Chapter 4
Articulation of the Reorganization Hypothesis

Leslie P. Steffe

When fractions are introduced in school mathematics, they are usually introduced in the context of continuous quantity. Number sequences are essentially excluded because, as quantitative schemes, they are thought to be relevant only in discrete quantitative situations. Even though I developed number sequences in Chap. 3 in the context of discrete quantity, I can see no principled reason to keep them separate from continuous quantity. Reserving number sequences for discrete quantity stands in opposition to the concept of the real number line in higher mathematics, and, in this chapter, I argue that it also stands in opposition to the development of quantitative schemes. In articulating the reorganization hypothesis, I establish that a composite unit of specific numerosity can be used to make a split in the way that Confrey (1994) explained. This involves more than simply indicating the possibility of transferring the operations involved in compounding discrete units together to splitting continuous units. I do a deeper developmental analysis of children's quantitative schemes in which I explore whether the operations that produce discrete quantity and the operations that produce continuous quantity can be regarded as unifying quantitative operations. If so, these quantitative operations would justify the reorganization hypothesis.

I start the chapter with analysis of the construction of continuous items of experience as well as connected but segmented items of experience and develop the notion of quantity as a property of an object concept that can be subjected to comparison. The question "What is quantity?" can be interpreted as a question about mathematical concepts that exist independently of the children who are to learn them rather than about children's quantitative concepts. As an adult, I can say that intensive quantity is nonadditive, that an extensive quantity is additive, that the quotient of two extensive quantities yields an intensive quantity, and that an extensive quantity arises as a result of counting or measuring (Schwartz 1988). These ideas of quantity are essential, but they do not specify the operations children use to generate quantity.

Davydov (1975), following Kagan (1963), formulated a definition of quantity that supports the idea of quantity as a property of a concept. According to Davydov (1975), a quantity is any set for which criteria of comparison have been established for the elements. The necessity to specify criteria of comparison assumes that some common property of the elements has been established. Davydov's (1975) idea of quantity, then, orients us to viewing the origins of quantity in properties of concepts

L.P. Steffe and J. Olive, *Children's Fractional Knowledge*,
DOI 10.1007/978-1-4419-0591-8_4, © Springer Science+Business Media, LLC 2010

in the way I regard numerosity as a property of a composite unit structure rather than as the composite structure. Quantitative properties of concepts such as the iteration of a unit structure are introduced by the knowing subject's actions in the construction of the concepts.

Perceptual and Figurative Length[1]

Children do construct "continuous" items of experience as well as discrete items of experience. Although I do not and cannot make a definitive distinction between these two kinds of items of experience, continuous items of experience involve motion of some kind.[2] Such motion might be moving the eyes, crawling along the floor, walking along a path, sweeping one's hand through a space, or scratching a path in the frost on a window with a fingernail. In so far that each of these motions have a beginning or an end, they can be isolated from the rest of one's experiential field and, along with sensory material from the visual or tactual mode, form what I call experiential continuous items.

I usually think of the path of the motion as being the experiential item when there are visual records of the path. However, I may overemphasize the visual perceptual records. For example, a unitary item that corresponds to something like what an adult would call "rod" [a rod template] contains records of the motion involved in moving the eyes in the construction of the unitary item. If the unitary item can be used in re-presentation, i.e., in "visualizing," the scanning motion that is recorded in the rod template could be reenacted and produce a regeneration of the recorded visual material that constitutes the path of the motion. If the child becomes aware of the visualized path including its endpoints, of the motion that produces the path, and of the duration of the motion, the child would be aware of figurative length.[3]

If an observer's rod is moved from one place to another, the child might know that it is the "same rod" because the rod template can be used in re-presentation as indicated above. This is nothing but object permanence – the rod "exists" for the child independently of its particular location in experience because the child is aware of the rod without it being in the immediate visual field. But, a child still might believe that a rod whose right most end-point, say, goes beyond that of

[1] The concepts of discrete and continuous quantity presented in this chapter have their origin in Steffe (1991).

[2] If a continuous item of experience is bounded, from that perspective it is also discrete. Similarly, if a discrete item of experience has an interior that would qualify it as a continuous item of experience. Of course, by an "item of experience" I refer to an implemented attentional pattern with the understanding that the experiential item is a permanent object.

[3] Excepting an awareness of duration would eliminate an awareness of the continuity of the scanning motion over the regenerated visual material.

another rod is "longer" regardless of the relative positions of the left most end-points. This judgment would be an indication that the child is aware at least of perceptual length. An awareness of perceptual length means that a child becomes aware of the duration of scanning as well as of the visual path of scanning.

Children do construct experientially connected but segmented items as well as experientially continuous items like a rod. A sidewalk is one example and a row of telephone poles is a complementary example.[4, 5] In the case of a sidewalk, when establishing a row of sections a child might experience a section as an object concept in a way that is similar to any other object concept. If so, the child could use this concept in assimilation and become aware of scanning more than one section in the assimilation. This introduces repetition into the object concept and an awareness of a plurality of sections. When coupled with an awareness of the perceptual length units of the sections, this produces what I call an awareness of perceptual length of an experientially connected but segmented unitary item. Such an awareness of perceptual length is a gross quantitative property of a row of perceptual unit items called segments.[6]

When a child uses its concept of a continuous but segmented unitary item in re-presentation, this opens the possibility of the child using its concept, segment, in reprocessing a re-presentation of a continuous but segmented unitary item. This process produces a segment as a figurative unit item and a row of segments as a figurative lot structure that is analogous to the figurative lot structure of Fig. 3.5 in Chap. 3.

Generically, the property of a segment that I call length is an awareness of the scanning action over an image of the segment along with an awareness of the duration of the scanning action. If the child is aware of a figurative row of segments (a figurative plurality of segments), then that awareness, when coupled with an awareness of the scanning action over the segments of the row, is what I mean by an awareness of figurative length of a row of segments. The construction of a figurative row of segments illustrates the construction of operations that children could use in future occasions to project units into experientially continuous units and provides a basis for a synthesis of discrete and continuous units. It is meant to illustrate how children's construction of continuous quantity at the most elementary level involves operations that are also involved in their construction of discrete quantity.

Piaget's Gross, Intensive, and Extensive Quantity

Quantitative properties of concepts like the iteration of a unit structure are introduced by the knowing subject's actions in the construction of the concepts. Hence a quantitative property of a concept can be viewed as an abstraction of the records of

[4] They are complementary because the former leads to length and the latter to distance.

[5] See Steffe (1991) for a conceptual analysis of this construction.

[6] Here, "segment" is not to be interpreted mathematically.

actions that were involved in the construction of the concept. This fits well with Piaget and Szeminska's (1952) idea of a gross quantity. An awareness of perceptual length is a gross quantitative property of a segment or of a row of segments. It is a gross quantitative property because the child abstracts it from the activity of scanning a segment or a row of segments by means of pseudo-empirical abstraction. The types of quantitative comparisons explained by Piaget and Szeminska (1952) help to understand the difference between an awareness of perceptual length and an awareness of figurative length.

Gross Quantitative Comparisons

In Piaget's work, if a child judges that six blocks arranged in a row are "more than" seven blocks aligned in a shorter row as shown in Fig. 4.1, this would indicate an awareness of the intervals between the blocks. In this case, the child is aware of a row of blocks as a segmented but connected unitary item. If there is no further indication that the child is aware of producing a figurative row of intervals, the child's comparison would be judged as a gross quantitative comparison, where the gross quantity is an awareness of perceptual length[7] of a row of intervals.

