Chapter 4
Modeling a Crucial Aspect of Students' Mathematical Modeling

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Abstract Through the use of three mathematical examples, our study explores the origins of students abilities to model (a) the process in which students acquire the ability to both model and learn transferable modeling abilities across modeling activities, and (b) the way in which the modeling cycle should be characterized. We conclude by suggesting that philosophical issues are present in understanding how the modeling ability emerges in students who have never modelled. This is linked to efforts to find activities and methods that will enable better modeling capabilities to be learned.

4.1 Introduction

In theoretical or empirical terms, attempts to model what actually happens (or what ought to happen) when students engage in mathematical modeling, are typically based on one or more of the following four components: (a) the phases of the modeling process (Blum and Niss, 1991), (b) the nature of the mathematical modeling competency (Blomhøj and Jensen, 2003) (c) other mathematical competencies (Niss, 2003), and (d) the interplay among these components. Often this is supported by some version of a diagrammatic representation of the modeling process, chosen from a multitude of such representations proposed in the literature. Our considerations in what follows refer to one such diagram. As most readers of the present volume are likely to be familiar with the interpretation of modeling diagrams, which are rather self-explanatory, this is not the place to invest in further explanations of it (Fig. 4.1).

Now, let us take a closer look at one crucial – if not simply *the key* – aspect of the modeling process as depicted in the diagram: the mathematization of the situation at issue. Two focal questions are of interest to this chapter: *How is mathematization actually generated and brought about?* And: *What are the main challenges and*

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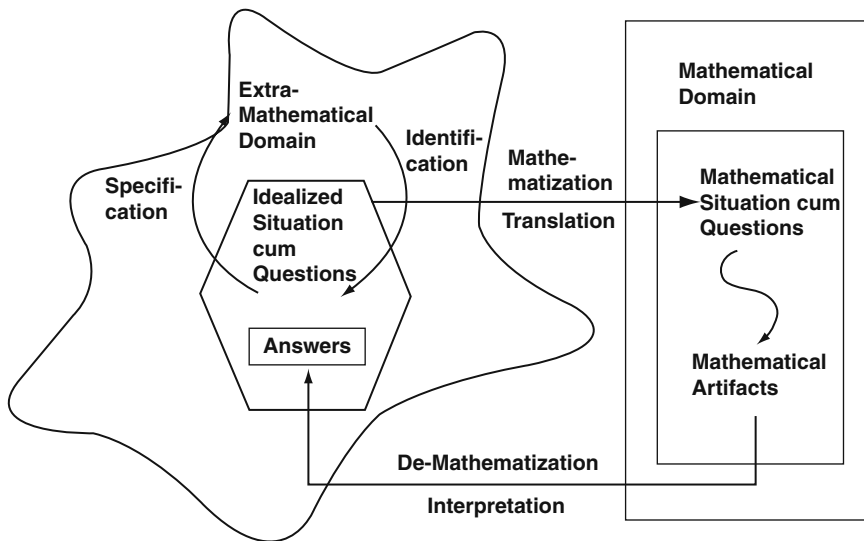IMFUFA, Department of Science, Roskilde University, Roskilde, Denmark
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Fig. 4.1 Modeling processes

obstacles that students encounter in the mathematization part of the modeling process? More specifically, we wish to model students' work on mathematization. For reasons that will become apparent later, our deliberations will primarily be of a theoretical nature. To assist and illuminate our analysis we shall begin by considering three concrete examples.

4.2 Three Examples

All of the examples presented below are cast as descriptions of extra-mathematical situations that lend themselves to posing (and answering) a given question.

Example 1: A conic glass

   Situation: Glasses in which the cup is of a conic shape are quite commonplace. At a party, people discuss how much the glass should be filled for the volume of liquid to be half of that of a full glass. One of the participants, A, suggests that the liquid should be poured to 2/3 of the full height of the glass. Another participant, B, objects by saying that while this might be true in some cases, it cannot be true in general, as the answer must depend on the opening angle of the glass. *Question*: How would you settle this discussion?

   This problem is somewhat stylised, but not at all unrealistic, at least not in bars selling drinks by volume. In order to help us analyse what is in play in the modeling process, we shall look at one way to deal with the problem.

The problem situation is already of a physico-geometrical nature, since we can think of the cup as a physical realisation (undoubtedly with several imperfections) of a geometric object, a cone. So, the first part of the mathematization is almost automatic. We simply represent the inner part of the cup by a cone whose base is a circle – the inner opening of the cup – and whose height is the distance from the centre of this circle to the bottom, which we model as a single point. The straight line through the bottom point and the centre of the base circle is what we call the axis of the cone. In formal terms the opening angle of the cone is the angle between two diametrically opposite generators of the cone, i.e., the (smallest) angle between the two half-lines determined by the intersection of a plane containing the axis and the surface of the cylinder. So far, mathematization has been straightforward.

