Chapter 11
Continuing Research on Students' Fraction Schemes

Anderson Norton and Amy J. Hackenberg

Directly or indirectly, *The Fractions Project* has launched several research programs in the area of students' operational development. Research has not been restricted to fractions, but has branched out to proportional reasoning (e.g., Nabors 2003), multiplicative reasoning in general (e.g., Thompson and Saldanha 2003), and the development of early algebra concepts (e.g., Hackenberg accepted). This chapter summarizes current findings and future directions from the growing nexus of related articles and projects, which can be roughly divided into four categories. First, there is an abundance of research on students' part-whole fraction schemes, much of which preceded *The Fractions Project*. The reorganization hypothesis contributes to such research by demonstrating how part-whole fraction schemes are based in part on students' whole number concepts and operations.

Second, several researchers have noted the limitations of part-whole conceptions and have advocated for greater curricular and instructional focus on more advanced conceptions of fractions (Mack 2001; Olive and Vomvoridi 2006; Saenz-Ludlow 1994; Streefland 1991). *The Fractions Project* has elucidated these limitations while articulating how advancement can be realized through the construction of key schemes and operations that transcend part-whole conceptions. In particular – and deserving of its own (third) category – research on fraction schemes has highlighted the necessity and power of the splitting operation in students' development of the more advanced fraction schemes, such as the reversible partitive fraction scheme and the iterative fraction scheme.

Finally, and more recently, researchers have used results from *The Fractions Project* to demonstrate how advanced fraction schemes can contribute to students' development toward algebraic reasoning. Although this research is in its infancy, one of the main findings so far is that the more advanced fraction schemes are critical in the construction of proportional reasoning (Nabors 2003), reciprocal reasoning (Hackenberg, accepted), and in solving basic linear equations of the form $ax = b$ (Tunc-Pekkan 2008).

L.P. Steffe and J. Olive, *Children's Fractional Knowledge*,
DOI 10.1007/978-1-4419-0591-8_11, © Springer Science+Business Media, LLC 2010

Research on Part-Whole Conceptions of Fractions

The reorganization hypothesis has roots in work by McClellan and Dewey (1895), who argued, "the psychological process by which number is formed is first to last essentially a process of 'fractioning' – making a whole into equal parts and remaking the whole from the parts" (p. 138). We see this in Steffe's (2002) work, as he has described numerical operations that become reorganized as vital components of fraction schemes, such as unitizing, partitioning, disembedding, and iterating. Working with Steffe, and building on the ideas of McClellan and Dewey, Hunting (1983) carefully examined the progress of a 9-year-old student named Alan, from whole-number concepts toward the development of fraction concepts. Hunting identified partitioning (for which "fractioning" might be an euphemism) as the key operation in Alan's development of a part-whole conception for fractions.

Before elaborating on Hunting's findings, we briefly comment on a subtle distinction between fraction schemes and fraction concepts, which was alluded to in previous chapters. We consider fraction concepts as fraction schemes whose results are available prior to engaging in the activity of the scheme. This implies that the activity of the scheme has been interiorized and that the child can engage in operating in the absence of material in the child's perceptual field. Tzur (2007) has made a similar distinction in terms of *participatory* and *anticipatory* schemes. In those terms, concepts are anticipatory schemes. Although we cannot elaborate further here, Tzur's study empirically demonstrated the negative consequences of classroom instruction that does not support students' development from the participatory stage of scheme construction to the anticipatory stage.

In a fraction concept, the operations of the fraction scheme are contained in the first part of the scheme (the recognition template, or "trigger"), which enables the scheme to become anticipatory; that is, the scheme can be activated prior to its enactment in the sense of a resonating tuning fork (Steffe, 2002), with no need for carrying out a sequence of mental actions to establish meaning for a particular situation or numeral. In the case of the part-whole fraction scheme, part-whole conceptions can be inferred once the part-whole fraction scheme is symbolized by any given fraction word or numeral. A child who has developed a part-whole conception of fractions immediately understands "¾," say, as three parts disembedded from a whole that has been partitioned into four equal parts. However, as we have pointed out, a part-whole conception of fractions is a bit of a misnomer because the *partitive fraction scheme* is the first genuine fraction scheme.

