Chapter 16
Evolving Geometric Proofs in the Seventeenth Century: From Icons to Symbols

Evelyne Barbin

16.1 Proof to Convince and Proof to Enlighten

The seventeenth century marked a major change in the meaning of proof[1]. It was a time of widespread dissatisfaction with the geometric proofs of the Greeks. For instance, Arnauld and Nicole listed their objections to Euclid in *The Logic or the Art of Thinking* (1662), writing that Euclid is "more concerned with certainty than with evidence, and more concerned with convincing the mind than with enlightenment."[2] This criticism was a consequence of the development of new methods by geometers in the seventeenth century, in particular the method of Descartes to translate geometry into algebra. The question is whether these new methods could be regarded as proofs. Descartes and Arnauld considered the new methods enlightening because they ahowed explicitly how the results are obtained.

When Descartes introduced the use of algebra in his book on *Geometry* in 1637, it was not his explicit intention to rewrite Euclid. However, following Descartes' method of building "compound" ideas on "simple" ideas, Arnauld considered it to be against the "natural order" to prove Proposition 2 of Book VI of Euclid (on proportionnal lines and parallelism), which is a proposition about simple lines, by using triangles, which are compound ideas. He therefore considered Euclid's *Elements* to be "confused and muddled" and set out to replace the logical order of propositions in Euclid by a new "natural order" based on the cartesian method in his *New Elements of Geometry* (Arnauld 1667). This involved deducing compound things from simple things using simple relations. Simple things are straight lines, compound

E. Barbin (✉)
Centre François Viète d'épistémologie et d'histoire des sciences et des techniques, Université de Nantes, Nantes, France
e-mail: evelyne.barbin@univ-nantes.fr

[1] Barbin, E., 'The meanings of Mathematical Proof', *In Eves' circles*, The Mathematical Association of America, n°34, 1994, p. 44.

[2] Arnauld A., Nicole, P., *La logique ou l'art de penser*, Paris, PUF, 1965, p. 325.

G. Hanna et al. (eds.), *Explanation and Proof in Mathematics: Philosophical and Educational Perpectives,* DOI 10.1007/978-1-4419-0576-5_16,

things include triangles and circles, while the simple relations include the four operations of arithmetic and the extraction of roots.

A few years later, Lamy published his *Elements of Geometry*, which also followed the method of Descartes, and subsequently gave an updated version of Thales proposition in a later edition.

Our purpose is to examine the treatment of magnitudes [lengths, areas, volumes] in these books. One of the major contribution of the geometry of Descartes is the "arithmetization of magnitudes" accomplished by introducing the notion of a unit in geometry. This makes it possible to perform calculations with magnitudes without the need to interpret them directly as numbers. The main question is how this calculation with magnitudes produces a proof that *enlightens*, rather than reproducing the kind of argument used in Euclidean geometry.

To compare Euclid with Arnauld and Lamy, we analyze five proofs of Thales' propositions on proportion. This proposition was given in two forms in Euclid and reformulated in the seventeenth century to state that, when four magnitudes are proportional, the product of the extremes is equal to the products of the means. We will examine the proof of Proposition 14 in Euclid's *Elements* Book VI, the proof of Proposition 19 in Euclid's *Elements* Book VII, the proof in Arnauld's *New elements of geometry* (1667), the proof in the second edition of Lamy's *Elements of geometry* (1695) and the proof in the fifth edition (1731).

In our analysis we use the terminology of Peirce to distinguish between icon, index, symbol, and diagram[3]. Peirce (1992–1998) gives various definitions in his writings; here we use the following: "An *icon* is a sign fit to be used as such because it possesses the quality signified."[4] For instance, Fig. 16.1 shows an icon for a parallelogram.

"An *index* is a sign which denotes a thing by focusing attention on it."[5] In Fig. 16.2, A, B, C and D are indices for the corners.

"A *symbol* is a sign that refers to the object that it denotes by virtue of a convention, usually an association of general ideas, which operates to cause the symbol to be interpreted as referring to that object."[6] For instance, in the discussion following, the letters AC are used as a symbol for the parallelogram in Fig. 16.2. "A *diagram*

Fig. 16.1 Icon for a parallelogram

[3] This classification is interesting to study mathematical writing and its understanding, but it is not often used. For instance, Fischbein (1993) uses psychological idea of mental images.