Ideally, if we neglect surface tension etc., for a full glass the liquid forms exactly this cone. For a glass half full, the liquid forms a smaller cone with the same bottom point and another inner circle (parallel to the opening circle), the radius of which is yet unknown, and whose centre has a distance to the bottom point which is also not known yet. The opening angle of the "half-full" glass is the same as for the full one (Fig. 4.2).
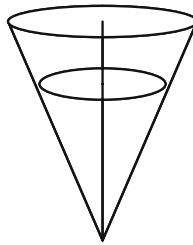

Fig. 4.2 A cone "cup"

Against this background we are now able to continue the mathematization process as follows: We represent the situation by a cut in the conic surface by a plane containing the axis, while the two liquid surfaces are represented by the base line in the resulting plane isosceles triangle and a parallel transversal, respectively. Let us illustrate this with a figure which labels the relevant pieces in the triangle: The radius of the base circle in the "big" cone, R, is half the length of the base side of the triangle. The height, H, of this cone is also the height of the isosceles triangle. Analogously, the radius of the base circle of the "small" cone is denoted by r, and the corresponding height by h. Finally, the common opening angle of the two cones is denoted by α (Fig. 4.3).

To complete the mathematization part of the modeling process, we have to specify the volumes of the bigger and the smaller cone, respectively. The volume of the big cone is one third of the area of the base circle multiplied by the height of the cone, i.e., $V = 1/3 \cdot \pi R^2 H$ while the volume of the small cone is $v = 1/3 \cdot \pi r^2 h$.

As the real situation has now been translated entirely into a mathematical situation, the mathematization part is complete when the question we were asking in the real situation has been translated into a question within the mathematical domain: What are the dimensions of the small cone if its volume is half that of the big cone?
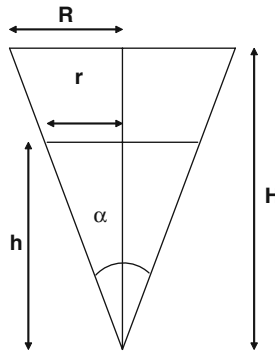
Fig. 4.3 A cut in a conic surface

To answer this question let us look for another suitable equation that relates the relevant components of the two cones. Exploiting similar triangles, we get $r/R = h/H$ which gives $r = R(h/H)$. Hence $v = 1/3 \cdot \pi R^2[h^2/H]^2 h = 1/3 \cdot \pi h^3 R^2/H^2$

The equation we are looking for results from equating volume v with $\frac{1}{2}V$, i.e., $v = 1/3 \cdot \pi h^3 R^2/H^2 = 1/2V = 1/2[1/3 \cdot \pi R^2 H]$. Getting rid of common factors on both sides, this equation is fulfilled exactly when $h^3 = \frac{1}{2}H^3$, i.e., exactly when $h = (1/2^{1/3})H \approx 0.79H$

We observe that this result does not neither depend on the opening angle of the cone (nor – equivalently – on the radius of the large cone).

Translating this mathematical answer back to addressing the real situation, we obtain the following answer to the initial question: Neither A nor B is right. The liquid poured into the glass is half the volume of the full glass when its poured to 79% of the full height, and this holds for any opening angle of the cone.

In order to analyse this example with regard to the modeling process, let us focus especially on what it takes to be able to (a) undertake the mathematization part and (b) to obtain answers to the mathematical questions posed within the mathematical domain.

As was hinted at above, the idealisation and specification part needed in preparation of the mathematization are almost ready from the outset. This is due to the fact that the shape of the glass is given in geometric terms. So, the mathematization part is straightforward and does not require much consideration beyond terminology (cone, angle, circle, height, radius etc.) The only additional thing needed is the formula for the volume of a cone. The modeller does not even need to know this formula by heart (or to be able to derive it him- or herself). It suffices to know that such a formula exists and where to find it. Anyone to whom the initial *problématique* is engaging would probably be able to mathematize the problem this far without any difficulty. The resulting mathematical model of the situation consists of the two cones and their interrelations, including their volumes.

However, having a clue as to what to do with this model is a different matter. Dealing with the mathematized situation within the mathematical domain is not quite as straightforward. It requires the activation of several mathematical competencies and related skills.

Firstly, the modeller has to identify which components of the two cones are relevant to solving the problem. These components are exactly the ones that occur in the two expressions for the big and the small volumes, respectively. Secondly, the modeller has to relate these by setting $v = \frac{1}{2}V$, and to see that as there are four variables in this equation from the outset, it is not possible to draw any conclusion without identifying and utilising further links between the variables. Such steps are known to be very difficult to make. Solving equations, once they exist, is one thing, establishing them in sufficient numbers so as to narrow down the variability of a given problem is even more demanding and requires the solver to possess/or to look for a clear strategy to pursue. A fairly high level of mathematical insight is involved in realising that more than the equation $v = \frac{1}{2}V$ is needed to solve the problem. If R and H are taken as givens, and r and h are to be determined in terms of R and H, we need one more equation. Where to find one?

In the approach taken here, the key step is to relate the radius and the height in the small cone to the radius and height of the big cone, in some tractable way. There are several ways to do this, all of which require insight into similar triangles and resulting proportionalities. To introduce such a relation into the $v = \frac{1}{2}V$ equation requires the solver to be able to express one of the variables – here r was chosen – in terms of the remaining three variables (which in turn requires the ability to manipulate basic algebraic expressions) and then insert the outcome into the volume equation, simplify and rearrange this equation and arrive at the equation $h^3 = \frac{1}{2}H^3$, and then – activating more algebra – to make the inference that $h = (1/2^{1/3})H$, and to finally conclude that $h \approx 0.79H$. To many a non-experienced modeller this is a non-trivial exercise that requires one to keep the goals clear in mind while, at the same time, being able to deal, in a satisfactory manner, with all the technical issues that occur along the road. From the huge literature on problem solving (e.g., Schoenfeld, 1992) we know that the strategic part poses major challenges, if not obstacles, even to problem solvers who are in possession of all the knowledge and technical skills involved in carrying out the strategy.

