Chapter 13
Proof as Experiment in Wittgenstein[1]

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Ludwig Wittgenstein famously declared that we should let the proof show us what was proved (e.g., PI II: xi, and PG II: V, 24). He also suggested that one can regard proof in two ways: namely, as a picture or as an experiment. In this paper I establish that, consequently, the proof also shows us in two different ways what is proved. This difference helps explain why interpreters of Wittgenstein's concept of proof have offered bewilderingly divergent accounts. However, the proposed reconciliation of these different interpretations poses a new problem for the philosophy of mathematics: Is it indeed the case that every proof can be regarded in both ways? Though he appears to take it for granted, Wittgenstein does not make this explicit or subject it to systematic questioning.

Briefly put, the two ways of regarding proof can be contrasted thus: On the one hand, a proof can and ought to be regarded as a picture that meets the requirement of being surveyable (Mühlhölzer 2005), as exemplified by a calculation on a sheet of paper. Here, what was proved serves as an identity-criterion for the proof; indeed, only the proof as a surveyable whole can tell us what was proved. On the other hand, a proof can be regarded as an experiment, necessarily so if one wants to understand the productive and creative aspects of proof. In analogy to scientific experiments, proof as experiment refers to the experience of undergoing the proof, as exemplified by *reductio ad absurdum* or negative proof.[2] Here, the conclusion of the proof does not add a conclusion to the premises but leads to the rejection of a premise and changes the domain of the imaginable. The proof shows us what was

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[1] This paper originated in an attempt to understand Wittgenstein's argument in the *Tractatus Logico-Philosophicus* – his "proof" that "there is indeed the inexpressible" (Nordmann 2005; TLP 6.522). An intermediary sketch appeared in a German web publication (Nordmann 2006). The present version benefited from a seminar on proof at Darmstadt Technical University (with Ulrich Kohlenbach and Johannes Lenhard) and from the workshop in Essen. However, as far as philosophy of mathematics is concerned, it still stands somewhere near the bottom of a steep learning curve.

[2] For the purposes of this paper, the terms *reductio ad absurdum*, negative or indirect proof will be used interchangeably.

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proved in that it implicates us in a certain experience at the end of which we see things differently: that is, we evaluate certain commitments, mathematical procedures or hypotheses differently and therefore, in a sense, live in a different world.[3]

If proof as picture is exemplified by written calculation and proof as experiment by *reductio ad absurdum*, the new problem for philosophy of mathematics comes to this: Can every proof be regarded as a calculation and as a *reductio ad absurdum*? Might one say, for example, that the discovery, establishment, and reenactment of a proof displays the experiential structure of a *reductio*-argument and leads one to see the world differently, but that the very same proof can be a picture written down in a surveyable manner for the validation of the proper logical relations between its various lines or propositions?

Given the heterogeneity of methods of proofs and their technical expansion far beyond individual human experience and surveyability, it might be neither feasible nor necessary to show that everything accepted as proof can indeed be regarded in both ways. Even Wittgenstein's suggestion that it holds for broadly shared normative conceptions of proof turns out to be challenging and fruitful enough. Hence, I will limit myself to establishing the complementary ways of regarding proof and, in particular, to explicating the oft neglected dimension of proof as experiment.

13.1 Proof as Picture

For the account of proofs as pictures, I need to merely refer the reader to Felix Mühlhölzer's exposition (2005). Mühlhölzer asks what Wittgenstein means when he demands that proofs be surveyable. He answers, in brief: Surveyability is a necessary condition for a proof being a proof[4]; it is a shared feature of proofs and pictures that permits reproducibility and an identity-criterion for what the proof is a proof of.[5] Taking the notion of proof as a picture literally (as Wittgenstein does), obviously implies that a proof is reproducible with certainty in its entirety: Rather than repeatedly "go through" the proof to see whether one can always reproduce its result, one can

[3] This complementarity has repercussions on a metamathematical level. Mathieu Marion points out that Wittgenstein had to rely on some doctrinal position and did rely on a constructivism of sorts: "There is no free lunch in these matters, those who think so do not know what is at stake" (Marion 2004: 221). Though the notion of proof as experiment relies on a moderate constructivism (see notes 10 and 14 below), the oscillation between proof as picture and proof as experiment indicates why Wittgenstein nevertheless did not have to commit to a foundational theory of mathematics.

