Chapter 5
Completions, Constructions, and Corollaries

Thomas Mormann

5.1 Introduction

In his paper *A Renaissance of Empiricism in the Recent Philosophy of Mathematics?* (Lakatos 1978), Lakatos painted the history of Western epistemology with a broad brush:

> Classical epistemology has for two thousand years modeled its ideal of a theory [...] on the conception of Euclidean geometry. The ideal theory is a deductive system with an indubitable truth-injection at the top (a finite conjunction of axioms) – so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system. (Lakatos 1978: 28)

The Euclidean perspective, as Lakatos defined it, has not much to say about proofs beyond the well-known characterization that they are deductively valid arguments that necessarily lead from true premises to true conclusions. In the case of Euclidean geometry, this means that the axioms of Euclidean geometry logically imply the theorems of Euclidean geometry. Today we take this assertion as a triviality. Philosophically, it might be less trivial than one thinks at first view. According to the founding father of modern epistemology – Kant – the just-mentioned "triviality" is no triviality but a blatant falsehood. More precisely, Kant proposed the thesis that the axioms of Euclidean geometry do *NOT* logically imply the theorems of Euclidean geometry. This sounds a bit surprising, to say the least. But Kant insisted that proof needs something more than just pure logic: namely, pure intuition.

If this is true, then Kant does not belong to the tradition of Euclidean epistemology as Lakatos defined it. Hence the question, "Whom else we can pick out as a good example of Lakatos's 'Euclidean tradition'?" A good choice would be Bertrand Russell, who vigorously argued for the Anti-Kantian thesis:

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The axioms of Euclidean geometry do logically imply the theorems of Euclidean geometry. More generally, proofs in mathematics must not contain any nonlogical ingredients. (Russell 1903, § 5)

Let's call this Russell's thesis. The first time Russell presented it was in *The Principles of Mathematics* (Russell 1903). *The Principles* are heavily influenced by the logical and mathematical achievements of Peano, Cantor, and Frege, but Russell may be credited as the first professional philosopher who argued for this logicist thesis. If one accepts Russell's thesis, the philosophy of mathematics and the philosophy of the empirical sciences become neatly separated: On the side of the empirical sciences, one has a variety of procedures to obtain scientific knowledge, ranging from deductive and inductive arguments to experiments of various kinds. On the other hand, mathematics has only one method of producing knowledge: proving theorems through using arguments of deductive logic. Not everybody subscribed to this neat "apartheid" between philosophy of mathematics and philosophy of empirical science. Among the dissenters, one may mention (1) Peirce's Semiotic Pragmatism, (2) Cassirer's Critical Idealism, and (3) Lakatos's Quasi-empiricism.

I'll say nothing about Lakatos but will concentrate on Cassirer, with some occasional glances at Peirce. I do not aim at elucidating the relation between Peirce's and Cassirer's philosophies in general; rather, I'd like to concentrate on one pertinent issue, namely the role in both of intuition and symbolic constructions for mathematical knowledge. Both accounts may be characterized as attempts to do justice to Kant's philosophy of mathematics and at the same time to overcome the limitations of the traditional Kantian account of pure intuition in the realm of mathematical proofs. Both meant to withstand Russell's radical logicist stance, according to which anything like intuition is completely obsolete for modern mathematical and scientific knowledge. In particular, his emphasis on the role of idealization[1] in mathematics *and* the sciences may be interpreted as an attempt to revive something like Kant's pure intuition, or so I want to argue. The outline of my paper is as follows:

1. The Role of Intuition in Mathematics according to Kant
2. Russell's Logicist Expulsion of Intuition
3. Cassirer's Critical Idealism
4. Idealizations, Constructions and Corollaries
5. Concluding Remarks

[1]"Idealization" points to the more general topic of the "symbolic" character of scientific and mathematical knowledge, a huge issue that involves epistemology, philosophy of science, and other disciplines. It cannot be adequately treated in a short paper like this; for further information, the reader may consult the following: Ferrari and Stamatescu (2002), Ihmig (1996, 1997) Rudolph and Stamatescu (1997), Ryckman (1991).