Intensive Quantitative Comparisons

If two rows of blocks were of equal length as in Fig. 4.2, and if a child made a judgment that the bottom row has more blocks than the top row because they are closer together and the rows are of the same length, Piaget and Szeminska (1952) called this an intensive quantitative comparison. In my terminology, I would say that the

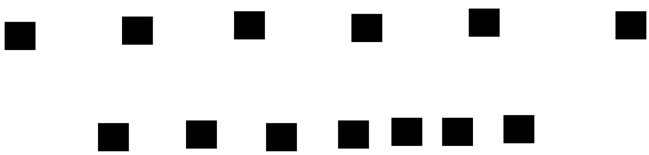

Fig. 4.1. Two rows of blocks: Endpoints not coincident.

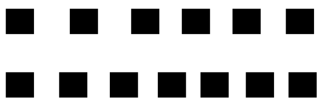

Fig. 4.2. Two rows of blocks: Endpoints coincident.

[7] In this case, the child is aware of a row of blocks as a segmented but connected unitary item. Nevertheless, the child may be aware also of perceptual plurality.

child coordinates an awareness of the perceptual length of the rows and an awareness of the perceptual plurality of the counters in the rows. To make this intensive quantitative comparison, the child does not make a comparison based solely on the perceptual length of the rows as in Fig. 4.1. Instead, the child would need to extract herself from the immediate here and now and not make a quantitative comparison of "the same." To do this, the child would need to be aware of the intervals between the blocks of each row of blocks and of the perceptual density[8] of the blocks in each row. Furthermore, the child would need to be aware that the increase in perceptual density of the blocks in the bottom row is compensated for by the decrease in the length of the intervals. Hence, the child would need to operate at least one level above the perceptual level as well as at the perceptual level. Otherwise, the child would not reflect on the intervals between the blocks as well as on the density of the blocks in a row. What this means is that the coordination of two gross quantities does not happen at the level of the gross quantities. Rather, it occurs in re-presentation.

A child who can make an intensive quantitative comparison, then, must be able to visualize a row of blocks to coordinate two gross quantities. I do not assume that the child is aware of a visualized row of blocks when using its recognition template in making the comparison because the visualizing experience may occur outside of the awareness of the child. But the template for producing a figurative row of blocks must be active in re-presentation for the child to "step back" from the rows of blocks in the immediate here-and-now and coordinate the perceptual length and density at the level of re-presentation when making the comparison.

The dual use of the template at the re-presentational level and the perceptual level allows the child to focus on either the intervals between the blocks or on the blocks. This means that, in a way, the possibility of a separation between the intervals of a row of blocks and the blocks was already "contained in" the recognition template prior to operating in the immediate here-and-now. This differentiation between the intervals of a row and the blocks of the row allows for the intuitive understanding that the blocks are more frequent in the bottom row and the intervals between them are shorter, which in turn leads to the judgment of "more blocks in the bottom row." The child can now make judgments in comparing the rows that were not previously possible. So, to reiterate the central point, by using a figurative row of blocks, a child can focus on either the intervals or the blocks and coordinate the two.

An Awareness of Figurative Plurality in Comparisons

In the construction of the figurative lot structure (Chap. 3, Fig. 3.5), I did not account for the intervals between the figurative unit items in those cases where the elements are arranged in an identifiable path. These intervals are not empty

[8] By "perceptual density," I mean an awareness of the frequency of instantiation of the block concept within definite boundaries. By "perceptual plurality" I also mean an awareness of more than one instantiation of the involved template.

in that they contain no perceptual material. Likewise, in a figurative row of segments, what is in between the segments is not vacuous. So I regard the structure of a figurative row of blocks as a more complete account of both a figurative lot structure and a figurative row of segments in special cases. As a coordination of two gross quantities, intensive quantity, then, is nothing more than an awareness of both figurative length and figurative density. The way in which I defined an awareness of figurative length in the case of a figurative row of segments necessarily included an awareness of figurative plurality in that an awareness of figurative length includes an awareness of more than one figurative unit of length. This generalizes Piaget and Szeminska's (1952) idea of intensive quantity and constitutes a step in unifying discrete and continuous quantitative schemes because it opens the possibility that a connected but segmented unit can be a situation of the child's counting scheme.

This possibility is clearly indicated in some of the experiments conducted by Piaget et al. (1960). They aligned two rows of five matches parallel to one another and then made a zigzag path using the matches in one of the rows. One of the youngest children they interviewed fits my idea of an awareness of perceptual length as a gross quantity.

Protocol I.

CHA (4; 0) Five matches in parallel alignment with five more: "Is it the same length from there to there as from there to there or are they different? – The same length. And like this (The right-hand row is re-arranged in a series of zigzags)? – No.–Which is longer?–I don't know which but they aren't the same. –Is there the same way to go? – No. – Is one a longer road to go? –Yes, there (indicating turnings) no, that one (straight row), because it's long. Here, it's nearer the end (zigzag row, showing relation between extremities). (Piaget et al. 1960, p. 106)

CHA's indecisiveness concerning which row was the longer after the zigzag path was made indicates an awareness of moving over the segments (matches). First, CHA judged the zigzag path to be longer apparently because of the up and down oscillating motion as opposed to the linear motion. Here, CHA focused on traversing the paths. When CHA judged the straight row to be longer, the endpoints of the rows became relevant and CHA made a comparison of the perceptual unit lengths between the endpoints. That there were five matches was irrelevant in his comparisons.

The authors report that some older children begin to become aware of the plurality of the matches in each row, but this is lost if the change in shape is excessive or if one of the matches is broken. This is exactly what I would expect in the case of an awareness of figurative length. JAN, in the Protocol II, was judged to be a level IIA child whereas CHA was judged to be a level I child.

Protocol II.

JAN (5; 10) Two straight rows, each of five matches, in parallel alignment: "It's the same length. And like this (both outlines forming right angles with two matches in one limb and three in the other)? – Also. – And like this (one outline made up of right angles and the other in zigzag formation)? – (He counts.) Also the same because it's four and four. (The experimenter breaks two matches in half. Four whole matches are arranged in the shape of a right angle and the remainder forms the outline of a staircase.) And like this? – Here it's longer. There are eight of them: that's more. – Yes but to walk along, is the road longer, or are they the same? – It's longer. (Piaget et al. 1960, p. 108).

That Jan counted when one of the two rows was arranged in an L shape and the other in a zigzag path indicates that his concept of length included an awareness of how many matches were in the rows. His "correct" comparison of length after counting indicates that he coordinated the figurative length of a row of matches and the figurative plurality of the matches because children who are gross quantitative comparers would have judged that one of the rows of matches was longer. After two matches were broken into half, he seemed to rely on the relative density of the matches in the two configurations and based his judgment of "longer" on his judgment of "more." Breaking two of the matches into two parts apparently destroyed the length units he established for the matches. He apparently did not conceptually reunite the two parts of each match back together to form a whole match, which is necessary to conserve the length of a match. This nonconservation of the length of a match solidly indicates that he established at most figurative length. Still, I see the beginnings of a unification of discrete and continuous quantity in the way in which Jan compared the two configurations of matchsticks.

Extensive Quantitative Comparisons

Extensive quantity enters into Piaget and Szeminska's (1952) system when the child introduces arithmetic units into intensive quantities. In this case, the child might regard the comparison in Fig. 4.1 as indeterminate because the blocks in the upper row are more spread out but the row is longer. I believe that the units of which Piaget and Szeminska wrote in the case of his extensive quantity are the result of a specific kind of unitizing activity that gives rise to abstract units. I have already shown that there are unitizing activities that precede Piaget's specific kind and, as I have shown in Chap. 3, that follow. Children who are gross quantitative comparers make perceptual unit items, and children who are intensive quantitative comparers make figurative unit items. The extensive quantitative comparers produce Piaget's (1970) arithmetic units, where: "Elements are stripped of their qualities…" (p. 37). Although these units are at the same level of abstraction as the arithmetical units I have identified, there is no requirement that they contain records of counting acts. They are parallel to the abstract unit items of von Glasersfeld's arithmetical lots.