This is an example of what we might call "classical applied problem solving". By applied problem solving I mean handling an extra-mathematical problem which has already been (pre)mathematized, either by its very nature – as in this case – or by preparations made by the presenter of the task. In other words, in classical applied problem solving, we do consider a problem sitting in an extra-mathematical domain. So by definition mathematical modeling is indeed at play but the mathematization part of the modeling process has already been completed or is quite straightforward. The real challenge lies in selecting the relevant information from the mathematized situation and in subjecting this information to mathematical treatment. Three crucial points are involved here:

- The selection of what information is considered relevant has to be based on an anticipation of a possible effective mathematical problem solving strategy.
- The selection has also to be based on the modeller's trust in his or her own ability to implement the strategy envisaged.

- The modeller/problem solver has to possess most of the mathematical knowledge and technical skills involved in implementing the strategy to a satisfactory degree.

*Example 2:* Day-length in Copenhagen

Situation: Table 4.1 is the total length of the day in Copenhagen for the 21st of each month of 2006. Each length is given in hours with decimals (in the original data set the length was given in hours and minutes)

Table 4.1 Total length of daylight hours in Copenhagen

21st of	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
		7.98	10.10	12.25	14.62	16.60	17.53	16.67	14.65	12.32	10.05	7.97	7.02

Question: What was the length of the day on 4th July 2006?

To get a feel for the situation we plot the data in a Graph 4.1, where one unit on the first axis is 1 month, so that the point k represents the 21st of month k, and one unit on the second axis represents a length of 1 h. Note that this implies that we have decided not to take the differences in month lengths from 28 to 31 days into consideration.
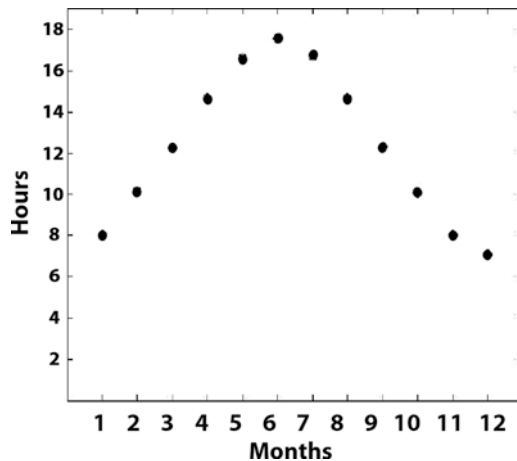
Graph 4.1 A graph of daylight hours

So far, by representing the data in a graph, we have only modelled the situation very preliminarily. The graph above can be perceived as a first model of the situation.

To proceed we now make four assumptions. Firstly, inspired by general astronomical knowledge, we assume that the data are a sample from a periodic function on the integers whose period is 12 months. Secondly we assume that this function is a restriction of a function of period 12 defined on the real line. Thirdly, the latter function is assumed to attain its minimum at $t = 12$ and its maximum at $t = 6$.

Finally, we go on to make the much stronger (mathematical) assumption that this function is actually an affine transformation of a sine function of an affine transformation of time, i.e., the day length function l of time t can be expressed as follows: $l(t) = a \sin(bt + c) + d$, with $a > 0$.

This is the first real step in the mathematization of the situation. The outcome is a second, general, model, which has to be specified further in order to give rise to a specific model of the situation. In other words, we have to choose a particular function from this large class of functions by deciding which parameter values to make use of. This means that we have to estimate, in some way or the other, the four parameters of l. As in the previous example this requires applied problem solving.

To that end we make use of our knowledge of the sine function. As the function attains its minimum value l_m for $t = 12$, $l(12) = l_m = l(0)$, corresponding to the minimum value (-1) of sine, (the latter part follows from periodicity), we have (since $a > 0$) that $-1 = \sin(b \cdot 0 + c) = \sin c$, from which we deduce that $c = -\pi/2$. Moreover, as l attains its maximum value (corresponding to sine attaining its maximum, 1), l_M, for $t = 6$, we have $1 = \sin(6b - \pi/2)$, which yields $6b - \pi/2 = \pi/2$, i.e., $b = \pi/6$. In other words, the function we are looking for is of the form $l(t) = a \sin(\pi/6 \cdot t - \pi/2) + d$.

We still have to estimate a and d. Utilising that $l_m = l(0) = -a + d$ and that $l_M = l(6) = a + d$, we conclude that $d = \frac{1}{2}(l_M + l_m)$, and $a = \frac{1}{2}(l_M - l_m)$. Combining everything we obtain our third model: $l(t) = 1/2(l_M - l_m)\sin(\pi/6 \cdot t - \pi/2) + 1/2(l_M + l_m)$. Inserting the minimum and maximum values l_m (7.02) and l_M (17.53), i.e., the day lengths of the 21st December and the 21st June, respectively, we obtain our fourth and final model:

$$l(t) = 1/2(17.53 - 7.02)\sin(\pi/6 \cdot t - \pi/2) + 1/2(17.53 + 7.02).$$

With this model in hand we can now mathematize the initiating question. As 4th July = 21st June + 13 days $\approx 6 + 13/30 = 6.433\ldots$ the mathematized question is: What is the value of l at $6 + 13/30 \approx 6.433\ldots$ The answer is straightforward: $l(6,433\ldots) = 1/2(17.53 - 7.02)\sin(0.5722\ldots\pi) + 1/2(17.53 + 7.02) \approx 17.39$

By the way, the registered length of the 4th July 2006 was 17.37 h. Translating our model-based mathematical answer back to the real domain we get: The length of the 4th July was 17 h 23 min.

