# Chapter 22
# Communication: The Essential Difference Between Mathematical Modeling and Problem Solving

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**Abstract** In this chapter, I discuss the formulation of tasks used as a communicative tool for developing someone's mathematical modeling competency and mathematical problem solving competency. These two competencies are characterized and their different crux is highlighted. This is exemplified by the formulation of different kind of tasks, and two hypotheses are offered for further debate and investigation concerning the kind of tasks that dominate in mathematics education and why.

## 22.1 Introduction and Conclusions

Consider the following tasks:

1. What is the relation between one's income and the tax paid?
2. How does the tax one pays depend on the income tax and the VAT?
3. Microorganisms breed by cell division. To slow down the cell division in a particular sample of microorganisms, a certain substance is added. After adding the substance, the number of microorganisms can be described as a function with the expression

$$f(t) = 12 + 3t - e^{0.5t},\ t \in [0;6.9]$$

where $t$ is the number of hours after the substance is added and $f(t)$ is the number of microorganisms, measured in millions, at the time $t$. Determine the number of microorganisms 3 h after the substance has been added. Also, determine $f'(t)$ and interpret it. Finally, use a graphics calculator to solve the equation $f(t) = 13$.

Determine the maximum number of microorganisms after the substance has been added.

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In this chapter, I will elaborate on and exemplify the following conclusions:

- Reflections among, and communication between, teachers and students about the crux of different mathematical competencies can be used to engender the kind of work processes aimed at in mathematics education.

  Therefore it is fruitful to identify mathematical modeling competency and mathematical problem solving competency as two competencies that do overlap, but have completely different cruxes.

- One of the benefits of identifying the different cruxes is that this identification can be used to sharpen the way teachers invite students to develop the competencies, for example, by the formulation and orchestration of written tasks.

  As examples of this, the first task above is an invitation to mathematical modeling, the second task is an invitation to mathematization and problem solving at upper secondary level, and the third task is – as a contrast – an unfocused task that does more harm than good when the aim is to develop the two competencies at play.

## 22.2 A Competency Perspective

My work with mathematical modeling and mathematical problem solving in relation to mathematics education are analytically biased. The bias comes from my involvement in the development of a more general idea: To use a set of mathematical competencies as a perspective on what it means to master mathematics and how the answer to this question can and should be used to develop mathematics education.

The scaffolding of this idea was the hub of the so called KOM project (KOM is an abbreviation for "competencies and mathematics learning" in Danish), which took place in the years 2000–2002 under the leadership of Mogens Niss from Roskilde University in Denmark and is thorougly reported in Niss and Jensen (2002 and to appear). The important analytical steps carried out in this project were to:

- move from a general understanding of the concept *competence*, which I – in semantical accordance with the KOM project – take to be someone's insightful readiness to act in response to the challenges of a given situation (Blomhøj and Jensen, 2007),
- to a focus on *a mathematical competency* defined as someone's insightful readiness to act in response to *a certain kind of mathematical challenge* of a given situation (Blomhøj and Jensen, 2007),
- and then identify, explicitly formulate and exemplify *a set of mathematical competencies* that can be agreed upon as independent dimensions in the spanning of what it means to master mathematics.

Such a set of mathematical competencies has the potential to replace the syllabus as the hub of the development of mathematics education, because it offers a vocabulary for a focused discussion of the aims of mathematics education that can make us feel comfortable for the same reasons that we presently are with the traditional specificity of the syllabus (Blomhøj and Jensen, 2007).

The result of the KOM analysis is visualized in condensed form in Fig. 22.1. I will now elaborate on the part of the KOM perspective that is the focus of this chapter: the picture of mathematical modeling competency and mathematical problem solving competency as two distinct, but overlapping constituents of mathematical mastery.

