations in his current recursive partitioning operations. So, this opens a possibility that Joe had constructed a scheme of recursive partitioning operations.

There is no doubt that Joe operated in a way that opens the possibility that he had indeed constructed such a scheme. However, it must be remembered that Joe was not asked in Protocol XV, or in either of its continuations, to explain how he could operate. For example, he was not asked how he would find how much the part would be if he and Melissa took, say, two more turns apiece. Had he been able to make such an explanation, this would be a solid indication that he was becoming aware of how he could operate on any but no particular turn,[24] which is essential in inferring a scheme of recursive partitioning operations.

The notational system that Melissa generated in Protocol XVI and its continuation is more of an indication of a scheme of recursive partitioning operations than was Joe's ability to produce the next fraction in the sequence of fractions being produced by the children's partitioning actions by multiplying the number of parts in the current partition and the number of parts produced by the preceding partitions. However, Melissa's making of her notational system is still not sufficient to infer a scheme of recursive partitioning operations even though we did infer that Melissa had constructed the operation of recursive partitioning based in part on her notational system. She definitely became aware of how she was operating because making such a notational system is based on an awareness of taking the results of

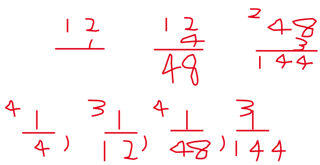
[22] A unit has two meanings. First, it is a unitizing operation, and, second, it is a proverbial slot in which records of experience are recorded.

[23] A scheme of recursive partitioning operations is different from a recursive partitioning scheme.

[24] This is akin to operating with a variable.

current operation as input for the next operation. This is the basis for my inference that she had finally constructed recursive partitioning.

To infer a scheme of recursive partitioning operations, it would be sufficient to be able to infer that Melissa was able to use her notational system as input for further operating without resorting to making drawings. However, in Protocol XVII, after both children were asked to find how many 1/144-pieces would fit into a 1/44-piece, Melissa resorted to using her drawing instead of her notational system. In that same protocol, she also made a drawing to find how many 1/144-pieces would fit into a 1/4-piece instead of simply resorting to her notational system that she had made and which we repeat below.



Melissa never realized that all she had to do was to "read" the numeral "3" from her notational system to find how many 1/144-pieces would fit into a 1/48-piece, etc., and to find the product of three, four, and three to find how many 1/144-pieces would fit into a 1/4-piece. Of course both "readings" of her notational system would entail reasoning reciprocally. In fact, an awareness of how she operated on any but no particular turn would be based on reciprocal reasoning – understanding that if a bar is partitioned into three parts, then each part is one-third of the original bar, and the original bar is three times any one of its three parts.[25] But, it involves more. She would also need to reason that if a 1/48-bar is partitioned into three equal parts, then each of the forty-eight 1/48-bars would be partitioned likewise, and so that would produce three times forty-eight, or one-hundred forty-four parts, and thus each one of the three equal parts would be a 1/144-bar. This reasoning is indicated by her notational system.

What her notational system does not indicate is an awareness that if she started with a 1/144-bar, to find how many of these bars are in a 1/48-bar, all she needed to do was reverse the steps she took to find how much of the whole bar each one of the three equal parts of the 1/48-bar was (creating a unit of units). This is the specific context of the reciprocal reasoning in which she needed to engage. To find how many 1/144-bars in a 1/12-bar, she needed to use the knowledge that there were three 1/144-bars in a 1/48-bar when finding how many 1/48-bars was in a 1/12-bar. This simply entailed finding the product of three and four had she correlated using her notational system with her reasoning, creating a unit of units of units. To finish the problem, she would need to reason reciprocally and use the numerosity of this unit structure, twelve, to find how many 1/144-bars were

[25] Such reciprocal reasoning is based on the operation of splitting.

in a 1/4-bar by finding the product of three and twelve. So, developing a scheme of recursive partitioning operations not only involves making units within units within units, which Melissa could do, and symbolizing these operations, but also involves using the most elemental unit produced through partitioning in uniting to make a unit of units and then a unit of unit of units. That is, partitioning and uniting must be constructed as reciprocal operations at three unit levels.