Once again, we take a closer look at what it takes to complete these mathematization and problem solving parts of the modeling process.

In order to get an overview of the situation, the modeller begins the mathematization by plotting the data given (albeit in a slightly modified, decimal, form) in a discrete graph in a coordinate system, resulting in the first model. Here the modeller makes the assumption that variation in month lengths is insignificant in relation to the question posed. The modeller has to be aware of at least some benefits that may be derived from plotting the data in a coordinate system in addition to knowing how to do so. The latter exercise requires that dates are represented by natural numbers on the first axis, indicating the month, such that k stands for the 21st of month k. It further requires day lengths to be converted from hours and minutes into hours with decimals so as to fit with the numbers on the second axis. Although neither

conceptually nor technically demanding this preparation of the data for graphical representation is not automatic but does require decision-making, overview and care.

So far, the mathematization has "only" consisted in sheer graphical representation of the situation. It is not possible on that basis alone to answer the initiating question. For that further mathematization is required. The next set of steps, involving the four assumptions made, serve the purpose of establishing day length as some periodic function of time, the graph of which fits the data points in some way or another. Our modeller knows that the sine function is a typical periodic oscillating function which is often well suited to capture exactly such relationships. Of course, this in turn requires the modeller to know something about the sine function and its fundamental properties. Otherwise this idea would never have occurred to him or her. Introducing two affine transformations in the time domain and in the length domain, respectively, so as to build in some flexibility to adjust the function to the data points, further requires some knowledge about and experience with such an approach. Expressing the situation by means of a parameterized family of transformed sine functions completes the first, most important, mathematization part of the modeling process resulting in a general model of the situation.

Now, the modeller is faced with a generic parameterized model, whose parameters have to be determined before the modeling process can be carried any further. There are several ways of proceeding. Our modeller has chosen to assume that maximum and minimum points in the data set are also the maximum and minimum points of the model function. (This might not be the case, however, and to see whether or not it is, alternative approaches involving statistical estimation techniques are available). This assumption can be justified by astronomical/everyday knowledge of the fact that the shortest and longest days of the year are the 21st or 22nd December and the 21st or 22nd of June, respectively, so that a possible difference of 1 day wouldn't matter a lot when determining the fit-function. Taking this assumption as the next point of departure allows one to deduce the exact values of two of the four parameters, b and c, right away from properties of the sine function. Again, this presupposes that the modeller knows these properties and how they can be exploited.

In that way the general function model has been specified to a two-parameter model, still too general. Utilising, once again, the properties of the sine function the modeller establishes two simple linear equations that link the two remaining parameters with the maximum and minimum values of the function. Knowing that the determination of two parameters usually requires the establishment of two combined equations is also a necessary prerequisite for the strategy adopted by the modeller. The actual solution of the combined equations presupposes some technical experience and skill in dealing with such equations. Once the equations have been solved, and known values of the maximum and minimum of the function have been inserted, the final model has been established.

On that basis the initiating question can now be mathematized and answered within the mathematical domain. For this, two steps are needed. The first – small – step is to represent the 4th July as a number on the time axis. The second step is to calculate the resulting value of the function. In addition to requiring access to the

specific values of the sine function (here at $0.5722\ldots \cdot \pi$), the calculation requires either personal technical skills or access to computational devices of some kind.

In this example, too, a successful modeller following the path just outlined has to be able to combine two things: Knowledge of a couple of specific modeling instruments (graphical representation and sine functions) as a basis for devising a modeling strategy. And an anticipation of the ways in which these instruments can be employed so as to result in an answer to the initial question. The modeller has to be able to envisage, at least in outlines, the main steps involved in implementing this strategy and to stipulate whether he or she is able to carry out these steps. The main difference to the previous example is that the crucial steps in the mathematization of the situation are not at all present, let alone manifest, from the outset. The modeller has to devise a model more or less from scratch. In doing so, he or she has to look for mathematical objects that are familiar to him or her and can serve as possible modeling instruments in the context at issue. In other words, a major challenge lies in establishing the model. Along the road to the final model, the modeller has to pose and solve mathematical questions – i.e., to perform applied problem solving. Once the model has been established, mathematising the question, answering it by mathematical means, and translating the answer back to the real situation involve only computational skills and techniques, most of which can in fact be left to computers or calculators.

Imagine a modeller who does not know anything about trigonometric functions. Such a person would, of course, not be able to do what we have just done. What other possibilities would he or she have to come up with to answer the initial question by way of modeling? Well, the most obvious alternative would be to make use of linear interpolation between the day lengths of the 21st June and 21st July. Between those two dates the day length decreases by $17.53 – 16.67 = 0.86$ h. As 4th July is 13/30 of a month away from the 21st June, we would assume that the day length decreases by the same factor, giving $13/30 \cdot 0.86 = 0.37$.