5.2 The Role of Intuition in Mathematics According to Kant

First we have to deal with Kant's claim that the axioms of Euclidean geometry do not logically imply the theorems of Euclidean geometry. Indeed, Kant contended that the theorem that the sum of the angles of a triangle is two right angles (180°) is *not* implied the Euclidean axioms. First I'll give the textual evidence, then explain why Kant made such a claim and why it is correct – even from our more modern point of view.

Kant's "antilogical" thesis is expressed most clearly in the "Discipline of Pure Reason in Its Dogmatic Employment" in the *Critique of Pure Reason*, where Kant contrasted philosophical with mathematical reasoning:

Philosophy confines itself to general concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept in concreto, although still not empirically, but only in an intuition which it presents a priori, that is, which it has constructed, and in which whatever follows from the general conditions of the construction must hold, in general for the object of the concept thus constructed.

Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angle bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, along with the concept of just as many angles. However long he meditates on these concepts, he will never produce anything new. He can analyze and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any properties not already contained in these concepts. Now let the geometer take up this question. He at once begins by constructing a triangle. Since he knows that the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of the triangle and obtains two adjacent angles which together equal two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a solution of the problem that is simultaneously fully evident and general. (Kant 1797/2006: B743–745)

According to Kant, the only kind of logic available for the philosopher to analyze concepts was traditional syllogistic logic. As Peirce and Russell already noted, syllogistic logic is not very helpful for proving theorems of geometry and other mathematical theories. Thus, Kant was quite right in claiming that the axioms of Euclidean geometry do not logically imply the theorems of Euclidean geometry. If we rely on syllogistic logic, we need help from a nonlogical source to carry out geometrical proofs. For Kant, this source was provided by pure intuition.

The experts on the Kantian philosophy of mathematics have formed no consensus about what exactly "Kantian Pure Intuition" means (cf. Friedman 1992). Here, I am not interested in parsing Kantian philology. Rather, I'd like to take Kant as a starting point.

The important thing about "pure intuition" in a broad Kantian sense is that it casts mathematical proofs as ideal spatio-temporal scenarios, in which certain constructions are carried out according to certain rules constituting the ideal domain in which this mathematical activity takes place. Something like this can already be found in the *Critique of Pure Reason*:

> I cannot represent to myself a line, however small, without drawing it in thought, that is gradually generating all its parts from a point. Only in this way can the intuition be obtained […] Geometry together with its axioms, is based upon this successive synthesis of the productive imagination in the generation of figures. (Kant 1787/2006: B 203–204)

This Kantian drawing of straight lines does not take place in real space-time; rather, it refers to an ideal space-time – more precisely, an idealized Newtonian space-time. The constructions guided by pure intuition take place in this idealized space-time, where *ideal* points, *ideal* trajectories, *ideal* straight lines and so on exist, and where an *ideal* subject is able to draw perfect geometrical figures. This ideal space is defined by Newtonian mechanics and thus, in some sense, geometry presupposes Newtonian mechanics. In other words, a "mixing" of physical and mathematical ideas was essential to the unity of Kant's philosophy of mathematics. As we shall see similar features may be discerned in Cassirer's and Peirce's accounts.

Summarizing, then, I propose to consider "pure intuition" as a faculty involved in checking proofs step by step to see that each rule has been correctly applied – in short, the intuition involved in "operating a calculus" (cf. Hintikka 1980). Kantian pure intuitions should be interpreted as having a strong operational or constructive component. Such a constructive version may help preserve a role for something like intuition even for modern mathematics.