Hunting (1983) found that Alan was able to develop a part-whole conception of fractions by applying his knowledge of numerical units to situations involving partitioning and sharing parts. Thus, he demonstrated the utility of partitioning operations and coordinating units at two levels in the construction of early fraction knowledge. However, Hunting was surprised to find that, although Alan seemed to understand one-fourth and one-eighth as one of four and eight equal parts, respectively, Alan did not understand that one-eighth was less than one-fourth (the "inverse order relationship," also addressed in Tzur 2007). Subsequent research, which we discuss in the next section, has indicated that such understanding requires the iterating operation and partitive conceptions of fractions.

Several researchers have affirmed the value of engaging students in situations involving sharing and partitioning, in support of students' construction of part-whole concepts (Behr et al. 1984; Empson 1999; Kieren 1988; Mack 2001, Saenz-Ludlow 1995). *The Fractions Project* has provided a theoretical basis to support such findings by identifying the role of the partitioning operation in early fraction concepts, and by explaining the construction of the partitioning operation in terms of the construction of composite wholes. Subsequent research by psychologists unfamiliar with *The Fractions Project* has independently affirmed its main theoretical underpinning – namely, the reorganization hypothesis. Working with three 7-year-old students using nonverbal whole-number and fractions tasks, Mix et al. (1999) came to a conclusion that contradicted earlier work by researchers who had advanced an interference hypothesis:

> There were striking parallels between the development of whole-number and fraction calculation. This is inconsistent with the hypothesis that early representations of quantity promote learning about whole numbers but interfere with learning about fractions. (p. 164)

At least in the mathematics education research community, it is now commonly accepted that numerical operations, such as partitioning and disembedding, constitute students' development of fraction concepts – when students have constructed these operations in continuous contexts, such as with connected numbers. This is clearly illustrated in recent work, even by researchers unaffiliated with *The Fractions Project*. In particular, Mack (2001) implicitly relied on the reorganization hypothesis in her study of six fifth-grade students, examining the development of fraction multiplication. Mack found that, indeed, students' informal knowledge of partitioning contributed to their construction of fraction concepts.

On the one hand, findings such as Mack's and Hunting's underscore the foundational importance of part-whole reasoning in developing fraction conceptions. On the other hand, to construct "genuine" fractions, students need to transcend part-whole conceptions. In fact, in the very same work cited above, Mack (2001) found that "students' informal knowledge of partitioning did not fully reflect the complexities underlying multiplication of fractions" (p. 291). The problem is confounded when we recognize that – as Streefland noted in 1991 – "teaching efforts have focused almost exclusively on the part-whole construct of a fraction" (p. 191).

The singular focus of curricula and instruction on part-whole concepts has contributed to students' difficulties in working with fractions operations and even algebraic reasoning. For example, in working with a student named Tim, Olive and Vomvoridi (2006) found that restriction to part-whole concepts hindered his ability to meaningfully engage in classroom activities that implicitly required more advanced conceptions. "Sparse conceptual structures limit students' understanding; once these conceptual structures had been modified and enriched, Tim was able to function within the context of classroom instruction" (p. 44). However, educators must recognize that they cannot change students' structures at will. Laura's case study (Chaps. 5 and 6) exemplifies this fact: Despite persistent efforts to provoke accommodations to her part-whole fraction scheme, Laura did not construct a partitive fraction scheme for over a year.

Transcending Part-Whole Conceptions

Olive (1999) and Steffe (2002) have demonstrated that numerical schemes contribute to the construction of fraction schemes, even beyond initial constructions such as the part-whole fraction scheme. Earlier work by Saenz-Ludlow (1994, 1995) elucidated those contributions by establishing explicit links between the numerical and fraction conceptions of two third-grade students named Michael and Anna. In this section, we build on the previous one by describing how students like Michael used numerical schemes to construct partitive fraction schemes. At the same time, we share findings on ways in which partitive reasoning transcends part-whole reasoning.