As before, I generalize Piaget and Szeminska's (1952) idea of extensive quantity in the case of a row of blocks to include the row of segments between the blocks. If a child reprocesses a figurative row of segments using the template that was involved in producing the row, this opens the possibility of "stripping" the figurative segment of its sensory-motor qualities and constructing an abstract unit segment on a par with our abstract unit. Contrary to what might be expected, the property of the segment that I called length is not eliminated. Rather, the records of motion are still contained in the abstract unit segment. These records of motion are now turned into an operation rather than a figurative action. That is, the child no longer has to run over the items, even in visualized imagination, to recreate the sense of motion and the resulting sense of length. Instead, the records of motion are implicit in the recognition template. This is reflected in the Protocol III of a child JEA, 6 years 3 months of age in the matches' experiment of Piaget et al. 1960. JEA was judged to be at stage IIB.

Protocol III.

JAE (6;3) They're the same because there's five and five. – And like this (straight row and staircase)? – It's still the same because it's still five and five. It's still just as far. – One match broken and the bits laid end to end.) – It isn't the same anymore because you've broken one. – But is one of the roads shorter than the other? – Yes, that one, because it has seven bits and this one has five. – And if I break more matches? – It'll be longer still. – And if I lay them out straight like this (two straight lines.)? – That path (with broken matches.) will be longer. (Piaget et al. 1960, p. 112).

In contrast to JAN, JEA maintained that there were still five matches in each row without counting after one row of matches was changed into a zigzag path[9] [JEA said that ,"It's still the same because it's still five and five. It's still just as far."]. This indicates that the conceptual structure that constituted a row of segments for JEA was at least at the same level as a numerical composite in the discrete case. Further, JEA's meaning of "five" included five units of length as indicated by the comment, "It's still just as far." Correspondingly, I call JEA's concept of "five" a connected numerical composite or, more simply, a connected number.

There was still a lacuna in JEA's reasoning as indicated by the experiment of breaking one match into pieces.[10] I interpret this lacuna in reasoning as indicating a lack of the uniting operation, which is that operation that JEA would have used to conceptually reunite the broken matches into the original matches. This indication of the lack of the uniting operation warrants interpreting JEA's connected number five at the level of a numerical composite in the discrete case. Her connected number consisted of a sequence of connected segments. Constructing connected numerical composites opens the way for the construction of a connected number sequence, which is a number sequence whose countable items are the elements of a connected but segmented continuous unit. The construction of a connected number sequence is an initial step in the construction of measurement as well as an important step that integrates discrete and continuous quantity. A child at this level has constructed an awareness of indefinite length as well as of indefinite numerosity as quantitative properties of a connected number. Hence, I regard both an awareness of length and an awareness of numerosity as extensive quantities, which generalizes the concept not only across the discrete and the continuous, but also across the schemes that constitute measurement and number.

[9] I assume that "staircase" is used for JEA rather than "zigzag path."

[10] I assume that one match in Protocol III was broken into three pieces because JEA said there were seven bits.

Composite Structures as Templates for Fragmenting

In my analysis, I found that of the operations that produce an awareness of indefinite length and indefinite numerosity are not of a different kind or a different genre. This fits well with the finding of Piaget et al. (1960, p. 149) that children's construction of the operations of measurement parallels their construction of number. To complete the analysis, I turn now to the construction of the operations involved in fragmenting a continuous unit.

Piaget et al. (1960) observed what they thought was a developmental lag in the construction of a unit of length. It is important to understand that a unit of length for Piaget et al. was an iterable unit.

> Unlike the unit of number, that of length is not the beginning stage but the final stage in the achievement of operational thinking. This is because the notion of a metric unit involves an arbitrary disintegration of a continuous whole. Hence, although the operations of measurement exactly parallel those involved in the child's construction of number, the elaboration of the former is far slower and unit iteration is, as it were, the coping (capping) stone to its construction (Piaget et al. 1960, p. 149).

They found that only one child in ten of those from six to seven years of age, half of those from seven to seven years six months, and three-quarters of those from seven years six months to eight years six months attained operational conservation of length. For them, operational conservation of length "entails the complete coordination of operations of subdivision and order or change of position" (Piaget et al. 1960, p. 114).

Although I have not presented such normative data for the construction of the explicitly nested number sequence, based on my observations in interviews, these ratios seem compatible with the ratios of children who have constructed the ENS at corresponding ages, with the possible exception of the children in the age range from 6 to 7 years (Steffe and Cobb 1988). For this youngest group, I would expect that a greater ratio than one in ten would have constructed the explicitly nested number sequence. Nevertheless, I find the compatibility striking and indicative of the common operations required for the construction of the ENS and subdivision (fragmenting) schemes.

In the context of the matchstick experiment, I have shown that children can construct an abstract unit of length and a connected number sequence, both integral to the construction of extensive quantity, which parallels the construction of the arithmetical unit and the initial number sequence. I will now show that operational fragmentation into equal fragments emerges for planar spatial regions at the level of the initial number sequence by interpreting the experiments of Piaget et al. (1960) on subdividing a continuous unit. That is, operational fragmentation of planar regions into equal units emerges upon the emergence of extensive quantity. For this reason, I would not say that there is a developmental lag in the establishment of a unit of length in the context of connected but segmented units. However, subdivision of nonlinear continuous units is apparently more difficult than the establishment of connected but segmented units. For this reason, I would expect to find that the operational segmenting of nonlinear continuous units lags behind establishing connected numbers. The main issue that is addressed in this section concerning arithmetical and length units in spontaneous development is the relation between the construction of operational fragmenting (up to and including partitioning) and the construction of number sequences.

Experiential Basis for Fragmenting

The construction of operational fragmenting is a product of spontaneous development in the same way that the number sequence is a product of "spontaneous development"[11] in that fragmenting has its own experiential foundations separate from the establishment of connected but segmented units. Consider the example of a plate dropped on a hard surface. I distinguish among an experience of the plate, of the plate shattering, and of the shattered plate. These experiences are separate and distinct one from the other, even though they are contiguous. The auditory and visual experience of the plate shattering would be an item of experience attentionally no different than any other experiential item. Nevertheless, experiences like the shattering of a plate would contribute to the spontaneous construction of fragmenting operations.

If a child establishes the fragments of the shattered plate as perceptual unit items, the attentional structure of a shattered plate would be no different than the attentional structure of a perceptual lot. In this, I assume that the child forms an attentional pattern that I would call "fragment," and uses this attentional pattern in reprocessing the fragments of the broken plate, creating a perceptual lot structure. This process of categorizing in fragmenting is the same process that makes possible the uniting operation and a unit of units.

The experiential contiguity of the plate, the shattering plate, and the shattered plate qualifies the records of the three experiences as possible elements of a scheme. A goal structure along with the possibility of enacting the breaking action upon the recognition of a plate might be missing, however. If a child intentionally drops a plate in an attempt to reenact the breaking action, I would say the child has established a fragmenting scheme! Forming a goal to break a plate and carrying out the actions necessary to break it are essential elements of a fragmenting scheme and thus of fragmenting operations.

In the establishment of a fragmenting scheme, there has to be a change in the breaking action from being simply observed to being intentionally executed. For the

[11] Number sequences develop during early childhood mathematics education, and so they do not develop independently of the children's mathematics education. For this reason, I have placed the phrase, "spontaneous development" in quotation marks to acknowledge the contribution of children's mathematics education in their construction of number sequences.

fragmenting scheme to be transformed into a fraction scheme, the breaking action has to change further to include fragmenting a continuous unit into so many equal fragments. I am interested in the development of the intentional fragmenting of a continuous unit into a specific plurality of equal fragments.