[4] Mühlhölzer acknowledges that the later Wittgenstein was aware, of course, that many accepted proofs are not surveyable (2005: 58 f.). How, then, could Wittgenstein argue in RFM III, 2 that the non-surveyable figure of a proof only becomes a proof when a change of notation renders it surveyable? Mühlhölzer (and Wittgenstein) suggest that to consider something an identifiable proof is just to render it in such a notation. (See below on the availability of the identity criterion only within a surveyable picture or sufficiently rich notation.)

[5] Mühlhölzer thus puts "proof as picture" in the place of "proof as grammatical or linguistic rule"; to serve as a paradigm is one feature of the proof as picture. In contrast, accounts according to which proofs establish and modify linguistic rules or paradigms do not require the notion of a picture at all (Frascolla 1994). These latter accounts, however, are haunted by rule-following arguments and their attendant difficulties.

reproduce it by copying the picture wholesale or "once and for all" (RFM III: 22). When recreating certain initial conditions, natural scientists must wait and see whether the same thing happens every time. Not so when a mathematician copies a picture or a surveyable proof and obtains the initial set-up together with the result, "the proof must be capable of being reproduced by mere copying" (RFM IV: 41). Obviously, this sets proof as a picture apart from a scientific experiment: "To repeat a proof means, not to reproduce the conditions under which a particular result was once obtained, but to repeat every step *and the result*" (RFM III: 55). Reproducibility, in other words, is tied to contemporaneous visibility (Mühlhölzer 2005: 68): All the symbols are arranged on paper or a reel of film and one can reproduce this arrangement in a purely formal fashion, without relying on causal or temporal processes.

It is less easy to grasp how surveyability offers an identity criterion for proofs. Surely, it is not enough for proofs to merely "look alike" to be considered identical, especially since new notations can introduce transformations that allow us to see a sameness of proof in a difference of signs.[6] Mühlhölzer argues *ex negativo*: In order to "establish the identity of proofs at the foundational level, the procedures of our normal counting, or similar procedures, are necessary." In other words, one has to go beyond the foundational level to the proof as a sufficiently detailed picture in order to *see* identity. For example, one cannot establish identity for all proofs that are generated in the same way so that the type of generation of proof secures identity among tokens. Since a proof would be different if it had another result, one can determine identity only at the level of the tokens, the pictures themselves (Mühlhölzer 2005: 60, 80). So, even believing that something is proven by the application of some principles or rules, one can be convinced and convince others only by the surveyable picture that is produced through the application of these rules. No matter what stands "behind" our proofs, the proof thus becomes a proof only within a notational system that can show us what was proved.[7]

To be a proof a proof needs to be convincing, of course. This account of surveyability leaves open whether and when seeing is not only necessary but also sufficient to produce conviction. For this, one has to conceive seeing as an activity of sorts, whether the act of accepting the picture as a paradigm or the act of studying relations between symbols. Either way, we see not just the symbols but also what the symbols yield; that is, how symbols lead to other combinations of symbols (Mühlhölzer 2005: 72). Of course, this way of looking at symbols is how one looks at calculations.[8]

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[6] If likeness or similarity were our guide, one might be stuck with the consequence that the color of the ink might be a criterion for the identity of a proof. Also, "looking alike" does not suffice, because it may take a kind of inferential procedure to ascertain that two sequences of strokes indicate the same number (Mühlhölzer 2005: 81); where such inferences are needed, the criterion of surveyability is not fulfilled. That's why the use of numerals can yield a proof where the use of strokes in the place of numerals produces merely a non-surveyable "figure of a proof".

[7] Here is one sense in which we can let the proof (as picture) show us what was proved. Inversely, if a proof is to induce a modification of concepts, rules, or paradigms, this is explained by the substitution of one picture for another. Especially, Wright (1980, 1991) adopts this replacement account of mathematical change.

[8] Mühlhölzer does not dwell on the fact that calculations are the standard case of proofs as surveyable and reproducible pictures but he appears to suppose as much (e.g. 2005: 72, 83 f.). Calculations play a central role especially in the interpretations of Wrigley (1993) and Frascolla (1994, 2004).