5.3 Russell's Logicist Expulsion of Intuition from Mathematics

For mathematicians, everything changed at the end of the nineteenth century, when modern relational logic arrived on the stage. For Russell, a paragon of an anti-Kantian philosopher of mathematics, the date of this change can be determined quite precisely. In a letter from 1910 to his friend Jourdain he wrote:

> Until I got hold of Peano, it had never struck me that Symbolic Logic would be of any use for the Principles of Mathematics, because I knew the Boolean stuff and found it useless. Peano's EPSILON, together with the discovery that relations could be fitted into this system, led me to adopt symbolic logic. (Cited in Proops 2006: 276)

"The Boolean stuff" Russell mentions was Boole's *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities* (1854). We may identify this "stuff" with standard syllogistic logic, which Russell rightly considered as rather useless for mathematics. At least, he was convinced that it would not do the job of deducing mathematical theorems from mathematical axioms. Thus, before he became acquainted with Peano's logic in 1900, Russell agreed with Kant that "logic" is not of much use for mathematical proofs. However, the work of Peano, Cantor, and Frege had made available a much more powerful logic that could do everything that in less fortunate times belonged in the ken of pure intuition. Russell's argument for expelling Kantian intuition from mathematics was simply that pure intuition was no longer needed:

All mathematics, we may say – and in proof of our assertion we have the actual development of the subject – is deducible from the primitive propositions of formal logic: these being admitted, no further assumptions are required.

Russell's *The Principles of Mathematics* (1903) may be considered as the source for a purely logicist conception of mathematical proofs. From Russell onwards, the mainstream philosophy of science conceptualized mathematical proofs as purely logical derivations. Of course, intuition might continue to play a restricted role, insofar as it might be considered as essential in determining which axioms are true. But intuition was expelled even from this last resort, when axioms lost their status of indubitable truths and became mere conventions or implicit definitions. Thereby, the logicist philosophy of mathematics established a neat boundary between the realm of mathematics on the one hand and the realm of empirical science on the other hand – because obviously, deductive logic was not the only method to produce knowledge in the empirical sciences.

Even though we may consider Kant and Russell antagonists with respect to the role of intuition in mathematics, in another sense they belonged to the same ilk. Both argued for a fixed and stable framework for doing mathematics: According to Kant, mathematics was based on some fixed pure intuition; Russell based it on some kind of equally fixed relational logic. Actually, matters never stabilized in the neat way Russell had hoped, since the new relational logic never achieved the fixed and unique character that Russell expected.

5.4 Cassirer's Critical Idealism

In contrast to Kant's stable intuition and Russell's stable logic, Cassirer's philosophy saw all science as an unending conceptual process of which the content and structure were not determined by armchair philosophy once and for all but unfolded in an unending process of scientific conceptualizations. Already in *Kant und die moderne Mathematik* (1907) Cassirer sketched out an attempt to overcome the logicist separation between mathematics and the sciences; he called it "critical idealism." He elaborated this Neo-Kantian approach in *Substance and Function* (1910) and later in the third volume of his opus magnum *The Philosophy of Symbolic Forms* (1929). The fundamental concept of Cassirer's unified philosophy of mathematics and science was the notion of *idealization*, or, more precisely, of *idealizing completion*. According to him, idealization plays a crucial role both in the formation of the concepts of empirical science and in the formation of mathematical concepts; idealizing completion was the common source of both mathematical and scientific concept formation.

Thus, Cassirer occupied a rather peculiar position among the attempts to philosophically understand modern mathematics and its place among the other sciences: On the one hand, he vigorously supported the then-new relational logic inaugurated by Frege, Peano, Russell, and others. In *Kant und die moderne Mathematik* he enthusiastically welcomed Russell's *The Principles of Mathematics* as an important achievement for the philosophical understanding of modern mathematics. On the Other hand, he thought that Russell and others had not fully grasped the philosophical consequences of the new logic and its rejection of intuition. For reasons of space I present only a brief and condensed description of Cassirer's main philosophical theses (for a fuller account see Mormann 2008).