[4] Peirce, C. S., 'New Elements', 1904, *The Essential Peirce, Selected Philosophical Writings*. vol. 2, p. 307.

[5] Peirce, C. S. , 'The Regenerated Logic', 1896, *Collected Papers of Charles Sanders Peirce*, vol. 3, p. 434.

[6] Peirce, C. S., 'A syllabus of certain Topics of Logic', 1903, *The Essential Peirce, Selected Philosophical Writings*. Vol. 2, p. 292.

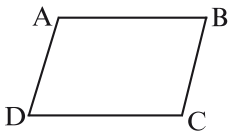


Fig. 16.2 Parallelogram with indices ABCD

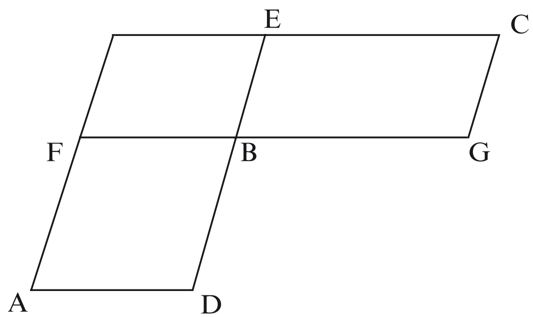


Fig. 16.3 Equiangular parallelograms

represents a definite form of relation. [...] A pure diagram is designed to represent, and to render intelligible, only the form of that relation. Consequently, diagrams are restricted to the representation of a certain class of relations; namely, those that are intelligible."[7] So, ideas of icon, index, symbol and diagram are respectively linked with those of resemblance, existence, convention, and relation.

16.2 Euclid's Book VI: Geometrical Icon and Diagrammatic Proof

Proposition 16 of Book VI states that "if four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means." It is a consequence of Proposition 14 that states that "equiangular parallelograms, in which the sides about the equal angles are reciprocally proportional, are equal." "Let GB be to BF as DB to BE; I say that the parallelogram AB is equal to the parallelogram BC (Fig. 16.3). For since, as DB is to BE, so is GB to BF, while, as DB is to BE, so is the parallelogram AB to the parallelogram FE. (VI, 1) and, as GB is to BF, so is the parallelogram BG to the parallelogram FE. (VI, 1) therefore also, as AB is to FE, so is BC to FE; (V, 11) therefore the parallelogram AB is equal to the parallelogram BC."[8] This is a consequence of geometrical propositions, mainly the

[7] Peirce, C. S., 'Prolegomena for an Apology to Pragmatism', 1906, *The New Elements of Mathematics*, vol. 4, pp. 315–316.

[8] Euclid, *Elements*, translated by Heath, vol. 2, second edition, Dover, pp. 216–217.

first proposition of Book VI. By this proposition, DB is to BE as is the parallelogram AB to the parallellogram FE.

Peirce (1978) explains that deduction consists of contructing a diagram. He writes, “Every act of deductive reasoning, even simple syllogism, implies an element of observation. Indeed, deduction consists in constructing an icon or a diagram so that the relations between parts of this icon present a complete analogy with parts of the object of reasoning, so that experimentation on this image in imagination and observation of the result occur in such a way that we can discover relations not noticed and hidden in the parts.”[9] Indeed, in Euclid’s (1956) proof, there is an analogy between the deductive reasoning “as DB is to BE, so is parallelogram AB to parallelogram FE” and the observation of the geometrical icons AB and FE. There is also an analogy between the deductive reasoning “as GB is to BF, so is parallelogram BG to parallelogram FE” and the observation of the geometrical icons BC and FE. The conclusion arises from the relations between different parts of the geometrical icon we discovered: “as DB is to BE, so is GB to BF, as AB is to FE, so is BC to FE; therefore the parallelogram AB is equal to the parallelogram BC.”