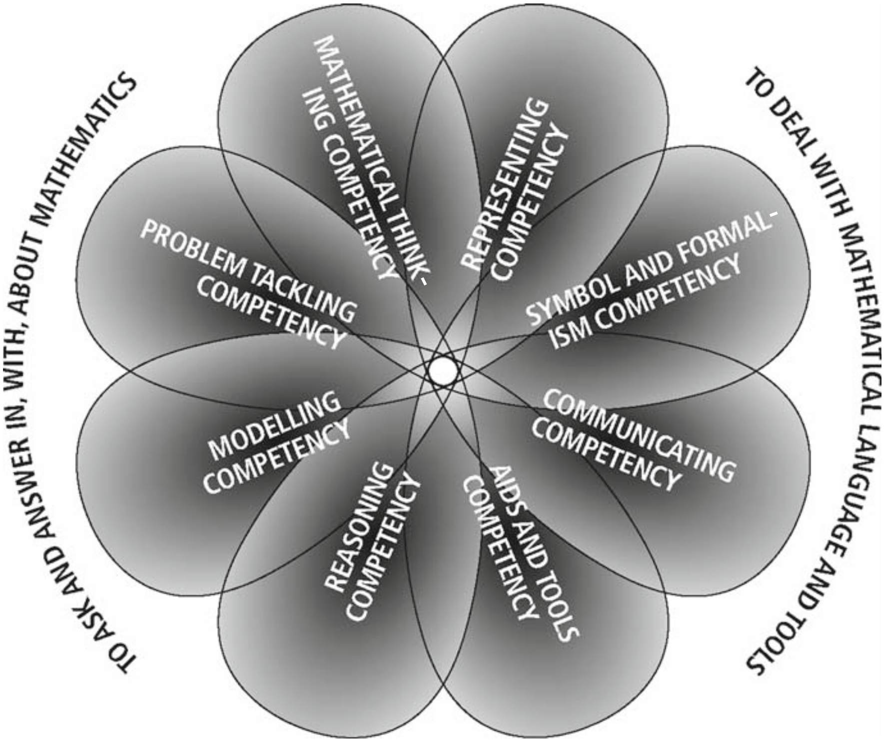


**Fig. 22.1** A visual representation – the "KOM flower" – of the eight mathematical competencies presented and exemplified in the KOM report (Niss and Jensen, to appear, Chapter 4)

## 22.2.1 Mathematical Modeling Competency

In my work I refer to a description of the creation and use of a mathematical model consisting of the following six sub-processes (Blomhøj and Jensen, 2003; Jensen, 2007a):

(a) Formulation of a task (more or less explicit) that guides you to identify the characteristics of the perceived reality that is to be modelled.
(b) Selection of the relevant objects, relations, etc. from the resulting domain of inquiry, and idealisation of these in order to allow a mathematical representation.
(c) Translation of these objects and relations from their initial mode of appearance to mathematics.
(d) Use of mathematical methods to achieve mathematical results and conclusions.
(e) Interpretation of these as results and conclusions regarding the initiating domain of inquiry.
(f) Evaluation of the validity of the model by comparison with observed or predicted data or with theoretically based knowledge.

Figure 22.2 is a visualization of this process. The figure contains a labelling of the sub-processes as well as an attempt to evaluate the six stages that frame them. It is not far from, and is indeed inspired by, many of the other models of the mathematical modeling process found in the mathematics education research literature, for example, the proceedings from the ICTMA conferences, of which Lamon et al. (2003) and Haines et al. (2007) are the two most recently published.

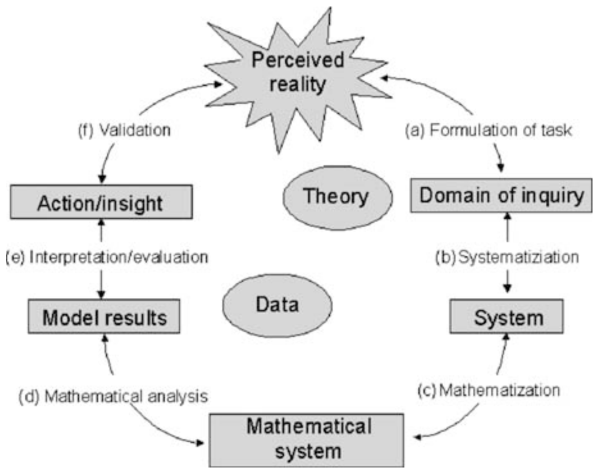


**Fig. 22.2** A visual representation of the mathematical modeling process (Blomhøj and Jensen, 2007)

I use *mathematical modeling competency* to describe someone's insightful readiness to carry through all parts of a mathematical modeling process in a certain context (Blomhøj and Jensen, 2003).

### *22.2.2 Mathematical Problem Solving Competency*

In the KOM report, the following characterization of mathematical problem-tackling competency is presented:

> This competence partly involves being able to *put forward*, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, "pure" as well as "applied", "open" as well as "closed", and partly being able to *solve* such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways. (Niss and Jensen, to appear, Chapter 4)
>
> [. . .] A (formulated) mathematical problem is a particular type of mathematical question, namely one where mathematical investigation is necessary to solve it. In a way, questions that can be answered by means of a (few) specific routine operations also fall under this definition of "problem". The types of questions that can be answered by activating routine skills are not included in the definition of mathematical problems in this context. The notion of a "mathematical problem" is therefore not absolute, but relative to the person faced with the problem. What may be a routine task for one person may be a problem for someone else and vice versa. (Niss and Jensen, to appear, Chapter 4)

Accordingly I use *mathematical problem solving competency* to describe someone's insightful readiness to solve different kinds of mathematical problems in their already formulated form (cf. Jensen, 2007a).