According to Cassirer, the philosophy of science is to be conceived as the theory of the formation of scientific concepts. These concepts do not yield pictures of reality; rather, they provide guidelines for the conceptualization of the world. For example, the fundamental concepts of theoretical physics are blueprints for possible experiences. In the endeavor to conceptualize the world, the factual and theoretical components of scientific knowledge cannot be neatly separated. A scientific theory inextricably interweaves "real" and "nonreal" components. Not a single concept but a whole system of concepts confronts reality. The unity of a concept is not to be found in a fixed group of properties, but in a rule which lawfully represents the diversity as a sequence of elements. The meaning of a concept depends on the system of concepts in which it occurs, which is no a single fixed system but rather a continuous series of systems unfolding in the course of history. Scientific knowledge is a "fact in becoming" ("Werdefaktum"). Our experience is always conceptually structured; there is no nonconceptually structured "given." Rather, the "given" is an artifact of bad metaphysics. Scientific knowledge does not cognize objects as ready-made entities. Rather, it is organized objectually: it objectifies cases of invariant relations in the continuous stream of experience. Thus, the concepts of mathematics and the concepts of the empirical sciences are of the same kind.

I'd like to concentrate on the last claim. As a start, it may be expedient to dwell upon it in more detail, quoting more fully from *Kant und die moderne Mathematik*:

> What "critical idealism" seeks and what it must demand is a *logic of objective knowledge* (gegenständliche Erkenntnis). Only when we have understood that the *same foundational syntheses* (Grundsynthesen) on which logic and mathematics rest also govern the scientific construction of experiential knowledge, that they first make it possible for us to speak of a strict, lawful ordering among appearances and therewith of their objective meaning: only then the true justification of the principles is attained. (Cassirer 1907: 44).

I'll refer to this thesis as the "sameness thesis." It lies at the heart of Cassirer's critical idealist philosophy of science (cf. Mormann 2008). If one subscribes to the sameness thesis, the logicist separation between mathematics and science is not acceptable. According to critical idealism, the philosophy of science should concentrate *neither* on mathematics, as an ideal science, *nor* on empirical science:

> If one is allowed to express the relation between philosophy and science in a blunt and paradoxical way, one may say: The eye of philosophy must be directed neither on mathematics nor on physics; it is to be directed solely on the connection of the two realms. (Cassirer 1907: 48)

More precisely, Cassirer contended that a philosophy of science had to look for the common root from which both physics *and* mathematics sprang: namely, the method of introducing ideal elements – which established the idealizing character of any scientific knowledge. In contrast to Russell, Cassirer did not attempt to neatly separate mathematics and the empirical sciences.

Today, when dealing with idealization in science, one implicitly assumes that idealization only concerns the empirical sciences. For instance, when discussing epistemological and ontological problems of idealization, one deals with ideal gasses, frictionless planes, ideal point masses and so on. One rarely takes into account idealization *within* mathematics, which is thought to be already on the ideal side, so to speak. Thus, we assume that idealization concerns solely the empirical realm. According to Cassirer such a theory of idealization starts too late: Since idealization has a role in both, a comprehensive theory of idealization must take into account both mathematics *and* the empirical sciences.

Moreover, Cassirer insisted that one should not tackle this problem armed with "philosophical" presuppositions of the correct methods of idealization. The methods of idealization should be studied empirically, so to speak; no philosophical intuition will give us the key, which has to be discovered by studying the history of science. Hence, the philosophy of science has to pay attention to the ongoing evolution of science; it has to investigate and explicate the formation of scientific concepts in the real history of science.

In a nutshell, then, the sameness thesis contends that the "common foundational syntheses," on which both mathematical knowledge *and* physical knowledge are based, are idealizing completions carried out by the introduction of "ideal elements." For Cassirer, idealization is a common mark of all sciences *qua* sciences.