Saenz-Ludlow began her teaching experiment with the hypothesis that "Michael's well-grounded conceptualization of natural-number units would facilitate the generation of fractional-number units" (1994, p. 63). In fact, Michael seemed to reorganize two key numerical operations – coordinating two levels of units and iterating – to construct a partitive fraction scheme for composite units. As Michael demonstrated, the new scheme transcended the power of his previously constructed part-whole fraction scheme. For example, consider Michael's response to the following task:

T: If I give you forty-fiftieths of 1,000 dollars, how much money will I give you?
M: (After some thinking.) Eight hundred dollars.
T: Why?
M: (Quickly.) Because one-fiftieth is 20 dollars and five 20s is 100, so five, ten, fifteen, twenty, twenty-five, thirty, thirty-five, forty (Keeping track of the counting of fives with his fingers and finally showing eight fingers.); that is 800.

Michael's ability to anticipate the value of forty-fiftieths before actually double-counting fives and hundreds on his fingers indicates that, in fact, he had interiorized three levels of units, at least for whole numbers. He was able to consider the given fraction (forty-fiftieths) as a quantity relative in size to the given whole (1,000). Moreover, the units he was iterating (100's) were each composed of five-fiftieths, which provides indication of a composite unit fraction. His overall way of operating resembles the partitive fraction scheme for connected numbers that Nathan constructed (cf. Chap. 9). It enabled Michael to perform such tasks, whereas, in using his part-whole fraction scheme, he would have been restricted to interpreting forty-fiftieths as 40 parts out of 50 equal parts within a referent whole.

Saenz-Ludlow (1994) alluded to partitive reasoning when she advocated student conceptions of fractions as quantities. Such conceptions enable comparisons of size between part and whole, or even part and part, through the iteration of units. However, the operations of the partitive fraction scheme remain constrained within the referent whole. Both of Jason and Laura (Chap. 5) experienced the necessary errors that result from this way of operating. Even late in their fourth grade, 9/8 became "nine-ninths" or "eight-ninths" or "one-eighth plus one, where the "eighth" referred to 8/8 (p. 406). So, "conceptualizing improper fractions is not a simple extension of iterating a unit fraction within the whole" (Tzur, 1999 p. 409). Thus, there are at least two developmental hurdles with regard to conceptualizing fractions: moving from part-whole to partitive conceptions, and moving from partitive conceptions of proper fractions to iterative conceptions of proper and improper fractions. Subsequent research has indicated the critical role splitting plays in clearing the latter hurdle.

The Splitting Operation

Several researchers have independently adopted the term "split" from their students (Confrey 1994; Empson 1999; Olive and Steffe 2002; Saenz-Ludlow 2004). Confrey was first in promoting use of the term in research, especially with regard to her splitting hypothesis. Her splitting hypothesis posits that children develop a multiplicative operation – splitting – in parallel with additive operations. According to Confrey (1994), splitting applies to actions of "sharing, folding, dividing symmetrically, and magnifying" (p. 292). "In its most primitive form, splitting can be defined as an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram… a one-to-many action" (p. 292).

According to Steffe (Chaps. 1 and 10), the splitting operation is the composition of partitioning and iterating, in which partitioning and iterating are understood as inverse operations. For example, a student with a splitting operation can solve tasks like the following: "The bar shown below is three times as big as your bar. Draw your bar" (see bar and student response in Fig. 11.1). Finding an appropriate solution requires more than sharing (or any other act of partitioning), and even more than sequentially applying acts of partitioning and iterating; the student must anticipate that she can use partitioning to resolve a situation that is iterative in nature. Namely, by partitioning "my" bar into three parts, the student obtained a part that could be iterated three times to reproduce the whole bar.