So, based on this model the day length on the 4th July is estimated to 17.16 h against the 17.39 h obtained by the sine model (and the official length of 17.37 h). Whether or not this is a significant difference of course depends on the purpose of estimating the day length of the 4th July. However, only the person who is able to carry through both modeling processes can assess the advantage of the trigonometric model over the linear interpolation model. This is even more true of a sophisticated modeller who doesn't hinge the determination of the four parameters of the sine model on the observed extreme points but on a regression analysis. Such a modeller would be in a position to compare the three models and the resulting differences in the answers to the initial question.

Example 3: Tempering a piano

Situation and question: It is often being claimed that it is impossible to temper a piano with absolute precision. Is that really true?

First of all, it is not clear at all what mathematics has to do with this question. What does "tempering" (and "tuning") mean, what is "absolute precision" in a musical context, what do "tones", "intervals", "musical scales", etc. have to do with

mathematics? So, a modeller who wants to activate mathematics to deal with this problem has to begin by settling on definitions of what all these concepts are supposed to mean. This typically happens by resorting to the literature on tones, musical scales, tempering etc. Equipped with such knowledge, the modeller specifies and idealises the extra-mathematical domain as follows:

What we want to model first is the chords in the piano, each of which sounds with a prime tone and a set of (more or less) harmonic overtones. We decide not to take these harmonics into account in our modeling here. Next, we want to model the musical scale generated by the sequence of chords of decreasing lengths (if we take the direction left-to-right on the piano keyboard). Tempering the piano then means to fix the lengths of all the chords. Of particular interest to us are classic musical intervals such as the octave, the fifth, the fourth, the third etc. A musical scale is a sequence of tones, played one after the other, that form intervals according to certain rules.

When moving on to mathematizing the situation, we want our modeling to take into account that musico-physical experiments, begun by the Pythagoreans and developed during several millennia, have shown that if we take a chord of a certain length, tightened and fixed at both ends, as the generator of a prime tone (by plucking it or beating it with a hammer), the fifth occurs if the chord is shortened to 2/3 of the original length, the octave (the same tone just "one higher") occurs if the original length is halved, and the fourth occurs if the length is shortened to 3/4 of the original length. We further want to found our modeling on a note as defined by one characteristic frequency (neglecting harmonics). Our model should also observe that playing two upward-going intervals in a row can be perceived as playing first a "local" prime tone, then a second tone corresponding to a certain fraction of the initial chord and playing the role of a new "local" prime tone, and then a third tone corresponding to a certain fraction of the previous chord. Then the third tone corresponds to a fraction of the initial chord – obtained by multiplying the two previous fractions. We shall call this "the multiplication principle". Finally, based on physical observations, we assume that if a tone corresponds to the fraction p/q (<1) of a given chord length giving rise to the frequency f, then the frequency of that tone – generated by the shorter chord – is $(q/p)f$.

Based on these considerations, made in an extra-mathematical domain consisting of a mixture of music and physics (which in itself contains mathematical features), we are now able to establish our mathematical model.

- Each tone T, is represented by a positive real number, its frequency, denoted by $f(T)$.
- A tone of pitch higher than T has frequency strictly larger than $f(T)$.
- The nearest fifth higher than T, denoted by $F(T)$, is assumed to have the frequency (1) $f(F(T)) = (3/2) \cdot f(T)$
- The nearest octave higher than T, denoted by $O(T)$, is assumed to have twice the frequency of T, i.e., (2) $f(O(T)) = 2 f(T)$.
- A musical scale is an increasing (finite or infinite) sequence of positive real numbers (frequencies)

Against this background we are now able to specify our question within the musical domain and then translate it into a mathematical question:

On a concert piano, the last key of 12 consecutive fifths is the same as the last key of 7 consecutive octaves. (Each fifth contains 7 half-tones, each octave contains 12 half-tones, and $12 \cdot 7 = 7 \cdot 12$.) *Is this exact in the sense that the frequencies of the two final tones are the same?* The mathematized counterpart of this question then becomes:

Is $f(F(F(\ldots(T)\ldots))) = f(O(O(\ldots(T)\ldots)))$, where the first term contains of 12 F's, while the last one contains 7 O's?... To answer this question within the mathematical domain we observe that (1) and (2) imply that $f(F^2(T)) = f(F(F(T))) = (3/2)f(F(T)) = (3/2) \cdot (3/2) \cdot f(T) = (3/2)^2 f(T)$, and that $f(O^2(T)) = f(O(O(T))) = 2f(O(T)) = 2 \cdot 2 \cdot f(T) = 2^2 f(T)$.

By induction we obtain $f(F^n(T)) = (3/2)^n f(T)$, and $f(O^p(T)) = 2^p f(T)$. Since for any $n > 0$ and any $p > 0$: $(3/2)^n \neq 2^p$, as $3^n \neq 2^{p+n}$, it is *not* the case that $f(F^{12}(T)) = (3/2)^{12} f(T) = 2^7 f(T) = f(O^7(T))$. Hence, the tempering of pianos is not exact – and can't possibly be.