The primary role of idealization in mathematics is to underwrite the constructive procedures used in mathematical argumentation, particularly in mathematical proofs. Idealizations aim to single out appropriate domains for doing mathematics, in that they warrant that certain symbolic constructions and procedures can be carried out smoothly. In the elementary case of geometry this means, for instance, that certain points exist – more generally, that certain constructions are feasible. Less elementary, and very generally, the axiom of choice may be interpreted as an often indispensable idealizing assumption that guarantees the construction of choice functions; that is, the possibility of picking out exactly one element of each set in a given set of nonempty sets.

Idealizing completions intend to provide conceptual domains that offer comfortable and promising realms for a variety of symbolic constructions, transactions, and calculations. For instance, in an obvious sense, the domain of natural numbers is less suited to carrying out less than elementary calculations than, say, the domain of real or complex numbers. The ideal character of a domain is to be assessed not by passively staring at its perfect, pure character but rather by exploring the variety of possible symbolic actions for which it offers an expedient frame. Or, to put it the other way round, a domain lacks ideal or conceptual completeness if we meet too many obstacles, exceptions, contradictions and ad hoc assumptions in the course of our conceptual activities within it. The completeness of a conceptual domain is particularly observable in the case of geometry, as manifested in the variety of geometrical constructions we can carry out that ensure us of the existence of certain points, lines, and other geometrical entities. For Kant, the warrant of the ideal completeness of the realm of geometry was pure intuition, which ensured us that the ideal points, lines, and planes of geometry possessed the properties that rendered possible certain constructions. For Cassirer, idealization became a multifaceted, pluralist endeavor that evolved in the ongoing process of science in which the unity of pure thought was constituted. In both cases the ideal character of geometry showed itself in the richness of possible symbolic actions and transactions.

5.5 Idealizations, Constructions, and Corollaries

Cassirer's paradigmatic example of an idealizing completion in mathematics was the construction of Dedekind cuts. To understand its guiding function for the general theory of idealization, I briefly discuss an elementary geometrical problem that shows how useful Dedekind completeness is in geometrical construction. Moreover, this example clearly exhibits the resemblances between Kant's pure intuition, Cassirer's idealization and Peirce's diagrammatic thinking for mathematics and the empirical sciences.

Consider the problem of constructing in the Euclidean plane E an equilateral triangle with a given side AB of length 1. A "naïve" construction proceeds as follows: Consider the circle C_A around A with radius of length 1 and the circle C_B around B with radius 1. Then the intersection of the two circles yields the third vertex X of the equilateral triangle ABX we were looking for. From a logicist point of view, this "intuitive construction" is flawed. Assuming Euclid's original axioms, the logicist will object that we do not know that the two circles C_A and C_B actually intersect. They may somehow avoid having a common point X, since one circle may slip through the other. This is more than a remote possibility. Indeed there are unintended models of Euclidean geometry showing that this indeed might happen. Consider the rational plane $\mathbf{Q}^2$ of ordered pairs of rational numbers (p, q) ∈ **Q**. The rational plane satisfies all geometrical axioms Euclid required, but for it the intersection point X does not exist. Assume A to have the coordinates (0, 0) and B the coordinates (0, 1). Then X has the coordinates (1, √3). But √3 is irrational and therefore (1, √3) does not belong to the rational plane $\mathbf{Q}^2$.

In order to ensure the existence of the intersection point X, one has to rely on a new axiom that does not appear in Euclid's *Elements* – namely, Hilbert's axiom of continuity, which is essentially equivalent to Dedekind's axiom ensuring the existence of sufficiently many Dedekind cuts. In sum, the construction of the equilateral triangle can be carried out successfully only if we are operating in a *completed* plane, which ensures that our constructions yield what we expect from them. In other words, the completion of the plane is a necessary presupposition to enable "naïve" constructions such as that of the vertex X above.

Completions of this kind are not restricted to elementary geometry. Cassirer convincingly argued that idealizing completions are typical for all areas of mathematics (for some modern examples, see Mormann 2008). For Kant, some kind of ideal Newtonian space-time determined the variety of these constructions. In contrast, for the Neo-Kantian Cassirer these conceptual frameworks no longer depend on some fixed ahistorical "pure intuitions," but emerge in the evolution of scientific knowledge itself; thus Cassirer's philosophy of science has a sort of Hegelian flavor (cf. Mormann 2008).