Using Specific Attentional Patterns in Fragmenting

My assumption is that fragmenting a continuous unit into a specific plurality of equal fragments originates in the context of using interiorized dyadic and triadic attentional patterns to project units into a continuous unit. Several studies do indicate the primacy of dyadic and triadic attentional patterns in children's development of number. For example, using 48 three-year-olds, 48 four-year-olds, and 48 five-year-olds, Gelman and Gallistel (1978) found that only after a 1 second exposure to two items, 33 three-year-olds, 44 four-year-olds, and all 48 five-year-olds recognized two items. For three items the corresponding numbers were 28, 37, and 43. In the case of four items, the numbers at each age level were 9, 23, and 33.[12] These data clearly indicate the primacy of dyadic and triadic attentional patterns, especially dyadic attentional patterns, in children's construction of discrete structures.

Fragmenting a circular cake: *Two dolls*. The emergence of dyadic attentional patterns in the context of fragmenting is not independent of the kind of continuous unit involved. Piaget et al. (1960) began their study of fragmenting continuous units using a circular slab of modeling clay together with two little dolls. The child was told that the clay is cake and the dolls are going "to eat it all up, but they've each got to have exactly the same amount as the other: how shall we do it?" (p. 302). The child was supplied with a wooden knife.

For 3-year-olds, the earliest response was cutting out little pieces of the cake without knowing when to stop; cutting the cake was an end in itself. "This fact is interesting because it shows that there cannot have been any anticipation of the aim, or if there was, it was not the sort of anticipatory schema which enables a

[12] When increasing the exposure time to 5 seconds for four items, the numbers at each age level were 21, 29, and 37, indicating that some children counted. Almost 50% of the 4-year-olds could recognize four items after an exposure of only 1 second. Recognizing four items after only a 1-second exposure is important because it is quite likely that the recognizing child would regenerate an image of the items after the exposure, which indicates that their quadriadic attentional patterns were constituted at least as figurative lots. However, the phenomenon of subitizing, instant recognition of numerosity, such as in the case of dyadic and triadic patterns, may have enabled many of these 4-year-olds to recognize the four items without their quadriatic attentional pattern being constituted as a figurative lot (von Glasersfeld 1981).

child to know in advance that he must cut two pieces, using up the whole cake" (Piaget et al. 1960, p. 304). Children's sensory-motor activity is essential for the formation of such an anticipatory fragmenting scheme, as shown in Protocol IV.

Protocol IV.

FRAN (4; 2) His final replies are at level IIA. …he begins by cutting two small pieces out of a clay cake and handing them to the two dolls – this in spite of the fact that the experimenter had warned him: "The mummy says that they can have all the cake." As he stops short after cutting those two small bits, the experimenter remarks: "Now don't cut such tiny bits. You divide up the whole cake. He is given another cake and cuts off first two pieces, then two more, and finally divides the rest in two. With a third cake he again cuts out little pieces and goes on to give as many as fifteen such bits to each doll. But with a rectangle, he immediately divides the 'cake' into two pieces (unequal but leaving no remainder). A square cake is cut into four and each doll is given two quarters. Finally, when given another round cake, he cuts it into two." (Piaget et al. 1960, p. 305)

FRAN initially used a dyadic attentional pattern (two dolls) to establish a goal, as indicated by cutting two small pieces out of the cake. FRAN's dyadic attentional pattern served as a guide throughout the fragmenting activity, which indicates that it was constituted at least as a figurative lot. Because FRAN eventually cut the whole of a round cake into two, I infer that he reprocessed the implemented dyadic attentional pattern, i.e., abstracted the dyadic pattern from its implementation. At the end of the protocol, FRAN knew that he had to share the whole of the cake between two people and he knew how to do it. But FRAN, according to Piaget et al. (1960), did not think of a part as being included in the whole from which it was made and to which it still relates in thought. It was simply a piece removed from the whole. In fractional schemes, the child can preserve the part in the whole.

Fragmenting a rope: *Two dolls*. According to Piaget et al. (1960), fragmenting a circular cake into two parts does not emerge until approximately 4 years of age. However, most of the 3-year-olds in a study by Hunting and Sharpley (1991) were able to cut a rope into two pieces. Two hundred and six children ranging in age from 3 years 4 months to 5 years 2 months were individually asked to share a piece of string 300-mm long between two dolls. Nearly all of these children cut the string just once (89%). Of these approximately 183 children, the difference in the two parts was less than 15 mm for 79 of them, which solidly indicates sensitivity to the equality of the parts. The difference in the two parts was between 15 mm and 30 mm for 28 other children, which can be also interpreted as indicating sensitivity to the equality of the parts. In sum, then, the difference in the two parts was less than 30 mm (1.2 in.) for 117 children, which is approximately 57% of the children interviewed. This finding establishes that a majority of children in the age range studied can share a linear object approximately equally between two people.

Of the remaining children who cut the string into two parts, 49 of them cut it so that the difference in the two parts was between 30 and 60 mm and 27 of them cut it so the difference was greater than 60 mm. For the 27 children for whom the difference was greater than 60 mm, equality of the parts did not seem to be an issue.

Whether the equality of the parts mattered for the 49 children for whom the difference was between 30 and 60 mm is more ambiguous.[13]

On purely conceptual grounds, in order to intentionally cut the string into two parts, I infer that the child's dyadic attentional pattern would be at least a figurative lot. That would allow children to generate and maintain an image of two dolls in the process of cutting the string, allowing the children to intentionally cut the string into parts, one for each doll. But sensitivity to the equality of the parts would be lacking despite a sense of twoness. Therefore, I infer that the 66 children who cut the rope into two parts, but whose parts were more than 30 mm different in length had a figurative attentional pattern.

When the dyadic attentional pattern is figurative, activity is guided by the twoness of the dolls. The intention to make equal parts specifies a property of the ropes, not the dolls. Hence, in order for the children to show greater sensitivity to the equality of the parts, the activity must be guided by properties of the rope parts. Since the dyadic attentional pattern, the twoness, initially referred to dolls, to use it to make fair shares of the rope involves a substitution of the rope parts for the dolls. This kind of flexibility in the referent of the units of the attentional pattern requires an arithmetical unit, where, in Piaget's terms, "elements are stripped of their qualities." Being stripped of their qualities, any sensory material can be used to "fill" the slots that the arithmetical units comprise and can stand-in for other sensory material. Therefore, children would not be able to intentionally break a continuous item of experience into two *equal* parts until they had constructed two as a numerical pattern – a pair of arithmetical units. Hence I infer that the dyadic attentional pattern of the 117 children for whom the difference in the two parts was less than 1.2 in. was numerical.

Once the dyadic attentional pattern is an arithmetical lot, further sensitivity to the equality of the parts would seem possible because the children could unitize the rope parts after cutting them and make an extensive quantitative comparison between them. Uniting the pair of items together permits setting the items "at a distance," which in turn permits awareness of the items, not only as distinct and separate but also as fair shares. This would lead to an inclination to review the two parts in an attempt to verify their equality.

After the children cut the rope, Hunting and Sharpley (1991) did ask the children if the dolls would be happy with their ropes. Out of 216 children, checking behavior was observed in the case of 94 children. Forty-seven simply looked at the two parts of the ropes in a visual check, 41 placed the two parts side-by-side, and 5 placed the ropes end-on. The behavior of the 41 children who did place the two parts of the ropes side-by-side indicates a review of the units comprising the rope parts. Whether it indicates two as unity is an open question. Had the children spontaneously checked, that would be a stronger indication of the construction of two as unity.

Subdividing a circular cake: *Three dolls*. Piaget et al. (1960) and Hunting and Sharpley (1991) agree that subdividing a circular cake does not require a more

[13] There is a discrepancy of 10 children between the total number of children who cut the string into two pieces and the number of children reported in the subcategories.

sophisticated composite structure than subdividing a rope into two parts, even though the former lags behind the latter by approximately 1 year. But there appears to be a large jump between using a dyadic attentional pattern in fragmenting and using a triadic attentional pattern in systematic subdivision of a circular slab of clay. Between the ages of 4 and 4½ years, according to Piaget et al. (1960), children usually succeed in sharing out two equal parts, but they cannot share into three equal parts. Fragmenting a circular slab of modeling clay into three parts apparently requires the development of the operations that produce the initial number sequence.