By the way, the (relative) inaccuracy between the 12 fifths and 7 octaves is $(3/2)^{12}/2^7 = 531441/524288 \approx 1.0136$ i.e., 1.36%. This difference, called the Pythagorean comma, is what the well-tempered musical scale is supposed to get rid of by distributing this error multiplicatively over the 84 half-tone intervals involved, making use of the fact that the inaccuracies thus introduced are not discernible to most human ears.

What did this modeller do as regards mathematization and answering the initiating question? Unlike the two previous examples, in which the extra-mathematical situation was either in itself (pre-)mathematical or contained some obvious pre-mathematical objects and relations, the state of affairs is different in this example. Here, the extra-mathematical situation does not contain, at the face level, any hints to pre-mathematical objects whatsoever. So, the modeller has to imagine (or know) that the situation does, nevertheless, lend itself to mathematical modeling of its key (sic!) elements and features. The real-world objects selected for mathematization are those that pertain to musical temperament, such as tone, frequency, interval, and scale. Moreover, we wanted to mathematize the particular musical scales that can be represented on the keyboard of a concert piano. The model arising from these considerations represented tones (frequencies) by (some) positive real numbers and musical intervals by rational proportions between the frequencies of the first and the last tone. The introduction of the multiplicative principle adopted in the model is based on the observation that the last of three consecutive tones has a frequency relative to that of the first tone that is the product of the frequency of the third tone relative to the second one and the frequency of that tone relative to the first one.

The initiating question is a rather vague one. It requires musico-physical knowledge to focus on frequency and on fifths and octaves, taking into account that on the piano seven octaves make twelve fifths etc., and that this might provide a sufficient basis for specifying the initial question so as to make it mathematizable – and eventually soluble – within the domain of elementary number theory.

All this is indeed very involved. The main load of the modeling process lies on its intricate mathematization part, in which a large number of idealizations,

assumptions, and simplifications had to be made, requiring the modeller to be able to navigate in muddled waters. Once the modeling is complete, the answering of the mathematized question only required the modeller to know/find out that no power of 3 equals any power of 2 (positive integer exponents).

4.3 The Intricacies of Mathematization

Assisted by the three examples presented above, we are now able to try to identify, in theoretical terms, the characteristics of the mathematization part of the modeling process. When students enter this part, the initial structuring – including specification and idealisation – of the situation to be modelled has been completed, either because this was already part of the initial setting (as in the conic glass and the day length examples) or as a result of the modellers own work (as in the musical scale example). The structuring of the extra-mathematical situation requires an anticipation of the potential involvement of mathematics, and the nature and usefulness of this involvement with regard to the modeling purpose. It further requires an initial anticipation of which mathematical domain(s) might be used to represent the situation and the questions posed about it. As the examples suggest, this is sometimes almost trivial, sometimes very demanding.

The outcomes of this structuring constitute the platform from which the mathematization process is to depart. This process comprises two connected pairs of sub-processes. The first pair consists of the extra-mathematical objects and relations that have been chosen for translation into mathematical objects and relations, as well as the corresponding mathematical objects, and the relations amongst them, selected to represent the extra-mathematical objects and relations. Similarly, the second pair consists of the extra-mathematical questions that are to be translated into mathematical questions, and the mathematical questions selected to represent the extra-mathematical ones. For these choices to be made, it is – once again – necessary to anticipate specific mathematical representations, suitable to capture the situation. In addition to requiring knowledge of a relevant mathematical apparatus, this also requires past experience with the capabilities of this apparatus to model other, more or less similar, situations. All this is known to be very demanding on the part of students (Blomhøj and Jensen, 2007; Kaiser and Maass, 2007; Rodriguez Gallego, 2007).

In the conic glass example, the overall framework of the mathematization was given from the outset: we model the glass cup and the liquid in it by two mathematical cones. Then, in order to select the objects and relations between them that are to be translated into mathematics, we focus on selecting objects that suffice to determine the two volumes, i.e., the radii and the heights of the two cones, taken as physical measurements. Given that the initiating question also referred to the angle of the cone, this is selected as well. The translation of the objects into mathematical ones is immediate, since physical measurements are represented by positive real numbers. The extra-mathematical questions selected for mathematization are two: "To what height should we pour liquid so as to obtain half the volume of the full

glass?", and "Does the answer to this question depend on the angle of the glass, and if so, how?". Translated into mathematics, these questions become: "Under what conditions is the volume of the smaller cone exactly equal to half of that of the bigger cone?", and "Do these conditions depend on the angle of the cone, and if so, how?"

The mathematization just summarized is a result of a knowledge-based mathematical anticipation of what is needed to find the two volumes, and another anticipation of the subsequent problem solving process that requires the establishment of sufficient – and sufficiently many – mathematical links between the translated objects so as to answer the initiating questions. I propose to refer to this as *implemented anticipation* of relevant future steps, projected "back" onto the current actions.

In the day lengths example, the first basis for the mathematization is in place from the outset, as the situation is defined by a set of 12 given (and 353 unknown) day lengths for each day of 2006. So, these dates and day lengths constitute the objects – already of a (pre)mathematical nature – that we are going to mathematize further. For simplicity we decide not to account for the variations in month lengths. For each day, the day length is linked to the date. This linking constitutes the real world relations to be modelled. In the data given only twelve such links have been provided. The day lengths are given by physical magnitudes (in the first place in hours and minutes), and each day is defined by its date.