Designing conceptual frameworks or settings for doing mathematics is, however, certainly not the entire story of the evolution of mathematics. The important part is putting these frameworks to work by formulating interesting problems and proving important theorems in them. Cassirer did not say much about these more concrete aspects of the idealizational practice of mathematics. Here Peirce's philosophy of mathematics comes to the rescue, in particular the insight that Peirce self-confidently characterized as his "first real discovery":

> My first real discovery about mathematical procedure was that there are two kinds of necessary reasoning, which I call the Corollarial and the Theorematic, because the corollaries affixed to the propositions of Euclid are usually arguments of one kind, which the more important theorems are of the other. The peculiarity of theorematic reasoning is that it considers something not implied at all in the conceptions so far gained, which neither the definition of the object of research nor anything yet known about could of themselves suggest, although they give room for it. Euclid for example, will add lines to his diagram which are not at all required or suggested by any previous proposition, and the conclusion that he reaches by this means says nothing about. I know that no considerable advance can be made in thought of any kind without theorematic reasoning. (Peirce 1976, vol 4: 49)

For reasons of space I can give only some brief hints why Peirce's distinction between theorematic and corollarial reasoning can be used to maintain for diagrammatic or symbolic reasoning an indispensable role in mathematics that can withstand the logicist criticism Russell put forward more than a century ago (for a detailed interpretation of Peirce's distinction see Hintikka 1980). First, according to Peirce, theorematic reasoning, which in geometry may be characterized through the introduction of new points, lines, and other geometrical objects not present in the original formulation of a problem, is not restricted to geometry. Rather, theorematic reasoning pervades all of mathematics. As Hintikka points out, what makes a deduction theorematic is not that it is based on some figures with some more or less well-defined properties but that we must take into account other objects than those needed to state the premise of the argument (cf. Hintikka 1980: 306). The new objects do not have to be visualized, but they do have to be mentioned and used in the argument. In contrast, an argument is corollarial, in Peirce's sense, if it is only necessary to imagine any case in which the premises are true in order to perceive immediately that the conclusion holds in that case (cf. Peirce 1976, vol. 4: 38). It seems appropriate, then, to contend that corollarial reasoning is based on what Russell called "the Boolean stuff"; that is, elementary propositional logic and syllogistic logic. Theorematic deduction, on the other hand, is deduction in which it is necessary to carry out some sort of imaginary experiment in order to bring about some useful effects that may allow drawing further corollarial deductions that finally lead to the desired conclusion (ibid.).

Conceived in this logical way (as Peirce and Hintikka do), the distinction between theorematic and corollarial argumentation does not fall prey to Russell's logicist criticism. Russell argued that there has been no role for intuitions and figures in serious mathematical arguments since the advent of modern relational logic, because valid geometrical reasoning could now be completely formalized. According to him, figures were thought of as indispensable simply because of the incompleteness of earlier axiomatizations. This incompleteness made it necessary for mathematicians to go beyond their own explicit assumptions and to appeal to some sort of Kantian "pure intuition." Peirce, as one of the founding fathers of modern relational logic, would be happy to subscribe to Russell's "complete formalization thesis." Nevertheless, he would insist on the necessity of distinguishing between different logical levels – to wit, corollarial and theorematic arguments. This distinction does not disappear even when geometrical arguments are "formalized." Moreover, as Hintikka has pointed out, if theorematic inference is characterized by the introduction of auxiliary individuals into the argument, one can consider the theorematic character of arguments as a gradual matter (cf. Hintikka 1980: 310).