As necessary as it seems for partitioning and uniting to be constructed as reciprocal operations in the construction of a scheme of recursive partitioning operations, it seems necessary for there to be a notational system produced that symbolizes these operations and their properties. For a notational system to be a symbol system, the symbol system must stand in for the operations actually carried out to produce the notational system and for the child to be able to reason using the notational system without actually carrying out the symbolized operations. This frees the child from the need to actually operate for there to be a result, because the result is symbolized. A symbol system is especially crucial when it become necessary to produce more than three levels of units, because the child can symbolize the numerosity of the third level of units in a unit of units of units and use this symbolized numerosity as if it referred to a unit of units that can be operated on further. My assumption that human beings can learn to operate in such a way that they can produce three levels of units and then use those three levels of units in producing three more levels of units, and etc., is justified by the Hindu-Arabic numeration system. A scheme of recursive partitioning operations is definitely involved in the production of that system especially in that case where it is extended to include decimals and their symbolization. So, if Melissa's notational scheme was in fact a symbol system, then she should have at least given some indication of wanting to use her notational system in her reasoning especially after she just used her diagram to find how many 1/144-pieces would fit into a 1/44-piece when finding how many 1/144-pieces would fit into a 1/4-piece. Instead, she made a whole new drawing and seemed to be in the process of constructing a symbol system rather than having completed the process.

The Children's Meaning of Fraction Multiplication

Both children had constructed a unit fraction composition scheme[26] and learned to use this scheme in the embedded recursive partitioning tasks. In the "recursive partitioning" activities from Protocol X forward, we did not emphasize multiplica-

[26] In the latter part of the teaching experiment, there was contraindication that both children could find, say, how much one-third of four-fifths of a unit bar is of the unit bar. This is a nontrivial generalization of finding how much one-third of one-fifth of a unit bar is of the unit bar because it involves distributive reasoning which the children were yet to construct.

tive language nor did we emphasize written notation. Rather, it was our goal to bring forth Melissa's recursive partitioning operations and, hence, her fraction composition scheme and to provide Joe with situations in which he could use his fraction composition scheme and modify it in the possible construction of a scheme of recursive partitioning operations. In retrospect, it would have been very easy for us to encourage the children to develop multiplicative fraction language in both spoken and written contexts as they worked in the embedded recursive partitioning tasks. For example, in the continuation of Protocol XVI, it would have been appropriate to suggest to her that she was in fact finding the part of the unit bar indicated by "1/3 × 1/48" to aid her in interpreting what she was doing using fraction language after Melissa made her sequence of written fractions:

$$ {}^{4}\frac{1}{4},\ {}^{3}\frac{1}{12},\ {}^{4}\frac{1}{48},\ {}^{3}\frac{1}{144} $$

It would have been possible then to ask her to write "one-third of one-forty-eighth" and help her in formulating the notation, "1/3 × 1/48." In finding how much this piece was of the whole bar, it would have been possible to take advantage of her product, $\frac{\overset{48}{3}}{144}$, and to ask her what this product meant, how many pieces were there of size 1/3 × 1/48, and how she used it to produce 1/144, using standard numeric notation. Of course, the product of forty-eight and three was the result of recursive partitioning operations, and this product, along with her use of its result to find the fractional part of the whole bar that was constituted by the bar of size 1/3 × 1/48, would constitute a child-generated algorithm for finding the product, 1/3 × 1/48. This child-generated algorithm could have been brought forth in Melissa as follows. To find how much the piece of size 1/3 × 1/48 is of the whole bar, find the product of three and forty-eight to find how many such pieces are in the whole bar. Since 3 times 48 is 144, 1/144 is how much the piece of size 1/3 × 1/48 is of the whole bar. So, 1/3 × 1/48 = 1/144.

It also would have been possible to induce a child-generated computational algorithm in Joe. The key in his operating is that he found the product of three and forty-eight just as did Melissa. Encouraging Joe to explain why he found this product would have led to a rational explanation on his part because he was aware of why he operated as he did, which is essential in a child constructing a child-generated algorithm for finding the product of two unit fractions. Further, Joe could have explained that the yellow piece was one-third of one-forty-eighth, so he could have been encouraged to write "1/3 × 1/48" as a *record of his operating*. From this point on, his operating was quite similar to the operations in which Melissa engaged, so making records of them should have followed approximately the same path as that followed by Melissa. Child-generated algorithms as they are manifest in notation are nothing but records of operating, and these records serve the function of constructive generalization.

A Child-Generated vs. a Procedural Scheme for Adding Fractions

In the first continuation of Protocol XVII, Melissa's writing of $1, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ is an example of a child-generated scheme for adding fractions. The sequence of numerals was a record of her production of combining a 1/2-bar and a 1/4-bar to produce a 3/4-bar, and then combining this configuration with a 1/8-bar to produce a 7/8-bar. Recursive partitioning was a key operation in her production of this sequence of numerals, and the sequence constituted a record of her operating. Joe also used recursive partitioning to produce the fractional part of the whole bar produced at each step of the partitioning. However, his paper and pencil algorithm for adding fractions, which was a procedural scheme, essentially excluded his conceptual solution of the task as well as his production of a child-generated scheme for finding how much of the whole bar was constituted by the 1/2-bar, the 1/4-bar, and the 1/8-bar combined when he was entirely capable of doing so. The investigation of a more general child-generated scheme for adding fractions than Melissa constructed was not possible because of the lack of distributive reasoning in the two children.