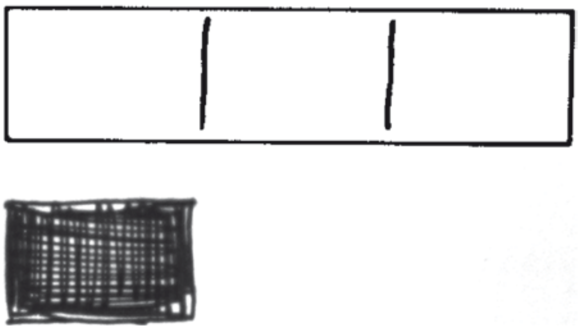

Fig. 11.1. Task response providing indication of a splitting operation.

As exemplified in Fig. 11.1, splitting involves partitioning, but it involves more. It supersedes even the levels of fragmenting identified in Chap. 10: simultaneous partitioning and equi-partitioning. As such, Confrey and Steffe's definitions for splitting contain similarities, but differ in one key regard: Confrey's definition makes no mention of iterating. In fact, Confrey (1994) intentionally juxtaposed splitting with iterating, which she viewed as contributing to repeated addition rather than the multiplicative reasoning that splitting supports. If she did not take exception to the inclusion of iterating operations, Confrey's splitting might include Steffe's splitting, as well as equipartitioning and simultaneous partitioning.

Splitting, as defined by Steffe, is especially powerful, as illustrated in the following case. During a semester-long teaching experiment with three pairs of sixth-grade students, Norton (2008) worked with a student name Josh who had constructed a splitting operation, but no genuine fraction schemes. That is to say, he could solve tasks like the one illustrated in Fig. 11.1 and he had developed a part-whole conception for fractions, but he had not yet constructed a partitive unit fraction scheme. Among the three pairs of students, only one other student, Hillary, had constructed a splitting operation (Norton and D'Ambrosio 2008). Relative to their peers, both students made impressive advancements in their constructions of fraction schemes, but we focus on Josh.

At the beginning of the teaching experiment, Josh was unable to unambiguously use fractional language. For example, when shown a 7/7-bar and asked what amount would remain if two-sevenths were removed, Josh answered, "5 pieces." When pressed for a fraction name, he could not decide between "five-sevenths," "fifty-sevenths," and "seven-fifths." Furthermore, when presented with a stick that had been partitioned in half, with the left half partitioned in half again, Josh thought the leftmost piece would be "one third." These responses indicate that Josh had not yet constructed a partitive unit fraction scheme. However, toward the end of the teaching experiment, Josh began estimating fractional sizes for proper fractions. Using the computer fractions software, TIMA: Sticks, Josh's partner produced an unpartitioned 2/9-stick along with its unpartitioned whole. When asked what the stick would measure, Josh lined up four copies of the fraction stick along the whole stick and estimated, "two-ninths." His estimate indicated that he had constructed a general partitive fraction scheme.

Throughout the teaching experiment, Josh formed conjectures that involved novel uses of his splitting operation. These conjectures seemed to support his construction of fraction schemes, including a partitive fraction scheme and a commensurate fraction scheme. Norton (2008) hypothesized that the splitting operation was particularly powerful in supporting his constructions because it composed two operations critical to the construction of fraction schemes: partitioning and iterating. In fact, studies cited in the previous two sections (e.g., Mack 2001) have illustrated the critical roles of those operations. Their composition then, provides powerful opportunities for growth, including the construction of more advanced fraction schemes.

In all of the fractions teaching experiments cited here, no student constructed an iterative fraction scheme – or a reversible partitive fractional scheme – without first constructing splitting. We have seen examples of this phenomenon from students mentioned in previous chapters, as well as Hillary, whose splitting operation supported her construction of a reversible partitive fractional scheme. In addition, all four of the students in Hackenberg's (2007) 8-month teaching experiment fit that pattern of development. All four students began sixth grade with splitting operations; all four constructed reversible partitive fraction schemes; and two of the students constructed iterative fraction schemes.