16.3 Four Proportional Numbers in Euclid’s Book VII

In Book VII, Euclid (1956) considers proportional numbers A, B, C, D, in which A is to B as C is to D. Proposition 19 states that “if four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third.” “Let A, B, C, D be four numbers (Fig. 16.4) in proportion, so that, as A is to B, so is C to D; and let A multiplied by D make E, and let B multiplied by C make F; I say that E is equal to F. For let A multiplied by C make G. Since, then A multiplied by C makes G, and multiplied by D makes E; the

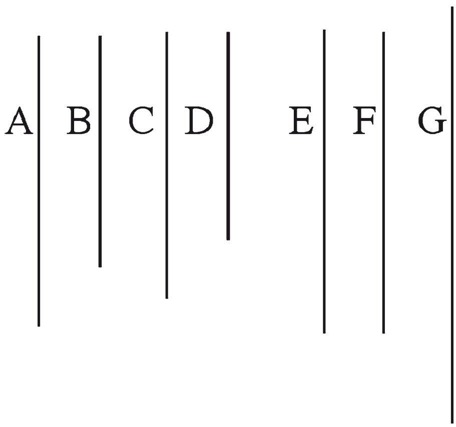


Fig. 16.4 Proportional numbers

[9]Peirce, C. S., ‘On the algebra of logic: a contribution to the philosophy of notation’, 1885, quotation in Peirce, *Écrits sur le signe*, Éditions du Seuil, Paris, p.146.

number A by multiplying the two numbers C, D make G, E. Therefore, as C is to D, so is G to E (VII, 17). But as C is to D, so is A to B; therefore also, as A is to B, so is G to E. Again, since A multiplied by C has made G, but further, B multiplied by C has also made F, the two numbers A, B by multiplying a certain number C has made G, F. Therefore, as A is to B, so is G to F (VII, 18). But further, as A is to B, so also is G to E; therefore also, as G is to E, so is G to F. Therefore, G has to each of the numbers E, F the same ratio; therefore E is equal to F (V, 9)."[10]

Here we have no icons, only symbols and indices. This proof is a consequence of arithmetical propositions, for instance, Proposition 17 of Book VII. By this proposition, if A multiplied by C gives G, A multiplied by D gives E, then as C is to D, so G is to E.

16.4 Euclid: Icons, Indices and Symbols for Straight Lines and Numbers

In Euclid's Books, signs for straight lines and for numbers are different. For a straight line, we have icon, index and symbol (Fig. 16.5).

For a number, we have a symbol and an index (Fig. 16.6).

A rectangle built from two straight lines AB and BC is given by an icon and a symbol (Fig. 16.7).

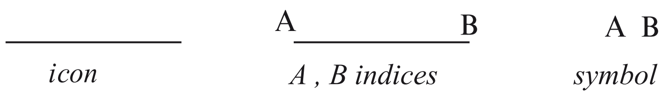


Fig. 16.5

Fig. 16.6

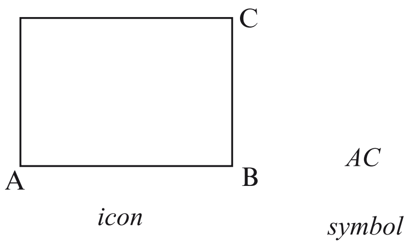


Fig. 16.7

[10] Euclid, *Elements*, translated by Heath, vol. 2, second edition, Dover, pp. 318–319.

But we have no icon and no symbol for the product of two numbers A and B (Fig. 16.8).

16.5 Arnauld: Multiplication of Magnitudes and Numbers

Peirce (1978) writes that "the reasoning of mathematicians principally rests on resemblances which are hinge-pins of the doors of their science."[11] In his *New elements of geometry*, Arnauld gives only one proposition for straight lines and numbers, because he establishes a resemblance between a rectangle and multiplication. He explains in Book I: "I suppose that multiplication can be applied to all magnitudes, and not only to numbers. Because, for example, we multiply length by width, when having a piece of ground of 4 *perches* for length and 3 for width, we say that this piece of ground has an area of 12 *perches*." [12](Fig. 16.9).

He gives as definition: "A plane magnitude is, for instance, the number 12, when we consider it is created from multiplication of 3 by 4"[13] (Fig. 16.10).

According to this resemblance, the same indices will be introduced for magnitudes and numbers. Arnauld writes, "I suppose that we are accustomed to conceive things by writing letters without seeking what they mean, because we use them only to conclude that *b* is *b,* that *c* is *c*, [...]."[14] There is one symbol for multiplication, "one

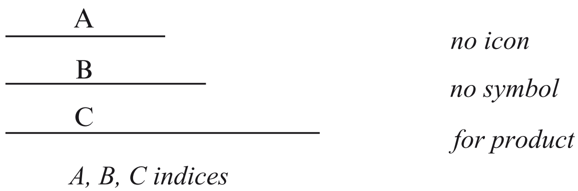


Fig. 16.8

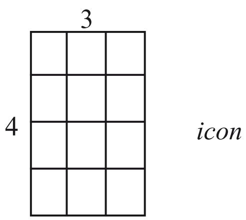


Fig. 16.9

[11]Peirce, C. S., 'The art of reasoning', 1895, quotation in Peirce, *Écrits sur le signe*, Éditions du Seuil, Paris, p. 151.

