Richard Lesh
Peter L. Galbraith
Christopher R. Haines
Andrew Hurford
EDITORS

Modeling Students' Mathematical Modeling Competencies

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Editors
Richard Lesh
School of Education
Indiana University
201 N. Rose Ave.
Bloomington IN 47405-1006
USA
ralesh@indiana.edu

Christopher R. Haines
Department Continuing Education
City University
Northampton Square
London
United Kingdom EC1V 0HB
c.r.haines@city.ac.uk

Peter L. Galbraith
Graduate School of Education
University of Queensland
Brisbane QLD 4074
Australia
p.galbraith@uq.edu.au

Dr. Andrew Hurford
School of Education
University of Utah
1705 Campus Center Drive
Salt Lake City UT 84112
USA
andy.hurford@utah.edu

ISBN 978-1-4419-0560-4 e-ISBN 978-1-4419-0561-1
DOI 10.1007/978-1-4419-0561-1
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2009941314

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

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Contributors

Sergei Abramovich State University of New York at Potsdam, Potsdam, NY, USA

Cynthia M. Ahn Thorp Scholastic Academy, Chicago, IL, USA

Hugo Alarcon Tecnologico de Monterrey, Monterrey, México

Burkhard Alpers University of Aalen, Aalen, Germany

Miriam Amit Ben-Gurion University of the Negev, Beer-Sheva, Israel

Cecilia C. Arias Rutgers University, Rutgers, NJ, USA

Jonei Cerqueira Barbosa State University of Feira De Santana, Feira De Santana, Brazil

Julio Benegas Universidad de San Luis, San Luis, Argentina

Jonas Bergman Ärlebäck Linköpings Universitet, Linköping, Sweden

Christer Bergsten Linköpings Universitet, Linköping, Sweden

Betsy Berry Indiana University-Purdue University, Fort Wayne, IN, USA

Mary Besterfield-Sacre University of Pittsburgh, Pittsburgh, PA, USA

Maria Salett Biembengut University Regional of Blumenau (FURB), Blumenau, Brasil

Werner Blum University of Kassel, Kassel, Germany

Cinzia Bonotto Department of Pure and Applied Mathematics, University of Padova, Padova, Italy

Jill Brown Australian Catholic University, Melbourne, VIC, Australia

Stephen R. Campbell Simon Fraser Simon Fraser University, British Columbia, Canada

Monica E. Cardella School of Engineering Education, Purdue University, West Lafayette, IN, USA,

Guadalupe Carmona The University of Texas, Austin, TX, USA

Theodore Chao The University of Texas at Austin, Austin, TX, USA

C. Christou University of Cyprus, Nicosia, Cyprus

Renee M. Clark University of Pittsburgh, Pittsburgh, PA, USA

Rosalind Crouch University of Hertfordshire, Hatfield, UK

Sarah M. Davis Learning Sciences Lab, National Institute of Education, Singapore

Angeles Dominguez Tecnologico de Monterrey, Monterrey, Mexico

Susan B. Empson The University of Texas at Austin, Austin, TX, USA

Lyn D. English Queensland University of Technology, Brisbane, Australia

Yakov M. Epstein Rutgers University, New Brunswick, NJ, USA

César Cristóbal Escalante Cinvestav-IPN, Mexico, Mexico

Shirley Fang Arizona State University, Tempe, AZ, USA

Thomas Fennewald Indiana University, Bloomington, IN, USA

Rita Borromeo University of Hamburg, Hamburg, Germany

Peter L. Galbraith University of Queensland, Brisbane, QLD, Australia

Tirupalavanam Ganesh Arizona State University, Tempe, AZ, USA

Fco. Javier García University of Jaén, Jaén, Spain

Gilbert Greefrath University of Cologne, Cologne, Germany

Steven Greenstein The University of Texas, Austin, TX, USA

Paula Guerra Arizona State University, Tucson, AZ, USA

Christopher R. Haines City University, London, UK

Guershon Harel University of California, San Diego, CA, USA

Juliet Hart Arizona State University, Phoenix, AZ, USA

Nelson Hein University Regional of Blumenau (FURB), Blumenau, Brasil

Brandon Helding Arizona State University, Tempe, AZ, USA

Linda Hernández Arizona State University, Tucson, AZ, USA

David Hestenes Arizona State University, Tempe, AZ, USA

Thomas Hills University of Basel, Basel, Switzerland

Celia Hoyles London Knowledge Lab, Institute of Education, University of London, London, UK

Nicholas Jackiw KCP Technologies, Emeryville, CA, USA

Irma Jan Ben-Gurion University of the Negev, Beer-Sheva, Israel

Tomas Højgaard The Danish School of Education, Århus University, Denmark

Gabriele Kaiser University of Hamburg, Hamburg, Germany

Sibel Kazak University of Massachusetts Amherst, Amherst, MA, USA

Ahyoung Kim Arizona State University, Tucson, AZ, USA

Christine Larson Indiana University; Bloomington, IN, USA

Della R. Leavitt University of Illinois, Chicago, IL, USA

Richard Lesh Indiana University, Bloomington, IN, USA

Katja Maass University of Education Freiburg, Freiburg, Germany

Colleen Megowan-Romanowicz Arizona State University, Tempe, AZ, USA

Muhsin Menekse Arizona State University, Tucson, AZ, USA

James Middleton Arizona State University, Tucson, AZ, USA

Judit Moschkovich University of California, Santa Cruz, CA, USA

Nicholas Mousoulides University of Cyprus, Nicosia, Cyprus

Dorit Neria Ben Gurion University of the Negev, Beer-Sheva, Israel

Mogens Niss IMFUFA, Department of Science, Roskilde University, Roskilde, Denmark

Richard Noss London Knowledge Lab, Institute of Education, University of London, London, UK

Michael Oehrtman Arizona State University, Tucson, AZ, USA

Andréia Maria Pereira de Oliveira State University of Feira de Santana, Feira de Santana, Brazil

M. Pittalis University of Cyprus, Nicosia, Cyprus

Chris Rasmussen San Diego State University, San Diego, CA, USA

Adolf J.I. Riede Ruprecht-Karls-University, Heidelberg, Germany

Lina Sanchez Rutgers University, Newark, NJ, USA

Roberta Y. Schorr Rutgers University, Newark, NJ, USA

Björn Schwarz University of Hamburg, Hamburg, Germany

Nicole Shechtman SRI International, Menlo Park, CA, USA

Larry J. Shuman University of Pittsburgh, Pittsburgh, PA, USA

Nathalie Sinclair Simon Fraser University, Burnaby, Canada

Robert Speiser Brigham Young University, Provo, UT, USA

Bob Speiser Brigham Young University, Provo, UT, USA

Bharath Sriraman The University of Montana, Missoula, MT, USA

Gloria Stillman University of Melbourne, Melbourne, VIC, Australia

Kelli Thomas The University of Kansas, Lawrence, KS, USA

Silke Tiedemann University of Hamburg, Hamburg, Germany

Pauline Vos University of Amsterdam, Amsterdam, Netherlands

Geoff Wake University of Manchester, Manchester, UK

Chuck Walter Brigham Young University, Provo, UT, USA

Lisa B. Warner Rutgers University, Rutgers, NJ, USA

Randall Young Indiana University, Bloomington, IN, USA

William Zahner University of California, Santa Cruz, CA, USA

Michelle Zandieh Arizona State University, Tucson, AZ, USA

Genaro Zavala Tecnologico de Monterrey, Monterrey, México

Judith Zawojewski Illinois Institute of Technology, Chicago, IL, USA

Chapter 1
Introduction: ICTMA and the Teaching of Modeling and Applications

Gabriele Kaiser

Applications and modeling and their learning and teaching in school and university have become a prominent topic in the last decades in view of the growing worldwide importance of the usage of mathematics in science, technology and everyday life. Given the worldwide impending shortage of youngsters who are interested in mathematics and science it is highly necessary to discuss possibilities to change mathematics education in school and tertiary education towards the inclusion of real world examples and the competencies to use mathematics to solve real world problems.

These concerns were the starting point for the establishment of the ICTMA group – an international community of scholars and researchers supporting the International Conferences on the Teaching of Mathematical Modeling and Applications. ICTMA has been concerned with research, teaching and practice of mathematical modeling since 1983. From the beginning, the aim of the ICTMA group has been to foster both teaching and research about mathematical modeling and the ability to apply mathematics to genuine real world problems – in primary, secondary, and tertiary educational environments – as well as in teacher education, and education in professional and workplace environments.

From the outset ICTMA has maintained the integrity of its focus, which has both a mathematical and an educational component. This makes a distinction from a mathematical focus on the one hand, and a mathematics education context in which the mathematics need not to have a connection with applications and modeling on the other hand. Thus a distinctive aspect of ICTMA is the interface it provides for collaboration between those whose main activity lies in applying mathematics, but who have an informed interest in sharing these abilities and developing relevant skills in others (an educational focus), and those whose principal affiliations are within education, but who have a commitment to supporting the effective application of mathematics to problems outside itself.

G. Kaiser (✉)
University of Hamburg, Hamburg, Germany
e-mail: gabriele.kaiser@uni-hamburg.de

Gabriele Kaiser is Current President of ICTMA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_1,

These goals have been promoted by biennial conferences – the International Conferences on the Teaching of Mathematical Modeling and Applications – taking place since 1983 in various parts of the world, namely Great Britain (1983, 1985, 1995, 2005), USA (1993, 2003, 2007), China (2001), Australia (1997), Germany (1987), Denmark (1989), Netherlands (1991), Portugal (1999).

At ICTMA conferences various themes and topics have been discussed, amongst others:

- How far are, at an international level applications and modeling examples, implemented in mathematics education at school and tertiary level?
- How is instruction to structure which is adequate for the teaching and learning of applications and modeling and considering country-specific differences?
- How can the development of modeling competencies be evaluated and promoted?
- How can authentic applications of mathematics in industry and technology be introduced into mathematics instruction at various levels?
- How can modern technologies, necessary for authentic modeling examples, be introduced in mathematics instruction at school and tertiary level?

At the last conferences and especially at the conference ICTMA13, of which important chapters are presented in this book, various answers to these questions and topics have been explored. However, it is obvious that real world and modeling examples still do not have a high level of importance in mathematics education at school and tertiary level, indispensable for a modern society. So, amongst other topics, it seems necessary to investigate affective barriers of teachers and students against the inclusion of these kinds of examples at various levels. Furthermore cognitive barriers which prevent students to develop modeling competencies need to be explored. First case-studies exist, but large-scale studies seem to be necessary in order to broaden and deepen our knowledge about these aspects.

In order to broaden the audience ICTMA became, in 2003, an affiliated study group of the International Commission on Mathematical Instruction, which gives the opportunity to run special sessions during ICME and to carry out ICMI-study 14 on that theme ICMI Study. With these strong relations to the mathematics education debate, and to the intense connection to the engineering and applied mathematics discussion, ICTMA has a unique flavour and will hopefully be able to bring in genuine real world and modeling examples into mathematics instruction around the world. This book expresses this flavour in a special way, it combines a design approach to mathematics instruction at various levels with many empirical studies from all over the world, not only from Western countries, but also in growing number from Asian countries and countries of the southern hemisphere. This growing internationality of the debate shows the joint worldwide concern about the lack of real world and modeling examples in mathematics education at school and tertiary level. But despite this growing internationality and the joint topics discussed all over the world, approaches from various parts of the world are still influenced by their underlying educational philosophies, their socio-cultural conditions and the studies and results presented in the articles of this book displays the variety of this debate. Future trends concerning the teaching and learning of applications and modeling should build on this variety and include it in its future debates.

Part I
The Nature of Models & Modeling

Chapter 2
Introduction to Part I Modeling: What Is It? Why Do It?

Richard Lesh and Thomas Fennewald

At ICTMA-13, where the chapters in this book were first presented, a variety of views were expressed about an appropriate definition of the term *model* – and about appropriate ways to think about the nature of *modeling activities*. So, it is not surprising that some participants would consider this lack of consensus to be a priority problem that should be solved by a research community that claims to be investigating models and modeling.

We certainly agree that conceptual fuzziness is not a virtue in a research community – especially if it impedes communication among members of the community. Furthermore, we agree that increasing clarity about key constructs is an important goal of research. Nonetheless, we also believe that, especially at early stages of theory development, a certain amount of diversity in thinking is as healthy for research communities as it is for (for example) engineers who are at early "brainstorming" stages in the design of space shuttles, sky scrapers, or transportation systems. Furthermore, we believe that the mathematics education research community in particular has suffered from more than enough pressure for premature ideological orthodoxy.

The theme of ICTMA-13 was *modeling students' modeling competencies*; and, in many of the research methodologies that are described throughout this book, students develop models to describe or design "real life" artifacts or tools; teachers develop models to describe students' modeling competencies – or to design productive learning environments; and, researchers develop models of interactions among students, teachers, and learning environments. So, the goal of many of our most productive studies focus on the development of powerful, sharable, and re-useable models – which then, in turn, influence theory development. But, model

R. Lesh (✉)
Indiana University, Bloomington, IN, USA
e-mail: ralesh@indiana.edu

T. Fennewald (✉)
Indiana University, Bloomington, IN, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_2, © Springer Science+Business Media, LLC 2010

development research is not the same as theory development research. For example, in theory-driven research, the theory determines what questions are appropriate to ask, what kind of evidence needs to be collected, how the information should be interpeted, analyzed, and assessed – and when the question has been answered or the issue resolved. But, in model-development research (or design research), the problem arises from a "real world" decision-making situation. Thus, this work is more like engineering than a "pure" science. As engineers (and other design scientists) often emphasize (see Zawojewski et al., 2009):

- Engineering is the science of solving "real life" problems where you don't have enough time, or money, or other resources – and when multiple stake holders often hold partly conflicting views about the nature success (low costs versus high quality, simplicity versus completeness, and so on).
- Realistic solutions to realistically complex problems usually need to integrate ideas and procedures drawn from more than a single discipline, or theory, or textbook topic area.
- When high stakes decision making issues arise in "real life" situations, it usually is important to design for success (power, sharability, re-usability) – not simply to test for it.
- In design research, many of the "things" that we most need to understand and explain are "things" that we ourselves are in the process of developing or designing. For example, in math/science education, these "things" range from students' conceptual systems to curriculum materials which include systems of activities for learning or assessment. And, in each of these cases, as soon as we come to better understand the "thing" being developed or designed, we tend to change them – so that another cycle of adaptation is needed. (Note: This is one reason why scientists do not speak of developing a single theory of space shuttles.)

In the book, *Beyond Constructivism: Models & Modeling Perspectives on Mathematics Problem Solving, Learning & Teaching* (Lesh and Doerr, 2003), we describe a variety of reasons why we consider MMP to be a "blue color" perspective which is like engineering more than it is like "pure" sciences such as physics, chemistry, or mathematics. In fact, for many of the same reasons why *Pragmatists* such as James, Pierce, or Dewey considered *Pragmatism* to be more of a framework for developing theories rather than being a theory in itself, we consider MMP to be a framework for developing models of students' modeling.

According to MMP, students, teachers, and researchers – all are in engaged in model development. Consequently, MMP research assumes that: (a) similar principles apply not only to the model development activities of students, but also to the model-development activities of teachers and researchers, and (b) researchers' early-iteration models are expected to be characterized by conceptual inadequacies similar to those that characterize the early-iteration interpretation systems of students or teachers.... MMP research recognizes that we – the researchers who study modeling – are humbly still in the early phases of *our own* model development. And, just as in the model development activities of the students and teachers that we study, we assume that we will only make progress if we go through multiple cycles of expressing > testing > revising our current ways of thinking. So just as Darwin emphasized in the development of other kinds of complex adaptive systems (1859), evolution of ways of thinking about teaching and students' learning is only likely to occur if provisions are made to encourage diversity, selection, communication, and accumulation (Sawyer, 2006).

In our own "blue collar" research on models and modeling, we take the following to be a useful first-iteration definition of a model. *A model is a system for describing (or explaining, or designing) another system(s) for some clearly specified purpose.*

The preceding definition seems simple and straightforward, but one thing that we like about it is that it has a clear history of pressing us toward important researchable questions. Furthermore, best of all, and unlike terms such as *cognitive structures* or *schemes* or other terms that are favored by cognitive scientists or constructivist philosophers, the preceding definition uses the term *model* in the same way that the term is used in mature science fields like physics or engineering. That is, a model is system that is used to describe (or interpret) another system of interest in a purposeful way. However, whereas physicists and engineers tend to focus on only the written symbolic aspects of the models they develop, when math/science educators investigate what it means to "understand" these models, it inevitably becomes apparent that the written and spoken embodiments that scientists emphasize scientists usually represent only something like the tip of an iceberg. For example, in order for scientific models to be useful, understanding them usually involves a variety of diagrams, concrete models, experience-based metaphors, and other expressive media – in addition to technical spoken language and symbol systems, each of which emphasize some aspects (but deemphasize others) for the "thing" that they are used to describe, or explain, or design. Furthermore, model development often involves dimensions of development such as intuition-to-formalization, concrete-to-abstract, situated-to-decontextualized, specific-to-general, implicit-to-explicit, global-to-analytic,[1] and so on. Furthermore, during early stages of development, models often function in ways that are more like "windows" that we look through rather than as "objects" that we look at; and, they often function in ways that are rather unstable. That is, when the model developer focuses on "forest-level" or "big picture" interpretations of the "thing" being described, they often neglect to keep in mind "tree-level" details about the "thing" being described. In other words, even for experienced scientists, but especially for young students, and especially during early stages of model development, a model developer's early interpretation of the "thing" being described is often far more situated, piecemeal, and non-analytic than most traditional theories of learning consider it to be.

Should early interpretations (such as those that function intuitively, informally, or in piecemeal or unstable ways) be referred to as models? We believe that such

[1] When we speak of these dimensions of development, we recognize that, in specific "real life" situations where model development is needed, the most useful model is not necessarily the one that is most complex, most formal, most abstract, or most decontextualized.

questions should be resolved through research – not through political consensus building. This point brings out a point that we especially like about the term "models and modeling". That is the term allows room for debates among researchers with opposing points of view. And, we believe that debates about the nature of various models have exhibited a clear history of leading to researchable questions, testable hypotheses, and a variety of model validation activities. This is why we say that Models and Modeling is not so much a view of how learning works as it is a methodological approach and framework for investigating learning.

Finally, we should not neglect to mention some other reasons why authors throughout this book view research on models and modeling to be especially important for mathematics education researchers. First, in nearly every field where researchers have investigated similarities and differences between experts and novices (or between successful learners or problem solvers versus those who are less successful), results have shown that expertise not only involves *doing* things differently (or better), it also involves *seeing* or *interpreting* things differently (or better). And, in mathematics and the natural sciences, interpretation development means model development. Furthermore, in students' lives beyond school, the ability to describe or explain things mathematically is one of the main factors that is needed in order for mathematics to be useful.

Throughout this volume, even though most of the authors generally support the ideas outlined above, conflicting views emerged – even during ICTMA-13. One reason why this happened is that the organizers of ICTMA-13 made special efforts to attract not only mathematics educators from more than 25 different countries, but equal efforts also were made to attract leading science educators, engineering educators, and mathematics educators focusing on mathematical and scientific thinking beyond school. Yet, even with such diversity, we know of no cases where differences in perspectives did not appear to be leading toward productive adaptations for the community as a whole.

This volume beings with chapters that aim to answer the fundamental question: what are models? Hestenes outlines important relationships among various forms of modeling and fields that apply modeling theory. In his plenary article, he distinguishes between uses of the term "model" to refer to a conceptual mental model and uses of "model" to refer to a publically shared model. By so doing, he sets the stage for a discussion that relates the act of modeling in professional and non-professional so-called "everyday life" to research in both science and mathematics education, thus giving a full view of modeling theory as describing the process by which a form of knowledge, modeling, is developed through lived acts that are both mental and social in nature. Niss continues this examination of the question of what models are and what modeling is with studies into how to model the learning of modeling itself. His study hints at perhaps the deepest and most philosophical of all questions related to the origin of modeling competencies and knowledge: What is the origin of understanding how to model? From his study of three different modeling problems, he suggests that the capability to model is a learned capacity. The teaching of modeling, he notes, can be approached through activities that encourage cognitive dissonance of a type that drives the emergent modeling of Gravemeijer –

or the model eliciting activities described by Zawojewski or Carmona, or the kind of scaffolded approaches described by Kaiser. Niss considers each as possible modes for teaching modeling. Larson and colleagues also address the question of what models are while also looking ahead to the future of modeling in the curriculum and examining ways modeling can meet the challenge to provide students opportunities to “create and refine ideas for themselves.”

With a notion about what models are, discussion then proceeds to examinations of where models and the modelers who make them are found. Noss and Hoyles take research about the pedagogical aspects of modeling beyond the classroom into the workplace, performing an investigation as to what “Techno-mathematical Literacies” are needed to understand manufacturing processes. Cardella examines the modeling of engineering undergraduates and graduates who might not otherwise see the connection of the math they are learning in the classroom to the real world work they will perform. This study is followed by Alpers who examines the modeling of mechanical engineers outside of school contexts, looking at math and modeling performed on the job. Together, these studies show both the application and real-world relevance of modeling in studies beyond the school setting – while at the same time demonsrating as well as to in-school modeling studies have to studies in other disciplines, and in higher level mathematics course.

The modeling process is examined by Larson by employing one of Lesh’s original modeling activities, “Summer Jobs”, in a study of how students develop quantitative reasoning needed in real-world problems, supporting concluding that changes in the perception students have of relationships among quantities leads to the development of better quantitative reasoning ability along a progression of Piagetian-like stages. Using a related activity, “University Cafeteria”, Mousoulides and Sriraman examine the progression in mathematical understandings made by middle school students over the course of their schooling.

The question of what is needed for modeling to occur is taken on by Galbraith, Stillman, and Brown, who provide insight into what has been recognized as one of the most critical aspects of setting up a modeling experience for learners – that is, creating a meaningful context in which it is hoped the modeling learners will engage in significant concepts. They explore this from a uniquely Australian perspective, investigating the response of Australian students to Australian themed modeling activities. Haines and Crouch further investigate this issue while exploring the intricacies of modeling cycles that distinguish modeling activities from other assessments, as they expand their discussion to include the assessment of modeling competencies in general. Amit and Jan then present an “extension of model-eliciting problems into model-eliciting environments which are designed to optimize the chances that significant modeling activities will occur.” Speiser and Walter then investigate another aspect critical to modeling activities and models in general. That is, models are tools created and applied for a purpose. And, purposes are strongly influenced by communities and societies in which designers and modelers operate. Clark and colleagues then discuss lessons learned from a pilot course they designed to elicit systems thinking in which they employed and modified MEAs designed to create a need for modeling. Davis then concludes the section by exploring the need for tools that can help teachers make sense of data collected in generative activities.

The question of how models develop is examined by Riede who studies students as they explore and rediscover Weber's law, progressing through modeling cycles, during their activities. Amit and Neria then examine the express > test > revise cycles of students engaged in generalizing pictorial linear pattern problems, finding that many students applied recursive strategies despite the appropriateness of global strategies. Work in the study of conceptual and model development is also important in the work of Dominguez who uses modeling activities to reveal multiplicity in students' ways of thinking – even when one final answer is agreed upon.

Finally the question of what ways modeling is different from solving traditional textbook word problems is addressed starting with Zawojewski, who asks what research implications and distinctions between the two can be drawn. She is followed by Carmona and Greenstein who find that modeling is suggestive of a spiral curriculum where powerful and underlying themes are revisited constantly – not arrived at permanently. Jensen then shows and investigates distinctions between modeling and problem solving competencies. Greefrath concludes with the examination of students planning processes, noting the differences in strategies used between modeling and problem solving questions.

These chapters reveal many facets of both modeling activities and the modeling research being conducted around student modeling activities. Although this collection of contributions is not intended to provide a comprehensive overview of the work being done to study the nature of modeling, this collection certainly gives the reader a diverse introduction to some of the most important frontiers in research on modeling and applications.

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Section 1
What Are Models?

Chapter 3
Modeling Theory for Math and Science Education

David Hestenes

Abstract Mathematics has been described as the science of patterns. Natural science can be characterized as the investigation of patterns in nature. Central to both domains is the notion of model as a unit of coherently structured knowledge. Modeling Theory is concerned with models as basic structures in cognition as well as scientific knowledge. It maintains a sharp distinction between mental models that people think with and conceptual models that are publicly shared. This supports a view that cognition in science, math, and everyday life is basically about making and using mental models. We review and extend elements of Modeling Theory as a foundation for R&D in math and science education.

3.1 Introduction

Why should a theoretical physicist be concerned about mathematics education? My answer will be a long one, but let me begin by introducing you to some of my esteemed colleagues in Box 3.1. These fellows are such good physicists that most if not all of them would be worthy candidates for a Nobel Prize if they were alive today. You may know that they are quite good at mathematics as well! Indeed, mathematics textbooks often count them as mathematicians without mentioning that they are physicists. I dare say, however, that they would be mightily offended to hear that they are not counted as physicists. Likewise, I am more than miffed when reviewers of my math-science education proposals discount my qualifications as a mathematician because my doctorate is in physics. Like the fellows in the list, I regard my scientific research as equal parts mathematics and physics. The fact that the education establishment does not recognize that theoretical physicists are uniquely well-qualified to address education at the interface between mathematics and science is traceable to a serious problem within the mathematics profession itself.

D. Hestenes (✉)
Arizona State University, Tempe, AZ, USA
e-mail: hestenes@asu.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_3,

Box 3.1 Some distinguished theoretical physicists

Newton	Cauchy
Euler	Poincaré
Gauss	Hilbert
Lagrange	Weyl
Laplace	Von Neumann

Of course, the list of *physicist/mathematicians* in Box 3.1 is far from complete. Many of my favorite colleagues are omitted. But there are at least two good reasons why the list ends in the middle of the twentieth century. The distinguished Russian mathematician Arnold (1997) put his finger on both in a widely circulated diatribe *On Teaching Mathematics*, wherein he asserts

> Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
>
> In the middle of the 20th century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course in total ignorance of other sciences.

Mathematician-cum-historian Morris Kline (1980) has thoroughly documented "the disastrous divorce" of the mathematics profession from physics, which began in the latter part of the nineteenth century. He estimated that, by 1980, eighty percent of active mathematicians were ignorant of science and perfectly happy to remain that way.

The divorce is thus an incontrovertible fact, but how disastrous can it be if only a minority of mathematicians like Arnold and Kline are alarmed? Isn't it a natural consequence of necessary specialization in an increasingly complex society? And isn't Arnold's claim of intimacy between math and physics merely a personal opinion? Surely the majority of mathematicians believe that mathematics is a completely autonomous discipline.

I claim that the answer to all these questions is a resounding NO! Indeed, *I submit that the single most serious deficiency in US math education is the divorce of mathematics from physics* in the education of mathematicians, in the training of math teachers, and in the structure of the K-12 (-16-20) curriculum. Moreover, this is not a simple deficiency in the breadth of education; it is a fundamental problem in conceptual learning and cognition. I claim that *cognitive processes for understanding math and physics are intimately linked and fundamentally the same*! Indeed, I claim that *Physics is cognitively basic to quantitative science in all domains!!*

Before delving into the deep cognitive issues, let us note some obvious academic consequences of the math/physics divorce. Training in mathematics is essential for all physicists, amounting to the equivalent of a dual major in mathematics for theoretical physicists. But math courses have become increasingly irrelevant to physics, so physics departments offer their own courses in "Methods of mathematical physics" at both graduate and undergraduate levels, with additional courses in more specialized topics like group theory. One consequence is a narrowing of the physicist's appreciation of mathematics. But a far more serious consequence is the reduction in opportunity for math majors to learn about vital connections to physics. This continues through graduate school, so the typical math PhD is ill-prepared for work in applied mathematics. Some math departments have attempted to remedy this deficiency with courses in mathematical modeling, but mathematical modeling without science is like the Cheshire cat: form without content! This is one of the deep cognitive issues that we need to address.

Far and away the most serious consequence of the math/physics divide is the deficient preparation of K-12 math teachers! The neglect of geometry and excess of formalism that Arnold (1997) deplores in the university math curriculum has propagated to teacher preparation. There is abundant evidence that most teachers see their job as teaching formal rules and algorithms. Few have even a minimal understanding of Newtonian physics, so most are inept at applying algebra and calculus even to simple problems of motion. Consequently, high school physics courses are forced to revisit the prerequisite math knowledge that students are supposed to bring from years of math instruction. As the math courses lack the intuitive base necessary for conceptual understanding, students are forced to rote learning, which has a short half-life, so their recollection of math has decayed to nearly to zero by the time they get to college.

I doubt that these crippling deficiencies in math education can be fully resolved without a "sea change" in the culture of mathematics. To drive such revolutionary change we need a coherent theory of mathematical learning and cognition supported by a substantial body of empirical evidence. My purpose here is to report on progress in that direction.

3.2 Origins of Modeling Theory

I have been investigating the epistemology of science and mathematics across the full range of academic disciplines for half a century. As that may sound implausible, let me describe the unusual initial conditions that got me started.

My father was an accomplished mathematician who helped organize American mathematicians to support the war effort in WWII. Consequently, he got to know mathematicians across the country on a first name basis. That served him well when, shortly after the war, he was wooed from the University of Chicago to build a first rank math department at UCLA. He was also appointed director of the Institute for Numerical Analysis (INA), where the National Bureau of Standards installed the first electronic computer in western United States. With a solid background in "pure mathematics" (Calculus of Variations), he blossomed then into a pioneer in the fledgling fields of Control Theory and Numerical Analysis, for which he was posthumously inducted into the Hall of Fame for Engineering, Science and Technology (HOFEST). The well-funded, vigorous research activity at the INA and the rapid emergence of the UCLA math department attracted a steady stream of distinguished mathematicians from around the world, for which my father was usually the host. He was at the acme of his career when I entered graduate school in 1956.

My undergraduate major in philosophy introduced me to the great conundrums of epistemology, and I was inspired by Bertrand Russell to switch to physics in search of answers. When I started graduate school, my father found me an office in the INA where I was surrounded by a whirlwind of excited activity about the beginnings of Computer Science and Artificial Intelligence. That prepared me to follow the evolution of both those fields throughout my career, though my main efforts were concentrated on physics and mathematics. By my third year I hit upon "Geometric Algebra" as the central theme of my scientific research. That induced me to reject Russell's logicist view of mathematics and sharpened my insight and interest in epistemological and cognitive foundations. Throughout my graduate years in physics I spent most of my time in the math department where I imbibed the culture of mathematics. This strong association with mathematics continued throughout my career in physics, anchored by relations with my father.

My diverse interests in cognitive science and theoretical physics converged in the 1980s when I got embroiled in problems of improving introductory physics instruction. Like any confirmed theoretician, I framed the problems in the context of developing a theory with testable empirical consequences. Largely from my own experience as a scientist I identified scientific models and modeling as the core of scientific knowledge and practice, and I proceeded to incorporate it into the design of physics instruction with the help of brilliant graduate students Ibrahim Halloun and Malcolm Wells. Thus began a program of educational R&D guided by an evolving research perspective that I called *Modeling Theory* in a 1987 chapter. That program has continued to evolve beyond all my expectations. An up-to-date review of Modeling Theory is available in a recent chapter (Hestenes, 2007). The present chapter is a continuation, introducing new material with only enough duplication to make it reasonably self-contained. Therefore, it contains many gaps, some of which can be filled by consulting the earlier chapter, and others that I hope will stimulate original research. For Modeling Theory is an enormous enterprise that amounts to a thematic approach to the whole of cognitive science. The best we can do here is sample the major themes.

As schematized in Fig. 3.1, research on Modeling Theory has developed along two complementary strands. The strand on the right investigates *scientific models and modeling practices* that are explicit and observable. It provides *a window to structure and process in scientific and mathematical thinking* that we aim to peek through. That involves us with the strand on the left, which will be our main concern.

You may ask, "Why should one adopt a model-centered epistemology of science?" There are three good reasons:

1. *Theoretical*: Models are *basic units of coherently structured knowledge*, from which one can make *logical inferences, predictions, explanations, plans and*

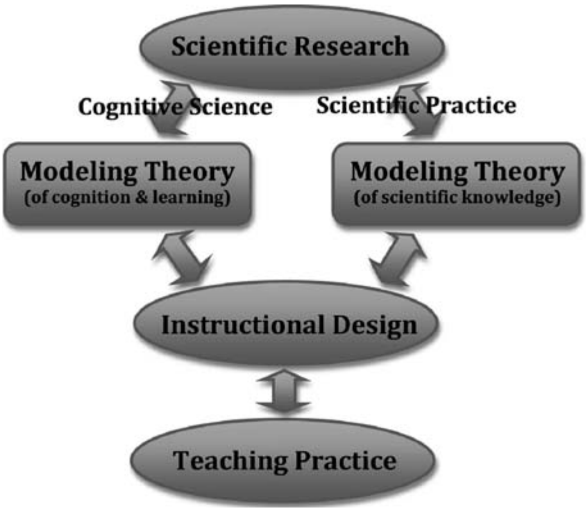

Fig. 3.1 Modeling theory – a research program

designs. One cannot make inferences from isolated facts or theoretical principles. A model can serve as inferential tool for the kind of structure it embodies.
2. *Empirical*: Models can be *directly compared with physical things and processes*. A theoretical hypothesis or general principle cannot be tested empirically except through incorporation in a model. Empirical data is meaningless without interpretation supplied by a model.
3. *Cognitive*: Model structure is concretely *embodied in physical intuition*, where it serves as an element of *physical understanding*.

The third reason is based on the Modeling Theory of cognition set forth in this chapter.

3.3 Models and Concepts

The term *model* is usually used informally (hence ambiguously), but to make crucial theoretical distinctions we need precise definitions. Although I have discussed this issue at length before, it is so important that I revisit it with a slightly different slant. I favor the following general definition:

A *model* is a *representation of structure* in a given *system*.

A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model.

We often identify the model with its *representation* in a concrete inscription of words, symbols or figures (such as graphs, diagrams or sketches). But it must not be forgotten that the inscription is supplemented by a system of (mostly tacit) rules and conventions for encoding model structure. As depicted in Fig. 3.2, I use the term *symbolic form* for the triad of elements defining a model. I chose the term deliberately to suggest association with the great work on symbolic forms by philosopher Ernst Cassirer (Cassirer, 1953).

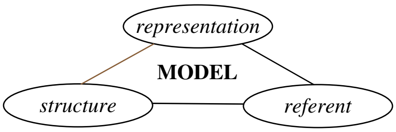

Fig. 3.2 Symbolic form of a model

We are especially interested in scientific models, for which I have often used the definition: A (scientific) *model* is a *representation of structure* in a physical system or process.

This differs from the general definition only in emphasis and scope. Its scope is limited by assuming that the objects in a physical system are physical *things*. Nevertheless, the definition applies to all the sciences (including biology and social sciences). Models in the various sciences differ in the *kinds of structure* that they attribute to systems. The term *process* is included in the definition only for emphasis; it refers to a change in the structure of a system. Thus a *process model* is an abstraction of structural change from a more complete model including objects in the system.

In most discussions of scientific models the crucial role of structure is overlooked or addressed only incidentally. In Modeling Theory *structure is central* to the concept of model. The *structure* of a system (hence structure in its model) is defined as a *set of relations* among the objects in the system (hence among parts in the model).

Universal structure types: From studying a wide variety of examples, I have concluded that *five types of structure suffice* to characterize any scientific model. As this seems to be an important empirical fact, a brief description of each type is in order here.

- *Systemic structure:* Its representation specifies (a) *composition* of the system (b) *links* among the parts (individual objects), (c) links to *external agents* (objects in the environment). A diagrammatic representation is usually best (with objects represented by nodes and links represented by connecting lines) because it provides a wholistic image of the entire structure. Examples: electric circuit diagrams, organization charts, family trees.
- *Geometric structure:* specifies (a) *configuration* (geometric relations among the parts), (b) *location* (position with respect to a reference frame)
- *Object structure: intrinsic properties* of the parts. For example, mass and charge if the objects are material things, or *roles* if the objects are *agents* with complex behaviors. The objects may themselves be systems (such as atoms composed of

electrons and nuclei), but their internal structure is not represented in the model, though it may be reflected in the attributed properties.

- *Interaction structure:* properties of the links (typically *causal* interactions). Usually represented as binary relations on object pairs. Examples of interactions: forces (momentum exchange), transport of materials in any form, information exchange.
- *Temporal (event) structure: temporal change in the state* of the system. Change in position (motion) is the most fundamental kind of change, as it provides the basic measure of time. Measurement theory specifies how to quantify the properties of a system into property variables. The state of a system is a set of values for its property variables (at a given time). Temporal change can be represented *descriptively* (as in graphs), or *dynamically* (by equations of motion or conservation laws).

Optimal precision in definition and analysis of structure is supplied by *mathematics, the science of structure*. This agrees with the usual notion of a *mathematical model* as a representation in terms of mathematical symbols.

Now, here is a *perplexing question* that bothered me for decades: If the meaning of a scientific model derives from its physical interpretation, *from whence comes the meaning of a mathematical model*? Mathematical models are *abstract*, which means they have *no physical referent!*

It dawned on me during the last decade that the emerging field of *cognitive linguistics* provides a *revolutionary answer*. Cognitive linguistics has revolutionized the field of semantics by maintaining that the actual *referents of language are mental models* in the mind rather than concrete objects in an external world. It follows that, if mathematics is "the language of science," then the *referents of mathematical models must be mental models.* Likewise, *the proper referents of scientific models must be mental models of physical situations*, which are only indirectly related to real physical systems through data, observation and experiment.

This implies a common cognitive foundation for math, science and language: Just as science is about making and using *objective models* of real things and events, so cognition (in mathematics and science as well as everyday life) is about making and manipulating *mental models* of imaginary objects and events!

Let me sum up this revolution in semantics with a modified definition:

A *conceptual model* is a *representation of structure* in a *mental model*.

As before, the representation in a conceptual model is a concrete inscription that encodes structure in the referent. However, we make no commitment as to what the structure of a mental model may represent. Henceforth, scientific and mathematical models are to be regarded as conceptual models. But the referent of a conceptual model is always a mental model, so its structure in the mind is inaccessible to direct observation. How, then, can this be an advance in Modeling Theory?

The answer is: It enables transfer from a Modeling Theory of scientific knowledge to a Modeling Theory of cognition in science and mathematics. Much is known about the structure of scientific models. We seek to solve the inverse problem of inferring the structure of mental models from the objective structure of scientific representations. If that seems like an impossible task, note that it is commonplace to infer thoughts in other minds from social interaction. Can we not make stronger inferences with the full resources of science? Here we have a modeling approach to the theory of cognition, so we can draw on the whole corpus of results in cognitive science for support and critique. I will not duplicate my previous reference to that enormous literature (Hestenes, 2007). However, I should emphasize the special relevance of cognitive linguistics and point out that two recent introductions to the field (Evans and Green, 2006; Croft and Cruse, 2004) provide a comprehensive overview that was difficult to put together only a few years ago.

Let me now propose the *First Principle for a Modeling Theory of Cognition:*

I. *Cognition is basically about making and manipulating mental models.*

I call this the *Primacy (of Models) Principle*, noting I have already tacitly invoked a variant of it for the Modeling Theory of scientific knowledge. Commitment to this principle might seem extreme, for I must admit that it is not to be found in the cognitive science literature from which I draw most of the supporting evidence. However, I contend that for a guiding research principle the standard is not that it is true but that it is *productive*, by which I mean that it leads to significant predictions that are empirically testable. Even if proved wrong, that would be quite an interesting result! In the meantime, we shall see that the primacy principle can carry us a long way.

For a start, the Primacy Principle helps sharpen the definition of a concept, as it implies that concepts must refer to mental models, at least indirectly. As done before, I define a *concept* as a *{form, meaning} pair* represented by a *symbol* (or assembly of symbols). In analogy to Fig. 3.2, I define the *symbolic form* of a concept as the triad in Fig. 3.3. Much like a model, the *form* of a concept is its conceptual structure, including relations among its parts and its place within a conceptual system. The *meaning* of a concept is its relation to mental models. All this is close enough to the usual loose definition of "concept" to conform to common parlance. It provides then a foundation for a more rigorous analysis of important concepts.

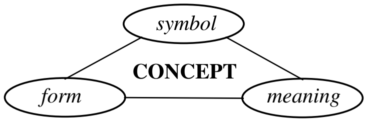

Fig. 3.3 Symbolic form of a concept

We are now prepared to propose the *Second Principle for a Modeling Theory of Cognition:*

II. *Mental models possess five basic types of structure: systemic, geometric, descriptive, interactive, temporal.*

I call this the *Principle of Universal Forms,* where the forms are the five types of structure. Obviously, this is direct transfer to mental models of the structural types identified above for scientific models. Thus, it provides us immediately with a rich system of conjectures about mental models to investigate and amend if necessary. Moreover, it brings along a rich system of basic concepts involved in characterizing the forms.

Scholars will note strong similarity of the Universal Forms to Immanuel Kant's "Forms of Intuition" and "Pure Concepts of the Understanding (Categories)" [1781, 1787]. This should not be surprising, since Kant engaged in a similar analysis of cognition with special attention to the mathematics and physics of his day. Kant proposes his Categories as a complete list of universal forms for logical inference. In Modeling Theory this should translate into universal forms for the *synthesis* (to use Kant's term) of mental models. A brief account of Kant's "transcendental" knowledge analysis is given below, but detailed comparison with Modeling Theory will not be attempted here. Today we have so much more factual information about the structure of science and cognition to guide and support our conjectures. Even so, the relevance of Kant's thinking to current cognitive science has been examined by Lakoff and Johnson (1999).

The Universal Forms are also similar to semantic structures identified in cognitive linguistics, especially in the work of Leonard Talmy (see Evans and Green, 2006). This is another rich area for comparative research that cannot be pursued here, though we shall touch on more ideas to throw into the mix.

Modeling Theory must ultimately account for the origins of structure in mental models and its elevation through the creation of symbolic forms into shareable concepts and conceptual systems. Let me comment on the second part of this ambitious research agenda. Taking for granted the existence of structured mental models in perceptual experience, we posit the human ability to make distinctions with respect to similarities and differences in model structure as the basic mechanism for creating *category concepts*.

Cognitive research has established that there are two general types of category concepts, which I shall distinguish by the non-standard terms *implicit* and *explicit* to emphasize an important point. *Implicit concepts* are determined by their mental referents, that is, they *derive meaning* from a web of associations with one or more mental models. For example, the concept *dog* derives meaning from a stored mental image of a prototypical dog. Most category concepts in natural language are of this type (Evans and Green, 2006), though my brief comments do not do justice to the subject. Implicit concepts could well be called *empirical concepts*, because their structures are built from experience in the mind of each individual.

In contrast to implicit concepts, which are grounded in private mental images, *explicit concepts are defined* by public representations. For explicit concepts, category membership is *defined by a set of necessary and sufficient conditions.* This is, of course, the classical concept of category that we inherited from Aristotle. It was only recently realized that ordinary (i.e. implicit) concepts are not of this type. Nevertheless, the crucial concepts of science and mathematics must be of the explicit type to qualify as objective knowledge.

3.4 Imagination and Intuition

Modeling Theory provides a foundation for precise definitions of important concepts in cognitive psychology. Human imagination is one such concept, important and familiar to everyone, but elusive in cognitive science. Let us reinvigorate it here with a definition that embodies the First Principle of Modeling Theory:

Imagination is *the faculty for making and manipulating mental models.*

This squares well with a well-established line of research on *narrative and discourse comprehension*, which supports the view that the *linguistic function of words is to* activate, elaborate and modify mental models of objects and events in an imaginary unfolding scene (Goldman et al., 2001). We are most interested here in the thesis that the very same cognitive process is involved in thinking mathematics and physics. To sustain that thesis we must account for the unique features of cognition in the scientific domain.

Since the latter part of the nineteenth century, mathematicians and philosophers have vigorously debated the foundations of mathematics with no sign of consensus (Shapiro, 1997). But all agree on a crucial role for *mathematical intuition*. Even the supreme formalist, David Hilbert, approvingly quoted Kant's famous aphorism: "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas." Though mathematical intuition is never mentioned in formal publications, it often comes up in informal discussion among mathematicians, and subtle hints of its presence appear in choices of mathematical terms and symbols. Recently, however, Lakoff and Núñez (2000) have dared to shine the light of cognitive science on the recesses of mathematical thought. My aim is to do the same from the perspective of Modeling Theory.

Physical intuition is privately held in the same high regard by physicists that mathematicians attribute to *mathematical intuition.* I submit these two kinds intuition are merely two different ways to relate products of imagination to the external world, as indicated in Fig. 3.4.

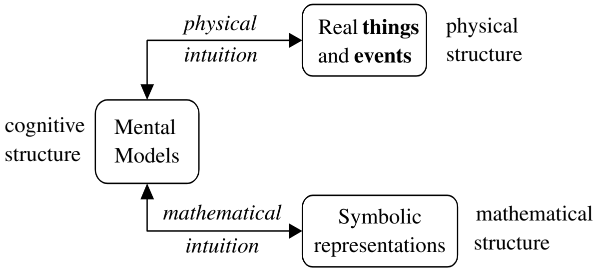

Fig. 3.4 Intuition of structure

Physical intuition matches structure in mental models with structure in physical systems. *Mathematical intuition* matches mental structure with symbolic structure. Thus, structure in the imagination is common ground for both physical and mathematical intuition.

I surmise that physical intuition is highly developed among experimental physicists, where they develop detailed mental images of experimental design, equipment, measurement procedures and data analysis. None of these abilities are involved in mathematical intuition, but theoretical physics requires integrating a good deal of both. Supporting evidence for this point will emerge as we move on.

Identification of *intuition* as the bridge between imagination and perception is a secure starting place for exploring the specifics of mathematical intuition. It would be helpful to have the testimony of proficient mathematicians to guide the exploration. Much anecdotal testimony is scattered throughout the literature, but it would take a major act of scholarship to bring it together. We shall have to be satisfied with a few telling examples.

Mathematician Jacques Hadamard (1945) surveyed 100 leading physicists and gives an introspective account of his own thinking, as well as that of others including Poincaré and Einstein. He documents two major facts about mathematical thinking: at the conscious level, much of it is imagistic without words; and, much of it is done unconsciously, with clear insights or solutions emerging with "sudden spontaneousness" into conscious thought. He does not discriminate between the thinking of mathematicians and physicists. He quotes from a letter by Einstein:

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought.... The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined.... The above-mentioned elements are, in my case, visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary state....

It is noteworthy that Einstein is famous for inventing thought experiments that proposed new relations between theory and experiment. When this is presented as evidence for his singular genius, it remains unremarked that invention of a thought experiment is an essential early step in the design of any experiment.

All this suggests that "free play of the imagination" (as Einstein put it) has a far more significant role in math/science thinking (and human reasoning in general) than is commonly recognized in educational circles. Most intuitive structure is represented subliminally in the cognitive unconscious and is often manifested in pattern recognition and conceptual construction skills. Finally, it should be noted that Einstein's description supports our view that intuition is grounded in the sensory-motor system; moreover, that ideas may be generated in the imagination before they are elevated to concepts by encoding in symbols.

Hadamard's report provides empirical support for general features of mathematical imagination and intuition, but it lacks the detail we need to describe its structure. To remedy that, we can do no better than turn to Kant's "transcendental analysis" in the *Critique of Pure Reason* (Kant, 1787). He was not a professional mathematician, but he did teach mathematics and physics for fifteen years before writing the Critique. Moreover, his analysis was greatly respected and highly influential among mathematicians throughout the nineteenth century and beyond. My attempt to present the nub of Kant's argument is indebted to clarifications by philosopher Quassim Cassam (2007).

Kant conceded to his empiricist predecessors Locke, Hume and Berkeley that all knowledge is derived from experience, but he rejected attempts to derive certain knowledge from that. Rather, he turned the problem of knowledge on its head and accepted Euclid's geometry and Newton's physics as objective facts. He then asked the trenchant question "How is such knowledge possible?" This posing of a "how-possible question" (as Cassam calls it) is the essential first step in Kant's "transcendental" approach to epistemology. He completes his argument with a multi-level answer.

Kant applies his transcendental method to a number of epistemological problems. But the test case is Euclidean geometry, as that was universally acknowledged as knowledge of the most certain kind. Thus, he asked: "How is geometrical knowledge possible?" This is just the kind of question we want to answer in detail. Kant begins his answer by identifying *construction in intuition* as *a means* for acquiring such knowledge:

> Thus we think of a triangle as an **object**, in that we are conscious of the combination of the straight lines according to a **rule** by which such an **intuition** can always be **represented**... This representation of a universal procedure of imagination in providing an image for a concept, I entitle the **schema of this concept**.

Kant did not stop there. Like any good scientist, he anticipated objections to his hypothesis. Specifically, he noted that his intuitive image of a triangle is always a *particular triangle.* How, he asks, can construction of a concept by means of a single figure "express universal validity for all possible intuitions which fall under the same concept?" This is the general epistemological *problem of universality* for the case of Kant's theory of geometrical proof. Kant's notion of geometrical proof is by construction of figures, and he argues that such proofs have universal validity as long as the figures are "determined by certain universal conditions of construction." In other words, construction in intuition is a *rule-governed activity* that makes it possible for geometry to discern "the universal in the particular."

Kant wants more. What still needs to be explained is the *capacity* of pure intuition to provide geometrical knowledge. Kant's argument leads ultimately to the conclusion that space itself is an "a priori intuition" that "has its seat in the subject only." He concludes famously that *space and time* are "a priori forms of intuition," intrinsic features of mind that shape all experience.

We need not follow Kant's argument to conclusions that have since proven to be untenable, such as the claim that geometry of the physical world must be Euclidean because our minds cannot conceive otherwise. We now know that there are many kinds of non-Euclidean geometry, and the geometric structure of space-time is an empirical matter to be settled by interplay between theory and experiment. The bottom line is that Kant's hypothesis of spatio-temporal constraints on cognition is still viable today, but it must be recognized as an empirical issue to be settled by research in cognitive science.

There is much more to be said in favor of Kant's analysis. First, his characterization of geometric intuition has been universally approved (or, at least, never challenged) by mathematicians even to present day, as it is easily adapted to any non-Euclidean geometry by simple changes in the rules. Second, his argument that *inference from the particular to the universal is governed by subsumption under rules* is a profound insight that has not attracted the attention it deserves, even, it seems, from devoted Kantian scholars. Its import is evident in Modeling Theory, for it determines a mapping of structure in mental constructions (models) to structure in drawn figures, propositions or equations. That is, rules for parsing and manipulating mental constructions correspond directly to rules for constructing and manipulating mathematical representations. This is evidently *a basic mechanism in mathematical intuition* as I defined it earlier. Moreover, it is *a means for constructing and sharing objective knowledge*, as the rules are publicly available to everyone, though it is nontrivial to learn how to employ them.

The power of rules was so evident to Kant that he posited a *faculty of judgment* to administer it: "If understanding as such is explicated as our power of rules, then the power of judgment is the ability to subsume under rules, i.e., to distinguish whether something does or does not fall under a given rule." Judgment developed into the central theme of Kant's philosophy, but in the abundance of its applications to morals, religion and aesthetics, its fundamental role in mathematical intuition and objective knowledge seems to have been lost.

We are now prepared for a more incisive comparison of mathematical and physical intuition. To begin with, the intuitive structure of Euclidean geometry is common knowledge for mathematicians and physicists. I submit, however, that their intuitions of geometry gradually diverge as they employ geometry in different ways. The mathematician concentrates on construction and analysis of formal structures. The physicist uses geometry for modeling rigid bodies and measurement of length, which is the foundation for physical measurements of every kind. Such developments in mathematics and physics do not have to go far before their common ground in Euclidean geometry is no longer obvious. With respect to the Kantian category of *causality*, intuitions of physicists and mathematicians diverge even more strongly, as we shall see.

It may be objected that our how-possible analysis of geometry is too limited for general conclusions about mathematical intuition. As a remedy, I recommend a how-possible analysis of set theory, group theory, algebra and any other mathematical system that the reader regards as fundamental. In fact, I submit that it would not be difficult and perhaps enlightening to frame the math concept analysis of Lakoff and Núñez (2000) in how-possible terms.

3.5 Mathematical Versus Physical Intuition

Let me reinforce our conclusions about mathematical intuition with testimony by Hilbert from an address delivered in 1927:

> No more than any other science can mathematics be founded on logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If

> logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from on another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else, nor requires reduction. This is the basic philosophical position that I regard as requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. This is the very least that must be presupposed, no scientific thinker can dispense with it, and therefore everyone must maintain it consciously or not.

Note the coupling between concrete signs and intuitions, with logical inference grounded on the intuitive side.

For comparison, let's hear testimony about physical intuition from an eminent physicist. Heinrich Hertz (1956) discovered the means to generate and detect electromagnetic radiation, surely one of the greatest experimental achievements of all time. He was equally accomplished as a theoretical physicist, though his tragic early death deprived the world of his genius. His profound grasp of cognitive processes in science is exhibited in the following passage (Hertz, 1956):

> The most direct, and in a sense the most important problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation.
>
> . . .We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the necessary consequents in nature of the things pictured [Predictability]. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought.
>
> . . .The images we form of things are not determined without ambiguity by the requirement of [Predictability].

[I have inserted the term [Predictability] to compress the link between his last two paragraphs.]

Hertz goes on to explain that images are constrained by certain *Conformability Conditions,* including

Admissibility: Images must not contradict the laws of thought.
Distinctiveness: Images should maximize essential relations of the thing.
Simplicity: Images should minimize superfluous or empty relations.

He adds that "Empty relations cannot be altogether avoided."

Hertz then explains that *scientific representations* (of our images) satisfy different postulates.

This passage (condensed for brevity) is studded with brilliant insights. First, note that it is consistent with Kant's account of geometric intuition (with which Hertz was surely familiar), but it surpasses Kant in original detail. Next, note how sharply Hertz distinguishes between images (mental models) and their scientific representations. He emphasizes that to have predictive value the images must satisfy certain rules, which he sharply distinguishes from rules governing their public representations. Finally, note the implication from Hertz's first paragraph that the faculty of intuition has evolved to guide effective action in the environment. This is currently a major theme in the emerging field of evolutionary psychology.

Differences between mathematical and physical intuitions emerge in meanings attributed to mathematical expressions. We often speak of mathematical symbols as though they have unique meanings that are the same for everyone. But we know that meanings are private constructions in the imagination of each individual, so tuning them to agree among individuals is a subtle social process. We have noted that public access to geometrical figures provides common ground for geometric intuitions of both mathematicians and physicists. Now let us consider an important concept where intuitions strongly diverge, namely, the concept of force.

The general concept of *interaction* has been identified as one of the universal forms of knowledge in Modeling Theory. It corresponds closely to Kant's *causality* category. Though that category is construed broadly enough to include human volition, there is no doubt that the *Newtonian concept of force* was centermost in Kant's thinking. *Force dynamics* also appears as a major category in cognitive linguistics, especially in the work of Talmy and Langacker (see Evans and Green, 2006). However, as I have explained before (Hestenes, 2007), linguistic research on the force concept has yet to be reconciled with physics education research (PER).

Divergence of student intuitions about force from the Newtonian (i.e. scientific) force concept are reliably measured by the *Force Concept Inventory* (FCI). FCI assessment on large populations of students from middle school to graduate school shows conclusively that before physics instruction student concepts diverge from Newtonian concepts in almost every dimension (Hestenes, 2007). Moreover, most students are far from Newtonian even after a year of university physics. I surmise from this that mathematics professors who have neglected physics in their education will likewise retain naïve concepts of force. To check that out, it would be interesting to test a representative sample of such subjects with the FCI. But who dares bell the cat?

Naïve concepts of force have often been dismissed as misconceptions to be replaced by the scientific Newtonian concept. But that is a serious mistake stemming from a naïve view of cognition and learning. It should be recognized instead that student intuitions are essential cognitive resources developed through years of real world experience. We understand the world by mapping events into the mental spaces of our imagination. The chief problem in learning physics is not to replace intuitions but to *tune* the mapping to produce a veridical image of the world in the imagination.

To thoroughly understand what learning the Newtonian force concept entails, we need an inventory of intuitive resources that students bring to the experience. Andrea diSessa (1993) has pioneered identification and classification of basic intuitions of physical mechanism that he calls *phenomenological primitives*, or *p-prims*.

Without going into detail that is readily accessible in the literature, I wish to explain how diSessa's theory of p-prims (or, at least, something very much like it) fits naturally into Modeling Theory. It will be sufficient to comment on the p-prims listed in Fig. 3.5.

Force and Agency	**Constraint Phenomena**
OHM'S P-PRIM	*BLOCKING*
SPONTANEOUS RESISTANCE	*SUPPORTING*
FORCE AS MOVER	*GUIDING*
DYING AWAY	
Balance and Equilibrium	
DYNAMIC BALANCE	
ABSTRACT BALANCE	

Fig. 3.5 Force p-prims (from Sherin, 2006)

Much like the image schemas in cognitive linguistics (Evans and Green, 2006), p-prims are stable units of mental structure employed in the construction of mental models. Though diSessa identified the p-prims largely from interviews of scientifically naïve students, I agree with Bruce Sherin (2001, 2006) that the same *p-prims are involved in structuring the physical intuition of mature physicists*. I regard this conclusion as a *major milestone in cognitive science*, so I will return to it after discussing the intuitive foundations.

diSessa found that, for naïve students, each p-prim is a simple, separate and distinct knowledge piece called forth for explanatory purposes by situational cues. Collectively, the p-prims compose a loose conceptual system that diSessa described as "knowledge in pieces." In contrast, Newtonian force is a complex, multidimensional concept (Hestenes, 2007). Let's consider how the p-prims can be integrated into an intuitive base for the Newtonian concept.

Many p-prims in Fig. 3.5 have familiar names. This should not be surprising, because they derive from common human experiences. However, diSessa has given the peculiar name *Ohm's p-prim* to the most important one in the lot. That does suggest a historical role in the creation of Ohm's law for electrical resistance. But its most basic role is in the intuition of force. Ohm's p-prim is schematized as an *agent working against a resistance to produce a result*. No doubt it originates in personal experience of pushing material objects, and it is projected metaphorically to other situations. diSessa notes that it serves as general intuitive schema for qualitative proportional reasoning (hence its applicability to Ohm's law by metaphorical projection). diSessa also suggests that it provides intuitive structure for the physicist's understanding of $\mathbf{F} = m\mathbf{a}$, where the result of applying a force is acceleration (but not velocity, as is common in naïve conceptions). Note the considerable adjustment in intuition required for a veridical match of mental model with physical events (in accordance with Hertz's conformability conditions). Indeed, ability to discriminate between velocity and acceleration (qualitatively as well as quantitatively) is already a major advance over naïve thinking. The upshot for a physicist is: the equation $\mathbf{F} = m\mathbf{a}$ serves as a symbolic form for reasoning about force and acceleration.

The *force as mover* p-prim holds that the response to an applied force is motion in the direction of the force, but after the vectorial concept of acceleration has been mastered, intuition can be adjusted to associate that direction with acceleration instead of velocity. It then becomes integrated into the intuition of $\mathbf{F} = m\mathbf{a}$ for the physicist.

There is an implicit principle in the Newtonian conceptual system that could be called *Universality of Force*. This principle, which holds that *motion is influenced by forces only*, is never mentioned in physics textbooks, so it is not surprising that many students who have completed a standard physics course have failed to reconcile all their p-prims (Fig. 3.5) with the Newtonian force concept. Initially, naïve students do not recognize the familiar motion effects of *resistance* and *dying away* as due to forces. Likewise, they do not associate effects known as *blocking, supporting* and *guiding* with forces. The physicist's intuition retains the p-prims for all these effects while integrating them into a universal force concept that associates forces with all these effects.

Reasonable as this account of the relation between p-prims and physics intuition may seem (to me, at least), we need more detail and stronger evidence to support it. Happily, Bruce Sherin (2001, 2006) has produced an impressive corpus of supporting evidence in a landmark study on the role of *intuitive knowledge in quantitative problems with physics equations*. Sherin's study is noteworthy not only for its originality and results, but for the quality of his data and data analysis. It provides an exemplar for a productive line of further research. Sherin's data come from videotaping pairs of moderately advanced students collaboratively solving physics problems of moderate difficulty. Consequently, it documents some behaviors characteristic of expertise along with revealing examples of ongoing learning. Readers are referred to the original articles for details. I will comment on Sherin's findings with a twist to fit them into Modeling Theory.

Sherin's first finding was supporting evidence for diSessa's conjecture that p-prims come to serve as heuristic cues for formal knowledge as expertise develops. Of course, we must distinguish situational (or verbal) cues for p-prims from their intuitive structure. The observable cues may be retained with little change, while the p-prim structure must be adapted to the expert's intuition. Modeling Theory enables a stronger inference, namely, that p-prim structure must be integrated consistently into the structure of the expert's mental model for a given physical situation. Actually, the p-prim may first cue construction of a mental model, which is in turn coordinated with construction of a formal representation for the model.

Sherin also found evidence that p-prims can drive problem solving in a fairly direct means. He looked for tuning of p-prims (refining the sense-of-mechanism) during equation construction and use. He suggests that happens when competing p-prims are cued to activation during analysis of a physical situation. Compromise then occurs when both p-prims are seen to have validity. In modeling terms this can be described as tuning the p-prims to fit a coherent mental model.

Sherin's most important finding came from observing students construct equations from ideas of what they want the equations to express. They were appealing to common sense, not just following formal rules. For example, in problems involving equilibrium of forces Sherin noted activation of *balancing p-prims* (Fig. 3.5) and strong linking of these p-prims to equations. This promoted qualitative reasoning from terms in an equation without actually solving the equation. Sherin concluded that the students were using the terms to represent p-prims; they were using equations to represent and coordinate common sense knowledge. He crystallized this brilliant insight by inventing the notion of a *symbolic form*, which he defined as a *symbol template* associated with a *conceptual schema* that specifies a few entities and relationships among them. Each template belongs to a symbolic system for representing the conceptual schema in an arrangement of symbols. His coding scheme for a catalog of symbolic forms is shown in Fig. 3.6.

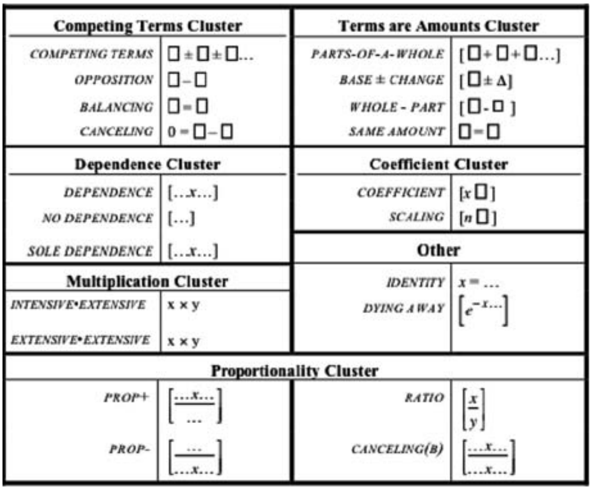

Fig. 3.6 Symbolic forms by cluster (from Sherin, 2001)

I was astounded when I first heard Sherin talk about his symbolic forms recently, for his term *symbolic form* is not only identical to the one I introduced to clarify the definition of *concept*, but its meaning and purpose strikes me as essentially the same as mine in the present context. The equivalence of terms is evidently a coincidence, but the equivalence of concepts bespeaks a convergence of independent lines of research. Consequently, it is a simple matter to merge Sherin's results into Modeling Theory, though I have not asked his permission.

The *Balancing Form* in Fig. 3.6 expresses equivalent effects of two competing influences, which is the essential structure of the *balancing p-prim*. The Form is the same whether the influences are forces or torques or whatever intuition suggests. The notion of *balancing an equation* probably originated from this p-prim. Though some mathematicians may dismiss that notion as a mathematically irrelevant metaphor, others hold that it is an indispensable intuitive foundation for understanding mathematical equality. This is an elementary instance of the tension between formalism and intuition in mathematical understanding.

The *Proportionality Forms* in Fig. 3.6 are especially important, as proportional reasoning is a critical skill in quantitative science, but it has remained distressingly difficult to teach. Sherin proposes separate forms $prop^+$ and $prop^-$ for direct and inverse proportionality, and he hypothesizes that they are strongly connected with intuitions of *effort* and *resistance* through Ohm's p-prim. This is quite interesting for many reasons. Math educators and psychologists have explored a number of ways to develop intuitive understanding of proportional relations. Historically, the first robust understanding was grounded in geometric intuition of similar triangles. Archimedes was probably the first to understand the proportionality of torques in balancing. Gradually the analogy "*a* is to *b* as *c* is to *d*" was transmuted into the proportionality symbolic form $a/b = c/d$. All of these facts are well known and influential in math education. I mention these facts because *Ohm's p-prim is not included,* so it is not among the most likely influences on prior understanding of proportional reasoning by the students that Sherin observed. Thus, Sherin has introduced new insight into cognition of proportional relations.

To reconcile the diverse intuitions of proportional relations, I suggest that Sherin's Proportionality Forms have other referents (intuitive meaning) besides Ohm's p-prim, including those in the historical list I mentioned. Multiple meanings for words are common in natural language; that is known as *polysemy* in cognitive linguistics, which has elevated it to fundamental status in linguistic theory. Accordingly, I submit that a physicist has a repertoire of many meanings that can be assigned to a mathematical form, depending on activation by situational cues. Representation of structure in diverse situations by a single mathematical form is no doubt a primary source of the great power in mathematical modeling, so instruction should be designed to cultivate it.

Most proposals for teaching proportional reasoning emphasize laying the intuitive foundation first, but Daniel Schwartz and Moore (1998) and Daniel Schwartz and colleagues (2005) argue for the reverse: using mathematical representation to refine intuition. Of course, it is fundamental in Modeling Theory that the mapping of structure between mental models and their mathematical representations goes both ways. Undoubtedly, mathematics plays a role in tuning intuition.

As a final point, Sherin suggests that "washing out of physical meaning is a fundamental feature of the move from intuitive physics to more expert knowledge." He notes what he calls a "fundamental tension" "between the homogenizing influence of algebra and the nuance inherent in intuitive physics." Though I don't subscribe to his "washing-out" metaphor, I submit that Sherin has observed a significant effect, namely, a special case of the *fundamental tension between abstract (mathematical) form and physical intuition* (or *physical interpretation*, if you will)! Let's call it the *Form-Content Tension* for future reference. I submit that this tension is basically about matching structure in mathematical models to observable structure in the world. It can be construed as tension between mathematical and physical intuition, as defined in connection with Fig. 3.4.

In understanding a physical equation, the physicist is always concerned with correlating the mathematical structure in the equation with intuitive structure in a mental model. This is known as *interpreting* the equation. The correspondence is a two-way mapping. In constructing an equation the physicist incorporates structure from a mental model of a physical situation (as Sherin observed students doing). Conversely, presuming that a given equation applies to a given physical situation, the physicist uses structure in the equation to structure a mental model of the situation. Let's call this *reading* the equation.

Sherin asserts that qualitative reasoning with equations is the hallmark of physics expertise! Perhaps so, but, as physicist Robert Romer (1993) emphasizes, reading physics equations for understanding is a prerequisite. If mathematics is the language of physics, then reading the equations of physics must be much like comprehending a narrative, namely, constructing meaning in a mental model. I submit that all qualitative reasoning is based on mental models, with terms in equations serving as cues for structure in the models. Reasoning from a mental model is necessarily qualitative. No model, no reasoning!

To be sure, equations also serve a quantitative role unlike statements in natural language. Semi-quantitative estimation and dimensional analysis are essential skills for matching models with data, much valued by physicists! However, overemphasis on the quantitative encourages students to look at equations purely algorithmically. Consequently, students often come to a first course in physics with something like a *vending machine model of algebraic equations*, wherein the variables are slots for inserting numbers and the equal sign spits out the answer. For them, equations have no more meaning than the nonsense phrase "Twas brilig and the slithy toves." For them, reading an equation is no more than reciting words. How can we engage students in making sense of equations?

3.6 Modeling Instruction

Let me quote myself on the objectives of science instruction:

> ***The great game of science is modeling the real world***, and ***each scientific theory lays down a system of rules for playing the game***. The object of the game is to construct valid models of real objects and processes. Such models comprise the content core of scientific knowledge. To understand science is to know how scientific models are constructed and validated. ***The main objective of science instruction should therefore be to teach the modeling game***. (Hestenes, 1992)

Modeling Theory has been developed with that purpose expressly in mind. Its implications for the design of curriculum and instruction have been thoroughly discussed in the literature reviewed in Hestenes (2007). Some of the highlights are reviewed here to make connection with the present chapter, which has addressed only structural aspects of scientific knowledge. Modeling Theory is equally concerned with procedural aspects of scientific knowledge, which it characterizes in terms of making and using scientific models.

Implications for Curriculum Design

- The *curriculum should be organized around models*, not topics! because models are *basic units of coherently structured knowledge*, from which one can make direct inferences about physical systems and comparisons with experimental data.
- Students should become familiar with *a small set of basic model as the content core* for each branch of science, along with selected extensions to more *complex models*.
- Theory should be introduced as a system of general principles for constructing models with a specified domain of validity.

Implications for Instructional Design
Students should learn *a modeling approach to scientific inquiry*, including

- proficiency with conceptual modeling tools
- qualitative reasoning with model representations
- procedures for quantitative measurement
- comparing models to data.

Implementation and Evaluation of Modeling Instruction
The above modeling principles for curriculum and instruction design have been fully implemented in a High School physics course, and intensive (3-week) summer workshops have been developed to train in-service teachers in the innovative techniques of *Modeling Instruction*. A series of such *Modeling Workshops* was continuously supported by the National Science Foundation for fifteen years, with unprecedented success on many measures, including student gains on the FCI, external evaluations, teacher satisfaction and buy-in. Although the Workshops are very demanding, their popularity is so great that more than 2,000 teachers have attended at least one; this is nearly 10% of all physics teachers in the U.S. Full details about the program and its evaluation are available in documents at the Modeling Instruction website (Hestenes, 2007).

A few more words about Modeling Instruction are needed to appreciate the unique features most responsible for its success. The *Modeling Method* of instruction is a student-centered inquiry approach guided by the teacher, as recommended by the National Science Education Standards. The big difference is that all stages of inquiry are *structured by modeling principles.* Typical inquiry activities (or investigations) are organized into *modeling cycles* about two weeks long.

Each investigation focuses on understanding a concrete physical system/process; for example, an oscillating block suspended by a spring. After class discussion to set the stage for the investigation, students are divided into research teams of three or four to design and carry out experiments, analyze results and prepare a report. The teacher subtly guides the entire inquiry process with questions, suggestions and challenges, introducing equipment, standard terms, conventions, and representational tools as needed. The students soon learn that the objective of the investigation is to formulate and evaluate a well-defined scientific model of the system in question. By the third time through a modeling cycle, the students have assimilated the procedural knowledge in modeling inquiry, and they proceed systematically in further investigations without help from the teacher. This leaves the teacher free to concentrate on guiding the students to a clear understanding of the conceptual structure in the models they develop. The primary guidance mechanism is *modeling discourse:* which means that the teacher promotes *framing all classroom discourse in terms of models and modeling.* The aim is to sensitize students to the structure of scientific knowledge, in both declarative and procedural aspects. Of course, *the skill and understanding of the teacher* are the main factors in success of the Modeling Method. Consequently, the Modeling Workshops are designed to promote and curriculum materials have been developed to support it.

Design of the modeling cycle needs to be described in more detail to see how modeling structure is incorporated. For instructional purposes, modeling inquiry can be decomposed into four major phases: model *construction, analysis, validation*, and *deployment*. Each phase deserves separate commentary. But it should be understood that emphasis on various phases in the cycle may vary greatly, depending on objectives of the inquiry. Moreover, the phases are not necessarily implemented in a linear order; for example, questions raised in the analysis or validation phase may lead to modifications in the construction phase.

Model construction (or *development*) incorporates into the design of a conceptual model some or all the universal forms delineated in Section 3.2. Students and even teachers are not informed of this fact. Rather, they are introduced to *representational tools* and engaged in using the tools to model structure in concrete systems. Of course, that is not unique to Modeling Instruction. Using the tools of analytic geometry to model geometric structure and differential equations to model temporal structure is commonplace and often indispensable. However, Modeling Theory coordinates application of all the various tools toward construction of a complete and coherent scientific model of any real situation. This has led to significant improvements in the conceptual process of model construction. For example, recognition that *specification of systemic structure is an essential first step* in constructing any model. That step consists of first identifying the composition and interactions of the system to be modeled, and second creating a diagram (which I call a *system schema*) to represent that information. The second part of that step is often overlooked, with the consequence that the model is representationally ill-defined. The value of system schemas such as circuit diagrams and organization charts is well-known, but system schemas are virtually unknown in such venerable domains as classical mechanics. The heuristic value of system schemas in any domain is immediately obvious to any teacher who instructs students to begin modeling or problem-solving by constructing a system schema. That is a universal solution to the common quandry of how to get started.

Model analysis is concerned with extracting information from a model, such as a physical explanation or an experimental prediction, or merely the answer to a question about the objects that are modeled. For simple linear models this phase can be relatively trivial, but beyond that it may involve solving differential equations or algebraic systems of many variables. In scientific research, model analysis may be a full time job for a theoretical physicist.

Model validation is concerned with assessing the adequacy of the model for characterizing the system/process under investigation. That may involve designing and conducting an experiment to test some prediction from the model. Or it may involve assessing consistency of the model with theoretical results or experimental facts from elsewhere in the scientific community. Students learn that the outcome of this phase must include clear answers to two questions: *What is their model, and how well does it work?* They learn gradually what constitutes good scientific answers, including theoretical limitations, sources and estimates of experimental error.

Model deployment consists in adapting a model developed in one context to characterize systems or processes in a totally different context. This serves to sensitize students to the fact that models embody universal structures that can be adapted to modeling in an essentially unlimited number of situations.

The culmination of student modeling activities is reporting and discussing outcomes in a whiteboard session. I believe this is where the deepest student learning takes place, because it stimulates assessing and consolidating the whole experience in recent modeling activities. *Whiteboard sessions* have become a signature feature of the Modeling Method, because they are so flexible and easy to implement, and so effective in supporting rich classroom interactions. Each student team summarizes its model and evidence on a small (2 × 2.5ft) whiteboard that is easily displayed to the entire class. This serves as a focus for the team's report and ensuing discussion. Comparison of whiteboards from different teams is often productively provocative. The main point is that class discussion is centered on visible symbolic inscriptions that serve as an anchor for shared understanding.

Of course, the pedagogical effectiveness of a whiteboard session depends on the skill and knowledge of the teacher. For implementing the Modeling Method, this is skill in facilitating *modeling discourse,* which has two major objectives: The first, as we have already noted, is framing discourse around models and modeling to promote structured understanding of science. The second, more subtle objective is to *engage student physical intuition for tuning to consistency with scientific concepts.* In preparation for that, the modeling workshops sensitize teachers to student intuitions (*aka* misconceptions) about force, as revealed by the FCI. They learn to amplify opportunities for students to articulate their intuitions for public comparison with scientific concepts and evidence. Whiteboard sessions have proved to be an exceptional arena for that. The teachers know that reconciling student intuition with scientific knowledge is a creative act that only students can do for themselves. The best the teacher can do is create the opportunity. From the perspective of Modeling Theory, this is instruction to promote the tuning of p-prims to be consistent with external evidence. This is where the principle of *Form-Content Tension* comes into play. Its implementation is a pedagogical art guided by a little bit of science. As Sherin (2006) says, "Instruction must nurture and refine intuitive physics, not confront and replace it, or simply build up a new set of frameworks." Physics education researchers David Hammer and Andy Elby (2003) emphasize that all students possess powerful *cognitive resources* that can be tapped by a skillful instructor. Their detailed accounts of how to do that have much in common with best practices in Modeling Instruction.

Primacy of modeling over problem solving. According to Modeling Theory, problem solving is a special case of modeling and model-based reasoning. The modeling cycle applies equally well to solving artificial textbook problems and significant real world problems of great complexity. Thus, the first step in solving a problem is constructing an explicit model of the situation implicit in conditions of the problem. The next step consists in extracting from the model an answer to the question posed in the problem. This is a special case of model analysis, and an example of model-based reasoning. The final step of "checking the answer" is a special case of model validation.

The modeling method, with its emphasis on coherence and self-consistency of the model, is especially well-suited to detection and correction of ill-posed problems, where the given information is either defective or insufficient. Moreover, students are thrilled when they realize that a single model generates solutions to an unlimited number of problems. Indeed, the Modeling Workshops teach that six basic models suffice to solve almost any mechanics problem in high school physics.

Implications of Modeling Theory for Math Education

The main problem with math education is that the link to physical intuition (the empirical source of mathematical ideas) is seriously degraded, if not broken altogether. *The problem is not with abstraction* in mathematics and mathematical modeling! *Formalization of mathematics* in terms of axioms, rules and algorithms is one of the greatest achievements of mankind, making computer modeling, simulation and data analysis possible, and facilitating construction of objective scientific knowledge

But thinking cannot be reduced to computation, and computers do not understand!! (at least not yet). *Mathematical understanding* requires development of both *physical and mathematical intuition*, which supply the essential *repertoire of mental structures* for constructing robust mathematical meaning. *Physical intuition is cognitively basic*, because it supplies the structural links to bodily experience from which all meaning ultimately derives.

I believe that the best way to address the divorce between math and science at the K-12 level is by integrating math and science instruction, especially in middle school. As pioneered by the Modeling Instruction program, workshops and instructional materials must be developed to enable teachers to enact the necessary reform. School district buy-in will be essential to permit reform. Of course, none of this can happen without substantial commitments and funding.

3.7 Conclusions

We have identified and analyzed three fundamental principles of Modeling Theory:

I. *Primacy of Models. II. Universal Forms. III. Form-Content Tension*. We have noted their non-trivial implications for the design of curriculum and instruction, with very robust implementation in Modeling Instruction. This is far from exhausting the content and implications of Modeling Theory, so let us dwell briefly on what has been omitted. Many gaps are filled in the literature already cited.

There is much more to be said about *levels of structure* in mental models. At the basic level we have models of *objects*, for cognition is fundamentally *object-oriented*, to use an expression from computer science that probably originates from reflection on intuition. No doubt the central role of objects in cognition derives from *perception*, for perception organizes sensory input into objects situated in an environment. Though objects are cognitively basic, they are not cognitively primitive; they have substructure. The catalog of cognitive primitives evidently includes p-prims and image schemas. These primitives also have structure; they are best described as structured wholes, or *Gestalts*, to use a term that suggests their origin in Gestalt perception. Turning from model substructure to superstructure, we note that mental objects are invariably situated in some mental context or *frame,* sometimes called a *script* or a *scenario* when action or change is modeled. The structure of frames and scripts that is evident in language use provides important clues to the structure of memory. Finally, at the grandest level we note the organization of concepts into conceptual systems.

There is no doubt more to mental models than we have considered so far. We have been concerned mainly with structure that can be represented by rules, as that is the kind of structure in mathematics and objective science. Let's call it *rational structure*, as it may be regarded as the foundation for rational thinking. As mathematician Saunders MacLane (1986) asserts, "Mathematics is not concerned with reality but with rule." Mental models also have subjective qualities, such as feeling or emotion, that express significance to the thinker. Emotion is known to play a crucial role in learning and memory. Its relation to rational thinking is yet to be nailed down.

To my mind, the bottom line of Modeling Theory is its implications for teaching and learning. We have seen substantial implications already, and directions for further research are clear. The most important research issues are perhaps in elucidating the mechanisms for creating, changing and maintaining mental structures. Ultimately, this reduces to research in cognitive neuroscience. But to identify brain mechanisms in cognition, it is necessary first to understand at a phenomenological level what cognition consists of.

Finally, before committing to an opinion on Modeling Theory, the reader may wish to ask: Do mental models really exist? Or are they merely figments of a theoretician's imagination? Certainly no one claims that they are directly observable, not even by introspection. The explanation that mental models are not observable because they are located (mostly, at least) in the cognitive unconscious does not answer the question of existence. Cognitive neuroscience even suggests that mental models are epiphenomena at best, for only distributed neural activation patterns occur in the brain.

To forge a scientific answer, comparison of cognitive science with elementary particle theory may be helpful. Physicists are quite confident about the existence of quarks, although quarks are not directly observable even in principle. The reason for the confidence is the explanatory power of quark theory. Likewise, I submit, the existence of mental models hinges on the explanatory power of Modeling Theory. Like quarks, mental models are theoretical constructs, and both exist in the sense that they provide coherence to diverse observations. In other words, both are *invariant objects*, invariant over a range of observations. Is that enough? Could there be more?

3.8 Epilogue: A New Generation of Mathematical Tools

The power of mathematics is derived in large part from the design of mathematical tools to think with. Like the tools of science and industry, mathematical tools are cultural creations. Many mathematicians would dispute this claim, and they may seem to be supported by standard textbooks, which give a clear impression that mathematics is a complete and permanent edifice that could hardly be improved. As evidence to the contrary, I offer a brief introduction to *geometric algebra*, introducing basic new tools with implications for the whole of mathematics. Though few mathematicians are aware of it, geometric algebra is already a fully developed unified mathematical language for all of physics (Hestenes, 2003). Though it has many advanced applications, I concentrate here on implications for mathematics education at the introductory level. First let's consider why that is important.

From the perspective of a practicing scientist, *the mathematics taught in high school and college is fragmented, out of date and inefficient*! The central problem is found in high school geometry. Many schools are dropping the course as irrelevant, and are thus oblivious to the following facts:

- Geometry is the starting place for physical science as well as providing the foundation for mathematical modeling in physics and engineering, including the science of measurement in the real world.
- Synthetic methods employed in the standard geometry course are centuries out of date; they are computationally and conceptually inferior to modern methods of analytic geometry, so they are only of marginal interest in real world applications.
- A reformulation of Euclidean geometry with modern vector methods centered on kinematics of particle and rigid body motions will simplify theorems and proofs, and vastly increase applicability to physics and engineering.

We see below how *geometric algebra* can save the day by *unifying high school geometry with algebra and trigonometry* and thereby simplifying and facilitating applications to physics and engineering.

The problem boils down to encoding the *geometric notion of vector as a directed magnitude* in suitable algebraic form. The standard concepts of *vector addition and scalar multiplication* constitute a partial encoding. What is missing in standard mathematics is a geometrically grounded rule for multiplying vectors. Here is how to fill that gap. Presume the standard concept of a real vector space and define the geometric product **ab** of vectors by the axioms:

(ab)c $=$ **a(bc)**	Associative	where $a = \|\mathbf{a}\|$ is a positive scalar called the *magnitude* of **a**, and $\|\mathbf{a}\| = 0$ implies that $\mathbf{a} = 0$.
a(b + c) $=$ **ab + ac**	Left distributive	
(b + c)a $=$ **ba + ca**	Right distributive	
$\mathrm{a}^2 = a^2$	Contraction	

These axioms are almost identical to the axioms for ordinary scalar algebra. The main difference is that we need two distributive rules because multiplication is not assumed to be commutative. It is the unassuming contraction rule that sets geometric algebra apart from other associative algebras. Among its many consequences, it implies that the zero scalar and the zero vector are one and the same:

Our main task is to elucidate the geometric meaning of the product **ab**, because that is what gives the algebra its unique power. Historically, the axioms were designed to encode geometric relations (Hestenes, 1999), so they are by no means arbitrary. We do the reverse here to take advantage of the reader's prior knowledge.

To move quickly to something familiar, we use the geometric product to define the familiar *inner product*: $\mathbf{a} \cdot \mathbf{b} \equiv \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \mathbf{b} \cdot \mathbf{a}$. And, to prove that this is identical to the usual Euclidian inner product, we use the axioms to derive the usual *law a of cosines*, thus $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + (\mathbf{ab} + \mathbf{ba}) = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b}$. So, I follows that we can write $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$ where, of course, θ is the angle between the vectors. This is, in fact, a convenient algebraic definition for the cosine.

Now, to see quickly that we have something genuinely new here, suppose that the vectors are *orthogonal*, which is to say that the inner product vanishes, whence **ab** = –**ba**. Thus, orthogonality is encoded as anticommutivity in geometric algebra. It is easy to prove also that collinearity is encoded as commutativity. But what is this new entity **ab**? It is neither scalar nor vector. To interpret it, let us first assume that **a** and **b** are orthogonal unit vectors and denote it by a suggestive symbol: **i** = **ab**. Then we can use anticommutivity to prove that: $\mathbf{i}^2 = (\mathbf{ab})^2 = (-\mathbf{ba})(\mathbf{ab}) = -\mathbf{a}^2\mathbf{b}^2 = -1$.

Thus, **i** is a truly geometric $\sqrt{-1}$. It is not a scalar, but it can be factored into a product of orthogonal unit vectors, and it can be proved that any such pair of vectors determine the same **i**. In other words **i** is a unique property of a Euclidean plain. To understand this better, we turn to the general case – where is convenient to define an antisymmetric *outer product* by $\mathbf{a}\wedge\mathbf{b} \equiv \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = -\mathbf{b}\wedge\mathbf{a}$. Next, we can assign a magnitude $|\mathbf{a}\wedge\mathbf{b}|$ to this quantity by $|\mathbf{a} \wedge \mathbf{b}|^2 = -(\mathbf{a} \wedge \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

The quantity **a**∧**b** is called a bivector, and it can be interpreted geometrically as an oriented plane segment, as shown in Fig. 3.7. It differs from the conventional

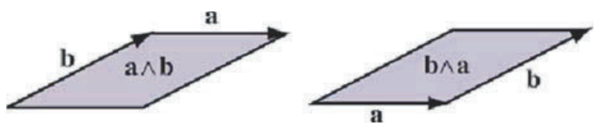

Fig. 3.7 Bivectors **a**ˆ**b** and **a**ˆ**b** represent plane segments of opposite orientation as specified by a "parallelogram rule" for drawing the segments

vector cross product $\mathbf{a}\times\mathbf{b}$ in being intrinsic to the plane. Note that the dimension of the vector space has been left unspecified, so all our considerations are quite general.

To make connection with trigonometry write

$$|\mathbf{a} \wedge \mathbf{b}| = ab \sin \theta \text{ and } \mathbf{a} \wedge \mathbf{b} = \mathbf{i}\,|\mathbf{a} \wedge \mathbf{b}| = ab\,\mathbf{i} \sin \theta,$$

where $\mathbf{i}$ has been introduced as a *unit oriented area for the plane* containing $\mathbf{a}$ and $\mathbf{b}$. Note that this can be regarded as defining $\sin \theta$.

Now we return to the geometric product note that it has the unique decomposition into symmetric and antisymmetric parts: $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$ We have noted the geometric meanings of the parts, but what is the meaning of the whole? To relate it to something familiar, we give it a symbol and a trigonometric expression: $z \equiv \mathbf{ab} = \lambda U$ where $U = \hat{\mathbf{a}}\hat{\mathbf{b}} = \cos\theta + \mathbf{i}\sin\theta = e^{\mathbf{i}\theta}$ with $\lambda = ab$. This is the familiar form for a *complex number*, with inner and outer products corresponding to real and imaginary parts. It has all the familiar properties of complex numbers fully integrated with additional properties relating to vectors. In particular, multiplication on any vector $\mathbf{c}$ in the plane of $\mathbf{i}$ rotates the vector by angle θ and rescales it by λ, as expressed by $z\,\mathbf{c} \equiv \mathbf{abc} = \lambda e^{\mathbf{i}\theta}\mathbf{c} = \mathbf{d}$.

Thus, *the product of two vectors is a complex number, which represents a rotation-dilation in a plane*. As shown in Fig. 3.8, it can be depicted geometrically as a directed arc (curved arrow), just as a vector is depicted as a straight arrow. See (Hestenes, 2003) for more details about this interpretation of complex numbers.

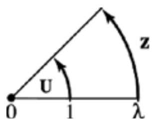

Fig. 3.8 A complex number depicted as a directed arc

We have skipped over some mathematical fine points, but the above account suffices to demonstrate that geometric algebra smoothly integrates the algebra of complex numbers with vectors. Thereby, the powerful tool of complex numbers for reasoning about rotations and plane trigonometry becomes available to students from the beginning. Thereby the artificial distinction between real and complex planes is obliterated, and coordinate-free mathematical modeling is enabled.

Geometric algebra extends all this to three dimensions and beyond. For example, it has been applied to reformulate the entire subject of Newtonian mechanics in coordinate-free form (Hestenes, 1999). This includes computation of rotations and rotational dynamics without matrices. Moreover, all this has been generalized to computations in linear algebra without matrices and applications to many other domains of mathematics.

I will not speculate here on the prospects for incorporating geometric algebra into the mathematics curriculum.

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Chapter 4
Modeling a Crucial Aspect of Students' Mathematical Modeling

Mogens Niss

Abstract Through the use of three mathematical examples, our study explores the origins of students abilities to model (a) the process in which students acquire the ability to both model and learn transferable modeling abilities across modeling activities, and (b) the way in which the modeling cycle should be characterized. We conclude by suggesting that philosophical issues are present in understanding how the modeling ability emerges in students who have never modelled. This is linked to efforts to find activities and methods that will enable better modeling capabilities to be learned.

4.1 Introduction

In theoretical or empirical terms, attempts to model what actually happens (or what ought to happen) when students engage in mathematical modeling, are typically based on one or more of the following four components: (a) the phases of the modeling process (Blum and Niss, 1991), (b) the nature of the mathematical modeling competency (Blomhøj and Jensen, 2003) (c) other mathematical competencies (Niss, 2003), and (d) the interplay among these components. Often this is supported by some version of a diagrammatic representation of the modeling process, chosen from a multitude of such representations proposed in the literature. Our considerations in what follows refer to one such diagram. As most readers of the present volume are likely to be familiar with the interpretation of modeling diagrams, which are rather self-explanatory, this is not the place to invest in further explanations of it (Fig. 4.1).