13.2 Proof as Experiment

According to Mühlhölzer, when he relates proof and picture Wittgenstein:

> alludes to a beautiful thought which he has already developed in Part I (and which he will develop further in Part VI) of the *Remarks*: that the real, temporal process of proving a mathematical theorem may very well be comparable to an experiment, but that the proof itself rather resembles the *picture* of such an experiment, in which the experiment is frozen, as it were, into something nontemporal. (Mühlhölzer 2005: 68)

Here, Mühlhölzer notes a complementarity overlooked by most readers of Wittgenstein's *Remarks*, many of whom take the consideration of experiments merely as a way to dissociate mathematics from empiricism and natural science: It is thought to be characteristic of mathematical proof that it is not an experiment (Frascolla 1994, Ramharter and Weiberg 2006; Weiberg 2008). Even Mühlhölzer describes that complementarity in rather weak terms. Although his paper explores Wittgenstein's suggestion that "the proof *is* a picture," the quoted passage speaks of proof *resembling* a picture and being *comparable* to an experiment. By stressing that a proof *is* a picture and also that it *is* an experiment, I would like not only to highlight that these are complementary aspects of proof for Wittgenstein but also to show that the complementarity is necessary.[9] This necessity is not due to foundational considerations, a theory of proof or the like, but arises simply from the fact that mathematicians move about in notational systems.[10] That they creatively produce a proof (experiment) and render it as a configuration of symbols (picture). To their readers, the proof appears as something to be gone through and re-enacted (experiment) or as something to be surveyed and seen (picture). Any movement in a notational system is an experience unfolding in time (experiment) and yields at any given moment a formal structure in space (picture). By enacting and reenacting proofs as experiments, mathematicians effect the modification of concepts; by surveying and

[9] Indeed, it would appear that Mühlhölzer requires a stronger notion of complementarity in order to arrive at a full account as sketched in note 4 above: Proofs as experiments are not surveyable and as such only figures or schemes of proof; they become surveyable and thus properly "proofs" only as they are rendered in an appropriate notation.

[10] This emphasis on notational systems places Wittgenstein in the proximity of formalism. To the extent, however, that the movements within a notational system go beyond the application of formal transformation rules, Wittgenstein also moves beyond formalism (Mühlhölzer 2008; Floyd 2008). Of the various extant reconstructions of Wittgenstein's philosophy of mathematics, the one proposed here is closest in letter and spirit to one of the earliest ones (Klenk 1976). Later interpretations tend to hold Wittgenstein answerable to question of realism vs. anti-realism, Platonism vs. formalism, constructivism, or empiricism, and to Kripke's discussion of rule-following. In contrast, see Klenk (1976: 124–126): "Wittgenstein is neither a finitist nor a radical conventionalist; he is willing to admit the full spectrum of mathematical techniques and results, and he has been able to do so without giving up the fundamental properties of mathematical propositions: their objectivity and necessity. […] Since Wittgenstein rejects the idea that mathematical statements refer to mathematical objects, for him these statements carry no ontological commitment at all, and he is thus able to enjoy the best of both worlds: the full range of classical mathematics, but without the ontological burden that usually goes with it."

beholding the proof as picture, they ascertain its certain and complete reproducibility and identity. This duality underwrites the oft-cited passage in which Wittgenstein compares the mathematician to an inventive garden architect who modifies the landscape to create the formal paths and tracks that the viewer then simply follows (RFM I: 167).[11]

In this duality of aspects, proof as picture and proof as experiment are strictly separate: "“The proof must be surveyable” really means nothing but: The proof is no experiment" (RFM III: 39). When a proof is surveyable, we see the entire garden path from beginning to end; whereas, in an experiment and in going through a proof, we may question whether the path will reliably take us from beginning to end (RFM I: App. 2, 2). "And thus I might say: The proof doesn't serve me as experiment but as the picture of an experiment" (RFM I: 36). Here again, Wittgenstein asserts surveyability as a necessary condition for proof. He makes clear, however, that this is not the whole story. If proof is a picture of an experiment, then proof is first of all an experiment that is distinguished from other experiments by becoming transformed into a picture. This transformation is possible because the proof is a movement among signs that culminates in a pictorial configuration of these signs.[12]

But what kind of movement among signs is a proof, and how does the recognition of this experimental movement account for the creativity and productivity of proof or for the way in which it effects a modification of concepts? Wittgenstein elucidates this primarily in reference to *reductio* arguments or negative proof. To the complementarity of proof as picture and proof as experiment therefore corresponds the complementarity of calculation and negative proof. Calculation exemplifies the proof as a picture or paradigm that works to establish identity, definition, and substitution. The *reductio* argument or negative proof exemplifies the proof as an experiment that probes commitments and establishes the connection between inference and decision. Yet, it is misleading to say that we look at *reductio* arguments differently than we look at calculations and their manner of yielding results. More appropriately, we should say that we don't *look* at them as *reductio* arguments or negative proofs at all; instead, we should say that we rehearse, enact, or go through *reductio* arguments: We undergo a negative proof just as we undergo an experience.