Consider the following exchange between the teacher–researcher and one of the pairs in Hackenberg's study, Carlos and Michael. The teacher–researcher asked Carlos to produce fourteen-thirteenths. Carlos began by partitioning a copy of the whole stick into 14 parts. Seeing this, Michael exclaimed, "No – no – no! You made fourteenths – (looks at Carlos) yours is thirteenths (gives a little laugh)." Carlos responded by asking the teacher–researcher, "didn't you say fourteen-thirteenths?" (Hackenberg 2007, p. 39). Carlos eventually produced the 14/13 by appending an extra piece to a 13/13 stick, but the exchange indicates Carlos's struggles in producing improper fractions. But then, he had much more success in solving tasks like the following: "Tanya has $16, which is 4/5 of what David has; how much does David have?" (p. 45).

Notice that the latter task requires a way of operating that is in reverse of the task Saenz-Ludlow (2004) posed to the third grade student named Michael (illustrated in the previous section). In particular, it requires a reversible partitive fraction scheme, which Carlos seemed to have constructed. However, Carlos had not yet constructed the kind of operating that his partner, Michael, used to solve the task involving 14/13. Namely, Michael had constructed an iterative fraction scheme. Both ways of operating require splitting because the students had to use partitioning in service of an iterative goal: Producing 14/13 required Carlos to partition the whole into 13 parts so that one of them could be iterated 14 times to produce the improper fraction; producing David's amount of money required Carlos to partition the given 4/5 part into four parts so that one of them could be iterated five times to produce the unknown whole. However, only Michael could readily solve the former task, and Hackenberg (2007) attributes this difference to the interiorization of three levels of units. A student would need to posit three levels of units prior to activity to purposefully produce a bar containing 14 thirteenths, with the understanding that the whole is produced from 13 iterations of any one of those thirteenths.

Findings from Norton's and Hackenberg's teaching experiments have challenged previous hypotheses about the origins and nature of splitting. Steffe (2004) had hypothesized that the splitting operation is based on the reversible partitive fraction scheme: "I presently consider the splitting operation to be the result of a developmental metamorphic accommodation of the reversible partitive fractional scheme" (p. 161). He revised this hypothesis after considering the case of Josh (Norton 2008) who had constructed a splitting operation even before constructing a partitive unit fraction scheme. We now understand that – to the contrary of the initial hypothesis – splitting is required for the construction of the reversible partitive fraction scheme.

Revising Steffe's (2004) hypothesis about the origin of the splitting operation begs the question: Where does splitting "come from" in students' constructive itineraries?

Confrey (1994) attributed the origins of splitting to abstractions from fair sharing activities, which makes sense given that for her splitting is based on making equal partitions. Saenz-Ludlow (1994) demonstrated how students transform Confrey's split from whole number to fractions contexts. However, in Steffe's splitting, which includes iterating as well as partitioning, fair sharing activities alone would likely be insufficient as an origin. Although Josh's case offers an exception, it seems that students' experiences in partitive fraction situations support their construction of splitting. After all, the partitive fraction scheme involves the sequential use of partitioning and iterating. Applying those operations as part of one scheme could plausibly contribute to their eventual composition as a single operation: splitting.

This view aligns with Steffe's revised splitting hypothesis expressed in Chap. 10: Construction of splitting results from interiorization of the equipartitioning scheme – a necessary prequel to the reversible partitive fraction scheme. In fact, the partitive unit fraction scheme is also a derivative of the equipartitioning scheme (cf. Chap. 10), so it makes sense that most students construct partitive unit fraction schemes prior to their construction of splitting (Norton and Wilkins, in press). The revised hypothesis also aligns with Hackenberg's (2007) findings regarding units coordination. Namely, students operating with a partitive fraction scheme should have constructed the two levels of interiorized units required to construct a splitting operation.