Now, let us take a closer look at one crucial – if not simply *the key* – aspect of the modeling process as depicted in the diagram: the mathematization of the situation at issue. Two focal questions are of interest to this chapter: *How is mathematization actually generated and brought about?* And: *What are the main challenges and*

M. Niss (✉)
IMFUFA, Department of Science, Roskilde University, Roskilde, Denmark
e-mail: mn@ruc.dk

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_4,

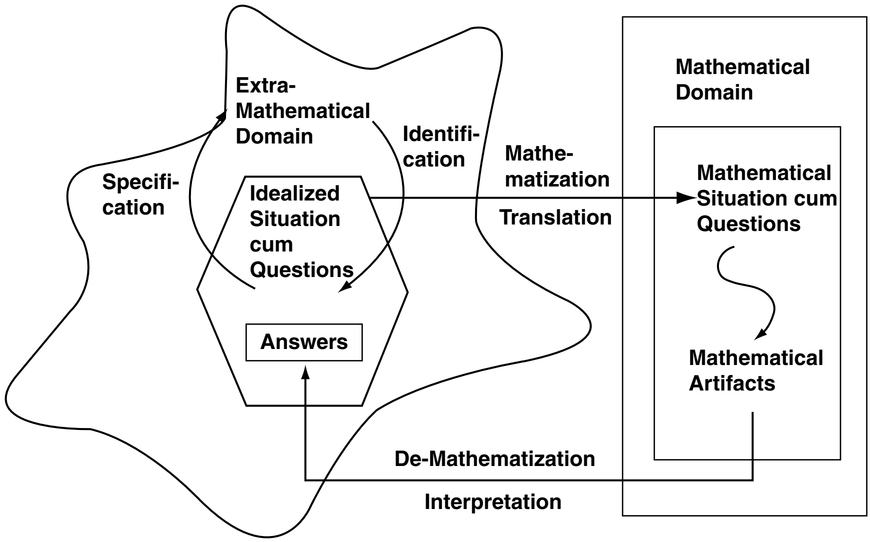


Fig. 4.1 Modeling processes

obstacles that students encounter in the mathematization part of the modeling process? More specifically, we wish to model students' work on mathematization. For reasons that will become apparent later, our deliberations will primarily be of a theoretical nature. To assist and illuminate our analysis we shall begin by considering three concrete examples.

4.2 Three Examples

All of the examples presented below are cast as descriptions of extra-mathematical situations that lend themselves to posing (and answering) a given question.

Example 1: A conic glass

Situation: Glasses in which the cup is of a conic shape are quite commonplace. At a party, people discuss how much the glass should be filled for the volume of liquid to be half of that of a full glass. One of the participants, A, suggests that the liquid should be poured to 2/3 of the full height of the glass. Another participant, B, objects by saying that while this might be true in some cases, it cannot be true in general, as the answer must depend on the opening angle of the glass. *Question*: How would you settle this discussion?

This problem is somewhat stylised, but not at all unrealistic, at least not in bars selling drinks by volume. In order to help us analyse what is in play in the modeling process, we shall look at one way to deal with the problem.

The problem situation is already of a physico-geometrical nature, since we can think of the cup as a physical realisation (undoubtedly with several imperfections) of a geometric object, a cone. So, the first part of the mathematization is almost automatic. We simply represent the inner part of the cup by a cone whose base is a circle – the inner opening of the cup – and whose height is the distance from the centre of this circle to the bottom, which we model as a single point. The straight line through the bottom point and the centre of the base circle is what we call the axis of the cone. In formal terms the opening angle of the cone is the angle between two diametrically opposite generators of the cone, i.e., the (smallest) angle between the two half-lines determined by the intersection of a plane containing the axis and the surface of the cylinder. So far, mathematization has been straightforward.

Ideally, if we neglect surface tension etc., for a full glass the liquid forms exactly this cone. For a glass half full, the liquid forms a smaller cone with the same bottom point and another inner circle (parallel to the opening circle), the radius of which is yet unknown, and whose centre has a distance to the bottom point which is also not known yet. The opening angle of the "half-full" glass is the same as for the full one (Fig. 4.2).

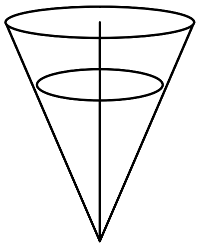


Fig. 4.2 A cone "cup"

Against this background we are now able to continue the mathematization process as follows: We represent the situation by a cut in the conic surface by a plane containing the axis, while the two liquid surfaces are represented by the base line in the resulting plane isosceles triangle and a parallel transversal, respectively. Let us illustrate this with a figure which labels the relevant pieces in the triangle: The radius of the base circle in the "big" cone, R, is half the length of the base side of the triangle. The height, H, of this cone is also the height of the isosceles triangle. Analogously, the radius of the base circle of the "small" cone is denoted by r, and the corresponding height by h. Finally, the common opening angle of the two cones is denoted by α (Fig. 4.3).

To complete the mathematization part of the modeling process, we have to specify the volumes of the bigger and the smaller cone, respectively. The volume of the big cone is one third of the area of the base circle multiplied by the height of the cone, i.e., $V = 1/3 \cdot \pi\ R^2\ H$ while the volume of the small cone is $v = 1/3 \cdot \pi\ r^2\ h$.

As the real situation has now been translated entirely into a mathematical situation, the mathematization part is complete when the question we were asking in the real situation has been translated into a question within the mathematical domain: What are the dimensions of the small cone if its volume is half that of the big cone?

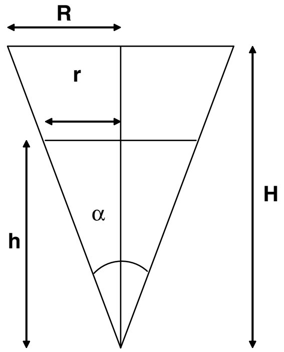


Fig. 4.3 A cut in a conic surface

To answer this question let us look for another suitable equation that relates the relevant components of the two cones. Exploiting similar triangles, we get $r/R = h/H$ which gives $r = R(h/H)$. Hence $v = 1/3 \cdot \pi R^2[h^2/H]^2\, h = 1/3 \cdot \pi h^3 R^2/H^2$

The equation we are looking for results from equating volume v with $\frac{1}{2}V$, i.e., $v = 1/3 \cdot \pi h^3 R^2/H^2 = 1/2V = 1/2[1/3 \cdot \pi R^2 H]$. Getting rid of common factors on both sides, this equation is fulfilled exactly when $h^3 = \frac{1}{2}H^3$, i.e., exactly when $h = (1/2^{1/3})H \approx 0.79H$

We observe that this result does not neither depend on the opening angle of the cone (nor – equivalently – on the radius of the large cone).

Translating this mathematical answer back to addressing the real situation, we obtain the following answer to the initial question: Neither A nor B is right. The liquid poured into the glass is half the volume of the full glass when its poured to 79% of the full height, and this holds for any opening angle of the cone.

In order to analyse this example with regard to the modeling process, let us focus especially on what it takes to be able to (a) undertake the mathematization part and (b) to obtain answers to the mathematical questions posed within the mathematical domain.

As was hinted at above, the idealisation and specification part needed in preparation of the mathematization are almost ready from the outset. This is due to the fact that the shape of the glass is given in geometric terms. So, the mathematization part is straightforward and does not require much consideration beyond terminology (cone, angle, circle, height, radius etc.) The only additional thing needed is the formula for the volume of a cone. The modeller does not even need to know this formula by heart (or to be able to derive it him- or herself). It suffices to know that such a formula exists and where to find it. Anyone to whom the initial *problématique* is engaging would probably be able to mathematize the problem this far without any difficulty. The resulting mathematical model of the situation consists of the two cones and their interrelations, including their volumes.

However, having a clue as to what to do with this model is a different matter. Dealing with the mathematized situation within the mathematical domain is not quite as straightforward. It requires the activation of several mathematical competencies and related skills.

Firstly, the modeller has to identify which components of the two cones are relevant to solving the problem. These components are exactly the ones that occur in the two expressions for the big and the small volumes, respectively. Secondly, the modeller has to relate these by setting $v = \frac{1}{2}V$, and to see that as there are four variables in this equation from the outset, it is not possible to draw any conclusion without identifying and utilising further links between the variables. Such steps are known to be very difficult to make. Solving equations, once they exist, is one thing, establishing them in sufficient numbers so as to narrow down the variability of a given problem is even more demanding and requires the solver to possess/or to look for a clear strategy to pursue. A fairly high level of mathematical insight is involved in realising that more than the equation $v = \frac{1}{2}V$ is needed to solve the problem. If R and H are taken as givens, and r and h are to be determined in terms of R and H, we need one more equation. Where to find one?

In the approach taken here, the key step is to relate the radius and the height in the small cone to the radius and height of the big cone, in some tractable way. There are several ways to do this, all of which require insight into similar triangles and resulting proportionalities. To introduce such a relation into the $v = \frac{1}{2}V$ equation requires the solver to be able to express one of the variables – here r was chosen – in terms of the remaining three variables (which in turn requires the ability to manipulate basic algebraic expressions) and then insert the outcome into the volume equation, simplify and rearrange this equation and arrive at the equation $h^3 = \frac{1}{2}H^3$, and then – activating more algebra – to make the inference that $h = (1/2^{1/3})H$, and to finally conclude that $h \approx 0.79H$. To many a non-experienced modeller this is a non-trivial exercise that requires one to keep the goals clear in mind while, at the same time, being able to deal, in a satisfactory manner, with all the technical issues that occur along the road. From the huge literature on problem solving (e.g., Schoenfeld, 1992) we know that the strategic part poses major challenges, if not obstacles, even to problem solvers who are in possession of all the knowledge and technical skills involved in carrying out the strategy.

This is an example of what we might call "classical applied problem solving". By applied problem solving I mean handling an extra-mathematical problem which has already been (pre)mathematized, either by its very nature – as in this case – or by preparations made by the presenter of the task. In other words, in classical applied problem solving, we do consider a problem sitting in an extra-mathematical domain. So by definition mathematical modeling is indeed at play but the mathematization part of the modeling process has already been completed or is quite straightforward. The real challenge lies in selecting the relevant information from the mathematized situation and in subjecting this information to mathematical treatment. Three crucial points are involved here:

- The selection of what information is considered relevant has to be based on an anticipation of a possible effective mathematical problem solving strategy.
- The selection has also to be based on the modeller's trust in his or her own ability to implement the strategy envisaged.

- The modeller/problem solver has to possess most of the mathematical knowledge and technical skills involved in implementing the strategy to a satisfactory degree.

Example 2: Day-length in Copenhagen

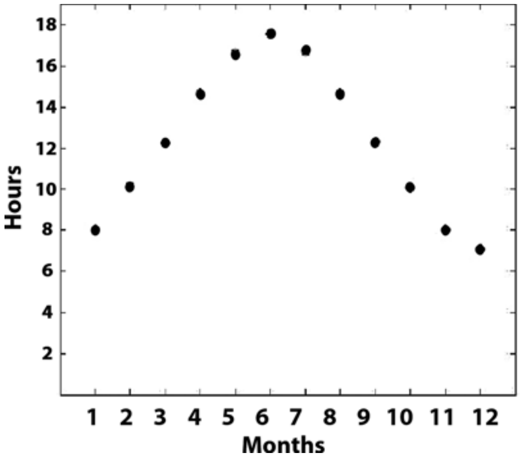
Situation: Table 4.1 is the total length of the day in Copenhagen for the 21st of each month of 2006. Each length is given in hours with decimals (in the original data set the length was given in hours and minutes)

Table 4.1 Total length of daylight hours in Copenhagen

21st of	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
	7.98	10.10	12.25	14.62	16.60	17.53	16.67	14.65	12.32	10.05	7.97	7.02

Question: What was the length of the day on 4th July 2006?

To get a feel for the situation we plot the data in a Graph 4.1, where one unit on the first axis is 1 month, so that the point k represents the 21st of month k, and one unit on the second axis represents a length of 1 h. Note that this implies that we have decided not to take the differences in month lengths from 28 to 31 days into consideration.



Graph 4.1 A graph of daylight hours

So far, by representing the data in a graph, we have only modelled the situation very preliminarily. The graph above can be perceived as a first model of the situation.

To proceed we now make four assumptions. Firstly, inspired by general astronomical knowledge, we assume that the data are a sample from a periodic function on the integers whose period is 12 months. Secondly we assume that this function is a restriction of a function of period 12 defined on the real line. Thirdly, the latter function is assumed to attain its minimum at $t = 12$ and its maximum at $t = 6$.

Finally, we go on to make the much stronger (mathematical) assumption that this function is actually an affine transformation of a sine function of an affine transformation of time, i.e., the day length function l of time t can be expressed as follows: $l(t) = a\sin(bt + c) + d$, with $a > 0$.

This is the first real step in the mathematization of the situation. The outcome is a second, general, model, which has to be specified further in order to give rise to a specific model of the situation. In other words, we have to choose a particular function from this large class of functions by deciding which parameter values to make use of. This means that we have to estimate, in some way or the other, the four parameters of l. As in the previous example this requires applied problem solving.

To that end we make use of our knowledge of the sine function. As the function attains its minimum value l_m for $t = 12$, $l(12) = l_m = l(0)$, corresponding to the minimum value (–1) of sine, (the latter part follows from periodicity), we have (since $a > 0$) that $-1 = \sin(b \cdot 0 + c) = \sin c$, from which we deduce that $c = -\pi/2$. Moreover, as l attains its maximum value (corresponding to sine attaining its maximum, 1), l_M, for $t = 6$, we have $1 = \sin(6b - \pi/2)$, which yields $6b - \pi/2 = \pi/2$, i.e., $b = \pi/6$. In other words, the function we are looking for is of the form $l(t) = a\sin(\pi/6 \cdot t - \pi/2) + d$.

We still have to estimate a and d. Utilising that $l_m = l(0) = -a + d$ and that $l_M = l(6) = a + d$, we conclude that $d = ½(l_M + l_m)$, and $a = ½(l_M - l_m)$. Combining everything we obtain our third model: $l(t) = 1/2(l_M - l_m)\sin(\pi/6 \cdot t - \pi/2) + 1/2(l_M + l_m)$. Inserting the minimum and maximum values l_m (7.02) and l_M (17.53), i.e., the day lengths of the 21st December and the 21st June, respectively, we obtain our fourth and final model:

$$l(t) = 1/2(17.53 - 7.02)\sin(\pi/6 \cdot t - \pi/2) + 1/2(17.53 + 7.02).$$

With this model in hand we can now mathematize the initiating question. As 4th July = 21st June + 13 days $\approx 6 + 13/30 = 6.433\ldots$ the mathematized question is: What is the value of l at $6 + 13/30 \approx 6.433\ldots$ The answer is straightforward: $l(6{,}433\ldots) = 1/2(17.53 - 7.02)\sin(0.5722\ldots\pi) + 1/2(17.53 + 7.02) \approx 17.39$

By the way, the registered length of the 4th July 2006 was 17.37 h. Translating our model-based mathematical answer back to the real domain we get: The length of the 4th July was 17 h 23 min.

Once again, we take a closer look at what it takes to complete these mathematization and problem solving parts of the modeling process.

In order to get an overview of the situation, the modeller begins the mathematization by plotting the data given (albeit in a slightly modified, decimal, form) in a discrete graph in a coordinate system, resulting in the first model. Here the modeller makes the assumption that variation in month lengths is insignificant in relation to the question posed. The modeller has to be aware of at least some benefits that may be derived from plotting the data in a coordinate system in addition to knowing how to do so. The latter exercise requires that dates are represented by natural numbers on the first axis, indicating the month, such that k stands for the 21st of month k. It further requires day lengths to be converted from hours and minutes into hours with decimals so as to fit with the numbers on the second axis. Although neither conceptually nor technically demanding this preparation of the data for graphical representation is not automatic but does require decision-making, overview and care.

So far, the mathematization has "only" consisted in sheer graphical representation of the situation. It is not possible on that basis alone to answer the initiating question. For that further mathematization is required. The next set of steps, involving the four assumptions made, serve the purpose of establishing day length as some periodic function of time, the graph of which fits the data points in some way or another. Our modeller knows that the sine function is a typical periodic oscillating function which is often well suited to capture exactly such relationships. Of course, this in turn requires the modeller to know something about the sine function and its fundamental properties. Otherwise this idea would never have occurred to him or her. Introducing two affine transformations in the time domain and in the length domain, respectively, so as to build in some flexibility to adjust the function to the data points, further requires some knowledge about and experience with such an approach. Expressing the situation by means of a parameterized family of transformed sine functions completes the first, most important, mathematization part of the modeling process resulting in a general model of the situation.

Now, the modeller is faced with a generic parameterized model, whose parameters have to be determined before the modeling process can be carried any further. There are several ways of proceeding. Our modeller has chosen to assume that maximum and minimum points in the data set are also the maximum and minimum points of the model function. (This might not be the case, however, and to see whether or not it is, alternative approaches involving statistical estimation techniques are available). This assumption can be justified by astronomical/everyday knowledge of the fact that the shortest and longest days of the year are the 21st or 22nd December and the 21st or 22nd of June, respectively, so that a possible difference of 1 day wouldn't matter a lot when determining the fit-function. Taking this assumption as the next point of departure allows one to deduce the exact values of two of the four parameters, b and c, right away from properties of the sine function. Again, this presupposes that the modeller knows these properties and how they can be exploited.

In that way the general function model has been specified to a two-parameter model, still too general. Utilising, once again, the properties of the sine function the modeller establishes two simple linear equations that link the two remaining parameters with the maximum and minimum values of the function. Knowing that the determination of two parameters usually requires the establishment of two combined equations is also a necessary prerequisite for the strategy adopted by the modeller. The actual solution of the combined equations presupposes some technical experience and skill in dealing with such equations. Once the equations have been solved, and known values of the maximum and minimum of the function have been inserted, the final model has been established.

On that basis the initiating question can now be mathematized and answered within the mathematical domain. For this, two steps are needed. The first – small – step is to represent the 4th July as a number on the time axis. The second step is to calculate the resulting value of the function. In addition to requiring access to the specific values of the sine function (here at $0.5722\ldots \cdot \pi$), the calculation requires either personal technical skills or access to computational devices of some kind.

In this example, too, a successful modeller following the path just outlined has to be able to combine two things: Knowledge of a couple of specific modeling instruments (graphical representation and sine functions) as a basis for devising a modeling strategy. And an anticipation of the ways in which these instruments can be employed so as to result in an answer to the initial question. The modeller has to be able to envisage, at least in outlines, the main steps involved in implementing this strategy and to stipulate whether he or she is able to carry out these steps. The main difference to the previous example is that the crucial steps in the mathematization of the situation are not at all present, let alone manifest, from the outset. The modeller has to devise a model more or less from scratch. In doing so, he or she has to look for mathematical objects that are familiar to him or her and can serve as possible modeling instruments in the context at issue. In other words, a major challenge lies in establishing the model. Along the road to the final model, the modeller has to pose and solve mathematical questions – i.e., to perform applied problem solving. Once the model has been established, mathematising the question, answering it by mathematical means, and translating the answer back to the real situation involve only computational skills and techniques, most of which can in fact be left to computers or calculators.

Imagine a modeller who does not know anything about trigonometric functions. Such a person would, of course, not be able to do what we have just done. What other possibilities would he or she have to come up with to answer the initial question by way of modeling? Well, the most obvious alternative would be to make use of linear interpolation between the day lengths of the 21st June and 21st July. Between those two dates the day length decreases by $17.53–16.67 = 0.86$ h. As 4th July is 13/30 of a month away from the 21st June, we would assume that the day length decreases by the same factor, giving $13/30 \cdot 0.86 = 0.37$.

So, based on this model the day length on the 4th July is estimated to 17.16 h against the 17.39 h obtained by the sine model (and the official length of 17.37 h). Whether or not this is a significant difference of course depends on the purpose of estimating the day length of the 4th July. However, only the person who is able to carry through both modeling processes can assess the advantage of the trigonometric model over the linear interpolation model. This is even more true of a sophisticated modeller who doesn't hinge the determination of the four parameters of the sine model on the observed extreme points but on a regression analysis. Such a modeller would be in a position to compare the three models and the resulting differences in the answers to the initial question.

Example 3: Tempering a piano

Situation and question: It is often being claimed that it is impossible to temper a piano with absolute precision. Is that really true?

First of all, it is not clear at all what mathematics has to do with this question. What does "tempering" (and "tuning") mean, what is "absolute precision" in a musical context, what do "tones", "intervals", "musical scales", etc. have to do with mathematics? So, a modeller who wants to activate mathematics to deal with this problem has to begin by settling on definitions of what all these concepts are supposed to mean. This typically happens by resorting to the literature on tones, musical scales, tempering etc. Equipped with such knowledge, the modeller specifies and idealises the extra-mathematical domain as follows:

What we want to model first is the chords in the piano, each of which sounds with a prime tone and a set of (more or less) harmonic overtones. We decide not to take these harmonics into account in our modeling here. Next, we want to model the musical scale generated by the sequence of chords of decreasing lengths (if we take the direction left-to-right on the piano keyboard). Tempering the piano then means to fix the lengths of all the chords. Of particular interest to us are classic musical intervals such as the octave, the fifth, the fourth, the third etc. A musical scale is a sequence of tones, played one after the other, that form intervals according to certain rules.

When moving on to mathematizing the situation, we want our modeling to take into account that musico-physical experiments, begun by the Pythagoreans and developed during several millennia, have shown that if we take a chord of a certain length, tightened and fixed at both ends, as the generator of a prime tone (by plucking it or beating it with a hammer), the fifth occurs if the chord is shortened to 2/3 of the original length, the octave (the same tone just "one higher") occurs if the original length is halved, and the fourth occurs if the length is shortened to 3/4 of the original length. We further want to found our modeling on a note as defined by one characteristic frequency (neglecting harmonics). Our model should also observe that playing two upward-going intervals in a row can be perceived as playing first a "local" prime tone, then a second tone corresponding to a certain fraction of the initial chord and playing the role of a new "local" prime tone, and then a third tone corresponding to a certain fraction of the previous chord. Then the third tone corresponds to a fraction of the initial chord – obtained by multiplying the two previous fractions. We shall call this "the multiplication principle". Finally, based on physical observations, we assume that if a tone corresponds to the fraction p/q (<1) of a given chord length giving rise to the frequency f, then the frequency of that tone – generated by the shorter chord – is $(q/p)f$.

Based on these considerations, made in an extra-mathematical domain consisting of a mixture of music and physics (which in itself contains mathematical features), we are now able to establish our mathematical model.

- Each tone T, is represented by a positive real number, its frequency, denoted by $f(T)$.
- A tone of pitch higher than T has frequency strictly larger than $f(T)$.
- The nearest fifth higher than T, denoted by $F(T)$, is assumed to have the frequency (1) $f(F(T)) = (3/2)\cdot f(T)$
- The nearest octave higher than T, denoted by $O(T)$, is assumed to have twice the frequency of T, i.e., (2) $f(O(T)) = 2\,f(T)$.
- A musical scale is an increasing (finite or infinite) sequence of positive real numbers (frequencies)

Against this background we are now able to specify our question within the musical domain and then translate it into a mathematical question:

On a concert piano, the last key of 12 consecutive fifths is the same as the last key of 7 consecutive octaves. (Each fifth contains 7 half-tones, each octave contains 12 half-tones, and $12{\cdot}7 = 7{\cdot}12$.) *Is this exact in the sense that the frequencies of the two final tones are the same?* The mathematized counterpart of this question then becomes:

Is $f(F(F(\ldots(T)\ldots)) = f(O(O(\ldots(T)\ldots))$, where the first term contains of 12 F's, while the last one contains 7 O's?... To answer this question within the mathematical domain we observe that (1) and (2) imply that $f(F^2(T)) = f(F(F(T))) = (3/2)f(F(T)) = (3/2)\cdot(3/2)\cdot f(T) = (3/2)^2 f(T)$, and that $f(O^2(T)) = f(O(O(T))) = 2f(O(T)) = 2\cdot 2\cdot f(T) = 2^2 f(T)$.

By induction we obtain $f(F^n(T)) = (3/2)^n\, f(T)$, and $f(O^p(T)) = 2^p\, f(T)$. Since for any $n > 0$ and any $p > 0$: $(3/2)^n \neq 2^p$, as $3^n \neq 2^{p+n}$, it is *not* the case that $f(F^{12}(T)) = (3/2)^{12} f(T) = 2^7 f(T) = f(O^7(T))$. Hence, the tempering of pianos is not exact – and can't possibly be.

By the way, the (relative) inaccuracy between the 12 fifths and 7 octaves is $(3/2)^{12}/2^7 = 531441/524288 \approx 1.0136$ i.e., 1.36%. This difference, called the Pythagorean comma, is what the well-tempered musical scale is supposed to get rid of by distributing this error multiplicatively over the 84 half-tone intervals involved, making use of the fact that the inaccuracies thus introduced are not discernible to most human ears.

What did this modeller do as regards mathematization and answering the initiating question? Unlike the two previous examples, in which the extra-mathematical situation was either in itself (pre-)mathematical or contained some obvious pre-mathematical objects and relations, the state of affairs is different in this example. Here, the extra-mathematical situation does not contain, at the face level, any hints to pre-mathematical objects whatsoever. So, the modeller has to imagine (or know) that the situation does, nevertheless, lend itself to mathematical modeling of its key (sic!) elements and features. The real-world objects selected for mathematization are those that pertain to musical temperament, such as tone, frequency, interval, and scale. Moreover, we wanted to mathematize the particular musical scales that can be represented on the keyboard of a concert piano. The model arising from these considerations represented tones (frequencies) by (some) positive real numbers and musical intervals by rational proportions between the frequencies of the first and the last tone. The introduction of the multiplicative principle adopted in the model is based on the observation that the last of three consecutive tones has a frequency relative to that of the first tone that is the product of the frequency of the third tone relative to the second one and the frequency of that tone relative to the first one.

The initiating question is a rather vague one. It requires musico-physical knowledge to focus on frequency and on fifths and octaves, taking into account that on the piano seven octaves make twelve fifths etc., and that this might provide a sufficient basis for specifying the initial question so as to make it mathematizable – and eventually soluble – within the domain of elementary number theory.

All this is indeed very involved. The main load of the modeling process lies on its intricate mathematization part, in which a large number of idealizations, assumptions, and simplifications had to be made, requiring the modeller to be able to navigate in muddled waters. Once the modeling is complete, the answering of the mathematized question only required the modeller to know/find out that no power of 3 equals any power of 2 (positive integer exponents).

4.3 The Intricacies of Mathematization

Assisted by the three examples presented above, we are now able to try to identify, in theoretical terms, the characteristics of the mathematization part of the modeling process. When students enter this part, the initial structuring – including specification and idealisation – of the situation to be modelled has been completed, either because this was already part of the initial setting (as in the conic glass and the day length examples) or as a result of the modellers own work (as in the musical scale example). The structuring of the extra-mathematical situation requires an anticipation of the potential involvement of mathematics, and the nature and usefulness of this involvement with regard to the modeling purpose. It further requires an initial anticipation of which mathematical domain(s) might be used to represent the situation and the questions posed about it. As the examples suggest, this is sometimes almost trivial, sometimes very demanding.

The outcomes of this structuring constitute the platform from which the mathematization process is to depart. This process comprises two connected pairs of sub-processes. The first pair consists of the extra-mathematical objects and relations that have been chosen for translation into mathematical objects and relations, as well as the corresponding mathematical objects, and the relations amongst them, selected to represent the extra-mathematical objects and relations. Similarly, the second pair consists of the extra-mathematical questions that are to be translated into mathematical questions, and the mathematical questions selected to represent the extra-mathematical ones. For these choices to be made, it is – once again – necessary to anticipate specific mathematical representations, suitable to capture the situation. In addition to requiring knowledge of a relevant mathematical apparatus, this also requires past experience with the capabilities of this apparatus to model other, more or less similar, situations. All this is known to be very demanding on the part of students (Blomhøj and Jensen, 2007; Kaiser and Maass, 2007; Rodriguez Gallego, 2007).

In the conic glass example, the overall framework of the mathematization was given from the outset: we model the glass cup and the liquid in it by two mathematical cones. Then, in order to select the objects and relations between them that are to be translated into mathematics, we focus on selecting objects that suffice to determine the two volumes, i.e., the radii and the heights of the two cones, taken as physical measurements. Given that the initiating question also referred to the angle of the cone, this is selected as well. The translation of the objects into mathematical ones is immediate, since physical measurements are represented by positive real numbers. The extra-mathematical questions selected for mathematization are two: “To what height should we pour liquid so as to obtain half the volume of the full glass?", and "Does the answer to this question depend on the angle of the glass, and if so, how?". Translated into mathematics, these questions become: "Under what conditions is the volume of the smaller cone exactly equal to half of that of the bigger cone?", and "Do these conditions depend on the angle of the cone, and if so, how?"

The mathematization just summarized is a result of a knowledge-based mathematical anticipation of what is needed to find the two volumes, and another anticipation of the subsequent problem solving process that requires the establishment of sufficient – and sufficiently many – mathematical links between the translated objects so as to answer the initiating questions. I propose to refer to this as *implemented anticipation* of relevant future steps, projected "back" onto the current actions.

In the day lengths example, the first basis for the mathematization is in place from the outset, as the situation is defined by a set of 12 given (and 353 unknown) day lengths for each day of 2006. So, these dates and day lengths constitute the objects – already of a (pre)mathematical nature – that we are going to mathematize further. For simplicity we decide not to account for the variations in month lengths. For each day, the day length is linked to the date. This linking constitutes the real world relations to be modelled. In the data given only twelve such links have been provided. The day lengths are given by physical magnitudes (in the first place in hours and minutes), and each day is defined by its date.

Having chosen the objects and relations to be translated into mathematics, we have to select mathematical objects and relations to represent them. The dates of 2006 are represented by positive rational numbers in the interval [0,13], such that the integers represent the given dates, while the day lengths are represented by rational numbers with two decimals in the interval [0,24]. This involves some, not necessarily trivial, further mathematization. The relations between the objects are represented by a function, where the 365 (date) numbers in the interval of definition are mapped into the y-interval. Now four mathematical assumptions are introduced to complete the mathematization. The first one is rooted in astronomical evidence, namely that this function is the restriction of a periodic function defined on all "date numbers" on the real line – considering all months to be of the same length. The second assumption is one of mathematical convenience, namely that this periodic function is the restriction of a continuous function defined on the real line. The third assumption, the strongest one, is that this function is some affinely transformed sine function, yet to be specified. So, the extra-mathematical objects and the relations between them are all translated into features of this function. The last assumption is that all the data points given lie on the graph of this function. The initiating question in the extra-mathematical domain is translated into the question "What is the value of the model function at the rational number representing the 4th July?"

In this example, too, implemented anticipation, projected back onto current actions, is crucial. It simply drives the establishment of the final model, including the applied problem solving involved in determining the parameters of this model. Once the final model is in place, its utilisation to answer the translated question is straightforward.

Implemented anticipation is of crucial importance in the piano tempering example as well. This example provides an excellent illustration of the two aspects involved in this anticipation. Firstly, the anticipation of what mathematical representations of musical tones, intervals and scales might at all be possible and relevant, is very involved indeed. It requires a fair amount of knowledge about musical phenomena and concepts and their physical roots. It further requires knowledge and intuition about potential inconsistencies in the conditions imposed on musical temperament, and the ability to pose a question that, if translated into mathematics, may be answered by mathematical means. Secondly, the mathematization at issue has to anticipate ways in which the mathematical question can be answered. Here, the latter part is the easier one, because once the mathematical question has been formulated, the answer follows rather immediately for anyone familiar with the fundamentals of elementary number theory. In other cases this part may be the most demanding one.

4.4 Modeling Students' Mathematizations

We are now ready to summarise our considerations into a theoretical model of the mathematization process. Successful mathematization requires *implemented anticipation* of a rather involved nature. At three key points in the mathematization process, the modeller has to be able to anticipate potentially difficult subsequent steps, and to implement this anticipation in terms of decisions and actions that frame the next step to be made:

(1) In structuring the extra-mathematical situation so as to prepare it for mathematization, the idealisation and specification of the situation, aimed at capturing the essential elements, features, and questions, have to be based on a first anticipation of their potential mathematization. This preparation of the extra-mathematical situation involves implemented anticipation of this potential.
(2) When subjecting the thus prepared extra-mathematical situation to mathematization, the modeller has to anticipate, in specific terms, relevant mathematical representations which are suitable for capturing the situation. This necessarily has to be based not only on the mathematical apparatus with which the modeller is familiar but also, perhaps even more importantly, on past experiences with the capabilities of this apparatus to mathematize similar situations. The resulting mathematization is an implementation of this anticipation.
(3) When anticipating the mathematical apparatus that might provide a suitable mathematization of the situation, the modeller also has to anticipate the use that may be made of the mathematization chosen and of the resulting model. Differently put, the modeller has to be able to envisage the ways in which the mathematical apparatus employed in the mathematization can provide answers to the mathematical questions posed. This implies that the modeller has to envisage mathematical problem solving strategies and procedures that will lead to a

solution of these questions. Again, the mathematization results from an implemented anticipation of the problem solving tools that can be activated after the mathematization has been completed.

The third of these steps, in which mathematical problem solving is anticipated for later implementation, is in itself very involved (Goldin and McClintock, 1984; Lesh and Doerr, 2003; Lesh and Zawojewski, 2007; Schoenfeld, 1985, 1992). However, the first two steps are in principle even more involved, as they aim at creating links, not yet established, between an extra-mathematical situation and mathematics, while at the same time anticipating the last step. The following figure is meant to represent an idealised version of the mental processes involved in mathematization (Fig. 4.4).

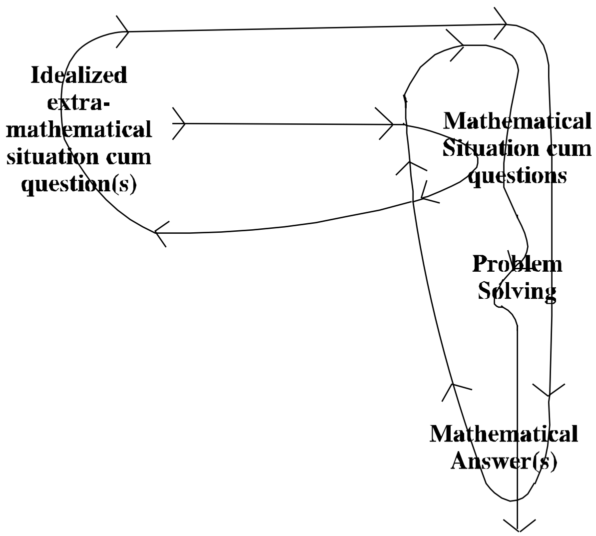


Fig. 4.4 An idealized version of mathematization processes

The key element in our proposed theoretical model of students' mathematization is implemented anticipation which presupposes

- mathematical knowledge relevant to the situation
- putting this knowledge to work for modeling
- application-oriented beliefs about mathematics on the part of the student
- "mathematical" self-confidence and perseverance.

How can students learn to anticipate putting mathematical knowledge to work in modeling before they have learnt modeling? In principle this leads to an infinite regression, leading to a learning paradox similar to that identified by Sfard (1991) in relation to the reification of process generated mathematical concepts – where an object resulting from reification of a process cannot be perceived as an object without considering it as being subjected to new processes operating on it – and Vollrath who in a more general context has considered paradoxes emerging from dualities of mathematical entities and characteristics (Vollrath, 1993).

There are basically two different approaches to the teaching of modeling aiming at overcoming these paradoxes. One is emergent modeling (Gravemeijer, 2007) and its close relative model-eliciting activities (Lesh and Zawojewski, 2007, 783ff) both of which focus on anchoring the very learning of mathematical concepts in modeling extra-mathematical worlds. Thus the paradox is supposed to be overcome by dissolving the demarcation line between pure mathematics and mathematical modeling. In the other approach, modeling (see e.g., Kaiser and Maass, 2007; Blomhøj and Jensen, 2007; Ottesen, 2001) is introduced by involving students in teacher supervised projects in which they are engaged in performing modeling tasks in situations where their existing mathematical knowledge is of relevance or where they are assisted in identifying and acquiring new mathematical knowledge potentially useful in the situation at issue. The latter approach is supposed to overcome the paradox by a stepwise reclamation of new modeling land by the students.

Now, is our proposed theoretical model of the mathematization process empirically valid? It seems that we don't really know, as only a very few empirical studies have focused specifically on the mathematization process and its intricacies. Firstly, nobody seems to have studied modellers engaged in a "post-arithmetic" modeling process outside the education system. Secondly, while there are several studies of students involved in modeling at various educational levels, most of these studies have a broader focus, at least when we are talking about the secondary and tertiary levels. Quite a few of these studies do, as an aside, provide circumstantial evidence that suggests that students do in fact experience the challenges posed by the need for implemented anticipation. However, focused empirical studies are needed to unveil the extent to which this theoretical model does in fact capture students' work with the mathematization process. Should our model turn out to be empirically valid, we would have access to an explanation of the fact that mathematical modeling does indeed constitute tough challenges for students.

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Chapter 5
Modeling Perspectives in Math Education Research

Christine Larson, Guershon Harel, Michael Oehrtman, Michelle Zandieh, Chris Rasmussen, Robert Speiser, and Chuck Walter

Abstract Too often powerful and beautiful mathematical ideas are learned (and taught) in a procedural manner, thus depriving students of an experience in which they create and refine ideas for themselves. As a first step toward improving the current undesirable situation in undergraduate mathematics education, this chapter describes several different modeling perspectives and their implications for teaching and learning.

C. Larson (✉)
Indiana University, Bloomington, IN, USA
e-mail: larson.christy@gmail.com

G. Harel (✉)
University of California, San Diego, CA, USA

M. Oehrtman (✉)
Arizona State University, Tucson, AZ, USA

M. Zandieh (✉)
Arizona State University, Tucson, AZ, USA

C. Rasmussen (✉)
San Diego State University, San Diego, CA, USA

R. Speiser (✉)
Brigham Young University, Provo, UT, USA

C. Walter (✉)
Brigham Young University, Provo, UT, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_5,

5.1 Introduction

A central and enduring problem in undergraduate mathematics education is how to structure and design learning environments so that students feel the need to develop for themselves (and in conjunction with their instructor), formal and abstract mathematical ideas. Too often powerful and beautiful mathematical ideas are learned (and taught) in a procedural manner, thus depriving students of an experience in which they create and refine ideas for themselves. A basic premise of our collective work is that theory and practice are mutually informative. Thus, to make progress on the actual teaching and learning of mathematics, one needs to deepen and further one's understanding of theoretical ideas that frame teaching and learning. In this chapter we examine different theoretical perspectives on how models and modeling can inform undergraduate mathematics education. Currently, the field is developing several such approaches, each of which is informed by somewhat different orientations. Our conjecture is that there are significant points of compatibility between these different perspectives, but also important differences.

This chapter takes a first step towards articulating and synthesizing some of these comparisons. In the sections that follow, each of the five different perspectives on modeling address the following three questions, with illustrative examples as space permits: (1) What is a model? (2) What are the research goals for students in classrooms? (3) How does the modeling perspective relate to the broad learning goals for students in classrooms? . . .Answers to these questions depend on the modeling perspective adopted.

5.2 Spesier and Walter on Models

For us, mathematics is something that one does. Specifically, one solves a problem. A model, in this context, can function as a tool to help make sense of something that we want to understand. We begin with the world, and in particular the world of the student's experience, where mathematics has the power to put order into that world and help us understand its behavior (Taylor, 1992, p. 7). The game gets interesting when learners feel impelled to build or reinvent the mathematics that might help. Our collective research focuses on how groups of learners build key mathematical ideas and understanding as they work on problems that demand new insights. To begin such work, we concentrate initially on task design in order to offer problems that give opportunities for key ideas or understandings to be built collectively. Given such tasks, our research concentrates on how groups of learners explore, reason, and communicate.

For concreteness, here is an example (Speiser and Walter, 2004) from undergraduate first-semester calculus in which students are provided enlarged copies of the photograph in Fig. 5.1. Students also were given metric rulers and a suggestion that they measure distances in millimeters and angles in radians. In this way, they had to choose an origin and polar axis, then decide for themselves how to meaningfully

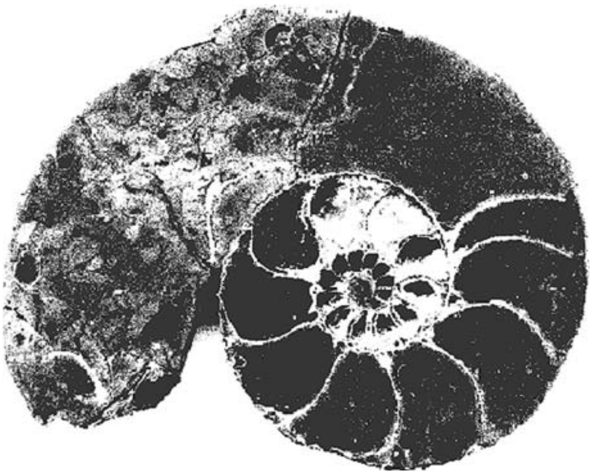

Fig. 5.1 *Placenticeras*. A fossil ammonite, 170 million years old, from Glendive, Montana. About half actual size. (Photo: Two Samurai Graphics)

organize and interpret the data to be investigated. In their text (a preliminary version of Hughes-Hallett et al., 2002), the students already had seen exponential functions, and they knew how to look for constant ratios to make sense of them. Working in groups, they built three models in succession.

Task (Speiser and Walter, 2004): It is natural to represent the spiral of this shell in polar coordinates (r, θ). What can you say about the relation between r and θ?

A consensus rapidly emerged that r should vary exponentially with θ. Hence they proposed $r = r_0 a^{\theta}$ as a first model, with reasonable values for the two constants, r_0 and a. Inspecting graphs (software at hand) most students thought they found reasonable fits. One student, however, thought this fit, though reasonable, might still deserve examination. "Have you noticed," he asked, "that the data points kind of snake back and forth across the exponential?" The students graphed the difference between the data points and what their model had predicted. They recognized an oscillation, whose period might well have been 2π. Hence it made sense to modify the initial model by adding a term of the form $b \sin(\theta + c)$, for suitable constants b and c to be determined from their data. For this second model, each group reported a somewhat closer fit. But then came *a* surprise: each group had found *significantly different* constants b and c.

Let's pause a moment to take stock. If we simply take the measured r-values *as given*, then each group obtained a closer fit with a model of the form $r = r_0 a^{\theta} + b \sin(\theta + c)$. Hence, perhaps, we should have stopped right there. (*Rationale*: the model fits the data.) To stop, however, would have left unresolved an interesting question: *Why* did different groups find different b and c? Could each group's pair of constants b and c depend on where one chose to place the origin? With this possibility in mind, each group used the specific b and c they had determined to *relocate the origin* and on this basis reconstruct their data. This relocation led to a third model, of the same form $r = r_0 a^{\theta}$ as before, but in each case with new r_0 and a. These final models gave the closest fits so far obtained (Speiser and Walter, 2004; Speiser et al., 2007; Speiser and Walter, in press).... We explore important psychological and epistemological implications of our particular approach to models in a further chapter in this volume.

5.3 Harel on Models

Rather than asking, "What is a model?", the DNR[1] theory that I have developed asks, "What is Mathematics?" The answer to this question determines in large part my research goals, both locally (Question 1) and globally (Question 2). *DNR-based instruction in mathematics* is a theoretical framework that can be thought of as a system consisting of three categories of constructs: *premises* – explicit assumptions underlying the *DNR* concepts and claims; *concepts* – referred to as *DNR determinants; and instructional principles* – claims about the potential effect of teaching actions on student learning.

Central to *DNR* is the distinction between "ways of understanding" and "ways of thinking." "Ways of understanding" refers to a cognitive product of a person's mental actions, whereas "way of thinking" refers to its cognitive characteristic. Accordingly, mathematics is defined as the union of two sets: the set WoU, which consists of all the institutionalized ways of understanding in mathematics throughout history, and the set WoT, which consists of all the ways of thinking that characterize the mental acts whose products comprise the first set.

The members of WoT are largely unidentified in the literature, though some significant work was done on the problem-solving act (e.g., Schoenfeld, 1985; Silver, 1985) and the proving act (see an extensive literature review in Harel and Sowder, 2007). The members of WoU include all the statements appearing in mathematical publications, such as books and research chapters, but it is not listable because individuals (e.g., mathematicians) have their idiosyncratic ways of understanding. A pedagogical consequence of this fact is that a way of understanding should not be treated by teachers as an absolute universal entity shared by all students, for it is inevitable that each individual student is likely to possess an idiosyncratic way of understanding that depends on her or his experience and background. Together with helping students develop desirable ways of understanding, the goal of the teacher should be to promote interactions among students so that their necessarily different ways of understanding become compatible with each other and with that of the mathematical community.

Since mathematics, according to the above definition, includes historical ways of understanding and ways of thinking, it must include ones that might be judged as imperfect or even erroneous by contemporary mathematicians. The boundaries as to what is included in mathematics are in harmony with the nature of the process

[1] DNR stands for duality, necessity, and repeated reasoning (Harel and Sowder, 2007).

of learning, which necessarily involves the construction of imperfect and erroneous ways of understanding and deficient and faulty ways of thinking. These boundaries, however, are not to imply acceptance of the radical view that particular mathematical statements could be true for some people and false for others.

My definition of mathematics implies that an important goal of research is to identify desirable ways of understanding and ways of thinking, recognize their development in the history of mathematics, and, accordingly, develop and implement mathematics curricula that aim at helping students construct them. As an example, I mention the algebraic invariance way of thinking. With it, students learn to manipulate symbols with a goal in mind – that of changing the form of an entity without changing a certain property of the entity. Another example involves the role of Aristotelian causality, as a way of thinking, in the development of mathematics. This raises the question of whether the development of students' proof schemes parallels those of the mathematicians of these periods. An answer to this question would likely have important curricular implications.

Pedagogically, the most critical question is how to achieve such a vital goal as helping students construct desirable ways of understanding and ways of thinking. DNR has been developed to achieve this very goal. As such, it is rooted in a perspective that positions the mathematical integrity of the content taught and the intellectual need of the student at the center of the instructional effort. The mathematical integrity of a curricular content is determined by the ways of understanding and ways of thinking that have evolved over many centuries of mathematical practice and continue to be the ground for scientific advances. To address the need of the student as a learner, a subjective approach to knowledge is necessary. For example, the definitions of the process of "proving" and "proof scheme" are deliberately student-centered (see Harel and Sowder, 1998). It is so because the construction of new knowledge does not take place in a vacuum but is shaped by one's current knowledge. What a learner knows now constitutes a basis for what he or she will know in the future. This fundamental, well-documented fact has far-reaching instructional implications. When applied to the concept of proof, for example, this fact requires that instruction takes into account students' current proof schemes, independent of their quality. Despite this subjective definition *the goal of instruction, according to DNR, must be unambiguous – namely, to gradually refine current students' proof schemes toward the proof scheme shared and practiced by contemporary mathematicians.* This claim is based on the premise that such a shared scheme exists and is part of the ground for advances in mathematics.

5.4 Larson on Models

What is a Model? Lesh and Doerr (2003) characterize models as

> . . .conceptual systems (consisting of elements, relations, operations, and rules governing interaction) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other systems(s) – perhaps so that the other system can be manipulated or predicted intelligently. (p. 10)

Or, even more specifically, Lesh et al. (2000) provide the following definition of a model:

> A model is a system that consists of (a) *elements*; (b) *relationships* among elements; (c) *operations* that describe how the elements interact; and (d) *patterns* or *rules* . . . that apply to the relationships and operations. However, not all systems function as models. To be a model, a system must be used to describe another system, or to think about it, or to make sense of it, or to explain it, or to make predictions about it. (p. 609)

According to this models and modeling perspective (MMP), the term *model* is used to refer to the conceptual system with which people make sense of their experiences (Lesh and Doerr, 2003). Models may be expressed externally using any combination of representational media ranging from physical actions (including gestures), to spoken words, to written words or symbols (including metaphors and equations), to images (including pictures, graphs, tables, and diagrams).

MMP Research Goals. Characterizing the research goals of an entire community of researchers who identify themselves with this theoretical perspective is a daunting and perhaps, in some sense, dangerous task. I do not claim to completely and exhaustively articulate the research goals of the broader community of researchers associated with and influenced by MMP. Rather, I offer a set of general impressions that are a reflection of my experiences as a graduate student and researcher who has spent several years working with Dick Lesh.

MMP is a broad perspective, and thus has the potential to inform a wide variety of research questions. Much of the work that has been done using MMP focuses on problem solving and local conceptual development (heavily influenced by Piaget), teacher development, and the nature of mathematical knowledge needed by students to succeed in an increasingly technologically sophisticated society. Central to much of this work has been the use and development of so-called *model-eliciting activities* (MEAs) which are "real-life" tasks that are designed to require students to invent, refine, and generalize powerful mathematical constructs (Lesh and Doerr, 2003). An MEA is a task that is carefully designed to act as a research tool to help document student thinking. In particular, these tasks are designed so that students are able to test the quality of their own solutions (and thus revise those solution strategies as they deem necessary), and so that students' solutions provide descriptions of the ways of thinking that they used to solve the problem. Thus, solutions provide trails of documentation of students' thinking (Lesh et al., 2000).

M&M Learning Goals. A central goal of MMP research is to facilitate students development and refinement of their own abilities to mathematize, ". . .by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities" (Lesh and Doerr, 2003, p. 5). This perspective places a particular emphasis on the importance of presenting students with opportunities to cyclically improve the mathematically significant systems that they develop to reason with and about "real-life" situations such as the contexts described in particular instances of model-eliciting activities.

5.5 Oehrtman on Models

My research has drawn significantly from Max Black's interactionist theory of metaphorical attribution to identify and characterize mental models constructed by students as they reason about new mathematical ideas. Employing a design research methodology, I assess which sets of student metaphors are amenable to instruction to form more mathematically rigorous and powerful conceptual foundations.

Black's description of strong metaphors, those accounting for new ways of understanding, differs slightly from recent influential characterizations of metaphor by cognitive linguists. Lakoff and Johnson (1980), Lakoff (1987), and Lakoff and Núñez (2000), for example, describe conceptual metaphors as mappings in which abstract concepts are generated and made meaningful through the projection of pre-conceptual structures from domains of "embodied experience." Black refers to this type of projection as a "theoretical model" (1962). This is the nature of reasoning employed, for example, by Clerk Maxwell in invoking the image of motion in an incompressible fluid to represent his ideas about an electrical field. Such a model provides a well-understood source domain (e.g., fluids) that is imagined to be isomorphic with respect to certain structures and properties of the new scientific domain (e.g., electrical fields) so that inferences may be transferred.

Black draws a distinction between theoretical models and metaphorical thinking in that the creator of a theoretical model "must have prior control of a well-knit scientific theory if he is to do more than hang an attractive picture on an algebraic formula. Systematic complexity of the source of the model and capacity for analogical development are of the essence" (Black, 1962, p. 239). This description of scientific systematicity is far stronger than what most introductory calculus students are likely to display with respect to building models for complex ideas such as limits. Furthermore, typical calculus instruction does not expect students to develop the solutions to (or be engaged by) the types of technical problems that gave rise to the formal structures that are the targets of such expert mappings.

Metaphorical attribution generating new ways of understanding does not simply involve antecedently formed concepts of the domains involved, but is achieved through an implicative dialectic between conceptual domains (Black, 1962, 1977, 1979). Thus, in identifying student metaphors, I focus on characterizing emerging concepts through changes in ways of viewing both the source and target domains. This perspective on metaphorical reasoning is consequently closer in form to characterizations of "conceptual blends" by Fauconnier and Turner (2002) than it is to the projections of conceptual metaphors.

Central properties of "strong" metaphors drive my methodological practices. Strong metaphors are "emphatic," commanding commitment to their particular structure. As a result, I look for a convergence of repeated application of a metaphor to any given concept, applications across a variety of situations involving the concept, and use of similar metaphors by a large number of students. Strong metaphors are also "resonant," supporting a substantial degree of implicative elaboration. Thus I also require the presence of conceptual consequences of the application of the metaphors. Finally, they are "ontologically creative," establishing new perspectives that would not have otherwise existed for the student. To capture this aspect of students' reasoning, I seek evidence that the claims being made within the application of a particular metaphor are unique to that way of reasoning.

The design research component of my work may best be described as choosing student metaphors that have potential to be restructured to guide students in more rigorous mathematical reasoning and developing, evaluating, and refining curricular activities that help students incorporate these new structures into their reasoning in a systematic way. In terms of Max Black's categorizations, I aim to assist students in converting their "metaphorical thinking" into "theoretical models."

As an example, consider the metaphor of limit as approximation. An expert version of such a metaphor applied to the limit of a sequence might include

- the limit, L, as a quantity to be approximated,
- the terms, a_n, as approximations,
- the magnitude of the difference, $|L - a_n|$, as the error
- epsilon as a bound on the size of the error, $|L - a_n| < \epsilon$,
- the condition $n > N$ for some N as a way of controlling the size of the error,
- convergence as a statement about obtainable accuracy,
- Cauchy convergence as a statement about obtainable precision, etc.

While students use many aspects of this structure in their own reasoning, their spontaneous approximation metaphors contain only fragments of these ideas plus idiosyncrasies such as approximations "always becoming more accurate" and errors being "mistakes" (Oehrtman, 2007). Research on the implementation of our activities indicates that using approximation as a thematic foundation for calculus instruction does help students develop more refined versions of these metaphors and systematic control over their use (Oehrtman, in press). Ongoing research also indicates that this approach strengthens students conceptual understanding of the Riemann integral (Sealey and Oehrtman, 2005, 2007), and subsequent studies are planned to investigate its impact on students' development of other central concepts in calculus defined in terms of limits such as continuity, derivatives, and series.

5.6 Rasmussen and Zandieh on Models

What is a model? We define models to be student generated ways of organizing their mathematical activity with physical and mental tools. Thus, models are not simply the "things" that one uses (e.g., a graph, an equation, etc.) but rather the ways in which learners structure their activity with and conceptions of graphs, equations, and even definitions. Moreover, from a Realistic Mathematics Education (RME) point of view, there is not just one model, but a sequence of models that develop (or emerge) through student activity. This progressive characteristic is typically referred to as the *Emergent Model* heuristic. As described by Gravemeijer (1999), students first develop *models-of* their mathematical activity, which later become *models-for* more sophisticated mathematical reasoning. This model-of/model-for transition is commensurate with the creation of a new mathematical reality (for learners).

Zandieh and Rasmussen (2007), drawing on their work with undergraduate students in a proof-oriented geometry class, extend the construct of emergent models to the activity of defining. In a case study that involves student work with both the plane and sphere, they detail the evolution of models in terms of four layers of activity, referred to as Situational, Referential, General, and Formal (Gravemeijer, 1999). The table below defines each of these four layers and gives a brief example in the case of defining.

Layers of activity	Defining example
Situational activity involves students working toward mathematical goals in an experientially real setting	Students create a definition for triangle on the plane and in the process revisit some of their concept images for planar triangles. Planar triangles are, for these students, experientially real in the sense that they have rich concept images of planar triangles
Referential activity involves models-of that refer to physical and mental activity in the original task setting	Students use the planar triangle definition (slightly modified for the sphere), along with the properties that they associate with this definition, to create examples of spherical triangles and to notice some of their properties. Students' organizing activity with the definition and associated concept images of planar triangle applied to the sphere functions as a model-of (or definition-of) the relevant physical and mental activity in the plane
General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting	Students' organizing activity with refined definitions and concept images of spherical triangles functions as models-for (or definitions-for) enlarging the new mathematical reality of spherical triangles, and making generalizations about these objects in ways that do not refer to the plane
Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require the support of prior models-for activity	Students reason about spherical triangles in ways that reflect new structural relationships between these objects and consequently use definitions as links in chains of reasoning without having to revisit or unpack the meaning of these definitions

What are our research and learning goals? We view our research as falling within the larger category of Design Research (Cobb and Gravemeijer, in press). Design Research has a dual focus in which research on teaching and learning guides the implementation of the instructional design, and the implementation of the instructional design provides the data for both an ongoing and a retrospective analysis. There are several different types of instructional design products that this process has the potential to generate. One overarching product is a local instructional theory (LIT) specific to the teaching and learning of a specific topic area, such as linear algebra or differential equations. A LIT is an essential support for teachers. This is not meant to be a step-by-step instruction for the teacher but rather a "description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic (Gravemeijer, 2004, p. 107)." For us, a LIT includes four related and revisable aspects: (1) learning goals about student reasoning, (2) a storyline of how students' mathematical learning experience will evolve, (3) the role of the teacher in the storyline, and (4) a sequence of instructional tasks that students will engage in.

These somewhat pragmatic products develop concurrently with our broader, theoretical goals. For example, in ongoing work we have three core research goals: (1) To create theoretical means for understanding and interpreting student learning (from both cognitive and sociocultural perspectives) as it relates to the model-of/model-for transition, (2) To develop methodological approaches that enable these kind of analyses, and (3) To apply these analyses to instructional design in the teaching of undergraduate mathematics in general, and in linear algebra in particular.

The strength of this work is the tight integration of basic research on student learning and instructional design work with students in their normal classroom settings to develop products that are both practical and theoretical.

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Section 2
Where Are Models & Modelers Found?

Chapter 6
Modeling to Address Techno-Mathematical Literacies in Work

Richard Noss and Celia Hoyles

Abstract We report the findings of a three-year investigation aimed at characterizing and developing the "Techno-mathematical Literacies" (TmL) needed for effective practice in modern, technology-rich workplaces. We characterize TmL as required mathematical knowledge that is shaped in terms of how it is expressed by the systems that govern workplace practices. Put simply, we wanted to know just what kinds of mathematical knowledge modern workplaces require, and to address this knowledge gap through co-designing with employers and employees, modeling activities that made visible and manipulable, the underlying mathematics of the artifacts that characterized the workplaces. We focused on four sectors: Financial Services, Pharmaceuticals Manufacturing, Packaging and Automotive Manufacturing.

Note
All the work reported here was carried out in close collaboration with our colleagues, Dr. Arthur Bakker and Dr. Phillip Kent.

Since the submission of this chapter, much of the material reported has been drawn together in an elaborated and comprehensive form: see Hoyles et al. (in press).

For a succinct overview of the project's findings, see Hoyles et al. (2007).

6.1 Introduction

There has been a radical shift in the mathematical skills required in modern workplaces, particularly those that have had to become more responsive to customer needs, and the demands of increasing global competition. Yet this change has yet to

R. Noss (✉)
London Knowledge Lab, Institute of Education, University of London, London, UK
e-mail: r.noss@ioe.ac.uk

C. Hoyles (✉)
London Knowledge Lab, Institute of Education, University of London, London

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_6,

be fully recognised by the formal education system or by employers. With the ubiquity of IT, employees now require new kinds of mathematical knowledge that are shaped by the systems that govern their work. Employees often lack these critical skills, and companies struggle to improve them.

We report the results of a three-year research project, investigating the nature of mathematical, statistical and technological skills that are needed in workplaces, based on field research in companies in both manufacturing and service industry sectors, and a series of prototype learning interventions within companies. We have proposed a new formulation for understanding the requirements of and roles for mathematics in work, "Techno-mathematical Literacies", TmL. Our research effort has been to characterise TmL in a variety of contexts, and to develop ways to support employees in developing appropriate TmL, which involves further elaboration of the characterisations through the design experiment methodology. This chapter will focus on the latter goal; for more on the former, see (Bakker et al., 2006; Hoyles et al., in press; Kent et al., 2007; Noss et al., 2007).

The idea that discipline knowledge such as mathematics is *transformed* in the transition to knowledge-to-be taught has been recognised from various perspectives. Two particularly pertinent examples are Bernstein's (1990) notion of pedagogic recontextualisation, and Chevellard's (1991) concept of "didactic transposition". Bernstein describes how school knowledge is transformed from academic knowledge to school knowledge, producing a new discourse with distinctive principles of selection, ordering and focusing. Chevellard's notion similarly analyses how subject knowledge is transformed into school knowledge, allowing us to interrogate the process by which disciplinary transformations take place. Both these viewpoints afford insight into the recontextualisation of knowledge as "it" crosses boundaries between communities. For example, they illuminate why traditional "training", in terms of standard mathematical skills (such as percentages and compound interest calculation, in the financial context described below), is largely ineffective in workplace settings. It simply fails to take account of the inevitability of recontextualisation by theorising (mostly implicitly) the transfer of knowledge from workplace to training or from school to workplace as unproblematic.

Our interest is, in some sense, a generalisation of these ideas – not so much looking at the boundary between school and workplace (although one outcome of our research may be to illuminate just that) but on the boundaries that inhere within workplaces, and between workplaces and those (e.g., customers) who interface with them. A promising notion for this purpose is *boundary crossing* (cf., Tuomi-Gröhn and Engeström, 2003), to which we add Star and Griesemer's (1989) notion of a *boundary object*, an object that can serve to coordinate different perspectives of several social worlds or communities of practice. Boundary crossing occurs when such boundary objects are used or developed across the boundaries of different activity systems and facilitate communication between them.

Wenger (1998) has used the concept of boundary object to analyse the working and learning practices in a medical insurance company where claim forms are boundary objects that are passed between different constituencies (such as customers) and communities of practice (different departments in the company).

Engeström (2001) and his colleagues have employed the term boundary crossing for their method of interventions that aim to change the organisation of work in companies; the core idea of their Boundary Crossing Laboratory is to (re)negotiate the objects of activity and thus to deal with contradictions that are disruptive to the existing work organisation. In this way, new artefacts or instruments such as a novel schedule for handling patients within a city healthcare system can be developed (Engeström, 1999); these can be seen as boundary objects, although Engeström himself does not employ this term.

Our focus is on mathematical knowledge, and our interest is on trying to understand how *symbolic* artefacts either do, or can be modified to, achieve boundary crossing. We are particularly interested in the questions of how employees can learn to "situate abstractions" (see Noss and Hoyles, 1996), such as the symbolic outputs of computational artefacts in the workplace, so they can become part of the work process, and a means of communication between groups. How can we support employees in developing the types of mathematical knowledge and skills that will actually benefit them in their work, and what kinds of learning are likely to be effective? These are, at present, under-researched questions, as most accounts of situated learning are framed by interpretive and non-interventionist research frameworks, with the notable exception of Engeström's (2001) Boundary Crossing Laboratory.

Before we describe our methodology, it is appropriate to define the idea of techno-mathematical literacies, TmL, an idea that emerged from earlier work (e.g., Hoyles et al., 2002; Noss et al., 1999), and which has become a fundamental idea within the present research. The prefix *techno* aims at highlighting the ways in which mathematics enters into the workplace activity system, simultaneously driving the models that underpin so much of the workings of current information technology systems, and yet largely invisible to those who use them. Work practices increasingly involve quantitative or symbolic data processed by IT, as part of the interactions between employees in different areas of work, and between employees and customers. By *literacies* we suggest an understanding of the role of "techno-mathematics" as a cultural tool mediating the social practices within workplaces, as much as we have understood the pervasive, mediating role of conventional literacy (reading and writing) for the past century and more. Thus being able to reason with these symbolic data, and to integrate such reasoning into decision-making and communication, is at the core of TmL.

6.2 Methodology

The research was divided into two phases: the ethnographic phase and the design/implementation phase. In Phase 1, we carried out ethnographic case studies in ten companies. We progressively focused on probing the meanings held by different groups involved in the work process, of the symbolic artefacts that were supposed to convey information between groups, such as computer input and outputs, production statistics, statistical analyses of processes, and other paper or computer-based documents that contained symbolically-expressed information.

These artefacts were putative boundary objects, although the question of how successfully they fulfilled this role in bridging meanings across communities was precisely at issue. We probed how these boundary objects were interpreted by different communities within the workplaces and how their meanings were communicated between them. Methods used included work-shadowing, analyses of documentation and semi-structured interviews with managers and employees in different positions of responsibility. We also joined team meetings and process improvement teams (in manufacturing) and eavesdropped on conversations with customers (in finance) to ascertain if and when problems of communication arose and how employees reacted to these challenges.

In Phase 2, we carried out iterative design-based research with our employer-partners, to design learning opportunities aimed at developing among our target group the TmL identified in the first phase. Learning opportunities incorporated interactive software tools – *technology-enhanced boundary objects* (TEBOs) that modelled elements of the work process or were reconstructions of the symbolic artefacts from workplace practice. All the TEBOs could be accessed through a web browser to ensure maximum availability in workplace settings. The learning opportunities were embedded in activity sequences largely derived from authentic episodes recorded in our ethnographic studies or reported by employer-partners. The tools and activities aimed to allow exploration and discussion at different levels of abstraction, of the normally-invisible models; thus to facilitate boundary crossing around the symbolic artefacts.

The effectiveness of our approach to TmL development was assessed through follow-up interviews with participants (face-to-face, telephone and email) and continuing communication with employer partners to assess its impact on workplace practices. The design-based approach, with its iterative cycles of collaborative design, testing and revision also led us to more nuanced understandings of TmL and their use in the workplace, since employees' interactions with our software tools opened further "windows" on their appreciation of underlying models.

6.3 Findings

In this short chapter, we will merely give a flavour of our results in two sectors and provide an example within each: (a) in manufacturing industry we choose an example based on our work in an automotive plant and (b) in the financial sector, our example involves communicating information about mortgages to customers. We begin, however, with the somewhat unusual step of stating our main results in four succinct points, in the hope that this will assist in orienting the reader in what is a set of findings, based on a complex mix of data.

6.4 Results

1. Techno-mathematical Literacies (TmL) are new skills needed in IT-rich workplaces that are striving for improvements in efficiency and customer

communication. It is simply not true that the IT presence removes the need for facilities to interpret and understand the output of the technologies: on the contrary, new kinds of knowledge, new ways of acquiring it, and new conceptualisations of workplace learning, becoming increasingly necessary. From a theoretical point of view, there is also a need to problematise aspects of situated theories of learning and in particular their lack of emphasis on the role of disciplinary knowledge, such as mathematics, in studies of workplace learning.

2. IT systems are based on models involving mathematics that is largely invisible. This means that TmL can seldom be picked up on the job and need to be developed explicitly. It is difficult for managers and trainers responsible for skills development to recognise the nature and scope of the TmL that impact on their business. Even in sites that are most involved in disruptive change arising through the introduction of e.g., process improvement techniques, we have found that the need for new knowledge in the form of TmL is not sufficiently recognised by all levels of management throughout the company. Looking across all the sectors that we studied, we found limited capacity amongst trainers and managers to recognise the need for TmL, and to communicate with companies' own technical experts in order to develop appropriate training.
3. Symbolic information in the form of numbers, tables and graphs are often understood by employees as "pseudo-mathematics" – as labels or pictures with limited appreciation of or connection to the underlying mathematical relationships. Information fails to fulfill its intended role in facilitating communication between communities, across boundaries within the workplace, or between the workplace and outside. The implication is that the development of relevant TmL is an effective challenge to the problem of pseudo-mathematics, which may facilitate employees to develop shared meanings for symbolic boundary objects.
4. Effective learning of TmL can follow from activities that make work process models more visible and manipulable through interactive software tools; i.e., engaging with technology enhanced boundary objects (TEBOs). This necessitates determined efforts in iterative co-design of TEBOs and authentic activities that use the complementary expertise of employers and educators are needed for effective and sustainable TmL upskilling.

6.4.1 Two Examples: Manufacturing and Statistical Process Control

(a) Modeling manufacturing processes: In a packaging factory making plastic film by an extrusion process, we investigated how the computer control and monitoring system served as a boundary object between managers, engineers and shopfloor machine operators. The extrusion process is complex, involving about twenty steps. The plastic starts from raw granules, is melted to form a thick tube, which travels through several stages as a flat "tape" and is then extruded (stretched) at

different temperatures and tensions that need to be very precisely controlled, becoming thinner at each stage until the desired thickness (gauge) is reached (e.g., 19 micrometres). The most sensitive stage of the process is at what is known as "the bubble" – where the tape is inflated with compressed air so that it rapidly expands and the film is thinned down to its final thickness.

Each extrusion line is controlled by a computer system that monitors and records process parameters – typical display screens (see Fig. 6.1) present flow diagrams representing actual quantities, and flows such as the temperatures and pressures at different points in the line. The computer system records these process data and stores them as historical data for several months (see Fig. 6.2). Although these records are accessible to all, our ethnography indicated that shopfloor operators and line managers rarely if ever engaged or even looked at them. Managers were convinced that if shopfloor employees were able to engage with these data, they would have much-improved models of the process, which would lead to more effective operator control and more efficient production for the company.

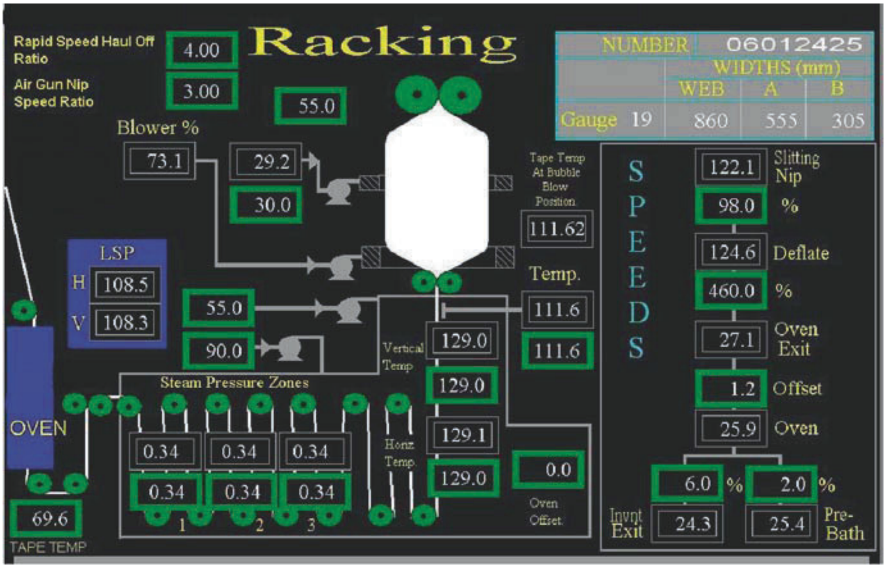

Fig. 6.1 A screen-shot of part of the computer control system for the film production process; white "thread" shows the flow of the film through various production stages, with temperatures, pressures, etc. displayed, terminating in the bubble, the white hexagonal shape in the centre

We identified the following TmL: understanding systematic measurement, data collection and display; appreciation of the complex effects of changing variables on the production system as a whole; being able to identify key variables and relationships in the work flow; and reading and interpreting time series data, graphs and charts, some of which are standard and some idiosyncratic and company-specific. We also noted the need for employees to be able to control the process for target mean and minimal variation and to communicate about these values with other

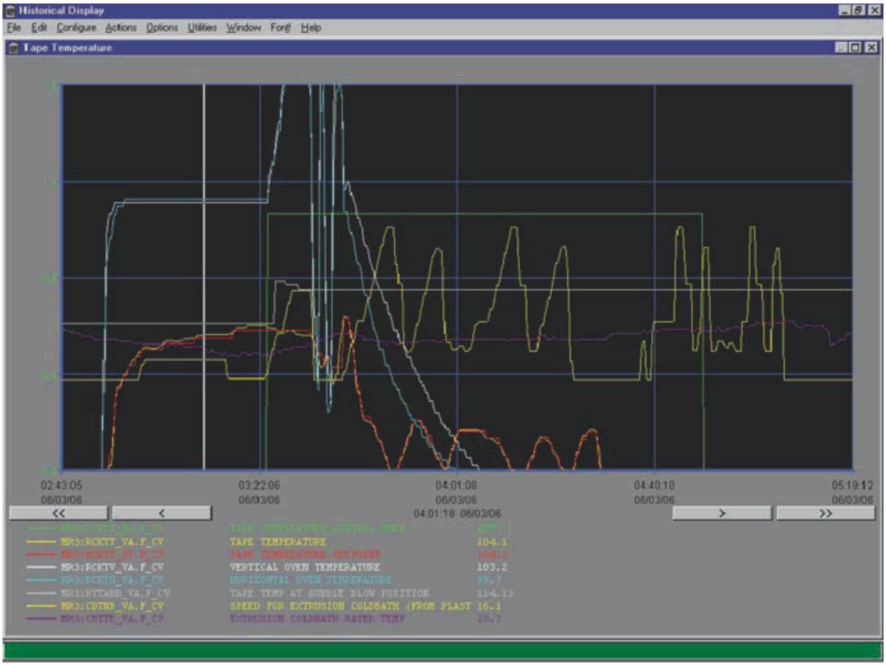

Fig. 6.2 Graphs of the "historic" data. Several graphs are displayed on a single scale: there are eight different scales implicit on the y-axis

employees and with management. Finally, we identified a need for employees to appreciate the role of invisible factors – such as the cost of raw materials and selling price of the product – in determining the target mean and variation of the physical film, none of which were evident in the computer-generated data available.

TEBO: Simulating and "opening up" a model of the process. We developed a software simulation of the production process of making plastic film by extrusion: see Fig. 6.3. The learning opportunities were highly rated by the participants, and will be adopted by the company in future induction and training. A process engineer commented: "If every operator and shift leader went through training using the tool there would be a base-line level of understanding that we risk not getting with the observational style training we currently use. I think the tool also helps identify people's strengths and weaknesses not only in terms of film-making process understanding but also logical problem solving ability. I didn't expect this. After going through the training activities with him, I now know far more about William and the way he thinks about things than I knew before. Speaking as someone who hates open-ended tasks, I'm pleased I can now give them all to William!"

(b) Manufacturing industry: the case of statistical process control. SPC is a set of techniques widely used in workplaces as part of process improvement activities, such as "Six Sigma". Using such techniques requires many employees to interpret and communicate one-number process measures, which we have observed to be

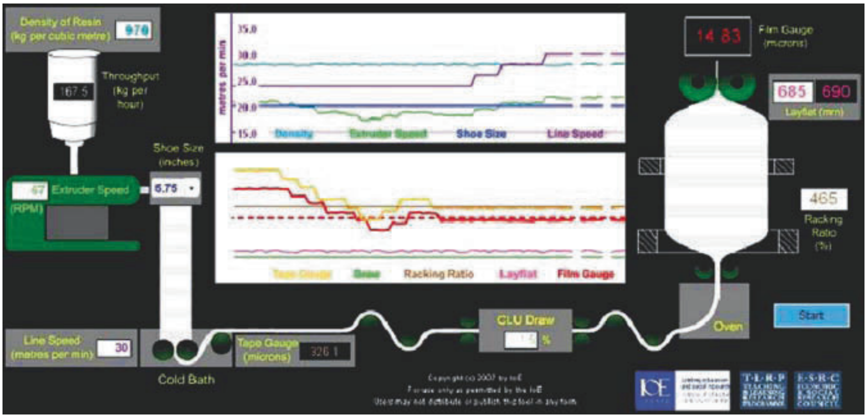

Fig. 6.3 Software tool to model the production of plastic film by extrusion; to reduce the complexity, just the start and end parts of the process are modelled, but the format of the real systems is imitated (cf. Figs. 6.1 and 6.2). Numbers in the *white boxes* are inputs or parameters that the user can modify. The goal is to achieve stable running of the process with film gauge (*thickness*) at a required target. It is crucial not to make changes that "burst the bubble" on the right (this in practice stops the production process). The graphs in the *middle* show historical data for 9 variables: the *top set* are inputs/parameters for the start of the process, the *bottom set* are for the end part of the process; on each graph, an appropriate scale appears when a given variable is selected

mathematically and contextually challenging. In several car factories we investigated how "process capability indices" were used and trained. We found that the usual introduction of such measures deploys statistical and algebraic symbolism as well as laborious manual calculations that seemed to hinder employees' understanding of the underlying mathematical relationships. The indices were mainly understood as "pseudo-mathematics": as labels for "good" or "bad" processes without meaningful connection to their basis in the manufacturing process, yet the company wanted the indices to be meaningful and as prompts for appropriate action. Working in partnership with the company trainers, we developed tools and activities to enhance existing training and shopfloor practice.

We identified the following TmL: understanding systematic measurement, data collection and display; appreciation of the complex effects of changing variables on the production system as a whole; being able to identify key variables and relationships in the work flow; reading and interpreting time series data, graphs and charts; distinguishing mean from target and specification and control limits; knowing about the relation between data and measures and process and model; understanding and reducing variation, and appreciating the basis of process capability indices and how they are calculated.

In Phase 2, we developed TEBOs that aimed at making the statistical concepts used in SPC training courses more understandable for intermediate-level employees by (a) simulating the physical experiments that trainees did to generate sample process data, so that after doing them manually they were able to generate much larger data sets in the software, making statistical patterns and trends

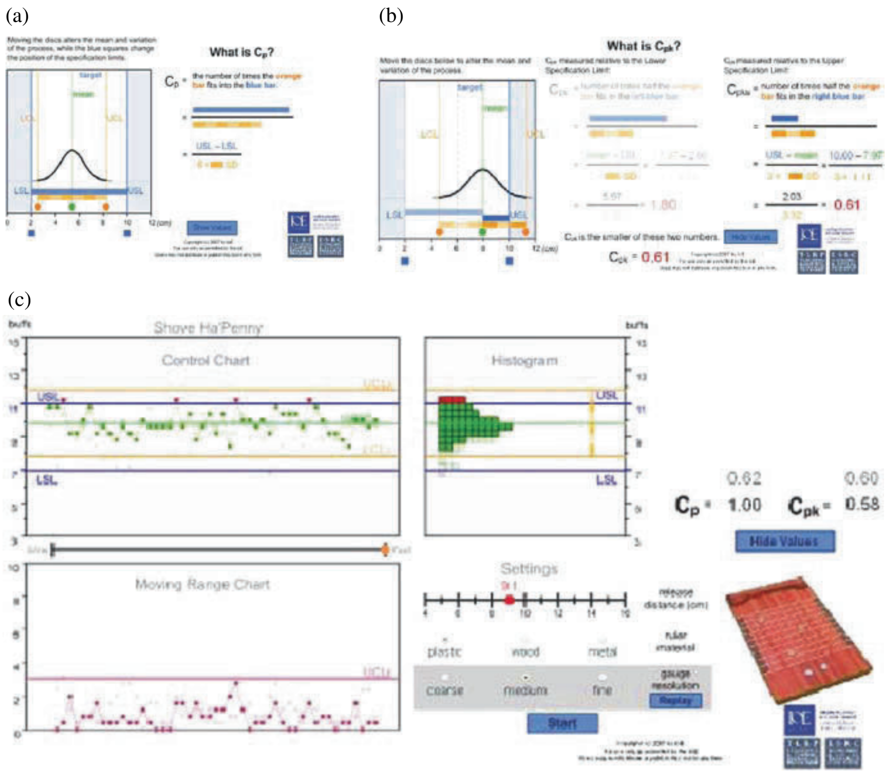

Fig. 6.4 (**a**) and (**b**): Software tools which model Cp and Cpk, "capability measures" for how well a process is under control and meeting required targets; a visual and interactive representation to help employees understand how these measures depend on the statistical behaviour of physical data and the imposed human specification of targets. (**c**): This tool provides a context for Cp and Cpk by demonstrating how they can be used in practice to improve a process; in this "process improvement" version of the pub game of "Shove ha'penny", a ruler is attached to a flat board and is used to flick coins along it. The simulation allows the user to see repeatedly generate trials of 50 "flicks", plotting where the coin lands each time on the Control Chart. The player attempts to improve the process by altering several key process parameters

easier to perceive and (b) allowing direct manipulation of capability measures with visual feedback alongside algebraic formulae that are quoted to trainees but hardly understood. Examples of these are shown in Fig. 6.4, but the reader can gain a much better impression of the approach by engaging with the software (see www.lkl.ac.uk/technomaths/tools).

Our activities and tools were positively evaluated by the in-house SPC team. They were adopted by other automotive companies; tools taken up were influential in supporting communication between SPC trainers and managers. One key idea was Cpk, a measure of how well a process was under control (see Figure 6.4). An SPC specialist engineer commented: "One of the hardest things we have to get across is what the Cpk means – once you're familiar it becomes trivial, but to translate that to someone who doesn't know is really difficult, the tool enables you to show in a dynamic way – if I move this then this moves. It's like creating a cartoon from a load of slides. When the operators chart data they are taking little snapshots in time and your tool brings it all together like a cartoon, animating it."

6.5 Conclusions

The major skills deficit that we perceive in workplaces is the understanding of systems, not the ability to calculate. Calculation and basic arithmetic are necessary, but for the intermediate-level employees with whom we worked, their importance was of subsidiary importance compared to a conceptual grasp of, for example, key variables and relationships in the work flow, how graphs and spreadsheets highlight relationships and trends, and how systematic data may be used with powerful, predictive tools to control and improve processes.

Our findings point to the ways in which the skills gap of techno-mathematical literacies needs to be systematically addressed by employers, working together with educators. It needs a commitment of time and resources on the part of employers to come to terms with the need for this new kind of mathematical understanding and to develop new pedagogical approaches for training, so as to make TmL more visible and available for exploration and development.

TmL is most evident in workplaces that are involved in changes in working practices. However, even in sites where the need for change is recognised and supported, we did not find that the need for TmL is, in general, sufficiently recognised throughout a company. Long-established preconceptions about mathematics are deeply ingrained, both in the world of work and the world of education. One way these reveal themselves is where employers complain about "poor numeracy skills" in practices, such as financial services, where employees have little need to do arithmetic, because of the total computerisation of work processes. A central challenge for skills development is to educate both employees and managers about important mathematical models and relationships that are rendered invisible by IT within workplaces.

We have mentioned several instances of the problem of "pseudo-mathematical" understanding: where numbers become labels that are attached to workplace artefacts, and are disconnected from the underlying mathematical models and relationships which make such numbers meaningful. It is important to stress that the problem of pseudo-maths is very common and is due in large part to the ubiquity of IT systems and *how they are designed* by mathematical experts with the workplace routines organised around them, such that employees are intentionally disconnected from the mathematical models which drive the IT systems. We propose that engaging employees in the development of TmL, through authentic modeling activities within TEBOs, can be an effective challenge to the growing problem of pseudo-maths.

Such an approach was endorsed at our project dissemination event, which involved 60 invited participants from industry, education and policy organisations. We were privileged to be able to present our research in the form of *joint presentations* between us and our employer-partners. At this event, it was confirmed that TmL development had achieved a very positive effect on employees' self image; a finding we did not anticipate. The final contribution of the day came from the chief executive (Europe) of one company, who remarked: "I have been very impressed today by what you have shown, and I am very pleased that our company has been involved in this project; the tools developed by the researchers have contributed significantly to our employees' skills and employees' sense of empowerment."

Acknowledgment Funding of the TmL research project October 2003–June 2007 by the United Kingdom ESRC Teaching and Learning Research Programme [www.tlrp.org] is gratefully acknowledged (Award Number L139-25-0119).

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Chapter 7
Mathematical Modeling in Engineering Design Projects

Monica E. Cardella

Abstract While engineering students are required to complete a number of mathematics courses, some engineering students and practitioners believe that they do not use the mathematics that they learned from their courses in engineering projects. This study investigates engineering students' use of mathematics through observations of two teams of students working on extensive design projects. The case studies presented in this chapter provide insights into situations when engineering students engage in modeling behavior and also explore ambiguity and precision in engineering design. These insights can inform engineering education as we help engineering students become more aware of the ways that mathematics is used in engineering. Additionally, understanding the ways that mathematics and mathematical thinking is used in professional applications can help us motivate and contextualize mathematics instruction as well as determine what should be taught to students in both college and pre-college settings.

7.1 Introduction

This study was originally motivated by the disparate perceptions of mathematics that are evident in engineering education. While some engineering students are able to recognize the value of the mathematics that they have studied, especially as it relates to their professional practice (Graves, 2005) and its utility as a "tool" (Satwicz, 1994), other students question the relevance of the courses they have completed (Cardella, 2006). Similarly, within professional practice, some engineers also question the usefulness of the mathematics they have studied. As Pearson (1991) reflected on the conversations he had engaged in with thousands of engineers, he estimated that only 30% actually used the calculus and differential equations that they studied in college during the course of their professional projects. However,

M.E. Cardella (✉)
School of Engineering Education, Purdue University, West Lafayette, IN, USA
e-mail: mcardell@purdue.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_7,

within engineering education, mathematical and analytical competence is generally considered to be a fundamental skill that all engineering students need (ABET, 2003; NRC, 2006). Because of these disparate perceptions, we might wonder why mathematics is important for engineering, and how it is that engineers actually use mathematics.

The study is also motivated by the opportunity to further understand how mathematics is used in authentic situations (Hutchins, 1996; Hoyles and Noss, 2007; Hall, 1999; Stevens and Hall, 1998). One study of mathematics in the workplace that is particularly relevant is Gainsburg's study of the mathematical practices of structural engineers (2003). Gainsburg observed structural engineers with varying levels of experience at two engineering firms as the engineers went about their usual work. She found that structural engineers live and think in a world of quantities, units, procedures, and concepts (Gainsburg, 2003). She also found that mathematical modeling was central to and ubiquitous in the engineers' work (Gainsburg, 2006) – the engineers in her study used, adapted, and created models of various representation forms and degrees of abstraction (2006). Gainsburg describes the practicing engineers' on-the-job knowledge acquisition (2003), and suggests that this constructivist form of learning and the types of models the engineers worked with are not well-reflected in the modeling tasks typically prescribed for the K-12 classroom (2006). This study builds on Gainsburg's work through similar unobtrusive observation of engineers at work in their normal practices – in this case engineering students working with industry partners on 5–9 month-long authentic design projects. In addition, this work builds on other studies of engineering students' modeling behavior and students' acquisition of modeling skills (Diefes-Dux et al., 2004; Moore and Diefes-Dux, 2004).

7.2 Methodology

The data and analysis build on a previous study of engineering students' uses of mathematical thinking in engineering design. The data were originally collected as part of a larger investigation of engineering students' uses of mathematical thinking in the context of capstone (or senior) design projects. In the previous study, the data were analyzed along five aspects of mathematical thinking derived from Schoenfeld's description of mathematical thinking (1992): knowledge base, problem solving strategies, use of resources, beliefs and affects and mathematical practices. These five aspects of mathematical thinking provided an opportunity to illuminate engineering students' mathematical knowledge bases in addition to the problem solving strategies they learned from their mathematics courses, the resources that they learned to use in their mathematics courses, the ways that engineering students learned to monitor their use of mathematical resources (i.e., metacognitive processes such as planning, monitoring and reflecting), beliefs and affects engineering students have developed about mathematics and the types of mathematical practices that students engage in (e.g., using estimation, being precise, using numbers to justify a design decision). These findings are discussed in more detail elsewhere (Cardella, 2006; Cardella and Atman, 2004). This chapter focuses on one specific mathematical thinking "theme" that emerged through grounded theory methodology (Strauss and Corbin, 1994; Coffey and Atkinson, 1996; Dick, 2004) from the previous analyses: mathematical modeling.

Two groups of engineering students participated in this study: one team of five Industrial Engineering undergraduates participated during their final year of college, and one team of four Mechanical Engineering students who were pursuing their Masters degrees. Each of these students agreed to be observed during their team meetings over the course of approximately 5 months. The students also shared copies of the documents that they created for their project – both interim design documents (such as rough sketches) and copies of the final reports. Eight of the students also agreed to be interviewed about their educational backgrounds (particularly their mathematical background) and their own estimation of how they used mathematics in engineering design projects. These data provide information about two teams' meetings – one of the undergraduate team's meetings and one of the graduate students' team meetings. These two case studies (Yin, 2003) provide a rich description of two scenarios when engineering students engaged in mathematical modeling as part of an extended design project. In particular, we are able to learn about the different types of models that engineering students create and use as well as the ways that engineering students grapple with ambiguity, uncertainty, and the need for precise information.

7.2.1 Industrial Engineering Undergraduates

A team of five Industrial Engineering undergraduates worked together to evaluate and change part of a supply chain system. Like their classmates, they had been paired up with an industry partner who had assigned the team an authentic problem that they would work on over the course of their 5-month capstone design project (see Fig. 7.1). During the course of the 5 months, the team met regularly with their industry partners – engineers representing the partner – in addition to their weekly meetings with their capstone design course instructor. In order to provide their industry partner with a final recommendation on whether or not they should establish a satellite center on Bellissima Island, the team determined that they needed to do a detailed cost analysis of each option (keeping the system as it was, changing the delivery routes, or adding a satellite distribution center). The team has three additional options to consider; there are three standard models for how the packages would be sorted into individual delivery routes that they might use for the satellite center.

During meeting 14, three of the five undergraduates have gathered to work on their project. One of the first tasks that they attend to is developing a plan for the paper that is due at the end of the project. The end is quickly approaching – they now have 2 weeks left to finish the project before presenting their solution to a

Currently, six to eleven package cars travel daily from the Supply Chain Solutions hub (SCS) to and from Bellissima Island. During a typical nine hour shift, each of the package car drivers will spend an average of 60 minutes commuting to Bellissima Island. An additional 50-60 minutes will be spent returning to the hub from Bellissima Island. Upon arriving onto Bellissima Island, each package car driver will then travel to their designated loop where they will complete their necessary deliveries and package pick-ups for the day. One of the big concerns SCS has with the Bellissima Island routing process is idle time.

A large amount of idle-time is incurred due to the commute to the ferry dock, the ferry queue, the 35 minute trip across the Sound, and the queue to get off of the ferry. Idle-time is defined as the time when a driver is *not* doing anything that is value adding to SCS. Due to the large amount of idle-time, SCS deems the driver utilization rates for the Bellissima Island delivery loops as unsatisfactory.

Ideally, package car drivers should deliver high volume over the minimum number of miles necessary to complete their tasks. In the case of the Bellissima Island routing process, too much of a driver's day is spent on commuting rather than delivering and picking-up packages. One way to achieve higher utilization rates is to deliver a high volume of packages using smaller distances. The reason for a volume over miles ratio is that it is inefficient and expensive to pay drivers to be idle.

Although the underlying goal of this project is to reduce the cost of the Bellissima Island delivery process, it is understood that in order to achieve this, the volume-miles ratio must be minimized.

Fig. 7.1 Problem statement for the industrial engineering students team

panel of judges and providing their industry partner with a recommendation. After developing a tertiary plan, they turn to the issue of utilities. In analyzing the cost of setting up a satellite center on Bellissima Island, one of the many costs they need to include is utilities. Diego reviews the utilities they need to consider: garbage rates (and the initial cost of purchasing garbage cans), electricity (which means they also need to determine how much power is required for the satellite center's daily operations), "honey bucket" prices (and the cleaning rate for the honey buckets, where the "math is: 5 guys working 8 h days")

Meeting 14 was the 14th meeting that I observed. Diego also noted that if they were to rent a facility, many of the utilities would be covered under a single service fee. They consider the potential for recycling – which could reduce the cost of garbage but would increase the initial cost associated with purchasing garbage cans. They also talk about the sewage services: if the satellite center is position near a gas station, perhaps they do not need to pay for portable toilet services.

Following the discussion of the utility costs, the team returns to an issue they have discussed in earlier meetings: the population growth rate. As the team evaluates the options for delivering packages on Bellissima Island, they have realized that they must consider not only the current patterns in the number of packages delivered at various times of the year, but also the future quantities of packages that will be delivered on the island. In conversations with one of their industry partners, they learned that the growth pattern for the package volume (the number of packages that are delivered) closely mirrors the growth pattern for the population of the island.

After revisiting the growth rate the team moves on to consider one of the project's major constraints. In addition to considering the costs of utilities and land, and quantities of packages involved in the system, the team must also consider human factors. In particular, the team must ensure that delivery drivers are not working unrealistic hours. Legally, no delivery driver can work more than 12 h in a single day. So, the team has considered having one driver transport the packages from the main distribution center hub to the satellite center at the beginning of the day and then have the driver return to the hub with all of the packages that have been picked up in the course of the day. In addition, the team also needs to consider other time constraints: the time in the morning when the packages will be ready to leave the hub for transit to the island, the time in the evening when the packages must be back at the hub if they are going to be able to be shipped out that night, and the guaranteed deadlines the company has established for its customers for package delivery and pick-up.