[12]Arnauld, A., *Nouveaux éléments de géométrie*, Savreux, Paris, 1667, p. 3.

[13]Arnauld, op. cit., p. 4.

[14]Arnauld, idem.

magnitude written with one character like *b* or *c* is called a linear magnitude. When we put them together as *bc,* it does not mean that they are added but that one is multiplied by the other, it is what we call the *product*."[15]

16.6 Four Proportional Magnitudes: Algebraic Icons and Algebraic Diagrams in Arnauld

Proofs of Arnauld use algebraic icons. He gives as definition, "when the ratio of an antecedent to its consequent is equal to the ratio of another antecedent to another consequent, this equality of ratios is called geometrical proportion, and the four terms proportionals. We say that as *b* is to *c,* so is *f* to *g* and we write $b \,.\, c :: f \,.\, g$."[16] So,

$$b \,.\, c :: f \,.\, g$$

is an algebraic icon. As Peirce (1978) notes: "every algebraic symbol is an icon because it shows, by means of algebraic signs, the relations of quantities in question."[17] For instance, proof of the second theorem uses icons, "when two magnitudes are multiplied by a same magnitude, they have the same ratio after being multiplied that they had before being multiplied."[18]

Arnauld examines two cases. In the first case, *b* and *c* have a common measure *x*:

$$b \,.\, c :: fb \,.\, fc$$

$$10x \,.\, 9x :: 10fx \,.\, 9fx$$

In the second case, *b* and *c* are not commensurable:

$$b \,.\, c :: fb \,.\, fc$$

$$10x \,.\, 9x + r :: 10fx \,.\, 9fx + fr$$

These icons allow manipulation of magnitudes and the conclusion comes from these manipulations. The importance of manipulation in algebra is emphasized by Peirce (1978): "For algebra, the idea of this art is that it presents a formula that we can manipulate and that by observation of the effects of this manipulation we discover properties which would be impossible to discern otherwise."[19]

---

[15] Arnauld, op. cit., p. 6.

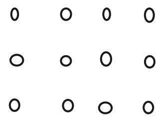
[16] Arnauld, op. cit., p. 26.

[17] Peirce, C. S., 'The short Logic', 1893, quotation in Peirce, *Écrits sur le signe*, Éditions du Seuil, Paris, p. 153.

[18] Arnauld, op. cit., p. 32.

[19] Peirce, C. S., 'On the algebra of logic: A contribution to the philosophy of notation, quotation in Peirce', *Écrits sur le signe*, Éditions du Seuil, Paris, p. 146.

Fig. 16.10 Multiplication: 3 × 4



icon

Arnauld's main proposition on proportion states that, "when four magnitudes are proportionals, the product of the extremes is equal to the products of the means."[20] It is not easy to prove, because Arnauld does not want use the complicated definition of equality of ratios that is given in Book V of Euclid's *Elements*. He introduces a way to prove it, which is a consequence of his first theorem that two magnitudes are equal when they have the same ratio to the same magnitude. Arnauld writes:

$$b\ .\ c :: f\ .\ g \quad \text{by hypothesis}$$

Therefore $$bf\ .\ bg :: f\ .\ g \text{ by 44 sup.}$$

$$bf\ .\ cf :: b\ .\ c \text{ by 44 sup.}$$

So $$bg = cf \text{ by 43 sup.}$$

These four lines constitute a diagram, because they render intelligible different relations between icons. Arnauld then writes, "because, by hypothesis, the ratio of $f\ .\ g$ (which is the same as the ratio of $b\,f\ .\ b\ g$) is equal to ratio of $b\ .\ c$ (which is the same as that of $b\,f\ .\ c\,f$) and consequently $b\ g$ and $c\,f$ have the same ratio with the same magnitude, $b\,f$. Consequently $b\ g$ is equal to $c\,f$."[21]

This proof is similar to the proof of Book VII of Euclid, but here we have algebraic icons and an algebraic diagram. His conclusion comes from an analogy between parts of reasoning and parts of a diagram. As Peirce (1976) writes, "the diagram not only represents the related correlates, but also, and much more definitely, represents the relations between them, as so many objects of the icon."[22] Here, the algebraic diagram uses a spatial disposition of signs.