## 22.3 Contrasting the Crux of the Competencies

I now return to the tasks mentioned in the introduction in order to characterize these in the light of the given competency descriptions.

The first task challenges the subject to work with all the mathematical modeling sub-processes, and I will therefore label it an invitation to mathematical modeling. By virtue of the "underdetermined" nature of the initial parts of the mathematical modeling process, the crux of this challenge is to learn to handle the many often equally sensible choices that needs to be made before mathematical concepts and techniques can be of any use, and the lack of a clearly defined strategy to use when making these choices. Or, to put it in a different way, to cope with a feeling of "perplexity due to too many roads to take and no compass given" (Blomhøj and Jensen, 2003). Seen through a didactical "competency lens," mathematical modeling is mainly interesting if it carries with it such a handling of "openness":

> Even though in principle we are concerned with mathematical modeling each time mathematics is applied outside it's own domain, here we use the terms model and modeling in those situations where there is a non evident cutting out of the modelled situation that implies decisions, assumptions, and the collection of information and data, etc.
> Dealing with mathematics-laden problems which do not seriously require working with elements from reality belongs to the above-mentioned problem tackling competency. Those aspects of the modeling process that concentrate on working within the models are closely

> linked to the above-mentioned problem tackling competency. However, the modeling competency also consists of other elements which are not primarily of a mathematical nature, e.g., knowledge of nonmathematical facts and considerations as well as decisions regarding the model's purpose, suitability, relevance to the questions, etc. (Niss and Jensen, to appear, Chapter 4)

The second task from the introduction is an example of a task that "concentrates on working within the models." It represents a kind of task that only challenges the subject to work with sub-process (c), (d) and (e) of the mathematical modeling process as it is characterized here. The delimitation of the context and task in (a) and (b) is already dealt with in the formulation of the task, and the inclination to work with (f) comes from having worked with (a) and (b). Tasks like this challenge the subject's competency to mathematize a more or less well-defined problem of a non-mathematical character, and therefore often give the subject a feeling of "knowing what the goal is without knowing how to achieve it" (Blomhøj and Jensen, 2003).

An ability to cope with such a quite frustrating feeling of being "cognitively stuck" is, in my view, the crux of mathematical problem solving competency (Jensen, 2007a). A task like "How does the tax one pays depend on the income tax and the VAT?" is therefore mainly to be seen as an invitation to develop this competency within the domain of applications of mathematics. It does not challenge – and can therefore not be used to completely develop – mathematical modeling competency and all the associated sub-competencies (Jensen, 2007b).

I will label the combination I am dealing with here *mathematization competency* to describe someone's insightful readiness to solve problems defined as such by a challenge to mathematize. More loosely speaking, mathematization competency is the combination of mathematical problem solving competency and mathematization.

## 22.4 Experiences with Challenging the Competencies

I have attempted to put the ideas and approaches laid out here into educational practice in various projects, of which I will mention two. The first is my involvement in the structuring and writing of a series of mathematics textbooks for grade k-9 (see Jensen et al., 2002, as an example). The presentation in these books is in two important ways influenced by my understanding of and emphasis on mathematical modeling competency and mathematical problem solving competency in mathematics education. Firstly, the tasks introduced in each chapter are in separate sections following whether they are meant to be competency-oriented problems (for some of the pupils) or drill-oriented exercises (cf. Fig. 22.3). Secondly, the books contain tasks meant to challenge mathematical modeling competency in two different ways. One is as activities of relatively short duration of up to one lesson, consisting of both mathematization tasks and modeling tasks that are relatively easy to systematize. These kinds of tasks are included in the problem section of each chapter. Another is as activities meant to last for weeks. These so-called investigations are initiated by tasks given in a special section in the back of each book,