To determine if they are meeting the time constraints, the team needs to understand how much time is required for each task that is part of the delivery and pick-up system: the amount of time required to transport packages from the hub to the island, the time required to sort the packages brought over in one large transit truck into five, six or seven delivery routes, the amount of time spent delivering all of the packages (and picking up new packages), the amount of time spent loading all of the picked-up packages onto the truck that will return to the hub, and finally the time required to return to the hub from the island. The students have received information from their industry partner that will help them answer some of these questions: they have been given an Excel spreadsheet that contains several months of data for the number of packages delivered each day and the amount of time spent on each individual delivery. With this data, and some additional information from the industry partner, they have been able to create simulations of the sorting process to determine a range of values for the amount of time spent sorting the packages into the different delivery routes. They have also analyzed the data to determine the amount of time spent delivering packages (they have determined the average time spent delivering packages, as well as the range of amounts of time drivers have spent).

During meeting 14, the team also is checking their previous work. *Is it possible that we overestimated the time needed to sort the packages into the different delivery routes?* They are finding that the constraint of the 12-h day is not easy to meet. They consider hiring additional part-time drivers to help sort the packages. *What happens if we run out of time? Will the delivery driver be stuck on the ferry?* John checks the ferry schedule for the fare information as well as the schedule of times; one concern is the frequency of the ferry trips, and the implications of the driver missing the appropriate ferry. They also consider all of the distances that will be driven. *Are these distances rectilinear or Euclidean? We need accurate distances.* Diego continues estimating the distances while Mei and John try to envision the daily schedule. John looks for information off the internet while Mei begins to write out the daily schedule for the system of all of the transit and delivery cars. At 1:00 a.m. John decides that he is no longer able to contribute; Mei continues work on the schedule on a whiteboard while John goes home. Mei considers the drivers' lunch breaks, and revises the schedule as she writes. She considers the drivers' current schedules, and the fact that the time that they currently spend on the ferry they can spend delivering packages in the satellite center scenario. Because the 12-h limit is not negotiable, she takes a conservative approach and plans the day so that the driver does not work more than 11 h each day. When she reaches a satisfactory schedule, she copies down her work and explains that she based much of her work on the current delivery schedule and specific time constraints, such as the ferry schedule and the hub's typical schedule (the time of day when packages are ready to leave the hub and the deadline for when they must return to the hub). Beyond that, she looked at "one section at a time" to make sure that all of the constraints were met. She and Diego are content with their progress and leave the undergraduate meeting space to go home.

7.2.2 Mechanical Engineering Graduate Students

A team of four Mechanical Engineering graduate students worked to design a portable dental unit that dentists could take to use in remote rural locations, or use in a small private practice. Like the undergraduates, the team worked with an industry partner – a company that manufactures compression and vacuum systems for dentists. The main criteria that the team considered in their design decisions were weight of the system, cost of the system, and performance of the system. Specifically, the team worked to ensure that the compressed air and vacuum delivery system was able to provide a certain air flow in order to power the dental tools: the drill, suction and spray. To ensure that their system met the air flow constraint, the team created two models of the system: physical, "experimental" models and mathematical "simulation" models. With the experimental models, the team connected motors, compressors, vacuums and dental tools to test the systems' ability to meet the constraints. Using Engineering Equation Solver, the team created mathematical models of the same systems to simulate the air flow through system.

During the February 20 meeting, the team is looking at the "theoretical" model (the simulation) and interpreting "what the numbers mean." They are investigating the ways that different piston sizes and layouts (in the motor powering the compressor) effect the rate of air flow. They realize that at any pressure, q[in]=q[out]: the air flow going in to the system should equal the air flow exiting the system. The team discovers two potential problems: (1) air flow in does not always equal air flow out and (2) the air flow they measured in the experimental models did not always match the airflow predicted by the simulation.

They run a simulation and are trying to understand the correlation between increases in pressure and decreases in air flow (scfm). Scfm is the air flow at standard conditions, and should be consistent – q[in] should equal q[out]. One of the course instructors, who acts as a technical advisor for each of the eleven teams in the class, explains one potential cause for the discrepancy: a gap in the chamber by the piston which leads to a growing air pocket as pressure increases, causing the air intake to decrease. When they finish compressing air, air that is trapped in this gap expands and also sucks in more air.

As the team talks with the instructor about the discrepancy between the experimental results and the simulation results, the instructor iteratively checks the experimental set-up for problems – tubes that are too long, mistakes in the order or direction of the equipment and mistakes in how air flow is measured – and checks for errors in the mathematical model. He checks the physical devices, questions the team on the mathematics and returns to the physical devices. He is called away to work with another team, and so our team continues the process of checking both the experimental set-up and the mathematical model for potential mistakes.

The team decides to collect more experimental data. Two of the teammates set up the equipment – they build a base to hold one measurement device, prepare the press and set up their laptops with LabView. As they begin collecting data, they have a target pressure of 100 psi and a maximum flow rate of 1.69 scfm. They start the drill press, look at the data that is being collected via LabView in an Excel spreadsheet and then stop the drill press when they have reached the target pressure. Initially they collect data at 1600 rpm, and decide to collect it at a few other speeds too – they collect some unplanned data because everything is already set up and they want to make sure they have everything they need. They collect data, reflect on the data – the starting values, the maximum values, etc. – then collect more data.

They revisit their constraints – the hand piece a dentist uses requires 1.5 scfm of air flow, and they are planning to supply a total of 3 scfm. They consider that the highest air flow needed would be a combination of a hand piece – which they now quote at 1.4 scfm – and a vacuum, which needs 1.97 scfm. They are not confident in their recollection of this constraint, and remind themselves that they need to look at the matrix of dental unit usage that the industry partner gave to them – they think there may be a maximum of three devices used at a time, where one is the vacuum ("always have the vacuum on") and the dental assistant may be running spray in addition to the dentist's use of another hand piece.

After considering these constraints, they turn to modeling tasks – one team member asks if they have modeled their newest compressor yet. They decide they need to open the compressor up to measure different parts, and they run the system with LabView to measure the air flow and pressure. One team member looks at the specifications for the compressor on the manufacturer's website, and realizes that the main air flow rate that is advertised is unrealistic; it is only achievable under specific, rarely realized circumstances.

They redirect their focus to the simulation they are creating using EES (Engineering Equation Solver). They check their units of measurement, and realize that they have not been consistent. They also check the mechanical deficiency that they have been using. Their instructor had previously shared an academic paper with them that served as a valuable reference for determining the appropriate mechanical deficiency. They work through by hand some of the calculations that relate air flow, mechanical deficiency and pressure, and debate whether they should only include the final value for the air flow output in the code, or if they should also include the pressure, mechanical deficiency and the air flow entering the system, since these three values would be captured in the value for air flow out. So, rather than having the code calculate that q[in] will be 7.5 scfm every time they run the script, they consider setting q[in]=7.5 scfm in the code because this is "what the code needs to know." "This dictates the geometry of our pistons."

This discussion around the calculation of air flow brings them back to their initial conversation about air flow: q[in]=q[out]. At this point, they realize that although their q[in] will always be the same, their q[out] will "always be changing" (though it will be consistent for a particular pressure). The deficiencies – which will cause q[in] to not equal q[out], will "correlate to the geometry of the pistons."

The final values that need to be inputted into the mathematical model for the simulation are the dimensions of this particular compressor. They begin by estimating the stroke length, piston diameter and speed (number of rotations per minute). They turn to the internet to see if they can confirm the estimates, but in the end determine that they need to open up the housing to measure the parts themselves. The stroke length is 1.2 in. (although when they write this number down, they change it to 6/5) and the piston diameter is 1.75 in.

As they continue to move in between their theoretical model and their measured experimental results, they compare the stroke length from the simulation – 1.8 in. – to the "actual" stroke length: 1.2 in. They note that they were off by half an inch, and decide that designing it with an extra half an inch is reasonable. However, they still double check the stroke length. They realize that the value in the simulation is not 1.8 in. but is actually 1.28 in.

In the end, they decide that the simulation is correct – or at least close enough. They turn to other tasks, such as the recent email they have received from their Industry Partner and further analysis of the data they collected near the beginning of this design session. While his teammates look at a graph they have created, Daniel revisits the results from the experimental and simulation models (he is still concerned about the discrepancy, even though he had earlier decided it was close enough) and realizes that he had forgotten something. He fixes the error, as the team concludes the design session.

7.3 Discussion

Both teams' projects involved multi-layered models. The team of industrial engineering undergraduates ultimately modeled the cost of each of the alternative solutions that they considered for the package distribution problem. To model the cost of the proposed satellite center solution, however, they needed to complete three other modeling tasks. One of the factors associated with the cost of the satellite package distribution center was the cost of drivers' salaries and wages. To include this cost in their analysis, and to ensure that the satellite center was a feasible alternative, the team created a model of the daily schedule for all aspects of this solution – the times spent on each subtask that needed to be completed for all packages to be delivered (and picked up) on the Island. To model the daily schedule, the team needed to know how much time would be spent each day on the sorting process (sorting the packages brought to the island by the transit car into the different delivery routes) and how much time would be spent during the delivery portion of the day. Each of these times – the time for sorting and the time for delivery – were determined by creating mathematical models of these processes using data and information the team acquired from their industry partner.

The team of mechanical engineering graduate students attended to the task of modeling air flow and pressure in the design of an air compression and vacuum delivery system that would be used to operate dental tools. To create this model, the graduate students, like the undergraduates, attended to several sub-tasks that involved modeling. In particular, the team's system consisted of a combination of a motor, compressor, vacuum and the dental tools. The team considered multiple options for most of these components and deemed that it was important to model each individual component in addition to the system (for example, the team created mathematical models for multiple compressors). The mechanical engineering graduate students also created multiple types of models for the entire air compression and vacuum delivery system – they created "experimental," physical models as well as "theoretical," simulation models.

One notable difference between the two projects was the tangible aspect of each project. The mechanical engineering students were able to construct physical models of the system to use to further understand the system as well as to check the accuracy of the mathematical models. The industrial engineering students were not able to build a satellite center or otherwise physically interact with prototypes of their designs. They were, however, able to observe the current package distribution system as it operated and took advantage of the opportunity to follow the delivery drivers through a day of work. Additionally, when the team approached the task of modeling the daily schedule for the proposed satellite center solution, they took a non-mathematical approach by writing and re-writing the schedule and using an intuitive sense of checking to see if constraints were violated, rather than creating a linear program to solve the scheduling problem.

One of the challenges that the undergraduate Industrial Engineering students commented on during the course of their project was the need for "precise information" in order to have a precise final cost for each alternative solution, which was frustrated by ambiguity and missing information. During meeting 14, some of the information that was missing was the cost of specific utilities as well as information about which utilities were actually necessary. In an interview after the project was over, one of the students commented that perhaps he and his teammates should have used a variable as a placeholder for the missing information, rather than allowing the missing information to impede their progress on the project. At other times, the team overcame the problem of missing information by using estimates. However, they later became frustrated that the estimates were not precise.

As the undergraduates struggled with ambiguity, uncertainty and estimation, it is interesting to consider what they may have believed "precise" meant. From their behavior, the Industrial Engineering students presumably envisioned an exact dollar amount that would not change. However, from a mathematical perspective, representing the costs of each utility with a variable (as the utility costs may vary over time and as the company grows) as well as the other factors such as employee wages, may be a more precise solution.

The graduate students' activity provides another interesting perspective on precision. As the mechanical engineering graduate students verified that the proposed compressed air and vacuum delivery system would provide enough air for the dental tools to function, they considered many sources of information. They evaluated the system by collecting "experimental" data and also evaluated the system through a mathematical model and "simulation" data. Later, when the two did not match, they iteratively checked each model – with an underlying question of which model is more accurate, more precise. Likewise, in creating the mathematical model, they gathered information from multiple sources: estimates of lengths and diameters, manufacturer specifications and their own measurements of the stroke length and piston diameter. Ultimately, they too needed to determine which source was most accurate, most precise. It is interesting to note too that the graduate students tended to gather information from multiple sources – perhaps suggesting that no single source was accurate or precise enough to rely on.

7.4 Conclusion

Like Gainsburg's study of structural engineers, this study provides evidence that mathematical modeling is central to engineering practice and a valuable tool for engineers. While the students may not have used every particular aspect of the mathematical knowledge base they had acquired throughout their undergraduate and pre-college studies, creating mathematical models was integral to their ability to complete their project. At times the students explored alternatives to mathematical models – at times to understand the project from a "hands-on" perspective, at times to simply get a second perspective on the problem, and perhaps at times the non-mathematical approach was more efficient. While one of the undergraduates reflected on ways that his team might have used mathematical modeling more in their project, it seems that each team had some understanding of how mathematical modeling could help them complete their project (and also understood some alternatives to mathematical modeling).

The cases in this chapter represent examples of how engineers use mathematics, which can provide context and motivation for mathematics learners: undergraduate engineering students, undergraduate mathematics students and pre-college students. The cases also provide insights into further opportunities in undergraduate engineering education. Both teams also grappled with the ambiguity and uncertainty inherent in their projects, and struggled to be accurate and precise. The two cases presented in this chapter provide insights into students' perceptions of "precision," and suggest that some undergraduate engineering students can become frustrated by the ambiguity and uncertainty that are normal for authentic engineering tasks. The data in this chapter provides further support for the notion of introducing students to complex, authentic tasks that will give students practice with responding to uncertainty and ambiguity.

Acknowledgment This research was made possible in part by National Science Foundation grant SBE-0354453. This work was supported by the National Academy of Engineering's Center for the Advancement of Scholarship in Engineering Education, the Center for Engineering Learning and Teaching at the University of Washington, and the Center for Design Research at Stanford University and the LIFE (Learning in Informal and Formal Environments) Center.

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Chapter 8
The Mathematical Expertise of Mechanical Engineers – The Case of Mechanism Design

Burkhard Alpers

Abstract In this contribution we present the results of a project that investigates the mathematical qualifications a mechanical engineer needs for working on practical tasks in his daily life. In particular, we report on the results concerning a mechanism design task where certain machine parts have to be moved in order to realize a cutting activity.

8.1 Introduction

The mathematical education of engineers has two major goals: first, to enable students to understand and use (and maybe develop on their own) mathematical models that are used in the application subjects like Engineering Mechanics or Control Theory; secondly, to provide a mathematical basis for their future professional life. The second goal is far more nebulous because it is much harder to obtain information on the mathematical expertise that is necessary in the daily life of an engineer. This contribution reports on a project that investigates the mathematical expertise of mechanical engineers. We restrict ourselves to "traditional tasks", which do not try to capture the research and development sector because our graduates are normally not employed for such work. In Alpers (2006), we reported on the results concerning a typical static construction task using a CAD program. Here, we deal with a typical mechanism design task for an industrial cutting device.

8.2 Method of Investigation

Although there are some studies on the mathematical content of workplace activity (cf. Bessot and Ridgway, 2000; Gainsburg, 2005), there are only very few studies on engineering professions. Kent and Noss (2003) and Gainsburg (2006) investigated

B. Alpers (✉)
University of Aalen, Aalen, Germany
e-mail: burkhard.alpers@htw-aalen.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_8,

civil engineering by visiting construction firms, interviewing engineers and managers and joining engineers in their daily life. Although this gives direct access to practical work, the method is very time-consuming, and for an outsider it is hard to get a good understanding in a short time. Therefore, the research presented in this contribution uses a different method. We hired two students during their last semester who already spent one or two semesters in industry. The students are given "typical" tasks which we identify together with a colleague who teaches machine elements, CAD and FEM and who worked for several years in the car industry. The students are paid for 100 h of work. They use industrial strength programs that are available at the university. The colleague acts as a mentor similar to a group leader in industry. The students are advised to make notes on their thoughts such that their thinking processes can be investigated. The author interviews the students and lets them explain and demonstrate their tool usage. These sessions are recorded with screen recording software.

Based on these data, the author investigates the mathematical concepts that are used to work effectively and efficiently on the task, in particular those ones which are important to make reasonable use of the tools involved. Since the author has some basic experience with the tools, he is able to re-perform the activities of the students for closer investigation. Particular attention is given to so-called "breakdown situations" (Kent and Noss, 2003) where a seemingly correct use of tools leads to problems or unreasonable results.

Cardella and Atman (2005a, b) apply a somewhat similar approach by investigating the role of mathematics in so-called "capstone design projects" where the students were mainly from Industrial Engineering (but also a few from other study courses). They capture the mathematical thinking activities using the more aggregated categories developed by Schoenfeld (1992) whereas the goal of our investigation consists of getting a deeper insight into the more detailed mathematical activities when using technology.

8.3 The Task: Design of Part of a Cutting Device

The task is a typical mechanism design task: a knife has to be moved up and down for cutting off a part of a foil which also has to be moved forward for a certain length. This mechanism is taken from a real machine which is used for producing halogen lamps.

Figure 8.1 shows the sketch the students received. It contains the essential dimensions and positions. The students have to construct cam disks (to be placed on the cam shaft) and some connecting rods to generate a mechanism for moving the knife up and down and the feeder back and forth. Since in real engineering life, there are often similar devices from which an engineer gets ideas (instead of developing everything from scratch), we provided them with the example in Fig. 8.2.

Finally, in order to synchronize with other parts of the system, a motion plan is given (Fig. 8.3) which indicates how the motions of the feeder and the knife depend on the rotation angle of the common cam shaft (used by all mechanisms). The plan

Fig. 8.1 Sketch

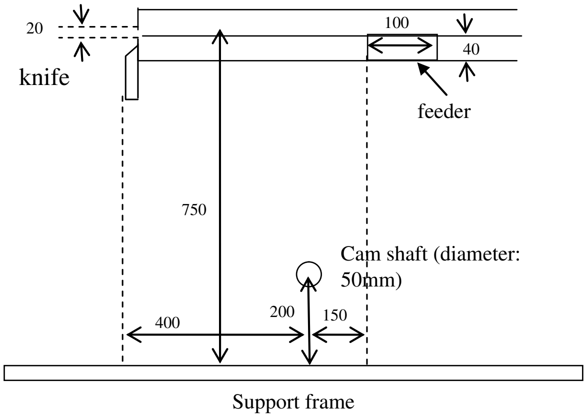

Fig. 8.2 Example

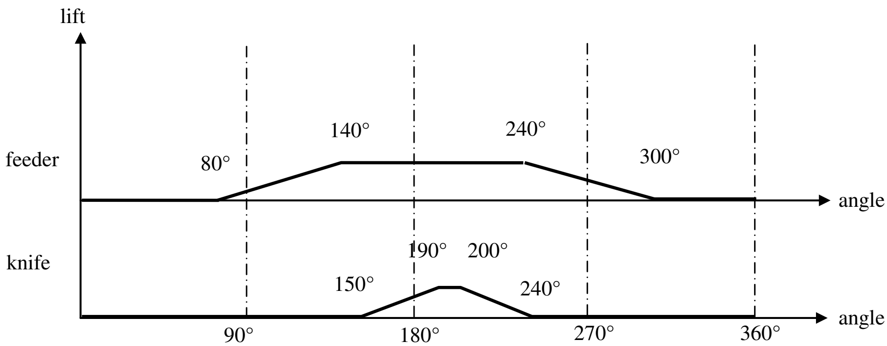

Fig. 8.3 Motion plan

only prescribes the intervals of constant lift. It is up to the engineer to find functions to get from one constant lift level to the next (e.g. from 80° to 140° in case of the feeder or from 150° to 190° in case of the knife). The linear interpolation is only inserted for optical reasons.

For the knife, a force of 100 N is required to perform the cutting process. For dimensioning the motor driving the cam shaft it is also necessary to compute or estimate the required driving torque.

8.4 Results and Discussion

The students knew in principle how to proceed by performing the following steps:

- Design of connecting rods using the CAD program Pro/Engineer®
- Design of motion functions for the knife and the feeder using a tool
- Design of the cam disk using the respective tools in Pro/Engineer®
- Design of the springs which guarantee that the rollers attached to the rods have contact with the cam disk
- Computation of the required driving torque for the motor driving the cam shaft.

Figure 8.4 depicts the final design results of the two students. We do not report on the geometrical qualifications when using the CAD program Pro/Engineer® because these were already described in Alpers (2006).

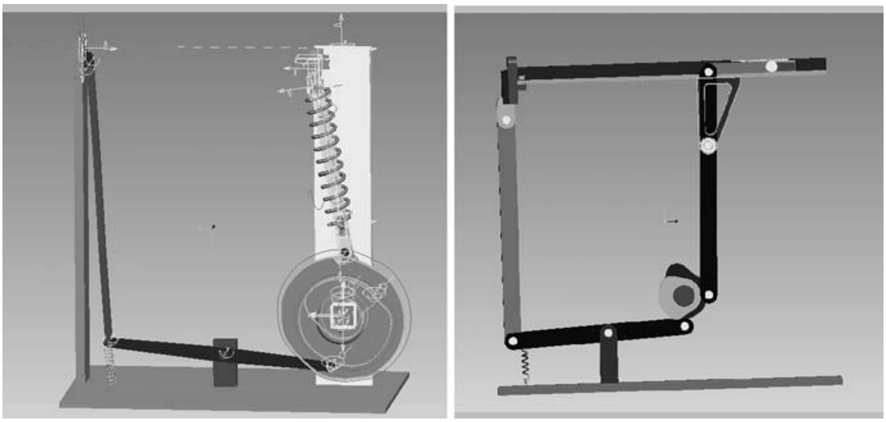

Fig. 8.4 Mechanisms designed by the students

When designing the linkage, one of the students started with a leverage of 1:1 without making any further investigations having in mind that the ratio could be changed later if the required driving torque was too high. The other one made some investigations by hand (with paper and pencil) in order to see how the ratio influences the necessary torque and the shape (largeness) of the cam disk. The student drew the rough shape of the cam disk for some ratios and computed an overly rough estimate for the torque. He realized that moving the swivel joint to the left reduced the necessary torque but makes the cam disk “steeper” and finally chose a ratio of

1:0.8 as a good compromise. According to the colleague involved in the project it is quite common in industry to start with a first guess coming from experience or knowledge of a similar device and make changes only when problems come up later on.

Having constructed the linkage, the students had to construct the “missing parts” of the motion functions for the knife and for the feeder, e.g. for the feeder the function piece on the interval [80°, 140°]. For such tasks, the German Association of Engineers (VDI) has set up guidelines where certain functions are proposed, e.g. a polynomial of degree 5 or a modified sine function (see Alpers and Steeb, 1998). There are tools for constructing such functions for a given situation (guaranteeing continuity of the first and second derivative, i.e. velocity and acceleration) and the students had access to such a tool which was programmed in a former diploma thesis. For using the tool the students had to understand its required input and the produced output. The user has to specify the start and end points of the resting phases and the functions to be used between two such phases. Here, functions according to the VDI guideline are offered. The VDI guideline additionally specifies several quality criteria for choosing a function (e.g. maximum absolute value of velocity or acceleration or of the product of these quantities). We told the students to use the modified sine function since this was the guideline for such mechanisms given by the company that uses the cutting mechanism. We will go back to the meaning of the quality criteria below when we discuss a rough model for estimating the necessary torque.

Finally, the tool needs the number of revolutions per minute in order to output the lift over time function in form of a table of values that can be stored in a file.

Such files are the input needed by Pro/Engineer® as motion functions of the knife and the feeder, respectively. Alternatively, Pro/Engineer® offers certain function types to specify the motion (e.g. polynomial, cosine function) and the user must input the parameters occurring in the function type (e.g. coefficients of a polynomial). Finally, the user can also insert piecewise defined functions where the pieces are expressions and the domains are intervals. The user has to take care himself to guarantee continuity with respect to position, velocity and acceleration. We mention this way of input since we also have a tool for producing the motion function according to the VDI guidelines symbolically and we could use this to check the numerical input (cf. remarks below on unwanted numerical effects).

After having specified the motion of the knife and the feeder in Pro/Engineer® by providing the respective files, the CAD program is able to simulate and compute the motion of the centers of the rollers. By constructing a plane which is parallel to the linkage, by rotating this plane with the number of revolutions used earlier on and by projecting the centers of the rollers onto this plane (orthogonally) one gets the motion curves of the centers on this plane. Using the CAD program to construct offset curves, one finally gets the boundary curves of the two cam disks. From this, the disks can be constructed easily by adding a certain depth.

The last design task consists of inserting springs such that the rollers do not leave the boundary of the cam disks. Only the first student had enough time to dimension the spring. He started with a constant value of 100 N/m and ended up with 1000 N/m when the roller did not leave the disk. He did not perform any optimisation which is quite usual according to the colleague involved.

Finally, for making a dynamic analysis, the force function for the knife had to be specified. Pro/Engineer® allows the same ways of input as in the case of motion functions described above. Only the first student had enough time to perform such an analysis. He specified the force function as being 0 for the period the knife rests and then it goes up to 100 N very quickly, remains at this level for a short time before going back to 0. The colleague involved and the author had a different perception of the cutting process: We thought that the cutting takes place when the knife is near its maximum velocity such that one gets a "clear cut". Like the student, we did not know how the force development in the cutting process really is. The author had a closer look into the force modelling of the student when the resulting maximum driving torque that was computed by Pro/Engineer® did not comply with a computation using a rough model we describe below.

Kent and Noss (2003) observed that mathematical thinking processes, particularly deficits in these processes, can best be investigated in "breakdown situations" where problems come up. We observed three such situations which we report in the sequel.

In the analysis of the driving torque performed with Pro/Engineer®, a strange oscillatory behaviour showed up which could not be explained in the given configuration. Since the work within Pro/Engineer® seemed to be correct, we (the author and the student) had a closer look at the input data. The students used the output table of the tool described above as input for describing the motion of the knife in Pro/Engineer®. Pro/Engineer® allows to graph also velocity and acceleration. Figure 8.5 depicts the acceleration function which already shows a rather "weird" behaviour. It should look like the one shown in Fig. 8.6 where the knife has first a positive and then a negative acceleration until its upper rest is reached. It was clear that Fig. 8.5 did not make any sense (although the s(t) function looked quite reasonable). Moreover, extremely high values for acceleration were shown (Figs. 8.5 and 8.6 have different scales).

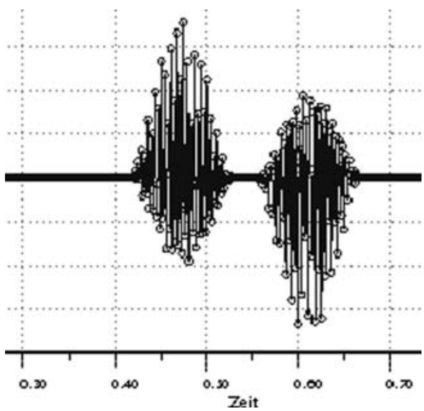

Fig. 8.5 Oscillation in acceleration

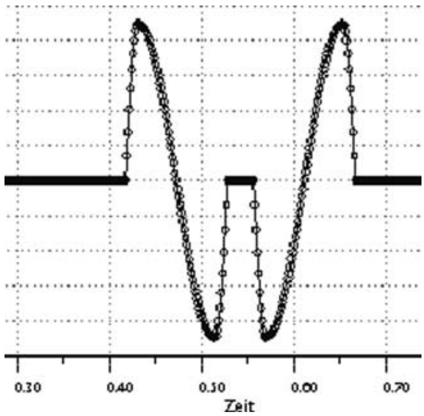

Fig. 8.6 No oscillation

A closer inspection of the input data revealed that the tool producing the table of values worked with a fixed output precision of 1/1000 (meters for distance and seconds for time). This was too coarse given a rotational frequency of 60 rpm, i.e. 1 rps, since the data table was constructed using $\Delta\varphi = 1°$ which corresponds to $\Delta t = 0.0028$ s. This still produced an acceptable s(t) function (distance over time) but then led to the faulty oscillation behaviour in the derivatives. After the precision of the tool had been extended, the reasonable graph shown in Fig. 8.6 was produced.

The situation described above shows how important it is to check the output (here: the driving torque graph) for plausibility. For this, a rough model can be used as will be described below, and the shape must reflect the mechanical situation. For example, when the knife cuts off a piece of the foil the required momentum has its maximum. The situation also showed how easily erroneous results are produced in high-level tools like Pro/Engineer® when wrong or imprecise input is fed in. Therefore, it is very important that the user is able to use all available facilities in order to check the input for reasonability. Here, a closer look at the also available plots of velocity and acceleration revealed the input problems. For interpreting these graphs, the user has to understand the relationships between the graphs (graph of derivative) and hence to know what to expect. The student did not perform these checks on his own but he was able to understand and then perform them after the author had presented them.

For checking the cam disk automatically produced by Pro/Engineer® (the light grey one in Fig. 8.7) one of the students constructed a cam disk directly in the CAD program by drawing circle sections for the resting phases and using the CAD program for interpolation (the dark grey one in Fig. 8.7). This way he recognized that the boundary curve produced automatically by Pro/Engineer® did not have a circle segment at the rest which is not possible, yet he could not find the reason since the input seemed to be okay. Figure 8.8 depicts that already the motion curve of the roller centre shows the same problem, so the problem has nothing to do with the construction of offset curves. This figure also gives an idea for the reason: the common points in both roller centre curves are at equidistant disk angles (this is no longer the case after the production of the offset curves!). This led the author to the

Fig. 8.7 Cam discs

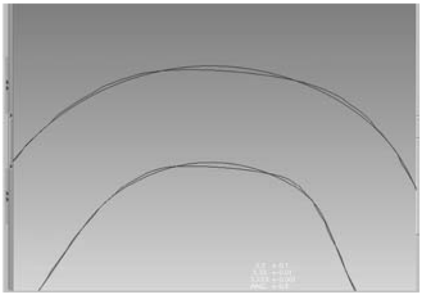

Fig. 8.8 Boundary curves

assumption that both curves might be interpolating the same set of points but differently. An investigation of how Pro/Engineer® produced the curve showed that for rapid production the student chose to compute 25 positions of the mechanism, so every 14.4° a point is computed and the curve is produced by using spline interpolation. Since the rest interval of the knife has only a length of 10° it is not surprising that the curve produced by Pro/Engineer® deviates from the circle segment shape. Another run using 1000 positions showed no such deviation.

From this breakdown situation one can conclude how important it is to recognize the effects of interpolation. Again, the situation shows how easy an inadequate usage of a high-level tool can lead to unacceptable results, so it is always important to be able do perform simple checks like the student did who compared the automatically constructed disk with a "hands-on approximation" of his own.

The author looked up a rough model in literature (Vollmer, 1989) in order to check the results on the required driving torque produced by one of the students. The model uses a power balance: "input power = –(output power + loss)" but the loss is neglected. The input power is the product of the input driving torque Min and the rotational velocity ωin and the output power is the product of the output force $\mathbf{F_{out}}$ and the output velocity $\mathbf{v_{out}}$ (we restrict ourselves to the cutting part of the overall mechanism, so $\mathbf{v_{out}}$ is the velocity of the knife):

$$M_{\text{in}} \cdot \omega_{\text{in}} = -F_{\text{out}} \cdot v_{\text{out}}, \quad \text{hence} \quad M_{\text{in}} = \frac{-F_{\text{out}} \cdot v_{\text{out}}}{\omega_{\text{in}}}$$

The output force is the sum of the applied force (here: the knife force of 100 N), the inertial force, the spring force and the gravitational force:

$F_N(t) \leq 100\,\text{N}$, $F_T = m \cdot a_{out}(t)$, $F_F = c \cdot s_{out}(t)$, $c = 1000\,\text{N/m}$, $F_S = m \cdot g$.

Hence:

$$\begin{aligned}
&F_{out} \cdot v_{out} = (F_N + F_T + F_F + F_S) \cdot v_{out} \\
&\leq 100 \cdot v_{out} + m \cdot a_{out} \cdot v_{out} + 1000 \cdot s_{out} \cdot v_{out} + m \cdot 9.81 \cdot v_{out} \\
&\leq 100 \cdot \max\{v_{out}\} + m \cdot \max\{a_{out} \cdot v_{out}\} + 1000 \cdot \max\{s_{out} \cdot v_{out}\} \\
&\qquad + m \cdot 9.81 \cdot \max\{v_{out}\}
\end{aligned}$$

This simple model shows quite well the influence of knife force, velocity and acceleration. The tool used for setting up the motion function already provides the maximum values of the velocity and the product of velocity and acceleration:

$$\max\{v_{out}\} = 0.317[m/s], \quad \max\{v_{out} \cdot a_{out}\} = 1.59[m^2/s^3]$$

With a mass of about 0.5 kg and a distance of 0...0.02 m (take 0.01 m where velocity has its maximum), we get the following upper estimate:

$$\begin{aligned}
&100 \cdot 0.317 + 0.5 \cdot 1.59 + 1000 \cdot 0.01 \cdot 0.317 + 0.5 \cdot 9.81 \cdot 0.317 \\
&\quad = 31.7 + 0.795 + 3.17 + 1.55 \approx 37.2
\end{aligned}$$

This shows the dominating influence of the power relating to the knife force. Division by the rotational velocity of 2π [1/s] gives a momentum of about 5.9 Nm. The student came up with a momentum of about 3 Nm. This has the same order of magnitude but only half the value which gave reason to investigate the causes. We found out that the student had used a knife force function where the force started much earlier and was finished before the velocity reached its maximum (for a clean cut it would be better to have a higher velocity when the cutting process starts). Since the computation presented above is dominated by the product of force and velocity it is quite understandable why the value computed by the student was lower. The model also explains why the company uses a so-called modified sine function for constructing the knife motion because the VDI guideline mentioned above recommends this type of function for getting a low maximum value for the velocity. This shows that simple models like the one given above can have high explanatory value and provide excellent opportunities for performing checks and possibly for detecting hidden assumptions.

8.5 Conclusions

The method we used for investigating the mathematical expertise within the daily work of a mechanical engineer proved to be very fruitful. It allowed probing deeply into the ways standard tools like CAD systems are used for performing practical tasks including the dangers and pitfalls that go with the usage. The colleague who acted as a guide confirmed that most of the work done by the students reflected practical work of junior engineers. He also found the situation quite usual that the students had to work with the mechanism design and analysis part of Pro/Engineer® without having had any introductory course.

The investigation showed that although most of the mathematical concepts and procedures are "buried in technology" for reasonable usage of the interface mathematical knowledge and understanding is still necessary. In the case of mechanism design, this refers in particular to a good understanding of the different representations of functions and their derivatives as well as of interpolation concepts.

The deeper analysis of breakdown situations (Kent and Noss, 2003) also revealed how easily faulty input or input relying on questionable assumptions produces erroneous analysis results or constructions. For the discovery of faults, it is very important to be able to compare the results with expected behaviour (oscillating torque) and to have instruments of checking at hand like simple constructions (with circle segments) or simple models (power balance). If one does not expect anything one will never see the unexpected. For finding the causes of faults, one should be able to use and interpret all the representation options in the tool (input functions and its derivatives) and also to understand how the tool produces its curves (computation of positions and interpolations). Here, the understanding of possible pitfalls of approximation methods like interpolation is also helpful. For performing variations efficiently, small models like the power model are very useful in order to identify the dominating factors of influence.

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Section 3
What Do Modeling Processes Look Like?

Chapter 9
Modeling and Quantitative Reasoning: The Summer Jobs Problem

Christine Larson

Abstract According to Smith and Thompson (Algebra in early grades, Mahwah, NJ: Lawrence Erlbaum, in press), "problem situations involving related quantities serve as the true source and ground for the development of algebraic methods" (p. 4). This study investigates the ways in which students reason about real-world problem situations involving related quantities. This study explores the role of quantities and quantitative reasoning in students' development of a solution to an open-ended real-world modeling task known as the "Summer Jobs" problem (Lesh and Lehrer, Handbook of research design in mathematics and science education (pp. 665–708). Mahwah, NJ: Lawrence Erlbaum Associates, 2000).

9.1 Theoretical Framework

My earliest experiences in the field of mathematics education involved watching a classroom full of pre-service teachers work on open-ended real-world tasks known as *model-eliciting activities* (MEAs). Lesh and Doerr (2003) characterize these activities according to the types of solutions they require of students: "… the products that students produce… involve sharable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, explaining, manipulating, predicting, or controlling mathematically significant systems" (p. 3). While the mathematics done by students as they engaged in these tasks was usually informal and at times unsophisticated, I was struck by the cleverness and personal meaningfulness of students' solutions, as well as the powerful ways of reasoning about the problem that they were able to develop by reasoning mathematically and repeatedly refining their solutions. As I continued to have experiences observing students working on these types of problems, I became curious about the strategies that students were using to develop their clever solutions. My initial intent was to characterize the

C. Larson (✉)
Indiana University, San Diego State University, Bloomington, IN, USA
e-mail: larson.christy@gmail.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_9,

strategies students developed so that those strategies might be leveraged to improve instruction. I approached this project from a *Models and Modeling Perspective* consistent with that articulated by Lesh and Doerr (2003). In this tradition, the term "model" is used to refer to the conceptual system by which students make sense of some problem situation. Thus, my attempt to characterize students' strategies was framed as my characterization of the students' models and the development of those models. As such, I appeal the following definition of a model offered by Lesh et al. (2000):

> A model is a system that consists of (a) *elements*; (b) *relationships* among elements; (c) *operations* that describe how the elements interact; and (d) *patterns* or *rules*... that apply to the relationships and operations. However, not all systems function as models. To be a model, a system must be used to describe another system, or to think about it, or to make sense of it, or to explain it, or to make predictions about it. (p. 609)

According to Harel and Lesh (2003), iterative refinement plays a central role in model development. In particular, watching students engage in MEAs provides insight into local conceptual development, as these activities require students to repeatedly express, test, and revise their ideas about the problem situation. As students repeatedly engage in these cycles, the cyclic and iterative refinement of model development can be observed. Harel and Lesh have characterized these cycles as often being situated versions of those identified by Van Hiele and Piaget.

As I began to examine the data, it became clear to me that I needed an additional lens with which to refine and articulate my characterization of students' models and the development of those models. There seemed to be a strong relationship between modeling and quantitative reasoning. Kaput (1998) went so far as to claim that "Quantitative reasoning... can be regarded as modeling – building, usually in several cycles of improvement and interpretation, mathematical systems that act to describe and help reasoning about phenomena arising in situations" (p. 16). As such, I elected to use for my analysis an additional lens: the notion of quantitative reasoning as articulated by Smith and Thompson (in press). In particular, I will define quantitative reasoning to be reasoning with and about quantities, where quantities "are measureable attributes of objects or phenomena; it is our capacity to measure them – whether we have carried out those measurements or not – that makes them quantities" (Smith and Thompson, in press, p. 10).

9.2 Methods

Participants in this study were linear algebra students at the university level. These students had a strong mathematics background and were required to complete two semesters of calculus before enrolling in this linear algebra course.

I observed four groups of students working on an MEA known as the summer jobs problem (Lesh and Lehrer, 2000). An MEA is a task that is carefully designed to act as a research tool to help document student thinking. In particular, these tasks are designed so that (1) students are able to test the quality of their own solutions (and thus revise those solution strategies as they deem necessary), and (2) students' solution to the problem is a description of the process they used to solve the problem and thus their solution provides a trail of documentation of their thinking (Lesh et al., 2000). Each group was composed of two or three students and was given 90 min to complete the task. The interviews were videotaped and student work was collected.

In the summer jobs problem, students are given data about a group of workers who sold concessions at an amusement park. This data describes the numbers of hours worked and amounts of money earned by each vendor, disaggregated by month and by how busy the park was at the time when the corresponding hours were worked and money was collected. The students' task is to write a letter to Maya, the owner of the concession business, in which they recommend who she should rehire based on the data provided. They are also to describe the process by which they determined their recommendations so that the owner can decide whether the students' method is one she wants to use, and so that she could reuse the method in the future with years when the numbers of employees under consideration may differ as well.

I began data analysis by first reducing the data. I watched the videotapes of the groups of students working on the task and examined student work, which included the letter written to Maya by each group. This gave me a sense of the nature of the strategies students were using to develop their solutions to the summer jobs problem. It seemed that tracking students' quantitative reasoning would provide a powerful lens for analyzing students' solution processes and their development. I then selected one group of two students, who I will refer to as Jeremy and Aaron, who I felt did a particularly nice job of articulating their solution process. I then performed a detailed analysis of their solution to the summer jobs problem. In this analysis, I attempt to: (a) characterize the students' model, (b) characterize the role and nature of quantities and quantitative reasoning in students' model for reasoning about the summer jobs problem, and (c) characterize the relationship between quantitative reasoning and model development (also within the context of the summer jobs problem).

9.3 Results

All four groups of students used similar strategies and quantitative reasoning in developing their solutions to the summer jobs problem. All chose to distinguish between busy, steady, and slow times but not by month, and all formed some sort of average number of dollars per hour earned during busy, steady, and slow times, respectively. (Some of the groups summed dollars and hours separately, then divided these quantities as a way of finding "average" dollars per hour; other groups formed ratios of dollars per hour first and then computed the average of these ratios as a way of finding "average" dollars per hour.) Several groups discussed the possibility of developing a scheme for weighting these quantities, but they all ended up choosing to use a ranking scheme instead as a way of "fairly" comparing dollars per hour earned during busy, steady, and slow times. Several groups also chose to compute the overall average dollars per hour earned by averaging, for each worker, the average dollars per hour earned from the busy, steady, and slow times.

I will now provide a detailed analysis of the solution developed by Jeremy and Aaron. Jeremy and Aaron computed the average dollars per hour earned during busy, steady, and slow times (respectively) by first computing the total dollars earned and total hours worked during each of these time periods and by then forming a ratio of the two quantities (see Figs. 9.1 and 9.2).

HOURS WORKED LAST SUMMER

	JUNE			JULY			AUGUST			SUMMER		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
MARIA	12.5	15	9	10	14	17.5	12.5	33.5	35	35	62.5	61.5
KIM	5.5	22	15.5	53.5	40	15.5	50	14	23.5	109	76	54.
TERRY	12	17	14.5	20	25	21.5	19.5	20.5	24.5	51.5	62.5	60.5
JOSE	19.5	30.5	34	20	31	14	22	19.5	36	61.5	81	84
CHAD	19.5	26	0	36	15.5	27	30	24	4.5	85.5	65.5	31.5
CHERI	13	4.5	12	33.5	37.5	6.5	16	24	16.5			
ROBIN	26.5	43.5	27	67	26	3	41.5	58	5.5			
TONY	7.5	16	25	16	45.5	51	7.5	42	84			
WILLY	0	3	4.5	38	17.5	39	37	22	12			

MONEY COLLECTED LAST SUMMER (IN DOLLARS)

	JUNE			JULY			AUGUST			SUMMER		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
MARIA	690	780	452	699	758	835	788	1732	1462	2177	3270	274
KIM	474	874	406	4612	2032	477	4500	834	712	9586	3740	1595
TERRY	1047	667	284	1389	804	450	1062	806	491	3498	2247	1225
JOSE	1263	1188	765	1584	1668	449	1822	1276	1358	4669	4132	257
CHAD	1264	1172	0	2477	681	548	1923	1130	89	5664	2983	637
CHERI	1115	278	574	2972	2399	231	1322	1594	577			
ROBIN	2253	1702	610	4470	993	75	2754	2327	87			
TONY	550	903	928	1296	2360	2610	615	2184	2518			
WILLY	0	125	64	3073	767	768	3005	1253	253			

Fig. 9.1 Jeremy and Aaron sum hours worked and money earned according to busy, steady, and slow times

After computing the average dollars per hour during busy, steady, and slow times, and averaging these three measures to obtain an overall "average dollars per hour," Jeremy and Aaron decide to rank the employees within each of these categories. They then summed the ranks for the busy, steady, and slow times, and finally rank the sum of the ranks. Hiring was based first off the rank of the overall "average dollars per hour," using the rank of the sum of the ranks as a "tie-breaker" if needed (see Fig. 9.3).

9.3.1 What Is the Students' Model?

Lesh's definition of a model identifies four crucial components: *elements*, *relationships*, *operations*, and *patterns or rules*. I used this framework to characterize Jeremy and Aaron's model in the summer jobs problem. I categorized the *elements*

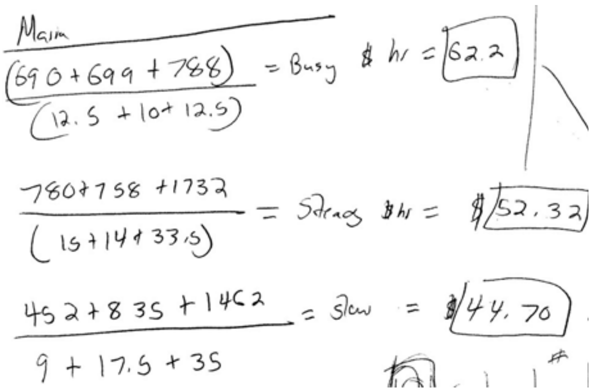

Fig. 9.2 Jeremy and Aaron divide total dollars earned by total hours worked for busy, steady, and slow times to determine dollars per hour earned

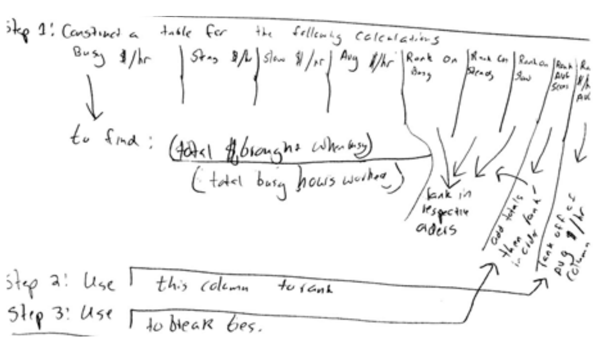

Fig. 9.3 Jeremy and Aaron then rank the employees according dollars per hour within the busy steady, and slow categories, sum those rankings, then rank the sum

in Jeremy and Aaron's model into two classes: non-quantitative elements and quantitative elements. More specifically, the non-quantitative elements were people, and each quantitative element provided information about the non-quantitative elements (people) to which it was associated. I identified four distinct types of quantitative elements, classified according to their units, that the students used to reason with and about the problem situation: (1) hours, (2) dollars, (3) dollars per hour, and (4) non-united quantities that originated from the ranking scheme developed by Jeremy and Aaron.

I observed four types of *operations* that Jeremy and Aaron invoked on the quantitative elements in their system: (1) summation, (2) division to form ratios, (3) averaging by computing the arithmetic mean, and (4) ranking. Jeremy and Aaron only summed and computed averages of quantities with like units; on the other hand, they never divided like-united quantities. To me, this suggests that the operations that Jeremy and Aaron chose to invoke on quantitative elements of their model reflected an awareness of certain *relationships* between those elements. In this way, I infer that part of Jeremy and Aaron's model involved relationships among quantitative elements as well as the non-quantitative elements. A more detailed characterization of these relationships is merited, but more sophisticated research methodologies are needed if these relationships are to be identified, documented, and generally characterized in a scientifically conscionable way. The *rules* in Jeremy and Aaron's model might be best characterized as the overall process they developed for determining, for each worker, each worker two rankings that ultimately determined the worker's fate – one that served as the primary decision-maker for re-hiring, and the second that served as the tiebreaker.

9.3.2 What Is the Role of Quantities in Students' Models?

Quantities play a central role in students' models, acting as "elements" in students' models that provide information about other, non-quantitative elements (in the case of the summer jobs problem, these other non-quantitative elements are people). Quantitative elements are special in that they can be operated on in ways that give rise to new quantitative elements (which I refer to as derived quantities) that provide different but related information about non-quantitative elements.

9.3.3 What Is the Role of Quantitative Reasoning in Students' Models?

Quantitative reasoning provides a language for characterizing the way in which students consider relationships between quantities and use these relationships to decide how to operate on these quantities in ways that will give rise to new, meaningful derived quantities (which are also then elements in the students' model) that will help them to reason with and about the problem situation.

9.3.4 What Is the Relationship Between Quantitative Reasoning and Model Development?

As students work on this problem, they repeatedly reason with and about quantities which they then operate on to create a new set of derived quantities. Furthermore, the way in which they choose to operate on existing quantities is often developed through a series of express→test→revise cycles in which the new derived quantities resulting from these operations then become focus for the students' next iteration in refining their solution process. Thus, the derived quantities that are the *products* of thought at one stage of development (often) become the *objects* of thought at the next stage of model development. In this way, quantitative reasoning can be seen as a central mechanism in model development.

9.4 Discussion

Overall, this study helped to identify quantitative reasoning as a useful lens to apply to a Models and Modeling Perspective. In terms of contributions to the broader mathematics education to a research community, this study had two key results, and I will attempt to articulate these in a way that generalizes beyond the particular language of the Models and Modeling framework:

Result #1: Operations on quantities reflect perceived quantitative relationships.... Quantities the students used were classified according to their units, and four types were identified: (1) hours, (2) dollars, (3) dollars per hour, and (4) non-united quantities. The students invoked four types of operations on those quantities: (1) summation, (2) division, (3) averaging by computation of arithmetic means, and (4) ranking schemes. The operations of summation, averaging, and ranking were only invoked on like-united quantities; the operation of division was only invoked on unlike-united quantities. This suggests that the operations students choose to invoke on quantities are reflective of the relationships they perceive among those quantities.

Result #2: Quantitative reasoning is a central mechanism for iterative refinement of solution processes to real-world problems ... As students work to develop a solution to this problem, they repeatedly reason with and about quantities which they then operate on to create new derived quantities. The students subsequently use these new derived quantities to reason with and about the problem situation, and they operate on these derived quantities to create even more, new, derived quantities with which they reason with and about the problem situation, and so on. Thus the derived quantities that are the products of thought at one stage of development (often) become the objects of thought at the next stage of development. In this way, quantitative reasoning can be seen as a central mechanism toward students' development of a solution.

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Chapter 10
Tracing Students' Modeling Processes in School

Nicholas Mousoulides, M. Pittalis, C. Christou, and Bharath Sriraman

Abstract In this study, we report on an analysis of the mathematization processes of one 6th and one 8th grade group, with emphasis on the similarities and differences between the two groups in solving a modeling problem. Results provide evidence that all students developed the necessary mathematical constructs and processes to actively solve the problem through meaningful problem solving. Eighth graders who were involved in a higher level of understanding the problem presented in the activity employed more sophisticated mathematical concepts and operations, better validated and communicated their results and reached more efficient models. Finally, a reflection on the differences in the diversity and sophistication of the constructed models and mathematization processes between the two groups raises issues regarding the design and implementation of modeling activities in elementary and lower secondary school level.

10.1 Introduction

A number of professional organizations (AAAS, 1998; NCTM, 2000) address the need for a change in school mathematics. They also propose reforms in mathematics education to fulfil the economy and work force's demands for school graduates that are able to possess flexible and creative mathematical problem solving abilities

N. Mousoulides (✉)
University of Cyprus, Nicosia, Cyprus
e-mail: n.mousoulides@ucy.ac.cy

M. Pittalis (✉)
University of Cyprus, Nicosia, Cyprus
e-mail: m.pittalis@ucy.ac.cy

C. Christou (✉)
University of Cyprus, Nicosia, Cyprus
e-mail: edchrist@ucy.ac.cy

B. Sriraman (✉)
University of Montana, Missoula, MT, USA
e-mail: sriramanb@mso.umt.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_10,

and to effectively use technological tools in working collaboratively in demanding projects (NCTM, 2000). An effective medium for achieving the above demand for a change in school mathematics is modeling; the study of mathematical concepts and operations within real world contexts and the construction of models in exploring and understanding real complex problem situations (Lesh and Doerr, 2003; English, 2006).

Although mathematical modeling has been reserved for the upper secondary and tertiary education levels (Greer et al., 2007), recent research shows that students in primary and lower secondary school level can effectively work with modeling activities; constructing appropriate and effective models for solving complex real world problems and effectively using where available technological tools (e.g., English and Watters, 2005; Mousoulides, 2007). The present study aims to further contribute to our understandings of students' modeling processes and mathematical developments in elementary and lower secondary school, by examining similarities and differences between 6th and 8th grade students' modeling processes and mathematical developments. This identification is expected to further contribute to the appropriate introduction of modeling in elementary school mathematics.

10.2 Theoretical Framework

Mathematical modeling has been considered to be an effective medium to prepare students to deal with unfamiliar situations by thinking flexibly and creatively and to solve real world problems (Lesh and Doerr, 2003; English, 2006). Although the National Council of Teachers of Mathematics (NCTM, 2000) calls for purposeful activities along with skilful questioning to promote the understanding of relationships among mathematical ideas, this recommendation can be pushed further and modeling activities can be used as a way to cultivate critical thinking and critical literacy (Mousoulides, 2007, The Modeling Perspective in the Teaching and Learning of Mathematical Problem Solving, Unpublished Doctoral Dissertation). Related research indicated that student work with modeling activities assisted students to build on their existing understandings and to be successfully engaged in thought-provoking, multifaceted complex problems (English, 2006). Modeling activities set within authentic contexts, allow for students' multiple interpretations and approaches, promoting intrinsic motivation and self regulation. Students are also engaged in important mathematical processes such as describing, analyzing, constructing, and reasoning as they mathematize objects, relations, patterns, or rules (Lesh and Doerr, 2003).

Research in the field of mathematical modeling listed six design principles for developing modeling activities that are based on the work of teachers and researchers and that have subsequently been refined by Lesh and his colleagues (2000). The Model Construction Principle ensures that the solution to the activity requires the construction of an explicit description, explanation, procedure, or justified prediction for a given mathematically significant situation. The Reality

Principle requires that students can interpret the activity meaningfully from their different levels of mathematical ability and prior knowledge. The Self-Assessment Principle ensures the inclusion of criteria that the students themselves can identify and use to test and revise their current ways of thinking. Specifically, the modeling activity should include information that students can use for assessing the usefulness of their alternative solutions, for judging when and how their solutions need to be improved, and for knowing when they are finished. The Model Documentation Principle ensures that while completing the modeling activity, the students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation. The fifth principle is the Construct Share-Ability and Re-Usability Principle, which requires students to produce share-able and re-usable solutions that can be used by others beyond the immediate situation. The Effective Prototype Principle ensures that the modeling activity is as simple as possible yet still mathematically significant. The goal is for students to develop solutions that will provide useful prototypes for interpreting other similar situations.

During the last years, an increasing number of researchers have focused their research efforts on mathematical modeling, especially at the school level (e.g., English, 2006, 2003). It is a necessity to implement worthwhile modeling experiences in the elementary and middle school years if teachers are to make mathematical modeling a successful way of problem solving for students (English, 2006). Modeling activities differ from traditional approaches to the teaching of elementary mathematics for a number of reasons. First, the mathematical concepts and operations that are needed to successfully solve the problems appear in the modeling activities go beyond what is taught and how in traditional mathematics classrooms. Additionally, mathematical concepts are presented with connections to real world situations and students have the opportunities to further elaborate on the related concepts and build on their prior understandings (English, 2006).

In modeling problems, students are offered rich learning opportunities. The problems presented are not carefully mathematized for the students, and therefore students have to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations. Students mathematize the problems in ways that are meaningful to them, and this process can result in improving competencies in using mathematics to solve problems beyond the classroom (English, 2003; Mousoulides, 2007). A second characteristic of modeling activities for elementary school students is that they encourage the development of generalizable solutions. These solutions (models) focus on the structural characteristics of the problems (systems) that they are referring to, and are expressed using a variety of representational media, including written symbols, diagrams or graphs. The latter is central in mathematics learning for elementary school students (Lesh and Doerr, 2003). In the activity presented in this study, student models for rates, for example, might include working hours and money collected, operations such as multiplication and division, and relationships between working hours and money collected that stand for productivity rate. In addition, models incorporate a number of external representations (e.g., graphs, tables). In constructing models, students identify, select and collect relevant data, interpret the solution in context, describe situations using a variety of representation media and document and communicate their solutions (Lesh and Doerr, 2003).

Students in the elementary and lower secondary school level can be benefited from working with authentic modeling problems (English and Watters, 2005; English, 2006; Mousoulides, 2007). The use of modeling activities encourage students to develop important mathematical ideas and processes that students normally would not meet in the traditional school curriculum. The mathematical ideas are embedded within meaningful real-world contexts and are elicited by the students as they work the problem. Furthermore, students can access these mathematical ideas at varying levels of sophistication (English, 2006). Student work in modeling activities facilitates student development of generalizable conceptual systems, as students move beyond just thinking about their models to thinking with them for solving real problems (Lesh and Doerr, 2003). English (2006) also reported that there was considerable evidence that students' mathematical ideas, mathematical language and fluency in using tables and data were improved after they worked in a sequence of modeling activities.

10.3 The Present Study

10.3.1 The Purpose of the Study

The purpose of the present study was twofold; first to examine students' modeling and mathematization processes as they worked on a mathematical modeling problem, and second to investigate similarities and differences between 6th and 8th grade (11 and 13 year olds) students' modeling and mathematization processes. The problem addressed in this study, "University Cafeteria", required students to construct models for selecting the best among a number of vendors. Specifically, students were given two tables presenting the hours worked and money collected for nine vendors in a university cafeteria (see Fig. 10.1). Based on the data provided, students needed to construct models for selecting three full time and three part time vendors for next year.

10.3.2 Participants, Modelling Activity, and Procedures

One intact class of 6th grade students (12 females and 7 males) and one intact class of 8th graders (10 females and 8 males) from two urban schools in Cyprus participated in the "University Cafeteria" modeling activity. The activity is a modified version of the "Summer Jobs" activity (see Lesh & Doerr, 2003). Students in both classes had prior experience working with modeling activities, since they participated in a larger project for implementing modeling activities in elementary and lower secondary school level (see Mousoulides, 2007). Specifically, prior working

Hours Worked Last Year									
	Autumn Semester			Spring Semester			Summer Period		
	Busy	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**
Maria	12.5	15	9	10	14	17.5	12.5	33.5	35
Kim	5.5	22	15.5	53.5	40	15.5	50	14	23.5
Terry	12	17	14.5	20	25	21.5	19.5	20.5	24.5
Jose	19.5	30.5	34	20	31	14	22	19.5	36
Chad	19.5	26	0	36	15.5	27	30	24	4.5
Cheri	13	4.5	12	33.5	37.5	6.5	16	24	16.5
Robin	26.5	43.5	27	67	26	3	41.5	58	5.5
Tony	7.5	16	25	16	45.5	51	7.5	42	84
Willy	0	3	4.5	38	17.5	39	37	22	12

Money Collected Last Year									
	Autumn Semester			Spring Semester			Summer Period		
	Busy	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**
Maria	690	780	452	699	758	835	788	1732	1462
Kim	474	874	406	4612	2032	477	4500	834	712
Terry	1047	667	284	1389	804	450	1062	806	491
Jose	1263	1188	765	1584	1668	449	1822	1276	1358
Chad	1264	1172	0	2477	681	548	1923	1130	89
Cheri	1115	278	574	2972	2399	231	1322	1594	577
Robin	2253	1702	610	4470	993	75	2754	2327	87
Tony	550	903	928	1296	2360	2610	615	2184	2518
Willy	0	125	64	3073	767	768	3005	1253	253

Fig. 10.1 The number of hours worked and money collected by each vendor

with the "University Cafeteria" students worked in four other modeling activities for a period of two months.

Students worked for four 40 minute sessions to find a solution for the problem presented in the activity. The purpose of the activity was to provide opportunities for students to organize and explore data, to use proportional reasoning and to develop appropriate models for solving the problem. Additionally, the activity provided a setting for students to work with the notions of ranking, selecting, aggregating ranked quantities and weighting ranks. During the first session, which lasted around 15 to 20 minutes, students were presented with a newspaper article about the different factors employers take into consideration when hiring employees, followed by readiness questions. Following, during the modeling stage of the activity (80-90 minutes) students worked in groups of three, using a spreadsheet software, to solve the "University cafeteria" problem. In the last session student presented their models in whole class presentations for reviewing and discussing with their peers. Finally, a whole class discussion focused on the key mathematical ideas and processes that were developed during the modeling activity. For the purposes of the present study the results from one representative group of students in each grade level during the modeling stage of the problem are presented here.

10.3.3 Data Sources and Analysis

The data for this study were collected through (a) audiotapes of students' work in their groups, (b) spreadsheet files from students' work, (c) students' worksheets, detailing the processes used in developing their models, and (d) researchers' field notes. The analysis of the data mainly focuses on the results of one group of students in each grade, as they worked on the modeling stage of the activity. The selected groups were representative of the two classes, in a way that other groups in their classes reported similar work.

The analysis of the data was completed in the following steps. First, the transcript was reviewed several times to identify the ways in which students interpreted the problem, their approaches to use data from the two tables, and their mathematization processes in linking data from the tables. The transcript was also reviewed to identify how students interacted in their group, and how discussion within the group resulted to a final model. Finally, all students' worksheets, spreadsheet files, and their final written letters were analyzed to identify and compare the mathematization processes used in their model development.

10.4 Results

The results of the study are presented in terms of the mathematization processes presented by each group of students. Specifically, the modeling processes, appeared in students' work, are presented for each group with regard to the steps of the modeling procedure (Description, Manipulation, Prediction of the problem, and Solution Verification). The respective students' mathematical developments are presented in three cycles, with regard of increased sophistication of mathematical thinking.

10.4.1 Modelling Processes

Sixth graders failed to fully understand the problem; they understood the core question of the problem but they did not succeed in connecting the core question with the provided data. As a result, they only focused on isolated parts of the data from the money collected table. Students' initial models were inadequate to solve the problem and only researchers' comments helped students to overcome these difficulties. On the contrary, eighth graders made the necessary connections and almost immediately merged data from both tables, and identified patterns and relations. However, their first attempts focused on specific vendors and they commented on either the amount of money collected or either the hours for each vendor. It was apparent that 8th grade students identified the necessary variables and relationships to describe and understand the problem. However, they failed to handle and relate these understandings with the core question of the problem.

During problem manipulation, both groups of students presented a number of interesting models. A number of differences can be tracked between 6th and 8th grade students' models. Sixth graders' initial model was based on ranking the nine vendors in each column and then finding a total ranking for autumn semester. Students proceeded on a second model, which was based on calculating the total amount of money each vendor earned, when they realized that their first model was not efficient enough. However, even this new second model was not appropriate since it was not based on data from both tables. Students' next (last) model was resulted after a discussion with the researchers; this "performance rate" model could answer (to some extent) the core question of the problem. Eighth graders' work resulted in better and more sophisticated models, which could answer the core question of the problem. Specifically, students easily reached a model based on the "performance rate" for each vendor and used this model in selecting the best six vendors. This model was based on calculating the total amount of money collected by each vendor and divided by the total number of hours each vendor worked. However, 8th graders' first model did not take into consideration the semester and/or time period factors. Eighth graders further improved their first model by firstly employing the semester dimension and second the time period dimension. These two final more sophisticated models extensively used the whole data set, including constraints, parameters and patterns in the data.

The fact that 6th graders did not interpret their results in the context of the real problem was one of the reasons that their solutions were not successful enough. Students failed to examine the appropriateness of their model and to discuss issues related to model's interpretation. In the case that two vendors were quite closed in terms of money collected, for example, there were long debates on deciding which vendor to select. However, 8th graders did not employ semester and/or time period factors. A possible reason was that students might lack the necessary mathematical concepts to better mathematize the real problem and therefore to construct a refined model. Eighth graders made significant efforts not only to interpret their model in the context of the real problem but also to examine different models and to make the selection based on this interpretation. Students' first interpretations questioned the appropriateness of their model, since that model could not help them in selecting the best vendors. A second dimension of models' interpretation and examination was presented in students' final discussion. Students' work resulted in two different but mathematically correct models. One model was based on the different semesters and the second on the different time periods. At that time, students asserted that being competent in different time periods was more important than being competent in different semesters and they concluded in adopting the time periods model.

A number of differences related to the verification of the solutions appeared between 6th and 8th grade students' work. Sixth graders did not actually verify their solution, but they only compared their last model to previous ones and made comments. This was a major disadvantage of their work and blocked their efforts to further improve their model. On the contrary, 8th graders not only reached two mathematically correct models, but they also based their final decision on the context of the real problem. Students finally chose the time periods based model and not the semester one, while documenting and supporting their decision. In terms of documentation and communication, students expressed their ideas and solutions not only verbally, but using a variety of representational media, including different graphs (bar chart, line graphs) and sketches. Students' efforts to convince the imaginary client (cafeteria manager) about the correctness of their solution encouraged students to reflect on their models. In their letters, students documented their results about the specific problem and tried to provide solutions for structurally similar problems. In comparing students' work in terms of communicating their results, three differences can be extracted. Eighth grade students used a variety of representations in documenting and explaining their results in their letters. The second difference was again located in students' letters to the cafeteria's manager. Eighth graders' letters were presented in details, and were based on students' previous approaches and models. The third difference relied on the discussions students had in their groups. Eighth grade students extensively discussed most of the issues that arose during their investigations. On the contrary, 6th graders' discussions were of less importance and in many times students just expressed their ideas, without trying to elaborate on their peers' ideas.

10.4.2 Mathematical Developments

Students' mathematical developments are summarised in terms of cycles of increased sophistication of mathematical thinking, with each cycle representing a shift in thinking. The analysis of students' mathematical developments is presented in the following order. First, students' efforts were limited to focusing on subsets of information. Second, students started using mathematical operations (e.g., finding ratios) and third students' work was based on more sophisticated mathematical ideas, such as identifying trends and relationships among the data.

Sixth graders commenced the activity by scanning the money table to find vendors who scored highly in one or more columns (i.e., money collected in busy, steady or slow time periods, in autumn or spring semester etc). Only limited mathematical thinking was displayed in students' unsystematic work. This was also evident in students' comments: "Jose and Chad both worked 19.5 h. Wow, Robin worked more. She worked 26.5 h. She is first". Students decided to use data from both tables only when they failed to find a solution. However, students' approach still remained unsystematic and isolated as they did not manage to "merge" data from both tables. As a result, students still used descriptive comments: "Robin worked more hours and earned more money than any other vendor". The inexistence of any systematic approach resulted in contradictions, which generated the need to further mathematize their approach. The group began to use two mathematical operations to aggregate the data for each vendor, namely: (a) simply totalling the amount of money each vendor earned and how many hours each vendor worked, and (b) finding the average for each category (money, hours) and classifying vendors above and below average.

Similarly to 6th grade group's work, 8th grade students commenced the activity by scanning the two tables to find the vendors who scored highly in one or more columns. This initial approach can be characterized as unsystematic. Students, for example, made comments like "Look! Robin worked more hours than anybody else" or "Tony might was on vacations (when Tony worked only for three hours)".

Students did not pay attention to the column headings. As a consequence, when they started mathematizing the problem, their first model was based on finding the total number of hours and money for each vendor. In contrast to 6th graders, 8th graders immediately realised that they had to use data from both tables. However, students' approach still remained unsystematic and limited as they only commented on specific vendors, without making any connections between the two tables. These results demonstrated the inefficiency of their approaches and encouraged students to mathematize their work. Students used mathematical operations to aggregate the data for each vendor, namely finding for each vendor the total amount of money earned and the total number of hours worked. They finally used the two rankings for selecting the six vendors.

The second cycle of students' mathematical developments can be characterized by the mathematical operations students used in constructing their models. Sixth graders acknowledged the weakness of their approach and suggested finding for each vendor the total amount of money collected in autumn semester. They justified their decision by explaining that: "It's difficult to find the best in each column. Maria is sometimes amongst the best ones and in other columns she is amongst the worst vendors". To resolve the issue and to better classify vendors, students decided to find the average and then classify vendors in two categories; above and below average. However, students only used data from the hours worked table. A second dimension of 6th graders' work was the misinterpretation of the hours worked table. Specifically, among two vendors who collected the same amount of money students chose the vendor who worked more hours. When 8th graders faced the need to rank the nine vendors, they decided to total the number of hours for each vendor and the total amount of money each vendor earned during the three semesters. They justified their decision by explaining that: "We can find one ranking for the hours worked and one ranking for the money collected. We can then use these rankings to select the three vendors that will work full time and the three that will work part time". However, the above model was not efficient for answering the core question of the problem, since the two rankings were contradictory. Terry, for example, was fifth in the first and ninth in the second ranking.

During the third cycle of mathematical developments, students' work can be characterized by the identification of trends and relationships in the data. Sixth graders realised, after discussing with the researcher, that their average based model was not appropriate enough and they focused their efforts on finding a relationship between money collected and hours worked for each vendor. The acknowledgement of these requirements led students to progress to the notion of rate. However, the notion or rate was partially employed, since students did not take into consideration the different time periods (busy, steady and slow). Eighth graders decided to proceed in finding the money per hour ratio for resolving the conflicts they faced and for classifying the nine vendors. At the same time, substantial discussion and argumentation took place when students tried to find a way to merge data from both tables. Students progressed to looking for a relation between money collected and hours worked. Students quite easily identified the relationship between money and hours and they constructed a model for calculating the money per hour ratio. Part of this discussion focused on “vendor’s productivity”. Specifically, students agreed upon selecting the most productive vendors and they defined productivity as the amount of money each vendor earned in 1 h. The acknowledgement of this new parameter directed students’ “performance rate” model.

Eighth grade students were concerned that using the “performance rate” model resulted in a totally different ranking. Although it was apparent that students were satisfied with the solution, they questioned the appropriateness of their model. Consequently, students refined their last model by finding money per hour ratio for each semester and for each time period. These two new rankings were completely different than the first one and there were also differences between them. Students decided to adopt this time period based model, since “it is reasonable to base our selection on the different time periods instead of the different semesters. It is important that one vendor is good in all time periods”. However, students failed to use these assumptions in further refining their model (e.g., weighting time periods).

10.5 Discussion

A significant finding of the present study, which is in line with other studies (e.g., English, 2003, 2006), is that elementary and lower secondary school students are able to successfully work with mathematical modeling activities when presented as meaningful, real world problems. The framework, within the “University Cafeteria” activity, helped students to realize and to get familiar with the problem situation and thus enhanced their understandings. Students in both groups progressed from focusing on subsets of information which resulted in not efficient models to applying the appropriate mathematical concepts and operations that helped them finding an appropriate mathematical model based on the “performance rate”. Their models were reusable, shareable and could serve to construct more sophisticated models for solving more demanding problems (Doerr and English, 2003). The results of the study provide evidence that students in both groups developed the necessary mathematical constructs and modeling processes to actively engage and solve the problem through meaningful problem solving. Students in both groups effectively used the spreadsheet software to represent their data in graphical and symbolic form and to find ratios for ranking the different vendors.

Among the differences between the two groups, 8th grade students more easily identified the necessary variables and relationships to describe and understand the problem, taking into consideration data from both tables. Additionally, 8th graders presented more refined and sophisticated solutions and successfully solved the problem by employing two “performance rate” models, based on the different semesters and time periods. A third difference between 6th and 8th graders was from the perspective of communication and assessment. Although students in both groups adequately communicated their ideas and solutions, only 8th graders undertook constructive assessment by listening to and reflecting on their peers’ suggestions and models. On the contrary, 6th graders mainly presented their personal ideas in their discussions and not reflected on others’ ideas and suggestions.

Although it is difficult to explain the differences between the two groups of students, given the plurality of possible reasons, it is nevertheless important to examine how these differences might influence the design and implementation of modeling activities in elementary and lower secondary school level. These differences could account in part for the variation in diversity and sophistication of the models the students created. The design of modeling activities should take into consideration students' mathematical developments and therefore do not constrain students' efforts in solving such problems. Additionally, what was evident from the present study was the importance of the researcher's – teacher's role in overcoming the difficulties that arose in, mainly, 6th graders' work. In concluding, the findings of the study shows that although modeling activities can be successfully implemented in elementary and lower secondary school and that can improve students' mathematical understandings; further research is needed towards this direction.

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Section 4
What Creates "The Need For Modeling"

Chapter 11
Turning Ideas into Modeling Problems

Peter L. Galbraith, Gloria Stillman, and Jill Brown

Abstract We show how the nucleus of an idea can be developed into modeling problems for secondary school using principles for problem design enunciated in Galbraith (2007). Once the germ has been developed for a task, the idea can be extended as necessary into related problems closer to the personal experience of the adolescents in secondary school. The issues and contexts secondary students choose to investigate, and the questions that they pose when given free reign or minimal constraints, are illustrated from an Australian modeling challenge. Finally, using these contexts as starting points, it is suggested such situations can be developed to engage students in important teaching issues involving necessary constituents of the modeling process.

11.1 Introduction

Over the years, a variety of modeling problems have been presented and discussed in forums such as this. While also referring to specific problems, our focus in this chapter is on the development process that leads from the germ of an idea to a problem that can be implemented in classrooms. The chapter has two parts. In the first part we articulate and apply principles to illustrate how a very general idea can be turned into a modeling problem. In the second part we illustrate from within a student workshop context, how examples relevant to their interests and environment can be used to both initiate modeling activity, and raise deeper issues associated with the practical implementation of modeling practices.

P.L. Galbraith (✉)
University of Queensland, Brisbane, QLD, Australia
e-mail: p.galbraith@uq.edu.au

G. Stillman (✉)
University of Melbourne, Melbourne, VIC, Australia
e-mail: g.stillman@unimelb.edu.au

J. Brown (✉)
Australian Catholic University, Melbourne, VIC, Australia
e-mail: jill.brown@acu.edu.au

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_11, © Springer Science+Business Media, LLC 2010

Wind Chill - The US National Weather Service uses the following formula to calculate wind chill:
WC = 91.4 – (0.474677 – 0.020425 * V + 0.303107 * SQRT (V)) * (91.4 -T)
where, WC = wind chill, V = wind speed (mph), T = temperature (° F)
Human Body Surface Area
Gehan & George formula: BSA (m^2) = $0.0235 \times \text{Height (cm)}^{0.42246} \times \text{Weight (kg)}^{0.51456}$

Fig. 11.1 Black box formulae using mathematics

To begin, consider an example (Fig. 11.1) of what we believe mathematical modeling is not. Merely using such formulae does not constitute mathematical modeling, although modeling was undoubtedly involved in their generation. Various such black box formulae exist that show how mathematics is used, not how to use mathematics.

Because various meanings are given to the term *mathematical modeling,* we begin by clarifying our aims with respect to modeling, and thence our purpose in designing modeling tasks.

11.2 Approaches to Mathematical Modeling

Julie (2002) identified two approaches to the teaching of modeling.

11.2.1 Modeling as Vehicle

For those using modeling as vehicle approach, mathematical modeling "serve[s] as a venue for mathematical learning in classrooms where learning goals focus on curricular mathematics" (Zbiek and Conner, 2006, p. 90), not "becoming proficient modelers" (p. 89). The purposes of such an approach are primarily to (a) service curricular needs, (b) motivate the need for particular content material and (c) use real world contexts to help to elicit and develop mathematical content. This follows in the tradition of Hickman (1986, p. 175) who used the term "didactic modeling" (for modeling used primarily for teaching purposes), when discussing some of the notions about mathematical modeling arising from the first ICTMA conference in 1983. Emergent modeling (Gravmeijer, 2007) is a related approach that focuses on opportunities to develop mathematical concepts by modeling realistic situations.

11.2.2 Modeling as Content

In the 'modelling as content' approach, "the modelling process is driven by the desire to obtain a mathematically productive outcome for a problem with genuine real-world motivation" (Galbraith and Stillman, 2006, p. 143). The purposes of this approach are to (a) develop student abilities to apply mathematics to problems in their world, (b) take mathematics beyond the classroom and (c) use the real world context as a key component.

In this latter approach, which we follow here, mathematical modeling is a process involved with solving problems arising in *other discipline areas* or in a *real* *world environment*. The process cannot live entirely in the mathematics classroom (it begins and ends in the real world or other subject contexts). We note further that a *mathematical model* (several may be involved) is only a *part* of the whole.

11.3 Educational Rationale

We see mathematical modeling as a significant activity, which can be useful in both developing and applying mathematical content. Students should be able to employ the mathematics they spend so much time learning, to address problems in their world. As Burkhardt (2007) points out, "modelling everyday life situations is at the heart of functional mathematical literacy" (p. 180).

11.3.1 A Framework for Mathematical Modeling

Modeling process diagrams such as Fig. 11.2 describe how problems are modeled and solved. Such a diagram, which will be familiar to many readers, is in the tradition of those originally designed and refined by modelers (e.g. Penrose, 1978). It is included here for completeness, noting that the diagram, as well as encapsulating the modeling process, can act as a metacognitive scaffolding aid for novice modelers.

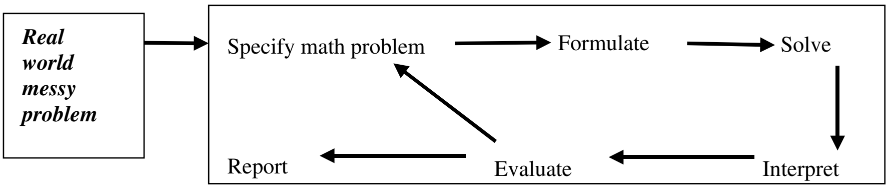

Fig. 11.2 Modeling process

11.4 Suitability Criteria for Modeling Problems

As outlined recently by Galbraith (2007, p. 55), the following principles can be used to identify a potential situation as suitable for model development.

- *Principle 1*: There is some genuine link with the real world of the students (RELEVANCE AND MOTIVATION).
- *Principle 2:* It is possible to identify and specify mathematically tractable questions from a general problem statement (ACCESSIBILITY).
- *Principle 3:* Formulation of a solution process is feasible, involving (a) the use of mathematics available to students, (b) the making of necessary assumptions, and (c) the assembly of necessary data (FEASIBILITY OF APPROACH).

- *Principle 4:* Solution of the mathematics for a basic problem is possible for the students, together with interpretation (FEASIBILITY OF OUTCOME).
- *Principle 5:* An evaluation procedure is available that enables solution(s) to be checked for (a) mathematical accuracy and (b) appropriateness with respect to the contextual setting (VALIDITY).
- *Principle 6:* The problem may be structured into sequential questions that retain the integrity of the real situation (DIDACTICAL FLEXIBILITY).

These principles, in providing a guide for turning ideas into modeling problems, represent a shift from an emphasis on a particular problem to the generic process of model generation – from being given a fish, to knowing how to catch fish. We now illustrate the application of these principles by developing modeling problems stimulated by the Morgan Spurlock Film (Spurlock, 2004), *Supersize Me*.

11.5 Example: Supersize Me (Morgan Spurlock Film)

The problem of obesity/diet/exercise is a much-publicised issue in a number of countries. In some places legislation has been enacted that is directed, among other things, at the kind of foods available in school tuck shops, and the amount of exercise timetabled for students on a daily or weekly basis.

In his film/documentary, Spurlock described his experiment of eating fast food.

Rules of the experiment

- For 30 consecutive days Morgan Spurlock ate three meals consisting of nothing but McDonalds food and beverages (approximately 5000 calories daily).
- If offered an "upsize" he must take it.
- He ate everything on the McDonalds menu at least once.
- Daily exercise was limited to that of the average American office worker.

Consequence: Spurlock went from 84 to 95.5 kg – there were other effects such as a dangerously high cholesterol level.

Open modeling problem: Build and *test* a model to describe Morgan Spurlock's weight gain.

Simplifying assumptions are that the (average) food intake may be taken as constant from day to day, as may be the rate at which energy is burned (sedentary lifestyle). The former is given (5000 calories) and the latter needs to be estimated. Many websites offer information on diet, exercise, and the energy value of foods. These differ a little on specifics but tell the same general story. Here is one simple method (Table 11.1) given for estimating the daily calorie intake to maintain body weight according to metabolic needs.

Males: Weight (kg) $\times$ 24 $\times$ Activity Factor $=$ Daily calorie needs
Females: Weight (kg) $\times$ 24 $\times$ 0.9 $\times$ Activity Factor $=$ Daily calorie needs

Table 11.1 Activity factors for various life styles

Life style	Activity factor	Exercise and activity
Very active	1.4–1.5	Daily intense exercise + construction work most of day
Active	1.3–1.4	Daily exercise + work on feet most of day
Light active	1.1–1.2	Exercise 3 times/week + desk job or at home most of day
Sedentary	1	No exercise + desk job or at home most of day

Nutritionists estimate a weight gain of roughly 1 kg for every 7800 extra calories.

An approach to the solution to this open problem is given in the Appendix. For other purposes it can be useful to design a structured format as in Fig. 11.3.

1. In general weight (today) = weight (yesterday) + excess calories consumed today converted to weight. If 'I' represents the average daily intake of calories, show that Morgan Spurlock's weight after day 1 of his 'diet' is estimated by:

$w_1 = w_0 + (I - 24\ w_0)/7800$. Express this in the form $w_1 = aI + bw_0$, and give the numerical values of 'a' and 'b'.

2. Design a spreadsheet that will calculate weight w_n for each successive day from n = 1 to 30, and use it to estimate his weight after 30 days. Comment on the result.

3. Show that in general $w_n = aI(1 + b + b^2 + \ldots + b^{n-1}) + b^n w_0 = aI(1-b^n)/(1-b) + b^n w_0$, and use it to check the result obtained in 2.

Fig. 11.3 Structured format of *Supersize Me*

Structuring may be used within an open modeling approach by releasing the detail selectively as hints when deemed necessary by the teacher. Alternatively, a structured format may enable creative applications of mathematics to be introduced that otherwise would never occur because of curriculum constraints. Then, once the germ of an idea like Morgan Spurlock's approach to sensitise the community to issues surrounding obesity has been developed into a task, the idea can be extended to generate related problems (see Fig. 11.4) closer to the personal experience of the audience, that is, adolescents in secondary school.

Ben weighs 72 kg and has an activity level at the top end of 'light active'.
For fun and additional exercise he skateboards for an hour a day, and his normal food intake maintains his weight at 72 kg. Some fast food outlets open near his home, and after skateboarding Ben now has a daily routine in which he eats a cheeseburger and drinks a 375 ml coke. The rest of his diet remains the same. What is the effect of this habit on Ben's weight after two months (60 days)? (Alternatively, Ben could be replaced with a personal choice of weight, food, and exercise using Tables 11.2 and 11.3.)

Fig. 11.4 Related problem for *Supersize Me*

Table 11.2 Calories burnt per kg per hour during selected exercise

Exercise type	Calories burned per kg per hour
Cycling, moderate effort	8.0
Running, on a track, team practice	10.0
Skateboarding, softball or baseball	5.0
Walking, 3.0 mph, moderate pace, walking dog	3.5
Walking, using crutches	4.0

Table 11.3 Calorific value of selected fast foods and drinks

Item	Calories
Cheeseburger: Ground beef patty, grilled (40 g), hamburger bun (65 g), lettuce, sliced tomato and onion, tomato sauce, cheese (16 g)	391
Domino's Pizza Deluxe, per slice (66 g)	171
Lemonade (375 ml), Coca Cola (375 ml)	158

11.6 Which Contexts Interest Students

Often it is claimed that mathematical modeling should be used "to motivate students to study mathematics by showing them the real-world applicability of mathematical ideas" (Zbiek and Connor, 2006, p. 89). But, which real-world contexts really interest students? As Julie (2007) points out, "the issues and contexts learners prefer for mathematics investigation is a largely under-researched area" (p. 193). What students might perceive as "personally relevant to them" is also "transitive and time-dependent" but it is often argued "that personal ownership of problem situations can be fostered by contexts found desirable by learners" (p. 201). Julie investigated the learners' context preferences of students in Years 8, 9 and 10 in low socio-economic areas in South Africa using a survey instrument. In contrast we look at the choices of Australian Year 10 and 11 students from a mix of private and public secondary schools working in mixed school teams at an annual two-day modeling challenge in Queensland, Australia, *The AB Patterson Gold Coast Modeling Challenge*. We are particularly interested in the type of topics and the questions that interest secondary school students when they are allowed to choose the modeling situations and pose the questions themselves.

A selection of topics and the questions posed by the students in the Years 10/11 section of the Challenge when allowed free rein in their choice are shown in Table 11.4. Even if a degree of restriction is imposed such as population modeling, there is still enough freedom for students to pursue their own interests so long as they possess enough mathematical tools within the group to complete the task in the time frame. Teams were expected to choose a real world situation, pose a problem, make and state any assumptions, clearly identify relevant variables and the basis of any estimates they found necessary, produce a model or models as the case may be, make predictions (as appropriate) and/or draw conclusions in answer to the question

Table 11.4 Modeling situation choices and questions posed

Topic	Question(s)
Man-made disasters	
Aral Sea in Uzbekistan	When will the Aral Sea dry up completely?
Catastrophic events	
Tsunamis	How can you predict the severity of the damage of a Tsunami from the Richter scale value based on energy?
Environmental problems	
Drought causing water shortages in cities	How much water do we really have left in the Hinze Dam? Will it cater for the current and future population of the Gold Coast?
Unrestricted spread of introduced biological control agent (the cane toad)	What is the culling rate needed to stabilise the cane toad population in Australia?
Disease (epidemics and pandemics)	
SARS	If another SARS epidemic breaks out in the future, what will be expected deaths and infected cases at the end of a month?
HIV/AIDS	What do the past and current trends in AIDS diagnosis and deaths suggest for the future number of people afflicted and their odds for surviving?
CJD (Human form of "Mad Cow" disease in the UK)	In 2001 the UK Chief Medical officer said the final death toll was likely to be "hundreds and hundreds of thousands". Is it likely that there is that number of people who are infected but not showing symptoms?
Sport	
Archery	What is the optimum angle of elevation for the release of an arrow in the sport of archery so that the arrow hits the perfect bullseye? How does air resistance subsequently influence this optimum release angle?
Population modeling in restricted habitats	
Banteng cattle (a feral species) on the Cobourg Peninsula in the Northern Territory	How can we describe mathematically the dynamics of a small population of feral cattle released into virgin land where there are few predators and not limiting resources?
Feral Pigs in Mt Kosciusko National Park	What culling rate would be needed to ensure the feral pig population died out in the national park?

posed and evaluate their model(s) specifying any limitations or revisions needed. As the time is restricted, the students may find once they evaluate their model that it is inadequate or in error and not have time to revise. In such circumstances they were expected to critique the model and suggest where they might explore next. All these elements have to be displayed on a poster by 3 pm on the second day for judging.

Caron and Bélair (2007) point out that from a modeling perspective there are benefits in choosing social science topics rather than purely scientific topics. Firstly, most students are familiar with them at the general level (e.g., a natural disaster) if not the specifics, thus alleviating the need for domain specific knowledge "where the formulation of the real-world situation itself requires demanding a priori knowledge" (p. 127). Secondly, social data (e.g., number of SARS cases reported in a particular time period in Singapore) is often readily available via the Internet. Thirdly, "social contexts seem to favour a critical analysis of the model(s) used" (p. 128). It is not surprising then that many of the contexts chosen by adolescents in the circumstances described for the challenge are social or ecological contexts. What is surprising is that in a nation extolling sports of all kinds few teams chose to use a sports context even when suggested.

11.6.1 Using Student Selected Topics for Teaching Modeling Competencies

"The development and description of the problem to be tackled is the most important and most ambitious part of a modeling process, mostly neglected in ordinary mathematics lessons" (Kaiser, 2007, p. 113). One reason that is given for this is the amount of time that must be invested to conduct a modeling investigation; however, it is not necessary to always focus on developing a complete solution to the problem especially if the pedagogical intention is to be continually developing independent modeling competencies.

Example: Stabilizing the cane toad population in Australia

One team chose to model the problem of an introduced pest species in Australia, the cane toad. Approximately 100 toads, native to Central and South America, were introduced into sugar cane fields at Gordonvale in Queensland in 1935 (altogether about 3000 were released in Australia at this time) in an effort to control the grey back cane beetle. We don't know exactly how many cane toads are in Australia, (over 200 million in 2006 according to Wikipedia) and they are spreading across northern Australia and down through New South Wales. According to the Invasive Animals Cooperative Research Centre (2006), cane toads have expanded their range across the north of Australia at a rate of 25–50 km/year. They occupy more than 500,000 km^2 of Australia and have reached densities of 2000 toads per hectare in newly colonised areas of the Northern Territory. However, the average density of toads in areas where they have been established for more than 20 years such as coastal Queensland townships is much lower – about 80/ha. By investigating and sourcing similar information on the Internet, the team decided they would investigate the question: What is the culling rate needed to stabilise the cane toad population in Australia? Below is part of their solution including their list of assumptions and their analysis of the effects of assumptions.

Determining the Cane Toad Population

Since the population of cane toads are not controllable, it would not be a logistic model. So, we decided to model the cane toad population with an exponential model since population itself has a behaviour of growing exponentially.

Assumptions:

* They don't get killed by predators.
* The increasing rate of cane toad growth is constant; according to Wikipedia, the growth rate of cane toads per year is 25%. This percentage takes into account death rates and the birth rates of the population.
* The initial population of cane toads released in Australia in 1935 was 3102 [sic].

Effects of Assumptions:

The predators would affect the death rates of cane toads because, for example, magpies now know to roll the toads on their backs so as to eat them rather than being poisoned by toxin excreted from the toads' back.

The increasing rate of cane toads would not be constant in reality because there would be some fluctuations. This would thereby take our model further away from reality.

There might be more toads released in Australia after 1935. This would affect the model as new toads are coming into the population.

Formulate:

Population (year) $= 3102(1.25)^n$

Culling rate:

By the growth rate according to Wikipedia; 25% every year

$$\begin{aligned}(1-h)\times 1.25 &= 1\\ 1-h &= 1/1.25\\ h &= 1-(1/1.25)\\ &= 0.2\end{aligned}$$

The culling rate according to this assumption is 20%. This means that every year there needs to be a killing of 20% of toad population so that the population is controllable and steady.

From this point, which is typically where initial time constraints curtail the first modeling effort, there is opportunity to examine the model output against some of the published data and to revisit assumptions.

For example the given growth rate of 25% would lead to a toad population in 2005 of almost 19 billion, or over 2400 toads for every square kilometre in Australia. This demonstrates that two of the pieces of data provided by Wikipedia (a constant growth rate of 25% and a present population of upwards of 200 million) are incompatible. A refinement of the assumed growth rate based on the estimate of "more than 200 million" for the toad population comes in at between 17 and 18% per annum, with a revised estimate for the necessary culling rate of around 15%.

This draws attention to another important aspect of the modeling competencies that can be developed. Traditionally in the formulation phase much is made of assumptions that need to be made in setting up the model for solution. However, it is often overlooked that assumptions need to be invoked at all stages of the modeling process. Furthermore, the assumptions are of different types and play different roles. Three types of assumptions were identified (Galbraith and Stillman, 2001), as those associated with (a) model formulation (b) mathematical processing, and (c) strategic choices in the solution process.