Arnauld introduces algebraic diagrams for his proofs. For instance, in the first corollary he writes, "from this proposition, it is easy to judge all the changes we can do between four proportionals terms without them ceasing to be proportional."[23] Then, he gives this diagram Fig. 16.11.

Peirce (1976) notes associations between diagrams and icons. "A diagram is essentially an icon, in the form of an icon of intelligible relations. It is true that what must be is not to be found by simple inspection. But when we say that deductive reasoning is necessary, we do not mean, of course, that it is infallible. But precisely what we do mean is that the conclusion follows from the form of the relations set

[20] Arnauld, op. cit., p. 39.

[21] Arnauld, op. cit., p. 40.

[22] Peirce, C. S., 'Prolegomena for an Apology to Pragmatism', 1906, *The New Elements of Mathematics*, vol. 4, p. 316.

[23] Arnauld, op. cit., p. 42.

1.	1. Ant. b.	1. Conſ. c.	::	2. Ant. f.	2. Conſ. g.	Principale.
2.	2. Ant. f.	2. Conſ. g.	::	1. Ant. b.	1. Conſ. c.	Equivalen- te.
3.	1. Conſ. c.	1. Ant. b.	::	2. Conſ. g.	2. Ant. f.	Permuta- tion.
4.	2. Conſ. g.	2. Ant. f.	::	1. Conſ. c.	1. Ant. b.	Equivalen- te.
5.	1. Ant. b.	2. Ant. f.	::	1. Conſ. c.	2. Conſ. g.	Alterne.
6.	1. Conſ. c.	2. Conſ. g.	::	1. Ant. b.	2. Ant. f.	Equivalen- te.
7.	2. Ant. f.	1. Ant. b.	::	2. Conſ. g.	1. Conſ. c.	Permutatiõ de l'Alterne
8.	2. Conſ. g.	1. Conſ. c.	::	2. Ant. f.	1. Ant. b.	Equivalen- te.

Fig. 16.11 Algebraic diagram for Arnauld's proof

forth in the premise."[24] This is the case in Arnauld, where the conclusion follows from a form of relations. Here, this is not a logical deduction of propositions, but a relational deduction of elements. This is typical of cartesian deduction[25] Logical deduction is a way to convince by discourse, but relational deduction may enlighten using the insight of a diagram.

16.7 Multiplication of Magnitudes in Lamy

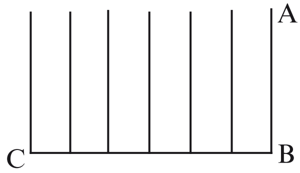
In the second edition of his *Elements of geometry* in 1695, Lamy introduces a metaphor between rectangles composed of lines and the multiplication of numbers. He writes: "To multiply *a* by *b*, is to take *a* as many times that *b* has parts; and we mark it by joining this two letters *a b*."[26] Then he adds (Fig. 16.12), "When we mark two

[24] Peirce, op. cit., vol. 4, p. 531.

[25] Barbin, E., *La révolution mathématique du XVIIe siècle*, chapter VII, Ellipses, Paris.

[26] Lamy, B., *Les éléments de géométrie ou de la mesure du corps*, seconde edition, Pralard, Paris, p.124.

Fig. 16.12 Lamy's multiplication of magnitudes



lines by two letters, for instance, $a\ b$ marks the multiplication of two lines AB and BC, we mean that these two lines make the rectangular shape ABC. It is evident that this shape is made by the motion of line AB moved from B to C, repeated or taken as many times as there are parts in BC."[27]

Following Peirce (1992–1998), here we say that we have a metaphor. Indeed, Peirce distinguishes three kinds of hypoicons (icons that can be any material image): an "image" has simple qualities, a "diagram" represents relations and a "metaphor" represents "a parallelism with something else."[28]

Lamy gives two symbols for multiplication of two straight lines. He writes, "To mark the multiplication of a line by a line, we have to use italics letters, and to mark each of these lines by only one letter, naming one b and the other c. The reason is that we usually denote lines by two capitals letters, as here, the line AB. Now this does not mean that A is multiplied by B, but only that A and B are the extremities of the line. The union of these two letters is not a sign of multiplication, to multiply the line AB by the line BC, we need another particular sign whose choice is arbitrary. We can multiply AB by BC putting a cross between them: AB × BC. This is the sign I use to express that AB is multiplied by BC."[29] The first symbol $a\ b$ is used when we have indices for lines and the second symbol AB × BC is used when we have symbols for lines.