Steffe had also previously hypothesized that, "upon the emergence of the splitting operation," the partitive fraction scheme would be reorganized as an iterative fraction scheme (2002, p. 299). He revised this hypothesis in light of Hackenberg's (2007) teaching experiment and one of its key findings:

> Although the splitting operation still seems to be instrumental in the construction of an iterative fraction scheme, it does not appear to be sufficient for it… Students can construct the splitting operation without also interiorizing the coordination of three levels of units, and this interiorized coordination appears to be necessary for constructing improper fractions, and therefore the improper fraction scheme. (p. 46).

Steffe revised his hypothesis to its present form: If the child's operations that produce three levels of units become assimilating operations of the partitive fraction scheme, then the partitive fraction scheme can be used in the construction of the iterative fraction scheme. In other words, the partitive fraction scheme requires two levels of interiorized units, but if it, furthermore, includes a structure for producing three levels of units, the splitting operation might indeed be used to reorganize the partitive fraction scheme into an iterative fraction scheme. In fact, Joe and Patricia's case studies (cf. Chap. 10) illustrate such constructions. Further, based on Melissa's case study, Steffe claims that children construct the splitting operation using three levels of units in activity, though the operations that produce these units are not necessarily assimilating operations.

Students' Development Toward Algebraic Reasoning

In the past decade, researchers have begun to work on the problem of how students' construction of fraction schemes and operations may support students' construction of algebraic reasoning. This research focus is part of a larger effort to understand how to help students base their construction of algebraic reasoning on robust quantitative reasoning (Kaput 2008; Smith and Thompson 2008; Thompson 1993). One thrust of this effort is to understand how students can develop significant, conceptually coherent quantitative reasoning that would actually warrant generating and using powerful symbolic tools of algebra (Smith and Thompson 2008). Researchers who study students' construction of fraction schemes have made some progress in this area, as we will outline below. In contrast, little research based in scheme theoretic approaches has as of yet made significant progress on how students construct algebraic symbol systems (cf. Tillema 2007; Tunc-Pekkan 2008).

Before discussing the research that has been done in this area, we give a brief outline of how we characterize algebraic reasoning. From a very broad perspective, Kaput (2008) posited that algebra has two core aspects: (A) systematically symbolizing generalizations of regularities and constraints, and (B) engaging in syntactically guided reasoning on generalizations expressed in conventional symbol systems. He envisioned these core aspects as embodied in three strands: (1) algebra as the study of structures and systems abstracted from computations and relations, including algebra as generalized arithmetic and quantitative reasoning; (2) algebra as the study of functions, relations, and joint variation; and (3) algebra as the application of a cluster of modeling languages. Much of the research on children's early algebraic reasoning focuses on Kaput's core aspect A and strand 1 (e.g., Carpenter et al. 2003; Carraher et al. 2006; Dougherty, 2004; Knuth et al. 2006). We do so as well, but, as Tunc-Pekkan (2008) has pointed out, we do not take for granted the quantitative operations that may be required to build algebraic reasoning – in fact, we aim to specify them in our work with students.

We also aim to specify how quantitative operations may be reorganized to produce algebraic operations, a potential extension of the reorganization hypothesis of *The Fractions Project* (Hackenberg 2006; Tunc-Pekkan 2008). One possible "bridge" from quantitative fraction schemes (with the partitive fraction scheme being the first of these) to algebraic reasoning lies in students' construction of ratios and proportional reasoning. Nabors (2003) investigated this arena in her teaching experiment. She worked with seventh grade students to help them construct fraction schemes prior to investigating how they constructed schemes to solve problems involving ratios and proportions and rates, such as the following:

> *Money Exchange Problem.* "In England, pounds are used rather than dollars. Four US dollars can be exchanged for three British pounds. How many pounds would you get in exchange for 28 US dollars? (adapted from Kaput and West 1994)" (p. 136).