The use of a triadic attentional pattern as a numerical pattern in the context of subdivision is exemplified in Protocol V.

Protocol V.

SES (6; 2) Starts off by cutting several series of small pieces and distributing these as he goes along. For the next cake he cuts off three large slices and leaves the remainder. Finally, he succeeds in cutting a third cake into three almost equal parts. (Piaget et al. 1960, p. 320)

The modification in SES's fragmenting activity was self-initiated, which solidly indicates a goal to share the cake among the three dolls. I believe his triadic attentional pattern was evoked and served as a template in fragmenting. That he succeeded in cutting the circular cake into three almost equal parts convinces me that he constituted the elements of his triadic attentional pattern as arithmetic units in the midst of his activity, if they were not already arithmetical units before he started.

Subdividing a circular cake: *Five dolls*. Another child, SIM (5; 9), who was also judged to be at the level IIB, succeeded in sharing a circular cake equally among three children, but proceeded as in Protocol VI for the case of five children.

Protocol VI.

What he does is to cut a series of small slices and deals them out as he goes along, leaving an unused remainder. When given another cake and told to finish it all, he cuts it into seven successive slices and distributes the first five only. Later, he finishes with six parts, but he never succeeds in dividing into five (Piaget et al. 1960, p. 323).

In this example, SIM initially focused on making fragments without coordinating the fragments with the whole. When he was asked to focus on the whole, he made enough fragments to exhaust the whole, but without coordinating the number of fragments with the whole. SIM definitely had a goal to share the cake among five children as indicated by distributing the first five of the seven slices. This goal could be made possible, I believe, by the activation of a numerical composite, five, and the projection of units of the numerical composite into the continuous unit, the cake. In actual fragmenting activity, SIM surely lost his sense of the numerosity of the composite unit, five; his goal was to make slices, where I emphasize an awareness of plurality rather than of specific numerosity. Initially, he may have intended to make five pieces, but in actual fragmenting, he focused on simply making pieces, an activity which was set in motion by his goal to make five pieces.

SIM's behavior is what I expect from a child who uses a *numerical composite*, as opposed to a *composite unit*, in fragmenting. A numerical composite is the result of generating unit elements, but because there is yet no unit containing the elements, that unit is not yet an object of reflection for the child and the child is not aware of the composite as one thing. Rather, the child operates at the level of the unit elements.

In a numerical composite, the child can produce the five cake pieces in visualized imagination, but cannot yet "hold them at a distance" to operate on these cake pieces by coordinating their size and shape with the whole they are to be part of. Similarly, when the child focuses on the whole of the continuous unitary item, the cake, he loses sight of the number of pieces, which is exactly what happened when SIM was asked to finish all of the cake (as indicated by cutting the cake into seven pieces and distributing five).

To reconstitute a numerical composite as a composite unit, the child must take the generated unit elements, "hold them at a distance," and reflect on them in order to make them inputs for the unitizing operation. Once the numerical composite has been unitized and becomes the elements of a composite unit, the child can simultaneously be aware of the five unit elements and the newly formed composite unit. In this sense, the child can now operate simultaneously with these two kinds of units. Once the unit elements of a composite are items of a reflection, the child can begin to mentally coordinate the size of the pieces, representing the unit elements, with the size of the whole. However, this may still require a bit of experimentation at first. An indication of mental as well as physical experimentation in cutting a cake into five pieces is contained in Protocol VII.

Protocol VII.

ROS (6; 8) Is asked to divide the cake into fifths. He begins by constructing a series of successive parts which together account for all the cake but which number seven instead of five. He therefore tries twofold dichotomy but soon rejects that hypothesis. Eventually, the cake is divided into five approximately equal pieces, but these are parallel as were his thirds (Piaget et al. 1960, p. 325).

ROS was judged by the authors to be at level IIIA. Of children at this level, Piaget et al. (1960) commented that, "While every one of these subjects is eventually successful in dividing into five or six equal fractions, they all begin with a certain amount of experimentation although this is no longer necessary to them when trisecting. Only at level IIIB do I find division into fifths or sixths carried out with the same assurance as trisection" (p. 326).

Given that ROS eventually made parallel cuts and approximately equal pieces of the circular region indicates that he mentally estimated where he should cut the cake. This ability to make mental estimates indicates that ROS mentally fragmented the cake and then united those fragments into a connected but segmented unit. He was clearly aware that the whole needed to be exhausted, and he coordinated the number and size of the pieces with that requirement as indicated by his starting over after making seven pieces. To do this, he would need to be aware that the pieces together comprised the whole. For this reason, I believe that ROS had constructed the uniting operation and could produce composite units using that operation. Consequently, I infer that ROS used his composite unit, five, as a template for fragmenting.

The basic difference in fragmenting into three pieces and into five pieces for children like SIM is that, due to the child's advanced triadic attentional pattern, the child can be aware of the cooccurrence of three individual unit items, while children generally do not have such an advanced quintic attentional pattern. The cooccurrence of the unit items can provide the child with a sense of a composite whole, even if the child has not constructed a composite unit of three. This allows the child to attack the problem similarly to the child who has constructed a composite unit. Even so, I would expect trial and error to be involved in fragmenting into three pieces, because the child cannot yet mentally experiment with fragmenting the cake. Remember that when three is constructed as a numerical composite, the child can use this structure to generate an image of three items, but when three is constructed as a composite unit, the child can also set the image of three "at a distance" and reflect on it. In the former case, the child is "in" the re-presentation whereas in the latter case, the child is "outside" of the re-presentation and so can set it at a distance and look at it. On the one hand, this allows children like ROS to fragment the cake into three equal slices without first experimenting. Five, on the other hand, is a large enough number of pieces that even children like ROS, who have constructed a composite unit of five, use a combination of mental and physical experimentation initially.

Number Sequences and Subdividing a Line

The distinctions between levels IIB, IIIA, and IIIB as articulated by Piaget et al. (1960) can be interpreted in terms of the three number sequence types. But before drawing the parallels between the levels and the number sequence types, I discuss tasks that Piaget et al. refer to as subdividing a line, because children's performance on these tasks are even more compatible with the number sequence types than the subdivision of cake tasks.

The authors devised a sequence of six experimental situations, the first two of which I discuss. The first situation concerned locating a point b_2 on a_2c_2 in Fig. 4.3 so that a_2b_2 would be just as long as a_1b_1. In the second part of this situation, the child was asked to start at c_2 rather than a_2.

In the second situation, illustrated in Fig. 4.4, the top segment was broken into the same two parts as Fig. 4.3. However, the endpoints of the top and bottom segments were no longer lined up.

Lack of subdivision. Piaget et al. (1960) observed that children in his stages I and IIA immediately solved the first part of the subdivision task of Fig. 4.3, but not the second subdivision task illustrated in Fig. 4.4. Of these children, the authors

Fig. 4.3. The first subdivision of a line task.