Assumptions (a) made during model formulation help define the interface between the type of mathematical model to be developed and the real situation – for example that as a first approximation a population growth rate that is not impacted by predators, or limited by area may be taken as constant over the time period of interest. Type (b) assumptions are mathematically based and apply within solution processes. For example, knowledge of the general properties of exponential functions, and specifically of how their output varies with different growth rates, feed into assumptions that determine the way they are employed. Assumptions (c) associated with strategic choices in the solution process are central in determining global choices available to the modeler and strongly influence the direction the solution takes at specific points (e.g., at a temporary impasse). They occur typically during the interpretation and evaluation phases. In this example the emergence of contradictory data causes a re-assessment of the exponential model used – is it sufficient to revisit the model using different parameters as assumed here – or is a different approach using a different formulation called for? An assumption will be made in determining such choices. This more analytical view of assumptions can be used to analyse and use the information and data given in a problem statement, or in associated documentation, obtained for example, from the Internet. Are they really assumptions, what types of assumptions are they (alerting students to the notion that there are several types they can make), are they sufficient for progress, and what roles do they play in the solution process?

11.7 Conclusion

In this paper, we have taken an analytical approach to the design and selection of tasks that involve topics found to interest adolescents in secondary school. The purpose has been to move beyond tasks as "given" elements in a modeling program, to make their selection and design a matter for student involvement, scholarly examination, creativity, and critique.

Appendix: A Possible Solution to Supersize Me

Assume average daily food intake (I) = 5000 calories (approx), and let w_n be weight after day n – Spurlock's original weight was $w_0 = 84$ (kg)

$$\text{weight}_{\text{today}} = \text{weight}_{\text{yesterday}} + (\text{energy intake}_{\text{today}} - \text{energy used}_{\text{today}})/7800$$

So

$$w_n = w_{n-1} + [\text{calorie intake(day } n) - \text{calories used (day } n)]/7800$$

$$w_1 = w_0 + (I - 24 \times 1 \times w_0)/7800 = I/7800 + (1 - 24/7800)w_0$$

as the lifestyle is sedentary.

Hence

$w_1 = aI + bw_0$

where $a = 1/7800 = 0.000128$, and $b = (1–24/7800) = 0.997$.

Similarly

$w_2 = aI + bw_1 \cdots w_{30} = aI + bw_{29}$

A Spreadsheet Solution readily obtains a final value of 95.15 kg.

Geometric Series Solution: Alternatively continuing the pattern: $w_2 = aI + bw_1 = aI(1 + b) + b^2 w_0$ leads to: $w_{30} = aI(1 + b + b^2 + \cdots + b^{29}) + b^{30} w_0$

Hence $w_{30} = aI(1 - b^{30})/(1 - b) + b^{30} w_0 = 95.15$(approx) – respectably close to the actual value.

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Chapter 12
Remarks on a Modeling Cycle and Interpreting Behaviours

Christopher R. Haines and Rosalind Crouch

Abstract Reported research and practice in mathematical modeling and applications has found it useful to describe student behaviours in terms of activity within a modeling cycle. This has enabled researchers to elucidate and to gain insights into processes deployed by students when faced with problems set in the real world and for which practical outcomes might be achieved by constructing a mathematical model. We consider various modeling cycles and we remark on individual modeling routes and other non-linear behaviours. Models of modeling in education raise questions as to the purpose of mathematical modeling and its assessment. We comment on the assessment of modeling competencies and we consider whether common methods of assessment address the mathematical model itself or more general competencies.

12.1 Introduction

In their recent paper on modeling perspectives, Kaiser and Sriraman (2006) put forward a classification in a mathematics education context as an aid to understanding the interrelations between the very different and complex approaches adopted by researchers and practitioners. Broadly, these five perspectives are:

A. realistic or applied modeling (using authentic examples and concerned with understanding of the real world and of modeling competencies);
B. contextual modeling (with subject-related goals such as solving word problems or psychological goals such as fostering learners' motivation);

C.R. Haines (✉)
City University, London, UK
e-mail: c.r.haines@city.ac.uk

R. Crouch (✉)
University of Hertfordshire, Hatfield, UK
e-mail: r.crouch@dsl.pipex.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_12,

C. educational modeling with a didactical or conceptual focus (the most popular approach, looking at the structure of the learning processes and introducing new mathematical concepts, methods and principles);
D. socio-critical modeling (promoting critical thinking about the role of mathematics in society);
E. epistemological or theoretical modeling (promoting theory development).

Other (meta-) perspectives adopted by researchers include cognitive modeling and the affective domain of learners (cf. Borromeo Ferri, 2007; Wake and Pampaka, 2007). The classification (Kaiser and Sriraman, 2006) should be viewed in the context that individual researchers and practitioners operate across several perspectives both concurrently and consecutively and so conclusions drawn or categories inferred would relate to behaviours at particular times or describe specific events. In this chapter we contribute to the discussion by remarking upon some models of modeling and associated behaviours, in particular models from perspectives A and C.

12.2 Student Behaviours in Cyclic Models of Modeling

In discussing mathematical modeling and associated activities, authors and researchers, as an aid to understanding student behaviours, often represent the modeling process as a cycle of activity. The cyclic representations developed in the late 1970s in undergraduate engineering mathematics courses could be said to come under perspective (A), describing stages modelers have to pass through and demonstrating a somewhat idealized approach to reality (though reality appears to be more complex and less linear). They focussed on student activity at six discrete stages (cf. Fig. 12.1) with the addition of a seventh reporting stage, and transition between the stages did not at that time receive much attention.

An interesting development by Voskoglou (2007) is also concerned with stages but treats the modeling cycle as a stochastic process, dependent upon the transition between successive discrete stages (Fig. 12.2). Each stage s_1 through s_5 is defined in terms of expected outcomes and transition from one stage to the next is wholly dependent upon the successful completion of the previous stage. This realistic modeling perspective treats the first five stages of Fig. 12.1 as a closed cycle.

Developments in educational mathematical modeling in schools, since the mid 1980s, prompted a didactical and/or conceptual focus on representations of modeling within a didactic or conceptual (idealized) approach from a cognitive perspective. The modeling cycle of Fig. 12.3 (cf. Blum and Leiβ, 2007) captures the essence of this approach, separating out the mathematical world from the extra-mathematical world (rest of the world) and in which the active parts of the process are themselves transitions between clearly identifiable “fixed points” in one or other of the worlds.

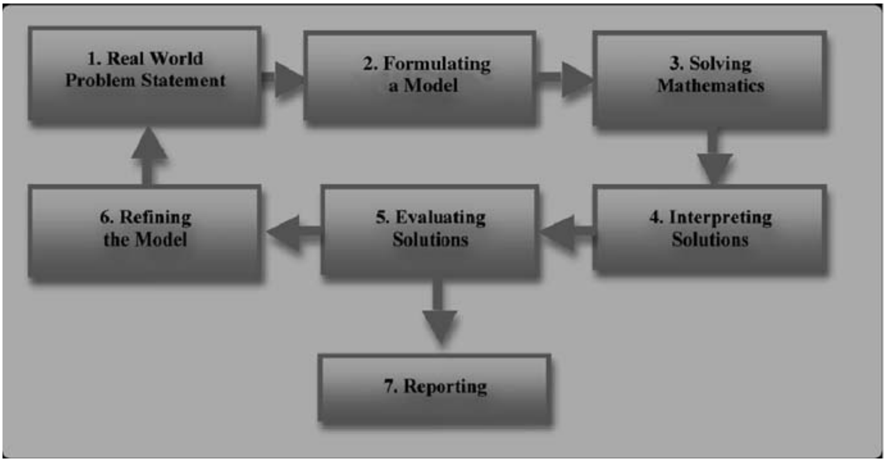
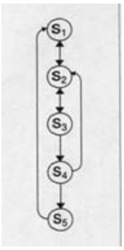


Fig. 12.1 Modeling represented as a cyclic process, originally developed from a realistic or applied modeling perspective (c.f. Berry and Davies, 1996)



s_1 = analysis of the problem (understanding the statement and the restrictions and requirements of the real system).
s_2 = mathematising, including the formulation of the real situation so that it will be ready for mathematical treatment and the construction of the model.
s_3 = solution of the model, achieved by proper mathematical manipulation.
s_4 = validation (control) of the model, which is usually achieved by reproducing, through the model, the behaviour of the real system under the conditions existing before the solution of the model.
s_5 = interpretation of the final mathematical results and relating them to the real system, in order to give the “answer” to our problem.

Fig. 12.2 Voskoglou’s (2007) stochastic model

The active parts, numbered 1–7 and representing: understanding/construction; simplifying/structuring; mathematising; working mathematically; interpreting; validating; exposing/reporting, are clearly related to the discrete stages of Fig. 12.1. A key feature of this representation (Fig. 12.3) is its emphasis on understanding the real situation and problem, moving through to a situation model (a kind of mental representation or mental model of the situation) and a real model and problem (a simplified and structured form of the real situation and problem, such as a word problem) whilst still in the extra-mathematical world.

Blomhøj and Jensen (2003) provide a helpful and comprehensive global visualisation of the mathematical modeling process (Fig. 12.4) linking idealized realistic approaches to a cognitive perspective, showing linkages between six key stages (a)–(f) and explicitly (rather than implicitly) with the mathematical knowledge and

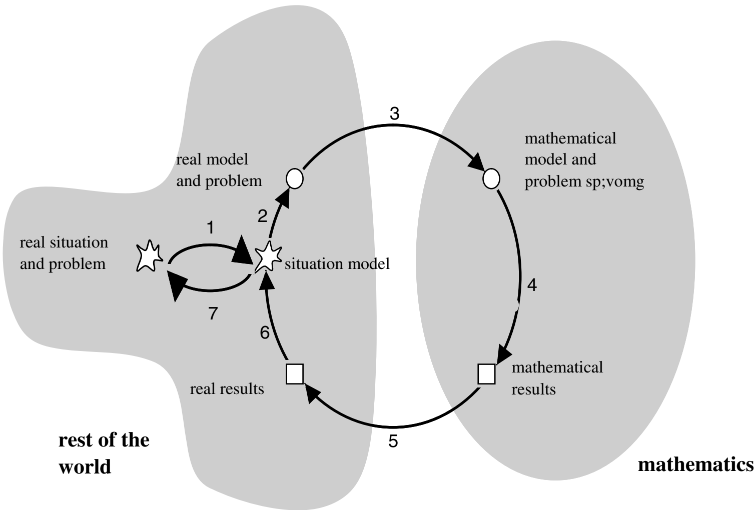


Fig. 12.3 The modeling cycle from a cognitive perspective (cf. Blum and Leiss, 2007)

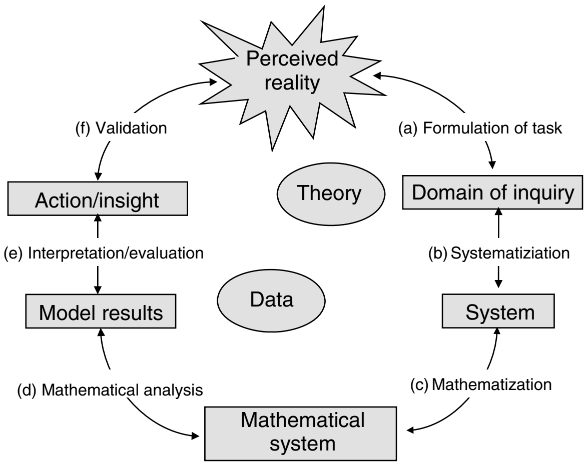


Fig. 12.4 A visual representation of mathematical modeling (Blomhøj and Jensen, 2003)

principles the learner needs to draw on. Again, stages (a)–(f) are related to the discrete stages of Fig. 12.1. Visualisations of the mathematical modeling process such as those provided by the four Figs. 12.1–12.4 are helpful in understanding what might be termed “ideal behaviour”, in which modelers proceed effortlessly from a real world problems through a mathematical model to acceptable solutions and report on them. However, life in the classroom and amongst modellers in industry and elsewhere, is not like that and is likely to be much more complex. For instance, modellers' ability to link the model to perceived reality will impact on the modelling process.

Novice modelers may have difficulty in understanding the problem and its context, and also in realizing when they have not done so (Crouch and Haines, 2004) and can be affected by personal experience with the context and also by the form of wording in word problems (Galbraith and Stillman, 2001). Voskoglou (2007) found that student modelers who could overcome all this and formulate the problem successfully went on to develop a successful model. Novice modelers also have difficulty deploying appropriate theory and linking it to the present problem (Crouch and Haines, 2004), and in handling data. It helps considerably if student modelers are exposed to a range of modeling activities (Doerr, 2007). These are some of the issues that will impact on the modeling process and the modeling route tbat modelers take, which is unlikely to be as linear as the idealized routes shown in the various models of modeling.

Recent research of Galbraith and Stillman (2001) showed that pupils, whilst ostensibly working at different stages of a modeling cycle (c.f. Fig. 12.1) continually returned to the real world problem, assumptions made and the context of the problem. This theme is evident in the work of Doerr (2007) who used the linear template of the Weigand and Weller (1998) steps in modeling: analyzing, simulating, modeling with equations, working experimentally, interpreting and explaining. She found that the process was far from linear, noticing a non-linear but cyclic view of the modeling process amongst pre-service teachers engaged on a modeling task. Conceptual and physical activity paths taken by modelers evidently do not necessarily follow the directions between stages indicated in Figs. 12.1–12.4. Borromeo Ferri (2007), reports that pupils in school take *individual modeling routes* when tackling mathematical modeling problems, associated with their individual learning styles. Her concept of individual modeling routes captures the essence of the behaviours noted by Galbraith and Stillman (2001) and Doerr (2007). These individual learning behaviours in mathematical modeling are important to consider, for Schoenfeld (1987) reports that in solving problems, experts continually return to underlying assumptions during a distinctly non-linear solution process. All these behaviours might help in the construction of meaning in relation to the modeling being undertaken and grounding that behaviour in reality.

Whilst Figs. 12.1–12.4 attempt to describe complete mathematical modeling processes, transitions between the real world and the mathematical model consistently present problems for students, which suggests that the transition is an extremely complex one and needs further research. Reassuringly, given the time devoted to mathematics, pupils in school and novice undergraduates are happier and more confident when in the mathematical world of the modeling cycle, working out possible solutions (c.f. Haines and Crouch, 2001; Crouch and Haines, 2004; Blum and Leiβ, 2007; Borromeo Ferri, 2007; Voskoglou, 2007).

The complexities involved in the mathematical modeling process raise issues as to how mathematical modeling can be effectively assessed. We will therefore briefly consider the purposes of mathematical modeling with a view to considering issues of assessment and relating this to the models of modeling behaviour.

12.3 Mathematical Modeling with a Purpose

The various contexts within which mathematics is used and applied in education, government, commerce and industry have associated with them different understandings of the nature and purpose of mathematical modeling. It is not just mathematicians that are interested in mathematical modeling. In his discourse on communicating big themes in applied mathematics, Hunt (2007) observes that:

> Mathematical analysis and modeling is evermore widely used by policy makers and industrialists for guiding and even making their decisions. However, by asking some basic questions they can understand better what reliance to place on the results and how to interpret them. (Hunt, 2007, p. 4)

in which the emphasis is on outcomes of the modeling process, focussing on stages 5 and 7 of Fig. 12.1, but with continual reference to the real world problem. However, mathematical modelers working with complex real world problems in a research and development environment spend a great deal of time and effort on establishing the boundaries for their problems and which variables it would be most useful to include in any developed model. Here there is a strong emphasis on the beginning stage 1 in the extra-mathematical world of Blum and Leiβ (2007) (Fig. 12.3). They remind us that, in schools, their first two steps in mathematical modeling are cognitively demanding (that is, reading a text and understanding both situation and what the problem is). It is clear that the very parts of modeling processes that assume critical importance outside education are those areas that present particular difficulties to students within education (c.f. Haines and Crouch, 2001; Crouch and Haines, 2004)

In a teaching and learning context, we ask the question: *What is the purpose of mathematical modeling?* In his wide-ranging review, Galbraith (2007) describes three models of modeling prevalent in education (i) where modeling may be subservient to the acquisition of mathematical knowledge (ii) where modeling provides students with mathematical learning skills and also useful skills for life outside the classroom, and (iii) where modeling is used in a more restricted sense such as data handling, statistics and modeling by curve fitting etc.

The development of mathematical understanding and the acquisition of mathematical knowledge drive the mathematics curriculum. Where mathematical modeling occurs as an activity it is unsurprising that the acquisition of mathematical knowledge and linking to mathematics already learned should dominate; assessment regimes and external/formal certification reinforce this view. The types of modeling, the range of problems and the extent of the activity common at primary and secondary levels often rely on recently introduced mathematics and they provide strong reinforcement to previously acquired mathematical knowledge. In terms of modeling cycles, or stages of mathematical modeling, they are more likely to comprise activity in the mathematical world. Although at some level, pupils in primary and secondary education can determine which factors to include in their model and assign variables appropriately and successfully, it is questionable that they have maturity to deal with these issues on other than a very narrow range of problems.

What modeling does with some success, is to provide pupils and students at all levels of education with mathematical learning skills and with useful skills for life outside the classroom. In higher education it does help students to learn a procedure for carrying out an investigation, in some case it is a rigorous scientific procedure. *Does mathematical modeling help pupils in schools to learn an investigative methodology? Are pupils and students able to abstract in the given problem and can they apply general principles?* The process of abstraction is complex and difficult for pupils and students to learn except by repeated application in a variety of different circumstances. *To what extent are useful skills for life learned and developed by modeling in schools?* Lesh (2003) reports that teachers do not emphasise such skills in association with mathematical modeling, they

> . . .seldom emphasize the modeling abilities that are needed when complex artifacts are produced using iterative design-test – revise cycles. Yet, these higher order understandings and modeling abilities tend to be precisely those that are needed for success beyond school. (Lesh, 2003, p. 51)

The third model of Galbraith (2007), where modeling is used in a more restricted sense such as data handling, statistics, modeling curve fitting and so on, is more directly related to how mathematics is useful and how and where it can be applied. In these cases the problems and contexts are often partially modeled for the students at the outset and the resulting modeling activity is identified with only parts of a modeling cycle (Figs. 12.1–12.4). Because some of the modeling is already done, this type of modeling will not address student difficulties in the transition from the real world to the mathematical model, although in interpreting and validating results it can assist students to understand and cope with transitions from the mathematical model to the real world.

There are other perspectives on the purpose of mathematical modeling in educational courses. *To what extent are we trying to produce good mathematical modelers?* If that is a current purpose, then it is interesting that employers, who recruit for "math modeling" jobs, emphasise "soft skills" pertaining to problem-solving and communication (Sodhi and Son, 2007). *Is problem solving in mathematical modeling courses appropriate? Are communication skills valued and assessed in mathematical modeling courses?* Some discussion is needed about the type of problem solving found in mathematics education and its relevance from an employer's perspective; further, we need to consider whether the assessment of communication skills within mathematics and mathematics education really does take place.

12.4 Perspectives on Assessing Mathematical Modeling

If modeling is subservient to the acquisition of knowledge: *Does the assessment of modeling as such take a lower priority that the assessment of mathematical competence and achievement?* If modeling provides useful skills for life outside the classroom then: *Where and how are those skills assessed for external/formal certification?* As Burton (1992) reminds us, the four purposes of assessment are: to inform the learner, to inform the teacher, to inform the next educational stage and to inform employers. Here "employers" is used as shorthand for the area to which the pupil or student moves outside school or university following the current course. In schools: *How is modeling assessed to these four purposes? Is it assessed at all?*

In order to increase expertise in mathematical modeling there has to be constant practice with models in order to understand and to use principles. The pupil or student needs to determine their own modeling routes and outcomes. They need to learn about various models for particular contexts, and when they may be applied. They build up expertise through a variety of detailed processes, recognising if they have seen one like this before. *How is this consistent with the demands of the assessed curriculum?*

Niss and Jensen (2006, Chapter 9) take a holistic approach to modeling which leads to overall estimates of modeling competence. They suggest a geometric model (Fig. 12.5) in which there are three dimensions of competence: Degree of Coverage, Radius of Action, Technical Level.

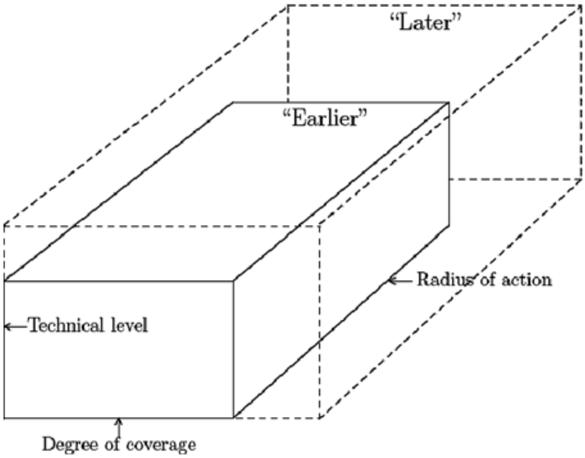


Fig. 12.5 A visual representation of three dimensions to work with when assessing modeling competence (Niss and Jensen, 2006, ch 9)

Degree of coverage relates to understanding stages of modeling and how they are linked. Haines and Crouch (2001) have demonstrated how assessment of this dimension could be done through a dissected approach using multiple-choice questions that focus on each stage of a modeling cycle (Fig. 12.1). They also suggest that such an approach could be used to develop rating scales of the dimension "Degree of coverage" applicable to postgraduate, undergraduate, pre-university and school levels (Haines and Crouch, 2007).

Assessing the radius of action is more problematic since it refers to the experience that the learner has in the variety and complexity of models. Such experience in schools will necessarily be more limited and narrower than that of students in undergraduate courses. But whilst that experience is necessarily restricted in comparison to an expert it might be extensive in respect of other children of the same age. Whilst there is a difficulty with the "Radius of action" dimension, Haines and

Crouch (2007) recognize that in a structure of a mathematical modeling expertise continuum, behaviours in school for example, might be qualitatively the same as in pre-university, undergraduate or post-graduate sectors, but differ substantially in terms of acquired expertise. *Is it possible to take a snapshot of a pupil's radius of action, his or her exposure to a variety of modeling problems and of differing complexity?* This area is extremely important because we know that a weak knowledge base in students and a lack of experience in abstraction cause difficulties in the transition from the real world to the mathematical world (Crouch and Haines, 2004). The knowledge base and experience can be improved by repeated graded exposure to models and modeling.

The third dimension, that of "Technical level of mathematics", is usually regarded as adequately assessed outside modeling activities. However, the technical level of mathematics influences the modeling that is accessible to the learner. In assessing mathematical modeling competencies. *To what extent is a particular level of mathematics competence assumed? How does the latent mathematical content of a modeling situation alter learning precedences for the pupil? If, when faced with a model for which the pupil's mathematics is inadequate, does learning the mathematics take precedence? Does this give greater purpose to modeling being subservient to the acquisition of knowledge?*

The models of modeling discussed in Figs. 12.1–12.4, of themselves do not help with assessing competence in other than the degree of coverage but they play an important part in developing good learning situations, enabling the teacher to draw out phases in specific modeling tasks and to introduce, for example, analogies to other situations. In that sense, by assessing those stages (Haines and Crouch, 2001), the assessment purpose of informing the teacher can be met (Burton, 1992).

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Chapter 13
Model Eliciting Environments as "Nurseries" for Modeling Probabilistic Situations

Miriam Amit and Irma Jan

Abstract This study presents an extension of model-eliciting problems into model-eliciting environments which are designed to optimize the chances that significant modeling activities will occur. Our experiment, conducted in such an environment, resulted in the modeling of a probabilistic situation. Students in grades 6–9 participated in competitive games involving rolling dice. These tasks dealt with the concept of fairness, and the desire to win connected students naturally to familiar "real life" situations. During a "meta-argumentation" process, results were generalized, and a model was formed. In this case, it was a model describing a "fair game" created by the differential compensation of different events to "even the odds." The strength of this model can be seen in its ability to first reject preexisting knowledge which is partial or incorrect, and second to verify the knowledge that survives the updating and refining process. Thus, a two-directional process is created – the knowledge development cycles lead to a model, and the model helps to retroactively examine the knowledge in previous stages of development.

13.1 Theoretical Background

Model Eliciting Activities (MEAs) are problem solving activities that lead to the formation of a model (Lesh et al., 2007c) which is a system that is used to construct, describe, or explain some other system for some purpose (Lesh et al., 1997). The following four characteristics of these models are especially relevant to our study. (1) Students' models often integrate thinking associated with a variety of disciplines or textbook topic areas. (2) Students' models often are expressed using a variety of interacting representational media. (3) Relevant concepts can be expected to be

M. Amit (✉)
Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: amit@bgu.ac.il

I. Jan (✉)
Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: irma@bgumail.bgu.ac.il

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_13,

at intermediate stages of development. (4) Model development usually involves a series of design cycles (or modeling cycles) which involve different ways of filtering, organizing, and interpreting the problem solving situation (Lesh and Yoon, 2004).

According to traditional views of learning and problem solving, learning to solve "real life" problems has been assumed to be more difficult than solving their counterparts in textbooks. Yet, it also is reasonable to expect that meaningful problems will be easier than problems about situations that are poorly understood. So, by emphasizing these latter characteristics, research on model-eliciting activities has shown consistently that, if students' tasks are designed using principles which have been specified for MEAs, then the chances are high that students will make significant adaptations to their initial ways of thinking about the problem solving situation (Lesh et al., 2007c; Lesh and Doerr, 2003a). In other words, model-eliciting activities often lead to significant forms of learning. Yet, no activity is model eliciting unless it actually elicits a model. So, tasks which satisfy the design principles for MEAs are never more than potentially model eliciting. For example, in addition to task variables which are addressed in the design principles for MEAs, it also is important for learning communities to form that have developed appropriate attitudes and dispositions that support inquiry; and, the general learning environment also can function to support or subvert productive inquiry. So, it is these latter characteristics that we have investigated in our research.

Because solutions to model-eliciting activities generally require students to express their current ways of thinking in forms that are tested and refined multiple times, such solutions tend to automatically generate auditable trails of documentation which reveal important information about changes in thinking that occur; and, because significant changes in thinking occur during relatively brief periods of time, it is possible for researchers to go beyond observing sequential states of understanding to also observe processes that lead from one state to another.

Probability Learning is culturally influenced by language, beliefs and everyday experiences (Van Dooren et al., 2003). For example, Amit and Williams (1999) documented links between children's cultures, their informal or constructed knowledge of probability, and their formal probabilistic knowledge; and, their findings revealed that the children's different cultures offer many interpretations of the term "chance," the most common being: "something which you are not sure will happen or not." They also found that children perceived the term "coincidence" in different ways, connecting it sometimes with luck and at others with superstition. Similarly, Amit and Jan (2006) found evidence that, even without the intervention of formal teaching, pupils (grades 6–9) who had no previous formal learning of probability often are able to create a linkage between sample size and the probability of an event, and construct a probabilistic language of their own for mutual communication purposes.

Wilensky (1997) describes another reason why probability is a difficult subject. He claims that the standard mathematical curriculum is not helpful in promoting students' probabilistic comprehension. In fact, he attributes many of students' difficulties to the traditional curriculum's failure to expose students to problems that motivate the invention of mathematical solution processes. Furthermore, Abrahamson and Wilensky (2005) provide evidence that students' biggest challenge with the domain of probability is not so much that the conceptual constructs per se are difficult, but that the domain is difficult *as seen through the lens of traditional mathematical representations* (Abrahamson and Wilensky, 2005).

Learning Environments for mathematics have been shown to have significant impacts on learning (Webster and Fisher, 2003). However, David Perkins' book, *Landscapes of the Mind* (1999) reminds us that the learning environment does not teach (does not do the teaching), but only acts as a scaffold for learning. So, an additional approach which we adopted in our research was suggested by Maher (2002) who investigated a variety of ways inquiry is supported by having students work in small groups. She found that working in small groups invites reflection and discussion among students about differences in their thinking, and that in justifying their ideas, students provided arguments that exemplify several important types of mathematical proof. Similarly, Amit and Jan (2007) found that challenging and supportive learning environments often involve exposing small groups to probability concepts through "hands-on" experiences that can serve as an infrastructure to build upon in a future, more formal approach to probability. The following study is an extension of the preceding study.

13.2 The Study

The aim of our study was to investigate the extent to which natural and familiar environments can contribute to the development and the understanding of concepts in probability – and the development of probabilistic models. The first task was a game in which students naturally encountered issues involving relationships between "fairness" and chance – and the need to quantify and compare probabilities of various events. In particular, they investigated how it is possible for a player with lower chances of success to win the game (which occurred several times during the experiment).

Game #1 (A Game for two players): Roll one die. If the die lands on 1, 2, 3, or 4, Player A gets a point (and player B gets 0). If the die lands on 5 or 6, Player B gets a point (and player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. (a) Is the game fair? (b) Play the game with your partner. Are the results confirming your answer to question a? (c) If you think that the game is unfair, change the rules and make the game fair (Vidakovic et al., 1998; Amit, 1999; Alston and Maher, 2003; Amit and Jan, 2006).

Game #2 (A Game for two players): Roll two dice. If the sum of the two is 2, 3, 4, 10, 11 or 12, player A gets a point (and player B gets 0). If the sum is 5, 6, 7, 8 or 9, player B gets a point (and player B gets 0). Continue rolling the dice. The first player to get 10 points is the winner. (a) Is the game fair? (b) Play the game with your partner. Are the results confirming your answer to question a? (c) If you think that the game is unfair, change the rules and make the game fair. (Vidakovic et al., 1998; Amit, 1999; Alston and Maher, 2003; Amit and Jan, 2006).

The Setting: The study involved two groups of students from grades six to eight. One group was composed of six talented students (three boys and three girls from grades six to eight) from the "Kidumatica" math club, which is part of the Ben-Gurion University of the Negev. The second group was composed of six students (three boys and three girls from grade nine) with high level achievements in mathematics from a comprehensive high school. All of the students were aware that they were participating in a study for research purposes and that they would not be receiving any grades. Nevertheless, all of them treated the experiment seriously throughout. Both groups had the same educational background – having received no formal instruction in probability, and being totally inexperienced in solving such "real life" problems in class. Note: No difference was found to exist in the behavior of the two groups and we shall not be relating to them separately in this article.

When the students participated in the games and associated tasks, they were required to document their work, to write their solution paths on forms, and to provide explanations for their answers. After writing down their respective personal solutions, the students held group discussions wherein each was required to justify his or her decision and persuade the other group members in the event of a disagreement.

At no time during the study were the students told if their conclusions were right or wrong, but they were always required to persuade other participants if they were convinced they were in the right. Heated debates often ensued among groups in their mutual efforts to persuade.

The two researchers were present throughout the experiment. Most of their time was spent closely watching events and walking among the students – but without interference. The researchers intervened actively only when students reported having completed the assignment or when a "critical event" in the learning process was perceived (see below explanation of "critical events"). The students were previously acquainted with the researchers, but the researchers were not their teachers and had no connection to their schools. So, the study posed no potential threat to their grades.

"Critical events" is a term used by Powell et al. (2003) to describe events in which students show a "leap" in the development of probabilistic comprehension, leading to the development of additional knowledge elements. In modeling contexts, these leaps generally correspond to times when students shift to a new modeling cycle (Lesh and Doerr, 2003a).

Data Collection: Data included video footage of the episodes, on student generated documentation and observation notes. These data were analyzed using an "analytical model" (Powell et al., 2003). Then, the video tapes were fully transcribed and the scripts were analyzed first by identifying "critical events," and second by identifying probabilistic thinking levels (Jones et al., 1999).

Important Relevant Features of a Model Eliciting Environment (MEE): (1) The problem solving situations involve familiar and unsophisticated paraphernalia from everyday life – coins and dice – both well known from traditional board games. (2) We presented the students with competitive game tasks in which the rules were accessible and simple. (3) The culturally valued motivation to win was used as a "driving force" for the model eliciting environment. (4) The tasks emphasized social interactions in a supportive and challenging atmosphere which was familiar and comfortable to the students. (5) The students entered the process with preexisting knowledge, which included primary assumptions and intuitions about fairness. So, when fairness was perceived to be violated, this created a conflict between the real and the expected which required the students to rethink and revise their initial ways of thinking. (6) Constant discussions and debates among students enabled them to rapidly put new ideas to the test. (7) As part of the game, the students were required to justify their claims and persuade their colleagues of the position. In particular, the processes of mutual persuasion led to the development of a *monitoring process,* which became an integral part of performing the tasks. (8) The MEE provided the students with a chain of tasks designed so that the results of each task expanded the students' knowledge base and led to the beginning of the next.

13.3 Analysis

The Evolution of a Probabilistic Model: This section describes the evolution of a probabilistic model that developed out of natural competition in game situations. The model was created by giving a differential compensation to different events in order to even the chances of an unfair situation. Model development involved two cycles. But, in the remainder of this chapter, we focus on how the students' first model was tested and refined to produce a second model.

13.3.1 The First Cycle – Four Steps of the Modeling Process

To begin, the students read the rules of game #1, and each student was assigned a role – player A or player B. Before beginning the task, we made sure the game rules were clear to the students. It was vital that all of the students understand the rules perfectly, so the researcher asked each student directly "when do you win?" The answers received were in accordance to the role assigned to the particular student: "when we roll 1, 2, 3, 4" or "when we roll 5, 6."

A. *Manipulation*: After clarifying the rules of the game, it was obvious to all of the students that the game was unfair, since one of the players had the advantage. The following quote indicates how the game was manipulated by a student to predict each player's odds of winning. Igal made a connection between the chances of getting each number on the die and the chances each player has to win the game:

> Igal: "The chance of getting each number is equal. So actually when someone gets 4 numbers, if they come out, then he gets a point (for each one), then he has an advantage over the other player who gets 2 numbers that if they come out he gets a point (for each one). So actually, the first player

has an advantage. Actually he has more chances of getting a point than the second player."

B. *Translation:* Translation involves carrying relevant results back to the real world. The students found that the practical results of experiencing the game tasks contradicted their assumptions (see above) several times – such as when the "disadvantaged players" won. The mathematical discussion carried out during the assignment showed that some students could translate the results from the reality of carrying out the game, and even justify the idea that despite the inequality of the chances for success, it is still possible for player B with the disadvantage to win. The following discussion took place between a researcher and the students helps to show the nature of this translation:

Liron: "The fact that you get 4 numbers (and not 2) means that you get an advantage but it doesn't mean you are definitely going to win."
Researcher: "Why do you say that, if you have an advantage, how could you not definitely win?"
Liron: "It's an advantage but it's more a matter of luck."
Researcher: "So even though he has an advantage, you can still win?"
Dana: "An advantage doesn't guarantee a victory. The first player has a probability of 4–6 that he will get these numbers and the second player has a probability of 2–6. Even though the firstplayer's probability is high it doesn't mean he'll definitely (certainly) win."

C. *Prediction and Verification*: Translations led to *prediction* – carrying relevant results back into the real world. For example, the following discussion ensued in which suggestions were offered for making the game fair. So, the students made a *translation* and carried relevant results into the game and made predictions to make the game fair.

Liron: *"Each player should have the same chances of getting points."*
Ronen: *"In a fair game the numbers of the die have to be divided evenly."*

D. *Verification:* The final step in the cycle was the *verification* of the usefulness of the students' actions and predictions. Igal inferred later that if player A has an "advantage," a greater chance of winning the game, then the game is unfair. In the *verification* process of justifying the game's unfairness, the students drew on previous knowledge acquired from their school curriculum and applied it to the new situation. With no guidance whatsoever, they quantified the "chances" using fractions, percentages and ratios, and the results of the quantification served to justify their claims.

Sharon: *"Player A has four options (1, 2, 3, 4) out of six options, which is 4/6=0.66=66%. Player B has two options (5, 6) out of six options, which is 2/6=0.33=33%. Player A has a 66% chance of winning and player B has a 33% chance of winning and that is why it is not fair."*

The preceding comments suggest how the need arose for the creation of a model during the first cycle of the study. That is, the game is unfair because player A has a greater chance of winning than player B. So, the next phase of study involved a four step cycle in which students connected the chances of throwing each number on the dice and each player's chances of winning the game.

13.3.2 The Second Cycle – Using "Inverse Proportions" to Balance Odds in an Unfair Situation

This stage marked the beginning of the second cycle in the process initiated by the model eliciting problem. According to Lesh and Yoon (2004), when new information is acquired, "the interpreted information requires certain aspects of the model to be refined or elaborated. The elaborated model allows new information to be noticed, thus giving rise to a new 'modeling' cycle". This second refining cycle led to a new, improved model, in which a differential compensation was awarded to different events, thus evening the odds of the unfair situation.

The following is a description of the stages leading to the construction of the model.

The students' suggestions for making the game fair included the even distribution of the numbers on the die between the two players (like 1, 2, 3 and 4, 5, 6) or the division into odd and even numbers. Tal, who was absolutely unsatisfied by the "simplistic solution" (in his terms), was looking for another way to balance the odds for the two players. He invented a new term – a new model that balances the unfair conditions in the game which he called – "Inverse Proportion" – as described below:

> Tal: *"I tried to do an inverse proportion. When one thing is bigger then there is another that's smaller. The odds for the first player are twice as big as for the second player 4/6 to 2/6. So I decided to double the amount of points the second player gets."*

Steps in the Second Cycle:

(1) *Description* – Tal describes the chances of player A in relation to those of B.
(2) *Manipulation* – Tal explains that since player A has double the chances of winning, to even out the situation he must double the points for player B, thus doubling his chances and equating his chances with those of player one.
(3) *Translation* – Later, Tal offers a suggestion for rules to make the game fair using "inverse proportion".

> Tal: "You throw the die. If you get 1,2,3,4, the player those numbers belong to gets one point. The other player, who got the numbers 5, 6, gets two points."

(4) *Verification* – A question comes up in group discussion about whether these rules are really fair to both players, since one player has a chance to win with less throws. The justification of the rules is below:

Igal: *"The second player will win in less throws while the first one will take longer to win."*

By calculating the "probability of fairness", the students conclude that the rules are fair to both players:

Dana: *"It could be that Tal's game really is fair. If the first player has a probability of 4 out of 6 and you multiply it each time by one point. 2 out of 6 is the probability times 2 so in the end it's 4 out of 6 for each one."*

The above quotes show that, by verifying the "inverse proportion" (i.e., *the model that was built for a fair game)*, students are showing the firsts signs of comprehending the notion of "expected value."

13.3.2.1 Refining the "Fair Game Model" Following Its Application to a New Game

An additional external cycle in the model eliciting problem process began when the students tried to apply the "inverse proportion model" to a to game with different rules. As the theory states: "The elaborated model allows new information to be noticed, thus giving rise to a new 'modeling' cycle; and, progressively more stable versions of a given model are able to take into account more information (Lesh and Yoon, 2004)". Towards the end of the above modeling cycle the "fair game model" undergoes further refinement, forming an improved model that meets the new reality of a game with different rules.

In light of the new game's rules it was difficult to determine immediately if the game was fair. There was a great deal of information to be organized before a decision about the game's fairness could be made (throwing two dice leads to many possible outcomes – 36 in all, while throwing a single die allows a far easier determination since there are only 6 possible results). Students were faced with a problem in organizing their data as well – how many different possibilities are there when throwing two dice (21 or 36?). This question was the subject of the most heated of the many debates that ensued among the students. The following short excerpt is taken from an animated discussion that ended in a general agreement that throwing two dice did indeed allow 36 possible results (and not 21 as some students originally thought). This discussion paved the way for the refinement and improvement of the "fair game model."

Researcher (to Ronen): *"Can you give an example of the ways of throwing an 11?"*

Ronen: *"6+5 and 5+6."*

Dana: *"That's the same result!"*

Ronen: *"It's two different results!"*
Shani: *"It's the same result because of the commutative law".*
Researcher: *"meaning?"*
Shani: *"6+5 is equal to 5+6."*
Ronen: *"Because it's two different ways. They may be the same numbers but it's not the same way. It can be that I got 5 first and then 6, or 6 first and then 5."*
Tal: "We're trying to find the sum and not the sources of the factors. The sum is the same; the factors need to be different."

The "argument" was a means of refining the meaning of the *commutative law*, a central point, as this allowed the representation of all the possible results of throwing two dice. Toward the end of this modeling cycle the model improved as it passed the test of application to a game with different rules.

Researcher: *"What have you learned?"*
Tal: *"Inverse proportion, the ratio between 2 things – when one grows the other shrinks."…"When one thing grows by X, the other shrinks by X."*
Researcher*: "What are the two amounts?"*
Dana: "*The probability and the number of points an event gets."*
Researcher: *"Can someone give me a generalization?"*
Dana: *"If the probability grows, the amount of points for the event shrinks and vice versa."*
Researcher*: "And what happens when the odds are equal?"*
Dana: "*Then there is an equal amount of points and the game is fair!"*

13.4 Discussion

A solution of a model-eliciting problem is naturally not achieved in isolation from the learning environment. In order to amplify student understanding we expanded and modified the environment into a model eliciting environment. In doing so, we attempted to form a parallel or an analogy between the model eliciting environment and the model eliciting problem on which it was based.

Just as solving a model eliciting problem includes several cycles (Lesh and Doerr, 2003), the model eliciting environment includes cycles as well. In the former, the cycles arise from within the problem and are motivated by its internal requirements. The cycles of the latter are motivated by a series of tasks "tailored" so as to lead to the creation of a model to anchor the students' knowledge. Just as in the model eliciting problem, the student also arranges the problem and documents the various steps of the solution in the model eliciting environment. The series of problems in the model eliciting environment must be built so as to allow the students to make adjustments and corrections in every step of the model they have created for themselves, by applying it repeatedly to every new problem.

In our study, we showed that an appropriate design environment can be used to amplify the modeling process. We showed two modeling cycles that were related to each other: "The Four Steps of the Modeling Process" and "The Inverse Proportion model." After participating in these two cycles, the students gained a new interpretation (or model) which led to the development and refinement of a probabilistic model for a fair game.

We created a series of problems, each with an inner model of its own, but which, as a group, lead to movement in a particular direction, e.g., the construction of new knowledge. In fact we created a "nursery" for growing and fostering probabilistic thinking, from which students learned the meaning of concepts such as probable chance, quantification of probability, comparing probabilities, a mathematical meaning of fairness and the quantification of fairness. By giving probabilistic significance to the concept of fairness and quantifying it, the students were able to crystallize concepts such as: *Possible vs. certain, theoretical vs. actual probability, probabilistic advantage and the interface between mathematics and chance, fairness etc.*

The learning promoting environment is a reconstruction and an expansion of the learning promoting problem, both in the cognitive and the social sense. In the whole course of the problem solving process, the students were required to justify their solutions. They had preliminary assumptions (regarding the game's fairness, for instance), born of their intuition or prior knowledge. This preexisting knowledge did not withstand the practical experience of the task, conflicting with the actual results of playing the game. The knowledge was then rethought, and through group discussion, which included complex argumentation processes, was updated, improved, and tested again by practical experimentation and scientific justification. This process – trial, conflict, rethinking, correction, and retrial – was repeated several times, until the arrival at an understanding that met with consensus and withstood the test of practice. This exact process recurred while learning in the model eliciting environment.

Working in a challenging environment, the students were not satisfied with empirically proving their knowledge, but required of themselves and their colleagues that ideas be subjected to scientific-logical testing. To arrive at group consensus, an additional process "*meta-argumentation*" was called for, during which the results were generalized and a model was formed. In this case, it was a *model for a fair game* created by a differential compensation of different events to "even the odds." In probabilistic terms, seeds were sown for an understanding of the concept "expected value." The strength of this model lies in its ability to first reject preexisting knowledge which is partial or incorrect, and then verify the knowledge that survives the updating and refining process. In solving a model eliciting problem, a two-directional process takes place – the knowledge development cycles formed a model, the model helped to retroactively examine the knowledge in its previous stages of development. The process described above, which is relevant to the model eliciting problem, underwent an expansion and a transformation into a model eliciting environment. The entire environment is a model of learning, in which a model eliciting problem is "grafted" in what can be called "higher ordered modeling."

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Chapter 14
Models as Tools, Especially for Making Sense of Problems

Bob Speiser and Chuck Walter

Abstract For us, a model is a tool to solve a problem. Our research concentrates on what learners actually do when they solve problems that demand fresh insight. We want learners to build ideas and understanding they can use to solve new problems. In relation to this goal, the ways specific models help make sense of novel situations can become important subjects for investigation. We anchor our discussion to concrete examples drawn from elementary arithmetic.

14.1 Introduction

In a related article (Larson et al., Chapter 5 of this volume) we have urged a quite specific view of models. More precisely, we propose to view a model simply as a tool, to help make sense of something that we seek to understand. In this study, we explore in detail how such models might support productive exploration and reflection. For concreteness, we will focus here on making sense of products of whole numbers. Even here, we find remarkable complexity. Our guiding theme is how we make decisions in the course of goal-directed action.

14.2 Discussion

Under behaviorist assumptions, we respond to stimuli in the environment based directly on internalized experience. John Dewey deconstructed this conception in a celebrated paper on the reflex arc (1896), the link between response and stimulus.

B. Speiser (✉)
Brigham Young University, Provo, UT, USA
e-mail: speiser@mathed.byu.edu

C. Walter (✉)
Brigham Young University, Provo, UT, USA
e-mail: cnw.zia@gmail.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_14,

In his view, the behaviorists could well have got the process backwards. For Dewey, problem situations evoke exploration in response to a perceived need, an explicit sense of something missing, something to clarify and understand as well as seek. While we seek a stimulus when we respond to a perceived need, we need to think, perhaps to change our understanding, as we try to move toward closure. From this perspective, we respond to felt needs through reflection on internalized experience. Dewey thus stressed how stimuli emerge, thorough purposeful reflection, as goals (not simply triggers) for unfolding acts.

> Neither mere sensation, nor mere movement, can ever be either stimulus or response; only an act can be that; the sensation as stimulus means the lack of and search for such an objective stimulus, or orderly placing of an act; just as mere movement as response means the lack of and search for the right act to complete a given co-ordination (p. 106.)

In this view, perception and understanding build together in the course of active, goal-directed exploration (Speiser and Walter, 2004; Speiser et al., 2004; Speiser and Walter, in press). Such exploration, as Dewey emphasized, might best be viewed as a *response* to a felt need, as a coordinated search to *find* a stimulus. More specifically, the sought-for stimulus might be an insight, anchored to a tool constructed for the purpose, by a group of learners. Such tools and insights may thus become available, at least potentially, for later use. In this way, internalized reflection on experience might help us recognize or pose (not simply solve) a future problem.

To be precise, consider a specific tool, the model shown in Fig. 14.1. It was put together in September 1998 by six elementary education undergraduates. Their immediate purpose was to make sense of the product 17×23. We had, however, asked them to consider a more complicated product.[1]

Task: How might you multiply 54×37 using blocks? Perhaps you might want to consider different possibilities.

In the model shown, we see two blocks with dimensions $10 \times 1 \times 1$ cm, called *longs*, together with three 1-cm^3. In more standard terminology, these would be denoted *units*, but our students, following a local usage (Speiser and Walter, 2000) preferred to call them *piggies*.

The students told us that they understood their product, 17×23, as a repeated sum of 23 copies of 17. Hence, in their model, each piggy counts as 17, not 1, so that each long counts as 170. In this way, their model did not simply represent their product as an iterated sum; it also gave that sum a *structure*. Specifically, we see a sum of simpler products: 2×170 (two longs) and 3×17 (three piggies). These simpler products could be found, the students told us, by any of several alternative approaches.

In Dewey's terms, the sought-for stimulus might be a combination of two insights: the structure *and* the recognition that important progress had been made. Because this model did not directly address the original problem (54×37) we had

[1] Source: Speiser and Walter (1998, Field notes and tasks for Mathematics 305, unpublished). We gave each group a set of standard base ten blocks.

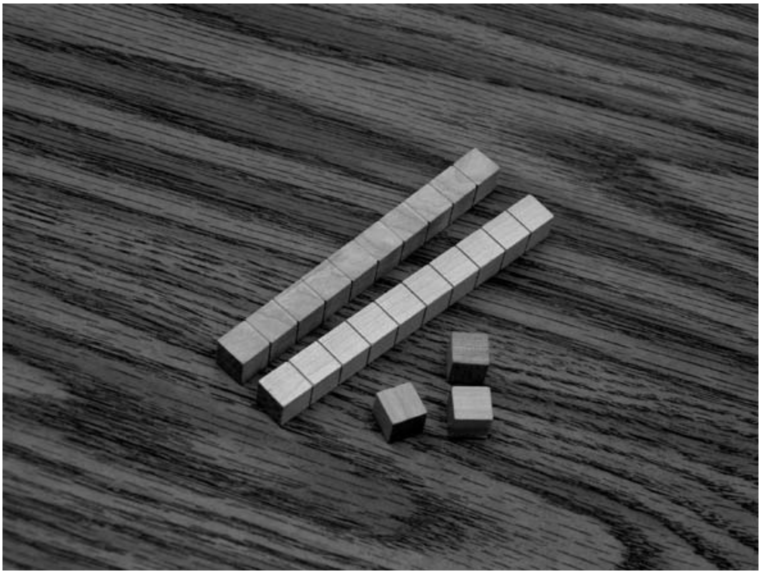

Fig. 14.1 Base ten blocks for the product 17 × 23

posed for them, but rather helped its builders make sense of a related, simpler problem, we can view its use for 17 × 23 as a trial run to test a *method* to find *any* product. For us, the model simply functions as a tool.

Indeed, the model shown in Fig. 14.1 does not, by itself, suggest a way to find the product. It simply indicates a *structure* for the product to be found. To find the indicated product, we still need to *use* that structure to make progress.

The inscription shown in Fig. 14.2 helps anchor one such way.[2] Reading from left to right, we can think of the arrangement of written calculations shown as a kind of flow chart, organized in columns. The first column indicates the blocks of Fig. 14.1, each labeled with the number that it represents. The second column shows two calculations: first a doubling of 17 to give 34, then the addition of a further 17 to give 51 for the three piggies. Now continue to the third column. There, to double 170, we see the prior doubling of 17, this time as a count of tens, used to obtain 340. The final result, 391, at the far right, emerges by direct numerical addition.

In contrast to the model shown in Fig. 14.1, which simply indicates a potential structure for the product, Fig. 14.2 records some of the further information needed for one person's computation to evaluate that product. In this way Fig. 14.2, a set of scratch notes, is *itself* a model as described above: a tool to help make sense of

[2] The approach shown here draws on a student-generated strategy (Speiser et al., 2004), based on strategically motivated doubling (Speiser and Walter, 2000), to make sense of a different calculation. We encourage readers to explore further alternatives.

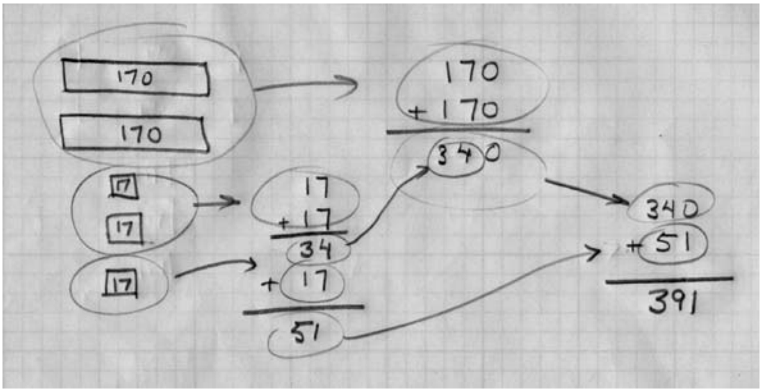

Fig. 14.2 From an author's working notes (RS): a scratch calculation starting from the model shown in Fig. 14.1

something that we wish to understand. That sense, as we have seen, was built up in progressive stages over time.

Based on such examples, we place central emphasis on methods, as distinguished from procedures (Speiser et al., 2004). In the analysis above, we have viewed the scratch notes shown in Fig. 14.2 as *just one possibility,* within a range of possibilities, to bring the model shown in Fig. 14.1 to bear on the initial problem. And here, thinking again of Dewey, we're after more than just a close analysis of given procedure steps, however helpful such analysis might be. Consider, for example the *initial act*, that is, the *choice* to work with base 10 blocks as shown in Fig. 14.1 to trigger a solution process. On what experience might such a choice depend?

We are also deeply interested in reasoning. The scratch notes shown in Fig. 14.2, structured in columns to record how information gathered at each step flows to the next, presents key details to facilitate a larger process. On the left we have the blocks of Fig. 14.1. As we progress from left to right in Fig. 14.2, we have already recognized how information found in partial calculations can eventually assemble in progressive stages to provide the claimed result. But now, given these calculations, attention needs to shift. Given the information sketched in Fig. 14.2, why is the claimed result *correct?* Given what we have so far, in other words, we must still explain precisely why the process sketched in Fig. 14.2, as distinguished from its calculation steps, *makes sense as a method* to find products.

To build such an explanation, to verify explicitly the process of the given calculation, we could track each long and piggy sketched at left to make clear its contribution to the final total. With this purpose in mind, we therefore read the Fig. 14.2 inscription horizontally, in rows. In this way, the model shown in Fig. 14.2 (that is, the pencil marks we see on paper) has been *restructured* to support an argument in favor of the given calculation. Indeed, to check a calculation, it can *never* be enough simply to verify that each procedure step was carried out correctly. One also needs to make clear why the *right* steps have been taken (Speiser et al., 2004). In the case at hand, the same tool, the same model, after suitable restructuring, supports investigation of a *qualitatively different question*, where the goal is now to make sense of a method rather than to find a number.

14.3 Conclusion

To summarize, we see *two* models here: the blocks configuration shown in Fig. 14.1, and the scratch calculation shown in Fig. 14.2. As presentations, each functioned initially as a presentation to oneself of parts of an ongoing computation, most of which is mental. In this first phase, the blocks facilitate a way to structure the initial problem, to reduce it to a pair of simpler problems. We have viewed this first phase, focusing especially on Fig. 14.2, as a flow of information through successive stages of assembly, organized from left to right by columns, where the wooden blocks of Fig. 14.1 recede into the background as numerical results accumulate. To build an explanation, in contrast, we folded back to the initial structure given by the blocks of Fig. 14.1, to initiate a second phase to help address the further need for explanation. While we initially viewed Fig. 14.2, in columns, as a record of progressively accumulating information, we then took the information shown as given, and restructured Fig. 14.2 in horizontal rows based on the structure given by the blocks, as a framework for an argument, not just a calculation. In each phase, in effect, we traced exactly the same flow of written information, but the way we structured and then understood that information clearly changed, because our purposes were different.

The initial choice or plan behind the blocks model, in both the phases we have traced, might reflect important past experience. Our students surely knew that smaller numbers lead to simpler calculations, and that multiples of ten, read suitably, reduce to more convenient multiples of one. Clearly such past experience might well have influenced the way the way these students' blocks model might have been designed. Nonetheless, based on the evidence at hand, we might emphasize instead the way key features of such models, and key features of the way that they have been used and justified, can lead to *new* experience, gained by reflection not just on steps taken, but perhaps most especially on how such steps were structured and hence understood.

Consider in more detail the change in how we interpreted the model shown in Fig. 14.2, as our main purpose shifted from finding a specific number toward building an argument in favor of a method to obtain such numbers. In the first phase, we traced one way the partial computations might have been written and assembled to obtain the desired product, in effect following the motion of the pencil as it wrote the numbers that we see. In the second phase, after the shift of emphasis, we began to think in greater generality, to make sense of a method to find any product of whole numbers, a method that the written calculation could be understood to implement in a test case (Speiser et al., 2007). In this way, the felt need to understand a method, not simply a specific number, can motivate a search not only for specific numbers, but also for reasons, arguments, or explanations that might be seen to hold in greater generality. The results of such a search, as lived experience, might therefore be new ways to structure and to reason. In keeping with Dewey's insight, we can view such ways as stimuli.

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Chapter 15
In-Depth Use of Modeling in Engineering Coursework to Enhance Problem Solving

Renee M. Clark, Larry J. Shuman, and Mary Besterfield-Sacre

Abstract There has been recent interest among engineering educators in the use of *models and modeling* as a means to promote vertical skills integration and problem solving within undergraduate engineering curriculums. We have extended the *MEA* (Model Eliciting Activities) construct to upper division engineering courses, reformulated the resultant exercises as *MIAs* (Model Integrating Activities). These were introduced as part of a pilot course focused on enhancing problem solving abilities for junior and senior level industrial engineering students. The course focused on developing systems thinking in order to solve unstructured problems, some of which incorporated global and ethical considerations. The course challenged students to practice various behavioral and professional skills, including ad-hoc teaming, written and verbal communication, revision and refinement of group work, and reflection. We learned valuable lessons from this unique, non-traditional class, which serves as a lead-in to an upcoming four-year research effort by six institutions to expand the application of *MEAs* to five engineering disciplines. One important lesson learned was the potential of a well-constructed *MIA* to uncover subject-area misconceptions held by students. We discuss this, other lessons learned, and challenges identified that should be addressed to better achieve our pedagogical objectives. This chapter discusses our experiences with this unique engineering course.

R.M. Clark (✉)
University of Pittsburgh, Pittsburgh, PA, USA
e-mail: rmwst1@pitt.edu

L.J. Shuman (✉)
University of Pittsburgh, Pittsburgh, PA, USA

M. Besterfield-Sacre (✉)
University of Pittsburgh, Pittsburgh, PA, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_15,

15.1 Introduction

At the University of Pittsburgh, we used *models and modeling* as an approach to drive vertical skills integration and enhanced problem solving in an upper-level industrial engineering course. Our methodology builds upon and extends the Model Eliciting Activity (*MEA*) concept. In doing this, we are, in essence, creating Model Integrating Activities (*MIAs*). Model Eliciting Activities (*MEAs*), which originated in mathematics education research, are open-ended problems set in a realistic context (Lesh et al., 2000; Zawojewsky and Lesh, 2003). The faculty at Purdue has successfully introduced *MEAs* into their freshman engineering courses (Diefes-Dux et al., 2004). In a pilot course at the University of Pittsburgh, we have extended the construct to use with upper-level engineering students as part of a process that requires them to use skills and concepts learned from prior coursework to solve posed problems. In this way, we feel we are better introducing a systems approach to engineering problem solving. We have designated our exercises as "model integrating activities" or *MIAs*, since, if the exercise is appropriately constructed, it should encourage the student team to integrate concepts previously learned (or at least covered). With a well-designed model integrating exercise, students can collaborate on real-world problems, applying their engineering knowledge in better developing both the mathematical system and accompanying procedural documentation to solve the problem at hand. One particular motivation for doing this has been the emphasis on testing, revising, refining, generalization, prototype development and formal documentation of solutions, all skills that future engineers should master that are core principles of a well-designed MEA. Indeed, formal documentation forces the student engineers to scrutinize or take a second look at procedural designs and therefore drives self assessment and reflection. Both of these latter two skills lead to refinement, robustness, and increased opportunity for reuse (generalization) for similar problems and/or by other engineers. In addition, a well-constructed MEA provides students with the opportunity to practice such professional skills, as teamwork and oral and written communication. In short, the *MEA* principles support the development of the abilities and skills required of graduates of accredited engineering programs as stated in ABET Criterion 3 a to k (ABET, 2007).

We are particularly interested in students' ability to reflect on the processes they have used in resolving our MIAs. Not only do we consider this skill an important way to enhance problem solving, but we also see it as providing a means that we as instructors and researchers can better assess the learning process, and understand how students do, in fact, develop good problem solving strategies. Consequently, we have incorporated exercises throughout the course that promote learning through self reflection, including the use of *Reflection Tools* proposed and piloted by Hamilton et al. (2007). Students completed a reflective exercise at both the beginning and end of the course that provided valuable insight into the ad hoc teaming model we used. In addition to insights gained about teaming (e.g., the need to better monitor teamwork), this pilot course afforded other lessons learned. These are valuable as we continue to apply a *models and modeling* framework in engineering education to achieve specific educational outcomes. For example, we found that *MIAs* serve to easily elicit various misconceptions held by students. The problem of misconceptions is widespread throughout STEM disciplines (Duit, 2006). Prior research at the Colorado School of Mines by Miller, Streveler, Olds and their colleagues in engineering student misconceptions of thermal science topics shows that senior-level chemical and mechanical engineering students retain a significant number of robust misconceptions even after completing courses in fluid mechanics, heat transfer, and thermodynamics (Miller et al. 2005, 2006). A list of commonly misunderstood concepts in thermal science has been developed by this research team (Streveler et al., 2003). We will further discuss the lessons learned and future challenges we must overcome in order to enhance the outcomes desired from this non-traditional engineering course.

The pilot course has served as a valuable lead-in to a multi-institutional research project on the use of *models and modeling* as a cornerstone of undergraduate STEM curriculum, especially within engineering disciplines. This comprehensive, four-year study by researchers from six universities has been funded by the National Science Foundation's CCLI (Course, Curriculum, and Laboratory Improvement) program (Phase III). Insight gained from our recent pilot course will be used to design, test, and implement new *MIAs* across multiple engineering disciplines, including chemical, civil and environmental, electrical and computer, mechanical, and industrial.

15.2 Methodology

15.2.1 In Depth, Serial Use of MIAs

The pilot course was offered in the summer of 2007; the class consisted of ten junior and senior-level industrial engineering students. It was designed for in-depth, serial use of *MIAs* to enhance team-based engineering problem solving. This involved the assignment of nine *MIAs*, most of which were in two-parts, throughout the course's twelve-weeks. Students were required to work in teams of three or four during class time to provide procedurally-based solutions to these posed, open-ended problems that addressed industrial engineering topics. For almost all of the exercises, the teams had to complete the *MIA* outside of class. For the two-part *MIAs*, two written reports were required, as the second part built upon the first part. Teams were randomly formed for each *MIA*, so that students gained experience with ad hoc teaming. Each student assessed both himself and his team members after each *MIA* to drive teamwork accountability.

Educational objectives included the development of both vertical skills integration and complex and creative reasoning. Students were challenged to consider: "What should we do in this situation? What body of knowledge should we pull from?" The modeling activities were specifically designed to challenge students to recall, (often relearn) and apply previously-learned concepts, including the use of supporting software, as might be done in actual practice. This included software for statistics, linear programming, and data mining. The nine *MIAs* consisted of exercises developed by colleagues at Purdue primarily for freshman engineering instruction (SGMM, 2007) as well as new exercises developed specifically for the pilot course. In some cases, we added content to the existing *MEAs* to better frame them for more advanced engineering students. We also created several new *MIAs* for the course. These nine modeling exercises are summarized in Table 15.1.

Table 15.1 Description of pilot class MIAs

	MIA	Concepts and skills targeted	Status
1	Supplier development	* Quantitative comparison of multiple suppliers * ANOVA techniques	New
2	Quality improvement	* Quality investigation plan to reduce variation and scrap * Overall product quality improvement process * Decision flow chart incorporating SPC tools	New
3	Compressor reliability	* Central Limit Theorem – application to failure time data * Confirmation of Wear-out vs. Burn-in failure * Distribution fitting (chi square)	New
4	CD compilation	* Optimization: heuristic algorithm or 0-1 integer LP	Existing content added
5	Gown manufacturing outsourcing	* Incorporation of multiple factors in decision methodology for global outsourcing choice	New
6	Trees	* Resolution of ethical dilemma: reducing accidents versus preserving old growth trees, including redwoods	Existing content added
7	Disaster decision modeling	* Decision modeling of events during a disaster and aftermath (including real-time information source)	New
8	Volleyball	* Synthesis of quantitative and qualitative variables to build equitable teams	Existing
9	Condo pricing	* Use of attributes for prediction of condo prices * Multiple linear regression	Existing

Several of the *MIAs* shown in Table 15.1 focused on the application of concepts from the students' two probability and statistics courses. These included *Supplier Development, Quality Improvement, Compressor Reliability, and Condo Pricing*. Specifically, skills. Table 15.1, including a description of the concepts and skills targeted. Experience gained from the course has already enabled us to begin the process of refining these *MIAs* so that they better address educational goals for upper-level engineering students. Clearly, more rigorous testing of an *MIA* is needed to fully ensure that it meets its intended goal of integrating certain acquired skills. As has been observed by Lesh (2007), in spite of the instructor/researcher's good intentions and preparation in designing an *MIA* to target the application of particular skills, students may come up with more creative alternatives, applying skills or methods that the designer had not considered in order to resolve the posed problem. As we learned in the case of *Condo Pricing*, such solutions are as good, if not better than what we had intended. In this case, the students developed a decision tree classifier to solve the problem, although the instructors had anticipated the use of multiple linear regression.

Such as the application (including underlying assumptions) of ANOVA, statistical process control, the Central Limit Theorem, distribution fitting, and multiple linear regression were targeted. For example, in part one of *Supplier Development*, the team was required to compare various suppliers based on delivery time, possibly using analysis of variance to better understand the data. (See Fig. 15.1 which shows part one of the *Supplier Development MIA.*)

To: Supplier Task Team
From: Purchasing Engineering Manager
Re: Supplier Evaluation

Washers-n-Boltz
Assemblies
Denver, CO

Dear Team:

We have many suppliers who provide goods and services to our operations. We are in the process of evaluating various suppliers to our manufacturing operations, who supply a range of goods and services. Our current evaluation plan centers on issues related to delivery lateness. We need your team to compare various suppliers, as we may decide to eliminate certain ones or work with them in a supplier development role for improvement. We also want to identify those companies who form our most dependable supply base.

I have attached some data for our 4 shipping suppliers. These companies ship in-process . . .

Shipper	HrsLate
1	1.5
1	3.1
1	4
1	8.7
1	2
1	0
1	2
1	3
1	6.5
1	5
1	6.5
1	7
1	2
1	0
1	2
1	1
1	2
1	5
1	4
1	1.5

Fig. 15.1 Supplier development MIA (part 1)

In part two, a second factor (product quality) was introduced, and the team was asked to determine whether this factor as well as the supplier had an effect on delivery time, this time possibly using two-way ANOVA for data analysis. Note that as shown in Fig. 15.1, each *MIA* (or part of an *MIA*) consisted of documentation from the client to the student team, typically in the form of a memo and/or news article. This memo is used to introduce the task, describing the unstructured problem and asking for a resolution and possibly additional engineering consultation. In some cases, the teams were asked to provide examples of how their mathematical system might be efficiently reused by the client. In order for students to begin analysis, a data set is typically provided as also shown in Fig. 15.1.

CD Compilation, originally developed at Purdue, was extended to challenge upper-level students who had already learned (or should have learned) the underlying theory and application (including software) of integer linear programming (LP) in a prior introduction to operations research course. The *Disaster Decision Modeling MIA* incorporated concepts from a prior probabilistic methods class, including decision tree analysis. For this latter exercise, we also introduced a real-time information component with this *MIA* in the form of a simulated expert on the scene of the disaster. The expert (i.e., picture the "Wizard of Oz") was available to respond to student queries and provide additional information for constructing the decision model; e.g., probability estimates. The student teams had to determine on their own the specific questions that they wanted to ask the expert. Again, proving Lesh's point, two of the three teams asked unanticipated questions involving factors not suggested in the introductory client memorandum. Interestingly, they used this additional information together with their decision tree results to determine an overall evacuation decision. This unanticipated approach most likely better simulated an actual decision making process, in which an analytic model would only be one source of information.

The engineering ethics-based *Trees MIA*, was developed from a short case by Harris, Pritchard, and Rabins. It requires the consideration of apparently conflicting concepts of environmental justice, resource conservation, and human safety in reaching a resolution concerning the removal of old growth trees bordering a road on which their have been a number of accidents in which most of the drivers were speeding (Harris et al., 2005). We added a second part to the original case, resetting the scenario in a national park where the trees were identified as California redwoods. *Gown Manufacturing Outsourcing* focused on a US company's off shoring decision making process. Students had to incorporate multiple types of information, including economic and demographic data, about three target countries to develop a decision methodology. These latter two *MIAs*, which were rooted in ethical, societal and global contexts, were intended to better prepare our students to better become world citizens.

An unresolved issue remains how to "prep" students for the *MIAs*, including how much background and review information should be provided, and when to provide it. For some exercises, students were assigned exploratory homework to complete individually both as preparation and as a mechanism reacquaint the students with the general subject area of the *MIA* without necessarily suggesting a particular technique or how it might be used to solve the upcoming *MIA*. For example, prior to *Quality Improvement*, we asked the students to explore a certain area of quality engineering to focus their efforts. Part one of *Quality Improvement* targeted the use of control charting, a technique used for statistical process control. For the homework, the students were asked to describe two related techniques from quality engineering, namely process capability and pre control. In studying process capability, they had to re-examine statistical process control, which included control charting, which typically precedes a process capability study. Part two of

Quality Improvement targeted the design of a quality improvement system, which would include using all three techniques in a specific order if designed appropriately (control charting followed by process capability and then pre-control). Thus, understanding the time dependencies among the techniques was critical for designing the system. Ideally, the preparatory work should have motivated the students to "pull together their research." Clearly, our goal with preparatory homework was for students to "connect" it to the *MIA*, and in some but not all cases, they did. In addition, their ability and willingness to research the subject area was a vital problem solving skill we wanted to emphasize and reinforce. However, this remains an area for further exploration.

15.2.2 Assessment

As a first cut at assessment, we have developed a rubric that measures the students on four dimensions in order to assess their *MIA* results. These four dimensions correspond to four of the *MEA* principles: (1) Re-usability, (2) Documentation, (3) Effective Prototype, and (4) Self Assessment (Testing and Improvement). The other two principles: Model Construction and Reality primarily relate to the design of the *MEA/MIA* and, consequently, reflect much more on the instructor rather than the students. By assessing the students on these four dimensions, we hoped to better emphasize these important problem solving attributes. The actual rubric is shown in the Appendix. In developing the rubric, we delineated each principle to coincide with the quality and level of work expected from upper-level engineering students. The delineation was based in part on Diefes-Dux et al. (2007). The Effective Prototype principle has been expanded to include refinement and elegance of the solution. Since this rubric was based on prior *MEA* research, it had a certain degree of content validity prior to use. A five-level scale is used to score each of the four dimensions, with the score indicating the degree to which the team achieved or executed the principle. The four dimensions can be summarized as follows:

- *Re-Usability*: Assesses the degree to which the resultant model is a working solution for the particular problem as well as similar cases. Is the model robust, and can it be easily "handed over" to others to apply in similar situations? Were ideas for efficient re-use of the procedure provided to the client, as requested?
- *Self-Assessment/Testing*: Assesses the extent to which the solution has been tested and procedure and team thought has been revised. Have nuances or special conditions in the data or problem been uncovered and accounted for in the procedure?
- *Model Documentation*: Evaluates the level of detail and explicitness in the written procedure. Clarity of expression, correct grammar, and ease of reading are also assessed. Have the assumptions that were made been clearly stated? Has all information specifically requested by the client been included?

- *Effective Prototype*: Measures the refinement and elegance of the solution procedure. Is the procedure based on thorough application of engineering concepts and methods? Have appropriate engineering ideas been used? Is the solution or implementation accurate and of high quality?

A score of a “1” on any given dimension would indicate that the principle was not achieved or executed in the solution. A score of “2” generally would indicate some, but insufficient, achievement or execution. A “3” would indicate a sufficient, or minimum, amount of achievement as well as satisfaction of the requirements. A score of a “4” indicates that the solution embodies the dimension or principle for the most part and that the solution has gone beyond the requirements; the team has achieved more than expected and has generally done a good job. In order to achieve a “5” on any given dimension, the principle must be executed in an outstanding and exceptional manner. While this is a first step at assessment, it only addresses the outcome and not the process. Below we discuss our initial efforts at assessing the solution process. As discussed in the section on future challenges, assessment of *MIAs* is an area that requires further study.

15.2.3 Reflective Exercises

In addition to using a rubric to assess *MIA* performance, we used “reflection tools” (RT) to assess students’ problem solving processes. In essence, reflection tools serve as a type of “process” assessment of problem solving activity. They provide a window into how students were thinking and provide a means to study the developmental process surrounding problem solving. As conceived by Hamilton et al., RTs are short debriefing surveys completed following an *MEA* exercise that assist students in recalling and then recording significant aspects of their activity. Reflection tools also provide a means for students to assess their own learning, including teaming. They were previously used at Purdue in the freshman-level (Moore et al., 2005). They have been explored by Hamilton et al. in a recent publication (Hamilton et al., 2007). We used the reflection tools as well as theory from this publication in providing the students with reflective exercises. The problem solving aspects assessed using Hamilton et al.’s RTs include the following: individual roles and changes in roles, problem solving strategies, team functioning, and levels and types of engagement. By using reflection tools throughout the course, we aimed to uncover changes in problem solving processes.

After two *MIAs* had been administered in the course, the students were also asked to answer a reflective, opened-ended question related to teamwork. The question posed was as follows: “Do you find a team useful or necessary for solving an MIA? Why?” This same open-ended question was later posed at the end of the class. As expected, on both occasions, all ten students responded that a team is typically or can be useful, as shown in Table 15.2.

Table 15.2 Summary of responses about teams

Summary	
Response	Count
Beginning	
Typically or can be useful	10
Not always useful	5
Necessary	3
Not necessary	4
End	
Typically or can be useful	10
Not always useful	4
Necessary	0
Not necessary	5

Table 15.3 Reasons for usefulness of teams

Usefulness of teams	
Reason	Count
Beginning	
Multiple views, inputs and ideas	7
Critique and error proofing	3
Division of labor and parallel activities	3
Complementary skills, experiences, strengths	2
Narrowing down ideas/approaches	2
End	
Multiple views, inputs and ideas	8
Division of labor and parallel activities	4
Higher quality solution	4
Critique and error proofing	2
Narrowing down ideas/approaches	2
Complementary skills, experiences, strengths	1

The reason provided most frequently for the usefulness of teams was the ability to obtain multiple views, inputs, and ideas. This reason was given by seven and eight students at the beginning and end of the class, respectively, as shown in Table 15.3. The following reasons were also stated frequently for the usefulness of teams: division of labor and parallel activities, higher quality solutions, and critique and error proofing.

Of concern, however, was a certain percentage of students who added that a team is *not* always useful in practice. Specifically, five and four students stated this at the beginning and end of the class, respectively, as shown in Table 15.4. Of these students, three stated this on both occasions. The reason given most frequently pertained to other team members not being motivated to contribute to the *MIA* solution. In addition, on both occasions, students stated that non-contributing members lead to inefficiencies or added stress. Other reasons are given in Table 15.4.

Table 15.4 Reasons for non-usefulness of teams

Non-usefulness of teams	
Reason	Count
Beginning	
Other members not motivated to contribute	4
Non-contributors lead to inefficiencies or added stress	2
Other members not knowledgeable of concepts/subject	1
Team too large	1
Team member who overpowers others	1
End	
Other members not motivated to contribute	3
Conflicting schedules	1
Non-contributors lead to inefficiencies or added stress	1

Interestingly, students who stated that a team is *not* always useful had higher average teamwork ratings than those who didn't state this. This difference in average teamwork rating was approximately 11%. However, students who did *not* expect teams to ever be non-useful were rated as relatively lower-performing team members. These results suggest that further work is needed to ensure that meaningful teaming and accountability occurs with *MIAs*, especially in a course which is heavily teamwork-based.

At the beginning of the course, three students stated that a team was necessary for solving an *MIA*, mostly for obtaining multiple views, inputs, and ideas but also for critiquing the solution and developing complementary skills. At the end of the course, this statement was not made. In addition, four and five students, respectively, felt a team was *not* necessary for solving an *MIA*, as shown in Table 15.2. Of these students, three said a team was *not* necessary on both occasions. These results suggest that our *MIAs* may need to be broadened in scope to focus on a wider range of industrial engineering skills. They also suggested that teamwork accountability is a factor that should be further studied and addressed with *MIAs*.

15.2.4 Lessons Learned

As with any new undertaking, we learned important lessons that will be of value in continuing this type of coursework. This course was non-traditional in that students were expected to use and integrate varied, previous coursework in determining a solution to an unstructured problem. The instructional approach was that of "guide but not tell or over-direct." Our interest was in challenging students and assessing their ability to determine a solution plan *on their own*, which was likely to produce multiple or unanticipated solution paths. In the case of *Condo Pricing*, in which the students did not receive a lecture on possible solution methods such as regression, one of the teams used a decision tree classifier, a type of data mining algorithm, to solve the problem. Had these students been instructed on a possible solution method, their critical and creative thought process may have been limited. Teams were expected to research and review subjects or topics they were not immediately familiar with. Further, the students were required to work as part of a different team each week. Thus, they could not choose their team members but had to learn to work with different people. Based on an open-ended question posed at the end of the course about changes in teamwork, there is evidence this type of learning occurred. Four of the ten students noted positive changes in teamwork, stating that they either learned to work in teams via practice or felt more comfortable working in teams as they got to know other students better.

In order for a course with these types of expectations to be as meaningful as possible to a student, the student must have a certain maturity level in terms of his/her attitude towards learning. The student must be willing to take an active role in his/her learning and teaming, being ultimately responsible and accountable for both. At a minimum, these expectations must be clearly stated at the beginning of the courses. Each student should understand that the course is designed to produce professional and behavioral growth. In addition, it requires the student to take responsibility for applying previously-learned concepts, as in a real world setting.

We determined that *MIAs* are quite viable for eliciting common misconceptions held by students. For example, with *Supplier Development*, although teams used an ANOVA approach, they did not test the assumptions required for conducting an analysis of variance, including for normality and equal variance. In *Condo Pricing*, the teams that chose a regression approach had to be guided to test the required assumptions and diagnostically assess their models, including consideration of multicollinearity. This was in contrast to examining R^2 only. *Compressor reliability* showed that the teams did not have a full understanding of the Central Limit Theorem. When a misconception surfaced, we documented it and attempted to repair it through in-depth written feedback to the team and often through a follow up lecture. Observation and interaction with students during in-class problem solving provided insight on the understanding of engineering concepts as well as misconceptions. It afforded the opportunity to guide the students in overcoming misconceptions so as to construct a better model. In using *MIAs* to address misconceptions, prevalent and robust misconceptions must therefore be identified and then meaningfully repaired. It remains a challenge to determine the most successful reparative methods, which if effective, should result in new pedagogical tools for promoting deeper understanding in our students.

We determined early in the course that assessing teamwork on a weekly basis was necessary to drive accountability among team members. To this end, each student completed a weekly teamwork assessment, in which he/she rated both him/herself and his/her teammates on a five-point scale. There were three dimensions rated, which were as follows: (1) Participation – Assumed a meaningful role, (2) Had or obtained pertinent skills, and (3) Overall fairness towards and contribution to team. These ratings, along with class attendance, were used in assessing the student's effort-based class participation.

15.2.5 Challenges

As mentioned previously, in a class intended to challenge and assess students' skills and integration abilities as applied to problem solving, the instructor should be careful not to direct students to certain solution paths or "tell" them how to solve the problem. This course was non-traditional in that lecture was not the primary instructional activity. Rather, team strategizing during in-class problem solving and instructor guidance comprised the main learning interactions. However, given our knowledge of misconceptions that surfaced and some unsatisfactory solution procedures that were submitted, it may be fruitful in this format to conduct a certain degree of "creative lecturing" to ensure better student preparation. This might entail lectures containing general topics, for example, the need to test assumptions when conducting a statistical test or to examine diagnostic measures of a model beyond its statistical significance. The challenge is in designing helpful, thought-provoking lectures that prompt students to recall concepts and methods they learned previously without suggesting specific methods they might use to solve the problem at hand.

Related to this is choosing the best time to conduct such lectures. Should all lecturing be done at the beginning of the course, before any *MIAs* are assigned? This approach offers the advantage of generality – the students still have to pull from among multiple topics discussed when solving a particular *MIA*. However, if such a method does not provide sufficient preparation, a general lecture given closer to the assigned *MIA* may be more beneficial. This might include a lecture given after the teams have worked with the *MIA* for a short period of time, as suggested by the STAR.Legacy learning cycle. That is, our tendency now is to use a modification of the STAR.Legacy Cycle used successfully by the VaNTH ERC team in adapting the "How People Learn" philosophy [(VaNTH – ERC STAR.Legacy]. This would start with the introduction to the *MIA* (challenge), require students to "generate ideas," followed by "multiple perspectives" when additional background and supplemental material would be presented by the instructor and then "research and review" by the students, before "testing your metal, going public" and "reflecting back." Note the similarities to these steps and those for the *MEA* process.

The assessment of *MIA* solutions is an area that requires additional study, including validation of the four-dimensional rubric. We noted that it was easier to assess the Model Documentation and Effective Prototype dimensions compared to the other two dimensions. In addition, these four dimensions are likely not independent. Based on scores assigned to the nine *MIAs*, Re-Usability was correlated with Effective Prototype ($r = 0.769$, $\rho = 0.682$, $N = 41$). This is reasonable, since an elegant procedure based on sound engineering methods and appropriate ideas will tend to be embraced and reused by other engineers. The relationship between Re-Usability and Effective Prototype was the strongest bivariate correlation among the various rubric dimensions. Based on this correlation, it may be possible to combine the assessment of these two dimensions.

In order to effectively assess the Self Assessment/Testing principle, the *MIA* should be specifically designed to elicit testing, revision, and refinement of the model. For example, in an *MIA* that targets the use of regression or a decision tree classifier, additional data should be included for validation purposes, without specifically noting the purpose of that data. In this way, the students would be challenged to determine *on their own* the need to perform model testing and validation. As a second example, in an *MIA* targeting operations research skills, a self assessment opportunity could be created by asking team members to consider the robustness or realism of their linear programming solution. The teams could then be challenged to determine ways to make their solutions more realistic, for example by adding constraints, decision variables, or objectives.

15.3 Conclusions and Future Plans

We have adopted a *models and modeling* framework to drive vertical skill integration and systems thinking with upper level industrial engineering students. In addition, we have targeted growth in various behavioral and professional skills, such as ad hoc teaming, written and verbal communications, collaborative writing and proofreading, procedural revision and refinement, and personal reflection on the modeling activities. The lessons learned and challenges uncovered during a recent pilot class will enable us to refine this new pedagogical approach to ensure it meets its full expectations. We feel this curricular approach holds great promise for educating future engineers in the highly competitive global market. *MIAs* will continue to be developed and implemented across various engineering disciplines as part of four-year, six-institution research project being funded by the National Science Foundation to expand this pedagogical approach.

In developing future *MIAs*, we have added a goal of incorporating increased multimedia interaction. This may be an effective means to address misconceptions. Specifically, we propose to introduce a laboratory or research component. For example, an *MIA* might be developed that engages students in a mini research study in which they must search for measurements or data relevant to a particular topic. They could use this data to formulate and test statistical hypothesis developed on their own. The benefit would be increased understanding of the origins of data and the implications of using it, thereby contributing to repair of misconceptions about the purpose of data and its use in hypothesis testing. The students would be encouraged to make full use of library resources, the internet, practitioners, and other experts, thereby making the exercise more of a real-world, interactive, and multimedia experience.

Acknowledgment Funded in part by NSF DUE – CCLI-Phase 3: Comprehensive, NSF 06-536, "Collaborative Research: Improving Engineering Students' Learning Strategies Through Models and Modeling."

Appendix: Rubric for assessing MIA performance

Level	Generalizability/ Re-usability	Documentation	Effective prototype	Testing and improvement
1	• Team on wrong track-needs to redirect efforts. • Model does not work or solve the problem.	• Lacks clarity or under standability. • No details or explicitness. • Lacks information specifi cally requested by client.	• Not accurate and will not work. • Not based on engineering concepts or principles. • Appropriate ideas/procedures not chosen.	• Readily apparent model not tested and steps not evaluated. • Readily apparent did not have a revision. • No feedback incorporated.
2	• Model is a start only. • Works only *partially* for specific situation.	*Some (but insufficient):* • Clarity. • Understandability. • Grammar. • Details or explicitness, including assumptions made. • Information specifically requested by client.	*Some (but insufficient):* • Engineering concepts/ principles used. • Appropriate ideas/procedures • Accuracy/quality of solution or implementation • Refinement.	*Some (but insufficient):* • Testing or evaluation. • Revision of thought or model. • Feedback incorporated. • Nuances in data or problem not uncovered.
3	• Model is a working solution, but only for specific situation. • Addressed re-use somewhat.	*Sufficient (minimum):* • Clarity (warrants improvement). • Detail or explicitness, including assumptions made. • Specific information requested by client.	*Sufficient (minimum):* • Engineering concepts/principles used. • Appropriate ideas or procedures. • Accuracy or quality of solution or implementation. • Refinement.	*Sufficient (minimum):* • Testing or evaluation. • Revision of thought or model. • Some nuances in data or problem uncovered.

Level	Generalizability/ Re-usability	Documentation	Effective prototype	Testing and improvement
4	• Working solution for specific situation and will work *partially* for similar situations. • Partially robust. • Addressed re-use well.	*Most or for most part:* • Necessary details. • Assumptions stated. • Specific information requested by client. • Clear, easy to read, understandable, grammar correct	*Most or for most part:* • Engineering concepts/principles used. • Appropriate ideas or procedures. • Accuracy or quality of solution or implementation. • Refinement/elegance.	*Most or for most part:* • Necessary testing and evaluation done. • Necessary revision of thought or model. • Not all nuances or special situations in data or problem accounted for.
5	• Working solution for this and similar problems. • Robust. • Can be handed over to and used by others for similar cases. • Addressed re-use very well.	• Exceptionally written. • Very clear and easy to read.Grammar correct. • Very highly detailed/explicit. • Most or all assumptions stated. • All specific info. requested by client.	• Model elegant and highly refined. • Thoroughly based on application of engineering concepts/principles, and appropriate ideas/procedures. • Highly accurate and high quality, including implementation.	• Very thorough, outstanding testing done. • Extensive revision of thought and model. • Uncovered several nuances or special situations in data or problem.
Score				

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Chapter 16
Generative Activities: Making Sense of 1098 Functions

Sarah M. Davis

Abstract This paper reports on early efforts to make sense of the new type of data created by Networked Function Based Algebra Generative Activities – specifically a data set from a single classroom activity consisting of over 1000 equivalent expressions. The data was collected during the first enactments of Networked Generative Activities in Singapore. The author led a series of networked generative, function-based algebra modeling activities with five classes of Secondary 1 students. The functions submitted by the students for a single activity were collected and analyzed. Findings showed that students created a range of correct functions utilizing many important mathematical concepts. Preliminary analysis of the functions highlighted the need for new and powerful tools for teachers to make sense of the vast quantity of data created in generative activities.

16.1 Introduction

This paper reports on the initial efforts to makes sense of the new type of data created by Networked Function-Based Algebra Generative Activities. Specifically a data set from a single classroom activity in which students are modeling algebraic functions using networked technologies that consists of over 1000 equivalent expressions. I will begin with an overview of why these activities were first created, move to a brief review of previous work and conclude with the current work being done in Singapore. The discussion of the current work will include the data analysis and thoughts on the types of tools needed to make these data sets more useful to teachers and researchers.

S.M. Davis (✉)
Learning Sciences Lab, National Institute of Education, Singapore
e-mail: sarah.davis@nie.edu.sg

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_16,

16.2 Why Generative Activities

Generative teaching as discussed by Wittrock, "is to teach relations among the concepts. . . by giving the structure and by getting students to generate associations and relations that elaborate that. . . structure, make it more precise, and extend it, always using their own words or images to show how. . . concepts are related to one another." (Wittrock, 1991, p. 182). Generative learning, in his model, takes place when students actively create their own artifact trying new ideas to existing knowledge structures. Function-based algebra uses the function as the organizing structure for the mathematics curriculum (Kaput, 1995). The function-based approach to teaching algebraic concepts has the potential to improve student understanding of the structural aspects of introductory algebraic concepts (Stroup et al., 2005). New networked technologies, technologies that harness the knowledge and ideas within the classroom (as opposed to the more typical use of networks which access knowledge outside the classroom i.e., finding information on the web), are becoming more widely available. These networks have the potential to increase student participation and engagement with the topics of instruction (Davis, 2003a, b; Roschelle et al., 2007, 2004). The combination of the three: Generative Design, function-based algebra and classroom networks allow for powerful new pedagogical practices to be created. Utilizing the power of the graphing calculator within a classroom network, new types of activities become possible. Generative Activities are activities where the data used for classroom discourse and instruction is generated by the students. These activities allow for "space creating play," increased participation and agency, and have a dynamic structure as they are enacted in the classroom (Stroup et al., 2004, 2002).

16.3 Previous Work

Previous work by Stroup, Carmona and Davis (2005), focused on creating a function-based algebra curriculum to be implemented in a Central Texas ninth grade (14–15 years old) Algebra 1 classroom. Foundational to this work was the belief that 80% of algebra could be covered in a curriculum that had an over arching structure which focused on concepts of equivalence, equals and aspects of the linear function. The treatment group consisted of 127 students taught by two relatively new teachers using a network-supported function-based algebra (NFBA) approach as integrated with the ongoing use of an existing school-wide algebra curriculum. The control group was comprised of 99 students taught by two more-senior teachers in the same school using only the school-wide algebra curriculum. The intervention consisted of implementing 20–25 class day's worth of NFBA materials over an eleven-week period in the spring of 2005. Because the students were not randomly assigned to the classes, the study is a quasi-experimental design. The analyses compared the results on the state-administered 8th Grade and 9th Grade algebra Texas Assessment of Knowledge and Skills (TAKS). Using a two sample paired t-test for means, statistically significant results for the treatment group (p-value one tail $=$ $0.000335 > \alpha = 0.05$) were obtained. For the control group scores from 8th to 9th grade showed no significant change whereas students in the treatment group did show significant improvement.

16.4 Generative Activities in Singapore (GENSING)

16.4.1 Setting

In a typical Singapore classroom the front wall will have a white board and a retractable projection screen and the back wall will have a bulletin board. Classrooms come equipped with a digital projector mounted on the ceiling, speakers on the front wall and an A/V junction where a laptop, document viewer (visualizer) and microphone can be attached. Class sizes are about 40 students and classrooms are assigned to a specific group of students who spend most of their day in that room while teachers move from room to room. The data for this study was collected at an upper performing secondary school in Singapore.

16.4.2 Participants

The participants for this study were 183 Secondary 1 students (12–13 years old). Data was collected from five sections of both normal and express track classes towards the end of the first semester of the school year. Students had been introduced, in the traditional curriculum, to concepts of simplifying algebraic expressions prior to the intervention. Class sizes in the study ranged from 38 to 42 students. In 2007, total enrollment at the school was 1584. Demographic breakdowns for the entire school were as follows; 77.5% Chinese, 18.3% Malay, 2.8% Indian, 1.3% Others. The genders were evenly represented with 48.7% of population boys and 51.3% girls. Of the school population, 26.8% of the students spoke English at home, 53.7% spoke Chinese at home, 16.2% spoke Malay and 0.5% spoke Tamil. English was the language of instruction.

16.4.3 Technology

The classroom network used for this study is the TI Navigator™ system created by Texas Instruments Inc. This system is comprised of a teacher computer running TI Navigator software, ten wireless hubs, a graphing calculator for each student and an access point for wireless communication. In this study the calculator used was the TI 84 Plus Silver. Each hub connects to four graphing calculators and transmits data to and from the teacher computer. The screen of the teacher computer is projected in front of the class, and displays input from all students synchronously. The graphing calculator is the student's private space and projection of the teacher computer is the group's space.