The first rule of Lamy establishes a parallel between algebraic icons and geometrical icons. First rule: when two given magnitudes each involve the sign +, their product must have the same sign +. We have to multiply $a+b$ by $f+g$. According to what we said about multiplication of simple magnitudes, we write $a\,f$, to denote the product of a by f; so making as many products as there are letters, we will have

$$a\,f + b\,f + a\ g + b\ g,$$

for the product of $a+b$ multiplied by $f+g$. Let $a+b=$AC, $a=$AB, $b=$BC. Let also $f+g=$AG, $f=$AH, $b=$HG. I suppose that ACEG is a rectangle cut by two parallels which make parallelograms ABIH, FGIH, BCDI, DEFI which are equal to ACEG, because the totality is equal to its parts (Fig. 16.13). Then it is evident that these four

[27] idem.

[28] Peirce, C. S., 'A syllabus of certain Topics of Logic', 1902, *The Essential Peirce. Selected Philosophical Writings*, vol. 2, p. 273.

[29] Lamy, op. cit., pp. 124–125.

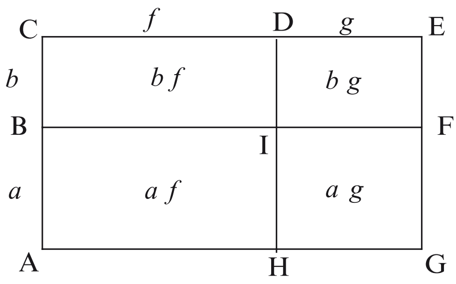


Fig. 16.13 Lamy's parallel between algebraic icons and geometrical icons

products $a\,f+b\,f+a\,g+b\,g$ are equal to the four parallelograms; they are also equal to AC × AG, which is parallelogram AC by AG, or $a+b$ by $f+g$.[30]

So, in the first place, using multiplication of algebraic icons, he obtains an algebraic icon, and in the second place, using a metaphor between rectangle and multiplication, he obtains a geometrical icon. In this way, we have two diagrammatic proofs with a parallelism between them.

16.8 Using metaphor: Book II of Euclid by Lamy

With his first rule, Lamy can obtain immediately all the propositions of Book II of Euclid by metaphor. Proposition 4 of Book II of Euclid states, "if a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by segments. For let the straight line AB be cut at random at C; I say that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB."[31] So, Euclid gives a construction using a geometrical icon and a geometrical proof of the equality by a diagram.

Lamy explains that everything that Euclid teachs, he can make "visually evident." For instance, his Proposition 4 is as follows. "One straight line being cut in two parts, the square on the whole line is equal to squares of its parts, and two times the product of its parts. Let z a straight line whose a and b are parts so $z=a+b$. Multiply $a+b$ by $a+b$, the product $a\,a+2\,a\,b+b\,b$ will be the value of the square, which contains the squares $a\,a$ and $b\,b$ of the parts a and b of z and $2\,a\,b$ which is two times the product of its parts a and b."[32] Here, he obtains an algebraic proof by a diagram:

$$(a+b)\,(a+b) = a\,a + 2\,a\,b + b\,b$$

He deduces the geometrical icon of Euclid in terms of a metaphor by a parallel between a geometrical rectangle and an arithmetical multiplication.

[30] Lamy, op. cit., pp. 128–129.

[31] Euclid, *Elements*, translation Heath, vol. 1, p. 379.

[32] Lamy, op. cit., p. 135.

16.9 Comparison Between Arnauld and Lamy: Resemblances and Metaphors

It is interesting to compare the ways that Arnauld and Lamy use to go from icons to symbols and to see how these ways differ from a diagrammatic viewpoint.

Arnauld establishes a resemblance between two icons: an icon of a rectangle for the product of straight lines and icon for the multiplication of numbers. Because of this resemblance, he uses the same signs for lines and numbers and he has only one symbol for the product of straight lines and the product of numbers.