Nabors hypothesized that the construction of what we have called more advanced fraction schemes (such as a reversible partitive fraction scheme, an iterative fraction scheme, and a reversible iterative fraction scheme) would be sufficient for students to reason with unit ratios to solve problems like the Money Exchange Problem (see Kaput and West's third level of proportional reasoning, 1994). This hypothesis was not confirmed – Nabors found that the fraction schemes were likely necessary, but not sufficient, for students' construction and use of unit ratios (cf. Davis 2003). Although the students in her study made progress in solving problems like the Money Exchange Problem and other problems involving rates, they used "build-up" strategies, both additive and multiplicative in nature (cf. Kaput and West's first and second levels of proportional reasoning, 1994).

In particular, Nabors (2003) found that students who had constructed a units coordinating scheme for composite units – in which composite units were iterating units – could solve problems like the Money Exchange Problem by repeatedly coordinating iterations of two composite units (in this case, units of four and units of three). Nabors agreed with Kaput and West (1994) that this solution strategy is primarily additive in nature, and her description of it indicates that students who have constructed the ENS can engage in it. In contrast, to solve the problem by anticipating that twenty-eight is some number of composite units of four, using division to determine that number, and then iterating the composite unit of three that number of times required that composite units were iterable units for the students. In other words, Nabors indicates that students had to be aware of the operation of iterating composite units prior to iterating them (p. 139). In essence, this finding implies that the operations that produce the GNS are needed for solving problems involving ratios and proportions with this "more advanced" build-up strategy. Even though some students in her study appeared to have constructed these operations, they did not produce solutions involving unit ratios (e.g., in which students determine that three-fourths of a pound corresponds to 1 dollar, and so three-fourths of 28 will yield the number of pounds that correspond to 28 dollars). Nabors did not hypothesize what operations are necessary to construct such solutions, except for noting that the interiorization of three levels of units is likely necessary.

Finally, Nabors (2003) found that students in her study could use some standard notational forms to solve problems involving ratios and proportions, but she could not claim that doing so meant they were engaging in reasoning beyond the two kinds of solutions discussed above. In fact, she notes that her study was an initial foray into this area, and that future research should investigate how students construct "numerical and algebraic representations of their reasoning processes" (p. 177) in these situations (cf. Kaput and West's fourth level of proportional reasoning, 1994).

Hackenberg's (2005, accepted) research is similar to Nabors' research in that she aimed to understand how students construct schemes and operations that underlie another traditional "component" of beginning algebra: the construction and solution of basic linear equations of the form $ax = b$. In her teaching experiment, she investigated how students reverse their quantitative reasoning with fractions to solve problems that can be solved with a basic linear equation of the form $ax = b$. A central finding of her study was the interiorization of three levels of units (i.e., the operations that produce the GNS) was critical for the construction of schemes to solve problems like this one:

> *Candy Bar Problem.* That collection of 7 inch-long candy bars [7 identical rectangles] is 3/5 of another collection. Could you make the other collection of bars and find its total length?

To solve this problem, one student, Michael, modified his splitting operation to include the units-coordinating activity of his multiplying scheme. That is, Michael had constructed a reversible iterative fraction scheme and he used it to assimilate this problem: He aimed to split the known quantity into three equal parts, each of which would be one-fifth of the unknown quantity. However, he had no immediate way of operating to use to split seven units into three equal parts – the *seven* seemed to be at the "heart" of his perturbation in solving the problem. He eliminated this perturbation by splitting each of the 7 in. into a number of mini-parts (three) that would create a total number of mini-parts (21) that he *could* split into three equal parts (each containing 7 mini-parts). Hackenberg proposed that Michael could operate in this way because he could flexibly switch between two three-levels-of-units structures. That is, he conceived of the collection as a unit of seven units each containing three units, and then he could reorganize (in thought) the 21 mini-parts into a unit of three units each containing seven units. This way of reasoning is based on the splitting scheme for composite units in which the distributive partitioning scheme is embedded.