16.4.4 Intervention

The intervention was done as a series of three researcher led activities with the participating classes. The three activities, a different one on each visit, were engaged in once every two weeks in order to mitigate interruption of the regular curriculum. Data presented in this paper was collected during the second activity. The activities used have been implemented in many classrooms in the United States and were created during previous studies on Generative Activities (Davis, 2002; Stroup et al., 2005, 2002).

The second activity, Function Activity, has students generate a set of points according to a rule and then find multiple functions that can model the data. The activity begins with each student controlling one point in a coordinate plane by pressing the arrow keys on their calculator. On their device each student sees only their own point. In the up front projected group space, they can see the points of everyone in the class, and the points are updated in real time as students move from coordinate to coordinate. Next, the students are then given a rule to guide their movements. For this example, the rule was to "create a point whose Y value is twice its X value". Students move to a location on the screen that they feel fulfils the rule and push a button to mark the point. With luck, the moving mass of points converges to form a line-like pattern. There will be points that do not fit the pattern, and these are opportunities for exploration of misconceptions. For example, there may be a point at (–4, 8). The teacher can ask, What's right about this point? What's wrong about it? What do you think the person was thinking? How do we fix it? Once all the points fit the pattern, the students are asked to find a function that goes through the data. After students have been successful finding one function to model the data they are challenged to find two more functions that also go through the data.

16.4.5 Data Collection

Students were asked to submit three functions for each rule as described in the activity above. The software allowed the teacher to save activity data in table format, and the data saved were Login ID, Function Name (Y1, Y2, Y3), and the student created function. At then end of each submission session, the activity was paused and the data was saved. This was done prior to class discussion and editing of functions. All classes worked with the following rules: Make your Y twice your X; and Make your Y 6 more than your X.

16.5 Data Analysis

Function-based algebra Generative Activities create a new type of data set – one composed of hundreds if not thousands of student-generated electronic artifacts that are both mathematical (points, lists, expressions) and text-based (answers to questions). The remainder of this chapter will discuss an emergent taxonomy for categorizing one of these data sets, equivalent expressions.

The original intent of the data analysis was to find examples in the student submitted data, of “basic” mathematical concepts that would be covered with direct instruction in normal practice. For this reason data analysis started from a central question, What do teachers care about?

16.5.1 Organizing the Data

The initial data analysis was done using Pivot Tables within Microsoft Excel. The data was analyzed in iterative passes starting by evaluating each function to verify if it was or was not equivalent to Y = 2X or Y = X+6. The data was sorted first by section, then by activity, then by student. This configuration allowed the viewing of all functions from one activity at one time and also to see whether the three functions submitted by a single student grouped together. There was a combined total of 1098 functions submitted for the two activities.

All functions had to be evaluated manually and this allowed me an intimate view of the data. As I went through 1098 expressions checking for their correctness, other features of the data emerged. Coordinating classroom observations and a review of the functions lead to the creation of Mathematical Strategies and Social Strategies as additional coding structures. These strategies will be further explained in the following text. The three coding columns created were Correctness, Mathematical Strategy and Social Strategy.

16.5.2 Correct/Incorrect

With the major levels of analysis identified, coding began (Table 16.1). In the column where Correct/Incorrect was coded there were five coding options; (1) Correct, (2) Incorrect, (3) Blank, (4) Incorrect-mathematical error, (5) Incorrect-thought process visible. ... Quantities for correct and incorrect were remarkably similar across both activities. For both activities combined 60% of the functions submitted were correct, 28% were blank and 12% were incorrect. While the number of blank functions may be disturbing, it is believed they are a result of the activities being run by a visitor to the classroom and the use of generic login names (rather than the students’ actual names). Still, even factoring in the blank functions, the clear majority of responses are correct. If the blank responses are eliminated from the data set, 84% of the submissions were correct.

16.5.3 Mathematical Strategies

In the Mathematical Strategy column there were ten different coding options: (1) Add/Sub/Mult/Div; (2) Variable A/S/M; (3) Simplest Form (where students

Table 16.1 Correct and incorrect strategies

Correct/incorrect strategies	Task	Total
Correct	Y2X	336
	YX+6	337
Incorrect	Y2X	47
	YX+6	49
Incorrect strategy:	Y2X	3
"Mathematical error"	YX+6	4
Incorrect strategy:	Y2X	11
"Thought process visible"	YX+6	9
Incorrect strategy: "Blank"	Y2X	155
	YX+6	147
Grand total		1098

submitted either 2X or X+6); (4) Rational (where students had a rational term in their expression); (5) Rational 1; (6) Additive 0; (7) Multiplicative 0; (8) Other; (9) Blank; (10) Incorrect. Items were coded as Add/Sub/Mult/Div if the operations took place only with numbers. Variable A/S/M was used for algebraic expressions that were not rational. Rational 1 items were rational expressions where the students used the concept that the numerator and denominator would cancel each other out and form a 1 (e.g., Y = 7878787878X/7878787878+6).

Additive 0 items were coded as being distinct from coding schemes 1 and 2 because the students used "chunks" of terms (X–X or 6–6) that totalled zero to create functions the same as 2X or X+6. An example of Additive 0 is; Y = X+6+6–6+6–6+6–6+6–6. In items coded as Multiplicative 0, the student would put in a term or a parenthetical group of terms and then multiply it by 0 to "make it disappear" as one student put it. Items coded as Other used strategies like square roots or the distributive property and they did not occur often enough to have a specific heading.

Manipulations of real numbers and non-rational algebraic expressions were the most frequent mathematical strategies adopted. None of these strategies was directly taught; they emerged from the students' experimentation in the private space of the calculator. From the first activity the students learned that if they created two functions that made the same graph, the functions were the same. This gave them the ability to try different combinations of terms to find things that worked.

Mathematical Strategies were heavily influenced by task. Note in Fig. 16.1 that the 2X task tended to elicit algebraic manipulations where the X+6 task elicited operations on integers. The 2X task was performed first. In the X+6 task, it was anticipated that the students would use strategies from the previous task to find things the same as X and then add 6 to the end of the functions. In reality, students focused on finding things the same as 6 and then added X to the end of the function.

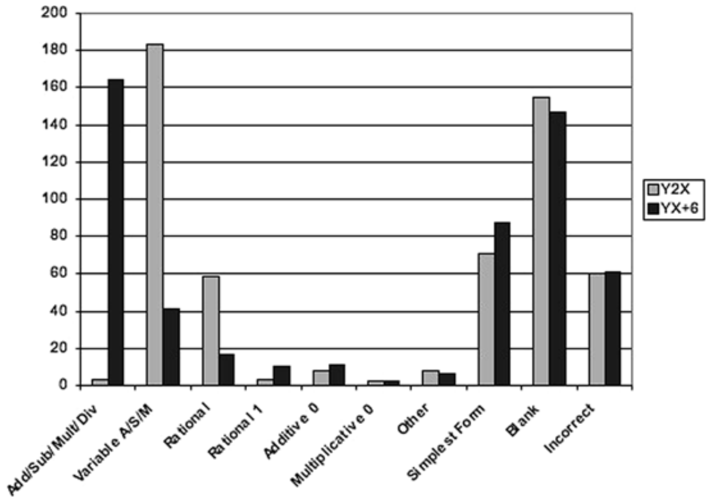

Fig. 16.1 Influence of task strategy

16.5.4 Social Strategies

The Social Strategies category builds on the idea that the upfront display can be considered a performance space. It is unclear if the students desired peer or teacher attention, or if they simply wanted to be different from the other students. The act of submitting their functions to the public space, opened their actions for interpretation. Examples of each of the Social Strategy designations can be seen in Table 16.2.

There were five coding options for Social Strategies: (1) Big Numbers (expressions using numbers with 5 or more digits); (2) Many Terms (expressions made up of 5 or more terms); (3) Unique Strategy (functions using mathematical elements that no one else in the section was using); (4) Humorous (As in theatre, humor may be in the eye of the beholder, but there were some functions that genuinely seemed to have been created tongue-in-cheek. While it could be argued that all of the Many Term functions, especially those with the large chains of X-X, were somewhat tongue-in-cheek, for this analysis, those will stay coded as Many Terms.); and coding option (5) None.

16.6 Discussion and Conclusions

There is still much work to be done in making sense of the vast amount of data created in the enactments of Generative Activities. This paper has discussed a first attempt to find meaningful ways to group the data created in just one activity. This analysis has discovered evidence of students using important mathematical procedures in their quest to create artefacts to submit to the public space. It has also created a new need for software that helps organize the data in meaningful ways and reduces the need for painstaking human assessment of such large data sets.

Table 16.2 Examples of social strategies

Big numbers		Humorous	
Y2X	123456789X–123456787X	Y2X	2.1X–0.1X
	X100000000000–99999999998X	YX+6	98652265256X*0+X+6
	1000000X/500000	Unique Strategy	
	15153X–15151X	Y2X	(85X–5X)/(50–10)
	800000X/400000		(sqrt(100)–8)*X
YX+6	X+123456789–123456783		10(X+5)–10(5)–8X
	(999999999*9999999)*0+X+6		2(4X–3X)
	100000000000X/100000000000+6	YX+6	X+sqrt(36)
	20202020–20202014+X		(–2–8)+X
	(123456789–123456783)+X		PI(X+6)/PI
			X+(5^2)–19
Many terms			
Y2X	(X*X*X*X*X*X*0)+2X		
	(X+X+X)/3*2/2*2		
	2(0.2X+0.2X+0.2X+0.2X+0.2X)		
	3X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X+X–X		
	X*X/X*X/X*X/X*X/X*2		
YX+6	XX–XX+X–X+XX–XX+6+X		
	(X^2)/X∗((X^2)^2)/X/X/X/X+6		
	2X/2+58–52+3–3–9780+9780*645/645		
	X*12/6*4/4/2+6		
	X+X–X+X–X+X–X+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6–6+6		

16.6.1 Covers the Core Topics

When approached about their willingness to try Generative Activities in their classrooms, teachers both in the US and in Singapore have expressed concern that they won't be able to cover core topics in their curriculum if they do. An analysis of the functions submitted by the students shows that generative activities are eliciting important mathematical concepts – concepts that are traditionally taught in discreet chunks of memorized material. Using examples from Table 16.2, students created functions using: the distributive property, e.g., Y = 10(X+5)–10(5)–8X; generalizations on mathematical objects, Y = PI(X+6)/PI; order of operations Y = (X^2)/X∗((X^2)^2)/X/X/X/X+6; identity property of multiplication, Y = 100000000000X/100000000000+6; combining like terms, Y = 2(0.2X+0.2X+0.2X+0.2X+0.2X); and many others.

16.6.2 Need for Tools

Powerful new classroom technologies require powerful new ways for teachers to analyze data. An analysis tool needs to be created that can evaluate large numbers of functions and sort those functions in ways that are meaningful to teachers. In that way, while class is running, the teacher could see how the class is progressing overall. Currently there are no tools like this with which to look at the data, either real-time or later, to look for trends in student responses or analyze how individual students are progressing. If there were such tools, teachers could profitably use the easily obtainable electronic data as a formative tool for real-time instructional decisions and for designing subsequent lessons. As a researcher, I had the time to go through and look, one by one, at 1098 functions. This is not a reality for classroom teachers. The Generative Activities used in this study are powerful activities on their own, and they could become even more powerful with the addition of software-base analytical tools to help the teacher make sense of the data and to use that information for planning future activities.

Acknowledgments This work was supported by in kind equipment donations from Texas Instruments Inc. and grant funding from the Learning Sciences Lab at the National Institute of Education in Singapore. Many thanks go to all of the teachers at the research site.

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Section 5
How Do Models Develop?

Chapter 17
Modeling the Sensorial Perception in the Classroom

Adolf J.I. Riede

Abstract This chapter shows how modeling sensorial perception can be an exciting experience in the classroom. Through experiments and exercises the students are included in the process of rediscovering the law of Weber and the law of Weber-Fechner. As the main example we deal in some detail with the sound intensity and its practical meaning in everyday life. We present here a very elementary generic introduction to the Weber-Fechner law that initially does not use the logarithm function such that this topic can be dealt with in early secondary school. This chapter contains historical aspects in sections 17.1 and 17.5. Furthermore, these sections interpret the historical development as running several times through a modeling cycle and they promote some points that the students could learn from the past. The reader will find further didactical remarks in the last paragraph of section 17.1 and sections 17.3.1, 17.5.1, and 17.6.

17.1 Introduction

"During the rush hour with its high noise level an increase of noise by a certain amount is not perceived at all, however in the silence of the night the same increment is perceived as very disturbing. This is a fact that the door banging car driver obviously does not know but which is very well known to the experienced housebreaker." This remark of Heuser (1984, p. 318) concerns the sensorial perception of external physical stimuli on the human body. More precisely, if I denotes the intensity of the stimulus, it concerns the *difference threshold* i.e. the increase ΔI of I which is just perceptible by the mind. ΔI is also called the *just noticeable difference*. We use the historical notation ΔI. It does not have the same meaning as a Δx in calculus as we shall see in section 17.2.2. The physiologist and anatomist *Ernst Heinrich Weber* (1795–1878) found a model for the difference threshold, which is now known under the name *Weber's law*. However the first scientist who thought

A.J.I. Riede (✉)
Ruprecht-Karls-University, Heidelberg, Germany
e-mail: riede@mathi.uni-heidelberg.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_17,

about sensorial perception and conjectured the Weber law was the physicist *Pierre Bouguer* (1698–1758). (Incidentally it was the same Bouguer who was the first to deal with curves of pursuit. See Riede, 2003.)

While the human senses can distinguish *qualitatively* differences of external physical stimuli on the body it is not possible to measure directly the *quantitative* magnitude of the difference in sensation. In particular sensorial perception cannot be measured directly. It was the physicist and philosopher *Gustav Theodor Fechner* (1801–1887) who had the crucial idea to derive from Weber's law a quantitative scale for sensorial perception. As an example we deal with this scale for sound intensity with the Decibel (dB) as its unit. Fechner used also the notion of an *absolute threshold.* This is the minimal stimulus intensity that according to Fechner, "lifts its sensation over the threshold of consciousness." (See W.H. Ehrenstein and A. Ehrenstein, 1999, p. 1214.) In an analogous way the difference threshold can be considered as the minimum difference intensity that lifts its sensation over the threshold of consciousness. Thus it is clear why the models are called *psycho*-physical laws. Historical data can be found on the webpages of wikipedia and in the articles by W.H. Ehrenstein and A. Ehrenstein (1999) and S. Hecht (1924). (Last accessed on September 7th 2009.)

To deal with Weber's and Fechner's basic psychophysical laws in the classroom we describe a generic introduction. The method includes the students in the modeling process. They carry out some steps within a thought experiment and a real experiment and in the process find the mathematical model.

17.2 The Weber Law

In our courses all participants were included in the process by the following exercise.

17.2.1 The Weight Experiment

Weber considered the sense of touch to distinguish between two weights. Think of putting a weight upon a person's hand who is aware of the weight only by its pressure on the hand perceived by the sense of touch. Then replace it step by step by slightly heavier weights and ask the person to say when he or she notices a difference of weight. Suppose for some starting weights you find the following data where the unit of weight is 1 g (gram).

In our courses for students of biology and in our teacher education or training courses the students or (future) teachers were included in the modeling process by the following exercise.

Exercise 1. Evaluate the measurements of Table 17.1. Which are meaningful quantities you could insert into the last columns? Which is the experimental law that the data suggest? (Note: If a student needs help you could suggest inserting the absolute and relative differences.)

Table 17.1 Data for the weight experiment

Starting weight I	Difference perceived from this weight on
20	21
40	42
60	63
80	84
100	105

17.2.2 Essential Observations

The difference threshold ΔI increases with the starting weight I. In particular ΔI is not constant but depends on the starting weight I; hence ΔI is a function of I. Moreover, these data suggest the assumption that ΔI is a fixed part of I or a fixed percentage of I, $\Delta I = 0.05\ I = 1/20\ I = 5\%\ I$. Weber (1834) found in experiments the values 1/30 or 2/30 i.e. 3 1/3% or 6 2/3% (Hecht 1924, p. 237).

17.2.3 The Length Experiment

Furthermore Weber considered the use of the sense of sight to distinguish between the lengths of two lines.

As in Fig. 17.1 project pairs of lines onto a screen and ask the students to decide for each pair which of the lines is longer than the other or whether they are equal.

Hand out a questionnaire to each student to fill in his or her decision. See Table 17.2. Analyse what the students have filled in.

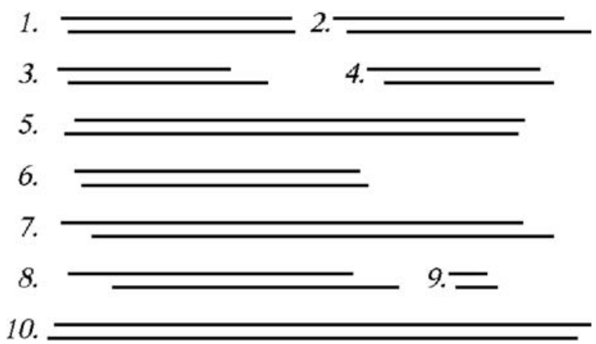

Fig. 17.1 Pairs of lines

Table 17.2 Questionnaire

Which line do you perceive to be longer? Mark with a cross!			
No.	Upper longer	Lower longer	Equally long
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Table 17.3 Evaluation

No.	*u*	*L*	|*u*–*l*|	*P* (%)	Upper longer	Lower longer	Equally long	Distinguishable
1	6.7	6.6	0.1	1.52	True 14	0	3	Yes
2	6.7	7.1	0.4	5.97	0	True 14	2	Yes
3	5	5.8	0.8	16.00	0	True 17	0	Yes
4	5	4.9	0.1	2.00	True 8	0	9	
5	13.15	13.35	0.2	1.52	2	True 10	5	
6	8.3	8.35	0.05	0.60	5	True 1	11	No
7	13.15	13.15	0	0.00	8	1	True 8	
8	8.3	8.1	0.2	2.47	True 15	0	2	Yes
9	1.05	1.15	0.1	9.52	0	True 5	12	
10	15.7	15.5	0.2	1.29	True 12	0	5	Yes

We present the evaluation of the experimental data in Table 17.3. The second and third columns show the lengths of the pairs of lines that a student had chosen in a seminar. The difference of the length of the longer line and the shorter one is denoted by $|u–l|$. The fifth column shows the difference as percentage P of the smaller line. Columns 6–8 indicate which answer was the true one and give the number of students who gave that answer. From the *numerical* data of Table 17.3 we have now to answer which conclusion for the difference threshold could be extracted. In the course we came to an ad hoc decision. This can be formulated in more detail as follows. If the lengths are different and a high majority of students choose the true answer then we can note this as distinguishable. If the lengths are different and a high majority choose equality then we note this as not distinguishable. Furthermore, if a high majority opt for the wrong answer we can note this case as not distinguishable. If we omit the ninth pair the lowest percentage of increase that could be distinguished was 1.29% of the length l of the smaller line for the tenth pair; the highest (the only one) that could not be distinguished was 0.6% l in the sixth pair. Thus we can note the result that the difference threshold lies between 0.6% l and 1.29% l. In other words:

$$\Delta l = 1\%\, l \pm 0.4\%\, l \tag{17.1}$$

Weber had found for this difference threshold $\Delta l = 1/100\, l = 1\%\ l$ (Hecht 1924, p. 237). Thus our experiment confirms the value as Weber.

Observe that the lines of the ninth pair differ in length by more than 9% but it was not noted as distinguishable. This seems to contradict the model. But a look at the length of these lines shows that they are very short lines in comparison with the others. Thus students can learn what is a quite common modeling experience namely that Weber's law is valid within certain limits. This experiment is different from the weight experiment. The particular choice for the lines was suitable for determining the percentage. But this choice turned out to be unsuitable for rediscovering that the percentage is independent of I. A follow up of this experiment could be choosing more appropriate pairs of lines to rediscover this independence.

The experiments on several senses led Weber to formulate a model:

Weber's law. For each type of sensorial perception there is a positive constant c such that an increase of an external physical stimulus I is perceived if the increase is at least $\Delta I = c \cdot I$.

Remember that this minimal perceived increase ΔI is called the *difference threshold*. (In practical mathematics we do not distinguish between minimum and infimum.) Hence Weber's law says that ΔI is proportional to I with constant of proportionality c. If we write

$$\Delta I = 1/aI = p\%I \quad \text{with } a = 1/c \text{ and } p = 100c \tag{17.2}$$

we can alternatively express the model in the following way:

The difference threshold is a fixed part, namely the ath part of I, or a fixed percentage p of I; fixed means independent of I within certain limits.

17.3 Levels of Sensation

17.3.1 Didactical Discussion

In the teacher education courses all students had attended a course on analysis for two semesters and were familiar with logarithm functions. Therefore we used the standard approach for the introduction of the Weber-Fechner law. However in a didactical discussion some students questioned whether it is a suitable topic for high school. On the other hand they demanded to learn as early as possible such applications of mathematics. To meet both issues we present here a very elementary introduction to the Weber-Fechner law that at first does not use the logarithm function and that uses the exponential function only with an integer argument. When the logarithm and exponential functions are needed with a real argument the students can initially think of them as certain functions that interpolate discrete data.

17.3.2 Fechner's Approach

Fechner chose a fixed intensity I_0 of the external physical stimulus and defined that the resulting sensation has level zero. For the perception of a weight you can take for example $I_0 = 20$ g. In practice you take often as I_0 the *absolute threshold.*

Furthermore, Fechner assigned to an increase of I by the difference threshold ΔI an increase of sensation by 1 unit. Fechner's crucial observation was that – because of Weber's law – the increased intensity could be described by a multiplication. More precisely, using $\Delta I = c{\cdot}I$, the following calculation shows that you get the increased intensity $I{+}\Delta I$ by multiplying I with a *constant* factor q.

$$I + \Delta I = I + c \cdot I = (1 + c) \cdot I = q \cdot I \quad \text{with } q{:} = 1 + c > 1. \tag{17.3}$$

Thus you can define that the sensation corresponding to the physical stimulus $I(1) = q{\cdot}I_0$ has level 1, a sensation corresponding to the stimulus $I(2) = q{\cdot}I(1) = q^2{\cdot}I_0$ has level 2, etc. At this stage students can be included in the procedure by the following exercise.

Exercise 2. Complete the Table 17.4; in particular derive a general formula for $I(n)$ with n a natural number. Which type does the dependence of $I(n)$ on n have?

Table 17.4 Relation of sensation and stimulus

Level of sensation	Corresponding physical stimulus
0	$I(0) = ?$
1	$I(1) = I_0 + \Delta I = qI_0$
2	$I(2) = I(1) + \Delta I = q^2 I_0$
3	?
etc.	etc.
n	?

We note by the way that an *arithmetic* sequence of levels of sensation corresponds to a *geometric* sequence of physical stimuli. As result we arrive at a formula that can be used as an indirect definition of the level n of sensation.

Definition The sensation has level n if it is evoked by the physical stimulus of intensity

$$I(n) = q^n \cdot I_0. \tag{17.4}$$

The constant c in Weber's law and hence also the constant q may differ from person to person. You can use an average of q in the population.

17.4 Sound Intensity

The formula for the levels of sound intensity had its origin not in medicine but in engineering. In engineering, when an intensity I was written in the form $I = 10^n{\cdot}I_0$, which says that I is 10^n times as high as I_0, then they saw in the exponent n a level to characterize the magnitude of I. They called the unit of this level 1 Bel in honour of *Alexander Graham Bell* (1847–1922), the inventor of the telephone (who of course had been engaged in sound intensity). Later the unit 1 Bel was subdivided into 10 parts and the new unit 1 dB = 1 decibel was introduced. Thus engineers used as constant q in formula 4 the basic number 10 of the decimal system.

The absolute threshold of hearing is $I_0 = 10^{-12}$ W/m^2 (Watt per square meter). (More precisely this is the definition for a tone of frequency 1 kHz (kilo Hertz). If a general sound is perceived equally loud as a 1 kHz tone of perceived audibility of S dB then it has also a sensorial audibility of S dB or S *phone* by definition of phone.) Table 17.5 shows some physical intensities of sound and noise in everyday life and their perception by the human mind, where S and s denote the perceived magnitude in dB and Bel respectively. Here are two exercises to include the students.

Table 17.5 Physical and perceived intensities of sound

Source of sound	Physical intensity	Level of sensation		Interpretation
	I in W/m^2	s in B	S in dB	
	10^{-12}	0	0	Absolute threshold of hearing
Normal breathing	10^{-11}	1	10	Scarcely audible
Whispering	10^{-9}	3	30	Very silent
Silent office	10^{-7}	5	50	Silent
Busy traffic	10^{-5}	7	70	
Heavy truck	10^{-3}	9	90	Permanent load damages the ear
Rock concert	10^{0}	12	120	Threshold of noise pain
Jackhammer	10^{1}	13	130	

Exercise 3. Present the data of Table 17.5 as points in an I–S-diagram. Join the successive points by straight lines so that you get a curve like a graph of a function. Is it possible to use the graph to determine the perceived intensity S of the physical intensity $\mathrm{I} = 5 \cdot 10^{-3}$ W/m^2?

The essential point of the solution to exercise 3 can be seen from Fig. 17.2.

Exercise 4. Present the data of Table 17.5 as points in a k–S-diagram where k is the exponent in the representation of I in the form $I = 10^k$W/m^2. Join the successive points by straight lines so that you get a curve like a graph of a function. What does the graph look like? Which type of function does the graph seem to describe? Guess a formula for the function $S = f(k)$.

The complete solution of exercise 4 is contained in Fig. 17.3.

The students should discover the formula (17.5) with Bel or decibel as unit.

$$s = k + 12 \quad \text{or} \quad S = 10k + 120 \tag{17.5}$$

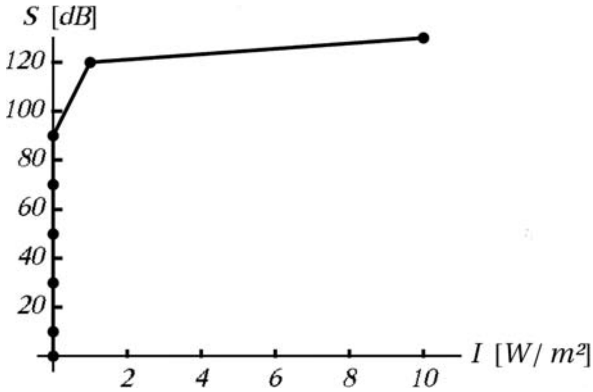

Fig. 17.2 Curve to Exercise 3

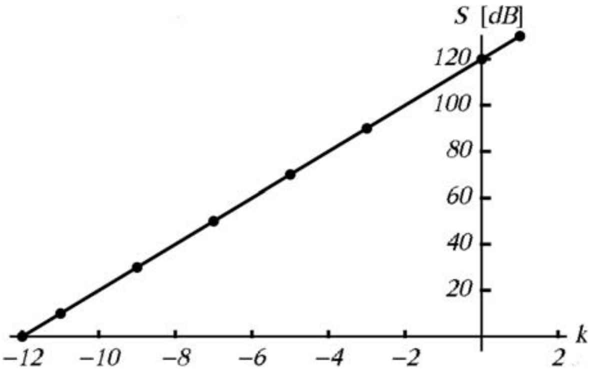

Fig. 17.3 Graph to Exercise 4

Furthermore the students should learn from the exercises that instead of the standard representation in an *I–S*-diagram the data need a new form of representation namely in a *k–S*-diagram.

To this point, we needed the exponential function only for integer arguments and did not need the logarithm function so that it should be possible so far to deal with this topic in early secondary school.

17.5 The Weber-Fechner Law

From now on we need the knowledge of the exponential function with real numbers as arguments, and we need the logarithm function. For simplicity throughout the chapter we use the logarithm to the base 10. If in later high school the logarithm function has been introduced the sensorial perception provides an application of the logarithm. At first, if we replace k by log(I) in formula (17.5) we get the *Weber-Fechner* law for the perception of the sound intensity:

$$S = 10 \log(I) + 120 \text{ [dB]} \tag{17.6}$$

For the general case regard the formula equation (17.4): $I = q^n \cdot I_0$. Remember that Fechner was a philosopher. As a philosopher he thought beyond the levels and thresholds that are perceived by the human mind. He postulated to put into equation (17.4) instead of the discrete variable n a variable s that varies continuously between some limits i.e. in an interval of real numbers. In words of modeling methodology Fechner regarded I as a continuous variable and hence s, too, and took the expression found for discrete values of I as a model also in the continuous case:

$$I(s) = q^s \cdot I_0 \tag{17.7}$$

We can find two forms of the solution of this equation with respect to the variable s:

$$s = \frac{1}{\log(q)} \log\left(\frac{I}{I_0}\right). \quad s \text{ is proportional to } \log\left(\frac{I}{I_0}\right) \tag{17.8}$$

$$s = \frac{1}{\log(q)} \log(I) - \frac{1}{\log(q)} \log(I_0). \quad s \text{ is a linear function of } \log(I). \tag{17.9}$$

These are two formulations of the general *Weber-Fechner law*.

17.5.1 Evaluation for the Perception of Brightness

As Hecht (1924) has reported the Weber-Fechner law was accepted for a long time as valid for all types of sensorial perception and for I in large intervals at least as a sufficiently accurate approximation to reality. For example for the perception of brightness by the sense of sight the Weber-Fechner law is correct for the magnitude classes of stars as levels of sensation. Hecht however points out that these magnitude classes correspond to physical intensities within a small interval in comparison to the whole interval of intensities that the human eye can perceive.

While up to the times of Weber the scientists had used candles in their experiments some decades later the scientists had better methods to produce light of a certain intensity. Thus they were able to collect precise data for the perception of brightness for all intensities. Hecht discussed the new data and found that the difference threshold ΔI can by no means be regarded as proportional to I in the whole interval of brightness. What Hecht then did can, in the sense of modeling, be interpreted as follows.

17.5.2 Rerunning the Modeling Cycle

Selig Hecht (1892–1947) considered the mechanism that transports the external physical stimulus to the mind. Between stimulus and mind there are the sense organ, the nerves and the brain, which hitherto were regarded as a black box. Hecht investigated this box in detail for the visual perception of brightness. In the sense of the modeling concept Hecht improved first the non-mathematical model and than ran again through the modeling cycle. Details of Hecht's investigations do not belong to the object of this chapter. But we should keep one point in mind: For some sensorial perceptions the Weber-Fechner law is valid only within very narrow limits.

17.5.3 The Purpose of a Model

Each modeling problem should have a certain purpose. Often the purpose is not explicitly mentioned because the author regards the purpose as obvious. However it may not be obvious to some readers and then it is possible that they miss the sense and meaning of the procedure. Furthermore the purpose is important because different purposes could lead to different models; each of them may be appropriate to the purpose for which it was constructed but not to the other purpose. Obviously for the sound intensity Bell's purpose was to construct a telephone. However, in medicine the purpose was to improve the treatment of illnesses. An implicit purpose of the model of sound intensity is to point to the health dangers of too loud sound. For the sound intensity we investigate the difference of the two models:

17.5.4 Question

The engineers used $q = 10$ and then subdivided the unit Bel into ten parts. In Weber-Fechner's approach q was determined by the difference threshold. *Are the engineers' model and Weber-Fechner's model very different?*

The answer is: *No.* The purely mathematical reason is that a calculation shows that the use of another q is equivalent to the use of another unit for the sensation s. Only if you choose q in Fechner's way do you have the difference threshold as unit of sensation. There is another reason by pure chance: It just turned out that 1 dB is nearly the difference threshold.

17.5.5 The Historical Development and the Modeling Concept

Let us look at the historical development of the Weber-Fechner Law from the times of Bouguer till the times of Hecht. From the last paragraphs and from the whole article we see that it has the features of a modeling process with its modeling cycle.

What benefits could students have from history? The author would like to promote the following points:

Students can comprehend that modeling has played an important role in scientific progress for a long time and that modeling is by no means a newly fashionable activity.

The identification of the powerful modeling method in historical scientific research should encourage the use of this method to cope with many problems in professional or personal life.

17.5.6 Two Numerical Examples for the Meaning of the Model

If you have to speak 3 dB louder to a deaf person you have to double your voice intensity.

This is correct because adding 0.3 to the exponent means multiplying by 10 to the 0.3th power. This power is with good accuracy equal to 2.

A diminution of noise by 10 dB requires reduction of the physical intensity to one tenth. This indicates why noise reduction requires high efforts.

17.6 Concluding Didactical Remarks

Look at the equation $k = \log(I)$ as a non-linear coordinate transformation leading to a logarithmic scale. Students having in mind exercise 4 and the attempt of exercise 3 will comprehend: *Quantities that vary over many powers of 10 are properly presented on a logarithmic scale.*

The model shows the importance of computing techniques and motivates carrying out the computations because they have a purpose and the results have a sense and a meaning.

The author wishes to hint of a difficulty observed in didactical courses, namely the difficulty to set out the calculation clearly *and* to communicate didactical explanations between the different steps of the calculation. Perhaps we as lecturers should have asked the speakers in the courses to practise both points.

In the didactical discussions the students supported the generic procedure as well as their inclusion in the modeling process. They argued that being involved in the development facilitates the concentration and the comprehension. They stated that being involved in experiments has the important pedagogical impact that it brings fun into the study of mathematics. Furthermore rediscovering the laws was regarded as more exciting than first stating the results and then testing them by means of experiments.

Acknowledgments The author thanks Hanspeter Eichhorn for the good teamwork during several courses on high school teacher education and training and J. Roy Dennett, formerly Department of Mathematics, University of Hull for advice in preparing this chapter.

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Chapter 18
Assessing a Modeling Process of a Linear Pattern Task

Miriam Amit and Dorit Neria

Abstract This research investigated the process of generalizing a pictorial linear pattern problem, as done by fifty-three mathematically promising students participating in an after school math club. The students' work revealed a range of solution paths and representations, and a cycle of expressing – testing – revising. While the majority of them found the constant difference property of the pattern, they experienced difficulties in expressing the general rule. The majority of students applied recursive strategies, even when more global strategies were called for. Although the aforementioned task lacks a real-life context that is essential for modeling problems, the advantages of such problems in multi-cultural classes are discussed.

18.1 Theoretical Background

Mathematical modeling problems are defined as problems that involve mathematizing objects, relations, operations, patterns, and regularities. Model-eliciting activities are characterized by their complex real-life context, and solving them engages students in a cycle of "expressing – testing – revising" (Lesh and Doerr, 2003; Lesh et al., 2006).

Krutetskii (1976) claimed that mathematical giftedness consists of several abilities, including pattern recognition, the ability to generalize, the ability to reason, and the flexibility of mental processes in mathematical activities. The process of generalizing is a crucial aspect of mathematical thinking, especially higher order mathematical thinking (Sriraman, 2004), and, of course, in mathematical modeling (Lesh and Doerr, 2003). (Note: there is a large theoretical foundation on generalization, e.g., Dorfler, 1991; Harel and Tall, 1991, which is beyond the scope of

M. Amit (✉)
Ben Gurion University of the Negev, Beer-Sheva, Israel
e-mail: amit@bgu.ac.il

D. Neria (✉)
Ben Gurion University of the Negev, Beer-Sheva, Israel
e-mail: nerya@bgu.ac.il

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_18,

this chapter). In order to generalize correctly one has to single out similarities in structures and relationships (Krutetskii, 1976).

Components of mathematical ability and the generalization process can be revealed in pattern problems. Linear patterns are patterns in which the n*th* element can be expressed as $an + b$. They can be presented in a range of representations, including figural or numerical (Zazkis and Liljedahl, 2002). These pattern problems have been found to be challenging for middle school students (English and Warren, 1998; Stacey, 1989).

There are four main strategies that correspond to the cognitive levels of mathematical development: *Procedural activity* – at this level the students recognize the constant difference of the linear pattern, and are focused on the procedural feature of the pattern, i.e., adding the constant difference. This is the level where students can successfully perform "near generalizations", that is, obtain a correct answer in a step by step approach, either by drawing or by calculating (Garcia-Cruz and Martinon, 1998; Stacey, 1989). *Procedural understanding* – students are able to establish an invariant from the figures and to apply the calculation rule. This stage is characterized by the students' ability to verbally express a rule (Garcia-Cruz and Martinon, 1998). *Searching for a functional relationship* – students at this stage can see the global structure of the pattern, can obtain a solution for "far generalizations", and can express the patterns' rule in a non-formal way. The fourth level involves not only understanding the *functional relationship* but also expressing it in a *formal algebraic* way.

This research investigated how mathematically promising middle-school students solve linear pictorial pattern problems. Although not a real-life situation, it offers insight into the development of generalization, which is crucial to all modeling activities.

18.2 Methodology

18.2.1 Population

The research population was made up of fifty-three mathematically promising middle school students who participate in "Kidumatica" – an after-school math club in southern Israel. Kidumatica offers a variety of mathematical activities for above average students. The students participating in Kidumatica come from diverse backgrounds – some are Israeli natives and some are immigrants from various countries. Some live in urban communities and others in villages, kibbutzim etc.

18.2.2 Settings and Research Instrument

The research instrument was a questionnaire comprised of six non-routine problems, including the pattern task discussed in this report. The task (Fig. 18.1) was based on the research of Rivera and Becker (2005), and contained four graduated questions dealing with a pictorial linear sequence. In their research, Rivera and Becker investigated the figural versus numerical modes of generalization of prospective school teachers. In this research, we used the same task to investigate the generalization process of middle school students participating in the math club. The students were specifically instructed to describe their solution path.

The following illustration presents the three first patterns in a sequence:

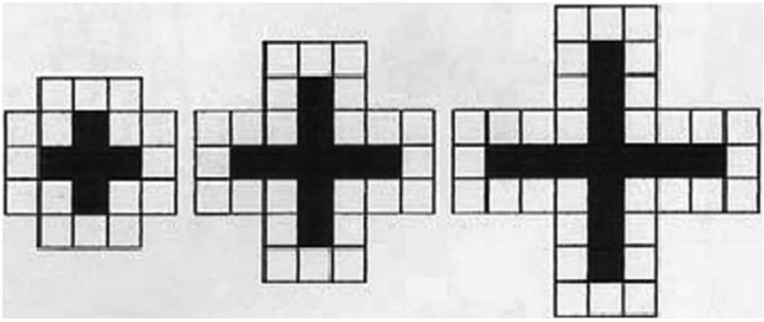


Fig. 18.1 The first patterns in a sequence

The pattern task

A. How many white tiles are needed to make the next pattern?
B. How many white tiles are needed to make pattern 10?
C. Suggest a method to calculate the number of white tiles needed to make any pattern in this sequence.
D. Suggest a method to calculate the number of white tiles needed to make the nth pattern in this sequence.

Although the students had a sufficient background to meet the challenge, the problem was considered non-routine because pattern tasks do not appear in the standard textbooks. Also important to note is that most of the students participating in this research did not have formal instruction in algebra.

Questions A and B could be solved in a step by step approach. Question A (the immediate pattern), was a "warm up" question, that enabled the solvers to examine and investigate the pattern. Obtaining a correct answer could be achieved either by drawing the next pattern, or by grasping the patterns' general rule.

Question B demanded pattern recognition as well as forming a generalization, although a correct answer could be obtained simply by adding up the numbers, or even by drawing.

Questions C and D dealt with pattern generalization. Question C enabled the students to represent the generalization in any form with which they felt comfortable. In Question D there was an explicit demand that the generalization be presented in a formal algebraic mode. These two questions provided the distinction between those students who can only "think algebraically" and those who can also "write algebraically".

18.2.3 Analysis Methods of Student Answers

The answers were analyzed qualitatively according to three criteria (Neria and Amit, 2004, 2006): correctness of the answers, solution strategy, and the mode of representation in the solution path. (Note: our chapter omits the third category).

Correctness of answers: The analysis included the right answer, the wrong answer, and no answer. This category referred only to the final answer, regardless of solution paths.

Solution strategy/method: The categories were based on previous studies (English and Warren, 1998; Ishida, 1997; Lee, 1996) and are described in Table 18.1. The two forms of the additive strategy (drawing or counting) are non-general methods. In contrast, searching for a functional relationship (from the figures or from a table/list) are general methods.

Table 18.1 Students' solution strategies for solving near-generalization problems

Strategies	Sub-categories	Example
Additive strategy (recursive approach): Students observed that each step increases by a constant difference	(I) Drawing a figure and counting to get an answer	See Fig. a in the Appendix
	(II) Setting up a table/list and completing it to get to the requested answer	See Figs. b, c, and d in the Appendix
Searching for the functional relationship: Students attempt to identify the function that describes the pattern		See Fig. e in the Appendix

18.3 Results

Correctness of answers: 32 students (60.4%) got the correct answer for Question A, 24 students (45.3%) were correct in questions B and C. Only 3 students succeeded in writing the correct pattern rule in a formal algebraic form (see Table 18.2).

Solution strategies: In the questionnaire, the students were specifically instructed to describe their solution path. This led to rich data in words, numbers and diagrams. Table 18.3 refers to the distribution of the strategies used in Questions A, B, and C. The majority of students employed additive strategies that usually entailed forming

Table 18.2 Distribution of the correctness of answers ($N = 53$)

	Question A: The next pattern	Question B: The 10th pattern	Question C: Intuitive generalization	Question D: Algebraic generalization
Correct answer	60.4%	45.3%	45.3%	5.7%
Wrong answer	33.9%	45.3%	35.8%	41.5%
No answer	5.7%	9.4%	18.9%	52.8%
Total	100%	100%	100%	100%

Table 18.3 Distribution of the strategies ($N = 53$)

	Question A: The next pattern 77.3%	Question B: The 10th pattern 64.2%	Question C: Intuitive generalization 49.0%
Additive functional	7.6%	15.1%	30.2%
No solution path demonstrated	9.4%	11.3%	1.9%
No answer	5.7%	9.4%	18.9%
Total	100%	100%	100%

a table or a list (see Figs. a, b, and c in the Appendix). A minority of students employed a functional strategy. They searched for the sequence rule and applied it (see in the Appendix Fig. d for application of the correct rule, and Fig. e for application of an incorrect rule).

18.4 Discussion

As seen in the results, 60% of the students knew how to calculate the number of tiles needed for the next pattern. Forty-five percent knew how to calculate the number of tiles needed for the 10th pattern (near generalization) and how to express the generalization intuitively. Far less (6%) succeeded in generalizing in a formal algebraic form.

It was no surprise that the students had difficulties in forming an exact algebraic expression, since most of them had not begun their algebra studies. As in former studies (English and Warren, 1998; Lee, 1996), the constant difference property was usually recognized, enabling students to find the *n*th element of a pattern from the (n–1)th element. However, the attempt to generalize was found to be difficult, even when generalizing in informal modes. Students engaged concrete strategies where more global strategies were called for, and preferred recursive approaches to the functional ones. The occurrence of fixation of a recursive approach found in this study supports previous researches (English and Warren, 1998; Lee, 1996).

The additive method was recognized and applied correctly by most of the students in the near generalization. Those who failed to generalize tended to start out with additive strategies, but it's possible they lacked the flexibility to switch to global strategies needed for reaching generalizations (English and Warren, 1998; Garcia-Cruz and Martinon, 1998; Zazkis and Liljedahl, 2002). Another possible explanation for the difficulties in generalizing is a phenomenon described by Rivera and Becker (2005). In their study, they found that some students are predominantly more numerical than figural, causing difficulties in employing visual strategies. The predominantly figural students employ visual strategies and manage to focus on identifying relationships toward forming a generalization, while the predominantly numerical ones lack the flexibility to tackle the figural patterns.

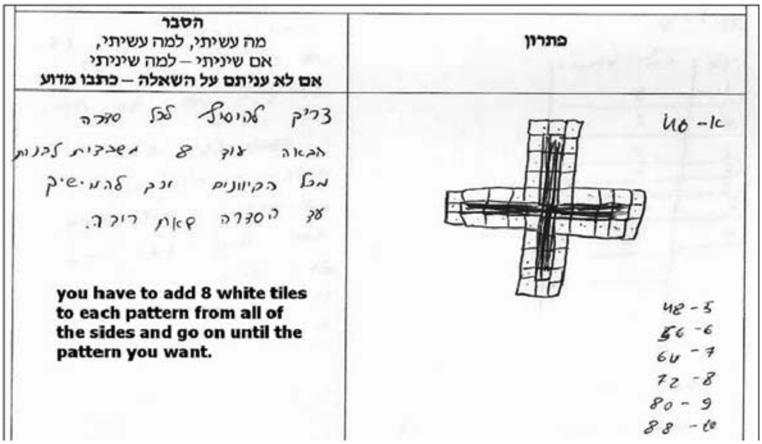
Although generalizing is a basic human activity and is demonstrated in young children (Lee, 1996), the type of generalizing needed while solving pattern problems generated several difficulties at different stages: at the perceptual level (recognizing the pattern), the verbal level (expressing the pattern) and at the symbolization level (using n to represent the nth pattern in a sequence) (Lee, 1996).

The ability to continue a pattern comes well before the ability to describe the general term. However, recognizing a pattern doesn't necessarily lead to a generalization (Zazkis and Liljedahl, 2002), and verbalizing a generalization doesn't necessarily lead to a generalization in algebraic form. It is reasonable to assume that students using functional strategies have the basis to develop algebraic thinking. On the other hand, one can't assume that those who did not use functional strategies do not have the basis and potential for developing algebraic thinking, mainly because the problem is not sophisticated enough.

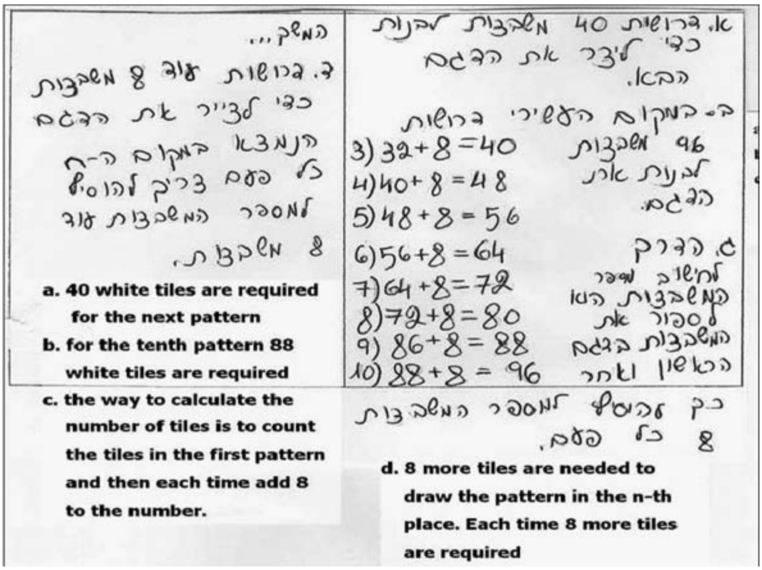
Did the students go through a modeling process? If we take the rigorous, somehow narrow modeling definition – then no. While the problem has some of the elements of modeling, such as a range of solution paths and representations, and students experiencing a cycle of expressing – testing – revising, the problem lacks some of the essential characteristics of model eliciting problems, such as real life context or no single correct solution (Lesh and Doerr, 2003).

Taking a wider perspective on modeling problems, particularly on the "*real-life context,*" raises a major question to consider – what kind of problems can be taken as "*real-life context*" and be meaningful for a wide range of students? Real life situations are dependent on culture and environment, so if we take into account multi-cultural populations, what kind of problems can be meaningful for *all* students participating in a math class? The problem presented in this chapter can be perceived as a prototype of a recursive activity towards generalizations of linear patterns. The problems' lack of a traditional real life context can be taken as an advantage, particularly when dealing with multi-cultural classes, and can increase the accessibility of model eliciting activities. This "tension" between the rigorous narrow stand of "real life" and the wider meanings that can incorporate more types of problem solving requires further discussion and elaboration.

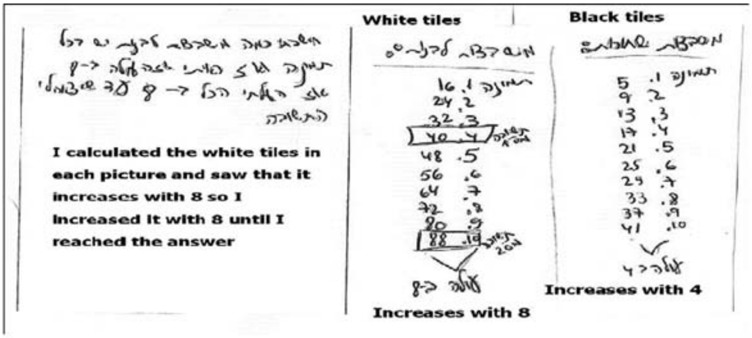
Appendix


הסבר
מה עשיתי, למה עשיתי,
אם שיניתי – למה שיניתי
אם לא עניתם על השאלה – כתבו מדוע
פתרון
you have to add 8 white tiles to each pattern from all of the sides and go on until the pattern you want.
48 - 5
56 - 6
64 - 7
72 - 8
80 - 9
88 - 10

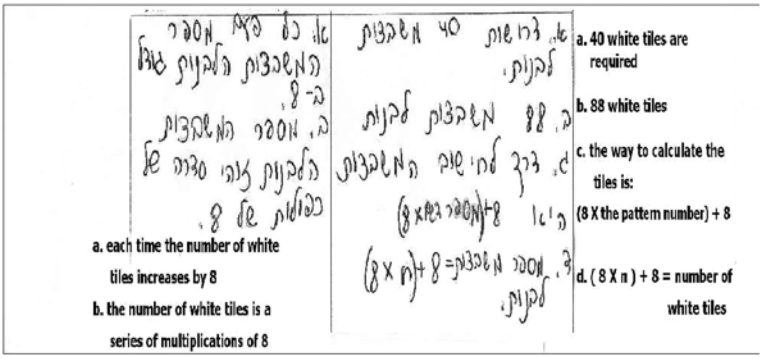
a.


3) 32+8=40
4) 40+8=48
5) 48+8=56
6) 56+8=64
7) 64+8=72
8) 72+8=80
9) 86+8=88
10) 88+8=96
a. 40 white tiles are required for the next pattern
b. for the tenth pattern 88 white tiles are required
c. the way to calculate the number of tiles is to count the tiles in the first pattern and then each time add 8 to the number.
d. 8 more tiles are needed to draw the pattern in the n-th place. Each time 8 more tiles are required

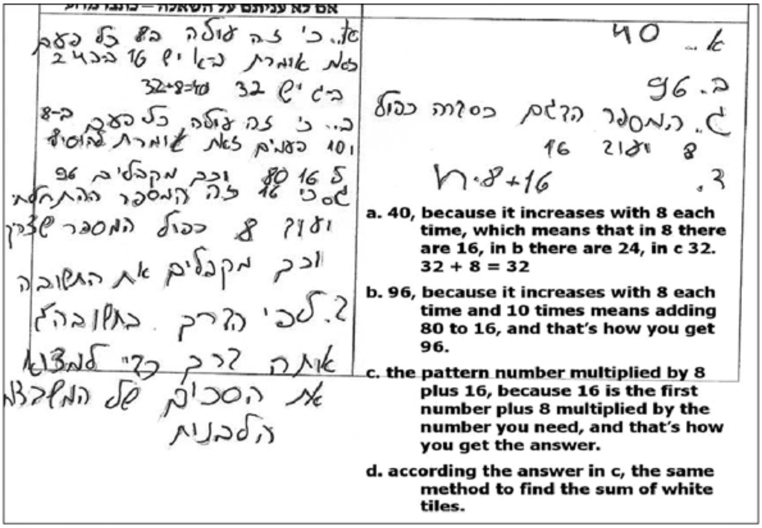
b.


White tiles
Black tiles
I calculated the white tiles in each picture and saw that it increases with 8 so I increased it with 8 until I reached the answer
Increases with 8
Increases with 4

c.


a. each time the number of white tiles increases by 8
b. the number of white tiles is a series of multiplications of 8
a. 40 white tiles are required
b. 88 white tiles
c. the way to calculate the tiles is:
(8 X the pattern number) + 8
d. (8 X n) + 8 = number of white tiles

d.


a. 40, because it increases with 8 each time, which means that in 8 there are 16, in b there are 24, in c 32. 32 + 8 = 32
b. 96, because it increases with 8 each time and 10 times means adding 80 to 16, and that's how you get 96.
c. the pattern number multiplied by 8 plus 16, because 16 is the first number plus 8 multiplied by the number you need, and that's how you get the answer.
d. according the answer in c, the same method to find the sum of white tiles.

e.

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Chapter 19
Single Solution, Multiple Perspectives

Angeles Dominguez

Abstract This article reviews a model-eliciting activity, with a single-numerical answer, implemented in a calculus class at the university level. A closer look at students' solutions reveals multiple perspectives in solving the problem, perspectives that present students' conceptual and representational systems. Most model-eliciting activities allow for multiple responses, and consequently multiple approaches. This article presents the wide variety of modeling perspectives, even though the activity has a single numerical answer.

19.1 Introduction

Model-eliciting activities (MEA) are thought-revealing activities. These activities are complex problem solving situations in which students engage in developing a mathematical model of a proposed real life situation. During the process of solving the problem, students engage in cycles of expressing, testing and revising their ideas in finding and interpreting their solution. According to Lesh et al. (2000), "the descriptions, explanations, and constructions that students generate while working on them directly reveal how they are interpreting the mathematical situations that they encounter by disclosing how these situations are being mathematized or interpreted" (p. 593).

As Lesh and Doerr (2003) stated, "models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behavior of other system(s)-perhaps so that the other system can be manipulated or predicted intelligently" (Lesh and Doerr, 2003, p. 10). In this study, this conception of model is used to identify the different perspectives the students used to solve a given problem.

A. Dominguez (✉)
Tecnologico de Monterrey, Monterrey, Mexico
e-mail: angeles.dominguez@itesm.mx

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_19,

The MEA that was used refers to an economic model (see appendix). The activity was designed by Aliprantis and Carmona (2003) in the context of a historic hotel in Indiana. They implemented the activity "in a mid-western public middle school; more precisely with three groups of typical seventh grade students who did not have previous instruction of algebra" (Aliprantis and Carmona, 2003, p. 258). In their study, Aliprantis and Carmona found powerful mathematical models developed by the students; especially taking into account that seventh grade students have not been exposed to quadratic equations, quadratic functions, or maximization.

In this chapter, I first describe the implementation of the Historic Hotel Activity to three groups of college calculus students. Then, I analyze the students solutions to identify the data analysis and coding procedures that the students developed. Next, I discuss some of the findings emphasizing the different approaches proposed by the participants of this study. Finally, I include some of the results reported by Aliprantis and Carmona (2003) as well as results from this study to make even more evident the richness and variety of perspectives. This is particularly important considering that the Historic Hotel Activity has a single-numerical answer. Yet, the product that the students produced was not simply an answer. The students also needed to produce a sharable and re-useable tool which could be used by others in other similar situations.

19.2 Methodology

A model-eliciting activity based on an economic model (see Appendix) was implemented in a college calculus class. At the time of the implementation the professor was collaborating as a visiting professor at a university in the South of the United States. A mathematics education research group of that university attended the implementation and took notes during the activity. The calculus class met 3 days a week (Monday, Wednesday and Friday) for a lecture (50 min each) and 2 days (Tuesday and Thursday) for a discussion session (50 min each). For the discussion session the class was divided into three smaller groups (about 40 students each). The professor had some experience implementing model-eliciting activities, but for the students participating in this study, it was the first time.

Three weeks before implementing the model-eliciting activity, the chapter corresponding to applications of differentiation was introduced in the calculus class. By the time the activity took place, students had already worked on and been evaluated with respect to related topics such as maximum and minimum, the relationship between the function and its derivative, and optimization problems. However, when the activity was implemented the lecture time was devoted to integrals. Since attendance to the discussion sessions was optional, to encourage participation the professor offered up to 10 extra points on a partial exam. This action motivated about 85% of the class to participate in the activity.

Students worked on the article, the readiness questions, and the problem statement on the first day of the activity. The groups presented their solution to the rest of the discussion group on the second day. Since not all groups were able to present their solution, the professor selected some groups to present during lecture time the following day. The groups selected to present to the entire class were asked to focus on specific elements of their solution, such as a graph, a table, modeling function, optimization strategy, or generalization. The intention was threefold: to share the solution with students who had not participated, to have time for all selected groups to present their solution, and to emphasize different modeling perspectives for the same problem.

Ninety-four students attended the discussion sessions in which the activity was implemented. Students worked in teams of four or five students each. So this resulted in a total of 23 teams in the three discussion sessions. At the end of the first day, each team collected all of their written work (articles, readiness questions, problem solutions, letters to the client, graphs, etc.) in a folder. For each discussion session the folders were labeled indicating discussion session (A, B, or C) and team number. The professor collected the folders for each session and brought them to the following meeting session. On the second day of the activity, teams picked up their folders, finished up final details of the letter and presented their solutions to the class. The classroom had a document camera that made it possible for groups to present any written worksheets during their presentation. Students were encouraged to asked questions of the presenting team. For the second day of the presentations, during lecture time, the professor selected nine groups to present. Since most of the students (about 85% of the class) participated in the activity, the professor gave a brief introduction to the problem statement and asked the selected teams to present their solution focusing on specific elements of their analysis. The order was chosen to link the work of the teams and present different perspectives in solving the same problem.

Data collection consisted of all of the students' written work collected in the folders. In addition, the research group that attended the discussion sessions discussed some of the main ideas elicited in each group, and notes from that discussion also formed part of the data.

For the data analysis, the folders were organized by section and by team number. Then, each material was reviewed in order to capture main ideas present in most of the teams' work. From that review emerged a first classification by numerical answer reached by the students. The folders were then organized into three groups, corresponding to the three different numerical solutions, 12, 68, and 72. These three numerical solutions have a different interpretation based on the definition of the variables in such a way that the final solution is the same for all. In order to maximize his profits, Mr. Graham needs to rent 68 rooms (12 vacant rooms) at $72 per room, generating a profit of $4624.

Next, for each of the numerical solutions it was observed that there were different modeling functions or optimization strategies, and generalizations. All of that information was concentrated in a table registering the following information by team: definition of variable, modeling function, optimization strategy, use of tables, graphical representation, generalization, and comments. From this analysis, a new coding emerged for definition of variable, modeling function, optimization strategy.

- Definition of variables

  $n =$ number of vacant rooms
  $r =$ dollar amount raise in rate per room
  $x =$ number of occupied rooms
  $y =$ daily rate per room

- Modeling functions

  I. Profit $= (80 - n)(60 - n) - 4(80 - n)$ or Profit $= (80 - r)(60 - r) - 4(80 - r)$
  II. Profit $= xy - 4x$ and $y = 140 - x$ to construct a function of profit with respect to x.
  III. Profit $= xy - 4x$ and $x = 140 - y$ to construct a function of profit with respect to y.
  IV. Profit $= (80 - (y - 60))y - 4(80 - (y - 60))$

- Optimization strategy

  D: Use of the first derivative test to determine a maximum (or minimum) point.
  G: Use of a graphing calculator to determine the maximum point of a function.
  T: Use of table to systematically compute the profit by decreasing the number of occupied rooms and increasing the daily rate per room.
  V: Use of the vertex formula, $\boldsymbol{x_V} = \frac{-\boldsymbol{b}}{2\boldsymbol{a}}$, for a quadratic function of the form $\boldsymbol{f(x) = ax^2 + bx + c}$.

Using this new code, I went back to the students' written work and constructed a new table using those labels. This time, I included the numerical solution (not the interpretation) the students arrived at by applying its optimization method. A summary of that table is presented in the Results section.

19.3 Results

College students enrolled in a first-year calculus course for natural and social sciences and engineering participated in this study. All students were enrolled in the same lecture and then divided intro three discussion sessions. The model-eliciting activity was introduced during the discussion sessions. Students worked in small groups of four or five students forming 23 teams. In one session students read the article, answered the readiness questions, worked on the problem and started to write the letter to the client. In the second session, some teams presented their solution to the rest of the class. In the third session (during lecture time) specific teams presented some elements of their analysis, mainly to emphasize the different models and modeling perspectives to solve the same problem with a single answer.

The following table summarizes the analysis for the definition of variable and modeling function based on the coding system explained in the methodology section. Variable/Function/Answer refers to three categories analyzed that were in common for each of the following triplets.

There are 26 responses because three teams provided two different approaches in their letters to the client. The numerical answer arrived at with that definition of

Variable/function/answer	Frequency
n/I/$n = 12$	4
r/I/$r = 12$	7
$n = r$/I/$n = r = 12$	8
x, y/II/$x = 68$	3
x, y/III/$y = 72$	2
y/IV/$y = 72$	2
Total	26

variable and modeling function is included to indicate that all teams were consistent in their procedures and arrived at the "correct" optimization values.

The modeling function I defines n as the number of unoccupied rooms and r as the increment in the daily rate per room. Since the problem statement indicates that for every one dollar increment in daily rate the number of occupied rooms decreased by one, then, $n = r$ in numerical value. All teams recognized this, but not all of them made this relationship explicit. Using the modeling function I, the numerical answer needs to be interpreted to obtain the number of occupied rooms or the required daily rate to maximize Mr. Graham's profit. The interpretation of the numerical answer was what took the students who used this function the most time. The construction of the function, simplifying, and computing the first derivative was relatively easy for all of those teams, but the answer obtained was 12. That is when they struggled. Some students said, "it cannot be, 12 what". They had to go back to the function and recognize that the variable was indicating an increment in one variable and a decrement in the other.

The modeling functions II and III start with the same equations (number of rooms) + (daily rate) = 140 and profit = (number of occupied rooms)(daily rate) – 4(number of occupied rooms). The difference is the choice of the independent variable (number of occupied rooms or daily rate). This decision produced different modeling functions.

The modeling function IV starts by defining y as the daily rate and the same relationship profit = (number of occupied rooms)(daily rate) – 4(number of occupied rooms). To write a function of profit with respect to daily rate, the number of occupied rooms is defined as the total number of rooms minus the unoccupied rooms, that is, $80 - (y - 60)$ because the number of unoccupied rooms is the same as the dollar amount raise in rate per room.

Notice how the definition of the variable is closely linked to the modeling function. The majority of the teams explicitly declared the meaning of the variables they were using. All teams constructed a modeling function for the problem and only one was consistent in the procedures.

Regarding the optimization strategy, the following table summarizes the methods that students chose.

There are 27 responses because three teams had two approaches and one team used the table to verify their solution obtained by the derivative method. As expected

Optimization strategy	Frequency
D	24
G	1
T	1
V	1
Total	27

from students enrolled in a calculus class who had been exposed to derivatives and its applications, students favored the derivative criteria as a strategy for determining a critical point.

Reviewing the letter and written worksheets of the members of the team that used the graphing calculator as the optimization method, it is evident that none of them attempted to use derivatives at any point during their discussion. In their worksheets there are some indications of computations such as listing a table of values; however, that strategy was not documented in the letter to Mr. Graham. Instead, they chose to use a graphing calculator (such as TI-83) to find the increment in the daily rate (x-coordinate) that would give the maximum profit (y-coordinate).

The team that used the vertex formula stated this method as an alternative way to find the maximum. They mentioned two methods in their letter, the vertex formula and derivatives. There was one team that used derivative test and then constructed a table to illustrate and verify their answer. From their written worksheet collected, it is clear that the construction of a table was their first attempt, but in the letter they only included the table as a check tool.

19.4 Discussion

The modeling functions and optimization strategies that this group of calculus students presented do not constitute all of the possible ways to solve this problem. Another modeling function and another optimization strategy that were not used by these students are presented below.

A modeling function not mentioned by these calculus students is the following. Let x represent the number of occupied rooms. Given the condition of the problem, we have that

Profit = (daily rate) (# of occupied rooms) – (maintenance fee)(# of occupied rooms)

Profit = (daily rate)$x - 4x$,

where daily rate = 60 + increment = 60 + $(80 - x)$, because the increment in the daily rate is the same as the decrement in the number of rooms. Then,

Profit = $(60 + (80 - x))x - 4x$. This function has a maximum value at $x = 68$ room, as expected.

An optimization function not mentioned by these calculus students is the following. Set the quadratic function (in any version) equal to zero and determine the roots (zeros) of the equation. This can be done by any algebraic method such as quadratic formula, factorization, or completing the square. Since the function in question refers to a parabola, we can use the symmetry property of parabolas to determine the maximum value by computing the midpoint between the two zeros. In this way, the value of the x-coordinate is found using basic concepts of algebra.

Even though the students constructed a variety of modeling functions and used different optimization strategies, it is clear that the modeling function I and the derivative criteria were used the most in their solutions. This tells us about the mathematical ideas and models that these calculus students have.

Now let us look at the results reported by Aliprantis and Carmona (2003) when the same model-eliciting activity was implemented in a middle school. Focusing on the representational system and mathematical procedures reported, we see that the middle school students often used a mixture of representations, and even though they did not know how to write their idea in formal mathematics, they were able to communicate them by creating their own representations.

Moreover, looking at the mathematical procedures reported (Aliprantis and Carmona, 2003), it can be observed that the students prefer "to multiply the price by the number of occupied rooms. Multiply the maintenance fee by the number of occupied rooms. Subtract the maintenance cost from the total cost of the occupied rooms" (p. 259). This procedure was also favored by the calculus students, only that it was expressed by using variables and functional relationships.

Next, I will use some of the principles for designing productive model-eliciting activities as a structure to argue the multiple perspectives that could be evoked and revealed by students working on the Historic Hotel Activity (see Appendix).

As the model construction principle states, this activity reveals "the conceptual systems that students use to construct or interpret structurally interesting systems.... [Students' solutions] include the development of an explicit construction, description, explanation, or justified prediction." (Kelly and Lesh, 2000, p. 608). Moreover, by the construct documentation principle we have evidence of such conceptual systems developed by the students. Recognition of these two principles in the calculus students' solutions to the Historic Hotel activity and in the middle school students' work reported in Aliprantis and Carmona (2003) exposes the multiple perspectives and highlights the different constructs elicited and revealed by the students.

Regarding the reusability principle we have that the problem statement asks students to help Mr. Graham to solve his current problem and extend the solution for future needs. As he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. This request opens the possibility for various levels of generalization and reusability for Mr. Graham. From the calculus students' solutions some generalizations have the same structure as the modeling function used, only with parameters instead of the data given, such as the daily rate and maintenance fee. These generalizations show, on the part of the students, the use of literal symbols to represent variables and the confidence that their optimization method works for a more general function. I did not find enough information on the middle school students' generalization of the economic problem reported by Aliprantis and Carmona (2003). From the calculus students' solutions I decided that the generalization and reusability aspect of the activity deserve to be discussed in a separate article. For the moment, I will only pose some questions that show the ample variety of expressing a generalization: How generalizable is the model provided? Does it account for a different daily rate, for a change in the maintenance fee? How is that generalization expressed? Do students' generalizations extend to the case of having a different number of rooms, different from 80?

The reality principle offers another window from multiple perspectives. The Historic Hotel Activity has a single-numerical answer, 68 rooms rented at $72 per day producing a profit of $4624. In real life, an administrator of a hotel will not settle for 12 unoccupied rooms. Thus, what to do with the unoccupied rooms offers a variety of responses. Some responses that the calculus students suggested include renting the 12 unoccupied rooms at a lower rate, using those rooms as storage, and constructing a game room in that space. This open-ended question brings multiple perspectives to a problem with a single-numerical answer.

19.5 Conclusion

The broad variety of responses speaks to the broad variety of models and modeling perspectives that this model-eliciting activity can evoke, even though the final solution is unique (68 rooms rented at $72 per day producing a profit of $4624). The previous discussion mentions different ways that the solution to the economic problem offers a full spectrum of approaches. I found this particularly important because this activity opens the door to generate more model-eliciting activities for mathematics or physics (to mention some areas) with a single answer but multiple perspectives.

Acknowledgments This study was conducted with the support of the Research Group of Dr. Guadalupe Carmona.

Appendix: Newspaper Article – An Enchanting Vacation

Going on vacation is something that everyone looks forward to. But staying in a historic hotel transforms any vacation into an enchantment. Finding these charming places is a task to which the National Trust of Historic Hotels of America is committed. To be recommended by the National Trust of Historic Hotels of America, hotels have to prove that they have faithfully maintained their historic architecture and ambience. Several of these hotels hold great pride in their stories, myths, and legends. For example, the French Lick Springs Resort and Hotel in Indiana was named after an early French outpost and its rich mineral springs that included a naturally produced salt lick (a lick is a deposit of exposed natural salt that is licked by passing animals). The hotel was originally built by Dr. William A Bowles, and it flourished during the mid-nineteenth century.

Whether for health reasons, or just for curiosity, visitors were compelled to visit the rich mineral springs, which were said to possess curative powers.

In 1897, the hotel burned down, and it was not rebuilt until 1902. The new owner, Thomas Taggart, built the new French Lick Springs Resort on the ruins of the original hotel. Mr. Taggart, mayor of Indianapolis, made the resort grow in size and reputation in the early decades of the twentieth century. Surrounded by lush gardens and landscaping, the six-story hotel, with its sprawling sitting veranda, was a more than relaxing environment many wished to enjoy. Among the most interesting celebrities that frequented the resort were John Barrymore, Clark Gable, Bing Crosby, The Trumans, The Reagans, Al Capone and President Franklin Roosevelt. In fact, Roosevelt even locked up the Democratic nomination for president in the hotel's Grand Colonnade Ballroom.

Maintenance for a hotel like The French Lick Springs Resort, with all its services, is not an easy task. In 1929, Mr. Taggart died, and inherited it to his son – the only boy among six children – Thomas D. Taggart. With the Depression, however, the popular French Lick Springs began to decline. World War II brought a monetary revival, but in 1946 young Tom Taggart sold the hotel to a New York syndicate.

Today, French Lick Springs Resort rests on some 2600 acres in the breathtaking Hoosier National Forest. Newly acquired by Boykin Lodging Company, the resort eagerly embraces a "New Beginning". It provides 470 rooms, full service spa, two golf courses, in-house bowling, a video arcade, indoor tennis center and outdoor courts, swimming, croquet, horseback riding, children's activities, skiing, boating, and fishing. Fine and casual dining are also available at a variety of restaurants. Two main meeting rooms, the Grand Colonnade Ballroom and the Exhibit Center, accommodate large-scale events.

Besides The French Lick Springs Resort, the National Trust of Historic Hotels of America has identified over 140 quality hotels located in 40 states, Canada, and Puerto Rico.

Readiness Questions

1. What do hotels have to accomplish in order to be recommended by the National Trust Historic Hotels of America?
2. What are the main features of The French Lick Springs Resort?
3. How many owners has the French Lick Springs Resort had since it opened?
4. What are some responsibilities that a hotel manager might have?

Problem Statement

Mr. Frank Graham, from Elkhart District in Indiana, has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is \$60 per room. He was also told that for every dollar increase in the daily \$60 rate, one less room is rented. So, for example, if he charged \$61 dollars per room, only 79 rooms would be occupied. If he charged \$62, only 78 rooms would be occupied. Each occupied room has a \$4 cost for service and maintenance per day.

Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

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Section 6
How is Modeling Different than Solving?

Chapter 20
Problem Solving Versus Modeling

Judith Zawojewski

Abstract This chapter is the third in a series intended to prompt discussion and debate in the Problem Solving vs. Modeling Theme Group. The chapter addresses distinctions between problem solving and modeling as a means to understand and conduct research by considering three main issues: What constitutes a problem-solving vs. modeling task?; What constitutes problem-solving vs. modeling processes?; and What are some implications for research?

20.1 Introduction

I have been wrestling with questions concerning distinctions between problem solving and modeling for a number of years, and in this chapter, I wish to share some of my current thoughts for the purpose of prompting discussion and debate. Are problem solving and modeling really different? Are there substantial distinctions between the *processes* involved in problem solving and modeling? Assuming distinctions do exist, then what are the implications for research?

20.2 Problem-Solving and Modeling Tasks

Definitions of problem solving have been posed over the years, with no one definition emerging as *the* accepted one in the field. In the process of co-writing a chapter with Richard Lesh on mathematical problem solving and modeling for *The Second Handbook of Research on Mathematics Teaching and Learning,* I read a large number of definitions and discussions about the characteristics of problem solving. Throughout the readings, I noticed that two predominant themes run across numerous views of problem solving. The first is that what constitutes a problem-solving task is generally defined with respect to the problem solver – what

J. Zawojewski (✉)
Illinois Institute of Technology, Chicago, IL, 60616, USA
e-mail: judiz@iit.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_20,

is a problem-solving task for one person may or may not be a problem-solving task for another. Further, the problem solver is assumed to *want* to engage in and solve the problem, yet does not have immediate access to a viable solution path. What follows is the second theme – that problem solving is comprised of a search for a means (i.e., procedures, steps) to solve the problem, where the goal is to find a correct way to get from the given information to the goal(s) set forth. These two characteristics have a great deal of intuitive appeal, because they do indeed describe what people often experience and observe in problem solving. However, they also bring about further questions.

Consider the characteristic – that a specific task for one person may constitute a problem-solving task, whereas the same task for another person may not. Such a definition rings true for all challenging tasks – those who are "experts" in a particular area may routinely solve the problems for which non-experts have no apparent means of solving the problem. However, while this common sense notion of problem solving may well describe the experience of solving a problem, as a theory or framework to guide research, there are problems. Consider, for example, the problem of designing research where the classification of task as "a problem-solving task" (vs. *not* a problem-solving task) depends on how the task is perceived and solved by the problem solver. The classification of the task as a problem-solving activity, then, could only be accomplished after the subject attempts to solve the problem, and would need to be accompanied by a protocol that elicits information about the subjects' perception of the problem and observations that confirm the subject's report of needing to search for an unobvious solution path. This problem for research design can be alleviated by considering a definition of problem solving that places emphasis on the nature of the task posed over the relationship between the process of solving the activity and the problem solver.

In the problem solving and modeling chapter that Lesh and I wrote, we proposed a definition of problem solving in terms of modeling: *A task, or goal-directed activity, becomes a problem (or problematic) when the "problem solver" (which may be a collaborating group of specialists) needs to develop a more productive mathematical way of thinking about the given situation* (p. 782). Developing a "productive way of thinking" means that the problem solver needs to engage in a process of interpreting the situation, which in mathematics means modeling. In other words, the task requires that a system of interest needs to be represented by a mathematical system – which will simplify some things, delete others, maintain some features, and distort other aspects. Thus, the defining characteristic is embedded more in the task than in the problem solver.

Activities can be *designed* that require a mathematical model be produced in response to the specification of a client's need (Lesh et al., 2000). These activities are described as thought-revealing and model-eliciting by Lesh et al., and thus research on small group modeling can be designed to study the development of mathematical models created by the group of modelers. Video tapes and/or on-site observations of the students' interactions during the episode can be used to identify different stages of model development, emphasizing a research focus on *model development* rather than problem solvers' processes and use of problem-solving strategies. Understanding how models develop has great potential to inform the field of mathematics education (and other fields), much like research done on the development of other mathematical ideas such as early number, rational number, and proportional reasoning.

Thus, a major distinction between problem solving and modeling is in the emphasis on and the importance of the nature of the task posed, and therefore task design becomes an important feature of research. No one set of design principles can be developed for the collection of types of problem-solving tasks (e.g., multi-step word problems, applied problems, non-routine problems, puzzles, etc.), whereas design principles have been developed for the production of model-eliciting activities (Lesh et al., 2000). From a modeling perspective, designing research (including the tasks, or sites for research) is not only possible, but has the potential to explain more about learning – as model development, or idea development.

20.3 Problem Solving and Modeling Processes

The definition of traditional problem solving (described above) suggests that during a problem-solving episode the primary process in which the problem solver engages is a search for a correct procedure(s) that will enable the identification of a solution path that proceeds from the "given information" to the "goals" of problem. Because the solver is assumed to have encountered a roadblock (i.e., no obvious solution path is immediately available), strategies for searching for and selecting appropriate procedures are of primary importance during the problem-solving process. That is, when faced with a problem that seems unsolvable, problem-solving strategies, or problem-solving heuristics, such as "identify a similar problem," or "simplify the problem" (e.g., by replacing the numbers with smaller numbers), or "draw a diagram," are used to help the problem solver search for and identify a correct procedure(s) that will reach the goal.

Model-eliciting tasks, on the other hand, require that the modeler interpret the information in the task and interpret the required outcomes (with respect to an articulated function) for the purpose of mathematically modeling the situation. Generally, most modelers (often small groups) have access to the task – albeit initial trial models may be extreme simplifications, represent misunderstandings about the task situation, include unwarranted expectations and biases that the solver brings to the task, and so forth. Therefore, the most important processes are those that facilitate the identification of flaws and "soft spots" in the model, which is accomplished by mapping back to the problem situation, testing the trial model, understanding the limitations and better understanding the problem situation, revising the model, and testing it again.

Compared to an emphasis on the search for correct procedures and the elimination of wrong procedures (as has been expected in traditional views of problem-solving processes), when engaged in model development, modelers start out with ideas that are somewhat right and often quite a bit "wrong." But the "wrong" part is simply part of a continuum toward improving models, rather than thought of as a "failing solution path." When engaged in the modeling process, modelers go through iterations of expressing, testing, and revising the trial model. In so doing, they simultaneously improve their model and also develop deeper understandings of the constraints and limitations that still exist at each stage of model development, and learn to articulate (to group members) the trade-offs and benefits of a particular model. Therefore, a very important component of individuals' development of modeling processes is learning to interpret and eventually produce different points of view in order to facilitate the model-revision process.

When the modeler is a small group, varied points of view about the nature of the model are usually generated, and the occurrence of multiple models provides opportunities for the members of the small group to compare and contrast, and consider trade-offs among the models posed. Thus, from a developmental perspective, students' learning to model initially involves the social activity of modeling, which can eventually be internalized, as described in Vygotsky's (1978) theory of development of higher psychological processes. In other words, as students encounter different perspectives from peers in their groups, they learn to interpret each other's points of views, engage in discussions that compare and contrast the proposed models, and work to reach a consensus on a group model. Over a series of modeling activities, an individual can begin to anticipate what others in the group might propose or might point out about a particular model, thus having internalized the process of the social dynamics of the group. While small group problem solving with conventional problem-solving tasks has some of the same benefits, especially when students' different points of view help in the generation and elimination of particular procedures to implement in the solving of a problem, the power in the modeling perspective is that the different perspectives often contribute to the iterative testing and refinement of a model, which is an essential ingredient of the modeling process

In summary, there exist some major distinctions between problem-solving and modeling processes. In problem solving, the "givens" and "goals" are considered static and unchanging, whereas in modeling the "givens" and "goals" are dynamic, constantly under reinterpretation, and able to be reformulated and modified depending the level and type of specification made concerning the function the model is to serve, and on the assumptions, conditions and limitations the problem solver brings to the process. Overall, research on problem-solving processes has, in a sense, emphasized understanding how problem solvers search for and identify appropriate mathematical things *to do,* whereas the emphasis in modeling research has been on how mathematical *interpretations* are generated and refined. Note that this characterization of problem-solving research oversimplifies the actual work done in conventional problem solving (e.g., see Schoenfeld, 1992 and Newell and Simon, 1972), but helps to distill a subtle distinction between the two paradigms.

20.4 Some Implications for Research

The study of problem solving has sought primarily to describe the problem solving strategies that are used by problem solvers, and then to use the identified successful strategies in subsequent research as instructional treatments, where the goal is to improve problem-solving performance of novice problem solvers. The overall findings have been inconclusive – the value of teaching expert strategies to novices for the purpose of improving problem solving performance has not yielded the hoped for results (Lesh and Zawojewski, 2007). In our handbook chapter, we suggest taking a models-and-modeling perspective as a way to frame research.

Since model-eliciting activities can be designed, and such problems are usually challenging enough to warrant small groups of students to work on them (resulting in the natural externalization of model development), conducting research on the processes involved in improving mathematical models is not only possible, but also enticing. The externalization of students' ideas often prompts argumentation and perspective taking on the part of the individuals in the small group – promoting growth and development of the mathematical model while also revealing the local model development for study. Therefore, the type of research conducted when using modeling activities is quite different from that of the studies commonly associated with problem solving (e.g., process-product, interview-based interpretation, think-aloud procedures coupled with observation). Recent perspectives on educational research promote design research (e.g., Kelly et al., 2008), and in so doing, provide visions for conducting research on problem solving *as modeling*. In particular, designing a model of students' modeling using Lesh and Kelly's (2000) multi-tiered teaching experiments (which assume that initially proposed models of student thinking will evolve just as the evolution of the students' mathematical models evolve as they engage in the activities) is a promising way to approach research in a complex, dynamic environment.

A models-and-modeling perspective is a conceptual framework *under development,* or *under design*. What is studied, what is produced, and the research methodologies are all taking shape and contributing to a growing body of literature. I conclude with some discussion about problem solving and modeling research in each of these three areas.

20.4.1 What Is Studied?

In a models-and-modeling perspective, what is studied is idea development, rather than strategies brought to bear by problem solvers. Understanding how conceptual models develop in local contexts has the potential to inform instruction, whereas the research on problem solving strategies has not yielded useful information for instruction. If the goal is to improve problem-solving performance, and if problem solving is viewed from a modeling perspective, then the types of behaviors that are desired are those that enable students to notice flaws and biases in premature models, and to seek alternative perspectives in order to break out of non-productive ways of thinking. Notice that relevant "problem-solving strategies" becomes less of a collection of behaviors to be learned, and more about the nature of experiences that students encounter in small-group modeling.

20.4.2 What Is Produced?

The assumption in much of the conventional problem-solving research is that students have learned mathematical concepts and processes that can be brought to bear on the problem, and therefore they need strategies for accessing the appropriate mathematical knowledge in order to solve the problem. However, the research literature reveals that higher performance on problem solving is much more closely related to the depth and breadth of one's mathematical knowledge than it is to their learning problem-solving and megacognitive strategies (e.g., Silver, 1985). Thus, a models-and-modeling perspective provides a different point of view: when students are generating iterations of their model for solving a problem, they are viewed as simultaneously solving a problem as they are *learning mathematics*. Given that students are generating, testing, and revising mathematical models to solve the problem, they are actually constructing mathematical knowledge as they engage in the process. Therefore one goal is to produce *mathematical learning* as a result of engaging in problem solving and modeling.

Another desired product of modeling research is to identify instructional devices for enhancing perspective taking, since the social process of taking on other points of view facilitates: (1) detecting flaws and deficiencies in current models; (2) comparing and contrasting the strengths and weaknesses of competing models; and (3) generating new ideas for revising models or posing alternative models. Since simply instructing students to "take someone else's perspective" will not work, the types of instructional devices produced are more closely tied to classroom practice. For example, one might *design*: guidelines for selecting students to be in small groups (e.g., identify the types of diversity desired; identify the ranges of abilities needed for productive groups), presentation reflection sheets (in which students react to each other's models after a student group makes a presentation), or a web-based peer assessment system (where groups post first draft models and other groups provide feedback on each other's model). These types of products tend to be instructional tools that enhance the express-test-revise cycles.

20.4.3 What Research Methodology Is Used?

As described in our handbook chapter, research on modeling taps a paradigm recently adapted from engineering – the notion of design research (Kelly, Lesh, & Baek, 2008). In research on modeling, the goal is to improve modeling performance, leading to two types of design research. One is the design of a model of students' modeling – which is essentially the design of a theory of student learning (which is currently under construction). The model of students' modeling is built upon the second type of design research – the design of instructional tools intended to improve student performance on modeling problems. The instructional tool is developed, tested, and redesigned through multiple iterations – each reflecting the researchers' current model of students' modeling. As a result of what is learned during the tool testing, the researchers' thinking about how students are modeling goes through iterative revision, which in turn informs the next round of tool design. Therefore, one of the most salient distinctions between the research methodology used in problem solving and in modeling concerns the role of changing conditions during a study. In a modeling perspective, the researcher *assumes* that his or her own way of thinking about students' modeling will evolve as part of the instructional tool design process, and therefore research in modeling necessarily embraces and exploits changing conditions by using multi-tiered teaching experiments (Lesh and Kelly, 2000) rather than working to insure pure treatments and controls in static conditions in traditional problem solving studies.

Acknowledgment Thank you to Frank K. Lester for his valuable feedback and suggestions, which led to revisions in this manuscript.

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Chapter 21
Investigating the Relationship Between the Problem and the Solver: Who Decides What Math Gets Used?

Guadalupe Carmona and Steven Greenstein

Abstract Tasks that are descriptive of a goal state and not prescriptive of the paths students must take to reach it inevitably generate spaces of possible interpretations of givens and goals, as well as possible paths from givens to goals, each featuring elements of a bounded space of mathematical concepts. When a sample comprised of students at elementary and post-baccalaureate levels of schooling was given one of these tasks, the solutions expressed rich and deep understandings of mathematical concepts that were common among groups at both levels of schooling. These findings are less supportive of the foundational metaphor of curriculum in which understandings serve to support the acquisition of more formal mathematics, and more supportive of the notion of a curriculum that "spirals" around central ideas that are revisited at multiple levels of schooling in order to provide learners with greater access to powerful ways of understanding mathematics.

21.1 Introduction

Typically the intended mathematical concepts addressed by a mathematical task are determined in advance, at the time the developer constructs the task. The mathematics lies in the question, it lies in the answer(s), and it lies along a path from the question to an answer. Consider the following closed task that illustrates this point: *Find the perimeter of a rectangle whose sides are 2x + 4 and 6 units in length.* The question prescribes the mathematics to be used to direct the problem solver from the givens to the goal: the perimeter of a rectangle is some function of its side lengths.

Incidentally, this feature of mathematical tasks is quite common among open-ended tasks, such as: *Vicki claims that you can find the area of any equilateral triangle given the length of only one side. Is Vicki correct or not? Justify your*

G. Carmona (✉)
The University of Texas, Austin, TX, USA
e-mail: lcarmona@mail.utexas.edu

S. Greenstein (✉)
The University of Texas, Austin, TX, USA
e-mail: greenstein@mail.utexas.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_21,

answer. This task is open because there are multiple ways to solve the problem. But each of these ways draws on a fairly small subset of geometric concepts: the area of an equilateral triangle is some function of its side length. Though the statement asking solvers to "justify their answers" may lead to different ways to support the solution, the problem *explicitly* requires students to provide a solution in which the area of an equilateral triangle is a function of the length of a side.

In this study we are interested in mathematical tasks that, similarly to the previous examples, also present clear givens and goals, but that allow for *multiple interpretations* of the problem, and can elicit *multiple solutions* that are mathematically valid to meet the goals of the problem. An example of these latter types of tasks will be described in the next section and is provided in the Appendix.

The purpose of this study was to investigate the mathematical concepts and representations underlying the solutions to a "non-prescriptive" problem of this type and to investigate how the concepts and their representations in the solutions vary according to the mathematical background of the problem solvers.

21.2 Theoretical Framework

Consistent with the ways classroom tasks have been characterized, we call those questions "closed-ended" when the expected answers are predetermined and specific. In contrast, we call those questions "open-ended" when they allow a variety of correct responses. Black and Wiliam (2004) place tasks on a spectrum, with closed tasks having a single well-defined outcome at one end, and open tasks with a wide range of acceptable outcomes at the other. Taking generative design (Stroup et al., 2007) and models and modeling perspectives (Lesh and Doerr, 2003) on learning and problem solving as our guide, we find Black and Wiliam's (2004) spectrum to be limited in its ability to capture the full range of mathematical tasks.

We consider Stroup et al.'s (2007) taxonomy of generative activities that differentiates a range of learning tasks in terms of a space that is more dimensional than a linear spectrum of varying openness (or "closedness"). Stroup and his colleagues suggest that "pathways are intellectual and/or behavioral routes for arriving at given endpoints" and "endpoints are outcomes or artifacts created by learners that represent some form of completion of the generative task" (p. 17–18). Closed tasks like the perimeter problem previously described are "nominally generative," because there's essentially a single pathway to a single correct answer. And open tasks like the area problem presented earlier are about the "exploration of kind and quality of pathways" (p. 37); the focus is on forms of reasoning and the quality of pathways.

The task that is featured in the present study is called The Team Ranking Problem (see Appendix). It is a generative activity, one with "multiple pathways and endpoints where fit with data is central" (Stroup et al., 2007, p. 26). Students engaged in activities such as these do not know the nature of the products they are to develop; they only know the criteria they have to satisfy (i.e., the data they have to fit). English and Lesh (2003) write about a class of structurally similar activities they call *ends-in-view* problems. Their example of a physician and a patient illustrates the concept of ends-in-view: "the physician's end-in-view is curing the patient, but exactly what this cure comprises is not yet known" (p. 299). The end-in-view is curing the patient, but the cure (product) is what is to be developed. Thus, there is a single end-in-view (curing the patient), but multiple (path-)ways to develop a cure (to fit the data).

The Team Ranking Problem also belongs to a genre of performance-based tasks called Model-eliciting activities (MEAs) (Lesh et al., 2000). An MEA presents a real-life problem to be solved by small groups of 3–4 students. The solution calls for a mathematical model to be used by an identified client, or a given person who needs to solve the problem. In order for the client to implement the model adequately, the students must clearly describe their thinking processes and justify their solution. Thus, they need to describe, explain, manipulate, or predict the behavior of the real-world system to support their solution as the best option for the client. As in real life, there is not a single solution, but there are optimal ways to solve the problem.

From a models and modeling perspective, models are *conceptual systems* embedded in *representational media* developed for a *particular purpose* (Lesh and Doerr, 2003; Lesh and Leher, 2003). Although MEAs are designed to focus on richer, deeper and higher-order understandings of mathematical concepts, they are not prescriptive of those concepts. For example, in the Big Foot Problem (Lesh and Doerr, 2003, p. 5), the concept of proportionality is not stated in the problem, although it almost inevitably appears in students' models.

Another intended feature of MEAs is that they make learners' thinking visible, thus allowing for aspects about how learners are interpreting the goals and givens to be revealed explicitly, "such as... what kind of relationships they believe are important and what kind of rules do they believe govern operations on these quantities and quantitative relationships... [And] these visible components are part of the students' models and conceptual systems; they are not part of researchers' models" (Lesh and Doerr, 2003, p. 9).

21.3 Method

Two groups of problem solvers were given the same model-eliciting activity: The Team Ranking Problem (Appendix). This activity presents students, working in small groups, with two perpendicular unit-less axes, labeled "Wins" and "Losses," that enclose an otherwise empty space containing twelve lettered points, each representing the win-loss record of one of twelve soccer teams. Each group's task is to use the record to develop a model for ranking the top five teams. The central mathematical idea that this activity was designed to elicit from students is any optimization model that uses a constrained space with the two-dimensional coordinate system.

The participants were 40 students in third grade at an elementary school in central Texas, and eight post-baccalaureates who were enrolled in a summer workshop at a large public university, also in central Texas. The elementary school students had nearly completed third grade, and the post-baccalaureates ranged in mathematics background from bachelor's degrees in science or mathematics to Ph.D.'s in mathematics or physics. In terms of the curriculum, the third graders had not been exposed to coordinate systems, and the only types of graphical representations that they had covered in previous instructional units involved the use of histograms.