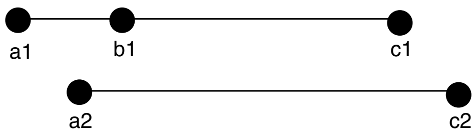

Fig. 4.4. The second subdivision of a line task.

stated, "They do not know how to transfer the distance a_1b_1 to the other string a_2c_2 and they simply put b_2 opposite b_1 without worrying about the inequality of the intervals so obtained (Piaget et al. 1960, p. 130). In my model, for the children to transfer the interval a_1b_1 to the other string, they would need to have at least constructed the concept of an interval as an arithmetical unit, which can occur at the level of the initial number sequence.[14]

Intuitive subdivision. The next step in children's progress identified by Piaget et al. (1960) is an intuitive solution of situation two; that is, children simply looked at the two lines and made visual estimates. These children do not measure, even though they were provided with a measuring instrument that was longer than the intervals to be measured. Making a visual estimate is a clear indication that the children were aware of the intervals and that this awareness is the result of projecting units into the segments. Piaget et al. confirm this when commenting on the fact that these children take the point of departure into account as well as the point of arrival: "This means that they are aware of the interval involved…That awareness leads to a second feature of their progress in that they begin to subdivide the line and relate the segments to its overall length" (p. 140). This major advancement is the result of the construction of extensive quantity and is compatible with how SIM in Protocol VI tried to fragment a circular cake into five equal pieces.

Measuring without unit iteration. Piaget et al. (1960) subdivided operational stage III into two parts that echo the construction of two number sequences beyond the initial number sequence. In substage IIIA, when the children had a measuring stick longer than the distance a_1b_1, success was assured, but when it is shorter, the children used auxiliary pieces of material to measure. This behavior is compatible with the construction of the tacitly nested number sequence, as I explain later.

Protocol VIII.

RAY (7; 10) Illustrates level IIIA. He gradually moves b_2 beyond b_1, saying: "I think it's right now." (Note: The ruler is shorter than a_1b_1). As the ruler is too short, he measures a_2b_2 by using his hand as well as the ruler, then in measuring a_1b_1 he uses a strip of paper to prolong the ruler, checking a_2b_2 in the same manner, and so achieving an accurate reproduction of the distance given (Piaget et al. pp. 142–143).

According to Piaget et al., for RAY, segment a_1b_1 would not symbolize the operation of iterating the segment to constitute a segment of length possibly equal to segment a_1c_1. However, the authors regarded RAY as quite capable of mentally fragmenting segment a_1c_1 into equal subsegments.

> *In the present experimental setup, AC is broken into the two portions AB and BC by the bead. Even here, subjects at levels I and IIA overlook the whole, AC, and fail to regard the interval AB as a part. But it is quite another matter to break up AB into a number of abstract segments, as given by successive momentary positions of a ruler or a strip of paper.*
> (Piaget et al. 1960, pp. 145–146)

From this comment, I infer that the authors regarded RAY as being able to break up segment a_1c_1 into a number of abstract segments as given by the segment a_1b_1.

[14] See discussion of the initial number sequence in Chap. 3.

I do agree with the authors that there is "nothing natural in subdividing a length where the parts are not perceptually given" (Piaget et al. 1960, p. 145). But RAY was wholly capable of making such an abstract subdivision. Like the children studied by Hunting and Sharpley (1991), who could use their dyadic and triadic attentional patterns in fragmenting, RAY could use his attentional pattern for making composite units as a template for fragmenting an unmarked segment. I believe that RAY could assemble and use composite units of unspecified numerosity as a template for fragmenting prior to action. In this, it is possible to appreciate the compatibility between how RAY and how ROS in Protocol VII experienced a composite unit.

If, as the authors claim, RAY did subdivide a_2b_2 as given by his ruler, he was yet to constitute that subsegment of a_2b_2 as being iterable. If it was iterable, he could have used it to reconstitute a_2b_2 as a partitioned segment by iterating the subsegment, and this would have been manifest by his iterating the ruler to measure a_2b_2. So, I find his use of his hand and a strip of paper to prolong the ruler as consistent with the interpretation that although he did subdivide a_2b_2 per the ruler, the subsegments were not identical subsegments. Rather, they were distinctly different subsegments that implied different perceptual material.

An important property of the tacitly nested number sequence that has not been mentioned is that because this number sequence is constructed by reprocessing the initial number sequence, children can, while operating with one number sequence, present another number sequence apart from the one with which they are operating and coordinate their use. What this means in Protocol VIII is that RAY could mentally project an indefinite number of units into segment a_1c_1 as determined by a_1b_1, and then form the goal of projecting segments of the same length into a_2c_2. This would be sufficient for RAY to spontaneously measure along a_2c_2 to find where b_2 should be placed.

Measuring with unit iteration. At substage IIIB, the concept of an iterable unit enables the children to apply iterative stepwise movements to a ruler shorter than the length to be measured.

Protocol IX.

BED (8: 7) Illustrates level IIIB. "If you go there (a_1b_1) then I must go here too (a_2b_2)." He measures the distance by hand and then by ruler, using several applications (Piaget et al., pp. 142–143).

The distinction in the two ways of measuring between RAY and BED is parallel to the distinction that was made between the tacitly and the explicitly nested number sequences. In fact, the advancement in measurement displayed by BED in measuring a_2b_2 is quite compatible with how Jason (Protocol V, Chap. 3) used his units of one as iterable during the latter part of his second grade in school. In this, it is possible to appreciate the difference between the conceptual structures implied by the measuring behavior of BED and RAY.

In the case of BED, I infer that segment a_1c_1 symbolized a composite unit of subsegments, which could be produced by iterating a_1b_1, and which, when joined together, were possibly of length equal to a_1c_1. In the case of RAY, I infer that segment

a_1b_1 was one of a number of different segments that could be made all of length equal to a_1b_1. This is a subtle distinction, but it is a critical one. For BED, there would be no felt necessity to perform the fragmenting actions to fragment the whole segment a_1c_1. For RAY, segment a_1b_1 was only one among other subsegments of a_1c_1, and units would need to be at least mentally projected into the whole segment in order to be "in" the segment.

Partitioning and Iterating

I can now understand how partitioning and iterating can be essential aspects of the same psychological structure. The unit structure diagrammed in Fig. 3.10 of Chap. 3 symbolizes a composite unit structure. This unit structure explains why a child can have a sense of simultaneity in the case of "seven" or any other such number word. If the number word "seven" activates the composite unit structure diagrammed in Fig. 3.10 of Chap. 3, the unit structure enables the child to experience seven as a unitary item rather than as a composite unit containing seven unit items. The child knows that seven unit items can be produced and is aware of a specific numerosity, but the child is not compelled to produce seven counted items to experience them. A sense of simultaneous cooccurrence of the seven unit items is thus produced as a byproduct of the symbolizing function of the iterable unit item of Fig. 3.10 of Chap. 3. The child can be also aware of sequentially producing the seven counted items of Fig. 3.10 of Chap. 3 for the same reason it can be aware of their simultaneous cooccurrence.

This sense of the simultaneous cooccurrence of seven unit items is involved when using the template of Fig. 3.10 of Chap. 3 as a partitioning structure. However, alone it is not sufficient because the child must disunite the unit items in partitioning. Heuristically, I think of disuniting what has been united as removing the boundaries (the unitariness) from the elements of the containing unit. But this understanding of disuniting essentially destroys the composite unit structure during the process of disuniting. Children who have constructed composite units can reunite the equal parts produced by fragmenting, but reuniting these fragments has to be carried out if only mentally to establish the fragments as elements of a composite unit (a connected but segmented unit).

In the case of the child who has constructed an iterable unit of one, I model the operation of disuniting in a way that preserves the composite unit structure. In the establishment of a composite unit at the experiential level, such a child produces an awareness [which could involve a figurative image] of the composite unit structure to be used in partitioning. Once the composite unit is used in partitioning, the unit items of the partition appear to the child to cooccur. This awareness of the cooccurrence of the unit items is made possible by the uniting operation, and it is essential for the items to be disunited.

The operation that produces a "removal" of the boundaries of the composite unit structure is reprocessing the unit items of the composite unit structure using the iterable unit.[15] The operation of reprocessing is already available to the child because that is precisely how the iterable unit was constructed. Disuniting, then, means that the child focuses his or her attention on the elements of the composite unit structure rather than on the unitary structure as one thing. In this, the child remains "above" the elements and can move back and forth between the elements and the unit structure that contains them. Experientially, the child retains a sense of the parts produced in partitioning as belonging to the original unit without needing to mentally reunite those parts. A part remains a part of the original unit as well as a unit in its own right.