However, Hackenberg (2005, accepted) also found that the interiorization of three levels of units was not sufficient to provoke or explain the construction of reciprocal reasoning – although it seems to be necessary for it. In particular, Michael did not reason reciprocally to solve problems like the Candy Bar Problem, but another student in the study, Deborah, at least began to do so. Hackenberg hypothesized that Deborah had abstracted a fraction as a multiplicative concept, i.e., as a program of operations that included those of Deborah's iterative fraction scheme and reversible iterative fraction scheme. As Tunc-Pekkan (2008) has identified, how Deborah produced this abstraction was not clear. A related limitation of Hackenberg's study was that she and her student-participants did not engage in operating explicitly on unknowns, an important characteristic of algebraic reasoning. In the context of solving problems like the Candy Bar Problem, reasoning reciprocally would facilitate operating on the unknown quantity.

Tunc-Pekkan (2008) conducted a teaching experiment specifically to investigate students' construction of reciprocal reasoning in stating and solving equations of the form $ax=b$ where a and b are both fractional numbers. She differentiated between reversible reasoning and inverse reasoning in this context. On the basis of her analysis of one of the two pairs of eighth grade students with whom she worked, she hypothesized that constructing an inverse relationship between two quantities requires (1) conceptualizing *both* quantities independently (rather than solely that the known can be used to make the unknown); (2) constructing explicit equivalencies between fractional parts of the known and unknown quantities; and (3) using operations such as disembedding and iterating parts of the known quantity to create the unknown quantity (i.e., using multiplicative reasoning to construct the unknown quantity).

Tunc-Pekkan's findings indicate that the construction of *measurement units* were critical for producing and operating with equivalency; constructing only identity relationships between parts of quantities, which is possible when composite units are iterable, was insufficient to construct standard measurement units as independent quantities. The construction of measurement units involves the coordination of sequences of units, and so surpasses the operations that produce the GNS (alone). This finding is important because it indicates that "numerical aspects" of reasoning with quantities must be included in what we call "quantitative reasoning" for it to be powerful enough to be a basis for algebraic reasoning, something that some advocates of quantitative reasoning as a basis for algebra have downplayed or ignored (cf. Smith and Thompson 2008).

In addition, Tunc-Pekkan (2008) found that the students' construction of a *symbolic* fraction multiplication scheme was critical for students' construction of reciprocal reasoning. By symbolic she meant that students need to construct "more" than an anticipatory scheme in which they can find (make) a fraction composition – i.e., in which they construct a new quantity (the composition) as a result of operating on known quantities. In addition, students need to be able to construct the measurement of those quantities using what she called *recursive distributive partitioning operations*. Constructing these operations involves constructing partitioning and iterating as inverse operations, as well as distributive partitioning.[1] Her conclusions are interesting in light of the central role that Steffe gives to splitting in the construction of fraction schemes: Constructing the splitting operation may be the first step in the construction of an awareness of partitioning and iterating as inverse operations (cf. Chap. 10). In this way, construction of the splitting operation is important in the development of algebraic reasoning.

A central message of Tunc-Pekkan's (2008) research is that the power of algebraic thinking comes from *not* being dependent on quantities produced through operating but from being able to think of and interpret quantitative situations in terms of measurement units. More needs to be understood regarding how students construct measurement units and recursive distributive partitioning operations. However, together the work we have reviewed in this section suggests that researchers have made progress in understanding two hallmarks of algebraic reasoning: how students build conceptual structures and operate on them further, and how students learn to operate explicitly on unknown quantities.

[1] For Tunc Pekkan (2008), distributive partitioning is the operation that a student might use to find, say, 1/7 of 3 in. The student might partition each of the 3 in. into seven equal parts, disembed three parts (e.g., one part from each of the three inches), and unite them together to make 3/7 of 1 in. Recursive distributive partitioning involves, further, being able to engage in distributive partitioning of parts of quantities that are not perceptually present in service of taking a fractional amount of a quantity. For example, consider taking 1/7 of 3/5 of a liter, when only the 3/5-liter is present in the student's visual field. If a student uses distributive partitioning but also applies it to the two fifths of the liter that are not present, to conclude that the result is 3/35 of a liter, then the student has used recursive distributive partitioning.