Students from both groups worked the problem in teams of two to four, and were given the same amount of time to solve the problem. The elementary school students were given two 1-hour sessions to solve the problem, present their solution to the rest of the teams, and engage in a classroom discussion of the different solutions presented. The post-baccalaureates were given one 2-hour session to solve the problem, present their solution, and engage in a discussion with the other teams of the solutions presented.

The data collected for this study included field notes and observations taken while students were solving and presenting the problem, as well as during the whole-class discussion. The focus of the observations was on the mathematical ideas that emerged and evolved during the full session. Students' written solutions to the problem, as well as other written or electronic artifacts they developed as part of their solution or presentation, were collected. Finally, transcriptions were prepared from the post-baccalaureate students problem-solving sessions from the time they received the problem until each team developed a final solution.

In order to describe the mathematical ideas developed by students when solving The Team Ranking Problem, these data were coded and analyzed by sorting the elicited conceptual systems and corresponding representations by school level (Miles and Huberman, 1994; Strauss and Corbin, 1998). The coding was accomplished via our working definition of a model as a conceptual system embedded in representational media for a particular purpose (Lesh and Doerr, 2003). According to this definition, a conceptual system consists of "elements, relations, operations, and rules governing interactions" (p. 10) among these elements. Thus, initial coding involved identifying the mathematical models developed by teams of students and coding the components of those models: representational media, elements of the conceptual system, and operations, relations, rules, and patterns among these elements.

Students' responses were sorted by identifying those responses that had common representations or conceptual systems. Categories emerging from the data were identified from the different representations and conceptual systems found in students' responses. Students' responses were re-coded with these emerging categories to establish validity (that the categories were appropriately describing students' mathematical ideas as elicited in The Team Ranking Problem) and completeness (that the categories were sufficient in describing each one of all students' solutions). These categories served in the analysis to describe students' different ways of approaching The Team Ranking Problem, and allowed for a systematic comparison between responses from students at the elementary and post-baccalaureate levels.

21.4 Analysis

Every group was presented with the same problem (Appendix), and every group solved that problem by producing a model. Every model consisted of the same two elements, wins and losses, and these elements were derived from information embedded in a coordinate system. In addition, every model featured a hierarchical decision system for breaking possible ties between teams that functioned in terms of these two elements, although models varied in the relations between, and operations on, those elements.

All student responses presented a "primary criterion" to rank order the teams. The researchers found three primary criteria among the responses: (a) the team with the greatest number of wins, (b) the team with the greatest difference between wins and losses, and (c) the team with the greatest ratio of wins to losses. If two or more teams tied according to the primary criterion, these ties were resolved according to a secondary criterion that favored the team who had played the greatest number of games. Student responses varied in the ways in which they operationalized these criteria, as will be described below.

Only at the elementary school level did elicited models depend on qualitative comparisons of the magnitudes of wins and losses. For example, one group would say that team B has more wins than team A because point B is higher than point B. Similarly, team J has more losses than team L, because point J is farther to the right than point L. Thus, in these models, "most wins" was represented as "highest" and "least losses" was represented as "farthest left," so higher-ranked teams are highest and farthest to the left.

Most of the models elicited by groups at both levels considered each team's actual numbers of wins and losses, and these numbers were found by numbering the axes to coordinatize the plane. This was a rather remarkable finding given that the elementary school students had never seen a coordinate plane as part of their curriculum.

Models elicited by groups at both levels of schooling featured conceptual systems based on the difference between wins and losses. In contrast to finding the greatest number of wins or the use of "highest" in models developed by the elementary school students, a model elicited by post-baccalaureate students broke ties by favoring the team with the greatest number of wins as determined by the team whose lettered point was "contained in the line with equation $y = c$" for the greatest value of c (see Fig. 21.1).

A model elicited by elementary school students represented this system symbolically in the form "W – L." A model elicited by the post-baccalaureate students

Elementary	*Post-baccalaureate*
• greatest number wins • highest point	• the lettered point contained in the line with equation $y = c$ for the greatest value of c:

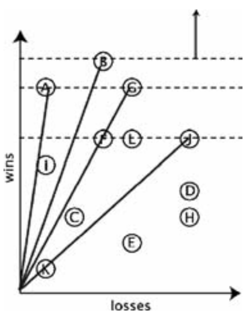


Fig. 21.1 Representations of conceptual systems based on greatest number of wins

represented their conceptual system in a more sophisticated manner; teams were grouped into equivalence classes based on the difference between their numbers of wins and losses. These equivalence classes are illustrated as segments containing the points that represent the teams with equal differences between wins and losses (see column 2, Fig. 21.2). During construction of this model, one of its authors explained that "A has the highest and it goes down from there toward the bottom right corner..." A similar representation in a model elicited by a group of elementary school students is presented in column 1, row 2 of Fig. 21.2, where students divided the space into four regions. When they presented their model to their classmates, they explained that "A is the best region and D is the worst."

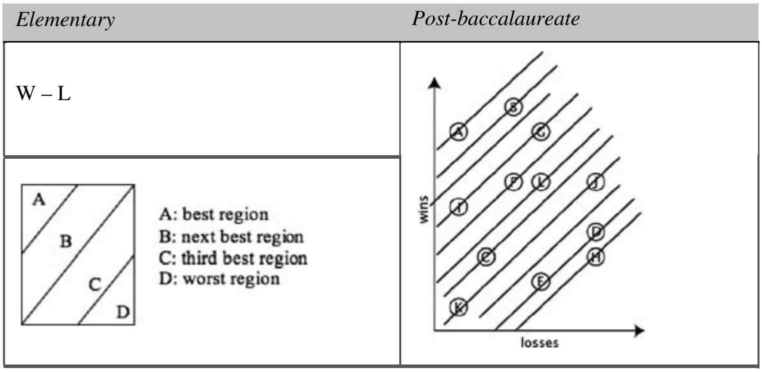


Fig. 21.2 Representations of conceptual systems based on the difference between wins and losses

Both groups elicited models featuring conceptual systems based on the ratio of wins to losses. Figure 21.3 contains illustrations of two different representations of these systems. It is not clear that one representation is more sophisticated than the other, and this is yet another significant finding in the elementary school students' work.

Finally, in a model developed by post-baccalaureate students, the total number of games played was considered in a tie-breaking strategy. Considering the other models produced by elementary school students, a likely representation of this strategy at that level of schooling is the sum "W + L." In contrast, a model developed by a group of post-baccalaureate students measured the lengths of segments connecting the origin to each of the lettered points (column 2, Fig. 21.3) using the formula, $\text{length}^2 = \text{wins}^2 + \text{losses}^2$. During development of the model, one of its authors noted that "since the magnitude of the vector position [of each point] is proportional to the number of games we can use it." Now, an exact measure of the number of games is the sum of the numbers of wins and losses, and the developers of the model realized this; they mention during model development that "the number of games is the sum of the coordinates." (Some students at the elementary level realized this, as well. They spent a significant amount of time finding the relationships between number of won games, the number of lost games, and the total number of games played by a team). Actually, the difference between one team's sum of wins and losses and

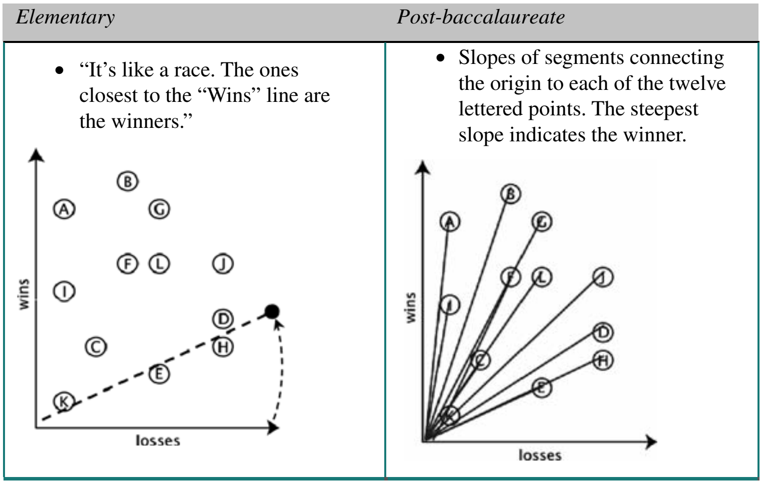


Fig. 21.3 Representations of conceptual systems based on the ratio of wins to losses

the length of that team's "segment" is enough so that rank orderings of these values are not equivalent. Because this group had already incorporated into their model a representation of the ratio of wins to losses (column 2, Fig. 21.3), we propose that the developers chose the "length" metric of the number of games played over the "sum of wins and losses" metric, because the "length" metric is more meaningful in terms of the representation that was already a part of their model. That is, all the information that is required to rank the teams is "contained" in each of the twelve segments: the ratio of wins to losses is the segment's slope, and the measure of the number of games played in the segment's length.

The model produced by one of the two groups of post-baccalaureate students ranked teams according to the ratio of wins to losses, and these ratios were represented as the slopes of segments connecting the origin to each of the twelve points. The other group's model ranked teams according to the difference between wins and losses. In contrast to the models produced by elementary school students where this difference was represented algebraically, the post-baccalaureate group of students represented these differences as segments drawn on the graph to correspond to equivalence classes of teams having the same difference.

To summarize, an *identical*, grade-neutral task was given to two groups of problem solvers of quite different levels of schooling and age. The task was challenging for both groups, and all sub-groups within those groups solved the problem in the same amount of time. Groups interpreted the givens in a variety of ways and the level of mathematics they used to justify their solutions also varied, but the power of the ideas embedded in these justifications did not. Groups that produced similar models used varied language to talk about concepts underlying their models and they accessed the very same concepts and the very same powerful ideas; but the way they developed those concepts varied as evidenced by the multiple representations used to support those conceptual systems. Groups composed of problem solvers with more advanced mathematics backgrounds produced models whose conceptual systems were structured by more-developed and better-supported mathematically formalized representations. In other words, elementary school students' ideas were embedded within the post-baccalaureate groups' models and the post-baccalaureate groups' models comprised formalizations of ideas elicited by groups at their own level *and also* at the elementary school level.

21.5 Conclusions

Among the several results of this study, some warrant further investigation. First, when The Team Ranking Problem was given in an identical form to groups of problem solvers ranging in level of schooling from elementary to post-baccalaureate, every group solved the problem by satisfying the goal state. Every solution featured conceptual systems embedded in representational media expressing problem solvers' rich and deep understandings of the mathematical concepts underlying those solutions. These models were *powerful* in the sense that a plane was coordinatized so that data embedded within it could be interpreted and used to operationalize criteria for the development of a hierarchical decision system for ranking. They were also *complex* in the sense that they served to organize and coordinate multiple elements (wins and losses). Compared to the elementary students' solutions, the post-baccalaureates' solutions were more sophisticated and better justified mathematically. Nevertheless, it was remarkable that students in the third grade were not only developmentally ready to construct mathematical models that could be generalized to situations beyond the current one, but their models embedded some of the very same powerful mathematical ideas contained in the models developed by the post-baccalaureate problem-solvers.

Second, given the very same givens and goals, all groups of problem solvers followed their own paths to a solution and landscaped those paths in collectively meaningful and powerful ways. Though all solutions could be sorted according to common characteristics, the processes by which each group arrived at its solution were qualitatively distinct in the initial ideas that were expressed, the means they used to test those ideas, and the criteria they used to revise them. Two important conclusions follow from this result. First, when students are engaged in open-ended modeling tasks like the one presented here, the processes by which they travel from givens to goals are *iterative* and *nonlinear*. Second, many powerful ideas were common among all students' solutions, and these ideas are a subset of the finite set of "big ideas" that exist in the *bounded solution space* of mathematical models that is defined by the problem and that can be matched straightforwardly to a mathematics curriculum.

Finally, starting with the assumption that all mathematical activity occurs within a space of mathematical ideas, we propose that closed tasks "close" that space to a single point that represents the single, linear (and replicable) pathway and its single, corresponding solution. In contrast, open-ended modeling problems like The Team Ranking Problem "open" up a bounded space in which problem solvers explore, develop, and refine particular mathematical ideas that lie within it. The mathematics is not defined only in the task. Rather, it is the problem solver who decides what mathematical ideas are to be used in the interpretations of both *givens* and *goals* and in the formation of the *path* that connects those givens and goals. Thus, the mathematics is defined in the interaction between the task and the problem solver.

21.6 Discussion

Typically, teachers who aim to facilitate learners' acquisition of a particular grade-specific standard will map that standard to tasks for which the solution space is constrained to the particular standard. Then, learning of that standard is measured via some sort of conventional assessment. The Team Ranking Problem permits a reversal of that strategy. The non-prescriptive nature of the task opens up a space in which problem solvers decide the math they find useful in establishing a path from givens to goals. For example, in this study, one group of elementary school students actually re-invented the coordinate plane, because they perceived a need to make quantitative comparisons among elements of the conceptual system they had developed.

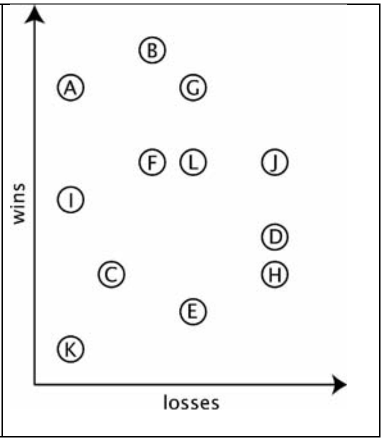
We have provided an exemplar of a kind of open-ended modeling task that elicits very complex models from individuals who have a broad span in terms of their mathematical level. Rather than teachers mapping standards to traditional tasks, problem solvers' solutions to alternative tasks like this one provide an opportunity for assessment of any number of standards, not necessarily at their grade level. This challenges the idea that a particular task or problem can be assigned a fixed level of difficulty. Rather, we utilize the results of this study to argue the level of difficulty is better determined by the solution the problem-solvers produced. Additional and important net effects of this sort of formative assessment strategies include opportunities for students to assume higher levels of agency in their own learning and, by doing so, find greater access to powerful ways of understanding mathematics.

Appendix: The Team Ranking Problem

Each point in the figure below represents the win-loss record of each of the twelve elementary school soccer teams in *Flat Mountain School District.*

READINESS QUESTIONS

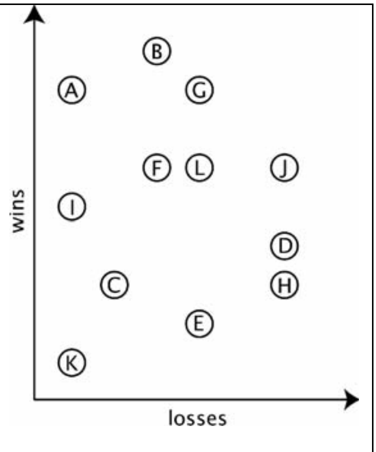
1. *Which team(s) won the most games? How do you know?*
2. *Which team(s) lost the most games? How do you know?*
3. *Did all teams play the same number of games? How do you know?*



THE PROBLEM

Your task is to develop a method for determining which of the twelve teams wins the first, second, third, fourth, and fifth place trophies. Make sure you explain very clearly why the teams should be ranked this way, so that the kids on all the teams will understand that your system is fair.

There are twelve teams in the *Flat Mountain School District*, but many, many more in the whole state! At the end of the season, the state soccer officials will need to find the top five teams for the state. Make sure the officials will understand how to use your ranking system for any number of teams.



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Chapter 22
Communication: The Essential Difference Between Mathematical Modeling and Problem Solving

Tomas Højgaard

Abstract In this chapter, I discuss the formulation of tasks used as a communicative tool for developing someone's mathematical modeling competency and mathematical problem solving competency. These two competencies are characterized and their different crux is highlighted. This is exemplified by the formulation of different kind of tasks, and two hypotheses are offered for further debate and investigation concerning the kind of tasks that dominate in mathematics education and why.

22.1 Introduction and Conclusions

Consider the following tasks:

1. What is the relation between one's income and the tax paid?
2. How does the tax one pays depend on the income tax and the VAT?
3. Microorganisms breed by cell division. To slow down the cell division in a particular sample of microorganisms, a certain substance is added. After adding the substance, the number of microorganisms can be described as a function with the expression

$$f(t) = 12 + 3t - e^{0.5t},\ t \in [0;6.9]$$

where t is the number of hours after the substance is added and $f(t)$ is the number of microorganisms, measured in millions, at the time t. Determine the number of microorganisms 3 h after the substance has been added. Also, determine $f'(t)$ and interpret it. Finally, use a graphics calculator to solve the equation $f(t) = 13$.

Determine the maximum number of microorganisms after the substance has been added.

T. Højgaard (✉)
The Danish School of Education, Århus University, Denmark
e-mail: tomas@dpu.dk

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_22,

In this chapter, I will elaborate on and exemplify the following conclusions:

- Reflections among, and communication between, teachers and students about the crux of different mathematical competencies can be used to engender the kind of work processes aimed at in mathematics education.

  Therefore it is fruitful to identify mathematical modeling competency and mathematical problem solving competency as two competencies that do overlap, but have completely different cruxes.

- One of the benefits of identifying the different cruxes is that this identification can be used to sharpen the way teachers invite students to develop the competencies, for example, by the formulation and orchestration of written tasks.

  As examples of this, the first task above is an invitation to mathematical modeling, the second task is an invitation to mathematization and problem solving at upper secondary level, and the third task is – as a contrast – an unfocused task that does more harm than good when the aim is to develop the two competencies at play.

22.2 A Competency Perspective

My work with mathematical modeling and mathematical problem solving in relation to mathematics education are analytically biased. The bias comes from my involvement in the development of a more general idea: To use a set of mathematical competencies as a perspective on what it means to master mathematics and how the answer to this question can and should be used to develop mathematics education.

The scaffolding of this idea was the hub of the so called KOM project (KOM is an abbreviation for "competencies and mathematics learning" in Danish), which took place in the years 2000–2002 under the leadership of Mogens Niss from Roskilde University in Denmark and is thorougly reported in Niss and Jensen (2002 and to appear). The important analytical steps carried out in this project were to:

- move from a general understanding of the concept *competence*, which I – in semantical accordance with the KOM project – take to be someone's insightful readiness to act in response to the challenges of a given situation (Blomhøj and Jensen, 2007),
- to a focus on *a mathematical competency* defined as someone's insightful readiness to act in response to *a certain kind of mathematical challenge* of a given situation (Blomhøj and Jensen, 2007),
- and then identify, explicitly formulate and exemplify *a set of mathematical competencies* that can be agreed upon as independent dimensions in the spanning of what it means to master mathematics.

Such a set of mathematical competencies has the potential to replace the syllabus as the hub of the development of mathematics education, because it offers a vocabulary for a focused discussion of the aims of mathematics education that can make us feel comfortable for the same reasons that we presently are with the traditional specificity of the syllabus (Blomhøj and Jensen, 2007).

The result of the KOM analysis is visualized in condensed form in Fig. 22.1. I will now elaborate on the part of the KOM perspective that is the focus of this chapter: the picture of mathematical modeling competency and mathematical problem solving competency as two distinct, but overlapping constituents of mathematical mastery.

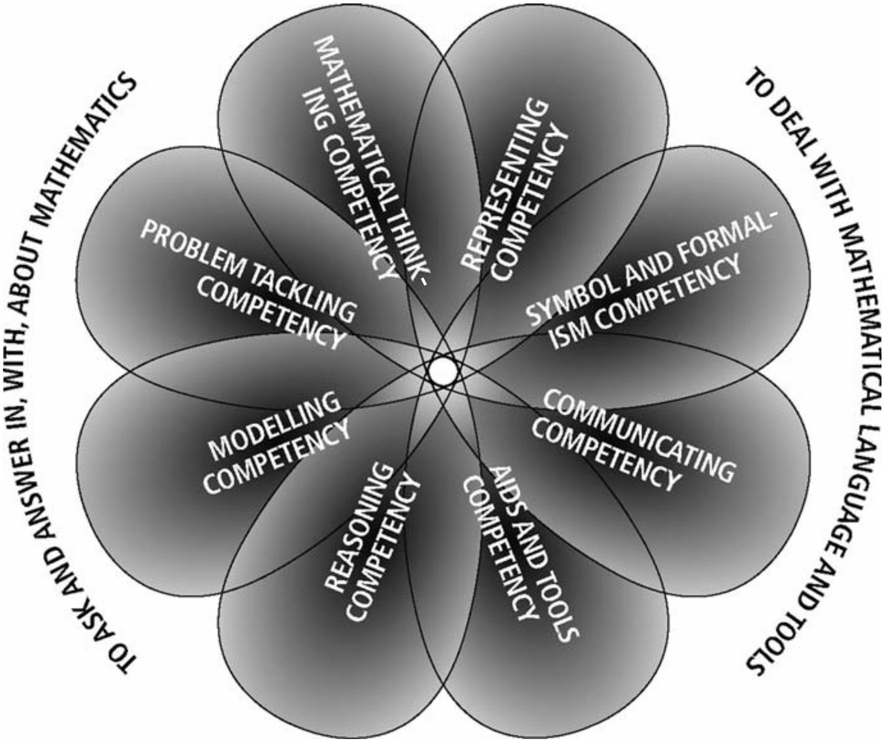

Fig. 22.1 A visual representation – the "KOM flower" – of the eight mathematical competencies presented and exemplified in the KOM report (Niss and Jensen, to appear, Chapter 4)

22.2.1 Mathematical Modeling Competency

In my work I refer to a description of the creation and use of a mathematical model consisting of the following six sub-processes (Blomhøj and Jensen, 2003; Jensen, 2007a):

(a) Formulation of a task (more or less explicit) that guides you to identify the characteristics of the perceived reality that is to be modelled.
(b) Selection of the relevant objects, relations, etc. from the resulting domain of inquiry, and idealisation of these in order to allow a mathematical representation.
(c) Translation of these objects and relations from their initial mode of appearance to mathematics.
(d) Use of mathematical methods to achieve mathematical results and conclusions.
(e) Interpretation of these as results and conclusions regarding the initiating domain of inquiry.
(f) Evaluation of the validity of the model by comparison with observed or predicted data or with theoretically based knowledge.

Figure 22.2 is a visualization of this process. The figure contains a labelling of the sub-processes as well as an attempt to evaluate the six stages that frame them. It is not far from, and is indeed inspired by, many of the other models of the mathematical modeling process found in the mathematics education research literature, for example, the proceedings from the ICTMA conferences, of which Lamon et al. (2003) and Haines et al. (2007) are the two most recently published.

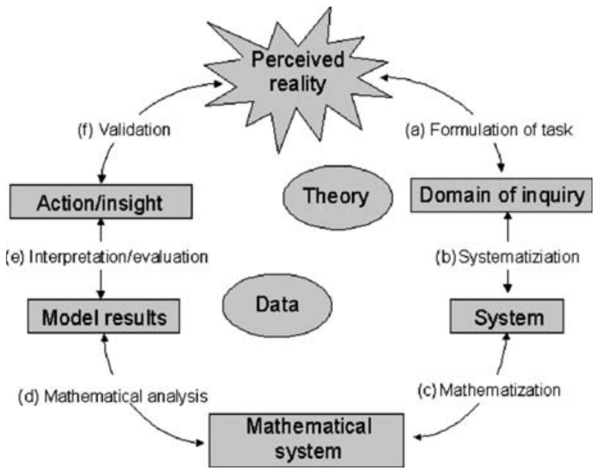

Fig. 22.2 A visual representation of the mathematical modeling process (Blomhøj and Jensen, 2007)

I use *mathematical modeling competency* to describe someone's insightful readiness to carry through all parts of a mathematical modeling process in a certain context (Blomhøj and Jensen, 2003).

22.2.2 Mathematical Problem Solving Competency

In the KOM report, the following characterization of mathematical problem-tackling competency is presented:

> This competence partly involves being able to *put forward*, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, "pure" as well as "applied", "open" as well as "closed", and partly being able to *solve* such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways. (Niss and Jensen, to appear, Chapter 4)
>
> [. . .] A (formulated) mathematical problem is a particular type of mathematical question, namely one where mathematical investigation is necessary to solve it. In a way, questions that can be answered by means of a (few) specific routine operations also fall under this definition of "problem". The types of questions that can be answered by activating routine skills are not included in the definition of mathematical problems in this context. The notion of a "mathematical problem" is therefore not absolute, but relative to the person faced with the problem. What may be a routine task for one person may be a problem for someone else and vice versa. (Niss and Jensen, to appear, Chapter 4)

Accordingly I use *mathematical problem solving competency* to describe someone's insightful readiness to solve different kinds of mathematical problems in their already formulated form (cf. Jensen, 2007a).

22.3 Contrasting the Crux of the Competencies

I now return to the tasks mentioned in the introduction in order to characterize these in the light of the given competency descriptions.

The first task challenges the subject to work with all the mathematical modeling sub-processes, and I will therefore label it an invitation to mathematical modeling. By virtue of the "underdetermined" nature of the initial parts of the mathematical modeling process, the crux of this challenge is to learn to handle the many often equally sensible choices that needs to be made before mathematical concepts and techniques can be of any use, and the lack of a clearly defined strategy to use when making these choices. Or, to put it in a different way, to cope with a feeling of "perplexity due to too many roads to take and no compass given" (Blomhøj and Jensen, 2003). Seen through a didactical "competency lens," mathematical modeling is mainly interesting if it carries with it such a handling of "openness":

> Even though in principle we are concerned with mathematical modeling each time mathematics is applied outside it's own domain, here we use the terms model and modeling in those situations where there is a non evident cutting out of the modelled situation that implies decisions, assumptions, and the collection of information and data, etc.
> Dealing with mathematics-laden problems which do not seriously require working with elements from reality belongs to the above-mentioned problem tackling competency. Those aspects of the modeling process that concentrate on working within the models are closely

> linked to the above-mentioned problem tackling competency. However, the modeling competency also consists of other elements which are not primarily of a mathematical nature, e.g., knowledge of nonmathematical facts and considerations as well as decisions regarding the model's purpose, suitability, relevance to the questions, etc. (Niss and Jensen, to appear, Chapter 4)

The second task from the introduction is an example of a task that "concentrates on working within the models." It represents a kind of task that only challenges the subject to work with sub-process (c), (d) and (e) of the mathematical modeling process as it is characterized here. The delimitation of the context and task in (a) and (b) is already dealt with in the formulation of the task, and the inclination to work with (f) comes from having worked with (a) and (b). Tasks like this challenge the subject's competency to mathematize a more or less well-defined problem of a non-mathematical character, and therefore often give the subject a feeling of "knowing what the goal is without knowing how to achieve it" (Blomhøj and Jensen, 2003).

An ability to cope with such a quite frustrating feeling of being "cognitively stuck" is, in my view, the crux of mathematical problem solving competency (Jensen, 2007a). A task like "How does the tax one pays depend on the income tax and the VAT?" is therefore mainly to be seen as an invitation to develop this competency within the domain of applications of mathematics. It does not challenge – and can therefore not be used to completely develop – mathematical modeling competency and all the associated sub-competencies (Jensen, 2007b).

I will label the combination I am dealing with here *mathematization competency* to describe someone's insightful readiness to solve problems defined as such by a challenge to mathematize. More loosely speaking, mathematization competency is the combination of mathematical problem solving competency and mathematization.

22.4 Experiences with Challenging the Competencies

I have attempted to put the ideas and approaches laid out here into educational practice in various projects, of which I will mention two. The first is my involvement in the structuring and writing of a series of mathematics textbooks for grade k-9 (see Jensen et al., 2002, as an example). The presentation in these books is in two important ways influenced by my understanding of and emphasis on mathematical modeling competency and mathematical problem solving competency in mathematics education. Firstly, the tasks introduced in each chapter are in separate sections following whether they are meant to be competency-oriented problems (for some of the pupils) or drill-oriented exercises (cf. Fig. 22.3). Secondly, the books contain tasks meant to challenge mathematical modeling competency in two different ways. One is as activities of relatively short duration of up to one lesson, consisting of both mathematization tasks and modeling tasks that are relatively easy to systematize. These kinds of tasks are included in the problem section of each chapter. Another is as activities meant to last for weeks. These so-called investigations are initiated by tasks given in a special section in the back of each book,

Invitations to…	*Mathematical problem solving*	*Mathematical exercising*	*Neither problem solving nor exercising*
Mathematical modeling	What is the relation between one's income and the tax paid?	How much fabric does one need to make a cloth for the dinner table?	Irrelevant category
Authentic mathematization	How does the tax one pays depend on the income tax and the VAT?	Draw a sketch of a 135 m^2 house.	Irrelevant category
Pseudo extra-mathematical orientation	The total length of The Loch Ness monster is 40 meters plus half its own length. How long is the monster?	Anna and Bob earn 20% off the sale of ice cream. How much do they earn if they sell for a) DKK 100? b) DKK 500? c) …	Microorganisms breed by cell division. To slow down the cell division in a particular sample of microorganisms, a certain substance is added. After adding the substance, the number of microorganisms can be described as a function with the expression $f(t) = 12 + 3t - e^{0,5t}, t \in [0;6.9]$ with t being the number of hours after the substance is added and $f(t)$ the number of microorganisms, measured in millions, at the time t. Determine the number of microorganisms 3 hours after the substance has been added. Determine $f'(t)$ and interpret it. Use the graphical calculator to solve the equation $f(t) = 13$. Determine the maximum number of microorganisms after the substance has been added.
No extra-mathematical orientation	A cube's volume is k times as big as the volume of another cube. What is the relation between the surface areas of the two cubes?	Solve the equations: a) $7 - x/3 = x + 1$ b) $x - 2 = \frac{x+2}{2}$ c) …	In a system of coordinates a parabola P and a line l aredetermined by $P: \ y = x^2 - 4x + 3$ $l: \ y = -x + b$ where b is a number. Determine the coordinates of the vertex T of the parabola P. Calculate the distance from T to l for $b = -2$. Determine the number of intersection points between P and l for every value of b.

Fig. 22.3 Examples of tasks spanning mathematical modeling competency and mathematical problem solving competency

dominated by very open invitations to mathematical modeling. Appendix contains several authentic examples of these different kinds of tasks.

This structuring of different kind of modeling tasks was also used in the so-called Allerød project (Jensen, 2007a). Here my ideas about mathematical modeling and problem solving were brought into an upper secondary mathematics classroom involving 25 students and their mathematics teacher. To make a long story short, my experience from being heavily involved in this project is that one of the main advantages of using a competency perspective on mathematics education is that reflections about the crux of different mathematical competencies can be used to engender the kind of work processes aimed at in mathematics education. More specifically, it became an important and shared part of the thoroughly developed classroom culture to distinguish between and work with the different kinds of tasks exemplified in Appendix, and to value their different contributions to the development of mathematical modeling competency.

22.5 Hypotheses to Promote Further Debate and Investigations

I have two hypotheses about the use of the different kind of tasks discussed in this chapter in general mathematics education. Neither of the hypotheses are grounded in solid research data, but are meant to promote further debate and search for more evidence.

One hypothesis is that invitations to mathematical modeling are far too often replaced with invitations to mathematization – because mathematization tasks are easier to formulate, orchestrate, work with, and assess.

Another hypothesis is that invitations to mathematization are far to often replaced with pseudo extra-mathematically orientated, neither problem solving nor exercise focused tasks (cf. Fig. 22.3) – because the latter kind of tasks are the easiest to formulate, orchestrate, work with, and assess. The third task from the introduction is an example of such a task (cf. Fig. 22.3). It also serves as a good illustration of the danger I hypothesize about here, since it is an authentic example of the only kind of application-oriented tasks given in the written math exam for the most common branch of upper secondary school (gymnasium) in Denmark.

Appendix: Examples of Tasks Developed for and Used in Grades 9–12 with the Specific Aim of Promoting the Development of Mathematical Modeling Competency (Jensen, 2002, 2007a)

Invitations to develop mathematical modeling competency – long duration (2–4 weeks):

1. What is the relation between one's income and the tax paid?
2. What is the cost of me?
3. Which means of transportation is the best?
4. How can one navigate?
5. Can one become slim by exercising?
6. How many windmills should Denmark have?
7. What is the best shape of a tin can?

Invitations to develop mathematical modeling competency – short duration (within a lesson):

8. How much fabric does one need to make a cloth for the dinner table?
9. How many times can one brush one's teeth with a tube of toothpaste?
10. Draw a sketch of a 135 m^2 house.
11. How far away is the horizon?
12. How far ahead must the road be clear for you to make a safe overtaking?
13. At what angle of incline does a tower topple?
14. What are the maximum sizes of a board if one is to turn a corner?

Invitations to develop mathematization competency – short duration (within a lesson):

15. How does the tax one pays depend on the income tax and the VAT?
16. When you buy something, is it better to get a percentage of the price in discount before or after the VAT has been added?
17. Which savings account do you prefer: The one that pays 8% in annual interest or the one that pays DKK 110 in annual interest?
18. A theater increases the ticket price by 30%, which causes the income from the sale of tickets to go up by 17%. By how many percentages has the size of the audience changed?
19. Between three cities of the same size, where should the only high school in the area be?
20. A liqueur glass is cone-shaped. What height of the liqueur served in the glass makes it halfway full?
21. An enclosure must have the shape of a rectangle with a semicircle at one end. How much land can you enclose with a given length of fence?

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Chapter 23
Analysis of Modeling Problem Solutions with Methods of Problem Solving

Gilbert Greefrath

Abstract This report describes the results of an empirical study of problem solving and mathematical modelling of pupils in secondary schools. Pupils of forms 8–10 were observed working on open, realistic problems. These observations were recorded and evaluated. The goal of the study is a detailed look at the planning processes of modelling problems. Of particular interest, are the problem-solving and modelling strategies used. In this context changes between real-life planning and mathematical planning during the planning phases are studied and evaluated. I describe in detail the sub-phases of planning and explain their connection to both modelling and problem solving processes. I illustrate different modelling types by certain courses of planning and sub phases of planning.

23.1 Introduction

To date, insights into the actual sequences of pupils' problem-solving and model-building processes are rare. Secondary school pupils (aged 10–16) were therefore observed within the framework of a qualitative empirical study while completing open, reality-based tasks. The aim of this study is to reconstruct and to adequately describe the model-building and problem-solving processes applied by these pupils.

23.2 Descriptions of Model-Building and Problem-Solving Processes

Model-building processes are often pictured in the idealized form of a cycle. A variety of descriptions of such cycles can be found in the literature. What all models

G. Greefrath (✉)
University of Cologne, Cologne, Germany
e-mail: g.greefrath@uni-koeln.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_23,

share as a common feature is that they start from a real situation and lead to a mathematical model. Müller and Wittmann (1984) present a cyclic model of modelling, which only uses one step to move from the situation to the model. An especially clear description of modelling showing one step from reality to the model is presented by Schupp (1988). This model separates mathematics and reality within one dimension, something which is customary for modelling models. Additionally, an equal distinction is made between problem and solution in a second dimension.

One well-known cycle of model building is described by Blum (1985, p. 200). This cycle represents in some respect a standard modelling model (Fig. 23.1). A more recent modelling model by Borromeo Ferri (2006, p. 92) has been designed from a cognitive point of view. Compared to the Blum model it has been extended by including the mental representation of the situation (situation model). The model by Fischer and Malle (1985, p. 101) also describes in detail the step from the situation to the mathematical model (Fig. 23.2). Especially the inclusion of the data collection process is of interest for the problems used in our study. With reference to the models mentioned above, it is not always possible to follow the entire cycle or to repeat it several times. Depending on the target group, the question to be researched or the special interest, the models mentioned earlier focus on different aspects of the modelling process.

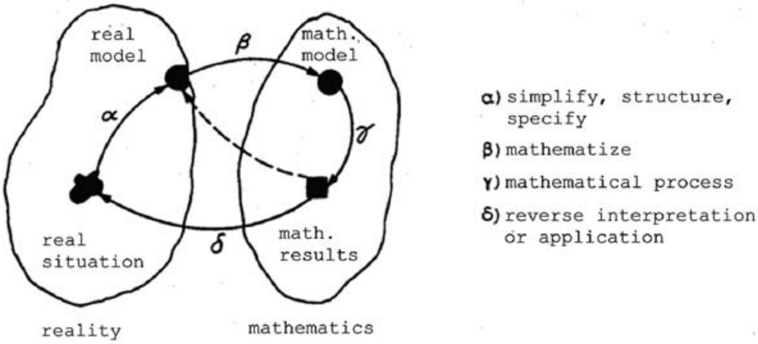

Fig. 23.1 Modelling cycle, Blum (1985, p. 200)

Problem-solving processes are also often described using a model. In his book *How to Solve It*, Polya (1971) developed a catalogue of heuristic questions designed to help in solving problems. For this purpose the problem-solving process is divided into four phases: Understanding the problem, devising a plan, carrying out the plan and looking back.

Within the literature there are many similar examples for the structuring of problem-solving processes. Schoenfeld (1985), for instance, describes the first phase as reading, followed by analysing, exploring and planning. Garofalo and Lester (1985) combine these first steps as orientation and organisation. In both models the next step is execution.

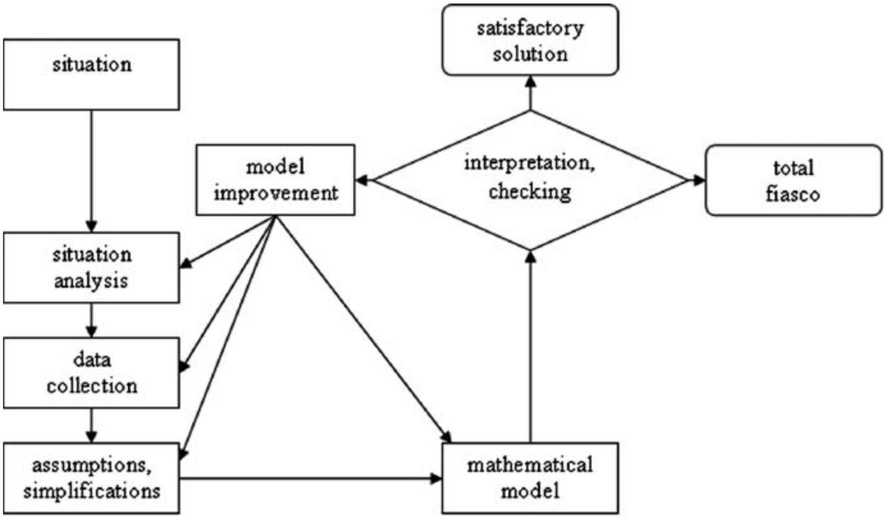

Fig. 23.2 Modelling cycle, Fischer and Malle (1985, p. 101)

The characterisations of problem solving thus refer to the description of the solution process. This is also the case with modelling models. Here, however, an additional content-related aspect (i.e. the distinction between reality and mathematics) is relevant.

23.3 Open, Reality-Related Problems

Open reality-related problems were used to analyse model-building and problem-solving processes. The following problems (Figs. 23.3and 23.4) are examples of the open and fuzzy problems with reality references used in the analysis.

In order to characterize the house-plastering problem, I use the description of a problem as initial state, target state and transformation, borrowed from the psychology of problem solving (Bruder, 2000, p. 70). The problem's initial state is unclear because the relevant information is missing. Also unclear is the transformation from initial state to target state which students can employ. However, the final state is clearly defined, for instance, by asking for a price.

23.4 Analysis Design and Evaluation Methods

Two pupils at a time were monitored while they worked on their problems. The students were asked to undertake the task in pairs – without any further help. The students' work was recorded using a video camera.

How much will it cost to plaster this house?

Fig. 23.3 The house-plastering problem

How many people are caught up in a traffic jam 180 km long?

Fig. 23.4 The traffic jam problem

For evaluation the entire video data were transcribed. Within the framework of open coding with three raters, the individual expressions of the pupils were allocated conceptual terms, which were discussed and modified during several runs through the data. These terms for individual text passages were then assigned to the following categories: planning, data acquisition, data processing and checking.

The process category "planning" describes text passages in which the pupils discuss the path to complete the task or which – in the broadest sense – relate to the path of completing the task. The process category "data acquisition" describes text passages in which pupils procure data for their further work on the problem. This can involve guessing, counting, estimating, measuring or recalling intermediate results that had been achieved earlier. The process category "data processing" describes the calculation with concrete values. This can be done either with or without a calculator. For all problems the pupils were provided with a (conventional scientific) calculator. The process category "checking" includes text passages in which the data processing, data acquisition or planning is questioned or controlled.

The choice of categories was done in such a way that the categories could be allocated in a consistent manner, independently from the problem. During this phase the preliminary categories were therefore combined and modified. Following this allocation, the entire transcripts were coded using the now finalized categories. Later we found that one rater was able to code all categories with adequate confidence on his/her own. The degree of agreement was checked by performing a sample correlation analysis (see Bortz et al., 1990, p. 460f), which showed statistically significant concordance at the 0.05 level. As a result, I then coded all of the remaining transcripts on the basis of the developed categories.

23.5 Results

Because we are interested in those planning processes that have special importance for the completion of modelling tasks and that have frequently appeared the completion of problem-solving tasks, we chose to correlate that descriptions of model-building and problem-solving processes. Consequently, these elements of the planning process are described and allocated to the phases of the modelling process.

23.5.1 Orientation Phases

Typically the orientation phases form the start of the work on the problem. They should relate directly to the material that is handed out and thus intrinsically belong to the domain of reality. If orientation phases appear again in the later stages of the completion process, it is a sign that some of the comprehension issues were not dealt with at the beginning of the work. These orientation phases can then be related to the situation model in the modelling cycle.

23.5.2 Transition of Planning in Reality and Mathematics

In some sections of the transcripts, planning processes in reality and in mathematics succeed each other closely. More precisely, there are three kinds of transition.

One type of transition is the one from reality to mathematics. Starting from the standard modelling model, this is the type that we would expect to appear most frequently, as the mathematical model is only supposed to be developed after the real model.

The second type is the transition from mathematics to reality. According to the standard modelling model this transition within the planning is rather unusual, since the creation of a mathematical model would be followed by carrying out the calculations and not by planning in reality. Such a step was found more frequently within the completion processes, especially for those completions, in which planning processes were discussed very precisely.

The third type is a multiple transition between reality and mathematics. This is a combination of the two types of transition mentioned above. It could be an indication of an intensive examination of the real and mathematical contents of the problem. In the standard modelling model this is the section between the real and the mathematical model, which cannot be separated any longer with this type of transition.

23.5.3 Partial Models

We speak of partial models in reality or mathematics, if simplifications and assumptions are made during the planning phases that lie in the area of reality or mathematics respectively. These planning steps can be identified with the creation of the real model or mathematical model, respectively, during the model-building cycle.

According to initial observations, the elements mentioned above are suitable for the characterization of the following planning types.

Type I: Following a target-led orientation phase pupils take time for planning and discuss simplifications of reality in depth. They often use mathematical terms and correctly apply them to reality. They also correctly associate objects from reality with the relevant mathematical actions and simplifications. Planning processes are discussed in depth and are generally successful.

Type II: The pupils' orientation phase is quiet and very short. In their discussions they refer mainly to the real situation. The mathematical models are used but not discussed. Reality is not consciously simplified or these simplifications are not expressed. Terms from reality are integrated into mathematical process descriptions. Here, no clear indications can be made regarding the eventual success of the students' modelling efforts.

Type III: Pupils frequently need orientation phases in order to see the problem in context. The discussion is very much reality-oriented. No mathematical terms are used and mathematical process description only takes place on a low level. Simplifications are rarely discussed. On the whole, the planning is not very abstract, but rather superficial and generally not successful.

As a consequence of our findings, we recommend that these different planning types provide indications for education. For example, pupils can discuss about mathematical and real simplifications of open problems. The Type I is a prototype for such a discussion. In our view the Type II is of special interest in diagnosing modelling competencies, because no clear indications can be made regarding the success. Whether the number of planning types is to be increased will be subject to future investigation.

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Part II
Modeling in School Classrooms

Chapter 24
Modeling in K-16 Mathematics Classrooms – and Beyond

Richard Lesh, Randall Young, and Thomas Fennewald

Abstract The first half of this book focused on research investigating what it means for students to "understand" modeling processes and models that are important in order for students to be able to use mathematics in real life situations beyond school. This second half shifts attention toward investigations about what is needed for modeling activities to be used productively to promote learning in mathematics classrooms. These investigations are not simplistic one-variable experimental studies intended to prove or disprove the success or failure of pre-defined "treatments" and "interventions." Rather, many of these chapters describe *teaching experiments* (Lesh, 2003) which are investigations designed to show how students' or teachers' thinking changes in situations which involve multiple iterative cycles of expressing, testing, and revising initial ways of thinking about (simulations of) "real life" decision-making situations. Or, other chapters describe *design studies* (Kelly, Lesh, and Baek 2008) in which students or teachers design artifacts or tools which need to be powerful (for some specific purpose in some specific situation), sharable (with other people), and reuseable (in other situations). Both of the preceding kinds of studies often involve *model-eliciting activities* in which: (a) one of the most important parts of the products that are produced are the underlying interpretation systems (or models), and (b) the interpretation systems (models) involve concepts that the researcher or teacher judges believe to be among the main "big ideas" that students should learn.

R. Lesh (✉)
Indiana University, Bloomington, IN, USA
e-mail: ralesh@indiana.edu

R. Young (✉)
Indiana University, Bloomington, IN, USA

T. Fennewald (✉)
Indiana University, Bloomington, IN, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_24,

In our own research, the preceding kinds of *teaching experiments* and *design studies* tend to involve multi-tier studies in which (for example): (a) students develop models for describing/explaining/designing artifacts or tools for some mathematical problem solving situation, (b) teachers develop models for making sense of students' evolving modeling competencies, and (c) researchers or other "evolving experts" develop models for making sense of interactions among students – or between students and teachers (Kelly and Lesh, 2000; Lesh et al., 2008). One reason why such multi-tier studies are needed is because of our recognition that the models (i.e., interpretation systems) that students develop are influenced by both: (a) the students' initial interpretation systems, and (b) the structure of the "things" that need to be described, or explained, or designed.

For many math/science educators, teaching that emphasizes models or applications simply means that the teacher uses fairly traditional methods of teaching mathematics topics which are known to have important uses beyond school.

For others, teaching that emphasizes models or applications means teaching all topics in ways so that the relevant ideas or abilities will be more likely to be useful beyond school. Usually this is done by using "real life" problems as contexts for introducing the relevant topics, and/or by pointing out "real life" situations where the topics are useful in "real life" situations beyond school.

For still others, teaching that emphasizes models or applications means helping students develop modeling abilities – which tend to be thought of as being similar to processes that have been emphasized in past attempts to "teach problem solving" (Lesh and Caylor, 2007).

In this book, few of the research studies that are reported fall cleanly into any single one of the preceding categories, and most of the studies are significantly different than research that has been conducted in any one of the preceding three categories. For example, even though it may be true that research on modeling is sometimes similar to research on problem solving, Lesh and Zawojewski (2007) describe a variety of ways that modeling processes tend to be significantly different than the kind of metacognitive processes, heuristics, strategies, dispositions, and beliefs that math/science educators have emphasized in research involving traditional textbook word problems. One reason this claim is true is because, in "real life" situations, students often begin by *making symbolic descriptions of meaningful situations* which have not yet been described symbolically (Lesh and Doerr, 2003). Whereas, when students try to solve textbook word problems, the opposite kinds of processes tend to be problematic. That is, one of the most common "challenges" that students face is to *make sense of a symbolically described situation*

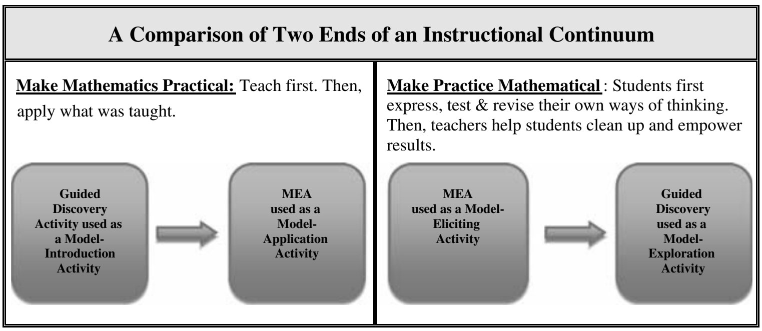
A second reason why modeling processes tend to be significantly different than the kinds of problem solving strategies or metacognitive processes that have been emphasized in research on textbook word problems is that, in mathematics and science education, traditional research on problem solving generally has characterized problems as situations which involve *getting from givens to goals when the path is not immediately apparent*. But, in research on modeling, *model-eliciting activities* are (simulations of) *meaningful goal-directed activities in which students need to* *make significant adaptations to their initial ways of thinking about givens, goals, and possible solution processes*. So:

- The products that students develop (which include models) tend to be more like powerful/sharable/reuseable artifacts and tools than they are like simple "answers" to questions posed in traditional word problems (Lesh, 2003).
- If *model-eliciting activities* are intended to be mathematically significant, then the interpretation systems (i.e., models) that students develop should focus on conceptual systems that underlie the half dozen to a dozen "big ideas" in any given course (English, 2007).
- Mathematical models (i.e., interpretation systems) develop similarly to other kinds of descriptive or explanatory systems. That is, they go through iterative sequences of first, second, and third drafts, where: (a) relevant ways of thinking tend to be at intermediate stages of development (not completely missing nor completely mastered), (b) early ways of thinking tend to involve fragmented collections of "fuzzy" (poorly differentiated and poorly integrated) interpretations of different parts of the situation, (c) model development involves iteratively expressing, testing, and revising current ways of thinking, about the nature of givens, goals, and possible solution processes, and (d) one of the most important goals is to get beyond early and inadequate ways of thinking (Lesh and Doerr, 2002).
- During solution process, what is being transformed is not simply data but also the interpretation systems that are used to interpret information that is available. Furthermore, the process of gradually refining, differentiating, and integrating partial interpretations seldom looks like the motion of a point along a path. Finally, when model development processes involve gradually sorting out, integrating, and revising early interpretations of problem solving situations, these solution processes tend to involve much more than simply putting together concepts and procedures which are assumed to have been mastered learned earlier – in the abstract, and unconnected to any meaningful problem solving situation at the time they were being learned (Kelly, Lesh, and Baek, 2008).
- Principles and rules that expert problem solvers use to describe past problem solving activities are not necessarily assumed to be useful to prescribe next steps for novice problem solvers. In fact, in research on the development of students' models and modeling competencies, the learning of useful problem solving strategies (as well as productive metacognitive processes, dispositions, or beliefs) is seldom reducible to learning of a list of explicit rules (Lesh and Zawojewski, 2007).
- The final products that students produce tend to integrate ideas and procedures drawn from a variety of textbook topic areas. So, model development activities typically result in chunks of knowledge which inherently include connected collections of ideas and procedures which are drawn from more than a single textbook topic area (Lesh et al., 2007).

Many of the preceding claims were emphasized long ago by John Dewey and other American Pragmatists. For example Dewey explained why students' understandings tend to be significantly different depending on whether their learning experiences focus on: (a) *making science practical, on* (b) *making practice scientific* (Lesh, Yoon, and Zawojewski, 2007c). So, it is not surprising when current research on modeling shows that students tend to develop significantly different kinds of understandings depending on whether their learning experiences focus on:

- *making mathematics practical* – by first guiding students along the teacher's ways of thinking about conceptual trajectories that lead toward a textbook's (or teacher's) cleaned-up version of the meaning of the relevant concepts or abilities, and then applying what was taught in "realistic" situations.
- *making practice mathematical* – by first putting students in simulations of "real life" sense-making situations where they express>test>revise their own relevant ways of thinking, and then by analyzing, decontextualizing, systematizing, or formalizing student-generated conceptual tools to endow them with more elegance, power, sharablity, and reuseablity.

To investigate differences between these two "treatments", Lesh, Yoon, and Zawojewski (2007c) created two curriculum units – both of which were assembled using the same two components: (a) a *model-eliciting activity* – in which students express, test, and revise their own ways of thinking, and (b) a *model-exploration activity* – in which students are guided toward their textbooks' or teachers' ways of thinking about the relevant concepts and abilities. Differences between these two treatments resulted from the *sequencing* of components. The treatments that focuses on *making mathematics practical* consist of a *model-exploration activity* followed by a *model-eliciting activity*; whereas, the treatment that focuses on *making practice mathematical* consists of a *model-eliciting activity* followed by a *model-exploration activity*



Our purpose in mentioning the preceding study here is that, even though our description makes the study sound like a typical "horse race" comparing *treatment groups* with *control groups*, it was in fact nothing of the sort. Although the two teaching methodologies are described as if they existed in "pure" forms, *in "real life" teaching and learning situations, we would expect any sensible teacher to use strategic mixes of these and other approaches*. In fact, the importance of mixed strategies and multi-disciplinary ways of thinking is a central point that was emphasized by Dewey and other *American Pragmatists* (William James, Charles Sanders Peirce, Oliver Wendel Holmes, George Herbert Mead). In general, *progmatists* rejected the notion that single "grand theories" should be expected to provide solutions to most "real life" problems – including those that arise for teachers or researchers who are trying to develop more useful ways of thinking about mathematics teaching, learning, and problem solving. Our goal was not to show that "it works" for some specific curriculum activity. Instead, our goal was to clarify the nature of understandings that students develop when they encounter certain kinds of learning or problem solving activities. Then, we leave it to others to develop curriculum activities that might optimize students' learning.

The fact that *model-eliciting activities* have proven to be useful to promote student learning is actually a pleasant but serendipitous consequence of the fact that they were developed first and foremost to provide tools for research. They were not developed to be ideal learning activities.

- In the late 1970's, the earliest MEAs were designed to be *simulations of "real life" situations* where mathematical thinking is useful beyond schools (Lesh and Landau, 1983). For specifically targeted "big ideas" in elementary mathematics, the goal was to investigate what sorts of understandings are needed, beyond those that are taught in school, so that the concepts will be useful in students' everyday lives.
- In the early 1980's, next-generation MEAs were redesigned to be *throught-revealing activities* in which researchers or teachers would be able to observe, document, and analyze or assess deeper and higher-order understandings of the type that we were able to investigate during one-to-one clinical interviews with students (Lesh and Kelly, 2001).
- Later in the 1980s, MEA's were again redesigned to be useful as *performance assessment activities* for assessing learning outcomes which are likely to develop using exemplary curriculum programs – but which are difficult or impossible to assess using traditional kinds of standardized tests (Lesh and Lamon, 1992). These activities also were useful in projects focusing on issues of equity and diversity because, when it is possible to document and assess a broader range of mathematical abilities, then a broader range of students tend to emerge as having extraordinary potential.
- In the 1990s, MEAs were again redesigned to investigate the nature of new types of situations were new types of mathematical thinking are needed beyond school. So, many of these MEAs were characterized as *case studies for kids* – because many of them were child-level versions of case studies that are used for

both learning and assessment in professional schools in fields such as medicine, business management, or engineering.

- Because MEAs are thought-revealing, and because one of the most powerful ways to help teachers become better is to help them become more insightful concerning the nature of their students' thinking, MEAs also tend to be useful to convert teachers regular teaching activities into *teacher-level MEAs* which function as powerful on-*the-job classroom-based teacher development experiences* (Kelly, Lesh and Baek 2008).
- Because students' thinking often develops through several different Piagetian stages during single 60–90 min problem solving episodes, MEAs often are referred to as *local conceptual development sessions* which function similarly to little Petrie Dish Activities in biochemistry laboratories (Lesh, 2003). This is because, during solutions to MEAs, significant conceptual adaptations often occur during sufficiently brief periods of time so that researchers can go beyond observing "snapshops" of development to also see "movies" where it is possible to directly observe processes that lead from one state of knowledge to another.

Throughout all of the preceding uses of *model-eliciting activities*, one of the main secrets to success was to put learners and problem solvers in meaningful goal-directed situations where they clearly recognize the need to develop specific types of interpretation systems (or models). And, the interpretation systems that we emphasized were those that underlie concepts which are regarded to be the most powerful, useful, and important in the K-16 mathematics curriculum.

Throughout the second half of this book, the studies span diverse topical themes and grade and age levels – 6 year old to adult. Yet, most of these studies investigate the development of ideas – not the success of treatments and interventions. As such they illustrate a central purpose behind research on models and modeling – that is, to understand how what it means to "understand" important concepts or abilities, and to understand how these understandings develop. The "subject(s)" of these studies range from investigating the development of ideas or modeling processes, to investigating the design of productive modeling activities. In general, the goals of these studies are not about having students reach predefined testing goals. Rather these studies aim to understand how students develop ideas and what, if anything, modeling activities can do to encourage ideas to develop.

In the modeling community, researchers recognize the importance of finding ways to investigate how students can recognize a need for modeling. For example, Lyn English investigates how *model-eliciting activities* often provide contexts in which average ability fourth graders routinely develop diverse approaches to problem solving and statistical understandings that are significantly beyond the typical 4th grade curriculum. Similarly, Guerra et al. (2007, Two case studies of 5th grade students reasoning about levers, unpublished manuscript) put students in situations where students recognize the need make significant adaptations to their existing understandings of levers. And, their results demonstrate extraordinary high levels of achievement from many students whose past histories of accomplishment in school have been unimpressive. Likewise, in a study of urban classroom environments, Schorr and her colleagues use *model-eliciting activities* to extend traditional conceptions of what is needed to motivate students and enlist unusually high levels of engagement and participation.

Modeling researchers also are interested in studying how modeling communities develop. Helding, Megowan-Romanowicz, Ganesh, and Fang show how discourse methodologies can be used by researchers to study student modeling, employing a discourse research method in their exploration of how teams of 5th graders as they learn properties related to levers at a whiteboard. Megowan-Romanowicz then examines modeling through the lens of discourse, around whiteboard activity, including in her analysis exploration of the many roles that teachers and students play. Next, Levitt and Ahn create a Teacher's Guide of implementation strategies for MEAs based on interviews and reflections of 8th grade mathematics teacher. Barbosa employs discourse analysis to analyse students' modelling activities as do Zahner and Moschkovich who study the patterns of interaction and discourse in groups as they develop models in response to simulations of "real life" decision making situations. Stillman, Brown, and Galbraith examine what causes blockages that hinder progress in transitions in the modeling process, finding there are differences between blockages due to a lack of reflection and those from a lack of knowledge. Bonotto then examines the role mathematics and modeling can play as socio-cultural tools for understanding and relating to one's community. Maass then concludes the discussion about how modeling communities develop with an examination of how students' beliefs about the usefulness of mathematics develop when modeling and application examples are integrated as part of day to day curriculum.

The models and ideas of teachers, who are themselves developing models of students modeling, are critically important to understand and investigate if modeling research is to truly impact student learning. The central issue of how teachers develop their models of modeling is examined by a host of contributing authors. Ferri and Blum begin with a study of if teachers are aware of how they are teaching modeling. Kaiser, Schwarz, and Tiedemann then show why it is necessary for future teachers to develop appropriate knowledge and competencies. Garcia and colleagues look to meet the challenge of reaching teachers at an international level by developing a teacher training program that can bridge international cultures and traditions. Escalante investigates the representations high school teachers have while working on problems that relate to change and variation. Warner, Schorr, Arias, and Sanchez look at how middle school teachers develop and change in their ability to interact with their students. Berry investigates the shared interpretations of teams of teachers as they develop tools and conceptions. Biembengut and Hein look at modeling activities as an agent for generation of positive change in teachers' teaching and students' learning. Zavala and Alarcon explore the development of a professional development course for high school and college physics teachers that includes modeling. Abramovich looks at the impact technology has on teacher education and modeling. de Oliveira and Barbosa explore one of the most important and also less often examined aspects of the modeling process, the tensions and experiences of the teachers who play the most critical role in the multi-tier research and implementation of modeling activities. Thomas and Hart examine the conceptions pre-service teachers have about model-eliciting activities. Warner, Schoor, Arias, and Sanchez examine ways in which two teachers change their styles of interaction with students concerning the kinds of opportunities they gave students to explain their solutions and opportunities to interact with peers. Pournara and Lampen investigate pre-service teachers in South Africa, where they examine challenges for implementing a course which they emphasize must address both teaching and mathematical modeling. Yu and colleagues bring another aspect of both teachers role in the modeling research and aspects of making modeling research and education international into light in their study of Taiwanese teachers who modify the classic "Big Foot" problem into a Taiwanese equivalent. West suggests Japanese teachers are ready to impliment modeling in their teaching practice. Campbell looks at connections between modeling and learning experiences in virtual environments (such as Second Life) and those in physical (real-life) environments. Carlson reports the findings of a multi-tier teaching experiment in which a new construct, the measurement heuristic, is identified.

How new technologies influence modeling in schools is addressed by several authors. Sinclair and Jackiw look at modeling using the *Geometer's Sketchpad*. Chao, Epsom, and Shechtman show how use of SimCalc mathworlds can enrich students' models. Kazak examined students developing models of probability as learning was supported by NetLogo software. Hills examined mathematical search behavior of students and illustrated it using network visualization software.

Finally the history and future of modeling is considered by Bergman and Christer Bergsten who look at the impact that a tool developed by them called the "modeling activity diagram" has on enhancing students' modeling abilities. Ending the volume in reflection, Vos accounts for 25 years of curriculum changes in the math curriculum in the Netherlands.

One striking observation about most of the preceding studies about models and modeling is that, even though most focused on investigating what it means to "understand" important models and modeling processes, most also led to learning gains that are quite remarkable. For example:

- Average ability students often develop powerful, sharable, and re-useable models which involve mathematical or scientific concepts which are far more sophisticated than would be expected from their past records of accomplishment in schools (e.g., English, this volume).
- Students who exhibit extraordinary achievements often are those from highly underprivileged segments of society (e.g., Schorr, this volume).
- When the products that students produce involve tools that are powerful (in the specific situation where it was developed), sharable (with other people), and re-useable (in other situations), then the ideas and procedures that are developed often exhibit remarkable characteristics of being both powerful, transferrable, and memorable (e.g., English, this volume; Lesh and Hurford, in press).

Therefore, for those who believe that the only kind of educational research that is of interest is research showing that something "works", then research that involves models and modeling is virtually unparalleled in the successes that it has produced.

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Section 7
How Can Students Recognize the Need for Modeling?

Chapter 25
Modeling with Complex Data in the Primary School

Lyn D. English

Abstract This chapter explores mathematical modeling for the primary grades with a focus on real-world problems involving complex data. To illustrate the diverse range of models children can create independently in working such problems, I report on some findings from two studies involving 7 classes of fourth-grade children (8–9 year-olds). Two of the problems the children completed involved the selection of Australian swimming teams for the 2004 Olympics and for the 2006 Commonwealth Games. The children's models exhibited diverse approaches to solutions and revealed how the children identified and dealt with key problem elements and their interactions, and how they operated on and transformed data. The latter included ranking and aggregating data, calculating and ranking means, and creating and working with weighted scores – all beyond what was expected in their regular curriculum.

25.1 Introduction

The need to deal effectively with complex data systems in our world has never been greater. Young children are very much a part of our data-driven society. They have early access to computer technology and daily exposure to the mass media where various displays of data and related reports can easily mystify or misinform, rather than inform, their young minds. More than ever before, we need to rethink the nature of the mathematical problem-solving experiences we present to children if we are to prepare them adequately for dealing with the complexity of our rapidly changing world (Lesh and Zawojewski, 2007). Traditional forms of problem solving constrain opportunities for children to explore complex, messy, real-world data and to generate their own constructs and processes for solving authentic problems (Hamilton,

L.D. English (✉)
Queensland University of Technology, Brisbane, Australia
e-mail: l.english@qut.edu.au

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_25,

2007). In contrast, mathematical modeling provides rich opportunities for children to experience complex data within challenging, yet meaningful contexts.

Mathematical modeling has traditionally been reserved for the secondary school years, with the assumption that primary school children are incapable of developing their own models and sense-making systems for dealing with complex situations. One only has to peruse the papers of the 14th ICMI Study on modelling and applications (Blum et al., 2007) to see the scant reference to mathematical modeling in the primary school years, despite a few authors' pleas to "take mathematical modelling seriously... at the elementary school level" (Greer et al., 2007, p. 89). This chapter argues for the integration of modeling problems within the primary mathematics curriculum – problems that capitalize on real-world, complex data and immerse young children in situations where they need to develop or invent mathematical ideas and processes. To illustrate children's potential in working such problems, I report on the diverse models created by seven classes of fourth graders (8–9 year-olds) for two problems involving the selection of swimming teams for two major world events.

25.1.1 Modeling for the Primary School

The terms, models and modeling, have been defined variously in the literature including with reference to solving word problems, conducting mathematical simulations, creating representations of problem situations (including constructing explanations of natural phenomena), and creating internal, psychological representations while solving a particular problem (e.g., Doerr and Tripp, 1999; English and Halford, 1995; Gravemeijer, 1999; Greer, 1997; Lesh and Doerr, 2003; Romberg et al., 2005; Van den Heuvel-Panhuizen, 2003). As used here, a model comprises "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (Doerr and English, 2003, p. 112). In contrast to traditional word problems, modeling problems adhere to a number of design principles that facilitate opportunities for extended learning for all children (e.g., Lesh et al., 2003, p. 43).

These principles include: (a) *The Personal Meaningfulness Principle* – The problems are designed so that children can relate to and make sense of the complex situation being presented; (b) *The Model Construction Principle* – A modelling problem should require children to develop an explicit mathematical construction, description, explanation, or prediction of a meaningful complex system. These models should be mathematically significant, focusing on underlying structural characteristics (key ideas and their relationships) of the system being addressed; (c) *The Model Documentation Principle* – Modeling problems should encourage children to externalise their thinking and reasoning as much as possible and in a variety of ways, including the creation of lists, tables, graphs, diagrams, and drawings. The requirement to include descriptions and explanations of the steps they took in constructing their models is also an important feature; (d) *The Self-Assessment Principle* – The design of the problem should include sufficient criteria to enable children to determine whether their final model is an effective one and adequately meets the client's needs in dealing with the given complex system (and related systems). Such criteria also enable children to progressively assess and revise their creations as they work the problem; and (e) *The Model Generalisation Principle* – The models created should be applicable to other related problem situations.

In addition, these modeling problems are ideally designed for small-group work where members of the group act as a "local community of practice" solving a complex situation. Numerous questions, issues, conflicts, revisions, and resolutions arise as the children develop, assess, and prepare to communicate their models to their peers (Doerr and English, 2003; Lesh and Zawojewski, 2007; Blum et al., 2007).

25.1.2 Interdisciplinary Modeling with Complex Data

My research has involved working with teachers to design interdisciplinary mathematical modeling problems that align themselves with the learning themes being implemented in their classrooms. Such themes have included, among others, natural disasters, the local environment, classroom gardens, early colonization, the gold-rush days, book reading clubs, class excursions to fun parks, and the Olympic and Commonwealth Games. Such interdisciplinary problems provide a powerful means of forging "strong and explicit connections between mathematical knowledge on the one hand, and the contexts within which that knowledge can be used on the other" (Swan et al., 2007, p. 276). A key feature of these problems is the presentation of complex data sets for realistic problem situations that draw upon other disciplines and that are meaningful and motivating to children (cf. Galbraith, 2007). The problems are multifaceted in their presentation and include background information on the problem context (e.g., the history of the Olympic Games, swimming records broken, etc.), "readiness" questions on this information, detailed problem criteria to be met, several tables of data, and supporting illustrations.

I report here on children's responses to two of these problems that involved selecting swimming teams for the Olympic Games (study A) and for the Commonwealth Games (study B). The main data set that was presented for the Commonwealth Games problem appears in the Appendix; a similar table of data was presented in the Olympic Games problem. My research questions included, among others, how the children would deal with the key elements in the problems including any interactions between elements, how they would generate and document their own mathematical ideas and processes in solving the problems, and how different forms of mathematical sophistication might be revealed in their final models.

25.1.3 Implementations of the Modeling Problems

The problems were implemented in seven 4th-grade classes in two schools (4 classes in study A completed the Olympics problem while 3 classes in study B worked the other problem). A teaching experiment involving multilevel collaboration (English, 2003; Lesh and Kelly, 2000) was employed. Such collaboration focuses on the developing knowledge of participants at different levels of learning (student, teacher, researcher). Given that the teachers' involvement in both studies was vital, I organized several half-day professional development workshops on modeling and its integration within the primary mathematics curriculum. Each problem formed a component of a sequence of modeling activities that enabled children to elicit a model, explore and apply the model to a similar problem, and adapt the model to solving a new, related problem. The present problems served as a model exploration/application activity in study A (Olympic Games) and as an adaptation problem in study B (Commonwealth Games). The children worked the problems in groups of 3–4 during four, 45–60 min sessions over 2 weeks.

The *background information* for the problems, which the children explored in the first session, included a history of the Games and some criteria for team selection, as illustrated in the following excerpts from the Commonwealth Games problem:

> The 18th Commonwealth Games will be held in Melbourne in March 2006. The Commonwealth Games were first held in Canada in 1930. Since then, the Commonwealth Games have been conducted every four years except for 1946 due to World War II.... The sports that are contested at the Commonwealth Games are Aquatics, Badminton, Basketball, Bowls, Boxing, Cycling, Gymnastics, Netball, Rugby 7 s, Shooting, Squash, Table Tennis, Triathlon and Weightlifting. Aquatics include diving, synchronized swimming, and swimming. Swimming in Australia has one of the highest participation levels of all sports and Australian swimmers have been among those athletes setting the highest standards in this sport. Competitors must strive to improve their performance, watch closely the performance of their rivals, and try to reach their peak whilst the games are on, not months before. 42 Gold Medals will be hotly contested in swimming alone in the 2006 Commonwealth Games. Over the years, World Records have been broken, setting the benchmark even higher. The 2000 Sydney Olympic Games led to a new Men's 100 m Freestyle World Record of 47.84 seconds. This still stands today. Take a look at the table showing the number of times the Men's 100 m Freestyle World Record has been broken... The Commonwealth Games Committee is working hard to select the most suited swimmers for the Melbourne Commonwealth Games. They need to make sure that the athletes selected for the 2006 Melbourne Team are in their prime condition if we are to have a chance of dominating the swimming events.

After addressing the background information and the readiness questions, the children explored in the next session the main problem criteria (below) and the accompanying table of data (Appendix for the Commonwealth Games problem).

> The Australian Commonwealth Games Committee is having difficulty selecting the most suited swimmers for competing in the Men's 100 m Freestyle event. They have collected data on the top nine male swimmers for the 100 m Freestyle event. Table 2 (Appendix) shows each of the swimmer's times over the last 11 competitions. As part of the Commonwealth Games Committee you need to use these data to develop a method to select the two most suited men for this event. Write a report to the Commonwealth Games Committee telling them who you selected for the Australian team and why. You need to explain the method you used to select your swimmers. The selectors will then be able to use your method to select the most suitable swimmers for all the other swimming events.

25.2 Data Collection and Analysis

Video- and audio-tapes were taken of the children's group work, their interactions with their teachers during group work, and their final class presentations in each study. Teacher-researcher meetings were also audio taped, as were end-of-year interviews conducted with the children in study A. Other data sources included all of the children's artefacts (including their written and oral reports) and classroom field notes. The video- and audio-taped data were transcribed, with subsequent iterative refinement cycles of conceptual change (Lesh and Lehrer, 2000) undertaken to identify the ways in which the children interpreted and re-interpreted the problem, their decisions on which elements to address, the interactions they explored among the elements, and how they operated on and transformed the data.

25.3 Findings

In reporting on a selection of findings from the two studies, I first consider how the children dealt with key problem elements and their interactions. I then describe the various models that the children created in solving both problems.

25.3.1 How Children Dealt with Key Problem Elements

In solving the two problems, children identified the following elements and their interactions although not all groups addressed all of these.

The swimmers' times. In identifying and comparing the swimmers' times, the children frequently discussed the relationship between the number of seconds and the "best time" (e.g., "The lower the better, or the higher the better?")

The DNCs ("did not compete"). This element generated a good deal of discussion, with some children choosing to ignore it while others considered it a significant factor in their model development. Issues that arose included how the number of DNCs impacted on the overall time totals of individual swimmers and how the number and recency of the DNCs impacted on a swimmer's potential performance.

The PBs ("personal best times"). Some groups ignored this element, especially as a couple of the PBs listed (as obtained from the website www.swimming.org.au) were higher than some of the individual race times. Issues that arose in the children's discussions included the recency of a swimmer's PB ("they didn't happen so recently;" "we don't know when the personal best time was done") and thus its relevance to future team selection, and the extent of variation of swimmers' times from their PBs.

The recency of events. The children addressed this element from a few perspectives. How recently a swimmer scored a best time was considered by some children to be an important factor in team selection: the more recently a best time was recorded, the better the indication of future performance. Hence, some children decided to ignore the 2003 and 2004 events.

The status/level of events. A swimmer's best time in relation to the level of the swimming event in which it was recorded was examined by a couple of groups. For example, a best time achieved in the 2002 Commonwealth Games would be considered more significant than a best time scored in the 2003 Telstra Australian Championships.

25.3.2 Children's Models

Across the two problems, the children displayed substantial variation in model creation, in terms of the elements they incorporated in their models, the data they chose to address, and how they operated on and transformed the data. Although the focus here is primarily on the children's final models, it is worth noting that the children displayed several cycles of modeling actions as they worked the problems. That is, they interpreted the problem information, expressed their ideas as to how to meet the problem goal, tested their approach against the given criteria and data, revisited the problem information, revised their approach, implemented a new version, tested this, and so on.

Model A: *A Focus on PBs*. This underdeveloped model involved a primary focus on the swimmers' personal best times to decide which swimmers should be chosen. However, a couple of groups recognized the limitations of sole reliance on PBs as can be seen in Kelly's group:

Kelly: Yeah, but Lana, they might just one day swim really, really well, like... they might have just had a really, really good day, yeah, or week or whatever.
Lana: Yeah, I know.
Kelly: They might be a really good swimmer and then they sort of you know they might have had an injury and gone back but they're not as good, so... it might have changed.
Tony: What we would have to do is look at the latest times, compare those, and then we will know.

Children in another class also questioned a reliance on the swimmers' PBs and began to consider possible variations from a given PB. After one group presented their class report, a boy asked, "How come you just compared their personal best because they don't do that all the time?" In subsequent class discussion the children explained that the swimmers' most recent times should be considered, "because they could have been slow when they first started, and they could have got stronger... and now they're going faster and faster." The children also commented that the swimmers "could get slower and slower," or "they could just stay on their personal best," or "they could go faster and then slower and then fast."

Model B: Aggregating Times, Ignoring DNCs. This model simply involved aggregating all the times for each swimmer, ignoring the DNCs "because we had no idea why they did not compete." The PBs were not taken into account as "they didn't happen so recently." The two swimmers with the fastest times were chosen (other children were aware of the difficulties with this approach, as evident in those who calculated mean times).

It is interesting to note that Kelly's group above initially contemplated just aggregating all the swimmers' times but the group decided this was not appropriate. As Lana commented, ". . . what I'm saying is. . . we can't add up the totals because there are so many Did Not Competes, and they've got uneven amounts, so that wouldn't be fair." Kelly added, "And they would get a lot lower (total)."

Model C: A Focus on Most Recent Events. In this model, children only considered the most recent events, namely, those of 2005. The two swimmers who had scored the fastest times in these events were selected.

Model D: A Focus on Swimmers' Fastest Times. Children who developed this model focused on the two or three fastest times of each swimmer, aggregated these times, and compared the totals to determine the swimmers to be chosen.

Model E: A Focus on Least Number of DNCs, Fastest Times, and PBs. This model involved choosing the swimmers with the least number of DNCs and, from this group, selecting three swimmers with the fastest times. In selecting the two best swimmers, the PBs of the three swimmers were examined and the two fastest PBs were chosen ("We were looking for the less time, because the quickest and fastest").

Model F: Eliminating Swimmers and Selecting Swimmers with Fastest Times. As a first step, children eliminated those swimmers who had scored times that were 50 s and over. Next, for each of the events listed, the children circled the fastest time. The swimmers with the greatest and second greatest number of circled times were chosen. One group noted, "You can use this method on all the swimming events, because we tried it on all the races and it worked."

Model G: Multifaceted Model Extending Previous Models. One group explained their model development for the Commonwealth Games as follows: Step 1 – We circled the three best times for each swimmer, not including their personal best times. We added their three best times together. Step 2 – We crossed off Ian Thorpe because he hasn't swum for more than a year. Step 3 – We next crossed off all the swimmers that had all of their three best times over 50 s. Step 4 – We looked at the lowest times and chose the two best swimmers. They were Ashley Callus and Michael Klim.

Model H: A Focus on the Mean. The use of mean scores was interesting given that the children in both studies had not been formally introduced to this notion. One group stated that they selected the two Commonwealth Games swimmers, Ian Thorpe and Michael Klim, by "investigating the averages. . .. It turned out that Ian Thorpe had the best average and Michael Klim had the second best. The averages calculate the time the swimmer would approximately get in the Commonwealth Games; that is why the averages are important." In study B, one teacher explored the children's understanding of the mean as follows:

Teacher: Why were the averages important? And how did you work out averages?
James: We worked it out by adding up all the scores, adding up all the times, and then dividing it by how many events they had competed in. In trying to probe further the children's understanding:
Teacher... you told me you divided the first by three because...
Luke: That's how many times the person actually swam.
Teacher: Did all swimmers do three races?
Luke: No.
Teacher: If somebody swam in four races what did you do then?
Luke: You had to divide it by four because it's not really fair if... someone competes three times and the other person competes like eight times because eight times could get a higher amount.

Model I: Use of Weighted Scores and Mean. For this model, one point was awarded to the swimmer who came first and one point to the swimmer who came second in each race. Each swimmer's scores were totalled and the mean calculated ("We divided it by the number of events they played in"). In identifying Ian Thorpe and Michael Klim as their choices for the Commonwealth Games, the children explained, "We were going to chose Ashley Callus, but he didn't get many good records on his races."

Model J: An Extension of Model I. This model involved the following processes: In each race, 2 points were awarded to the swimmer with the fastest time and 1 point to the swimmer with the second fastest time. Each swimmer's scores were totalled and recorded on a table as shown in Fig. 25.1. Next, the PB of each swimmer was considered, with 2 points awarded to the swimmer with the fastest PB and 1 point to next fastest; these results were added to the table. Each swimmer's scores were then aggregated to find "a total rank."

Model K: Calculating Means and Aggregating Differences in Each Swimmer's Successive Scores. This model, comprising two approaches to solution, was developed by one group of children in study B. The children explained that they decided on the two best Commonwealth Games swimmers in two ways: First, the children calculated the mean time for each swimmer. Second, the group considered each swimmer in turn, commencing with the swimmer's 2002 Commonwealth Games time and finding the difference between it and the next event (2003 Telstra Australian Championships). This process was repeated for each successive event. If there was an improvement, the children added the difference between the two times

	Ashley Callus	Michael Klim	Eamon Sullivan	Ian Thorpe	Todd Pearson	Grant Hackett	Adam Pine
Rank	10	10	0	11	4	1	1
PB	0	2	0	1	0	0	0
Total Rank	10	12	0	12	4	1	1

Fig. 25.1 Table created for model J

(ignoring the decimal point); if there was a decline, the difference was subtracted from the running total. The group reported as follows:

> Dear Commonwealth Games Committee, We have chosen two of the best swimmers for the Commonwealth Games. Those swimmers are Michael Klim and Andrew Mewing. We have decided by two ways. The first way was doing the mean. The mean is when you add all the numbers in the section of numbers and divide it by the numbers you add. We did not count the DNC. The second way was we started from the Commonwealth Games (and progressed) to the FINA World Championships. For example, we added the score when they get quicker and subtracted points if they were slower in the next race. For DNC we gave it a zero. If you calculate it, you will find that it is Michael Klim and Andrew Mewing.

25.4 Discussion and Concluding Points

The foregoing diversity in model creation illustrates primary school children's potential in solving meaningful, timely problems involving complex data – and in the absence of direct instruction. More than ever before, we need to prepare younger students for dealing with our increasingly complex, data-drenched world. Powerful technological tools for computation, conceptualization, and communication are leading to fundamental changes in the types and levels of mathematical understandings and abilities that are needed for success beyond school (Lesh and Zawojewski, 2007; National Research Council, 1999). Yet, as Lesh and Zawojewski (2007) noted, there seems little change in the nature of the problems children continue to encounter in the mathematics curriculum. Mostly, these problem-solving experiences involve dealing with countable or measurable items, with the problems centered on conventional teaching topics (e.g., teaching of the four operations) in contrived, not-too-real-world contexts. The powerful mathematical processes that are needed for solving problems now and in the future – processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, organizing, and transforming data – are lacking in these existing problem experiences. Clearly, a more future-oriented approach to problem solving in the elementary school and beyond is needed. Mathematical modeling, as addressed here, offers one such approach.

Mathematical modeling traditionally has been confined to the secondary school and beyond. Yet, the findings reported here and in other research (e.g., English, 2007a; English and Watters, 2005) has shown that such problems contribute effectively to primary school children's learning in several domains. Such problems allow for a diversity of solution approaches and enable children of all achievement levels to participate in, and benefit from, these experiences. In contrast to traditional classroom problem solving, these modeling problems facilitate different trajectories of learning, with children's mathematical understandings developing along multiple pathways. Importantly, children direct their own mathematical learning. That is, they elicit key ideas and processes from the problem as they work towards model construction. In the present studies, the children identified a range of elements and considered interactions among these. For example, the DNCs were ignored by some groups but were considered by others to be key factors. The interaction between the number of DNCs and a swimmer's overall total time was addressed by several groups. As the children worked with these key elements they initiated a range of mathematization processes including eliminating data, ranking and aggregating data, assigning scores and weighted scores, calculating and ranking means, and aggregating differences between successive sets of data.

Children's approaches to these modeling problems suggest that real-world, complex problem solving goes beyond a single mapping from givens to goals. Rather, such problem solving involves multiple cycles of interpretation and re-interpretation where conceptual tools evolve to become increasingly powerful in describing, explaining, and making decisions about the phenomena in question (Doerr and English, 2003; Lesh and Zawojewski, 2007). Furthermore, these phenomena can be drawn from a wide range of disciplines, including science, studies of society and environment, history, and literature (English, 2007b). In essence, modeling problems in the primary schools serve to not only foster future-oriented mathematical problem-solving abilities but also to enable children to apply their learning in multidisciplinary, authentic contexts.

Acknowledgments The research reported here was funded by grants from the Australian Research Council. Any claims in this article are mine and do not necessarily represent the position of the Council. The participation of the classroom teachers and students is gratefully acknowledged, as is the support given by research assistants Jo Macri, Larissa Fennell, Alecia Tinworth, and Fiona Maclachlan.

Appendix: Mens 100 m Freestyle Results Recorded by Australian Competitors (Seconds)

Competition	Ashley Callus	Michael Klim	Eamon Sullivan	Ian Thorpe	Andrew Mewing	Antony Matkovich	Patrick Murphy	Casey Flouch	Cameron Prosser
2005 FINA World Championships	DNC	49.32	DNC	DNC	49.99	DNC	DNC	DNC	DNC
2005 Telstra Grand Prix	DNC	50.04	51.64	DNC	50.95	DNC	50.96	DNC	51.28
2005 Telstra Trials	50.24	49.02	50.05	DNC	49.72	50.25	50.38	51.30	51.43
2004 Telstra FINA World Cup	DNC	DNC	49.82	DNC	48.96	49.69	DNC	49.08	DNC
2004 Athens Olympics	50.56	DNC	DNC	48.56	DNC	DNC	DNC	DNC	DNC
2004 Telstra Grand Prix	DNC	50.44	50.35	49.23	51.09	52.17	51.20	50.91	51.28
2004 Telstra Olympics Team Trials	49.31	49.78	50.06	48.83	49.98	50.15	50.48	50.51	51.57
2003 Telstra FINA World Cup	47.93	DNC	50.24	49.36	DNC	49.50	49.46	49.11	50.55
2003 Telstra Australian Championships	49.07	DNC	51.86	49.07	50.52	50.58	50.95	50.20	DNC
2002 Commonwealth Games	49.45	DNC	DNC	48.73	DNC	DNC	DNC	DNC	DNC
Personal Best Times*	48.92	48.18	50.06	48.71	49.72	50.15	50.29	50.20	49.38

DNC – Did not compete, * Best time across heat, semi final, and final.

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Chapter 26
Two Cases Studies of Fifth Grade Students Reasoning About Levers

Paula Guerra, Linda Hernández, Ahyoung Kim, Muhsin Menekse, and James Middleton

Abstract The purpose of this teaching experiment was to give fifth grade students activities using simple machines to see how they would use proportional reasoning to mathematize those activities. We chose a series of activities using two types of levers, because of their experiences with seesaws and because this combined mathematics with science. We hoped that they would develop models of the lever. We also hoped that they would recognize the inverse multiplicative relationship between distance and weight. The students did demonstrate evidence of preliminary models about the relationship between weights and distances on a lever. On the final day, the students, in a thought experiment, were able to discover the multiplicative relationship between distance and weight, but they did not realize the inverse nature of this relationship.

P. Guerra (✉)
Arizona State University, Tucson, AZ, USA

L. Hernández (✉)
Arizona State University, Tucson, AZ, USA

A. Kim (✉)
Arizona State University, Tucson, AZ, USA

M. Menekse (✉)
Arizona State University, Tucson, AZ, USA

J. Middleton (✉)
Arizona State University, Tucson, AZ, USA
e-mail: james.middleton@asu.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_26,

26.1 Introduction

This chapter is intended to describe two models constructed by two groups of fifth graders the teaching experiment as they interacted with a first class lever: a hanging lever sometimes called a Chinese scale. The students also worked with a seesaw type lever, but our focus will be on the hanging lever.

First, we will give an overview of the whole experiment. We observed them while they performed three activities involving these types of levers to see how they would use proportional reasoning to mathematize those activities. We chose the lever first because of the students' familiarity with the seesaw and secondly because these activities combined mathematics with science. We hoped that, while working with these two apparatuses – the lever as a model of the seesaw and the hanging lever, they would form mental models of the lever which would be illustrated by their manipulations of the apparatuses, their conversations about their experimentation and their inscriptions on whiteboard and in journals illustrating their understanding. We also hoped that they would recognize the inverse multiplicative relationship between distance and weight on a lever. In the first case there is some evidence that they seemed to have an intuitive understanding of the inverse relationship between distance and weight on a lever but their inscriptions did not demonstrate this relationship. (For the full description of the experiment see Section 26.3.)

Through these series of activities, students did demonstrate evidence of preliminary models about the relationship between weights and distances on a lever but time did not permit them to more fully develop their models. On the final day, the students, with the help of a fellow student in a thought experiment, were able to discover the multiplicative relationship between distance and weight, but they did not explicitly realize the inverse nature of this relationship.

This chapter will look at the model development of two cases in the second activity, a group of three girls and a group of four boys. We will closely examine their first attempt at balancing a bag with a notebook with 10 washers, by moving the fulcrum to achieve balance.

26.2 Theoretical Framework

Our theoretical framework comes from Lesh, Hoover, Hole, and Post (2000) where they describe characteristics of a model as elements, relations, operations and rules. We adapted their framework as expressed in the chart below. We used this framework to describe in detail the models the students constructed of the second activity which was with the hanging lever. (For this description see Appendix.)

Operations	Elements	Relations	Students' rules	Outcomes
What do kids do?	Do to what?	To achieve, to test etc	Using what rules or common practices Common constancies	What was the result?

26.3 Methodology

26.3.1 Participants

The participants were 25 fifth grade students from a lower middle class school in the south west of the United States. They were mostly Hispanic and bilingual. In the two cases that we are examining, there were three girls in the girls' group who spoke in both English and Spanish and four boys who spoke only in English in the other group.

26.3.2 Materials

Lever (seesaw): Paint stir stick, Lego for fulcrum, 10 washers of the same weight and size.

Hanging lever: 36′ stick, a handle with a spool of string for a fulcrum, 10 washers, a cup with a string on it, a grocery bag with a notebook in it.

26.3.3 Procedures

There were 4 days of video taped data collection over a two weeks period. Due to the fact that this teaching experiment was part of a university class, we were restricted to two days where the normal time period would have been two or three weeks. For more information on the teaching experiment from the researcher and teacher point of view, see Helding et al. (2007). The students did three activities with two apparatuses: the lever (seesaw) and the hanging lever.

First they did a thought experiment about a see saw. Then the teacher-researcher demonstrated the first device they were going to use: a lever, in this case, a model of a teeter totter. They discussed the parts of the system which the students could identify. They also discussed what it means is to be balanced and unbalanced. Then the teacher introduced a new word: fulcrum and a demonstrated how to do the first activity with the washers. The students manipulated the lever using metal washers as weights, which were all the same size and weight. The students tried to balance the lever using different numbers of washers on each side of the lever arm with a fixed fulcrum. Then they recorded in their journals the pertinent data: distance on both sides from the washers to the fulcrum, and the number of washers used.

The second apparatus they worked with was a hanging lever. The teacher-researcher then introduced the hanging lever by demonstrating it with a fixed fulcrum. She put a bag with one notebook on one side and asked the students where to put a bag with two notebooks with the fulcrum fixed in the middle. She drew diagrams on the board to illustrate the students' answers. The next day the teacher-researcher demonstrated how to achieve balance by moving the fulcrum instead of moving the weights. The students put the weights (books on one side and washers on the other) fixed on the ends of the stick of the hanging lever and had to move the fulcrum. It was not possible to balance the lever with the fulcrum in the middle, because the students only had 10 washers and the notebook weighed 19 washers, so the only way they could balance the lever was to move the fulcrum. When their lever balanced, they had to record their data in any way they thought was appropriate.

In the last activity, they returned to the seesaw type lever. The teacher-researcher engaged the students in a thought experiment with a lever in which they used their whiteboards. Then they went outside and experimented with balancing people on a giant lever.

For this chapter we will focus on two cases where the students are moving the fulcrum with the weights fixed at the ends of the hanging lever.

26.4 Results

We will describe the preliminary models two groups of students developed while working with the hanging lever. In Appendix, we provide a full description of the operations, elements, relations, rules that they employ and their outcomes. We recorded the most salient of these characteristics of their models. Also we assume that the students carry the characteristics from one attempt at balancing to the next.

26.4.1 Boys' Case

In the first case, we examine the boys' group as they grapple with the hanging lever (see Appendix, Table 26.1). Right off the bat, they balanced the lever with nothing on it intuitively finding the middle of the lever arm. Next, they put the weights on the lever arm and estimate the location of the middle of the lever. Suddenly, one student stopped them and measured to find out where the center of the lever arm was. He became distracted and made another measurement that he admitted that he did not know why. Then he measured half of the lever are and placed the fulcrum there. In the previous activity, the fulcrum was fixed in the middle of the lever arm so that is where they started. The system did not balance, so another boy questioned if there were really 10 washers in the cup. They counted the washers and repeated the procedure with the fulcrum in the middle. Of course, it failed again. They abandoned the idea that the fulcrum should be in the middle and moved the fulcrum toward the bag which was the heavier side. Some of the boys argued that it was the wrong direction, but the boy who was measuring insisted. It was very nearly balanced. They moved the fulcrum back and forth in very small increments until it balanced. One boy was so excited he jumped up and down and danced around. Then other boys said, "Measure, measure." So he measured from the fulcrum to both weights. When one boy wrote down only numbers on the whiteboard, he quickly erased the numbers and wrote, "45 cm notebook and 74 cm and 10 washers."