Lamy uses a metaphor by which a rectangle is taken as the product of two straight lines. There is a parallelism between, on the one hand, a geometrical icon and a diagram, and on the other, an algebraic icon and a diagram. So he introduces two symbols for multiplication. The first is for algebraic multiplication where lines are represented by indices: $a\ b$. The index a is a sign which indicates a line. The second one is for a geometric rectangle taken as a multiplication of lines represented by symbols: AB $\times$ BC where the symbol AB is a convention to represent a line by its endpoints.

16.10 Main Proposition in Lamy's Elements of 1695

Lamy uses the symbol of division of numbers to express a ratio of magnitudes. He writes: "Definition I. Ratio is the manner that a magnitude contains or is contained in the magnitude that we compare. To express a ratio, for instance, the ratio of a to b, we put a on a line, and b under in this manner

$$\frac{a}{b}.$$

This expression is natural because, as we saw before, it is the sign of division. Then division enables us to calculate how many times a magnitude is contained in another magnitude; so the sign of this operation expresses the value of a ratio which is called a quotient in arithmetic, which denotes the way in which one magnitude is contained in another."[33] It is important to note that this symbol represents not a ratio between two objects but a value for the operation of division.

However, Lamy uses an icon for proportion: "Definition IV. Equality of ratios is called proportion. If there is the same ratio for A to B as for C to D, we say there is the same proportion between these four magnitudes, or that there are proportionals; we can denote this with four points[34]

A B : : C D"

[33] Lamy, op. cit., pp. 139–140.

[34] Lamy, op. cit., p. 140.

This icon is similar the usual notation A : B :: C : D.

The main proposition on proportion is given in Theorem VI: "When four magnitudes are proportionals, the product of extremes is equal to product of means." The proof is as follows. "Let A B :: C D, then I say that A × D = B × C and I will prove it. Multiplying A and B by D, we make two products or planes, and, by Proposition 52, A × D B × D:: A B. In the same manner, multiply C and D by B, then B × C B × D :: C D. Consequently the ratio between B × C and A × D is the same as the ratio with B × D, they are equal. (50).[35] " Lamy uses Proposition 52 which establishes that if we multiply A and B by X then AX BX :: A B. To justify this assertion, Lamy explains once more that multiplication is a kind of addition. So if A is, for instance, three times B and X equals 6, "it is evident" that AX is three times BX. We see that Lamy's proof is similar of Arnauld's proof but without an algebraic diagram.

16.11 Main Proposition in Lamy's Elements of 1731

In the fifth edition of his book, Lamy names the value of a ratio (in French) as its *exposant*[36]. "Definition I. The ratio of a line to a line [and so on] is the number of times that a line contains or is contained in the line compared; [etc]. We know how many times a line is contained in another one by division. So to express the ratio of a line to another, as the line *A* to line *B,* we divide *B* by *A*, writing one under the other,

$$\frac{B}{A} \text{ or } \frac{A}{B}.$$

This expression shows or expresses the ratio of *A* to *B,* [...], and is called the *exposant* of the ratio of *A* to *B*."[37]

He gives the usual icon to represent a proportion: "Definition IV. The equality of ratios is called proportion. If there is same ratio of *A* to *B* as of *C* to *D*, we say these four magnitudes are proportionals, which we write thus:

$$A \,.\, B :: C \,.\, D'$$

But now, as he has given a name to the value of the ratio, he can give also a diagram: "We said that the ratio of A to B is expressed by $\frac{A}{B}$ so the ratio of C to D in same manner is $\frac{C}{D}$. Consequently the proportion of these four lines can also be expressed by

[35] Lamy, op. cit., p. 150.

[36] The French word «exposer» comes from the latin «exponere», it means to show or to exhibit. So, here Lamy introduces the word «exposant» to exhibit the ratio of two lines by an expression, which «shows» it.

[37] Lamy, *Géométrie ou de la mesure de l'étendue*, Nion, Paris, 1731, p. 153.

$$\frac{A}{B} = \frac{C}{D}.$$

So, he replaces an icon, which expresses a resemblance, by a diagram, which expresses a relation.