[11] Wittgenstein may be referring to just this duality when he speaks of experiment (invention, creation, experience) and calculation (survey of the tracks that have been laid) as the poles between which human activities move (RFM VII: 30). Klenk also speaks of "two aspects of proof": "the fact that we are brought to a new way of looking at things [proof as experiment], and given a new prescription of our language [proof as picture]" (1976: 82).

[12] Indeed, what distinguishes mathematics from empirical science is just this: In mathematics, there is no shift of medium as one moves from the experiment to its representation; the experiment takes place in the very same notational system which pictures it (RFM I: 36, cf. I: 165). This would indicate why no inductive process is required to judge the reproduction of proofs as pictures (compare Wright 1980: 466).

In order to substantiate all this, I present a somewhat more detailed reconstruction of Wittgenstein's reflections on *reductio* arguments and negative proofs.[13] Already in the *Tractatus,* Wittgenstein juxtaposed calculation and experiment:

> 6.233 To the question whether we need intuition [*Anschauung,* perception] for the solution of mathematical problems it must be answered that language itself here provides the necessary intuition [*Anschauung*, perspicuity].
>
> 6.2331 The process of *calculation* brings about just this *Anschauung*.
> Calculation is not an experiment.

If language itself provides the necessary perspicuity, a calculation is no experiment, because it does nothing to change the language or how things are seen. Instead, a calculation serves only to articulate and clarify relations within the notational system. After thus assimilating mathematics to logic in the *Tractatus*, Wittgenstein came to reconsider his early work and to introduce the notion of language games in the context of a broadened conception of mathematical practice (Epple 1994). Some language games are conservative and serve primarily to guarantee a result, others are experimental and might introduce change.

> "Proof must be surveyable" really serves to direct our attention at the difference between the notions: "to repeat a proof," "to repeat an experiment." To repeat a proof means, not to reproduce the conditions under which a particular result was once obtained, but to repeat every step *and the result*. (RFM III: 55)

The distinction applies to the difference between a calculation and a *reductio ad absurdum*. As we have seen above, the calculation assures reproducibility and identity of the proof by reproducing the result along with the "*compulsion* to preserve it" (RFM III: 55), a compulsion exerted by the proof in that it serves as a paradigm within the notational system. In contrast, the *reductio ad absurdum* provides the conditions under which the result could be obtained again and again but each time without necessity, since the *reductio* proves only that the conjunction of its various, more or less hypothetical premises cannot be maintained insofar as it leads into contradiction. If the *reductio* argument results in the denial of just one element of the conjunct, and if the selection of this element involves a decision, the repetition of the reductio argument does not necessarily include the repetition of the result.[14]

[13] The following reconstruction is adapted from Nordmann (2006).

[14] In his discussion of *reductio* arguments Wittgenstein nowhere distinguished between two cases that are often held apart. First, there are *reductio* arguments that feature among their premises only one explicitly hypothetical assumption. Since all the other premises are deeply entrenched axioms and theorems, the contradiction is here taken to force the denial of the hypothesis. In the second kind of *reductio* argument, the other premises or background assumptions are only taken to be relatively more secure than the hypothesis. In this case, the contradiction calls into question only the conjunction of all those assumptions and hypotheses, leaving at least a residue of choice in the determination of the conclusion. Wittgenstein did not recognize this distinction and thereby indicated that the language which provides perspicuity is always assumed and always subject to change, including even its deeply entrenched axioms and theorems (see VC: 181). Wittgenstein was not thereby arguing the finitist claim that we are constantly deciding whether to change the language or not, let alone that we ought to consider it as merely contingent; on the contrary, it is part of our natural history that we implicitly commit ourselves again and again to a received use of language (see RFM I: 118, IV: 11, or I: 63).

> If one considers a proof as an experiment, the result of the experiment is at any rate not what one calls the result of a proof. The result of calculation is the sentence with which it concludes; the result of the experiment is: that I was led by these rules from these sentences to that one. (RFM I: 162)

Here, proof and experiment are not opposed to each other. Instead, Wittgenstein invites us to consider the proof as a proof (a surveyable picture) or to consider the proof as an experiment (pictured by the proof as proof). Since these are two ways of considering proof rather than two types of proof, they cannot be distinguished as necessary on the one hand versus empirical on the other. The experiments of the mathematician and of the empirical scientist have in common that both researchers don't know what the result will be, but they differ in that the mathematician's experiment immediately yields a surveyable picture of itself – so that showing something and showing its paradigmatic necessity can collapse into a single step, which the empirical scientist's does not.[15]