Having chosen the objects and relations to be translated into mathematics, we have to select mathematical objects and relations to represent them. The dates of 2006 are represented by positive rational numbers in the interval [0,13], such that the integers represent the given dates, while the day lengths are represented by rational numbers with two decimals in the interval [0,24]. This involves some, not necessarily trivial, further mathematization. The relations between the objects are represented by a function, where the 365 (date) numbers in the interval of definition are mapped into the y-interval. Now four mathematical assumptions are introduced to complete the mathematization. The first one is rooted in astronomical evidence, namely that this function is the restriction of a periodic function defined on all "date numbers" on the real line – considering all months to be of the same length. The second assumption is one of mathematical convenience, namely that this periodic function is the restriction of a continuous function defined on the real line. The third assumption, the strongest one, is that this function is some affinely transformed sine function, yet to be specified. So, the extra-mathematical objects and the relations between them are all translated into features of this function. The last assumption is that all the data points given lie on the graph of this function. The initiating question in the extra-mathematical domain is translated into the question "What is the value of the model function at the rational number representing the 4th July?"

In this example, too, implemented anticipation, projected back onto current actions, is crucial. It simply drives the establishment of the final model, including the applied problem solving involved in determining the parameters of this model. Once the final model is in place, its utilisation to answer the translated question is straightforward.

Implemented anticipation is of crucial importance in the piano tempering example as well. This example provides an excellent illustration of the two aspects involved in this anticipation. Firstly, the anticipation of what mathematical representations of musical tones, intervals and scales might at all be possible and relevant, is very involved indeed. It requires a fair amount of knowledge about musical phenomena and concepts and their physical roots. It further requires knowledge and intuition about potential inconsistencies in the conditions imposed on musical temperament, and the ability to pose a question that, if translated into mathematics, may be answered by mathematical means. Secondly, the mathematization at issue has to anticipate ways in which the mathematical question can be answered. Here, the latter part is the easier one, because once the mathematical question has been formulated, the answer follows rather immediately for anyone familiar with the fundamentals of elementary number theory. In other cases this part may be the most demanding one.
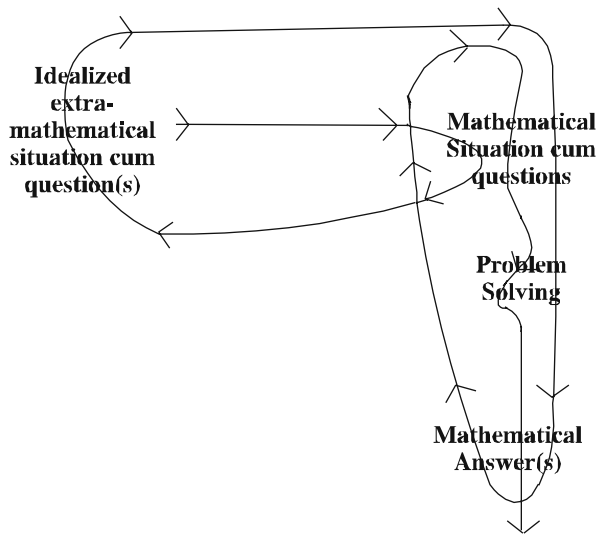
4.4 Modeling Students' Mathematizations

We are now ready to summarise our considerations into a theoretical model of the mathematization process. Successful mathematization requires *implemented anticipation* of a rather involved nature. At three key points in the mathematization process, the modeller has to be able to anticipate potentially difficult subsequent steps, and to implement this anticipation in terms of decisions and actions that frame the next step to be made:

(1) In structuring the extra-mathematical situation so as to prepare it for mathematization, the idealisation and specification of the situation, aimed at capturing the essential elements, features, and questions, have to be based on a first anticipation of their potential mathematization. This preparation of the extra-mathematical situation involves implemented anticipation of this potential.

(2) When subjecting the thus prepared extra-mathematical situation to mathematization, the modeller has to anticipate, in specific terms, relevant mathematical representations which are suitable for capturing the situation. This necessarily has to be based not only on the mathematical apparatus with which the modeller is familiar but also, perhaps even more importantly, on past experiences with the capabilities of this apparatus to mathematize similar situations. The resulting mathematization is an implementation of this anticipation.

(3) When anticipating the mathematical apparatus that might provide a suitable mathematization of the situation, the modeller also has to anticipate the use that may be made of the mathematization chosen and of the resulting model. Differently put, the modeller has to be able to envisage the ways in which the mathematical apparatus employed in the mathematization can provide answers to the mathematical questions posed. This implies that the modeller has to envisage mathematical problem solving strategies and procedures that will lead to a

solution of these questions. Again, the mathematization results from an imple-
mented anticipation of the problem solving tools that can be activated after the
mathematization has been completed.

The third of these steps, in which mathematical problem solving is anticipated for
later implementation, is in itself very involved (Goldin and McClintock, 1984; Lesh
and Doerr, 2003; Lesh and Zawojewski, 2007; Schoenfeld, 1985, 1992). However,
the first two steps are in principle even more involved, as they aim at creating
links, not yet established, between an extra-mathematical situation and mathemat-
ics, while at the same time anticipating the last step. The following figure is meant to
represent an idealised version of the mental processes involved in mathematization
(Fig. 4.4).