The statement of the main proposition on proportion is: "Proposition XV. When four magnitudes are proportionals, the product or rectangle of the extremes is equal to the product of the means." Lamy gives a new proof for this proposition: "*A* . *B* :: *C*. *D*. We have to prove that A × D=B × C. Let x be the *exposant* of the two equal ratios: so $A\ x=B$ and $C\ x=D$; so I can express this proportion in the form A . $A\ x$ :: C . Cx. We have to determine that $A\ Cx=A\ Cx$; which is evident."[38] The proof arises by expressing the *exposant* of a ratio by an index x.

If we compare these two proofs of Lamy, we can observe an evolution in the metaphor between ratio and division. In the second edition, Lamy explains that a ratio is the manner that a magnitude contains or is contained in another magnitude and that division enables us to know how many times a magnitude is contained in another magnitude. So the expression

$$\frac{a}{b}$$

is natural because it is the sign of division. In the fifth edition, he uses the same sign, but he also names this expression, as *exposant*. So, in this fifth edition he can replace the usual icon for proportion

A . B : : C . D

with a diagram that express the equality of two *exposants* :

$$\frac{A}{B} = \frac{C}{D}.$$

His new proof consists in representing this common *exposant* by an index, as we do when we manipulate numbers. As we said already, "arithmetization of magnitudes" does not mean that magnitudes are numbers, it means that it is now possible to perform calculations with them.

16.12 Conclusion

Calculation with magnitudes is a consequence of the arithmetization of geometry by Descartes. This calculation is accomplished by Arnauld and Lamy in two different ways. Arnauld is close to Descartes: he uses algebraic symbols to represent straight

[38] Lamy, op. cit., p. 163.

lines, he uses resemblances between geometrical icons for rectangles and for the multiplication of numbers he uses a similar index for straight lines and numbers. The calculation of Lamy is more radical, because, by a metaphor between rectangles and multiplication of numbers, he establishes a parallelism between magnitudes and numbers.

It is clear that with their calculation on magnitudes, Arnauld and Lamy avoid many of the difficulties of Book V of Euclid. The problem remains to know whether their reasoning can be taken as legitimate proofs by them and by their readers. The use of the cartesian word "evident" by Arnauld and, above all, by Lamy furnishes an answer. Their proofs are evident and so they are sure, because as the Italian Nardi says, "All evidence is certain, but all certainty is not evident." Descartes also declares in his *Discourse on Method*, "All we conceive very clearly and very distinctly to be true." Peircian analysis with icons and symbols shows what kind of evidence is required here, because, of course, the idea of evidence is not always the same in history.[39]

References

Arnauld, A. (1667). *Nouveaux éléments de géométrie*. Paris: Savreux.

Arnauld, A., & Nicole, P. (1662, reissued 1965). *La logique ou l'art de penser*. Paris: PUF.

Barbin, E. (1994). The meanings of mathematical proof. In J. M. Anthony (Ed.) *Eves' circles*. The Mathematical Association of America, *34*, 41–52.

Barbin, E. (2000). The historicity of the notion of what is obvious in geometry. In V. Katz (Ed.) *Using history to teach mathematics*. The Mathematical Association of America, *51*, 89–98.

Barbin, E. (2006). *La révolution mathématique du XVIIe siècle*. Paris: Ellipses.

Descartes, R. (1637). *La Géométrie*. Reprinted in facsimile with English translation by D. E. Smith, & M. L. Latham (1954) as *The Geometry of Descartes*. New York: Dover.

Euclid (translated by Heath, second edition) (1956). *Elements*, 3 volumes. New York: Dover.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.

Lamy, B. (1695). *Les éléments de géométrie ou de la mesure du corps*, 2nd edn. Paris: Pralard.

Lamy, B. (1731). *Géométrie ou de la mesure de l'étendue*, 5th edn. Paris: Nion.

Peirce, C. S. (1931–1958). *Collected Papers of Charles Sanders Peirce*, 8 volumes. Cambridge: Harvard University Press.

Peirce, C. S. (1976). *The new elements of mathematics, 4 volumes*. La Hague: Mouton Publishes.

Peirce, C. S. (1992–1998). *The Essential Peirce, Selected Philosophical Writings*, 2 volumes. Bloomington and Indianapolis: Indiana University Press.

Peirce, C. S. (1978). *Écrits sur le signe*. Paris: Éditions du Seuil.

[39] Barbin, E., "The historicity of the Notion of What is Obvious in Geometry', in *Using history to teach mathematics*, Katz, V. ed., The Mathematical Association of America, Notes 51, 2000, pp. 89–98.