| Invitations to… | *Mathematical problem solving* | *Mathematical exercising* | *Neither problem solving nor exercising* |
|---|---|---|---|
| *Mathematical modeling* | What is the relation between one's income and the tax paid? | How much fabric does one need to make a cloth for the dinner table? | Irrelevant category |
| *Authentic mathematization* | How does the tax one pays depend on the income tax and the VAT? | Draw a sketch of a 135 $m^2$ house. | Irrelevant category |
| *Pseudo extra-mathematical orientation* | The total length of The Loch Ness monster is 40 meters plus half its own length.<br>How long is the monster? | Anna and Bob earn 20% off the sale of ice cream.<br>How much do they earn if they sell for<br>a) DKK 100?<br>b) DKK 500?<br>c) … | Microorganisms breed by cell division. To slow down the cell division in a particular sample of microorganisms, a certain substance is added. After adding the substance, the number of microorganisms can be described as a function with the expression<br>$f(t) = 12 + 3t - e^{0,5t}, t \in [0;6.9]$<br>with $t$ being the number of hours after the substance is added and $f(t)$ the number of microorganisms, measured in millions, at the time $t$.<br>Determine the number of microorganisms 3 hours after the substance has been added.<br>Determine $f'(t)$ and interpret it.<br>Use the graphical calculator to solve the equation $f(t) = 13$.<br>Determine the maximum number of microorganisms after the substance has been added. |
| *No extra-mathematical orientation* | A cube's volume is $k$ times as big as the volume of another cube.<br>What is the relation between the surface areas of the two cubes? | Solve the equations:<br>a) $7 - x/3 = x + 1$<br>b) $x - 2 = \frac{x+2}{2}$<br>c) … | In a system of coordinates a parabola $P$ and a line $l$ aredetermined by<br>$P\colon\ y = x^2 - 4x + 3$<br>$l\colon\ y = -x + b$<br>where $b$ is a number.<br>Determine the coordinates of the vertex $T$ of the parabola $P$.<br>Calculate the distance from $T$ to $l$ for $b = -2$.<br>Determine the number of intersection points between $P$ and $l$ for every value of $b$. |

**Fig. 22.3** Examples of tasks spanning mathematical modeling competency and mathematical problem solving competency

dominated by very open invitations to mathematical modeling. Appendix contains several authentic examples of these different kinds of tasks.

This structuring of different kind of modeling tasks was also used in the so-called Allerød project (Jensen, 2007a). Here my ideas about mathematical modeling and problem solving were brought into an upper secondary mathematics classroom involving 25 students and their mathematics teacher. To make a long story short, my experience from being heavily involved in this project is that one of the main advantages of using a competency perspective on mathematics education is that reflections about the crux of different mathematical competencies can be used to engender the kind of work processes aimed at in mathematics education. More specifically, it became an important and shared part of the thoroughly developed classroom culture to distinguish between and work with the different kinds of tasks exemplified in Appendix, and to value their different contributions to the development of mathematical modeling competency.

## 22.5 Hypotheses to Promote Further Debate and Investigations

I have two hypotheses about the use of the different kind of tasks discussed in this chapter in general mathematics education. Neither of the hypotheses are grounded in solid research data, but are meant to promote further debate and search for more evidence.

One hypothesis is that invitations to mathematical modeling are far too often replaced with invitations to mathematization – because mathematization tasks are easier to formulate, orchestrate, work with, and assess.

Another hypothesis is that invitations to mathematization are far to often replaced with pseudo extra-mathematically orientated, neither problem solving nor exercise focused tasks (cf. Fig. 22.3) – because the latter kind of tasks are the easiest to formulate, orchestrate, work with, and assess. The third task from the introduction is an example of such a task (cf. Fig. 22.3). It also serves as a good illustration of the danger I hypothesize about here, since it is an authentic example of the only kind of application-oriented tasks given in the written math exam for the most common branch of upper secondary school (gymnasium) in Denmark.

## Appendix: Examples of Tasks Developed for and Used in Grades 9–12 with the Specific Aim of Promoting the Development of Mathematical Modeling Competency (Jensen, 2002, 2007a)

Invitations to develop mathematical modeling competency – long duration (2–4 weeks):

1. What is the relation between one's income and the tax paid?
2. What is the cost of me?
3. Which means of transportation is the best?
4. How can one navigate?
5. Can one become slim by exercising?
6. How many windmills should Denmark have?
7. What is the best shape of a tin can?

Invitations to develop mathematical modeling competency – short duration (within a lesson):

8. How much fabric does one need to make a cloth for the dinner table?
9. How many times can one brush one's teeth with a tube of toothpaste?
10. Draw a sketch of a 135 $m^2$ house.
11. How far away is the horizon?
12. How far ahead must the road be clear for you to make a safe overtaking?
13. At what angle of incline does a tower topple?
14. What are the maximum sizes of a board if one is to turn a corner?

Invitations to develop mathematization competency – short duration (within a lesson):

15. How does the tax one pays depend on the income tax and the VAT?
16. When you buy something, is it better to get a percentage of the price in discount before or after the VAT has been added?
17. Which savings account do you prefer: The one that pays 8% in annual interest or the one that pays DKK 110 in annual interest?
18. A theater increases the ticket price by 30%, which causes the income from the sale of tickets to go up by 17%. By how many percentages has the size of the audience changed?
19. Between three cities of the same size, where should the only high school in the area be?
20. A liqueur glass is cone-shaped. What height of the liqueur served in the glass makes it halfway full?
21. An enclosure must have the shape of a rectangle with a semicircle at one end. How much land can you enclose with a given length of fence?

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