The boys started with a model that required the fulcrum in the middle. They found the middle both by balancing the lever arm with nothing and by measurement. They tested this model twice and then left it to try moving the fulcrum toward the heavier side. Within a few minutes, this resulted in balance. They recorded their data with the appropriate units, but it did not further the main idea of the teaching experiment – the inverse relationship between distance and weight on a lever. Perhaps had they only dealt with one type of weight they may have been able to see the multiplicative relationship. But instead they could only see the additive relationship between the weight of the washers and their distances and the weight of the notebook and its distances. They saw them individually and did not relate them to each other.

26.4.2 Girls' Case

In the second case, we also focus on the hanging lever activity. As we can observe in the chart (see Appendix, Table 26.2), they had no a clear plan, and it took them several attempts to get their first measure. By the time they recorded the data for their first balance situation, the teacher asked if the groups in general had 2 measurements.

The idea of a system where the fulcrum is fixed in the middle, and the weights are equidistant from it is recurrent. Each time they needed to start over because they found what they were doing was not going to help them find balance for the system. They went back to this idea. We can see this clearly when before starting a new round of attempts they moved the fulcrum to the middle. Not only that, but they tried to change the rules of the activity, by adding (or subtracting) weights from the cups. Doing this, they took their system to a situation that is basic, an idea of a "symmetrical" system, where the fulcrum is in the middle of the stick, and on the extremes we have the same weights.

In their attempts, we saw the girls move the fulcrum towards the heavier side first, and then to the lighter side neither achieving balance. And later on, they started moving the fulcrum in a more systematical way, but when they passed the balance point, and the unbalance change side, they kept going to the same direction, proving that it was not clear for them any rule about the weights and the position of the fulcrum in order to balance the system.

Finally the students stopped and worked on the whiteboard. They considered what they have experienced so far, and then they came up with plan. When they went back to the lever though, they chose not to follow that plan, and put the fulcrum in the opposite side of the plan. Because in their original plan they were going to systematically move the fulcrum, they still were able to observe how the lever slowly got balanced, and this time, when the unbalanced changed direction, they were ready to move the fulcrum the opposite way, and finally attained balance. Their first experience with the lever helped them get the second measure faster and more efficiently.

We believe there are some points to highlight. Regarding this group of students, we believe they kept going back to the idea of the "symmetrical" situation of the lever, because this is the case was a familiar situation they had experienced also outside school, and was also the first one that was worked in our teaching experiment. The idea of the see saw with the weights also fixed on the extremes of it is the idea they can always go back to when nothing else works. We believe they feel they can go from there to build new knowledge, that this is a safe point where they can stand to start all over again, from simple and known, to complex and new.

26.5 Suggestions for Teachers

Students' models of the proportional relationship between weight and distance are extended and refined from their existing models of their experience with seesaws. However, our data shows the gap between a teacher's model and students' models at the beginning of an activity which makes the students and the teacher-researcher look for different aspects of data. It is hard for students to catch up a teacher's model. Therefore, to understand students' modeling cycle is important for a teacher to scaffold the students' development of their modeling and this cannot be done in a single round.

From our data, we suggest a teacher these tips for scaffolding students:

(1) Supply a classroom environment where students are allowed to question
(2) State the purpose of an activity and present it explicitly
(3) Help students to see pattern from collected data by providing practice with graphic organizers
(4) Make experiment simple and changes should be minimal from one activity to another
(5) Give students enough time to think
(6) Observe and collect students' work to guide them more accurately and for the improvement of a future lessons
(7) Discuss explicitly students' thinking using examples and non-examples

Due to the time constraint of the teaching experiment, we were not able to use sufficient time for the students' development of their model of the lever. We also realized that the order of the activities could have made the objective clearer to the students. The main objective was for the students to see the inverse relationship between distance and mass while using a class one lever.

We suggest keeping the first activity the same using the thought experiment and then allowing the students to explore the lever with a fixed fulcrum. We believe that the second activity might have not been for the best interest of the experiment. We think too many factors changed from the previous activities, distracting students from the main point. Keeping the fulcrum fixed like in the previous activities would have helped the students to really build upon the previous tasks instead of starting all over again. Also working with just books or just washers would have helped them to mathematize the relationship in the lever and in this way be able to see this multiplicative relationship more clearly.

We also have a suggestion for the material of the hanging lever. Since the weights will be washers for both sides, there should be cups for both sides and when the cups are supposed to be fixed they should be securely attached to the stick by the teacher before giving them to the students. The cups and bags kept falling off the stick even during the teacher-researcher demonstration. This took up a lot of time that could have been used more productively.

Acknowledgment The research reported in this chapter was supported, in part, by grants from the National Science Foundation (#0337795). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the N.

Appendix

Table 26.1 Boys' group

Operations	Elements	Relations	Students' rules	Outcome
Construct hanging lever Fix cup and bag at ends of stick	Fulcrum (handle), weights (cup with 10 washers and a bag with book) and measuring tape	To achieve "balance"	Fulcrum must be in the "middle" of the stick Using all 10 weights in the cup	The lever
Balance stick without weights	Fulcrum and lever arm	Balance means stick is horizontal	If fulcrum is in the middle, stick will balance with no weight on it	Balance achieved
Coordination of weights	Fulcrum in center and weights on ends of lever arm	Weights are on the ends of stick	If weights are on the ends of the stick, it will balance	Action halted to take measurement
Locate center by measuring the stick	Location of the center of the stick	Balance means stick is horizontal	Fulcrum always should be in center of stick.	Center not located
Measures again and marks a point about 3′ from the cup end of stick	Lever arm	Relationship is unclear	We must measure	Confusion about why to measure
Measures about $^1/_2$ of the stick and marks the point	Lever arm and location of fulcrum	Coordinating distance Locating the fulcrum in the middle	We must find the center to put the fulcrum there so system will balance	There are 10 weights
Fulcrum placed on mark and weights put on the ends of the stick	Lever arm, fulcrum and weights	Finding balance for the system	If the fulcrum is in the middle and the weights are at the ends the system will balance	Balance not achieved

Table 26.1 (continued)

Operations	Elements	Relations	Students' rules	Outcome
Verify if there are 10 weights by counting	Weights	Not enough weight on cup side to balance	If there are not 10 weights, it might not balance	There are 10 weights
Move the fulcrum towards the bag (heavier side)	Fulcrum and lever arm	Fulcrum too close to heavier side	Move fulcrum not weights to balance	Near balance
Move fulcrum back and forth in small increments to achieve balance	Fulcrum and lever arm and small increments of distance in both directions	Locating point of balance between 2 points of unbalance, adjusting in response to the direction of the unbalance	When unbalance changes direction, move fulcrum the opposite way When system is close to balance, move fulcrum in smaller increments	Balance achieved
Measure from fulcrum to bag and from fulcrum to cup	Measuring tape, centimeters (units)	Comparing distances from the fulcrum not focusing on the relationships between weights and distance	Measure from the weights to the fulcrum	Measuring of distances from the weights to the fulcrum
Record the results on the whiteboard	Numbers without units	Numbers are measurements	Write down the numbers for the measurements	45, 74
Erase and write distances with centimeters and weights in a chart	Chart, distances in centimeters, weights	Comparing distances of each weight serially not comparing distances of one weight compared to the other	Record the measurements of the distance with centimeters and their respective weights in a chart	45 cm bag 74 cm washers

Table 26.2 Girls' group

Operations	Elements	Relations	Students' rules	Outcome
Construct hanging lever Fix cup and bag at ends of stick	Fulcrum (handle), weights (cup with 10 washers and bag with book)	To achieve "balance"	Fulcrum must be in the "middle" of the stick Using all 10 weights in the cup	The lever
Estimate the middle (coordination of distance)	Location of the handle on stick	Balance means the stick is horizontal	If the handle is in the middle, it will balance	Balance is not attained
Coordination of weights	Fulcrum in the center, and addition and subtraction of weights	Changing the weights will establish balance	Systematically adding weights leads to balance	Number of weights exhausted. Balance is not attained
Reestablishment of the center	Fulcrum and the lever arm	Coordinating distance Locating the fulcrum in the middle	If the fulcrum is in the middle, it will balance	Balance is not attained
Move the fulcrum to the heavier side	Fulcrum and the lever arm	Establish balance by moving the fulcrum	Moving fulcrum toward the heavier side will achieve balance.	Balance is not attained
Move the fulcrum to the lighter side (several attempts)	Fulcrum and the lever arm	Moving fulcrum towards the opposite side of the previous attempt	Moving the fulcrum to the lighter side will balance	Further out of balance
Reestablishment of the center	Fulcrum and the lever arm	Coordinating distance Locating the fulcrum in the middle	Start over at the beginning	Balance is not attained
Move the fulcrum too close to the opposite heavier side (two attempt)	Fulcrum and the lever arm	Unbalanced, changes sides when fulcrum was moved		Moving the fulcrum to the heavier side will achieve balance

Table 26.2 (continued)

Operations	Elements	Relations	Students' rules	Outcome
Draw diagram of lever to mathematizing their experiences showing possible positions for fulcrum	Diagram of hanging lever, white board, markers and their experiences with the lever system	Move location of fulcrum to predict the outcome of different positions on the lever	Need to organize our thinking about the system with a diagram	Plan for systematic placement of the fulcrum
Place the fulcrum close to heavier side and move it twice closer to lighter side	Physical elements of lever and the plan	Fulcrum is moving in the direction of the balance point	Place fulcrum as close to the heavier side as possible, and move it towards the lighter side	Moved too far towards the lighter side. No balance
Move fulcrum by small increments in both directions	Small distances in both directions	Locating point of balance between 2 points of unbalance, adjusting in response to the direction of the unbalance	When unbalance changes direction, move fulcrum the opposite way When system is close to balance, move fulcrum in smaller increments	Balance achieved
Mark the point of balance Take measurements from ends of the stick	Measuring tape, centimeters (units), and pencil	Comparing distances from the fulcrum not focusing on the relationships between weights and distance	Mark balance point and measure distance from the fulcrum to ends of lever arm	Measurements of distances from weights to fulcrum
Record results with diagram of lever on the whiteboard	Diagram, distances in centimeters, balance point, weights and white board	Comparing only distances from the fulcrum	Make a picture of the real system on the whiteboard, adding the numerical information	Diagram of the lever

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Chapter 27
Don't Disrespect Me: Affect in an Urban Math Class

Roberta Y. Schorr, Yakov M. Epstein, Lisa B. Warner, and Cecilia C. Arias

Abstract We use a models and modeling approach to analyze affect as it occurs in several urban classrooms in New Jersey. We propose the concept of an "archetypal affective structure" based upon our observations and interviews with students and teachers. An archetypal affective structure refers to a recurring pattern that is a kind of behavioral/affective/social constellation. Such structures include typical patterns of behavior, indicative of affective pathways that have important cognitive interpretations and implications by students. In this chapter, we document a particular archetypal affective structure that underscores our hypothesis that, at times, the motivation to maintain "face" can become stronger than the motivation to engage in mathematical inquiry. We underscore the implications of this for teachers as well as for students.

R.Y. Schorr (✉)
Rutgers University, Newark, NJ, USA
e-mail: schorr@rci.rutgers.edu

Y.M. Epstein (✉)
Rutgers University, New Brunswick, NJ, USA
e-mail: yakov.epstein@gmail.com

L.B. Warner (✉)
Rutgers University, Newark, NJ, USA
e-mail: lwarnerb@gmail.com

C.C. Arias (✉)
Rutgers University, Newark, NJ, USA
e-mail: carias@pegasus.rutgers.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_27,

27.1 Introduction[1]

The purpose of this research[2] is to document the development of *powerful affect* as it relates to student engagement in *conceptually challenging mathematics* in urban classrooms – one of the major research initiatives of MetroMath, a Center for Learning and Teaching funded by the National Science Foundation (NSF) and the Newark Public Schools Systemic Initiative in Mathematics (NPSSIM), also funded by the NSF[3]. By *conceptually challenging mathematics,* we mean mathematical concepts that often are not offered to students because of their perceived difficulty level, or because of the cognitive or conceptual hurdles that they may pose to students. We also mean learning and teaching for conceptual understanding, including making connections, constructing and exploring representations, mathematical abstraction, modeling, defending and justifying solutions, and non-routine problem solving. Our ultimate goal is to better understand how to help students achieve at higher levels, particularly those whose achievements and abilities may go unnoticed in urban mathematics classrooms.

Our research suggests that effective classroom activity surrounding conceptually challenging mathematics is likely to involve mathematical discussion, exploration, individual students expressing their own ideas, disagreements, "wrong answers" and "blind alleys", as well as fruitful suggestions, challenges, and thoughtful questions posed not only by the teacher but also by students to each other (Lesh & Doerr, 2003; Schorr, 2004; Schoenfeld, 1992; Warner, Schorr & Davis, 2009). As this occurs, students and teachers are bound to experience a variety of emotions including, but not limited to, vulnerability, curiosity, puzzlement, bewilderment, confusion, frustration, annoyance, anger, fear, a sense of threat, defensiveness, suspicion, pleasure, satisfaction, etc. (DeBellis & Goldin, 2006; Goldin, 2000, 2007; Goldin & Schorr, 2008; McLeod, 1992, 1994). Many of these involve emotional risk-taking and related changes in affect with important cognitive and social consequences. In this study, we approach the challenging problem of characterizing the powerful affect that occurs, especially as it relates to the associated social and cognitive aspects, in the context of an urban mathematics classroom. We focus on several students, and their interactions with their peers and their teacher in order to highlight some of our findings.

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[1] We gratefully acknowledge the work of several other members of the Affect Research Project and their contributions to this chapter. They include all of the authors listed in footnote 2. We especially thank Gerald Goldin for his support, advice, and guidance in the preparation of this material.

[2] For further information, please see Epstein et al. (2007) and Alston et al. (2007).

[3] This research is supported by the US National Science Foundation (NSF), grants no. ESI-0333753 (MetroMath: The Center for Mathematics in America's Cities), and 0138806 (The Newark Public Schools Systemic Initiative in Mathematics). Any opinions, findings, and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF, Rutgers, or the participating school districts.

27.2 Theoretical Framework

Our framework for analysis is informed primarily by a models and modeling perspective, and additionally by several other inter-disciplinary perspectives described below. A model, as we use the term, is a "conceptual system (consisting of elements, relations, operations and rules governing interactions) that are expressed using external notation systems, and that are used to construct describe, or explain the behaviors of other system(s)..." (Lesh & Doerr, 2003, p. 10). We extend this definition to include *either* (or both) internal or external systems for describing, understanding, explaining, manipulating, making sense of, or interpreting an event, problem (mathematical or otherwise) or situation. Our primary assumption is that an individual's models are not static; that is, they are evolving and dynamic. However, they may be somewhat predictable, and consequently, in this research, we note that students' actions or reactions to a particular stimulus complex in mathematics classrooms are not isolated or de-contextualized, but rather the result of their social, cultural, contextual, cognitive, and affective models. In order to better understand how the particular cultural and contextual aspects of the urban environments in which the students live affect the students and their interactions with each other, the teacher, and their mathematics learning, we also draw upon the research base in socio-cultural theory (e.g., Cole, 1996; Cobb and Yackel, 1998; Wertsch, 1985), situated learning theory (e.g., Brown et al., 1989; Anderson et al., 1996), cognitive science (e.g., Greeno et al., 1995), and mathematics education (e.g., Lesh et al., 2007). In addition, we attend to the literature base focusing on the specific challenges faced by urban middle schools (e.g., Anderson, 2000; Dance, 2002).

As part of our framework, we also draw upon Cohen's (1978) capacity model of attention. According to this model, attention is a limited commodity that needs to be allocated amongst tasks. Circumstances demanding more attention get more attention, while those needing less, get less. Consequently, tasks or occurrences addressing high priority goals are given greater attention than tasks associated with lower priority goals. Accordingly, one important stimulus for many youngsters in urban environments is *danger* that can arise from an insult (tacitly or explicitly expressed) by another, an act that makes one look wrong or foolish, or lose "face" (Anderson, 2000; Dance, 2002; Devine, 1996; Fine, 1991). According to Anderson (2000), in his book *Code of the Street*, "life in public often features an intense competition for scarce social goods in which 'winners' totally dominate 'losers' and in which losing can be a fate worse than death" (Anderson, p. 37). He further notes that an important aspect of this "code" is to avoid ever being perceived as weak or a "loser". We conjecture, based upon our preliminary analysis of our data, that many of the students we are studying are hyper-vigilant for incidents in which their honor or "face" will be challenged during discussions about mathematical ideas. Thus, they allocate a significant portion of their attention (per Cohen's model) to monitoring for these stimuli at the expense of allocating this attention to the mathematical ideas involved in the task that they are working on.

There is no doubt that most middle school students, whether living in urban, suburban, or rural environments have a variety of issues and concerns that compete for their attention. In particular, social issues such as peer group acceptance are high priority goals. Consequently, actions instrumental to attaining acceptance are likely to be allocated a large share of any student's attention. We are not claiming that these issues are unique to urban students, but we do know that in interviews during our study, students often mention these issues.

Dance (2002) in her book *Tough Fronts: The Impact of Street Culture on Schooling* notes that "...life beyond school walls, life on the streets presents challenges with which urban students must contend. The streets are full of trials and tribulations, challenges of which school officials are unaware, or realities in which school officials have little interest" (p. 37). It is our stated goal to better recognize these realities in order to better understand how to create situations that maximize opportunities to help more urban students do well in school mathematics. Indeed, we know that many urban students *are* invested in mathematical problem solving and achieve at high levels (e.g., Schorr, 2003; Schorr et al., 2007; Silver and Stein, 1996). It is therefore important to understand the circumstances that surround this achievement. Consequently, in this study we focus on understanding the constellation of affective/social/cognitive structures that encompass the development of mathematical success by focusing on teachers who have created classroom environments where productive mathematical problem solving has been known to occur with a relatively high frequency.

Goldin (2000, 2007) has developed a framework in which essential affective structures include mathematical integrity, mathematical self-identity, and mathematical intimacy. Integrity entails commitment to truth and understanding in mathematical activity, self-awareness at any point of the limitations of one's mathematical understanding, and willingness to work to increase or deepen understanding. Self-identity encodes one's personal sense of oneself in relation to mathematics, "who I am" as a doer or user or learner of mathematics. And mathematical intimacy pertains to structures of emotions, attitudes, beliefs and values associated with intense, engaged, and vulnerable interaction in doing mathematics, possibly characterizing one's personal relationship with mathematics. These structures, self-identity, integrity, and intimacy, are key features in the concept of an "archetypal affective structure", which we have formulated as part of our analysis. This refers to a recurring pattern, inferred from observing the classroom and interview tapes, that is a kind of *behavioral/affective/social constellation.* Included are characteristic patterns of behavior, indicative of affective pathways and models (structures) that have important cognitive interpretations and implications by the students. In part, the interpretations and implications take the form of what we describe as "self-talk," and associated affective responses. Such affective structures can lead to important day-by-day choices, with powerful consequences – positive or negative – for students and teachers. The term "archetypal" is intended to suggest the idealized nature of the patterns, abstracted from the observed instances of their occurrence. An example of this will be highlighted below; others are documented in Epstein et al. (2007).

27.3 Methods and Procedures

Subjects: Two of the classrooms that are the focus of this research are located in schools in the largest city in the state of New Jersey. The other classroom is located in a smaller town several miles away. The schools are classified as "low income", and most of the students are members of minority groups. Students in the observed classrooms work in small groups varying in size from three to five students. The composition of the groups sometimes changes from session to session, affording us an opportunity to observe our *focus* students interacting with different students and in different groups. The eighth grade classroom that is the focus of this research report consists of 20 students and is 93% African American and 7% Hispanic or Latino.

Procedure: Classroom interactions for each of the entire classes were videotaped using three separate cameras. Classrooms were observed in each of four "cycles" with each cycle spanning a period of two consecutive days. The first cycle occurred approximately one month into the school year and subsequent cycles occurred approximately one month after the preceding ones. Prior to the start of a cycle, an interview was conducted with the classroom teacher to ascertain her plans for the lesson and her ideas about what she expected to happen when she taught the lesson, and a follow-up interview with the teacher took place after each cycle. At the end of each cycle, focus students were also interviewed using a stimulated recall protocol. Clips containing what the researchers believed were key affective events were selected from the videotapes and shown to the interviewee, to evoke the student's perspective on what had transpired and the emotional responses the student recalled having to those events. Transcripts were created from videotapes and student work about problem solutions were scanned. A team of researchers from the fields of mathematics education, social psychology, mathematics, and cognitive science reviewed and analyzed the results.

27.4 Results

As mentioned above, we have formulated the concept of an *archetypal affective structure.* Our hypothesis that the motivation to maintain face can become stronger than the motivation to engage in mathematical inquiry will be illustrated by the following example involving several young students. We focus, in particular, on one archetypal affective structure. In this case, we refer to the archetype as *Don't Disrespect Me*. It is illustrated below.

Don't Disrespect Me: The essential ingredient in this archetype is the experience of a challenge that poses a threat to one's actual or perceived safety or well being, and therefore must be resisted. The process of resisting elevates the struggle above the task, which serves as the arena for the struggle. The need to maintain face is paramount.

Behavior: Someone disagrees with my idea. I interpret that as a social *challenge* (self-talk: "He thinks my idea is wrong," together possibly with, "He wants to make me look bad and make himself look good"; note this is different than just disagreeing with my idea) → (self-talk: "I may be seen as weak or a loser") therefore feeling *threatened* → (self-talk: "People seen as weak get picked on or harassed by others") therefore experiencing *fear* → (self-talk: "How dare he disrespect my ideas") therefore feeling *anger* → (self-talk: "I can't let him get away with that") therefore aggression (fighting or arguing with the other person) → (self-talk: "What does he know about math anyway") therefore feeling *contempt* (for the other person and his ideas) and unwillingness to consider the ideas.

The following example will illustrate an instance of this archetype. In this case, all of the students in the class were working in groups to find a solution to the following problem: *Farmer Joe has a cow named Bessie. He bought 100 feet of fencing. He needs you to help him create a rectangular fenced-in space with the maximum area for Bessie to graze. Draw a diagram with the length and the width to show the maximum area. Explain how you found the maximum area. How many poles would you have for this area if you need 1 pole every 5 feet?* After working with their group-mates, the students were instructed to share their work with the other groups. One young female student, Dana, perceives[4] that a male student, Shay, from another group may be "disrespecting" her work and responds in a manner typical of the above archetype. We note that there is a difference between disagreeing and disrespecting. In the former case, a student may be quite comfortable acknowledging that her answer is different, even to the extent that she will change her mind about her own work. In the latter case, the student feels that the way in which the differences are brought to the fore disrespect her.

In the situation that we are about to highlight, Dana's group miscalculated the area in the problem below. When she saw that another group's work differed from hers, she appeared to be comfortable acknowledging that the method that they had used was correct while hers was not. However, the "Don't Disrespect Me" structure was triggered when she felt that a fellow student, Shay, was deliberately trying to prove her wrong, and in so doing, posed a *challenge* to her, and was disrespecting her.

An Overview of Dana's Story: On the first day of this problem solving exploration, Dana assumed (on her own initiative) the position of leading her group in the solution process. Essentially, the group discussed different potential perimeters for the fence. Ultimately, under Dana's direction, they decided to use a 40×10 ft configuration for the dimensions of the fence. In order to calculate the area, Dana incorrectly assumed that she needed to multiply lengths [side a + side c] by widths [side b + side d]) – therefore calculating the area as $80 \times 20 = 1600$ [square feet].

The following day, all groups of students were asked to prepare a poster documenting their solution strategies so that their classmates could walk around the room to observe and comment, in writing and/or verbally, on each other's work.

[4]Obviously, we cannot know for sure exactly what Dana or any of the other students were thinking at any given moment. The inferences that are made with respect to student's perceptions are based upon extensive analysis of the data and interviews.

Generally, under similar circumstances, the students would write their comments or ideas on post-it notes and place them on the respective posters.

Dana was very interested in the preparation of her group's poster, often directing her group mates in this task. It appeared that being an effective leader of her group was a very important feature of Dana's self-identity. She stated that she wanted the work to "be perfect". As the different groups walked around the room, Dana seemed interested in seeing what others had chosen for the dimensions of the rectangular fences and the associated areas. As she looked at another group's work (Shay's group), she noticed that the area was computed to be 624 [square feet] with the associated dimensions of 26 × 24 ft. Other members of her group appeared to be only casually interested in the differences. However, Dana appeared to be quite perplexed and intent on understanding the nature of the differences. She stated: "I don't understand. I want to know how they did that." ... Dana's intense mathematical engagement with the task seemed to be guiding her actions here.

> [Dana] But I don't know how they got this, how they got maximum area. . .(inaudible). . . I don't get that. (She picked up a calculator and began using it.) But, I want to know how do they get the answer. I want to know how they get this. . .they got this. I don't understand how they got this. [Von] You add it up, you do it and see what you get. [Dana] I'm talking about this, and why did they do this? [Von] I don't feel like sitting here. . .(inaudible) [Dana] So I put. . .(meaning the comment she writes on the Post-It note. [Von] "You did an outstanding job". . .(the comment Von suggests for the note). [Will] No, just this (i.e., the following note) "we don't understand. . ."(inaudible).

Dana continued in her quest to understand the nature of the difference, despite Von's encouragement. She is focused on how the group calculated the area.

> [Dana] Look, they said length times the width here; they multiplied the length and the width. [Von] Could we just agree on something? [Will] No, just write, "We don't understand how they got the maximum area." Write that. [Von] Write that? "We don't understand how y'all got your maximum area." [Dana] Ok, yes I do. I got it. They all right. 'Cause they said they got 26, they multiplied 26, so they multiplied this one for, um. . .

Dana realized that their method for calculating area was correct. Her commitment to the *truth* is evident in this episode, which we believe demonstrates her mathematical integrity. At this point, the teacher joined in the conversation.

> [Teacher] So is this the same as yours or different from yours? [Dana] They are different. [Teacher] It's different. [Dana] Yeah, 'cause we multiplied. . .(inaudible) [Teacher] So what do you guys have to say about what you see? [Dana] That they was right.

It is very important to note that Dana agreed that Shay's group's solution was correct. This is further supported by data taken from her interview several days later, where she noted that at this point in time she realized how to calculate area and knew that what Shay's group did was in fact correct. However, shortly after the above dialogue, Dana noticed that Shay's group was looking at *her* group's work. She listened to their comments and realized that they were saying that her group's work was wrong. In the excerpt that follows, the exchanges between Dana and Shay become quite heated. Their gestures and facial expressions were indicative of the feelings of fear, anger, and contempt associated with this archetype.

[Dana] (inaudible)...where it is wrong? [Shay] Cause you put, when you finding, um, the area, you timesed the width times the (inaudible). [Dana] All right, but we timesed all that up. [Shay] But you're not supposed to. [Dana] Alright, but we did it though, so... [Shay] But you're not supposed to, so it's wrong. [Dana] No, it's not wrong. Actually, no it's not. [Shay] It's wrong, it's wrong, it's wrong. [Dana] No, it's not. [Ghee] How is it wrong? [Shay] Look at that (showing on calculator), that's what y'all got, 16. [Dana] Yes. [Shay] That's what y'all got? [Dana] Yes. [Shay] So the width times the 40. [Dana] Well, we didn't do that, it equals 400, but we didn't do it. [Shay] 40, yes, that's how you do it. 40 times 10, that's how you get that. [Dana] Well, we didn't do it that, so, oh well.

[Shay] That's the right answer. [Ghee] 40 times 10? [Shay] Yes, that's the same thing they got, too. [Dana] So, for, you telling me that...whatever... [Ghee] 40 times 10? [Shay] 40 times 10 equals 400. [Dana] Oh well... [Dana] We already know 40 times 10 equals... [Shay] That's what your area is. Ok, that's what your area is. [Ghee] 40 times 10...You said 40 times 10? 40 times 10, 40 and 10...none of that add up to 100. [Lar] Yeah, so... [Shay] You add...you add this, that's 40 (*raising his voice*). That's 80 right there. And that's 100. Yeah, so you don't know what you're talking about. That's the perimeter, that's the perimeter (*almost shouting and moving closer to Gee's face as he points to the work*). And area is 400. [Ghee] No, 80...(inaudible)...that's exactly what we did. [Shay] And that was wrong. [Dana] 80 times 20 is not wrong from 100 (*raising her voice and speaking with conviction to Shay*). [Unknown] 80 plus 20 is 100. [Dana] Thank you! Thank you! 80 times...I mean, 80 plus 10...I mean, 20, is 100. [Ghee] So now who got ripped? Shut up, man! [Dana] So wait...80... [Shay] Exactly, tell 'em you gotta multiply length times width (*shouting at Dana*). Not what y'all did. [Dana] That's what we did! That's what we did (*shouting at Shay*)! [Ghee] That's exactly what we did! (inaudible) (makes angry motion)

Several days later, during the follow-up interview, Dana was asked to watch a video clip of the above interaction and reflect on it. The following transcription comes from that interview:

[Int.] So what do you think? [Dana] That was a funny clip [Int.] How were you feeling during that [Dana] Mad angry. Everything. [Int.] Well, why? [Dana] Because, Shay was proving me wrong. [Int.] So how did he prove you wrong? [Dana] He was trying to prove me wrong. Maybe he was right, and then maybe I wasn't right. Or maybe I was right, and then he wasn't right. I don't know. Cause he multiplied the length and the width, but I multiplied both of the lengths and both of the widths. So. . . [Int.] So what do you think? [Dana] That. . . maybe his answer is wrong. Maybe he multiplied the length and the width, because on the paper it said multiply the length, and not the width.

Note that, at this time, Dana stated that Shay's answer *could* have been wrong, which is in direct contrast to her comments to the teacher during the actual class. However, in the next excerpt, she stated that it was possible that she was wrong. She underscored the fact that she was "mad at him [Shay]".

[Int.] So what do you think? What did you understand from that day?
[Dana] That, maybe I was wrong. I don't know whether I was wrong or right. I was just, that day. He was, I couldn't say nothing to him cause he, I was mad at him.
[Int.] Were you feeling comfortable that day? [Dana] No. [Int.] Why?
[Dana] Cause he was trying to prove me wrong. [Int.] Do you feel that way often? Does this happen often? [Dana] Yes. [Int.] So you're uncomfortable often?
[Dana] Not all the time. But when I'm right, I'm not uncomfortable. But when I'm wrong, when they try to prove me wrong, I'm uncomfortable. [Int.] That's interesting. So when is it that, when are the moments when you get mad? When what's happening? [Dana] When they, when people in my group really aren't doing nothing at all, that makes me mad. [Int.] Is there anything else that makes you mad? [Dana] Uh. . . yeah, when people try to prove me wrong too.

During the interview Dana said that she was uncomfortable ". . .when people try to prove *me* wrong." Notice that she chose to use the word "me" rather than saying that she is uncomfortable when people try to prove her *idea* wrong. Shay's actions or words gave her the feeling of being challenged, thereby changing her primary focus from inquiry to saving face. Herein lies the heart of this archetype. A student becomes interested in a mathematical idea but then encounters a situation where it is critical to save face.

Dana initially appeared to be interested in exploring a mathematical idea, seeking mathematical truth (integrity), and trying to understand how Shay's group solved the problem (mathematical intimacy). Her primary emotion changed to one that is fearful of looking foolish, or publicly losing face (a challenge to her self-identity). Shay's response to her group's work was the behavior that seemed to trigger the "don't disrespect me" archetype and derail her attempt to continue to productively explore the problem. Instead of meaningful exploration, an affective structure designed to protect honor was elicited. As Shay pointed out the flaws in her group's solution, Dana seemed to lose face and her desired identity as an effective group leader (who was good at leading them toward a productive and complete solution) appeared to be damaged.

We believe that the above is a case where self-identity and integrity come into conflict. We suggest that Dana's investment in her identity as an effective leader of her group conflicted with her ability to elevate the mathematical integrity of the solution to its proper place. Investing in identity enhancement was done at the cost of sacrificing the integrity of the solution. Any solution that enhanced her identity as an effective leader (who could lead her group to a correct solution) was better than a solution with integrity that jeopardized her identity as an effective group leader (especially since she was now in the "public" position of leading her group toward an incorrect solution). So rather than the intense engagement (intimacy) functioning in the service of uncovering mathematical truth, it was invested instead in protecting a vulnerable self-identity, which would have been damaged because of a loss of face.

We speculate that, if Shay had not made his comments, Dana, upon her realization of the error in her group's work, could have effectively preserved her identity *and* integrity, as she led her group to a major revision of the work. Once the error was made public, the "Don't disrespect me" structure was triggered.

27.5 Conclusions

We hypothesize that much of the strongly expressed affect that we observed in this classroom and elsewhere is connected to the need to maintain a sense of "respect" or "face," which is a key element in the street culture of the urban[5] environment (e.g., Anderson, 2000; Dance, 2002). At times, it engulfs and overwhelms all else during mathematical group discourse, focusing students' attention on "looking good" or "saving face" as a higher priority than discovering mathematical truths or achieving success.

It is our intent that by defining such archetypes, we can better understand how to help teachers recognize and anticipate occurrences in which feelings of anger, fear, contempt, or hostility (each of which are associated with loss of "face" or respect) may occur. In doing so, they can ward off counterproductive interactions, thereby changing the atmosphere from latent resentment or hostility, with the consequent disinvestment from the task, to a feeling of justice and fairness, with the consequent ability to pay attention to the intellectual and practical aspects of the task. In addition, by recognizing such occurrences, teachers may be able to deal with them, as

[5] We are not suggesting that such issues do not occur in suburban or rural environments; however, our comments are directly related to our observations in urban environments and the related literature that vividly described such occurrences.

they occur, in ways that effectively help all involved save face, and maintain respect. By deflecting such comments or interactions, the teacher can minimize the attention paid to the situation or statement, and allow the student to refocus attention on the mathematics. This type of refocus would allow students to (re)engage in productive and meaningful mathematical problem solving experiences in what we term an "emotionally safe environment". We believe that this is a necessary (although not sufficient) condition for student engagement in mathematical problem solving.

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Section 8
How Do Classroom Modeling Communities Develop?

Chapter 28
Interdisciplinary Modeling Instruction: Helping Fifth Graders Learn About Levers

Brandon Helding, Colleen Megowan-Romanowicz, Tirupalavanam Ganesh, and Shirley Fang

Abstract This chapter examines and comments on a brief teaching experiment in which fifth graders explored levers. The teacher–researcher employed modeling instruction (Hestenes, 1996), an inquiry based pedagogical approach that is widely utilized in high school physics instruction. The method relies on collaborative meaning making by students who work together in small groups on laboratory exercises or problems, whiteboard their findings and share and compare their thinking with other groups at "board meetings" – whole class discussions facilitated by the teacher–researcher. We analyze the teacher–researcher's choices in designing and managing the learning environment in this setting in an attempt to identify how this instructional method can best be used with students of this age and mathematical sophistication.

28.1 Introduction

Modeling Instruction is a teaching method that was developed over 20 years ago by Malcolm Wells, a high school physics teacher who was intent upon finding a way to

B. Helding (✉)
Arizona State University, Tempe, AZ, USA
e-mail: brandon.helding@asu.edu

C. Megowan-Romanowicz (✉)
Arizona State University, Tempe, AZ, USA
e-mail: megowan@asu.edu

T. Ganesh (✉)
Arizona State University, Tempe, AZ, USA
e-mail: tganesh@asu.edu

S. Fang (✉)
Arizona State University, Tempe, AZ, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_28,

improve his students' performance on the pre-cursor of the Force Concept Inventory (FCI) (Hestenes et al., 1992). At the foundation of this pedagogical approach is a small collection of models that embody the fundamental physical relationships of mechanics. The method Wells developed was so successful that his high school juniors and seniors outperformed college physics students at large respected universities across the country – most of which were enrolled in traditional lecture-based courses – by a significant margin (Hestenes et al., 1992).

Thanks in part to a series of National Science Foundation (NSF)-funded grants over the last 20 years, the modeling method of instruction has been codified and provided to over 2000 teachers nationwide via summer workshops and courses. The method has been extended and curriculum materials developed for physical science, chemistry, and mathematics, and college physics courses.

Modeling instruction also has been used in some middle school science and mathematics classrooms. One study indicated that, despite their limited mathematical skills, the method works well with middle school students. One unanticipated consequence of introducing modeling instruction to students of this age (12–13 year olds) is that it stimulated students who had previously been observed to be disengaged from the learning experience to re-engage and participate meaningfully in learning processes (Megowan, 2007).

The brief teaching experiment described below explores the efficacy of the modeling method with still younger students, 11 year olds. We focus in particular on the dynamics of the learning environment, examine and critique the choices that were made, and identify the opportunities for student learning that were afforded or constrained.

28.2 Literature Review

Modeling instruction is an approach to teaching and learning that involves the collaborative construction, validation and use of scientific models (coherent knowledge structures) by students. Small groups, usually 3 or 4 students, engage in a series of tasks whose structure reflects the structure of the model under investigation. Students work together to identify, explore and elaborate fundamental physical relationships – models – and then generalize these relationships for use in solving novel problems with similar structure. They present and defend their thinking to their classmates via whiteboarded presentations that ideally involve multiple representational modes of collected and interpreted data (i.e., graphs, diagrams, equations, etc.). Central features of the classroom culture are inquiry, observation, collaboration, communication, and reasoning. The teacher becomes the asker of questions rather than the giver of knowledge (Megowan, 2007).

A model is a representation of structure. Hestenes calls models conceptual representations of real things, which consists of elements, operations, relations and rules (Hestenes, 2006; Lesh and Doerr, 2003). Conceptual models in physics typically have four kinds of structure that can be specified: systemic, spatial, temporal, and interactional structure.

Modeling is, thus, an activity in which students engage that involves the construction, validation, and application of models. A learning environment that is centered on the activity of modeling presents students with an event or series of related events of which students attempt to make sense by identifying, observing, measuring, and manipulating. In the case of the teaching experiment described below, students were given three situations to explore. These situations involved first class levers, and exploited students' notions of equilibrium and equivalence to construct a model of the multiplicative relationship between weight and distance of objects on opposite sides of a fulcrum.

As students observed and operated various kinds of first class levers, they identified the elements of the physical system that were relevant, i.e., fulcrum, lever arms, weights. They measured various combinations of weight and distance on opposite sides of the fulcrum that produced a "balanced" system – a system in equilibrium. They represented their results in an attempt to find an equivalence relationship between weight and distance for the two lever arms.

The success of a modeling instruction unit depends on the teacher's grasp of the model under investigation, on the students' representational skills, and on the classroom discourse environment that has been established in the course of instruction. Teachers can only effectively scaffold ideas for which they themselves have a coherent model. The theoretical perspective that framed lesson design in this case was that of Realistic Mathematics Education (RME) which moves students along a continuum from situational activity – reasoning that is contextually grounded – to referential activity – reduction and abstraction of contextualized information to manipulatable symbols or representations – to general activity – the application of mathematical operations, relations and rules. The success of students' model construction and validation relies in large measure on their ability to communicate and discuss their findings with one another. Students' communication skills, such as the orderly representation of data using tables, graphs and diagrams, become critical, and the belief that knowledge can reside in the verbal and written discourse of their peers plays a vital role in the model building and testing process.

An investigation of simple machines was chosen for this study for several reasons. Little has been published about elementary and middle school students' conceptions of mechanics in general, and of their understanding of simple machines in particular. While Newtonian physics is considered too abstract to be taught in the elementary grades (Bar, 1989), simple machines are a common element in the elementary school science curriculum. Simple machines have become a popular vehicle for introducing engineering education in the middle grades (McKenna and Agogino, 1998) with the advent of Lego kits produced especially for classroom use. Bar found that although students often confuse the concepts of force, mass and weight, they can still compare forces and distances to come to some conclusions about the principles that govern the behavior of levers (1989).

The instructional unit devised for this teaching experiment integrates ideas from mathematics and physics in the context of a familiar experience – the teeter-totter – in an attempt to help students construct, test and apply a simple model of the multiplicative relationship that exists between force and distance in a first class lever.

Research Question: How might modeling instruction be used to help 5th grade students acquire a coherent and useful model of the lever?

28.3 Methodology

Background: This study was conducted as a research project for a graduate seminar on modeling and cognition. On one hand, it was intended to give the participants in the seminar a classroom research experience with children engaged in modeling activities based in part on a longitudinal study of children's rational number understanding (Middleton et al., 2007, A Longitudinal Study of the Development of Rational Number Knowledge in the Middle Grades, unpublished manuscript). On the other hand, it was an opportunity for graduate students to see how modeling instruction is delivered in real time in a naturalistic setting, and how classroom representation and discourse norms are negotiated between students and their teacher. It was a glimpse of what a model-centered learning environment looks like in a 5th grade science and mathematics classroom.

Design: Because this teaching experiment was designed, in part, based on the efforts associated with a graduate seminar, and modeling instruction is adaptive to students' models and responses, a classical experimental "method" is not applicable. Instead, the graduate students generated curricular materials that were filtered by the teacher–researcher for classroom application. In particular, the teacher–researcher's 20 years of teaching experience led her to alter some elements of the modeling activities so that 5th grade students would find content and instruction engaging, accessible and comprehensible. Her choices were mediated by the constraints of the classroom and limited number of class periods over which the investigation was performed.

One overriding concern that affected the design and delivery of instruction in this instance was the shortage of available time. A typical modeling cycle lasts 2 weeks with over 600 min of classroom instruction. The participating classroom could only afford 240 min over 4 days. This was especially problematic since younger students take longer to traverse the modeling cycle. For example, an average "board meeting" (a whole-group discussion of small group work that students present on whiteboards) lasts 9 min. The 5th graders in this study spent over twice as long on board meetings. The modeling cycle, consequently, required additional time to refashion classroom expectations around communication and collaboration.

First we present the teacher researcher's preconceptions about the design of instruction, followed by a discussion of the three modeling activities, in conjunction with rationales and descriptions thereof. The teacher–researcher's final reflections will be presented with special emphasis on those elements of modeling instruction and student responses that were surprising or unexpected, and which accordingly changed the teacher–researcher's understanding of modeling instruction when applied to children of such young ages. These findings will be oriented in terms of the theoretical tenets associated with modeling instruction, and be offered as an impetus and onus for future research that uses modeling instruction with young children.

28.4 Teacher–Researcher's Initial Perspectives

An important point should be emphasized: the teacher–researcher in this instance was not simply teaching the 5th graders. Her efforts were divided between effective instruction for participating students and a demonstration of modeling instruction for the graduate seminarians. Her goals were to demonstrate how modeling instruction is used by teachers, how the practice of modeling is instantiated in the classroom, and how modeling activities are linked instructionally through the modeling cycle. To use her words, she wanted to "ensure an authentic experience" for fledgling researchers, the classroom teacher, and the participating 5th graders because she believed that "modeling is good". She concluded the best way to demonstrate modeling instruction was to design and deliver a modeling instruction cycle to students.

The first challenge was to adapt the instruction to the classroom and time constraints. Since the instructional activities to be used were designed by graduate students with little actual classroom teaching experience, the teacher-research made them more suitable for use with the students in the available time. Another factor that shaped the outcome of this teaching experiment was the teacher–researchers' initial expectations about students' representational skills and their primitive models of levers. The students, she thought, would most likely think about levers in the context of previous experiences with teeter-totters, and in this context, point masses would most likely be moveable, but not necessarily of equal magnitude. Critical foci of student thinking for this modeling unit of instruction were the relationship of weight and length of lever arm on each side of the lever, balance versus imbalance, with horizontal balance as an ultimate goal, and mathematization techniques. She expected measurement, quantification, and interpretation of measurements to be vital to development of students' mathematical models of balance.

Activities: Each activity will be discussed with underlying rationale for activity selection and descriptions thereof. Each modeling activity will be situated as part of the larger modeling cycle that underpins the entire teaching experiment. In terms of the modeling cycle, activity 1 was a model identification activity, activity 2 was a model elaboration activity, and activity 3 was a model deployment activity.

Description of Activity 1: Students used paint mixing sticks as the lever, Lego blocks as the fulcrum, and measuring tapes to construct first-class levers. Washers were used as weights, and determined by class consensus to be of identical mass. During the prelab discussion, the teacher–researcher presented the fulcrum (Lego) as static and bisecting the lever (paint mixing stick). Because the center was marked on each paint stick prior to instruction, it is impossible to determine if the class consensus of static fulcrums was a product of teacher–researcher intention or true class negotiation. The modeling instruction result, however, was the same despite the intent. Students explored the levers in small groups, varying weights and distances to achieve balance, recorded findings in journals, whiteboarded findings, and presented their whiteboards during board meetings.

Rationale: The lever was chosen for many reasons, some practical and some mathematical. It was a physical experiment made accessible by a graduate student's tinker-toy kits. The graduate seminar spent several weeks arguing over and devising possible physical experiments that would address the interdisciplinary needs of mathematics and physics. Simple machines, first-class levers in particular, provided students a medium through which to explore relationships of balance, weight, distance, and equivalence. It was the hope of researchers that a static fulcrum would allow students to alter weights and distances, achieve balance, and accordingly induce mathematical relationships between weight, distance, and balance. A first-class lever was also selected for the first activity because students would presumably have had previous experiences with teeter-totters, and would be able to make sense of the activity by mapping it onto previous experience.

Description of Activity 2: Students used one 36-inch wood rod as lever, string as a moveable fulcrum, one plastic grocery bag to hold a textbook that functioned as a uniform weight, one Styrofoam cup to hold washers as variable weight, ten washers to vary weight, two rubber bands to suspend weights from the lever, and a tape measure to measure distances between weights and fulcrum. Again, students explored the levers in small groups, varying number of washers and position of fulcrum to achieve balance, recorded findings in journals, whiteboarded findings, and presented their whiteboards during board meetings.

Rationale: The second activity involved a hanging balance. The weight at one end of the hanging balance was held constant (the textbook). The other weight and the position of the fulcrum were varied by students as demonstrated by the teacher–researcher during prelab. The hope of researchers was that by allowing students to alter the position of the fulcrum to achieve balance, rather than simply varying the number of weights and their position along the lever arm as done in Activity 1, it would encourage students to realize the underlying and consistent relationship between weight, distance to fulcrum, and balance.

Description of Activity 3: The students used a 20-foot plank of wood as a lever, pieces of plywood as platforms on either side of the lever to hold weights, and a wooden block as a fulcrum. The students balanced their Principal with their own weight, and the weight of several groups of students. They varied both the position and magnitude of their weights to obtain their balance. They did the same with their regular classroom teacher, the teacher–researcher, and various groups of students (e.g., boys versus girls). Their predictions, experiments with varied weights and distances, and findings were recorded by students in their notebooks, and by researchers with video cameras and digital cameras.

Rationale: The final activity was the lifting of an actual person. It was selected because it would presumably be engaging and enjoyable for the students. Also, for the research team, it would potentially provide evidence of students' ability to apply their models to accomplish a real-life task. In Lesh's conceptualization of modeling tasks, it is critical that such modeling tasks be used to solve realistic problems. In this case, the realistic problem was lifting the school Principal. In particular, researchers documented students' ability to map their understanding of the relationship between weight, distance, and balance to a giant wooden contraption, to making predictions, and to balancing their lesser weights with their Principal's greater weight (Fig. 28.1).

Fig. 28.1 Students prepare to lift the principal with their giant lever

28.5 Teacher–Researcher Final Reflections

Model of Modeling Instruction for Graduate Students: While the teaching experiment was designed and implemented as much for the graduate students enrolled in the modeling and cognition seminar as it was for the 5th graders, few graduate students actually looked at the video of the entire 4 h of instruction. The teacher researcher felt that this was a lost opportunity and a flaw in the design of the graduate seminar.

Model of Modeling Instruction for Students: The teacher–researcher noted students' challenges in adapting to the new social norms that modeling instruction introduced into their classroom. She found that initially students rarely connected one day's instruction with what they had learned on the previous day. Students appeared to enter math class, she stated, with the notion that each day's instruction stood alone. Also, she commented on students' tendency to interpret class instruction as an opportunity for performance rather than meaning-making. They demonstrated a "fill-in-the-blanks" mentality when recording the data they collected, as if writing down numbers were the object of the exercise rather than making an attempt to organize the "data" so that they might find clues to the meaning of the system under investigation. Citing students' talk about whiteboarded findings, within- and between-group communication, student references to previous day's activities and prior experiences, she concluded that, albeit slowly, students' view, that each day's instruction was independent from that of any other day, was beginning to change by the end of the teaching experiment.

Initially, students engaged one another in discourse only when prompted by the teacher–researcher. When the teacher–researcher failed to prompt communication between students during board meetings they reverted to traditional teacher question-response patterns, as illustrated in the following vignette:

Teacher–Researcher:	We are going to start with this group and we are going to, one by one, present... Ok, do we know what we are going to do?
Gary:	Yes.
Teacher–Researcher:	(turns to first group on her right) Okay, so, Belle, Joan, and Jenny, what did you do?
Belle:	We, um, we did it first with our paint sticks, Lego, and washers, and we put nine washers on one side and one on the other. And we wanted to make them equal. So we put the nine washers closer to the fulcrum line and that made the whole thing equal, and it was balanced so if we had added one a little bit closer to the fulcrum line that would be equal...
Teacher–Researcher:	Anybody want to ask Belle something about it?
Class:	(silence)
Teacher–Researcher:	So all of us understood what she did, so every time I ask you later about how she moved it... (there is a pause and then a boy raises his hand to ask a question). Okay, go ahead...
Jose:	How could it be balanced like that though?
Teacher–Researcher:	(turns to the group of girls) Ok do you know that? (They do not answer immediately.)
Jose:	(Asks his question again.) How could it be balanced with nine washers?
Belle:	(hesitantly at first) Um, we put nine washers, we just put nine washers on one side and one, and one was at the [other] end. And we put them like that, and we moved it closer and closer 'til it equaled, so it was like on the fulcrum.

Teacher–Researcher: (questioning the rest of the class) So what was she changing to make it balanced? She had nine on one side and one on the other, how could she make it balanced?

Gabriel . . .'cause she moved it.

In this excerpt, the conversation was mediated almost exclusively by the teacher. One student did finally question another, but only when prompted by the teacher–researcher to do so. When the teacher–researcher turned her attention away from the questioner and answer-provider, the class quickly lapsed back into waiting for the teacher–researcher's next question or direction. This was indicative of most students' behaviors during the initial board meeting. Their perception that the teacher was in control of the conversation shifted, somewhat, as the teaching experiment progressed, however, and by day 4, students were observed to question or challenge one another during board meetings.

Additive vs. multiplicative thinking: The teacher–researcher reported that students developed a good grasp of the connection between an object's distance from the fulcrum and its weight, when balanced by a weight on the opposite end of the lever. On a number of occasions they justified reasoning by saying that if the weight was increased it must be moved closer to the fulcrum. However, it became clear to her that this intuition was not based on any mathematical rationale. On day 4, she abandoned the use of graphical representations of the data in favor of presenting a simple table of the data, along with teeter-totter diagrams to provide the context for a discussion about patterns they could look for in the measurements (Fig. 28.2).

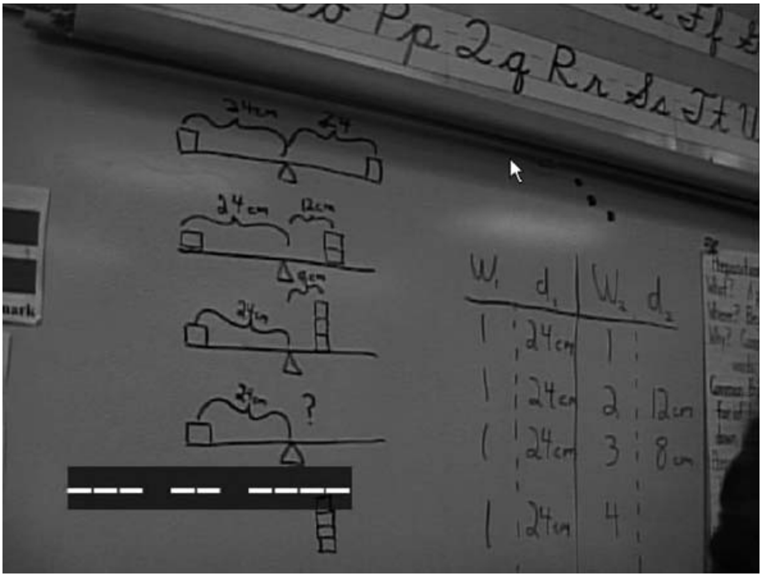

Fig. 28.2 Teacher generated inscription for class discussion

Students were asked to work together in their small groups to predict the distance at which they would have to place 4 weights from the fulcrum in order to balance a single weight, 24 cm from the fulcrum, on the other side. The teacher researcher was surprised at the difficulty the students' demonstrated in thinking multiplicatively. All of the groups predicted that the answer would be 4 cm and all of them used as their rationale that the pattern of the data was additive – that is, for every weight you add to the stack, you needed to subtract 4 cm from the distance to the fulcrum in order to achieve a balanced system. When they were asked if this pattern worked when there was only a single weight on each side, they demurred. After a few moments, Maria called out, "The answer is 6 cm!". When asked to explain she promptly replied, "because six times four is twenty-four." There was a collective "aha!" from her classmates, as different people chimed in "…and three times eight is twenty-four", "…and two times twelve is twenty-four". The class appeared to have an instantaneous appreciation and acceptance of her reasoning and from that time on, there appeared to be a fundamental shift in students' reasoning from additive thinking to multiplicative thinking. Once they made this shift, they stayed with it, at least through the end of the giant lever deployment activity. At one point, when students were asked to predict where their regular classroom teacher and the teacher–researcher should stand on the giant lever in order to produce a balanced system, one student stated that because their regular classroom teacher weighed more than the teacher–researcher she (teacher–researcher) should stand "further back". This student's hypothesis was verified when the teacher–researcher balanced the greater weight of the regular classroom teacher with the lesser weight of herself by increasing her distance from the fulcrum.

Student Difficulties: Random Measurements, Poor Data Organization. Measurement was not used meaningfully, that is, it was procedurally obtained, but not systematically organized and used to identify mathematical relationships. Multiple measurements of weight and distance were collected for balanced levers, but the data were not organized into data tables or Cartesian graphs (see Fig. 28.3).

The teacher–researcher reported that measuring problems were so persistent, that she was tempted to spend additional time during the teaching experiment helping students develop their measuring skills. Because such additional time was not available, and students were so lacking in measurement skills, the teacher–researcher returned to the class following the teaching experiment to teach measurement content specifically.

The whiteboard in Fig. 28.3 demonstrates a number of measurements that were not organized or systematically arranged such that patterns could be discerned. The disorganization of students' data, the teacher–researcher speculated, was critical to their difficulty in discovering the mathematical relationship between weight and distance in a balanced system. Measurement was used, but only after balance was achieved through guess-check-retry strategies, rather than to predict *a priori* which weights at what distances should balance. Cartesian graphing presented such difficulty to the 5th graders that this strategy was ultimately abandoned by the teacher–researcher as a tool for analyzing the data that students collected.

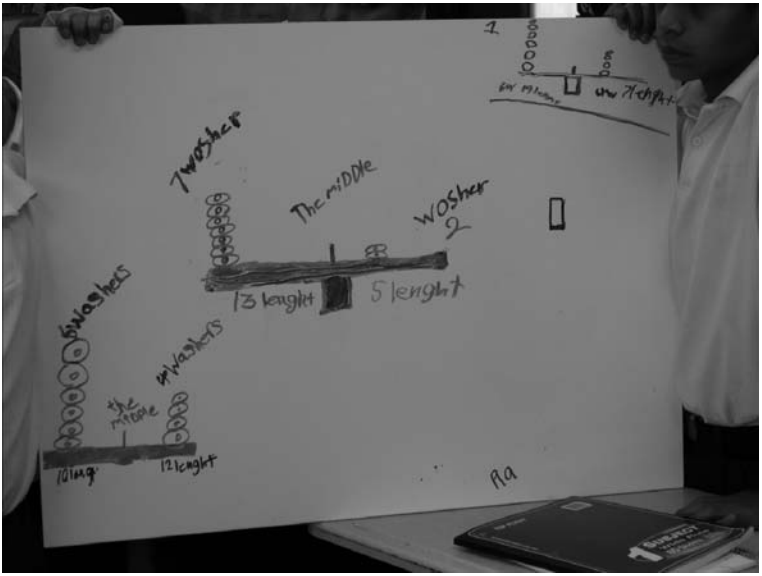

Fig. 28.3 One group's whiteboard

28.6 Conclusions

To a large degree, students' success in model construction hinges on their ability to collect, organize and represent information about the system under investigation. Once they received some scaffolding that bridged these difficulties, they were able to recognize and generalize the underlying mathematical relationship that characterized the model and then apply it successfully in a novel context.

According to the RME framework of model development, students began with situational models. They primarily applied their understandings and mathematizations of levers to their understandings and mathematizations of teeter-totters. That board meetings featured specific, balanced levers, rather than families of levers that balanced, or data tables indicating mathematical relationships among weights and distances to fulcrum, indicate students' grounded their thinking in episodic memory of prior situations. As the modeling cycle progressed, students did seem to, to varying, limited degrees, and based on teacher scaffolding, abstract the contextual information of teeter-totters and the balanced levers they found to drawings of prototypical levers. It was not until one students "ah ha" moment, however, that students discovered the general, informal relationship between weight and distance to fulcrum. To conclude that they had acquired general models may be overstating the situation, but their models held the seeds of generality that with further instruction could be buttressed with physical evidence as well as mathematical substance.

One significant flaw in the design of this unit stemmed from the fact that the instructional designers had no first hand knowledge of the students for whom they were designing lessons. The designers assumed that 5th graders had certain skills. Surely students would know how to measure. The designers assumed that if students wanted to measure the distance from a group of weights to the fulcrum, they would know that the weights needed to be stacked on top of one another and not spread out along the lever arm. The designers also assumed that students knew how to construct tables of data and scatter plots on coordinate axes. As a result of these misconceptions, instructional activities took longer than intended, and board meetings often did not produce the intended results.

Another unanticipated consequence of the teaching experiment was the novelty value of whiteboards themselves. The introduction of a new instructional medium into the learning environment was, at times, distracting to students. They took pleasure in spending time choosing the color of their whiteboard marker, carrying their boards from place to place, erasing and redrawing their inscriptions. In short, everything took longer than it might have if the representational medium had been more familiar. Whiteboards were not a bad thing – in fact, they were the focal point of a great deal of valuable and interesting student-to student-discourse, but once again, they accentuated the problem of having too little time.

There is no substitute for knowing the strengths and weaknesses of your students, their habits of mind, and the social norms that prevail in their learning environment. In the absence of this knowledge, it is fair to assume that everything will take longer than intended, and that adjustments in the design of the learning experience will have to be made on a continuous basis as instruction progresses.

28.7 Implications for Future Research

This brief episode suggested that modeling activities provide a context for important skills such as graphing, diagramming, tabling and measuring, and affords opportunities for children to practice presenting and defending their reasoning. It might be worthwhile to investigate whether children who learn these skills via modeling activities compare favorably on standardized performance measures with those who learn them in the traditional way.

Also, because students demonstrated at least embryonic general models, additional studies investigating instruction could offer insights into instructional techniques that help students further develop their primarily referential and partially general models to formal models. Along with developing formal models, future studies can investigate how students cognitively deduce and conceptualize axiomatic mathematical knowledge and develop that knowledge from instruction that focuses on particular apparati and ultimately prior experiences. Such studies of student movement from situational to axiomatic knowledge, in a whole class setting, would enable teachers to consistently teach formal mathematics and physics sensibly and meaningfully.

Acknowledgment The research reported in this chapter was supported, in part, by a grant from the National Science Foundation (#033--7795). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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Chapter 29
Modeling Discourse in Secondary Science and Mathematics Classrooms

M. Colleen Megowan-Romanowicz

Abstract This paper describes the structure of student-student and teacher-student discourse in four different classrooms in which modeling instruction was employed and it highlights characteristic features of this discourse. I describe the roles students adopt during whiteboard-mediated discourse and the ways in which these roles shape student reasoning and engagement in these very different learning communities.

29.1 Introduction

Modeling instruction has been in use in physics classrooms across the country for over 20 years (Hestenes, 1987). Based on a method first developed by Malcolm Wells in the 1980s, this pedagogical approach has students spending most of their time working together in small groups on either laboratory activities or paper-and-pencil exercises – and then sharing their thinking with their peers via whiteboards in whole group discussions.

Most teachers who espouse this teaching method learned about it by attending one or more Modeling Workshops where they assumed the role of a student, worked their way through the curriculum materials in cooperation with other teacher/students, and participated in whiteboard-mediated group discussions, which are referred to as "board meetings" and that were managed by teachers not unlike themselves who were considered expert practitioners of modeling instruction. After 3–4 weeks' immersion in this learning environment they returned to their own classrooms and attempted to replicate the experience with their own students.

It takes time to master the art of discourse management for most teachers, but eventually they find a "comfort zone" that works for them and for their students.

This study takes a close look at the various discourse environments that evolve and are eventually established within that comfort zone.

M.C. Megowan-Romanowicz (✉)
Arizona State University, Tempe, AZ, USA
e-mail: megowan@asu.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_29,

29.2 Review of Literature

Learning is a cultural process, situated in a context, and mediated through social interaction, activity and representation. A classroom learning community is considered to be a legitimate unit of analysis when examining student reasoning (Cobb, 2002; Doerr Tripp, 1999). Two easily observable exteriorizations of this reasoning process are the inscriptions they produce – written representations (Roth and McGinn, 1998) – and students' discourse with one another and with the teacher.

In modeling instruction, both inscriptions and discourse are collaboratively constructed by groups of students working together to develop representations that encode meanings of concepts they are exploring. When reasoning processes are distributed across individuals, their tools, artifacts and representations, it is referred to as distributed cognition (Hollan et al., 2000).

This paper explores a particular type of distributed cognition that is a hallmark of students engaged in modeling instruction: whiteboard-mediated cognition (Megowan, 2007). As mentioned above, students who learn via modeling instruction engage in a series of tasks – laboratory exercises, problem solving, practicing with the model – that they work on in small groups. This small group work centers on the negotiation of a collaboratively constructed shared visual representation (a whiteboard) of the problem space as they see it, and the discourse that surrounds the construction and sharing of this whiteboard can reveal a great deal about the coherence of the models that students rely on to assist them in this process.

What follows is a description of four different classroom cultures (Brown et al., 1989) that all have as their frame the Modeling Method of instruction. I take a careful look at the discourse environments they have produced.

29.3 Research Method

Data for this study were collected in four classrooms. Each class meeting of a third semester community college engineering physics class was videotaped for an entire semester; an 11th/12th grade honors physics class was videotaped daily during its first semester; a 9th grade physics class was videotaped daily for half a semester; and a 7th and 8th grade mathematics resource class was videotaped twice weekly for a month. All four of these classes were taught by experienced practitioners of modeling instruction – so-called "expert modelers". Three of the four classes were demographically similar, consisting of a majority of working class Hispanic students. The honors physics students attended an upper middle class suburban public high school.

Selected episodes of this enormous video library were transcribed – in particular, those showing students engaged in small group or whole group whiteboard-mediated discourse. The purpose of this study was to document how collaboratively constructed, shared visual representations aid students' collective meaning-making in physics and mathematics; to determine the structure of the discourse that surrounds the preparing and sharing of these representations; and to shed light on what teachers can do to optimize the ways students use inscriptions and discourse to aid their reasoning in physics and mathematics (Megowan, 2007).

Since modeling instruction relies heavily on whiteboard-mediated discourse, the videocamera observed a single group of three or four students during whiteboard preparation, or it recorded the entire class during board meetings or whole group discussions. In observing different groups of students engaged in these activities, day after day, it became apparent that, although no a priori framework appeared to structure the sharing of whiteboarded information, individual classrooms developed implicit norms for constructing and sharing visual displays. These involved the assumption of identifiable roles by students in both small and whole group discussions, and a predictable format for the discourse that, in some cases created spaces for potential reasoning and in other cases closed off those spaces. What follows is an account of the structure of discourse that developed around the modeling tasks on which these students collaborated.

29.4 Research Context

A brief profile of each of the four classrooms and their teachers is necessary in order to provide context for what follows. In each of the four cases, I have chosen a metaphor to characterize the teacher's demeanor and relationship with students. This is meant as neither critique nor endorsement of any particular style. It is merely a device I used to help me make distinctions among classes.

Middle School Mathematics – teacher as scoutmaster: This combination 7th and 8th grade mathematics resource class met twice weekly throughout the school year, and consisted of higher performing 7th graders and lower performing 8th graders in an urban, working class, largely Hispanic K-8 elementary school. The data were collected during a 5-week teaching experiment I conducted that involved introducing students to the use of whiteboards for collaborative student work to motivate more participation and engagement in the learning process.

A typical class opened with a journal prompt that students wrote about for 5 min, which foreshadowed the day's activity. This was followed by a whole group discussion, often accompanied by a demonstration to set the stage for the modeling task in which students would engage, i.e., finding a relationship between two quantities of some sort. On average, more than half the students present participated in some way in this initial discussion. As different students contributed ideas to the discussion, I jotted abbreviated versions of their suggestions on the board and then the group narrowed them down and finally selected the parameters that made the most sense to investigate. Then I turned students loose to work in groups of three or four students each, and as they gathered data and assembled their thoughts, each group prepared a whiteboard to present to the class. Since collaborative whiteboarding was new to these students, I was explicit about some parameters: their whiteboards should show multiple representations of the problem space, they should be collaboratively constructed, and every member of the group should be able to explain whatever was on the board.

When working in small groups, these students often switched back and forth between Spanish and English in conversation with each other. Typically, there was a student who led the discussion and this was often also the student who did the bulk of the writing on the whiteboard, albeit with significant input and, at times, censorship from his or her teammates. Almost all students participated to some degree in the discussions surrounding whiteboard preparation, although some students offered little input. I moved from group to group, asking or answering questions about what they were putting on their boards, and occasionally dropping hints about ideas I hoped would emerge in the whole-group discussion that followed. After a period of time, students were called together to share their results with the rest of the class in a "board meeting". Students stood in a circle with their whiteboards in front of them so that they could all see each other's whiteboards, and either a formal presentation of what appeared on some board or a comparison of two or more boards ensued. Initially I had to ask for students to describe their boards, and then ask questions myself about their presentation or request questions or comments from other students in the class. After a few repetitions of this routine, students grew more comfortable questioning and challenging each other.

In general, the atmosphere in this classroom was relaxed and congenial. Students expressed no overt concern for grades, perhaps because they were being taught by a guest teacher. Students did not appear to be concerned about whether they looked smart to me or to each other. They listened to each other, took turns speaking, and even when they disagreed they did so politely for the most part. They accepted responsibility for the work assigned and they appeared to approach the work they did as if it were a creative activity.

Ninth Grade Physics – teacher as parent: This class was a required course at a large, working class, majority Hispanic suburban public high school. During the 7-week period in second semester that I videotaped in this classroom, two students enrolled in the course dropped out and two new students entered to the class.

As these students entered their classroom each afternoon, it was apparent that they were very socially aware. A number of the girls were overtly flirty and many of the boys affected a macho demeanor. Their public persona seemed to be very important to them. Even so, most routinely made a visible effort to engage in each day's lesson, which opened with a 5-min journal prompt that asked them to answer some question. For a handful of boys, this was their cue to stroll over to the electric pencil sharpener and survey the room to see who was watching as they put an even finer point on their already sharp pencil. After the 5 min passed, the teacher immediately launched a discussion of the question, calling at random on members of the class, prodding them gently when they were unresponsive or not fully engaged, and writing their responses on the board. Once he had more than one response to his question, hands usually went up and students began critiquing or extending one another's contributions.

When this discussion was complete, the teacher described what students would be doing for the remainder of the class period, and then often mentioned briefly what was planned for the remainder of the week, i.e., collect and whiteboard data today, board meeting tomorrow, review exercises the following day, quiz on Friday. This rundown usually concluded with, "…so make sure you come to class so you know what's going on." Then he divided them into groups of two to four students to begin work on the day's activity, which might involve whiteboarding homework problems, collecting data in a laboratory exercise, or negotiating a solution strategy for a question that he posed to the whole class.

Students worked together in small groups, some writing in their notebooks, others writing directly on the whiteboard that they would be sharing with the class as they talked with each other. Many of the groups had a member or two who did not engage in conversations. In some cases, that person was left alone by his teammates, but in other cases, he or she was prodded to take part. The teacher circulated among the groups as their work progressed, asking about their reasoning, engaging students who appeared distracted, coaxing students to get something in writing before time was up. This was followed by a board meeting as described above, or by a series of presentations by each group in which they described what was on their board, walked the class through their solution strategy, and then answered questions. During this questioning, the teacher often probed members of the group who had not yet spoken in the presentation, asking them to explain or clarify their teammates' words or images that appeared on the whiteboard. He did not let them off the hook if they claimed that they "didn't get it" and students appeared to accept that they would be held responsible for at least having an opinion about how to approach the problem. The whole group sharing of whiteboards usually ended with a summary of the key ideas discussed and a critique of the session by the teacher.

The atmosphere in the classroom was familial. The students seemed aware that they would be held accountable for their words and actions (or inactions), and they appeared comfortable with one another and with the teacher (despite occasional posturing or bids for attention), even to the extent that they would challenge his assertions (about physics) when they conflicted with their own commonsense concepts. They believed that knowledge resided in their peers as well as their teacher, and to some extent, in the representations and tools they used.

Honors Physics – teacher as coach: This class was an elective course in a large, upper middle class, majority white, suburban high school. Students were punctual, well behaved, polite and diligent. Most appeared confident in their ability to do well in class. There was intermittent concern expressed by many about grades and points.

Each day's class opened with a polite greeting from the teacher that was reflected back to him by students, and then a rundown of the day's plan, which was written on the board. Sometimes this was followed by a class discussion led by the teacher that summarized what they had learned previously. Other times students were sent immediately to work in groups of three or four at lab tables in the back of the room, each group whiteboarding some problem from the previous night's homework. The teacher circulated during this process. These whiteboards were then presented to the whole group one at a time, each presentation followed by an opportunity for questions from the teacher or students. This was followed by some summary remarks from the teacher and then directions for the next activity.

If the day's plan called for a laboratory activity, this was preceded by a pre-lab discussion in which the apparatus, which often involved computer assisted data collection, was demonstrated and students were guided to identify a particular relationship between quantities that they could explore using this set-up. The discussion ended with the teacher writing the purpose for the laboratory activity on the board, and then students were sent back to their lab tables to collect data and prepare whiteboards. Laboratory results were shared via a board meeting, as described above, in which students compared their results with one another and attempted to come to a consensus on what conclusion they should write in their lab reports. The teacher had particular expectations about the proper use of language, and students made an effort to "speak physics correctly" in these board meetings. The teacher guided and prompted them to express their thoughts using the correct terminology to explain and justify their answers.

The atmosphere in this class was disciplined and focused on performing well. Students were comfortable with each other, worked well together, and appeared to care about their work and about their grades. The teacher, as coach, worked hard to help student learn and practice good form, and expected them to cooperate to perform their best, while student played to each others' strengths and endeavored to turn in a good performance.

Community College Engineering Physics – teacher as general contractor: This was a small laboratory-based physics course at a suburban community college that shrank from 15 to 12 students during the course of the semester in which I collected data. The topic was electricity and magnetism. The enrollment was majority Hispanic, and boys slightly outnumbered girls. The students in this class had all taken the first two semesters of this series of courses from the same professor, so the design of instruction and the expectations of the teacher were very familiar to them.

Class usually opened with a brief description of a task that the students were to given to explore and whiteboard, i.e., describe the electric field on a cylinder embedded in an infinite plane of charge density σ. Students moved into their regular small working groups of two or three without being told, retrieved a whiteboard and markers, and got busy. Conversations around the developing whiteboards were focused. They almost always started with a diagram. In some groups, the writing on the whiteboard was left to a single student. In others, two students shared the work, but the discourse was spread evenly across group. They worked deliberatively, probing one another and critiquing the choices that they made as they moved toward a solution. The teacher circulated while boards were being prepared and probed their thinking. When students asked him direct questions he responded by reflecting their question back to them in a series of simpler questions that were intended to help them think aloud.

When he saw that most groups were finished he called students to pull their chairs into a circle for a board meeting, and a discussion ensued. One of the groups started the discussion by describing what they had written on their board and then they gave up the floor to their classmates who asked questions or compared what they had shown with what appeared on other boards. The teacher was notably absent from these board meetings. He did not sit in the circle with the students, but rather sat apart from them at his desk and often appeared to be checking his email. In spite of appearances, however, he was tuned-in to the discussion, because when he heard it winding down he would probe individuals who did not yet appear to have a complete grasp of the concept under discussion. He pushed other members of the group to help clear up their confusion, often uncovering the confusion of others that lurked beneath a façade of understanding. Sometimes these discussions would extend for 30 min or more and involve thought experiments and ad hoc demonstrations. They would not end until the students' reasoning satisfied him. Occasionally students tried to "give up" trying to answer his question, but he never let them off the hook by answering it for them. He just keep prodding them and refusing to go on to the next task until they worked it through. In the end, he would wrap up the meeting with a summary of the key points that they had made in their discussion.

Class consisted of a series of such tasks or laboratory exercises, followed by board meetings. Students appeared to expect that the teacher would not make it easy for them. They were accustomed to the routine and accepted responsibility as a team for moving the discussion forward and persisting in their efforts until the task was complete, even when they were not entirely sure where it was going. They were not cocky but they seemed confident in their mathematical and reasoning abilities, and they appeared to expect that eventually, they would be able to competently use the models they were constructing. Students regarded the work they were doing as a creative activity and believed that knowledge resided in their inscriptions, in their peers and in their teacher.

29.5 Findings

Although learning environment and teaching practice differed, the structure of whiteboard-mediated discourse across these four classrooms shared certain similarities. The social organization of the groups formed a cognitive architecture that determined how information flowed through the group (Hollan et al., 2000; Megowan, 2007). This gave rise to the existence of certain roles that channeled and structured this information flow.

Small group laboratory investigations: The small groups I observed engaged in laboratory activities invariably had a student, whom I will call the Operator, who took the lead in setting up and operating the apparatus. Often this was the same student in lab after lab. Another role I observed repeatedly was that of the Measurer. Again, this task often fell to the same student in each lab. She read the ruler, spring scale, or ammeter, sometimes in consultation with another student but often alone, she was typically the one who decided how many and at what intervals measurements would be taken. Another common role was that of the Data Manipulator. In the high school honors physics class this usually fell to the student who was most confident in the use of the software used for data analysis. At the community college, it was taken up by whatever student was not otherwise occupied. In the 9th grade and middle school classes, the Data Manipulator was almost always the same person each time. This individual frequently also served the group as the Measurer, and when it came to determining what should be written on their whiteboard, this person usually functioned as the Decider – the person who determined what would be written on the group's whiteboard. Sometimes the Decider controlled this process by writing the whiteboard himself, and other times he directed some other student who was acting as Scribe for the group, what to write.

Although small groups worked together, there were clearly leaders and followers. This correlated with the adoption of the roles mentioned in Table 29.1. As indicated, students often played multiple roles in their group, and there tended to be pairs of roles that were assumed by students, i.e., the Measurer often also acted as the Data Manipulator. However, leaders were rarely followers in the same activity.

Table 29.1 Students roles in small group laboratory activities (Megowan, 2007)

Title	Role	Leader or follower?	Proactive,reactive or passive?
Decider	Determines information content of whiteboard	Leader	Proactive
Operator	Manipulates apparatus	Leader	Proactive
Measurer	Uses/reads measuring tools	Either	Reactive
Scribe	Records information	Follower	Passive
Data Manipulator	Manipulates/represents data	Leader	Proactive or reactive

Small group problem solving: The dynamic of small group problem solving activities was somewhat different, as there were fewer distinct roles available. There were two different types of problem solving activities that routinely occurred: Going Over Homework, and Practicing with the Model.

In Going Over Homework the group member who assumed leadership was most often the person who determined what version of the solution appeared on the whiteboard. It was this Decider's approach to the solution that was written down and it was the decider that presented the solution to the whole group. Sometimes the Decider created the whiteboarded representation himself. Other times he directed some other student, acting as Scribe, what to write (Table 29.2)

Other members of the group contributed to the solution, but the Decider seemed to have the ultimate veto power over whether their contributions were recorded for sharing with the class. I have come to think of this phenomenon as the Power of the Marker. Controlling the marker was, for all practical purposes, controlling the floor in a whiteboard discussion. The Decider determined what counted out of all the contributions that group members made to the discussion, and he did this by either writing down or not writing down their contributions. If the Decider did not give sufficient credit to some teammate's suggestion, another role emerged – that of the Eraser – a person who simply erased what the Decider had caused to be written. Sometimes this resulted in a transfer of the marker to the Eraser, but more often it resulted in a re-creation of an inscription by the Decider that was more in line with what the Eraser or other members of the group (Discussers) had in mind.

Table 29.2 Students roles in small group laboratory activities (Megowan, 2007)

Title	Role	Leader or follower?	Proactive, reactive or passive
Decider	Decides what should appear on whiteboard	Leader	Proactive
Scribe	Writes on whiteboard	Follower (when this is not the same person as the Decider)	Reactive
Eraser	Erases what is shown on whiteboard	Leader (turn as leader begins with erasing the whiteboard)	Proactive
Discusser	Discusses what appears or should appear on the whiteboard	Follower	Passive

In Practicing with the Model, most of the same roles were seen, except for the role of Eraser. This was probably because in this activity structure, the role of Decider frequently changed hands over the course of the discussion. The older the students, the more often the role of Decider changed hands. Group members tended to participate in the discussion as co-equals rather than leaders and followers. In the honors physics and community college classes, in most Practicing with the Model exercises this was particularly true. In the middle school and 9th grade classes, there was usually a single individual who functioned as Decider throughout the episode.

Whole group presentations and board meetings: The activity structure of formal whiteboard presentations appeared to be fairly rigidly scripted in all the classrooms that used them. A group brought their board to the front of the room and one member, usually the one who had functioned as Decider, initiated the presentation. Often the speaker began by reading the text of the problem being presented. This is called External Text Dialogue (Lemke, 1990), where the text or worksheet itself is given a voice in the presentation. Then the Presentation began. The speaker's tone shifted audibly as they ceased acting as "the voice of the worksheet" and began speaking in their own voice. In a typical presentation the speaker identified the formula that they used, the pieces of known information that were used to solve the problem, the type of information that the Answer was, and then they gave a brief explanation of how the problem was solved – usually a description of the computation process. Sometimes they explained how they knew the various bits of information that they used, i.e., from a graph or from the answer to some other problem. They finished with the Answer, which was followed by the Pause, typically a 2–5 s interval in which Questioning was expected to begin. In whole group whiteboard presentations, the speaker was almost always looking at and talking to the teacher, in spite of the fact that the presentation was ostensibly for the class as a whole. This was true in all three classes that utilized this format for whiteboard sharing (the community college did not do formal whiteboard presentations).

Classes handled the Pause differently. The middle school students raised their hands and waited for the teacher to recognize them before they spoke. In the high school classes, students occasionally called out their questions and at other times raised their hands, but most of the time they sat silently, avoiding eye contact with the presenter or the teacher, apparently waiting for either the teacher to ask the questions or for him to permit the presenters to take their seats. Teacher questions, when there were any, followed the Pause and often took the form of a triadic dialogue – Question-Answer-Evaluation (Lemke, 1990) – in which the teacher would ask a question, the student would respond, and the teacher would offer some evaluation of their response. This triadic dialogue occasionally took on the appearance of "target practice", where the teacher had a particular answer in mind and kept rephrasing the question until the responding student gave him the answer he was looking for. In most cases, the signal to end the presentation came from the teacher, either by way of an expression of thanks to the presenters or directions to the class to "give this group a hand". Sometimes the students "stole" this signal from the teacher by beginning to clap during the Pause, before the teacher had a chance to take back the floor and move the discussion along. Once in a while, the teacher would override this by beginning the questioning but other times he would simply call up the next group of presenters.

Board meetings were reserved for occasions when the whole class worked on the same problem – usually a laboratory activity or Practicing with the Model exercise. These might open with an initial Presentation similar to the formal whiteboard presentation described above, or they might simply begin with some student asking another group a question about something shown on their board. The Pause was followed by a comparison of groups' boards with one another. In the middle school and the two high school classes, the teacher usually retained the floor throughout most of the discussion, calling on different groups to explain differences or justify what appeared on their board. In the community college class, the Pause was typically brief and student questioning of each other was spontaneous and often intense. At times students interrupted each other's explanations to ask for clarifications. This seemed to be more a function of classroom norms than of student maturity. Students knew that if they did not probe one another until they understood, the teacher would almost certainly corner them for an explanation that they could not give.

Implications of the various architectures: Participation, in the form of contribution to the reasoning or problem solving process, was constrained in all cases by the dominance of some person (sometimes a students but often the teacher) who was perceived to have the floor, and by whether or not that person was willing to give it up their control of the floor to someone else. In small group whiteboarding, it was the Decider.

The Decider took the lead in creating the presentation in the small group, and in making the presentation to the whole class. Unless otherwise specified, the Decider was most often the one who answered whatever questions were posed by teacher or classmates after the group presented. In the high school classes, the Decider was the group member that had the best grasp of the algebraic solution of the problem. They were expert equation writers and solvers and this was a skill that was valued by students. In whiteboard presentations and in small group whiteboard preparation, more time, more whiteboard space, and far more discourse was devoted to algebraic manipulation than to interpretation of diagrammatic, graphical or spatial representations. These were only relied upon to justify how a student knew something or where they got some bit of information used in their algebraic solution of their problem.

More time and attention were given to procedural concerns and algebraic reasoning in cases where students valued getting numeric answers over representing the problem space. This may have been a function of the types of problems they were assigned, i.e., if the problem asked for the acceleration of a 500 g block sliding down the frictionless surface of a 30 degree incline, students knew that their goal was a numeric value. When they found that value, they were "done." Problems of this type were standard fare in the middle school and high school classrooms where worksheets that had been assigned for homework were used as classroom whiteboard exercises. Paradigm labs were exceptions to this, but even these usually resulted in some numeric answer, i.e., "the slope of my graph, which demonstrates that the relationship between change in position and change in time in this situation is directly proportional, is 22 cm per second, and this represents the velocity of my battery powered car." With the exception of the community college students, whose whiteboard exercises most often resulted in diagrams and derivations of formulas rather than "answers", the students observed showed a strong preference for answers.

29.6 Conclusion

In general, the most egalitarian student-to-student discourse occurred in the course of board meetings about problem solving tasks that involved Practicing with the Model. These discussions were not dominated by a few Deciders, but involved most of the students in the class. Moreover, the students, rather than the teacher, tended to control who had the floor in these conversations. In answering the demand for justification of answers, students were more apt to rely upon spatial representations to justify and clarify their thinking, and share the ways in which they made sense of the problem space. Although these Practicing with the Model board meetings occurred most often in the community college class, they were also seen in the middle school classroom and therefore it is not reasonable to assume that they are dependent on students' age or mathematical sophistication. The richness of the discourse environment is more likely a function of how well the modeling tasks are designed, and the norms for student engagement that are established and enforced in the course of instruction.

Acknowledgment The research reported in this paper was supported, in part, by grants from the National Science Foundation (#0337795 and #0312038). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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Chapter 30
A Middle Grade Teacher's Guide to Model Eliciting Activities

Della R. Leavitt and Cynthia M. Ahn

Abstract This Teacher's Guide of implementation strategies for Model Eliciting Activities (MEAs) was developed through a collaboration between an 8th grade mathematics teacher and a university partner who served as one of the "clients" during four week-long MEAs in the 2006–2007 school year. The guide provides principles which may be useful to teachers and researchers who wish to conduct investigations of students' or teachers' actions during modeling activities in middle grades or other contexts. Neither the mathematics teacher nor this Guide's collaborator had had prior experience with MEAs. Thirty-two 8th grade students from a magnet school in an urban U.S. district took part in these activities. They were combined from two classes: one regular mathematics class and one special education class. The heart of this Guide comes from reflections by the regular mathematics teacher explored during a May, 2007 interview with the first author who was also one of the university partners. From these experiences, four strategic categories emerged: (1) Group composition, (2) Relevant MEA selection, (3) Teachers' roles during group work, and, (4) Culminating group presentations and individual written work. During the activities, both the regular and special education teacher observed active engagement by several students who previously rarely participated in mathematics class.

30.1 Introduction to the Teacher's Guide

A team composed of middle grade mathematics and special education teachers and university partners worked together to implement four Model Eliciting Activities (MEA's) in a combined 8th grade class of thirty-two regular and special education students during the 2006–2007 school year. This k-8 magnet school is located in an

D.R. Leavitt (✉)
University of Illinois, Chicago, IL, USA
e-mail: leavittd@uic.edu

C.M. Ahn (✉)
Thorp Scholastic Academy, Chicago, IL, USA
e-mail: cmchao@cps.k12.il.us

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_30,

urban Midwestern district in the United States where over 50% of the 800 students qualify for free or reduced lunch. The diverse student today is comprised of approximately 30% Latino/a, 30% White, 23% Black, 11% Asian, 1% Native American, and 5% unknown or other.

The end product of these efforts is this Teacher's Guide. There are four categories of recommendations included in this Guide: (1) Student group composition, (2) Selection of relevant MEAs, (3) Teacher's roles during group work, and (4) Culminating group presentations and students' written work. These guidelines are written with the understanding that they may be adapted to your individual environments. A key observation noted by both the regular and special education teachers in this blended class: several students who did not typically engage during mathematics class became active participants during many Model Eliciting Activities.

30.2 Definitions: Modeling and Modeling Cycles

At the core of a Model Eliciting Activity (MEA) is the model that students aim to create through "modeling". "Modeling" is the *process* where students construct a symbolic system also known as mathematical model that is described by a sequence of steps. The model is the students' translation of their interpretation of a real world dilemma posed within the MEA into a mathematical representation. Students' learning takes place during repeated "modeling cycles" where students work in groups to identify mathematical constructs to explain the situation described by the Model Eliciting Activity. Students' participation during modeling cycles often follows this pattern: (1) Expression of possible mathematical approaches, (2) Tests, and (3) Revision of students' approaches based on their tests, discussions, and interpretation of results. Additional revisions occur after they are exposed to other groups' models during the culminating presentations. Students repeat this cycle many times as they build their mathematical model.

A test for each model is to follow the descriptive steps to reproduce the students' conclusion. Generalizability is tested by using different data tables for the same Model Eliciting Activity.

For more information about Models, Modeling Processes and Model Eliciting Activities (MEAs), consult the References at the end of this chapter.

30.3 Differences Between Model Eliciting Activities and Traditional Word Problems

The modeling process where students create a mathematically-justifiable model to fit the Model Eliciting Activity is the key difference between an MEA and how students solve a standard, familiar "word problems."

In teaching traditional "word problems" many teachers follow three phases: (1) introduction to a set of desired mathematical skills, (2) teaching companion procedures, (3) students apply these procedures with the goal of finding the problem's solution. Traditional word problems may be based in real world contexts; however, different from MEAs, "word problems" typically define static assumptions or "Givens" to guide students to one definitive solution. Although there are often several ways to reach that solution, there is usually only one "right answer". Often an answer key accompanies the text.

Model Eliciting Activities (MEAs) differ from these traditional problems in several ways. Students usually wrestle with ambiguities when they approach the dilemma posed in the MEA. There is often a lengthy phase while they struggle to create constructs to fit their interpretation of the MEA dilemma. Students often discuss, paraphrase, or draw diagrams to try to make sense of the MEA to create the mathematical model that best describes their approach. Students must explain their model by a set of step-by-step instructions that will make sense when applied to a different data table for the same problem scenario.

Students' mathematical models are the end result of their cycles of interpretation. These are repeated cycles where students express ideas, test, revise and extend their interpretations. The ambiguity of the problem situation may lead to a variety of solutions; however, be aware that not every solution is valid. Students must show mathematical justifications for the model's constructs and interpretations.

30.4 Recommendations for Group Composition: New Leaders Emerge

The size of the student groups is important. If possible, keep each group's size to three or four students. This size more easily affords each student an opportunity to participate in the modeling process and at the same time make it less likely that one student will dominate the discussions. It is important to the development of a robust mathematical model that all students have a chance to argue their ideas and explore interpretations with the other group members. If the group is too large, some students may not participate fully. If students work only in pairs, they may generate fewer initial ideas leading to less debate and revisions during modeling cycles. Keep in mind that the total number of groups must not exceed the limit that the team of teachers and partners can reach during the activity phase.

Compose groups with students who have different strengths. Students with different mathematical talents and work styles will add to each group's effectiveness. Students must feel comfortable to speak up and to express their ideas, yet not be so far apart in their understandings that they cannot progress together toward creation of a working model. The exchanges between students during the initial sense-making phase are key to the development of comprehensive models.

The mix of students within each group is also important for effective working groups. For example, this Guide is based on the experiences of a mathematics teacher and a special education teacher who combined classes to work on four one-week Model Eliciting Activities in November, January, April and May. Before the activities began, the teachers ranked the thirty-two students with the goal to compose mixed ability groups. The students' rankings were based on their standardized test scores and the teachers' previous classroom observations.

Upon completion of each of the four MEAs, the two teachers met to revise the group assignments. Their activities mirrored a cycle much like the one described in this Guide. The teachers (1) discussed their observations of the students from the combined classes in the new MEA environment, (2) "tested" their placement of individual students by assessing students' participation and individual write-ups, and (3) revised the group assignments before the next MEA week.

One surprise to the teachers was the students who became leaders during Model Eliciting Activities. The teachers noticed that many students' type and level of involvement was markedly different from their participation styles in their usual mathematics classes. Several Special Education students emerged as leaders within their groups.

30.5 Recommendations for Selection of Relevant MEA'S

It is important to select an activity context that middle grade students find personally meaningful. Relevancy helps students to understand the end goals of the activity. Students are more likely to imaginatively express their ideas during the modeling process of the development of a mathematical model to fit a relevant problem context. Relevancy helps students to freely make sense of the problem, move swiftly to iterative testing and helps them to envision additional revisions.

In the second MEA of the school year, Bigfoot (see Appendix 3), students engaged in active participation. Students developed a variety of creative models after lively sessions where many students measured each other with the curiosity of middle school students. Students found many ways to relate feet, shoe sizes, and other body parts to proportionally estimate the height of the person who left the foot print.[1]

Conversely, Green Thumb Gardens, the third Model Eliciting Activity, did not seem to engage the urban students as much as did the two previous MEA's. Perhaps, the urban students were not familiar with the intricacies of lawn-mowing. Also, the Green Thumb Gardens accompanying photograph was not representative of many of the students' home neighborhoods (see Appendix 4).