Fig. 4.4 An idealized version of mathematization processes

The key element in our proposed theoretical model of students' mathematization
is implemented anticipation which presupposes

- mathematical knowledge relevant to the situation
- putting this knowledge to work for modeling
- application-oriented beliefs about mathematics on the part of the student
- "mathematical" self-confidence and perseverance.

How can students learn to anticipate putting mathematical knowledge to work
in modeling before they have learnt modeling? In principle this leads to an infinite
regression, leading to a learning paradox similar to that identified by Sfard (1991)
in relation to the reification of process generated mathematical concepts – where

an object resulting from reification of a process cannot be perceived as an object without considering it as being subjected to new processes operating on it – and Vollrath who in a more general context has considered paradoxes emerging from dualities of mathematical entities and characteristics (Vollrath, 1993).

There are basically two different approaches to the teaching of modeling aiming at overcoming these paradoxes. One is emergent modeling (Gravemeijer, 2007) and its close relative model-eliciting activities (Lesh and Zawojewski, 2007, 783ff) both of which focus on anchoring the very learning of mathematical concepts in modeling extra-mathematical worlds. Thus the paradox is supposed to be overcome by dissolving the demarcation line between pure mathematics and mathematical modeling. In the other approach, modeling (see e.g., Kaiser and Maass, 2007; Blomhøj and Jensen, 2007; Ottesen, 2001) is introduced by involving students in teacher supervised projects in which they are engaged in performing modeling tasks in situations where their existing mathematical knowledge is of relevance or where they are assisted in identifying and acquiring new mathematical knowledge potentially useful in the situation at issue. The latter approach is supposed to overcome the paradox by a stepwise reclamation of new modeling land by the students.

Now, is our proposed theoretical model of the mathematization process empirically valid? It seems that we don't really know, as only a very few empirical studies have focused specifically on the mathematization process and its intricacies. Firstly, nobody seems to have studied modellers engaged in a "post-arithmetic" modeling process outside the education system. Secondly, while there are several studies of students involved in modeling at various educational levels, most of these studies have a broader focus, at least when we are talking about the secondary and tertiary levels. Quite a few of these studies do, as an aside, provide circumstantial evidence that suggests that students do in fact experience the challenges posed by the need for implemented anticipation. However, focused empirical studies are needed to unveil the extent to which this theoretical model does in fact capture students' work with the mathematization process. Should our model turn out to be empirically valid, we would have access to an explanation of the fact that mathematical modeling does indeed constitute tough challenges for students.

References

Blomhøj, M., and Jensen, T. H. (2003). Developing mathematical modeling competence: Conceptual clarification and educational planning. *Teaching Mathematics and Its Applications*, 22(3), 123–139.

Blomhøj, M., and Jensen, T. H. (2007). What's all the fuss about competencies? In Blum, W., Galbraith, P., Henn, H.-W., and Niss, M. (Eds.), *Modeling and Applications in Mathematics Education – The 14th ICMI Study* (pp. 45–56). New York: Springer Science+Business Media.

Blum, W., and Niss, M. (1991). Applied mathematical problem solving, modeling, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, 22(1), 37–68.

Goldin, G., and McClintock, C. E. (Eds.) (1984). *Task Variables in Mathematical Problem Solving*. Hillsdale, NJ: Erlbaum.

Gravemeijer, K. (2007). Emergent modeling as a precursor to mathematical modeling. In W. Blum, P. Galbraith, H.-W. Henn, and M. Niss (Eds.), *Modeling and Applications in Mathematics*

Education – The 14th ICMI Study (pp. 137–144). New York: Springer Science+Business Media.

Kaiser, G., and Maass, K. (2007). Modeling in lower secondary mathematics classroom – Problems and opportunities. In W. Blum, P. Galbraith, H.-W. Henn, and M. Niss (Eds.), *Modeling and Applications in Mathematics Education – The 14th ICMI Study* (pp. 99–108). New York: Springer + Business Media.

Lesh, R., and Doerr, H. (Eds.) (2003). *Beyond Constructivism: Models and Modeling Perspectives on Mathematical Problem Solving, Learning and Teaching*. Mahwah, NJ: Erlbaum.

Lesh, R., and Zawojewski, J. (2007). Problem solving and modeling. In F.K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 763–804). Charlotte, NC: Information Age Publishing.

Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis, and S. Papastavridis (Eds.), *3rd Mediterranean Conference on Mathematical Education* (pp. 115–124). Athens, Greece: Hellenic Mathematical Society and Cyprus Mathematical Society.

Ottesen, J. (2001). Do not ask what mathematics can do for modeling. Ask what modeling can do for mathematics! In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 335–346). Dordrecht: Kluwer Academic Publishers.

Rodriguez Gallegos, R. (2007). *Les équations differentielles comme outil de modélisation mathématique en Classe de Physique et de Mathématiques au lycée: une etude de manuels et de processus de modélisation d'élèves en Terminale S.* Doctoral Dissertation. Grenoble: Université Joseph Fourier, I.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. New York: Academic Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In G. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 334–370). New York: Macmillan Publishing Company.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1–36.

Vollrath, H.-J. (1993). Paradoxien des Verstehens von Mathmatik. *Journal für Mathematik-Didaktik*, 14(1), 35–58.