It is important to note that focusing attention on the elements of the composite unit structure does not remove the boundaries. Rather, the unit structure of the composite unit is constituted as background and the elements as foreground. This is a critical aspect of using a composite unit that has been produced by an iterable unit in partitioning. If the elements of the composite unit are applied to a continuous unit in such a way that equal parts of the continuous unit are produced, this activity would be framed by the unit structure of the composite unit. The parts could be mentally reunited to establish a connected but segmented unit in that case where the child brings the background unit structure into the foreground, but the child understands that this can be done and therefore does not need to perform the operation to establish an awareness of the result.

But more is possible. Each part of the disunited composite unit is an instantiation of a unit item that was produced using the iterable unit. As such, each part maintains its iterable quality. What this means is that the partitioning child can use any part of the partition in iteration to produce a continuous but segmented unit of the same size as the original unpartitioned continuous unit.

So I do not refer to a fragmentation of a continuous unit as an equipartitioning unless, first, the operating child intends to fragment the continuous unit into equal-sized parts and, second, unless the operating child can use any one of these equal-sized parts in iteration to produce a connected but segmented unit of the same size as the original unit.

Levels of Fragmenting

I have identified four levels of the fragmenting scheme that correspond to the construction of number sequences. First, there is the primitive fragmentation of a continuous unit into two parts made possible by the dyadic attentional pattern. At this level, children can begin to share a rope, a string, or some other kind of linear object into two parts using a figurative dyadic attentional pattern. Such sharing activity, especially in the case of planar regions, opens the possibility for reprocessing

[15] I assume that the operations of partitioning at the experiential level are simply those operations that such a child can use at the re-presentational level.

the produced fragments using the same pattern that was used in fragmenting. This, in turn, produces a numerical pattern, two, and children can become sensitive to the equality of the parts. Social practices involved in sharing, such as children always wanting a "bigger half," can obscure this precocious use of dyadic patterns in sharing, but it does not exclude it. When children understand that an item is to be shared equally, the rather artificial sharing situations used by Hunting and Sharpley (1991) and by Piaget et al. (1960) do indicate that a majority of children from the age of four years can indeed make two equal shares.

Although sharing into three parts in the case of planar regions requires the development of the operations that produce the initial number sequence, the study by Hunting and Sharpley (1991) does indicate an earlier emergence of sharing a linear object among three dolls. So I would expect children who can establish figurative quantity to be able to share a linear continuous unit into two or three parts, and I would expect them to be able, in the process of sharing, to reconstitute the involved patterns as numerical patterns in an attempt to make fair shares. However, I would not expect these children to share linear continuous units into five or more parts regardless of the equality of the units.

In the second level of fragmenting, there is the use of numerical composites to project units into continuous units. This scheme is characterized by fragmenting continuous units into three parts after some trial and error and by attempting to fragment a continuous unit into five or more parts. In the latter case, there is a lack of coordination of the number and size of the parts with using the whole of the continuous unit. I believe that this coordination might be achieved after some trial and error, but that the child would need to actually carry out the fragmenting activity. There would be no a priori necessity to share the whole of the continuous unit into so many equal parts.

In the third level of fragmenting, there is the use of composite units to project units into continuous units. Here, sharing a continuous whole into three parts entails an a priori necessity to share the whole of the continuous unit into three equal parts. This certainty is made possible by the simultaneous awareness of the unit items and of the composite unit to which they belong, along with the simultaneous cooccurrence of the three unit items. In other words, the three unit items seem to cooccur, and the child can present them in that way while remaining aware of their unitary wholeness. I believe that the elements of any other composite unit that has been established as a numerical pattern could be used in the same way. But, for those composite units whose elements do not occur in a pattern, the child uses mental and physical experimentation to partition a continuous unit using those composite units.

In the fourth level, there is the use of composite units whose elements are iterable to project units into a continuous unit. Here, the child understands that any one of the units can be used to reconstitute a connected but segmented unit equivalent to the original unit. It is here that I would say that the child has constructed an equipartitioning scheme.

There is a fifth level involving partitioning *n* items among *m* children exemplified in a study of children's partitioning behavior conducted by Lamon (1996). She presented 11 partitioning tasks to 346 students distributed throughout grades four to eight. Three of the tasks were to share 4 pepperoni pizzas among 3 children, to share 4 chocolate chip cookies among 3 children, and to share 4 oatmeal cookies among 6 children. To partition the *n* items among the *m* children, a child would have to distribute the operation of projecting a composite unit of numerosity *m* across the *n* items. The construction of this distribution operation should have been a distinct possibility for almost all of the children in Lamon's sample. For example, in the case where the child intends to share four pizzas among three children, if the child's concept of three is constructed in such a way that the number word "three" refers to a unit that can be iterated three times, the child would intend to share all of the pizza equally among three children prior to actually making the shares. That is, it would be the child's goal to split all of the pizza into three equal parts. Upon experimenting and finding that there is no easy or practical way of cutting the four pizzas into three equal pieces by making just two cuts, the child might then search for another way of cutting the pizza. The child might restructure the pizza into a unit containing three pizzas and another unit containing one pizza with the intention of giving one whole pizza to each child and then splitting the remaining pizza into three equal shares. If the restructuring is the result of productive thinking rather than an accidental restructuring, the child would also understand that partitioning each pizza of the original unit containing all four pizzas into three pieces and then combining the parts would produce three equal shares of all of the pizza. This, of course, constitutes a distribution of the operation of partitioning over parts of the original unitary whole to produce a partitioning of the whole. This would seem to be a crucial operation in the construction of fraction schemes.

Of the 123 children in the fourth and fifth grades, Lamon (1996) reports that the following percentage of children displayed incomplete, incomprehensible, or invalid strategies: approximately 50% of the children in the case of the pepperoni pizzas; approximately 48% of the children in the case of the chocolate chip cookies; and approximately 57% of the children in the case of the oatmeal cookies. The difficulty in sharing *n* items among *m* children was not because there were more items to share than there were children, because the greatest percentage of children who experienced difficulty in sharing occurred in the case of sharing four oatmeal cookies among six children.

These percentages are alarmingly high when considering that the children in the sample were at least 9 years of age, an age where I would expect over 75% of the children would have constructed the explicitly nested number sequence and thus iterable units of one. Hence I suspect that these children lack the distribution operation. In support of that theory, Lamon reported the successful children operating precisely in a way that would indicate use of the distribution operation.

Final Comments

The operations symbolized by children's number sequences play an instrumental role in their construction of connected number sequences and fragmenting schemes. This finding unifies children's quantitative operations across the discrete and the continuous.

I have indicated that the composite structures children use to establish situations of their number sequences serve as templates for fragmenting unmarked continuous units into fragments. Sharing a continuous unit among so many dolls provided an excellent context for the natural use of discrete structures as templates for fragmenting. The particular case of sharing a circular cake fairly among three dolls proved to discriminate children who had constructed the initial number sequence from children who had constructed only the figurative number sequence. It may seem strange that this apparently simple task discriminated so well between the prenumerical and numerical children. But there is a good reason for its discrimination. Cutting a circular region into three approximately equal parts requires a concentrated effort regardless of whether the child makes parallel cuts or locates the center of the region and makes radial cuts. To be successful, at some point in the fragmenting process, the child is obliged to make mental estimates of where to cut and mentally compare the parts that would be made if the cake was actually cut. This involves a coordination of the number and size of the parts with exhausting the whole of the cake. Certainly, operations that make this possible emerge at the level of the tacitly nested number sequence. Then children can use the numerical composite, three, as if it involved the uniting operation because they can be aware of three parts simultaneously without uniting them into a composite whole. However, due to the special or early production of a triadic figurative pattern, it can be used as a pseudo composite unit before the construction of the tacitly nested number sequence.