In other words, one should not consider logic as a monolithic tool but allow for different degrees of complexity, in contrast to Russell's sweeping logicism that lumped all logic together. Following the insights of Peirce and Cassirer, we obtain three different levels of "logical" reasoning in mathematics (and the sciences) ordered by degree of complexity:

(1). Corollarial Reasoning
(2). Theorematic Reasoning
(3). Completional Reasoning

All three levels are involved in mathematical reasoning. The most elementary level is corollarial reasoning, in Peirce's sense, characterized logically by the employment of elementary propositional and syllogistic logics. On the second level, one finds the realm of theorematic reasoning, which has often been characterized as the realm of some kind of "Kantian intuition." It is important, however, to conceive this kind of intuition not as a capacity of perceiving some kind of platonic reality but as the ability to carry out diverse symbolic or ideal constructions. Logically, these constructions can be described as the introduction of new individuals and relations, leading to an increased level of quantificational complexity. Finally, on the highest level, one finds what may be called the completional or idealizing reasoning directed to the design of appropriate "settings" or frameworks in which successful diagrammatic or symbolic constructions, in Peirce's sense, can be carried out. In other words, the axiom systems are proposals or blueprints of how to produce useful constructions.

Idealizing completions offer the framework for theorematic constructions, in Peirce's sense. Frameworks are proposals whose "correctness" has to be assessed pragmatically. Hence, Cassirer may be considered as subscribing to a "theoretical pragmatism" according to which:

> … The truth of concepts rests on the capacity [to lead] to new and fruitful consequences. Its real justification is the effect, which it produces in the tendency toward progressive unification. Each hypothesis of knowledge has its justification merely with reference to this fundamental task (Cassirer 1910: 318ff.)

Cassirer's theoretical pragmatism fits well with the implicit pragmatism upheld by working mathematicians, who prefer settings in which theorems "one likes to be true" are actually true (see Mormann 2008). Similarly, just as it has accused theorematic reasoning of being based on vague intuitions of psychological interest only, a narrow logicist philosophy of mathematics often relegated the choice of "appropriate settings" to the realm of subjective whims and matters of taste. The evolution of twentieth century mathematics has shown that this assessment is hardly tenable. Constructing idealizing completions has become a routine activity, and there is now an explicit theory that deals with these problems: Category theory offers a general framework in which mathematicians can discuss problems of appropriate settings in a manner that goes beyond subjectivist presentations and preferences. In category theory, problems of idealization, completion and the development of mathematical concepts become explicit topics on the agenda of mathematics. These questions are no longer restricted to informal philosophical considerations but have obtained the status of well-defined mathematical problems.

5.6 Concluding Remarks

One of Cassirer's most fruitful philosophical insights in the philosophy of mathematics was that idealizing completions such as Dedekind's were more than just mathematically interesting technical achievements. Rather, these constructions belonged to the conceptual core of modern mathematics, being prototypes for the idealizational constructions essential for twentieth century mathematics *and* for idealizational constructions in the empirical sciences too.

Evidence for this sweeping claim comes not from a priori considerations but from the empirical observation that idealizations and completions have become routine parts of the mathematician's daily work (cf. Mormann 2008). How these completed, idealized frameworks organize the practice of mathematics may be studied by relying on the conceptual apparatus centering around the distinction between theorematic and corollarial reasoning introduced by Peirce, Hintikka, and others.

In sum, the role of idealization may be taken into account as contributing to a more realist philosophy of mathematics. This philosophical approach takes real mathematics seriously, in contrast to the traditional approaches that too closely stick to over-simplified logical models of mathematics. Cassirer took one step on this new road by emphasizing the role of idealizing completions. Peirce took another one by pointing out the importance of diagrammatic constructions. Not that the thoughts of these authors are fully in agreement with Kant's original idealist *Ansatz*. Rather, Kant, Peirce, and Cassirer all still have useful ideas to offer in the philosophical task of explicating the roles of idealization and conceptual constructions in the formation of mathematical concepts. This endeavor falls in line with the general Neo-Kantian attitude that philosophy has the task not of providing secure and unshakable foundations for mathematics, science or any other symbolic endeavor but rather of understanding how they work and elucidating their ongoing evolution.

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