> *Wittgenstein*: [...] Suppose I say, "I have found that the prime numbers often come in pairs." Is this the result of an experiment? – Here it looks just like an experiment. I didn't know what the result would be, and I found out by going through some divisions.
>
> *Wisdom*: In this case you have shown it not by experiment but by proof.
>
> *Wittgenstein*: Yes – but why do we say this here? – There is no difference between showing that they come in pairs and showing that they *must* come in pairs, just as there is no difference between showing that 17 is a prime number and showing that it *must* be a prime. [...] It has often been said – and there is something true in it and something absurd – that a mathematician sometimes makes what one might call experiments, and then proves what he has found out by experiment. But is this true? Is not the figure itself – the curve or the division – a proof? (LFM: 121)

This rather open-ended exchange hints at the "beautiful thought" mentioned by Mühlhölzer (2005: 68): "A proof, one could say, must originally have been a kind of experiment – but is then simply taken as a picture" (RFM III: 23). The picture of the proof would thus embody the compulsion by which the result was obtained and must be obtained again and again. When written down, a *reductio ad absurdum* also becomes such a picture and becomes a commitment to a certain use of signs where the axioms and theorems are clearly set off against the mere hypothesis denied by the conclusion. The pictured experiment thus displaces the experience of the experiment; that is, "that I was led by these rules from these sentences to that one" and that I thus came to reject the hypothesis.

> *Wittgenstein:* […] What is indirect proof? An action performed with signs. But that is not quite all. There is a further rule telling me what to do when an indirect proof has been

[15] See note 12 above and compare Bloor (1997: 41 ff.) If I understand correctly, Bloor offers the following account: Wittgenstein's "assimilation of calculation to experiment" cannot be understood in terms of empiricism versus Platonism but it can be understood if one looks at the establishment of social institutions, such as the "institution of measuring", where facts become standards and standards are facts under self-referential conditions. Mathematicians act within a system of signs that represents their actions; therefore, if they use something as the measure of something, it *is* the measure of that thing (cf. RFM I: 161-165, III: 67-77).

given. (This rule may read, for example: If an indirect proof has been given, the assumptions from which the proof starts are to be deleted.) *Here nothing is self-evident. Everything must be said explicitly.* […]

Waismann: [...] You could retain the refuted proposition by changing the stipulation regarding the application of indirect proof, and then our proposition would no longer be refuted.

Wittgenstein: Of course we could do that. We should then have destroyed the character of the indirect proof and only its schematic representation would remain. (VC: 180 f.)

By going behind the mere schematic representation and appreciating the character of proof as an action performed with signs, Wittgenstein considers it as a structured experience undergone by the person who invents or re-enacts a proof. A somewhat more detailed example helps to introduce this notion:

Suppose that we have a method of constructing polygons [...]. We are only allowed a ruler and a pair of compasses whose radius is fixed. We draw two diameters at right angles to one another in a circle; this gives us an inscribed square. We then draw arcs from the intersection points of the drawn diameters. Whether we call this bisecting or not doesn't matter. This is what we do. Thus we get the octagon, for instance. Similarly we could get a polygon with 16 sides, and so on.

Now someone is asked to produce the 100-gon this way. At first he goes on trying and trying, keeps on bisecting smaller and smaller angles and doesn't get any satisfactory result. Then in the end we prove to him that the 100-gon cannot be constructed in this way.

It seems as if we first of all made an experiment which showed that Smith, Jones, etc. could not construct a 100-gon in that way, and then a mathematician shows that it can't be done. We get apparently an experimental result, and then prove that it could not have been otherwise at all.

But there is something queer about this: For how could the man try to do what could not be done? (LFM: 86 f.)

Like all *reductio*-arguments and, indeed, like all mathematical proofs, this proof is an impossibility proof: In light of background assumptions, commitments, or rules it proves impossible to hold on to an intention, to claim a possibility, or to assert a proposition. In the ideal case, this impossibility manifests itself in the form of a contradiction, but it can also manifest itself in the form of defeat: "It can't be done."[16] Either way, such impossibility proofs raise the fundamental question whether one can even try to do what turns out to be impossible. Wittgenstein never questions that it is impossible even to conceive a contradiction (see already TLP 3.03 and 5.61). How then can it be so easy to posit, think through, even insist for a while on a set of premises that turns out to be contradictory? Wittgenstein expresses this concern in the following passage:

The difficulty which one senses in regard to *reductio ad absurdum* in mathematics is this: What goes on in this proof? Something mathematically absurd, and hence unmathematical?

[16] The difference between these cases can be as inconsequential as that between showing that "17" is a prime and that it must be a prime.