Although it is possible to find children who have constructed three as a numerical pattern without having constructed the initial number sequence (Steffe and Cobb 1988), the latter construction follows on soon after the former, so I would expect subdivision of a circular region into three approximately equal parts to indicate the construction of the initial number sequence. Because of their nature, situations involving sharing a continuous unit into so many equal parts involve the use of discrete structures to project units into the continuous unit. However, in the tasks that involved the subdivision of a line, the only requirement was that the child place b_2 at a place on a_2c_2 in such a way that a_2b_2 was of length equal to a_1b_1. In that the tasks did not involve a specific numerosity like three or four, they provided a good test of whether the operations involved in constructing number sequences were also involved in fragmenting. It was a surprise that the operations involved in subdividing a line were even more compatible with the operations involved in constructing number sequences than were the sharing tasks. It was a surprise because the children's discrete structures were not activated by the utterance of a number word or by the presence of so many dolls. Yet, making visual estimates to find where b_2 should be placed and the corresponding two levels of measuring behavior were easily interpretable in terms of the three levels of number sequences.

The finding that child's quantitative operations emerge in both the continuous and the discrete case in the same time frame, and in quite similar ways, provides solid support for the reorganization hypothesis. However, the quantitative operations that are constructed in the continuous case are not accompanied by the concomitant establishment of a natural language notational system in a way that is analogous to the discrete case. When attempts are made to establish a notational system for continuous quantitative operations in the children's mathematics education, a fractional language notational system is emphasized, reserving development of the number sequence notational system to discrete quantity. I believe that this practice serves to separate the children's construction of fractional schemes from their number sequences. It places great demands on the continuous quantitative operations that are available to the children because it is very easy to go beyond the Stage IIIB quantitative operations in teaching fractions. It also serves to retard the elaboration and reorganization of the discrete quantitative operations. I believe that these practices only contribute to the separation of fractions and whole numbers in the mathematical experiences of children and may very well lead to whole number knowledge interfering with the construction of fractional schemes. Hence this interference, when it is present, is not due to the nature of learning, but to the nature of teaching.

Operational Subdivision and Partitioning

When numerical structures are used as templates for fragmenting, the items that are established (at least in the case of the explicitly nested number sequence) are construed as parts of the unit from which they originated. In fact, five of the seven aspects of operational subdivision identified by Piaget et al. (1960, pp. 309–311) are satisfied by the fragmenting operations of a composite unit at the stage of the tacitly nested number sequence. These are aspects 1–5, which I outline below. Six and seven await the construction of more advanced number sequences.

First, the continuous unit to which the composite unit is applied as a template for fragmenting is a "divisible whole, one which is composed of separable elements" (p. 309) because the elements of the composite unit comprise the "separable elements" when projected into the continuous unit. Second, there are a determinate number of parts in the case of a composite unit of specific numerosity. Children can also use a composite unit of indefinite numerosity in fragmenting, which provides them with even more possibilities than implied by Piaget et al. (1960). Moreover, when a composite unit is used as a template for fragmenting, the whole of the continuous unit is exhausted, and there is a coordination of the number and size of the parts with exhausting the whole of the continuous unit, which is the third aspect.

Establishing the relationship between the number of parts and the number of cuts is a possibility for children who have constructed the tacitly nested number sequence. In the case of a row of blocks, I indicated that children can coordinate a sequence of interiorized intervals and a sequence of interiorized blocks at the level of extensive quantity. Because the tacitly nested number sequence is one level above extensive quantity, the possibility is present for children to abstract this relationship, which is the fourth aspect.

Children are sensitive to the equality of the parts, the fifth aspect, even at the level of numerical composite, one level below the tacitly nested number sequence. Moreover, even though the children remove the uniting boundaries of the continuous unit in subdivision, these children can reunite the parts produced into a continuous but segmented unit that is equivalent to the original continuous unit. However, these are sequential operations and the children may regard the continuous but segmented unit as a result that is unrelated to the unit with which they began. It is only later, after the emergence of the explicitly nested number sequence, does the possibility open for the children to regard the whole as invariant and the sum of the parts to equal the original whole, which is the seventh aspect. Upon construction of the explicitly nested number sequence, children also construe the parts as units in their own right. But it is yet unknown if children consider them as units to be subdivided further, which is the sixth aspect.[16] It would seem as if children who have constructed the generalized number sequence can take a units of units of units as a given and use this unit structure in establishing a relation between any one of the subparts produced on the second subdivision and the original whole.

Children who use their number sequence in subdivision are also not restricted to subdividing into only a small number of parts. They can conceptualize the possibility of subdividing a continuous whole into as many parts as they know number words. Thus, they can establish meaning for fractions such as one one-hundred seventy-fifth, or any other fraction corresponding to a number word of their number sequence. This is especially the case for children who have constructed the explicitly nested number sequence. Finally, relationships among the number of parts and the size of the parts are within reach of these children.

Partitioning and Splitting

I now return to Confrey's idea of splitting and show how our analysis is compatible with how she defines splitting. In her definition, she commented that the "focus in splitting is on the one-to-many action" (Confrey 1994, p. 292). When a child has constructed the iterable unit of one, the child can focus on the continuous unit that is to be partitioned,[17] which is the unit item implied by "one" in "one-to-many," as well as on the number of parts into which the child intends to partition the continuous unit, which is the "many" in "one-to-many." This is made possible by the child being two levels removed from the elements of the composite unit that the child projects into the continuous unit. That is, the child can focus on the unit structure of the composite unit that is projected into the continuous unit in such a way that it both contains and partitions the continuous unit, and then focus on what is inside of the unit structure without destroying the unit structure. This produces an initial experience of one-to-many.

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[16] This will be one of the major issues that are investigated in the case studies that follow.

[17] Here, I use "partitioning" in the sense in which Confrey uses "splitting." Hereafter, I use "fragmenting" and "partitioning" rather than "splitting" to maintain the distinctions between partitioning and the earlier forms of fragmenting that I have identified.

But the child can do more in partitioning. If the child focuses on the elements of the composite unit into which the continuous unit is to be partitioned, the child can move back again to the unit structure. Moreover, the child can unite any subcollection of elements together and disembed them from the composite unit and constitute it as a composite unit in its own right without destroying these elements in the containing composite unit formed by partitioning. This establishes the classical numerical part-to-whole operation that serves as a fundamental operation in the construction of fractional schemes.

Finally, the child can use any singular part of the original partitioning in iteration to establish a connected but segmented unit equivalent to the original continuous unit. The child can also use any singular part of the original partitioning in iteration to produce a composite unit of elements of numerosity less than the numerosity of the composite unit used in partitioning and then compare that composite unit with the original composite unit. In this way, a child can establish meaning for, say, four-sevenths as four of one-seventh. The child can also compare four-sevenths with seven-sevenths and understand four-sevenths as four units out of seven units.

These are all crucial operations in the construction of fractional schemes. Although I do not deemphasize partitioning operations, neither do I focus on units of units as an end in themselves, as Kieren (1994) suggests. Rather, I use units of units as templates for fragmenting actions. By acknowledging the distinctions among numerical composites, composite units, and composite units that contain an iterable unit, I am able to make distinctions in what else the child might be able to do after making the initial fragmentation. Making the fragmentation is essential, but it does not supply the mental operations that are necessary for the construction of fractional schemes.

A more detailed analysis of children's fractional schemes is carried out in the context of the analysis of the fractional schemes the children in the teaching experiment actually constructed. There are many more distinctions that I have yet to make. What I have tried to do in this chapter is to establish a developmental rationale for the reorganization hypothesis. It seems to have great advantages for school mathematics. But, rather than leave the hypothesis unexplored, its implications for the teaching of fractions is thoroughly investigated in the next four chapters.