> How can one – one would like to ask – even hypothesize what is mathematically absurd? That I can assume what is physically false and lead it to absurdity creates no difficulties for me. But how to think what is so-to-speak unthinkable?! (RFM V: 28)[17]

The question admits of only one answer: No one is thinking the unthinkable. In the case at hand, we might just be misunderstanding or misapprehending the conjunction of premises because we cannot fully survey the situation that will lead us from the beginning of our experiment to a contradiction. In other words, we are not yet seeing the proof as a proof. However, the term "misunderstanding" might give rise to a misunderstanding of its own, because it suggests that the mistake or misapprehension is avoidable. We should more appropriately say that we do not and cannot understand the conjunction of premises until we have undergone the experience and conducted the proof as experiment. What makes the proof a proof is precisely that it leads us to see the impossibility even of trying what we set out to do only a little while ago: The proof effects a revision of the domain of the imaginable.

> The question arises: Can't we be mistaken in thinking that we understand a question?
>
> For many mathematical proofs do lead us to say that we *cannot* imagine something which we believed we could imagine. (E.g., the construction of the heptagon.) They lead us to revise what counts as the domain of the imaginable. (PI: 517)

What we were once able to imagine (the construction of a 100-gon) has now moved into the domain of the unimaginable. Indirect proofs or *reductio* arguments bring about just such revisions. This is neither the discovery of something new nor the mere exhibition of a meaning that is implicit in the conjunction of premises. Instead, it is a critical intervention or an action that alters the language and thus the form of intuition that provides perspicuity.[18]

Using as his example the impossibility of trisecting an angle by geometrical means, Wittgenstein details how this critical intervention unfolds: where our original confidence originates, when we encounter defeat and finally how we arrive at the insight that we wanted something unimaginable. Here, the revision of the domain of the imaginable consists in the experiment changing "our idea of trisection":

> Again, the importance of the proof that trisection is impossible is that it changes our idea of trisection. – The idea of trisection of an angle comes in this way: that we can bisect an angle, divide into four equal parts, and so on. And this leads to the problem of trisecting an angle. You are led on here by *sentences*. You have the sentence "I bisect this angle" and

[17] Michael Nedo shows how this passage originally appeared in Wittgenstein's manuscript 126 in the context of a sustained discussion of G.H. Hardy's *Course of Pure Mathematics*. Hardy would open an indirect proof with "suppose, if possible, that …" (Nedo 2008: 86-97; Hardy 1941: 6).

[18] Proof as picture displays the relation between sentences, showing how certain sentences are transformed to yield others (conclusions). Proof as experiment does not add or subtract sentences but concludes with a new way of looking at sentences. As we will see, this new way of looking at sentences alters the language by probing certain linguistic commitments and thus by playing off one part of language against another, without presupposing a strict separation between the prose that surrounds a formal mathematical core and the proofs themselves. (On prose vs. proof, see e.g., RFM IV: 27; cf. Floyd 2008 vs. Lampert 2008.)

> you form a similar expression: "trisecting". And so you ask, "What about the sentence, 'I trisect this angle'?" […] If we had learned from the beginning the series of constructions of *n*-gons, then nobody would ever have asked whether the heptagon is constructible. It's none of these, that's all.
>
> […] The problem arose because our idea at first was a different idea of the construction of *n*-gons, and then was *changed* by the proof. (LFM: 88 f.)

One quickly recognizes in this account a central theme of Wittgenstein's critique of language in the *Tractatus* as well as in the *Philosophical Investigations*. Led on by language, we imagine that every noun is a name, that every grammatical sentence pictures a fact. This is how we move so effortlessly from "This door is blue" to "This person is good" or from expressions of fact to expressions of value. However, had we learned from the beginning the proper sectioning of angles, the series of constructions of *n*-gons, or the way in which truth-conditions make for meaningful sentences, nobody would ever have asked whether trisection is possible or whether an absolute value is expressible in our language. If one wants to know how this shift from what can be imagined to what is unimaginable came about, one needs to understand what was proven. Also, inversely, if one wants to know what was proven, one must understand the revision in the domain of the imaginable that was effected by the proof. Thus, "let the *proof* teach you *what* was being proved" (PI II: xi).[19]

In an indirect or negative proof, one begins with something conceivable and ties it to a specific employment of signs. As we attempt to trisect an angle or to construct a 100-gon, we commit ourselves to certain rules of construction and then discover that they leave out the case of trisection or of the 100-gon; in other words, the rules simply don't provide for those[20]:

> The proof might be this: we go on constructing polygons and being very careful to observe certain rules. We should then find that the 100-gon is left out. If we want to construct the *n*-gon in that way, *n* has to be a power of 2. The last power of 2 before 100 is 64, after that is 128, and so 100 is left out. This would have the result of dissuading intelligent people from trying this game. (LFM: 87)

[19] This temporal and experiential dimension (only the proof can tell you what was proven) is not sufficiently appreciated by Jaako Hintikka's incisive critique of Wittgenstein. Hintikka recognizes that Wittgenstein rejects "the idea that statements of the *possibility* of geometrical constructions [the domain of the imaginable] belong to the same language game as the constructions themselves" (Hintikka 1993: 37). But why should they (as Hintikka assumes they should) belong to the same language game in the first place? The tools and rules that constitute the game are not surveyable while certain pictures constructible within the game are. And thus, I can be mistaken in what I understand and do not understand, what I can do (what is possible) and what I can't do (what is impossible) in my language.

[20] This is why Timm Lampert insists that, for Wittgenstein, proof is not a matter of logical deduction but of defining operations: Do the rules of construction provide or leave out a certain case? Contrary to Lampert, this does not imply that "mathematics completely dispenses with logic" and that Wittgenstein "rejects the use of certain deduction rules such as *reductio ad absurdum*" (Lampert 2008: 63). He only rejects certain construals of deduction rules.

If people are very careful to observe certain rules and discover that these rules do not allow them to pursue a plan or maintain a hypothesis, they will abandon their plan and deny the hypothesis – as long as they want to stick to their rules.[21] Indeed, by abandoning the plan and denying the hypothesis, they not only revise their conception of what they can hope for or what they can maintain within the game they are playing, they also reaffirm their commitment to the rules of the game itself: "Every proof is as it were a commitment to a specific use of signs." (RFM III: 41).

> The indirect proof says, however: "If you want it like *that*, you may not assume *this*: for *with this* is compatible only the opposite of that which you want to hold on to." (RFM V: 28)

The clause "if you want it like *that*" points to the conditional structure of the indirect proof, and thus to another aspect of the proof as experiment. To enter into the experiment is to be prepared to reevaluate its basic assumptions. An outward sign of this preparedness is the hypothetical beginning of the indirect proof. It places the experiment in the subjunctive mood: "If I were to assume this, what would follow?"[22] The experiment thus involves a sense of possibility that is ready to change or act. Wittgenstein describes this state of readiness in the *Philosophical Investigations*:

> The if-feeling is not a feeling which accompanies the word "if."
>
> The if-feeling would have to be compared with the special 'feeling' which a musical phrase gives us. (One sometimes describes such a feeling by saying: "Here it is, as if a conclusion were being drawn" or "I should like to say, '*hence*....' ", or "Here I should always like to make a gesture –" and then one makes it.) (PI II: vi)

Accordingly, *reductio ad absurdum* corresponds to a structured experience that makes sense. It allows us to shift from an old to a new state, from the wrong way of seeing the world to the right way.[23] But a way of seeing the world stands only at

[21] Similarly, the author and readers of the *Tractatus* are committed to certain rules of using sentences to picture facts. Probing these rules, one discovers that they do not provide for the expression of absolute value: This case is omitted by the notational system that is designed to describe the world truthfully (Nordmann 2005). This discovery needs to be actively made, e.g., by running up against a contradiction in TLP 6.41. In recent years, Cora Diamond and James Conant advanced a similar argument: "Thus the elucidatory strategy of the *Tractatus* depends on the reader's provisionally taking himself to be participating in the traditional philosophical activity of establishing theses through a procedure of reasoned argument; but it only succeeds if the reader fully comes to understand what the work means to say about itself when it says that philosophy, as this works seeks to practice it, results not in doctrine, but in elucidation, not in [philosophical sentences] but in [the becoming clear of sentences]. And the attainment of this recognition depends upon the reader's actually undergoing a certain *experience* – the attainment of which is identified in 6.54 as the sign that the reader has understood the author of the work: the reader's experience of having his illusion of sense (in the 'premises' and 'conclusions' of the "argument") dissipate through its becoming clear to him that (what he took to be) the [philosophical sentences] of the work are [nonsense]" (Conant 2000: 196 f.).

[22] See note 14 above regarding the conditional structure also of "direct" proof.

[23] Compare this language to the last remarks of the *Tractatus* (see Nordmann 2005).

the very beginning and end of the experiment. The experiment itself is characterized by Wittgenstein in terms of practical commitment, experiment, movement and change. To the question "What is indirect proof?" he answered, "An action performed with signs." (WVC: 180) The action of the *reductio* argument consists of its showing us something, and what it shows makes sense in the context of action but is not expressed by a sentence as a picture with propositional content and truth-conditions.

> There is a particular mathematical method, the method of *reductio ad absurdum*, which we might call "avoid the contradiction." In this method one shows a contradiction and then shows the way from it. But this doesn't mean that a contradiction is a sort of devil. (LFM: 209)

Quite the contrary, instead of being a sort of devil, the contradiction is an integral turning-point of a structured experience. The *reductio* argument shows the way from the contradiction to the conclusion, and the conclusion exhibits or reveals, in turn, the specific commitment that directs the avoidance of the contradiction.[24] So, the contradiction turns out to be creative: It is the vehicle by which our commitments disclose a new perspective from which to see the world aright.[25]

13.3 Conclusion

In the *Tractatus*, Wittgenstein distinguished between calculation and experiment (6.233 and 6.2331). In his later work, the distinction is that between proof considered as picture and proof considered as an experiment – calculation is an exemplary picture, the *reductio* argument an exemplary experiment. There is something appealing, of course, to the consideration of these two complementary aspects of proof. Pictures seem to be static, experiments dynamic; pictures stand for a synchronic and experiments for a diachronic dimension; pictures are objects in the context of justification and experiments belong to the context of discovery. It is important, however, to resist this easy and appealing view of the complementarity between pictures and experiments.

First of all, pictures and experiments are not aspects of proof. When we *see* a proof, we see a picture. We do not see the proof at all when we are engaged in an experiment. Then, we are trying to do something that, perhaps, cannot be done, and we learn from our failure when we run into a contradiction and use it as a prompt for a creative decision that changes the domain of the imaginable. Only the proof as picture is a proof to behold, but this is not to say that it is static and unchangeable;

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[24] Wittgenstein identifies this as the reason it makes sense to have multiple proofs of the same proposition. Further proofs do not render the proposition more secure. Each proof highlights some antecedent commitment or some mathematical context that would lead us into contradiction if we were to deny the conclusion (RFM VII: 10; also manuscript 126: 124 f. cited by Nedo 2008: 90).

[25] Louis Caruana identified three instrumental uses of contradictions (Caruana 2004: 232). This one is not among them.

the picture is an object of investigation par excellence, one that allows us to make discoveries about the relation of its elements. We might say, then, that the opposition between picture and experiment expresses well what is only clumsily hinted at by opposing static versus dynamic, synchronic versus diachronic, justificatory versus exploratory aspects of proof.

Indeed, the conception of proof as experiment is most informative to those who are already thinking about invention and change in mathematics but see this change only as the displacement of one picture by another and thereby neglect the experiential structure of change.[26] Accordingly, Wittgenstein's dictum that we should look at the proof in order to know what was proved (PI II: xi; PG II, V: 24) takes on a different meaning for the proof as picture and for the proof as experiment. In a proof considered as a surveyable picture, every step and the result tell us what was proved. Wittgenstein's injunction refers to identity-conditions: A proof with a different result is a different proof, whereas a scientific experiment with a different outcome can still be the same experiment. In a proof considered as an experiment, the experience of failure tells us what was proved, namely that we cannot have this if we want to hold on to that. The proof thus renders salient some piece of our language and some of our commitments, allowing us to settle into a domain of the imaginable. Here, our conclusion dissolves an irritation of doubt by transforming the situation so that our initial problem goes away. This experiential conception of proof moves Wittgenstein into the proximity of pragmatist epistemologies like those of Peirce and Dewey, and yet further from Frege's and Russell's conceptions of language, logic and thought.

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[26] Crispin Wright's (1991) argument against Kripke advances just such a narrow conception, which does not acknowledge the temporary suspension of rules as the domain of the imaginable is changed: "changing and extending [mathematical discourse] […] is a notion of which we can make sense only under the aegis of a distinction between practices which conform to the rules as they were before, and practices which reflect a modification in those rules generated by some pure mathematical development. Unless, then, there is such a thing as practice which is in line with a rule, contrasting with practice which is not, there is simply no chance of a competitive construal of Wittgenstein's positive proposals" (Wright 1991: 88; cf. Wright 1980). Cesare Cozzo presses the issue further: "Can we consistently endorse both the plasticity of meaning and the objectivity of proof?" (Cozzo 2004: 193). This is where one needs to insist that, to become a proof to behold, a proof has to become a surveyable picture without plasticity of meaning.

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