Related warm-up activities before each activity began added important context for students to better understand the upcoming Model Eliciting Activity. Before the first MEA, Field Fitness Day (See Appendix 1), students ran races and measured their vertical jumps. From this work, many students noticed one aspect of this activity's complexity. Students saw that it is difficult to divide a class into "equal" teams when students may do well in some events, but not in others.

Another set of meaningful set of pre-work activities took place in science class. Here students designed and flew paper airplanes to model the requirements for

[1]Before the Bigfoot problem, the teachers anticipated that students might want to measure body parts. Teachers assigned students to single-gender groups, because they wanted to minimize middle grade students awkward feelings.

the fourth MEA, the Paper Airplane Competition (see Appendix 2). A follow-up suggestion might be for another middle grade class to use the students' newly-developed models to judge an actual paper airplane competition.

Record students' data from the pre-work activities in the same format as in the tables they will encounter in the Model Eliciting Activity. This may help to alleviate confusion in the interpretation of the MEA data table. For example, in the Paper Airplane activity (Appendix 2), many students were confused about the possible meanings of the "Distance from Target" table. This affects how they will interpret the data during model building.

All of the pre-work served the important function of team-building before the students came together in the intensive modeling process work. Team-building was particularly important for this blended class.

30.6 Recommendations: Teachers' Roles During Group Work

Read the Model Eliciting Activities many times before you introduce the activity to the students. Work with the actual data to become familiar with the many ways that the data could be used to address the dilemma posed by the MEA. This work will also help you to anticipate the mathematics needed for the paths that the students might explore. Also, this work will help you, the teacher, to understand any clarifications for your students, and for you to understand what will be expected from them.

Review the written materials of the Model Eliciting Activity with the whole class. In this class, the teacher read aloud the introduction of the MEA, and student volunteers took turns reading aloud the activity to the whole class. Next, students wrote individual answers to short comprehension "readiness questions". (See Appendix 3). It is important for students to have an overview of the whole problem to spark students' initial small group discussions. Be sure to watch for any confusion between the "readiness questions" and the scenario described in the MEA that the students' model must address. Minimize the time spent in a teacher-directed whole group discussion. This will allot more time for students to freely explore the problem and to begin to discuss their ideas.

The beginning of the cycle of model-making is critical. Students will test and revise their ideas several times before they arrive at their mathematical model. It is important that they have time to go through several cycles for a well-considered end-product. Be sure to remind students of the goal to write down each step that describes their model so that someone else may clearly reproduce their conclusion.

Try to anticipate challenges that are not substantive parts of the problem. It was challenging for the middle school students to translate their models to a sequential set of steps. Creating and testing a model that is generalizable to different data may also be difficult.

Be willing to change how you as a teacher respond to students during small group work. Keep in mind the differences between MEAs and traditional word problems. Resist guiding students toward one specific method. Students need to spend time discussing, debating, testing and revising their ideas to develop their team's model.

Students may spend less time with calculations during the Model Eliciting Activities than they do in traditional problems. More time might be spent debating how to approach and mathematically represent the dilemma posed by the Model Eliciting Activity.

Allow students to be frustrated with the complexity and ambiguity of how to use the data to create a meaningful model of the dilemma posed by the Model Eliciting Activity. Listen to students' explanations of their thinking. When teachers circulate to each group, ask students to explain their reasoning. It helps students to clarify or revise their thinking when they verbally express their ideas to roving teachers, clients or to fellow group members. If you determine that students are heading in an ill-defined or mathematically unjustifiable direction, do not directly tell them that they are wrong. Perhaps, saying "I'm not sure what you mean" may prompt students to re-interpret their ideas. Sometimes, a student might notice their own inconsistencies or get ideas from other group members for needed changes.

In an example from the Paper Airplane competition MEA (see Appendix 2), students were asked to give one team the "Most Accurate" award. Many students calculated averages both for the longest time in the air and for the closest distance to the target to use as selection criteria. Sometimes students computed averages without demonstrating reasoning. Here, a teacher might ask students to explain why they used the average in this context, or whether adding data measured in seconds together with data measured in meters makes sense.

Ideally, a teacher or university partner should only answer specific questions from the activity. Resist offering directions for how to apply the data. For example, it may be appropriate for a teacher to offer an answer to a student's question such as "What is a high jump event?" within the Field Fitness Day MEA (see Appendix 1). However, how the student group decides to use the high jump data to create equal teams should be explored exclusively by the group of students during modeling cycles.

30.7 Recommendations for Group Presentations and Individual Written Work

Offer a presentation rubric for student presentations. A rubric will help students to know what will be expected from their presentation. Some suggestions for rubric categories are: (1) Clear definition of the sequential steps to apply the model, (2) Presentation by each group member with demonstrated understanding, and (3) Generalizability of the group's model. Did the students show examples of tests with other data? Could other students apply the presenting group's model to reach similar conclusions?

When students create a group poster to present, circulate to each group, and ask questions that might be posed during the formal Question & Answer session. This will help students to prepare for the Question & Answer session while helping them to consider previously unaddressed aspects of their model. One goal of this advance questioning is to increase students' confidence as they prepare for the upcoming group presentation.

After the presentation, engage each group in a Q & A session where teachers, partners, and students ask questions about the presented model. When students respond to questions, they have the opportunity to elaborate their solutions in a less formal manner which may demonstrate understanding differently from what they showed in their prepared presentations. Encourage students in the "audience" to ask questions of the presenting group.

After the presentations, assign students to write an individual solution addressed to the partners who are posing as clients. This assignment allows students to think through the activity cycle and to make their own conclusions to model a solution to the MEA "dilemma". Other groups' presentations may contribute to students' understanding, and students may go through another "Express, Test, Revise" modeling cycle. Allocate approximately one hour of class time within a few days following the Model Eliciting Activity presentations for students to write their individual model descriptions. Display all groups' presentation posters during class time for students' individual write-ups. Give each student access to the notes and calculations that were kept safely in their team's folder. A key advantage of writing during class time, and not at home, is that students can view the classroom display of all of the groups' posters and put together ideas from many groups.

The "clients" responded to each student's write-up in the form of a memorandum. Return the clients' written replies to individual students in a timely manner. Allow time for class discussion of the clients' replies. Timeliness will help students to better recall their own write-ups when they consider the clients' suggestions. When you spend class time in discussion of the clients' responses, this might prompt students for yet another cycle of revision to their models.

30.8 Bringing Out the Best in Your Students

Teaching Model Eliciting Activities can be a rewarding experience with many benefits as students communicate ideas and engage in a model-building process to respond to real world dilemmas. The MEAs are different from teaching mathematics using "word problems". Many students are excited about these learning opportunities. You, too, might find surprises in the creativity and depth of students' work. These activities often bring out the best in a wide range of students.

Ackowledgments Thank you to the 8th grade students who enthusiastically participated in these Model Eliciting Activities. Our sincere thanks go to Mrs. Linda Hunt, school mathematics specialist and this project's manager, for her timely direction to keep the activities running smoothly. Thank you also to Dr. Keith Bowman for his measured insights. Essential to these efforts was Principal Kathleen Bandolik's support. We recognize the collaborating special education teacher, Ms. Clare Hourican, for her shared role in this work amidst her many teaching demands. And a special note of gratitude to Dr. Judith Zawojewski for her invitation to the first author to engage in this worthwhile project.

Appendix 1 Fun on the Field: Model Eliciting Activity

Problem:	Student	100 Meter Run	800 Meter Run	High jump	Fitness test*
Use the data here to develop a method to split the 6th grade class into three equal teams. Write a letter to the organizers of the Fitness Field Day explaining the method you used to divide the class. The organizers will use your method for other grade levels and for the annual local-level competition among all district schools, where they will need to divide a large number of players into equal teams.	Betsy	17.3 s	3 min 38 s	5’3’’	Pass
	Caroline	16.0 s	3 min 1 s	3’5’’	Fail
	Daniel	19.89 s	2 min 42 s	5’5’’	Pass
	Dick	18.52 s	2 min 55 s	4’4’’	Pass
	Jason	16.48 s	2 min 55 s	3’9’’	Pass
	Judi	17.2 s	3 min 22 s	3’6’’	Fail
	Lupita	20.2 s	4 min 0 s	5’0’’	Pass
	Mack	18.25 s	3 min 16 s	5’6’’	Pass
	Manuel	17.1 s	3 min 11 s	4’2’’	Fail
	Margret	20.32 s	2 min 51 s	5’7’’	Pass
	Michelle	16.44 s	2 min 45 s	4’5’’	Fail
	Rob	19.2 s	3 min 12 s	4’10’’	Fail
	Sandra	17.34 s	3 min 50 s	5’1’	Fail
	Scott	17.0 s	3 min 30 s	4’11’’	Pass
	Susan	18.3 s	3 min 0 s	5’3’’	Pass

*Students either passed or failed the fitness test, which included 30 push ups, 50 jumping jacks, and 20 sit-ups

Appendix 2 Paper Airplane Model Eliciting Activity

Beginner competition problem	Straight path

In past competitions, the judges have had problems deciding how to select a winner for each of the four awards (Most Accurate, Best Floater, Best Boomerang, and Best Overall). The judges don't know which measurements to consider from each path to determine who wins each award. Your team is being asked to consider how to use the measurements for the Beginner Competition only. Recall that the Beginner Competition involves only a straight throw; so Best Boomerang is not an option. Some sample data from last year will be provided. Write a memo to the judges of the contest. In your memo include procedures to determine Most Accurate, Best Floater, and Best Overall. Within the procedure, clearly state the reason for each step, heuristic (i.e. rule), or consideration in your procedure. Use your procedure and the sample data provided to determine the winners in each category. Make a note of the winners but do not include this in your memo.

Team	Amount of time in air (seconds)	Length of throw (meters)	Distance from target (meters)
Team 1	3.1	11	1.8
	0.1	1.5	8.7
	2.7	7.6	4.5
Team 2	3.8	10.9	1.7
	4.2	13.1	5.4
	1.7	3.4	8.1
Team 3	4.2	12.6	4.5
	5.1	14.9	6.7
	3.7	11.3	3.9
Team 4	2.3	7.3	3.3
	2.7	9.1	4.9
	0.2	1.6	9.1
Team 5	4.9	7.9	2.8
	2.5	10.8	1.7
	5.1	12.8	5.7

Sample data table for beginner competition

Appendix 3 The "Big Foot" Model Eliciting Activity

Monrovia – Someone who lives in this small rural Illinois town just outside of Chicago cares about the town's children. While most residents will do anything for the children, from volunteering for tutoring at the elementary school to umpiring the children's baseball league, one resident went beyond the call of duty by fixing and repainting the playground equipment. Where there is no problem in making sure the equipment is in good shape, the problem is finding out which resident took the time and resources to fix the equipment. Park District Superintendent Ruth Spears would like to thank the person. Spears said the park district would like to thank the person who replaced the chains on the swings, repaired missing or loose boards on the merry-go-round and repainted the monkey bars and swings. "They didn't have to do that, but we thank them and we would like to honor them," Spears said.

Monrovia town marshal Samuel Rose first noticed the new park equipment when he and police officer Reggie Perkins were called to the town park around 11 p.m. on Monday when neighboring residents reported funny noises, such as rattling chains, nailing of wood and other sounds associated with repairs. "We saw the subject's shadow as he was leaving the park," said Rose. "We did not get a clear look at the person, so we have no idea what the person looked like." Officer Perkins, upon further investigation of the repaired and repainted equipment, noticed footprints around the swings, the merry-go-round and the monkey bars. "That is really the only clue we have to go by at this point in the identification process," said Perkins. Rose, along with Spears, said that no charges would be filed against the Good Samaritan. "We're just happy we have a resident who is willing to help the community," said Rose. "I just wish we knew who they were."

Readiness questions:

1. What is the problem in Monrovia?
2. Who is Ruth Spears?
3. Why does Ruth Spears want to contact this mysterious person?
4. Who first noticed the new park equipment?
5. What evidence did the mysterious person leave behind?
6. How can Ruth Spears identify this person?

The Problem: Sometime late last night, some nice person rebuilt the old brick drinking fountain in the park. The mayor wants to thank the person who did it. But nobody saw who it was. All the police could find were lots of footprints. One of the footprints is shown here. [Note: Students were given a cutout of a 22 inch footprint]

The person who made this footprint seems to be very big. But, to find this person, we need to figure out how big he or she probably is. Your job is to make a "HOW TO" direction sheet (i.e., a set of step-by-step directions) that the police can use to figure out how big people are just by looking at their footprints. Your direction sheet should work for footprints like the one that is shown here, but it also should work for other footprints.

Appendix 4
Photograph accompanying Tom Thumb Gardens Model Eliciting Activity

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Chapter 31
The Students' Discussions in the Modeling Environment

Joneia Cerqueira Barbosa

Abstract This paper describes a framework to analyse students' practices from a discursive point of view for mathematical modelling environments. The theoretical notion of interaction spaces is taken as the main analysis unit. Then, the students' actions are described as modelling routes and parallel discussions. The first is composed over mathematical, technological and reflexive discussions. These notions are presented and brought together to outline the framework.

31.1 Introduction

This paper summarises results from several previous studies (Barbosa, 2006a, b, 2007a, b, c). One way to conceptualise mathematical modelling is as a learning environment in which students are invited to investigate situations from daily life and other sciences by using mathematics (Barbosa, 2003, 2006b). Sometimes I one use the name "modelling" in this paper to avoid repetitions. The above concept puts emphasis on two aspects: the task should be taken from day-to-day or other disciplines, and students don't have previous schemes to solve the problem. However, the concept of modelling doesn't provide all necessary information to understand the student's practices in ways that support teachers in their classrooms.

Inspired in the discursive sociocultural approach for human action, the notions of interaction spaces, modelling routes, and mathematical, technological, reflexive and parallel discussions are presented and linked. Some short classroom episodes are going to serve as illustrative.

J.C. Barbosa (✉)
State University of Feira De Santana, Feira De Santana, Brazil
e-mail: joneicb@uol.com.br

This paper is part of a wider project entitled "Mathematical modelling and the students' discussions", carried out by NUPEMM Research Group on Mathematical Modelling. Home: http://www.uefs.br/nupemm. A reviewed version of this paper is published in Portuguese as: Barbosa (2007).

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_31, © Springer Science+Business Media, LLC 2010

31.2 A Discursive Sociocultural Perspective for the Students' Practices

The starting point to be approached is a social practice that refers to the actions developed by the people engaged in a modelling environment (in this case, students and the teacher). Then, this is taken as a basic unit for analyzing the sort of mathematical modelling that happens in classrooms.

When students are engaged in mathematical modelling tasks, different things might be done such as arithmetical operations, equation solving, drawings and graphs, and mainly discourse production. Students usually do it organised in groups by the teacher (Antonius et al., 2007); and, actions are carried out through mediational means such as pencil and paper, calculator or computer, or language. In fact, "the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of 'individual(s)-acting-with-mediational-means' rather than speaking simply of 'individual(s)'" (Wertsch, 1993, p. 12). For instance, when an individual carries out the action of computing an arithmetical operation with pencil and paper, this person is limited by the constraints and the affordances that pencil and paper medium provide. Analogously, we are able to produce a discourse just by considering the constraints and affordances that spoken language makes available.

My argument is that a way to understand the students' modelling is to focus on the discourses produced by them while they are developing models. I am going to consider the classroom as a discursive space in which some voices are positioned, but some are more privileged than others. These voices, or multivoice (Wertsch, 1993), include each individual voice.

The concept of appropriation used by Wertsch (1998) captures the process of taking other voices to constitute an individual voice. The author recalls the origin of the word in Russian, *prisvoenie*, from the verb *prisvoit*, which means "something like the process of making something one's own" (p. 53). This notion emphasizes the social nature of discourse. The point is that the discourse produced by students and teachers during their social interactions are crucial to understanding the modelling actions, because these discourses are constituted by those voices considered as legitimate at that context. In Barbosa (2007b), the notion of *interaction spaces* has also been proposed to express such aspects in modelling environments. It refers to the encounters among students and teacher or students by themselves to discuss modelling tasks. Analysing interaction spaces allow us gain sense about how the rules of action are working at any context.

In this section, I have described the approach of analysing students' practices in terms of discourses; and, interaction spaces have been proposed as the unit of analysis. Next I examine this idea in greater detail.

31.3 Students' Modelling Routes

As it is not possible to anticipate many aspects of students' actions in a modelling environment, I will be more descriptive than prescriptive in this study. Borromeo Ferri (2006) seems to share this argument by providing the notion of *modelling routes* to name internal and external processes carried out by students in modelling.

Ferri's theoretical notion will be considered in a slightly different manner in order to be compatible with the discursive socio-cultural approach. I understand modelling routes as a sequence of discourses produced by students and the teacher during the process of building mathematical models; and, I emphasize the idea that all discourse is more an action than an expression of a cognitive content. From a methodological view, discourses are not the data with which to infer what is happening in the mind, instead they are the focus for analysis.

These considerations brought me to investigate discursive practices as something carried out using available mediational means. In other words, I analyze what students say when engaged in modelling environments, and why they say what they say. To try to understand these issues, I have associated the idea of modelling routes to mathematical, technological, and reflexive discussions. This approach was developed during previous empirical studies about students' modelling practices (Barbosa, 2006a, b); and, it was inspired in Skovsmose's (1990) approach to modelling.

Mathematical discussions refer to concepts and ideas from pure mathematics; and technological discussions refer to the process of representing any situation into mathematical terms. In turn, reflexive discussions refer to connections between the assumptions used and the final mathematical model.

In order to illustrate these notions better, consider two pre-service mathematics teachers, Carlos and Rui, both students at the State University of Feira de Santana, Brazil. They were invited to analyse a rise in fares for public buses from R$ 1.40 to R$ 1.50 (Real – R$ – is the Brazilian currency), which generated strong protests among the citizens at that time. The task was to discuss the impact of these increases on family expenditures. The discussion produced by Carlos and Rui was recorded and transcribed. At starting to run the task, they made some assumptions:

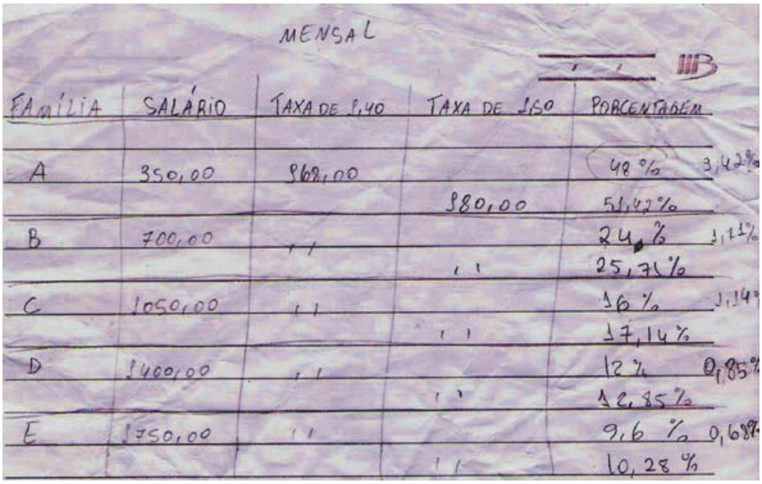
Carlos: The minimum wage is R$ 350.00.
Rui: This is the family [writing on the paper that it has 4 members, the parents and two children]. Let's verify how much they spent when the fare was R$ 1.40, and now that it is R$ 1.50.
Carlos: Let's assume they usually work from Monday to Saturday.
Rui: No, let's consider from Monday to Friday [for the parents], it's easier. The two children go to school in the same period, from Monday to Friday.

Carlos: Sure. They buy one bus ticket to go to work or school, and another to come back home.

Rui: Perfect! Each one would buy 10 tickets a week... so 40 tickets per person...equal 160 tickets a month.

The students were making these assumptions based on an ideal family with 4 members that would buy 2 bus tickets a day. Their plan was to compare expenditure before and after the fare went up. In this discussion, they structured the situation in order to represent it into mathematical terms. This refers to the technological discussion as defined above.

Later, Carlos and Rui made a table (Fig. 31.1) in which listed 5 family members' earnings each month and the expenditures on public transportation per month.



MENSAL

FAMÍLIA	SALÁRIO	TAXA DE 1,40	TAXA DE 1,50	PORCENTAGEM	
A	350,00	168,00		48 %	3,42%
			180,00	51,42 %	
B	700,00	"		24 %	1,71%
			"	25,71 %	
C	1050,00	"		16 %	1,14%
			"	17,14 %	
D	1400,00	"		12 %	0,85%
			"	12,85 %	
E	1750,00	"		9,6 %	0,68%
			"	10,28 %	

Fig. 31.1 Table by Carlos and Rui

Considering the table that follows, the students represented family expenditures by using the expression 350x, in which x refers to the number of minimum wages. Next, they defined the impact f(x) as the ratio between the total increase in fares monthly – R$ 12.00 – on family earnings: $f(x)=12/350x$. Then, Carlos and Rui decided to draw a function graph:

Carlos: As x is denominator, the graph is not a line.

Rui: If there was only the x so we may trace a hyperbole [drawing a hyperbole with the finger in the air].

Carlos: I see... we could consider the table values as points, and to mark them on the Cartesian Plan [drawing the Cartesian Plan on his notebook]

Rui: I think this is a hyperbole right now.

In this episode, the students were discussing about drawing a function graph, which is a pure mathematics subject. This sort of discussion is called mathematical. Later, all student groups were discussing their strategies for the problem with the teacher. Carlos and Rui noticed that other groups had found different solutions because they had made different assumptions:

Rui: Your solution presents a bigger impact because you considered a family with 5 people [talking to a colleague].
Teacher: Which is the right?
Rui: . . . depends on the family.
Carlos: We considered a family with 4 people, each one spending 2 tickets a day. . . so we have found R$ 12.00 and this graph.

In this episode, Carlos and Rui analysed the link between the assumptions they had made for the situation and their mathematical model, which characterises the so-called reflexive discussions.

The three types of discussions – mathematical, technological and reflexive – play a clear role in building a mathematical model. This doesn't mean that students always produce such discussions. For example, in Barbosa (2006a), a classroom episode is presented in which students do not carry out reflexive discussions. Analogously, it is possible that mathematical discussions don't appear in the classroom, mainly when students are more focalized in qualitative aspects such as described in Maaβ (2006).

If students produce three types of different discussions, then there are transitions among them (see Fig. 31.2). For example, in Barbosa (2006a), I analysed what makes students change the type of discussion; and, "blockages" were identified as a possible motivation. The example that was described in that paper came from a discussion among students about alternative ways of modelling a phenomenon that involves 9 variables. The students were trying to draw a graph. As soon as they realized that they could not represent 9 variables in a graph, they moved their attention to pure mathematical aspects of such job.

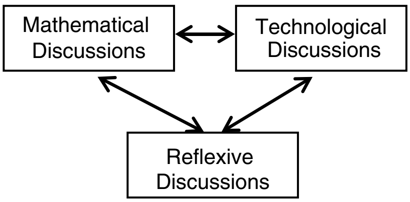


Fig. 31.2 The types of discussions that might compose the modelling routes

Then, discussions produce heterogeneity of voices in the modelling environment; and, from a didactical point of view, different goals for modelling privilege different discussions (Barbosa, 2007c) such as shown in the Table 31.1. The notion of privileging refers to the fact that some languages are considered more appropriated than others in different sociocultural contexts (Wertsch, 1993).

The idea of privileging doesn't make other discussions impossible; however, it does position them as relatively less important. Privileged discussions constitute, at the microsocial level, the general perspectives for Modelling. For example, the

Table. 31.1 Links between Goal for Modelling and Privileged Types of Discussions

Goal for mathematical modelling	Discussion privileged
To develop some mathematical concept	Mathematical
To promote skills and competences of modelling real problems	Technological
To analyse the nature of mathematical models and their uses in society	Reflexive

perspectives summarised by Kaiser and Sriraman (2006) might be seen in action during the discussions produced by students and the teacher. This point cannot be expanded here, but the reader may check out more in Barbosa (2007c).

31.4 Parallel Discussions

In a recent study (Barbosa, 2007a), I realized that students often produce discussions that cannot be classified neither mathematical, nor technological nor reflexive, because they have no clear role in building the mathematical model. For example, consider an episode in which Carlos and Rui considered the impact of raising bus fares.

Carlos: Sure. They buy one bus ticket to go to work or school, and another to come back home.
Rui: Perfect! Each one would buy 10 tickets a week... so 40 tickets per person...equal 160 tickets a month.
Carlos: My God, the wage is not enough!
Rui: Ok, 2 tickets a day [making annotations on the notebook]

Carlos' second utterance suggests the family expenditure is on rise. This is an important commentary, but its role is not clear in building the mathematical model. Rui and Carlos did not consider it for their problem solving strategy. If Carlos did not produce such utterance, the resulting model would be the same. I mean they did not take it in consideration as long as they were approach the situation. I called cases like this one as parallel discussions (Barbosa, 2007a).

Other types of parallel discussions are reported in Barbosa (op cit.). For instance, students might get interested in discussing mathematical or algorithmic ideas as a result of being engaged in a modelling task. Also, as illustrated above, modelling tasks might give rise to discussions about certain social situations.

31.5 Drawing a Framework

Some of the notions presented in this paper have been examined partially in previous papers (Barbosa, 2006a, b, 20072007a, b, c) in which particular aspects are addressed and detailed. However, this paper reported all of them together as part of attempts in outlining a framework to have used discursive terms to describe students' practices in mathematical analyse the students' modelling. In particular, I have used discursive terms to describe students' practices in modelling environments. Inspired

In the discursive sociocultural approach for human action, the initial point was the notion of *interaction spaces*. The notion of modelling routes proposed by Borromeo Ferri (2006) was adopted to name the discussions that play a clear role in building the mathematical models at school context, and the discussions are categorized in mathematical, technological and reflexive. The other sorts of discussions are called *parallel discussions*. By establishing relations among these theoretical notions, I am able to outline a framework that structures the students' modelling in order to support future researches. The Fig. 31.3 represents a schema for these notions.

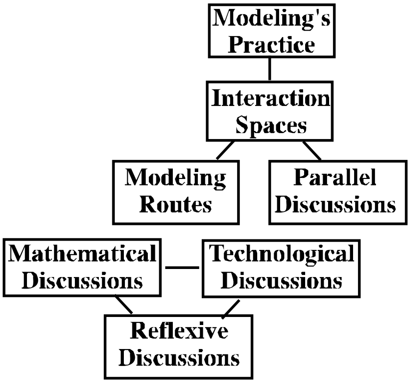


Fig. 31.3 Schema describing students' practices in modelling environments

Acknowledgements Although they are not responsible for arguments outlined here, I thank my colleagues from NUPEMM Research Group on Mathematical Modelling at the State University of Feira de Santana for comments to the previous versions of this paper. I thank the Research Foundation at the State of Bahia, Brazil (in Portuguese, Fundação de Amparo à Pesquisa do Estado da Bahia – FAPESB), for financial support to this research project (Number PPP0031/2006).

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Chapter 32
The Social Organization of a Middle School Mathematics Group Discussion

William Zahner and Judit Moschkovich

Abstract This study analyzes patterns of interaction among bilingual middle school students while they engaged in peer mathematical discussions. Using a socio-cultural lens on learning the practice of school mathematics, this study addresses three questions: (1) What kind of mathematical discourse practices did the students engage in? (2) What discourse patterns emerged during these mathematics conversations? (3) What are the implicit "rules" that appear shape students' interactions, what are the pragmatic implications of these rules, and which students benefit from these rules? Using conversation analysis and discourse analysis we show that the students primarily engaged in "calculational" conversations, that mathematics conversations followed rules distinct from the rules of everyday conversations, and "intellectual authority" emerged as an important construct for understanding students' mathematical discourse practices.

32.1 Introduction

Learning to communicate about mathematics and through mathematics is emphasized as a key element of developing mathematical competence in schools (National Council of Teachers of Mathematics, 1989, 2000). One popular tool for teaching mathematical communication among peers in a mathematics classroom is the use of small group activities. Researchers have documented the benefits associated with small-group discussions among peers in terms of the development of problem-solving skills (Mercer, 2005) and also in terms of the development of participation in mathematical discourse practices (Yackel et al., 1991; Yackel et al.,

W. Zahner (✉)
University of California, Santa Cruz, CA, USA
e-mail: wzahner@ucsc.edu

J. Moschkovich (✉)
University of California, Santa Cruz, CA, USA
e-mail: jmoschko@ucsc.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_32,

1990). This study analyzes patterns of interaction among a small group of bilingual middle school students while they engaged in peer mathematical discussions, and proposes a model for analyzing regularities in students' peer mathematics discussions. We conjecture that mathematics discussions in school are structured by implicit "rules[1]," and that these rules are distinct from the norms of everyday, or casual conversations (Eggins and Slade, 1997). Sociolinguists have documented some of the implicit social norms that govern the multiple elements of a conversation including turn-taking (Sacks et al., 1974), norms of politeness (Lakoff, 1973), and cooperation (Grice, 1975). Researchers such as Linde situated in the Conversation Analysis (CA) tradition have shown that specialized conversations – such as cockpit conversations between two police officers in a police helicopter – have specialized rules governing interaction among participants, and these rules reveal institutional structures that frame the interactions (Linde, 1988). Through this study, we seek to broaden our understanding of the regularities in a school mathematics discussion among peers, and we show that a peer discussion in school mathematics is a hybrid conversation that draws upon norms for everyday conversations as well as the discourse of school mathematics. This, in turn will broaden our understanding of how students manage the task of communicating while solving mathematics problems in collaborative groups.

32.2 Literature Review & Theoretical Framework

Prior research in mathematics education has called attention to the benefits of mathematics discussions in both whole class (Goos, 2004; Lampert, 1990) and small group settings (Yackel et al., 1991). For small-group discussions in mathematics classrooms there is extensive research that analyzes both the content and form of students' peer discussions in dyads (Forman and Larreamendy-Joerns, 1995; Moschkovich, 1996, 1998; Sfard and Kieran, 2001; Yackel et al., 1991). These studies have greatly expanded our understanding of how peer discussions mediate students' learning of mathematics. However with the exception of a few studies (Jurow, 2005; Pirie, 1991), analyses of students' mathematics discussions in small groups with more than two students located in typical classroom interactions have not received much attention. Studies such as this one, which employ an ethnomethodological approach, will enhance our understanding the nature of students' mathematical discussions within typical classroom settings.

This study of students' interactions during small group learning is situated in a sociocultural theoretical framework. Within this tradition we treat learning as a process of appropriating (Rogoff, 1990) the meanings and perspectives enacted by

[1] We am following Rowland's (2000) use of "rules" to refer to the regularities of interaction that point to a collection of implicit norms of behavior and speech that members of a speech community use while engaging in interactions. This is not meant to imply these rules are mandatory (in fact they are often made visible when they are broken) or that these rules are to be interpreted as normative judgments of what students *ought* to do.

members of the learning community (Moschkovich, 1996, 2004). Therefore, learning is taken to be an active process, and the corresponding emphasis in analysis is on activities such as "knowing" and "representing" rather than static entities such as "knowledge" or "representations" (Sfard and McClain, 2002). Furthermore this orientation highlights that roles (such as intellectual authority or "teacher") are dynamic and continually re-negotiated through discourse.

This study investigates the following questions: (1) What kind of mathematical discourse practices (Moschkovich, 2007) did the students engaged in?; (2) What discourse patterns did students use while engaging in a mathematics conversation, and how are these similar to or different from the discourses of casual conversation? (3) What are the implicit "rules" that shape students' interactions, what are the pragmatic implications of these rules, and who appears to benefit from these rules?

32.3 Participants and Setting

To answer these questions, we gathered data for this study by video-recording one group of seven students in a dual-immersion bilingual sixth grade classroom in California for approximately one month. The class used a standard textbook (Maletsky, 2002), and the teacher followed the prescribed curriculum and state standards for sixth grade mathematics. The number of students sitting in the focus group ranged from three to seven. For three class days in one week we videotaped all of the group's interactions. Given our orientation toward conversation analysis, our data collection reflected an ethnomethodlogical approach (Schiffrin, 1994). We sought to capture "everyday" student discourse in a naturalistic setting and we made efforts to not disrupt the "normal" flow of activities within the school setting (so, for example, we did not alter the type of prompts the students worked on). After taping, we made content logs for all 7 h of video we recorded, and then we selected the sections in which the students were engaging in sustained mathematical discussions (Pirie, 1991) for closer analysis. We then transcribed those sections of the video (approximately 56 min) in detail using conventions from conversation analysis. Besides the video and transcripts, our other data sources include ethnographic field notes, a semi-structured interview with the teacher, a focus-group interview with the students in the group, and a copy of the students' written work for the week.

32.4 Findings

Three preliminary findings have emerged from this study. First, the students' group discourse practices reflected the discourse practices commonly associated with school mathematics. Specifically, the students engaged in discussions that generally reflected the "calculation(al)" orientation of the prompts they discussed. Second, the students' discussions exhibited patterns of interaction distinct from interactional patterns that researchers have documented in casual conversations. Finally, "intellectual authority" in the group was a dynamic construct that appeared to mediate the interaction. The construction of intellectual authority was subject to negotiation during a discussion, and the role of "intellectual authority" is reflected in sociolinguistic aspects of the students' discussions. Below, we will elaborate on these findings and we will illustrate them with examples.

Calculational Orientation. The students engaged in mathematics discussions that reflect the patterns of "typical" school mathematics (Moschkovich, 2007). Further, the types of questions the students answered appeared to shape the type of mathematical discourse practices in which they engaged. A majority of the prompts for the students' group work were "exercises" (Schoenfeld, 1992), and the students used a subset of school mathematics discourse (Moschkovich, 2007) reflecting a "calculational" orientation (Thompson et al., 1994) while they engaged in their discussions. For example, one prompt asked students to consider the problem, "If six pairs of socks cost $4.50, how much will 9 pairs cost?" The text then listed four answer choices: "F $6.75 G $7.25 H $9.00 J Not Here." (p. 393 Exercise #4) Given this prompt, the students engaged in the following discussion:[2]

(1) [Monday, 45:05–45:31]

1. Lorenzo A:	((Amber walks around the table and stands to the right of Lorenzo A. who is writing on a shared paper)) How would I do?
...	
4. Amber:	Ah a um let me see
5. Lorenzo A:	Number four
6. Amber:	Number four (for the) (1) then you put (2) nine up xxx equals nine (3.5)=
...	
8. Amber:	=Now put a:: number-a letter (1) alright, and then, let's se[e:: six
9. Francisco:	[Oh my God, you guys have the problem up there ((points to blackboard))
10. Lorenzo A:	Six divided by (four-fifty)?

In this excerpt, Amber walked Lorenzo through step-by-step instructions on how to set-up the problem using equivalent ratios, and these instructions mirror the steps for setting up a proportion:

$$\frac{9}{x} = \frac{6}{4.50}$$

[2]Transcripts follow Jefferson's conventions as detailed in Schiffrin (1994) with minor revisions. The... symbol denotes one or more turns omitted from this presentation because those turns refer to a side conversation that occurred in parallel to the focal interaction. Also, English translations of utterances in Spanish are give in double parenthesis and quotations ((" ")).

In line 8 Francisco interrupted Amber's step-by-step instructions to tell Lorenzo and Amber that the solution to the problem was displayed on a poster on the wall. This interruption illustrates that the students perceived the goal of this activity to be finding the answer. Prior to Francisco' interruption, however, Amber responded to Lorenzo's original question "How would I do [the problem]" (line 1) with detailed step-by-step instructions for setting up a proportion. As Amber gave directions to Lorenzo she did not elaborate a rationale for each step, and she demonstrated to Lorenzo the mechanical steps necessary to complete the problem.

Throughout the data set, the students' mathematical discourse commonly focused on the computational steps necessary to arrive at an answer. The students frequently answered peers' questions and requests for help with calculational responses (this is how you. . .), rather than conceptual answers (this is why you. . .). This observation is not meant to impugn the mathematical practices or level of competence of the students in the focus group. Rather, their use of a calculational orientation is simply a demonstration of the type of mathematical discourse practices in which the students engaged in this setting in response to the given prompts, all of which might be characterized as "exercises." As Webb et al. (2006) argue, the students use the discourse practices that they have appropriated from their experiences of school mathematics. Furthermore, the prompt for this specific discussion, a multiple choice question, explicitly highlights the importance of finding the correct answer by doing the steps and then choosing the correct answer choice.

Distinctions between mathematical discussions and everyday conversation. A second finding is that several interactional regularities emerged in the students' discussion, and these patterns are distinct from patterns of interaction that sociolinguists have documented in everyday conversations (Eggins and Slade, 1997). Three such patterns unique to these mathematical discussions are: (1) verbalized inner speech, (2) the frequent use of direct contradiction or correction, and (3) the joint construction of meaning by two or more people verbalizing calculations in fragments while focusing on a shared artifact. This third point is illustrated in following excerpt where four students collaboratively do the operation 6 divided by 8 using long division:

(2) [Wednesday, session 1, 25:20–25:55]

1. Francisco:	ah- why six over eight seventy five percent?
2. Claudia:	((bobbing head slowly)) Divide it.
3. Amber:	OK watch. Six goes inside divide by eight [xxx=
4. Claudia:	[Por eso 'ira (("that's why, look"))
5. Amber:	=and [that . . .
6. Claudia:	goes in the casita (("little house")), you take out eight you put a decimal porque no se puede (("because you cannot")) [put a decim- you put a zero=
7. Amber:	[Uh-huh]
8. Claudia:	=ocho por siete (("eight times seven")) put a seven here it's fifty-six

9....
10. Dennis: four=
11. Francisco: =(and that's) four
12. Dennis: [[four
13. Claudia: [[five it's ten is four
14. Francisco: Oh yeah, seventy five percent
15. Claudia: y (luego) ((“and then”)) [xxx zero
16. Dennis: [Zero then five
17. Claudia: y luego es ((“and then it is”)) seventy five percent

Once again, looking carefully at the calculations the students are verbalizing may help clarify what they are saying:

$$\begin{array}{r} 0.75 \\ 8\overline{)6.00} \\ \underline{56\downarrow} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

There are two places in this excerpt where the students jointly constructed next steps in the algorithm as Claudia worked through it. In lines 10, 11, and 12, both Dennis and Francisco contribute to Claudia's implementation of the long division algorithm by supplying the four after she sets up 60-56. Interestingly, despite the fact that Dennis and Francisco have given her the solution to the subtraction, Claudia insists on verbalizing the intermediate “borrow” step necessary to do this subtraction in line 13. In line 16 Dennis again supplied a number, and this time Claudia appeared to pick it up his suggestion without the need to re-state the operation as she did in line 13.

Throughout the body of data, interactions like this are common, especially while the students maintain a joint focus of attention (Moschkovich, 2004) on the same problem or inscriptions. While working through exercises on a worksheet, or while jointly writing the group's solution to a prompt, the students were frequently able to interject the next step of the operation, or to verbalize calculations without noticeably disrupting the flow of the jointly constructed conversation. Such fluid constructions are not always present, however. When the students do not maintain a shared focus, or when they do not write out what they are trying to explain their interactions can appear disjointed. In the following excerpt, Francisco asks Amber for help solving the problem “At 6:00 AM the temperature was –4 F. By 6:00 PM, the temperature had risen 17°. What was the temperature at 6:00 PM?” (p. 393, Exercise #8). Here the students' adjacent statements do not sound coherent, and the lack of a common inscription for a point of reference may be part of the reason their utterances appear so disjointed.

(3) [Monday class 44:27–44:56]

1. Francisco:	But who ge-Do you get number eight at all?
...	
2. Amber:	Seventeen minus six (.) Du:h
3. Francisco:	Oye ((“listen” or an interjection)), I found it already
...	
4. Francisco:	Look. Subtract ss seventeen minus negative four [xxx
5. Amber:	[Thats what I said
8. Francisco:	You said subtract thirteen!
9. Amber:	No, I said seventeen!
10. Teacher:	They’re almost done ((addressing class))
11. Amber:	(minus) negative four
12. Francisco:	Now I get [it
13. Lorenzo A.:	Ahhh
14. Amber:	I got number eight ((looking up at teacher))
15. Francisco:	I got number eight too, but (kinda) she’s helped me

The construction of intellectual authority within the group. A final regularity that emerged from this data is that students’ participation in collaborative mathematical discussions revealed a dynamic where the students continually constructed and negotiated who (or what) served as an “intellectual authority” (Lampert, 1990) for the group. One result of our theoretical orientation – which highlights the social construction of knowing – is that no individual student can “have” (or lack) intellectual authority. Rather, the students continually (re)construct intellectual authority based on their active interpretation of the goals and meanings within the activity setting.

At some times the negotiation of intellectual authority was explicit. For example, during the first session on Wednesday, prior to the interaction detailed in excerpt (1) above, Amber and Claudia had a conflict when they had different answers to the same question. Ultimately Claudia convinced her group-mates of the correctness of her solution (through an appeal to the teacher’s authority), and after that point, the students treated Claudia as an authority for most of this session. In lines 2–6 of except (1) we see how a dispute over intellectual authority was enacted when Amber and Claudia both began to answer Francisco’s question from line 1. Though Amber began answering the question, Claudia interrupted her and talked through Amber’s explanation. By line 7, Amber ceded the “floor” to Claudia, and her back-channeling agreement (“uh-huh”) appears to affirm both Claudia’s solution as well as her right to give a solution. Claudia also appears to guard her position as an authority in this exchange by ignoring Francisco and Dennis’s quick answers to 60-56, and she carries out the subtraction algorithm with all of its steps.

At other times the negotiation of intellectual authority occurred implicitly, and it is a difficult construct to identify precisely. One way to identify a source of intellectual authority may be to analyze who is asked questions about mathematics. (This presumes that the students are not asking each other “known-answer” questions). Table 32.1 shows that with the exception of the first Wednesday session, Amber

Table 32.1 Number of questions about mathematics asked to each student during a group discussion

	Mon.	Wed. 1	Wed. 2	Fri.	Total
Amber	16	4	22	5	47
Claudia[a]	–	7	6	1	14
Dennis	3	0	1	0	4
Francisco	2	4	2	5	13
Joaquin	0	1	0	0	1
Lorenzo A.	0	0	0	1	1
Lorenzo Y.[b]	0	0	0	0	0
Teacher[c]	5	5	0	9	19
Totals	26	21	31	21	99

Note there were two separate math sessions on Wednesday
[a]Claudia did not participate in the group during Monday's session
[b]Lorenzo Y. missed a significant portion of the Monday and Wednesday 1 sessions
[c]The teacher only joined the group on an occasional basis

was asked significantly more questions about mathematics than all of the other students combined. What is also striking is the extent to which the students tried to rely on the teacher as an intellectual authority as well. Though the teacher only checked in with the group occasionally, they tended to ask her many questions. Also, during the "W2" session, Amber asked questions of the teacher off-camera, and that is why the number of questions to the teacher is coded as 0 in the table.

In addition to being constructed through the students' discourse, the role of intellectual authority also appears to have an impact on the sociolinguistic characteristics the peer discussion. For example, during excerpt (3) above, Amber is positioned in the role of the intellectual authority, and the shape of her interaction with Francisco reflects her assumed role. In her responses to Francisco's question, Amber constructs her authority first by providing Francisco with a very quick procedure to arrive at the answer (line 3). Second, she passes a judgment on the quality of the question through her use of the tag "Du:h". In both tone (with the elongated syllable) and content, this tag serves the purposes of making Francisco's question seem silly, and reifies the "unassailable" status of her answer. Francisco's response in line 6 is a challenge to Amber's intellectual authority, but it is interrupted by Amber, and again she uses an elongated syllable to make her opinion of the quality of his contribution known. In line 8 Francisco contradicts Amber, but she forcefully re-asserts her dominant position in lines 9 and 11. Perhaps the most revealing part of this exchange is in line 15, where Francis ceded authority to Amber, and he tells the teacher that he got the answer, but with Amber's help (and all of this despite the fact that he had the answer in line 6).

32.5 Discussion

Below we will highlight three possible implications of this work: (1) the importance of using of shared artifacts to build shared focus of attention during a group discussion, (2) the need for teachers and students to explicitly recognize the differences between a mathematical discussion and an everyday conversation, and (3) the need to address the emergent role of intellectual authority as a mediator of students' mathematical discussions.

This group of students' discussions while they worked on the exercises in the book and from worksheets varied considerably in terms of their coherence and their focus. For example, contrast the coherence of the conversation in excerpts (2) and (3) above. When the students appeared to have a unified joint focus of attention, their conversations exhibited far more coherence, and there is more evidence of intersubjectivity among the conversational participants. When students did not share a focusing artifact, the conversation quickly degenerated into apparently nonsensical talk, or into several concurrent side conversations. Therefore both an effective pedagogical tool and methodological tool may be the use of a single shared paper for the recording of group responses to shared prompts. This configuration appeared to create the most coherent group discussions, and the students were forced to respond to one another's contradictions as they attempt to coordinate the writing of responses.

Second, mathematics as a cultural practice and specifically the practices associated with traditional school mathematics (Lampert, 1990; Moschkovich, 2007) appeared to influence the conversational "rules of interaction" observed in this study. In everyday conversations, direct contradiction or the passing of judgment on a person's contribution usually signals a breakdown in the flow of conversation, and a flagrant violation of the "rules of politeness" (Eggins and Slade, 1997; Lakoff, 1973). Within the discussions observed in this study, however, such directly confrontational moves appeared with some regularity, and the right to make such moves appeared to be mediated by "intellectual authority." At times the use of confrontational language bordered on teasing or insult (see excerpt 3 above). Fortunately, as a cultural practice, school mathematics is also changeable, and it may be worth considering how the practices enacted in school mathematics may be transformed to make the conversations less directly confrontational. By shifting our conception of mathematics from a stable body of knowledge that must be acquired to a cultural practice that is appropriated by students, we believe that we can transform the practice of (school) mathematics and address issues of authority leading to unproductive (and at times potentially harmful) interactions. One possible further study may be to document how the use of open-ended modeling prompts results in shifts in students peer mathematics discussions.

Following on this point, giving explicit attention to how intellectual authority is constructed and negotiated within the practice of (school) mathematics may be a way to transform how relations of power are enacted among students, and between students and teachers during mathematical conversations in schools. Yackel and Cobb illustrated how "sociomathematical norms" developed in a lower grade classroom over the course of the year (Yackel and Cobb, 1996). Like the development of sociomathematical norms, if intellectual authority is explicitly negotiated and built in a collaborative classroom environment, then it may be possible to reconfigure the nature of school mathematical conversations.

32.6 Summary

This research has illustrated the characteristics of peer-group mathematics discussion among bilingual middle school students. Through conversation analysis and mathematical discourse analysis we have shown that the interactional norms in these discussions differed from the norms of everyday conversation. We also developed the construct of "intellectual authority" as a continually negotiated role in school mathematics conversations, and we demonstrated how intellectual authority mediated interactions between students. There are a number of avenues of inquiry that we have left unexplored in this study. Below we detail two possible follow-up inquiries. One group of question pertains to the use of national languages, and students' language choices. In an interview after the activity, all of the students expressed a preference for learning mathematics in Spanish, so it may be interesting to consider how the use of various national languages influences the type of mathematical discussions the students engage in. A second possible avenue of exploration is to consider how changing the type of prompts that students work on might alter the conversational norms of a peer-mathematics discussion. Our decision to leave the prompts for this study up to the teacher's discretion reflected the ethnomethodological approach to studying conversation. Future studies might consider how Model Eliciting Activities with multiple plausible answers change the norms of a peer mathematics discussion, and transform the position of intellectual authority. Ultimately, consideration of how the "rules" of a school mathematics conversation come into being will be a valuable tool for teachers who seek to implement cooperative learning in the mathematics classroom.

Acknowledgment The preparation of this paper was supported in part by a grant from the National Science Foundation to the Center for the Mathematics Education of Latinos (No. ESI-0424983). The findings and opinions expressed here are those of the author(s) and do notnecessarily reflect the views of the funding agency.

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Chapter 33
Identifying Challenges within Transition Phases of Mathematical Modeling Activities at Year 9

Gloria Stillman, Jill Brown, and Peter Galbraith

Abstract The Galbraith, Stillman, Brown, and Edwards Framework (2007) for identifying blockages hindering progress in transitions in the modeling process is applied to a modeling task undertaken by 21 Year 9 students. The Framework identified where challenges occurred; but, because some blockages proved to be more robust than others, another construct "level of intensity" was added. The blockages described here occurred during the formulation phase of the modeling cycle. We infer that blockages induced by lack of reflection, or by incorrect or incomplete knowledge, are different in nature and cognitive demand from those involving the revision of mental schemas (i.e., cognitive dissonance). The nature and intensity of the blockage have consequences for teacher intervention and task implementation.

33.1 Introduction

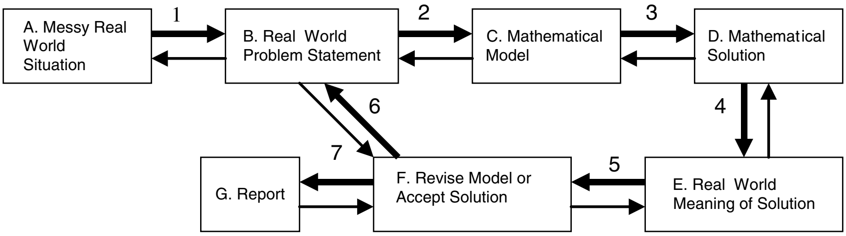
Various representations of the modeling process occur in the literature. Such devices provide a scaffolding infrastructure to help beginning modelers through this challenging task. When teaching and learning issues are also a focus, a version more oriented towards the *problem solving individual* is desirable (Borromeo Ferri, 2006) to give both a better understanding of what students do when modeling, and a better basis for diagnosis and intervention by teachers. We use a diagrammatic representation (see Fig. 33.1), which encompasses both the task orientation, and the need to

G. Stillman (✉)
The University of Melbourne, Melbourne, VIC, Australia
e-mail: g.stillman@unimelb.edu.au

J. Brown (✉)
Australian Catholic University, Melbourne, VIC, Australia
e-mail: jill.brown@acu.edu.au

P. Galbraith (✉)
University of Queensland, Brisbane, QLD, Australia
e-mail: p.galbraith@uq.edu.au

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_33,



1. Understanding, structuring, simplifying, interpreting context
2. Assuming, formulating, mathematizing
3. Working mathematically
4. Interpreting mathematical output
5. Comparing, critiquing, validating
6. Communicating, justifying (if model is deemed satisfactory)
7. Revisiting the modelling process (if model is deemed unsatisfactory).

Fig. 33.1 Modeling process

capture what is going on in the minds of individuals as they work collaboratively, but often idiosyncratically, on modeling problems.

In Fig. 33.1, the entries (A–G) represent stages in the modeling process with heavy arrows signifying transitions between stages. The total problem solving process is described by following these heavy arrows clockwise around the diagram from the messy real world situation. It culminates in either the report of a successful modeling outcome, or a further cycle of modeling if evaluation indicates that the solution is unsatisfactory. Light arrows (in the reverse direction) emphasise that the process is far from linear, or unidirectional, and indicate the presence of metacognitive activity that suffuses every part of the process (Stillman, 1998, p. 245).

In earlier studies (e.g., Galbraith and Stillman, 2006; Galbraith et al., 2007; Stillman et al., 2007), we have identified characteristic behaviours within the transition phases of the modeling process, where a particular interest has been to identify blockages that hinder progress towards a successful solution. Given the availability of technology, our research simultaneously focuses on the interplay between modeling techniques, mathematical content and technology.

Our earlier reported findings have been based on students' written solutions, clarified by videotaped observation of the students working in groups, with associated discussion to clarify with students what they had done. The present paper also incorporates data from another school where modeling tasks were undertaken but supplemented by reflective interviews with students in which they recall, and at times re-create, their experiences after modeling has been completed. In this way we obtain further student clarification of the nature and type of blockages and difficulties that emerged during the solution process. The purpose of this paper is to elaborate more precisely the nature and challenges associated with transitions between different phases of the modeling process.

33.2 The Study

Data were generated within the RITEMATHS project, an Australian Research Council funded Linkage project of the University of Melbourne and the University of Ballarat. The project involves six schools and Texas Instruments as an industry partner (http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/). Several modeling problems have been used with Year 9 students. The part of the study reported here involved the implementation of the final of three tasks in one Year 9 (14–15 years old) class (21 students, 10 female and 11 male). Wayne, the teacher of the class, has been providing opportunities for his students to engage in extended tasks set in real-world situations. The task, *Shot on Goal*, (Fig. 33.2) uses a soccer context – where the modeling problem involves optimising a position for an attacking player to attempt a shot on goal whilst running parallel to the sideline. The task wording included reflective questions designed to provoke students to pause and reflect on their immediate progress in relation to the total purpose. (This pertains to the purpose of the reverse arrows in Fig. 33.1 representing the role of reflective activity in the modeling process.) Data collected included 21 student scripts; video and audio recordings of the teacher, whole class, and collaborative student groups during task implementation; and 14 post-task student interviews. Three lessons (70 min + 2 × 50 min) were devoted to the task. Students worked on the task in self selected groups of 2–4 students. Five groups were subsequently selected by the researchers for video taping on the basis of representing the gender proportion of the class. Wayne allocated a different distance from the run line to the near goal post for each group.

The researchers were cognisant of the need for students to show both development and self-regulation in their modeling over the series of tasks. As Leiß (2005) notes, teacher intervention occurs in two contexts, "from his own wish to intervene or a student might ask for help" (p. 87). In this classroom, extended tasks were usually set by the mathematics co-ordinator, and the teacher worked in a context where he highly scaffolded the students' learning. In the first modeling task, as observed by the researchers, there was a substantial level of teacher intervention, often pre-empting student questions, despite the fact that not all students needed this. In the final task, the teacher was asked to allow students to be more independent. Two questions were added at the end of the *Shot on Goal* task (Fig. 33.2) without scaffolding to suggest how to approach them. A possible solution is offered in the appendix for the main part of the task for a run line distance from the near goal post of 20 m.

33.2.1 Facilitating Formulation

It was anticipated by the teacher that students would find the transition from the real world situation to a mathematical model difficult, so Wayne used the nearby school soccer field to provide an outdoor demonstration. Several students actively

Shot on Goal[1]: You have become a strategy advisor to the new football recruits. Their field of dreams will be the FOOTBALL FIELD. Your task is to educate them about the positions on the field that maximise their chance of scoring. This means—when they are taking the ball down the field, running parallel to the SIDE LINE, where is the position that allows them to have the maximum amount of the goal exposed for their shot on the goal?
Initially you will assume the player is running on the wing (that is, close to the side line) and is not running in the GOAL-to-GOAL corridor (that is, running from one goal mouth to the other). Find the position for the maximum goal opening if the run line is a given distance from the near post. (Additional suggestions were provided as to how the work was to be set out, and for intermediate calculations especially in the area of graphing calculator use providing extra task scaffolding.)

CHANGING THE RUN LINE: Investigate whether the position of the spot for the maximum shot on goal changes as you move closer or further away from the near post. What does the relationship between position of the spot for the maximum shot on goal and the distance of the run line from the near post reveal?

CHANGING THE RULES: Soccer is often a low scoring game. Some have suggested that it would be a better game if the attackers had more chance of scoring, so the width of the goal mouth should be increased. Others claim it would be a more skilful game if the goal keeper was given more of a chance to stop goals by reducing the width of the goal mouth. Investigate what effect changing the width of the goal mouth would have on the position of the maximum shot on goal for the run lines and give your recommendation.

[Image Source: http://images.sportsnetwork.com/soccer/wc/2006/stadium/gelsenkirchen.jpg]

Fig. 33.2 The *shot on goal* task (Image Source: http://images.sportsnetwork.com/soccer/wc/2006/stadium/gelsenkirchen.jpg)

participated in the activity, which was intended to support students in clarifying their thinking, whilst others watched. Wayne indicated a particular run line (see Fig. 33.2) to the students using a rope and asked them, one at a time, to run down the run line and stop when they thought they had the best shot on goal. When six of the students had done this, the class joined them and had a discussion as the individuals on the run line tried to explain why they had chosen a particular spot to stop. Jim, for example, said his determining factor was angle size.

Wayne: Okay, Jim, why did you stop there for?
Jim: Because the goal is big and you can make use of it.
Wayne: When you say the goal is big, how do you measure whether it is big or small or?
Jim: The angle.

[1]This task was refined by the researchers from a task originally designed by Ian Edwards, Luther College, based on observations of implementations of versions of the task in three different school settings.

One student on the run line, Jim, was asked to remain by the teacher as a representative of the average of their estimates. The experience was repeated with a second group of students for a run line closer to the goal. Another student, Mia, was chosen as the average of this second set. This time the class discussion focused on whether moving the run line had made a difference as this exchange with Stella shows.

Wayne: So did it make a difference, where our run line was, as to where the best possible shot on goal occurred?
Stella: [pointing with outstretched arm] Yes, she [Mia] is down that way.
Wayne: It did because Mia is a little bit closer to the goal line than what Jim is, isn't she?
Stella: Yes.
Wayne: It does make a difference but the concept was the same, trying to get the best, or the biggest possible angle at the goal.

For the remainder of this paper we focus on the three student groups who encountered challenges to their progress, a group of girls (Stella, Mia, and Gabi) and two pairs of boys (Len and Ned, and Jim and Ahmed) who all experienced blockages but such that the robustness of these differed in intensity. In all cases these blockages were in the early transitions (i.e., 1 and 2) with the former impacting on their enactive interaction with the real context. All seven of these students, with the exception of Gabi who missed the first lesson on the task, participated in the demonstration, which they saw as either clarifying or confirming their understanding. Stella, for example, found the task clarified her understanding.

Stella: That kind of, that showed me what we were actually doing and it showed us that when people were going really close to the goals [meaning closer to the goal line along the run line] there was a pretty bad angle so I am like I kept that in my mind so when we came back in and do it, it was like "oh good".
Interviewer: Now did you actually go down one of the run lines?
Stella: Yeah.
Interviewer: That helped more than when you were watching?
Stella: Yeah, because when I was walking along I was like, "too far, too far away, too close," so I stepped back a bit and I kind of mentioned to myself where it should be.

Mia also believed the demonstration was essential for her understanding, however her group's formulation of the problem called her understanding into question later.

Mia: No, I didn't understand it, like before we went out and did the, like physically went out and the teacher demonstrated it to us on the soccer field . . . What it showed us was like the angles, like the different angles of the shots, the different angles to the goal from where you were. And the further forward you went, well what I thought but it wasn't actually true, because I thought the more forward we went towards the goal the angle would get smaller as opposed to, like the goal was there and you are standing here [directly in front of the goal] you have like all this, right, and then I thought it gets smaller but it was actually getting bigger, I think, so yeah.

For others such as Len, the demonstration confirmed their understanding.

Len: I walked along the run line, from this I got a feel for the "best" position to shoot for goal. Seeing this demo helped me realise that I did understand what the task was about.

Some domain knowledge (i.e., of soccer) was of benefit. Ned, for example, did not know which was the corner post and which the goal post, however, his partner Len did. Mia who had watched soccer and played it in the backyard with her brother, also did not know this distinction. However, her partner Stella was quite knowledgeable even claiming to be a "professional soccer player" (meaning she played in a soccer competition outside of school sport). This is seen as Stella interprets the diagram of the field that they were given.

Mia: The goal post, is that that?
Stella: No, they are the goal posts [pause] no, wait, they are the goal posts, that is the goal box [pointing]. Okay, that is the penalty box [pointing].
Mia: Isn't the area really small?
Stella: Yeah, and the goalie is allowed to run up to there and kick it.

Even though domain knowledge is of benefit in modeling, at times it can also be a source of hindrance when modellers use this knowledge to reinforce the status of their justifications and the explanations with which they cling to current models in the face of contradictory evidence as happened in this task with Stella's group.

33.2.2 Blockages During Formulation

Ned and Len worked quickly, totally confident in their understanding of the situation and their mathematization of the situation. They were so confident that they decided to leave out the reflective questions that may have given them food for thought about their interim results. However, close to the end of the first lesson as they generated values for their "angle on goal" from various distances along the run line (1–50 m from the goal line), Len noticed that the size of the angle was not what he expected.

Len: Wait a minute. This doesn't seem right. Are you sure this is right?
Ned: That's the maximum. 65.56 [Angle at 44 m from goal line as they are still generating values.]
Len: Hang on, put in... Wait a minute!

Ned also seemed puzzled by the values they generated as they did not mesh with his visualisation of the situation as he stated "the angle would be sharper even if from further away."

Ned and Len had begun by attempting to find the angle of the shot at the shot spot as the angle from the run line to the line joining the spot to the near goal post (θ in Fig. 33.3). They also incorrectly identified the adjacent and opposite sides of the triangle they were using and actually calculated the angle at the near post (ϕ). In interview Ned reported it was not until he was mulling over the task at the beginning of the second lesson that he became fully aware of their errors in formulation. As he reflected on what they had done the previous day, it occurred to him that "there is no way it could be a whole 50° angle … when the goal isn't that wide". He then identified the angle they should be finding as the angle from the shot spot subtended by the goal mouth (α). He asked Wayne to confirm this, then he revised their mathematical representation of the problem. During this process, he realised they had also interchanged the adjacent and opposite sides when using trigonometric formulae previously.

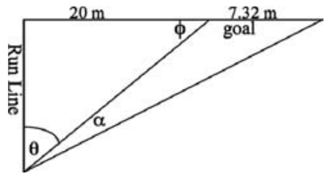


Fig. 33.3 Run line diagram for *Shot on Goal*

Analysis of several implementations of *Shot on Goal* has resulted in the development of the instantiated framework in Fig. 33.4 identifying places where student blockages potentially occur for the first two transitions of the modeling cycle as shown in Fig. 33.1. Ned and Len correctly identified that they were looking for an angle (1.3) but then specified the incorrect angle for their model (1.4). This was really their major blockage as their incorrect representation of the geometric elements for their mathematical formula (2.3) was quickly noticed once the removal of this blockage within transition 1 was effected.

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Jim and Ahmed also experienced their blockage during formulation. They modelled the situation with the distance from the run line to the near goal post varying, rather than the distance along the run line from the goal line of the shot spot varying, as Jim points out.

1. MESSY REAL WORLD SITUATION → REAL WORLD PROBLEM STATEMENT
1.1 Clarifying context of problem [WATCHING/ACTING DEMONSTRATION & DISCUSSING PROBLEM SITUATION]
1.2 Making simplifying assumptions [NO GOAL KEEPER; STRAIGHT RUN LINE, BALL TRAVELS STRAIGHT LINE]
1.3 Identifying strategic entity [RECOGNISING ANGLE AS THE KEY ENTITY]
1.4 Specifying the correct elements of strategic entity [IDENTIFYING *DIFFERENCE* OF THE *TWO CORRECT ANGLES]*
2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL
2.1 Identifying dependent and independent variables for inclusion in algebraic model [ANGLE, AND DISTANCES MEASURED FROM RUNLINE]
2.2 Realising independent variable must be uniquely defined [CANNOT USE SAME SYMBOL AS DISTANCE FROM TWO DIFFERENT POINTS]
2.3 Representing elements mathematically so formulae can be applied [A MEANS TO FIND THE ANGLE THAT HAS BEEN IDENTIFIED]
2.4 Making relevant assumptions [MAINTAIN STRAIGHT LINE PATH ALONG RUNLINE]
2.5 Choosing technology/mathematical tables to enable calculation [RECOGNISING HAND METHODS ALONE ARE IMPRACTICAL]
2.6 Choosing technology to automate application of formulae to multiple cases [*LISTS* HANDLE *MULTIPLE DISTANCE -VALUES*]
2.7 Choosing technology to produce graphical representation of model [SPREADSHEET OR GRAPHING CALCULATOR WILL GENERATE PLOT OF ANGLE FOR DIFFERENT DISTANCE VALUES]
2.8 Choosing to use technology to verify algebraic equation [RECOGNISING GRAPHING CALCULATOR FACILITY TO GRAPH ANGLE VERSUS DISTANCE]
2.9 Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation [GRAPHING CALCULATOR CAN PRODUCE GRAPH OF AN ANGLE FUNCTION TO FIT POINTS – SPREADSHEET CANNOT]

Fig. 33.4 Elements of first two transitions of blockages Framework

Jim: Before we started we messed it up, before we were changing that angle [sic, length] along there [along the goal line] that length, instead of that length that way [perpendicular to goal line]. So we had to do that all over again.

Interviewer: So your model was that a person was on a run line but they were running towards the goal parallel to the goal line?

Jim: Yeah.

They also had difficulties related to the mathematical formulation (using radians instead of degrees and occasionally interchanging opposite and adjacent sides in inverse tan formulae). The teacher corrected their error with the radians when asked to intervene by the boys; but they eventually detected the other errors themselves from Ahmed's observation of, and reflection on, the teacher's discussion of another pair of students' formulation of the situation. Jim and Ahmed had been alerted to their error by negative values of the shot angle when they subtracted the appropriate angles specifying it. Their blockages were associated with element 2.3 of the framework but also 2.1 as they were varying the dependent, rather than the independent, variable.

33.2.3 Blockage Intensity

Our initial framework focused upon the identification of blockages in transitions between the several modeling processes. Examination of data, from this implementation of *Shot on Goal,* has confirmed that the Framework is sufficient for identifying where blockages occur. However, these data indicate that another construct, the "level of intensity" needs to be added. *Level of intensity* refers to the robustness of a blockage – that is how difficult it is to remove. We have noted, from the data, that blockages of high intensity frequently occur in transitions 1 and 2 – associated with the specification of a mathematically tractable problem from a messy real-world situation, and the formulation of a model to address the articulated problem. The formulation phase has long been recognised as one of the most challenging aspects of mathematical modeling (Crouch and Haines, 2004; Potari, 1993). Stella, Mia, and Gabi were so convinced that their particular model was correct they did everything possible other than review their model. Unlike Len and Ned, this trio seriously attempted all reflection questions, however the end result was that they forced their data into fitting with their formulation, rather than revising their model.

It was not until the end of the second lesson that any crack in the armour of the defence of their model began to appear. At this point the group was somewhat concerned to see the task booklet suggest that they may need to use more than one formula. They called over the teacher, "Well we only did one formula. We only went straight in to get the answer". Wayne drew a geometric representation of the situation on the board, marking in the shot angle (Fig. 33.5a). He asked them to explain their by-hand calculations. "We were trying to find that angle there. Weren't we? . . . So you could do that all in one hit could you?" They agreed they were finding this angle even though they had been finding the angle from the run line to the broken line in Fig. 33.5c (angle RSG), from the shot spot to the centre of the goal.

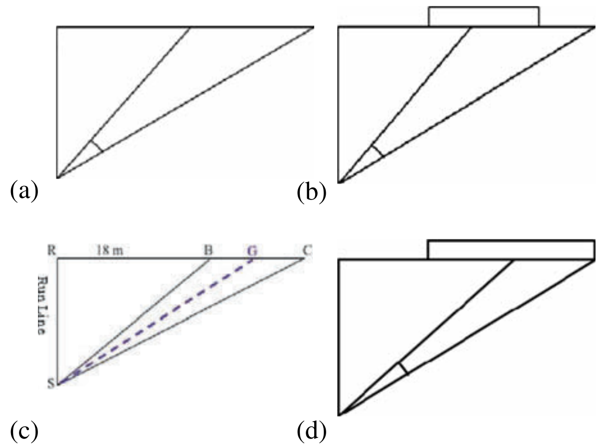

Fig. 33.5 *Shot on Goal* diagrams

[Fig. 33.5c is the diagram drawn in Mia's task booklet.] At this point Stella came to the board and added a representation of the goal to the diagram where Wayne's line to the near post now was roughly to the middle of the goal (Fig. 33.5b). Stella then realised that she and Wayne were talking about different angles but was unable to articulate their mathematization. Mia came to the board and carefully redrew the diagram to show that the line was to the centre of the goal mouth (see Fig. 33.5d). This group had previously decided that "we are shooting there, right in the middle" and used this line element in identifying the angle that they were endeavouring to calculate. After some further comment from Mia, Wayne then drew in line segments from the shot spot to what was now for all concerned a clear representation of the goal posts (and not the midpoint of the goal). [Wayne's diagram was similar to Fig. 33.5c].

Wayne: Right that is the goals. [He erased the lines from the shot spot. Then he drew new segments to Mia's goal posts] So that has to go like that. So this angle [marks the shot angle] has to measure the angle of the goal face [marks segment of goal mouth].

Mia: Ooh!

Stella: So that is BSC.

Wayne: Okay?

Mia: [Turns to the other girls] But we didn't do it like that, we didn't do it for the whole goal.

Stella: I know.

However, the bell rang, and it was not actually until the beginning of the third lesson that Mia, Stella, and Gabi fully comprehended the inappropriateness of their model. The next day Wayne came prepared with a dynamic geometry diagram to represent their model including the midpoint of the goal mouth. He used this dynamically to allow them to see for themselves how unsatisfactory their model was by making it clear that the maximum for their angle RSG approached 90° as the shot spot approached the goal line. Wayne's intervention did not focus on changing their actions to build the "right model." Instead he focussed firstly on allowing the students to examine and alter their mental model and its external representation in the diagram before considering what remedial actions should be taken. They were thus able to see the need for the change in their mental model and make it themselves. The focus of his intervention was therefore on engendering "reflective learning" (Warwick, 2007, p. 36) through the visual feedback afforded by the dynamic geometry software.

Although Stella and Mia had appeared to understand the task, as evidenced by their comments about participation in the outdoor demonstration, mathematizing the situation was problematic. In their initial representation, they marked G, the centre of the goal mouth, as shown in Fig. 33.5c, as this gave the best shot. They subsequently measured with a protractor angles RSG, RSB, and RSC. Towards the end of lesson one, Mia and Stella began to use trigonometry to find angles. In the second lesson, their third group member Gabi arrived. Although Mia and Stella calculated angles RSG, RSB, and RSC for a run line distance of 20 m, when considering points on the run line of 15, 10, and 5 m respectively, for some unknown reason, they only calculated angle RSG. Although Stella questioned the pattern in their interim results, “Why is it that the closer you get, the bigger the angle”, Mia’s response “the face of the goal cannot widen” was sufficient to satisfy them both that they were still on the right track. When they stopped to answer the reflective questions, Gabi convinced the group through discussion that the desired model was for the best angle at every spot along the run line, that is, the angle the kicker turns from facing directly forward on the run line to facing the centre of the goal (the centre being their considered best spot for scoring a goal). Mia and Stella realised this was different from what they had observed on the soccer field, but Mia explained this difference away when she wrote their “beliefs [when on the soccer field] were that [they] were aiming not at the centre of the goal”. From this point on they persisted in finding one angle until Wayne intervened as above.

In terms of blockages, Stella, Mia, and Gabi identified the strategic entity they were looking for was an angle (1.3), however, they then used an incorrect specification for their model (1.4) but conflicting results only served to increase their belief in its validity.

33.3 Discussion

It remains to identify factors responsible for variation in the intensity of the various blockages. Blockages of low intensity (that prove relatively straightforward to deal with) include those encountered by Ned and Len as described previously. These involved identification of an incorrect angle, and the interchange of *opposite* and *adjacent* sides in the use of formulae. Proper attention to reflection – light arrows in Fig. 33.1 – might have enabled them to achieve the necessary adjustments earlier as Jim and Ahmed did.

On the other hand the blockage encountered by Stella, Mia, and Gabi, proved much more robust. While they engaged in reflection their concept of the appropriate representation proved so resilient that they did not question its appropriateness until a cleverly crafted intervention by Wayne demonstrated that their conception led to an impossible situation. We conjecture that we see here an example of resistance of schemas to the need for accommodation in Piagetian terms. These students persisted in attempting to assimilate, rather than accommodate, (Piaget, 1950) new contradictory information into their chosen structure, and resisted the necessity to consider the structure itself as deficient—the students engaged in cognitive dissonance (Atherton, 2003; Festinger, 1957) which prevented them from activating procedures to unblock their progress. From a teaching perspective we infer that the distinction between blockages induced by lack of reflection, or incorrect or incomplete knowledge, and those involving the need to revise schemas are different in type, and in cognitive demand. It remains to examine the nature of other blockages to build a cumulative picture of the different kinds of teaching challenges they provide.

33.4 Conclusion

It has been illustrated in this paper that the Galbraith et al. (2007) was useful in identifying where challenges occurred and progress was subsequently potentially hindered. In addition, the application of the Framework to blockages in the most challenging part of the modeling cycle, that is, the formulation phase, has led to a further construct being identified, namely "the level of intensity". Furthermore, a focus on the modeling cycle that explicitly includes recognition of a modeller's non-linear process and the presence of metacognitive activity at each part of the process has identified a potential means to overcome blockages of low intensity – namely, genuine reflection. This reflection would include the potential to recognise and hence rectify the application of incorrect or incomplete knowledge.

In addition, the analysis has led the researchers to infer that the cause of the more robust blockages are different in type and cognitive demand. These blockages result from a resistance to accommodate new contradictory information. Cognitive dissonance was evident here as the students seemed unable to consider that their model was somehow lacking. Understanding and identifying blockages and their intensity is important for teachers as they facilitate students' development as independent mathematical modellers. Providing the means to recognise the intensity of blockages and the differing options to scaffold students in overcoming these will prove invaluable for teachers.

Acknowledgment This research was funded by Australian Research Council linkage project, RITEMATHS(LP0453701), industry partner secondary schools and Texas Instruments. Jill Brown was a PhD student for the University of Melbourne at the time of this data collection.

Appendix: A Possible solution

An outline of essential steps in the solution follows. Table 33.1 shows calculations obtained using the LIST facility of a TI-83 Plus graphing calculator. Calculations are shown for positions of the goal shooter at (typical) distances from the goal line of between 2 and 30 m; along a run line that is 20 m from the near goalpost (see Fig. 33.6). Width of goalmouth is 7.32 m. (The students encompassed more calculations than these, increasing the distance along the run line beyond 26 m, and the lateral position of the run line varied for each group.) The maximum angle and its reference points are highlighted in the table, which was generated by the LIST facility of the calculator (following hand calculations to establish a method).

A graph (Fig. 33.6), showing angle against distance along the run line is drawn, using the graph plotting facility of the calculator. Additional points near the maximum can then be calculated, to provide a numerical approach to the optimum position (8.90 ° at 24 m) from the goal line – a suitable approach for early or middle secondary students – or an algebraic model can be constructed and the maximum found using the graphing calculator operations. An interpretive statement of advice to an attacking player as strategy adviser could be "your optimal angle will be at 24

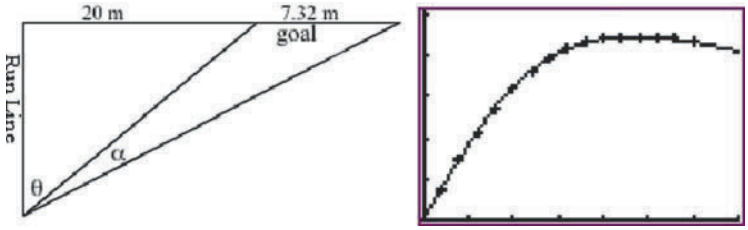

Fig. 33.6 Angle (α) to be maximised, and function for (α) passing through scatter plot for run line 20 m from near goal post

Table 33.1 Sample calculations from a typical solution to *Shot at Goal*

Distance (m) L1	Angle1 (°) L2	Angle2 (°) L3	Shot angle (°) L4
2.00	84.29	85.81	1.52
4.00	78.69	81.67	2.98
6.00	73.30	77.61	4.31
8.00	68.20	73.68	5.48
10.00	63.43	69.90	6.46
12.00	59.04	66.29	7.25
14.00	55.01	62.87	7.86
16.00	51.34	59.64	8.30
18.00	48.01	56.62	8.61
20.00	45.00	53.79	8.79
22.00	42.27	51.16	8.88
24.00	**39.81**	**48.70**	**8.90**
26.00	37.57	46.42	8.85
28.00	35.54	44.30	8.76
30.00	33.69	42.32	8.63

Note: Calculator LIST formulae used were L2 = "$\tan^{-1}$(20/L1)", L3 = "$\tan^{-1}$((20 + 7.32)/L1)", L4 = "L3 – L2".

m but shooting in the range 30–18 m would be fine. The closer you are towards the goal with a clear shot in this range is best as the further away when you shoot the more chance your shot will be intercepted".

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Chapter 34
Realistic Mathematical Modeling and Problem Posing

Cinzia Bonotto

Abstract In this paper I present a teaching experiment characterized by a sequence of activities based on the use of suitable cultural artifacts, interactive teaching methods, and the introduction of new socio-mathematical norms in order to create a substantially modified teaching/learning environment. The focus is on fostering (i) a mindful approach toward realistic mathematical modelling, and (ii) a problem posing attitude. I argue for modelling as a means of recognizing the potential of mathematics as a critical tool to interpret and understand the communities children live, or society in general. Teaching students to interpret critically the reality in which they live and to understand its codes and messages so as not to be excluded or misled, should be an important goal for compulsory education.

34.1 Introduction

In previous studies (see e.g. Bonotto, 1995) I have analyzed some difficulties regarding the understanding of the structure of decimal numbers. These include conceptual obstacles that elementary and middle school students encounter in mastering the meaning of decimal numbers and in ordering sequences of decimal numbers (Nesher and Peled, 1986; Resnick et al., 1989; Irwin, 2001). Results from two questionnaires, each involving elementary and middle school Italian teachers and each concerning the way they teach the topic of decimal numbers in class shed light on the way the usual instructional practice seems totally extraneous to the richness of the experiences students develop outside school (Bonotto, 1996). Many teachers introduce decimal numbers by extending the place-value convention. They tend to spend little time allowing children to understand the meaning of decimal numeration

C. Bonotto (✉)
Department of Pure and Applied Mathematics, University of Padova, Padova, Italy
e-mail: bonnoto@math.uni.pd.i

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_34,

or reflect on decimal number properties and relationships. As a consequence, children learn to carry out the required computations, but have difficulty in mastering the relationship between symbols and their referents, and between fractional and decimal representations.

In agreement with other researchers (e.g. Hiebert, 1985; Irwin, 2001) I believe that the decimal numbers concepts need to be anchored in students' existing knowledge; and, one way to do this is to help students integrate their everyday knowledge with school mathematics.

The study presented in this paper involves a teaching experiment based on a sequence of classroom activities in upper elementary school aimed at enhancing the understanding of the structure of decimal numbers in a way that was meaningful and consistent with a disposition towards making sense of numbers (Sowder, 1992). As in other studies (Bonotto, 2003; 2005 and 20072007a), the classroom activities are based on an extensive use of suitable cultural artifacts – in this case some menu of restaurants and pizzerias. The classroom activities also are based on the use of a variety of complementary, integrated, and interactive teaching methods, and on the introduction of new *socio-mathematical norms*, in an attempt to create a substantially modified teaching/learning environment. This environment is focused on fostering a mindful approach toward realistic mathematical modeling and a problem posing attitude.

34.2 Theoretical and Empirical Background

The habit of connecting mathematics classroom activities with everyday-life experience is still substantially delegated to word problems. However, besides representing the interplay between in- and out-of-school contexts, word problems are often the only examples that are provided to students to cultivate basic sense experiences in mathematization and mathematical modeling. Yet, word problem rarely reaches the idea of mathematical modelling; and, they often promote in students a "suspension" of realistic considerations and sense-making.

> Rather than functioning as realistic contexts that invite or even force pupils to use their common-sense knowledge and experience about the real world, school arithmetic word problems have become artificial, puzzle-like tasks that are perceived as being separate from the real world. Thus, pupils learn that relying on common-sense knowledge and making realistic considerations about the problem context – as one typically does in real-life problem situations encountered outside school – is harmful rather than helpful in arriving at the 'correct' answer of a typical school word problem, Verschaffel etal. (1997, p.339).

Several studies point to two reasons for this lack of use of everyday-life knowledge: (a) textual factors relating to the stereotypical nature of the most frequently used textbook problems: "*When problem solving is routinised in stereotypical patterns, it will in many cases be easier for the student to solve the problem than to understand the solution and why it fits the problem*" (Wyndhamn and Säljö, 1997, p.364), and (b) contextual factors associated with practices, environments and expectations related to the classroom culture of mathematical problem solving: "*In general the classroom climate is one that endorses separation between school mathematics and every-day life reality*" (Gravemeijer, 1997, p.389).

Finally, in my opinion, another reason for not using realistic considerations is that the practice of word problem solving is that they have meaning only within the school. Rarely will students encounter these activities in this form outside of school (Bonotto, 2007a).

I claim that an early introduction in schools of fundamental ideas about modelling is not only possible but also desirable even at the primary school level. Mathematical modelling has received much more curricular attention over recent decades (e.g., Blum et al., 2007b). A particularly sustained and theoretically highly developed program has been carried out by Lesh and his colleagues (Lesh, 2003; Lesh and Doerr, 2003; Lesh and Zawojewski, 2007). In this contribution the term mathematical modeling is not only used to refer to a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated. Besides this type of modeling, which requires that the student already has at his disposal at least some mathematical models to use to mathematize, there is another kind of modeling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts. This second type of modeling is called "emergent modeling" (Gravemeijer, 2007). Although it is very difficult, if not impossible, to make a sharp distinction between the two aspects of mathematical modeling, it is clear that they are associated with different phases in the teaching/learning process and with different kinds of instructional activities (Greer et al., 2007). However, in this contribution the focus will be more addressed to the second type of mathematical modeling.

In elementary schools, to introduce fundamental ideas about realistic mathematical modeling, and lay foundations for developing a "mathematization disposition", I believe that we must create more realistic and less stereotyped problem situations which are more closely related to children's experiential world. In particular, an extensive use of suitable cultural artifacts, with their incorporated mathematics, can play a fundamental role in bringing students' out-of-school reasoning experiences into play, by creating a new tension between school mathematics and everyday-life knowledge (Bonotto, 2007b).

The cultural artifacts that we introduced into classroom activities (see for example Bonotto, 2003; 2005 and 2007a) are concrete materials, real or reproduced, which children typically meet in real-life situations. In this way we offered students the opportunity of making connections between the mathematics incorporated in real-life situations and school mathematics. These artifacts are part of their real life experience, offering significant references to out-of-school experiences. In this way we can enable children to keep their reasoning processes meaningful and to monitor their inferences. Finally, I believe that certain cultural artifacts lend themselves naturally to helping students with problem posing activities.

Problem posing also is an important aspect of both pure and applied mathematics, as well as being an integral part of modelling cycles which require the mathematical idealization of real world phenomenon (Christou et al., 2005). For this reason, problem posing is as important as problem solving (Silver et al., 1996; Ellerton and Clarkson, 1996; English, 1998 and 2003; Christou et al., 2005) and is of central importance in mathematical thinking (e.g. NCTM, 2000). Some of the preceding studies provided evidence that problem posing has a positive influence on students' ability to solve word problems and those activities also provide a chance to gain insight into students' understanding of mathematical concepts and processes. In particular, it was found that students' experience with problem posing enhances their perception of the subject, provides good opportunities for children to link their own interests with all aspects of their mathematics education, and can prepare students' to be intelligent users of mathematics in their everyday lives.

Problem posing is seen here as a classroom activity which is important both from the cognitive and the metacognitive viewpoint. Children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also their perception of the nature of mathematics (Ellerton and Clarkson, 1996) and their attitude towards this discipline.

Problem posing has been defined by researchers from different perspectives (see e.g. Silver et al., 1996). In this paper I consider mathematical problem posing as the process by which students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. This process is similar to situations to be mathematized, which students have encountered or will encounter outside school. According to English (1998) "we need to broaden the types of problem experiences we present to children . . . and, in so doing, help children "connect" with school mathematics by encouraging everyday problem posing (Resnick et al., 1991). We can capitalize on the informal activities situated in children's daily lives and get children in the habit of recognizing mathematical situations wherever they might be".

34.3 The Study

The Basic Characteristics of the Teaching/Learning Environment

The basic characteristics of the teaching/learning environments that we use have been described in other recent publications [see e.g. Bonotto, 2003, 2005, 2007b]. Our design principles include the following.

1. Create more realistic and less stereotyped problem situations based on the use of suitable cultural artifacts – e.g. labels or supermarket receipts – which can provide a didactic interface between in and out-of-school knowledge, and encourage out-of-school reasoning experiences to come into play.
2. Use a variety of complementary, integrated and interactive instructional techniques (involving children's own written descriptions of the methods they use, individual or in pairs working, whole class discussions, . . .).

3. Establish a new classroom culture also through new socio-mathematical norms, for example the norms about what counts as a good or acceptable response or solution procedure, in order to undermine some deeply rooted and counterproductive beliefs such as mathematics problems have only one right answer or there is only one correct way to solve any mathematical problem.

34.3.1 Participants/Materials/Procedure

The study was carried out in two fourth-grade classes (children 9–10 years of age) in a lakeside resort in the north of Italy by the official logic-mathematics teacher, in the presence of a research-teacher. As a control, two fourth-grade classes were chosen in the same town. Most of the local population are involved in tourism and catering, and most of the children's parents either owned or worked in restaurants, bars, ice-cream parlors or pizza shops; for this reason it was found that the pricelists and menus of pizza shops, fast food and normal restaurants were part of the children's experiential reality.

The teaching experiment was subdivided into six sessions, each lasting two hours, at weekly intervals. The first session was devoted to the administration of the pre-test and the introduction of various kinds of artifacts which the children, divided into groups, had to analyze, by reading and interpreting all the data present, whether numerical or not. Sessions 2–6 concerned five experiences involving different opportunities offered by the artifacts. The sixth session was also devoted to administration of the post-test. Sessions 2–6 were divided into two phases. In the first, each pupil was given an assignment to carry out individually or in pairs. In the second phase, the results obtained were discussed collectively and the various answers and strategies compared.

34.3.2 Data

The research method was both qualitative and quantitative. The qualitative data consisted of students' written work, field notes of classroom observations, and mini-interviews with students after the experiences. Quantitative data were collected using pre- and post-tests which administered to two experimental classes and two control classes. The two tests were constructed by taking items normally used in the bimonthly tests utilized by the same teachers or usually present in the textbook.

34.3.3 Research Questions and Hypotheses

The first general hypothesis was that the teaching experiment class would foster the understanding of some aspects of the multiplicative structure of decimal numbers in a way that was meaningful and consistent with a disposition towards making sense of numbers (Sowder, 1992). This would be expected due to the opportunity children had to refer to a concrete reality (via the cultural artifact), explore their strategies and compare them with those of their schoolmates, and use estimation and approximation processes, as well as both problem posing and problem solving abilities.

A second hypothesis was that, contrary to the common practice of word-problem solving, children in this teaching experiment would not ignore the relevant, plausible and familiar aspects of reality, nor would they exclude real-world knowledge from their observations and reasoning (hypothesis II). Furthermore, children also would exhibit flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was meaningful and closer to the procedures emerging from out-of-school mathematics practice (hypothesis III).

Finally I wanted to evaluate the impact on problem solving of problem posing activities and the use of suitable cultural artifacts.

34.4 Some Results

By presenting the students with activities that were meaningful and that involved the use of material familiar to them, motivation was increased even among less able students. A good example is the case of an immigrant child with learning difficulties related chiefly to linguistic problems. For her, as for many others, being confronted with a well-known everyday object with "*few words and lots of numbers*" acted as a stimulus. Indeed, it led her to say *"It's easier than the problems in the book because we already know how things work at a restaurant!"*

An analysis of class discussion showed that a process of *problem critiquing* (English, 1998) was set up whereby the children attempted to solve, criticize and make suggestions or correct the problems created by their classmates. Here is an example taken from the second session in which the children read and interpreted the data and information in the various menus (products on offer, prices, ingredients, cover charges etc.). Working individually on one of these menus, the children compiled a hypothetical order, which they themselves chose according to their experience outside school. In so doing, they had to follow the structural features of a blank receipt (description of goods, quantity, cost etc.) provided by the teacher. Finally, the children had to calculate how much they would have to pay, by adding up the bill.

I. Look at this. Do you think it could be a kind of problem?
P 737. It's not written...
I. You're right... It's not written, but all the data is there. We could write the text ... Shall we try?
P 734. A man goes to a restaurant and orders 2 dishes of sea-food, 1plate of escallops in lemon sauce, 1 mixed salad, 1 fresh fruit, 1 still mineral water, 1 medium Coke. How much does he spend?

I. Does everyone agree?
P 740. You can't do it like that because there are no prices ... We have to put a menu underneath the problem or put the prices in the text.
P 725. And then... how do we know how many people were eating if they didn't put the cover charge?

The idea of "*pretending to be at a restaurant*" and "*acting like grown-ups" ["grown-ups don't take a calculator or a pencil and paper with them to see if they can afford to order this or that .... They work it out in their heads*" said one child, P725] helped the children, including those with greater difficulties, to reason more freely and adopt calculation strategies they had never used before.

Another example, taken from the fifth session, shows impact of using a more complex menu where students were asked: (i) to analyze the menu and to read and interpret all the data and information contained therein; (ii) to choose what to order knowing that they had only 15 euros to spend; (iii) to make a mental estimate of what they would have to pay and whether it would come within their budget; (iv) to write out the bill in full to check their estimate. Here is an example, P734, of a child considered "low level" who had been placed in the "extra help group".

> First of all, I take away the money for the cover charge, which is obligatory. So € 15.00 – € 1.25 is like doing € 15.00 – € 1.00 which makes 14.00 but it's a bit less because it was € 1.25... so we can pretend that I've still got € 13.50 for example. Then I decided to have the cheapest pizza, the pizza marinara, so I have to take away € 3.35. A quick way is to take away € 3.50 and so I'd still have about € 10.00 and I can add something on the pizza, for example ham which costs € 1.44. Let's pretend it's 1.50, so € 10.00 – € 1.00 makes € 9.00 and then taking away another € 0.50 it makes € 8.50. In the end, I can have a Coke as well which costs € 3.00 and that makes € 5.50... (a brief pause)... But I wanted a dessert and now I can't afford it... I have to leave something out... (he thinks for a few seconds)... I need another euro because the desserts cost € 6.50.... Perhaps it's better to have water instead of a Coke. That way instead of € 3.00 I only spend € 2.00 for drinks and now I've got € 6.50 which is the price of a 'semifreddo' dessert.

As can be seen, P 734 is unable to calculate mentally € 15.00 – € 1.25 but by approximating upwards all the prices, he is able to find a solution to the problem after his first failed attempt. Like P 734, all the other children, without exception, managed to carry out this exercise. Most of them preferred to use subtraction and calculate the amount they had left in order to decide if they should order something else and what to order. Only two children preferred to use addition and calculate as they went along the amount of the final bill. Here is an example:

> I started with the cover charge which is € 1.25 then I added the pizza siciliana which is € 5.94... but to do it quickly I pretended it was €6.00 and so the total was € 7.25. Now I can get something to drink. I decided to have water which costs less so that I could perhaps manage to have a dessert afterwards. So I say € 7.25 + € 2.00 = € 9. 25. All the desserts cost € 6.50 each. If I do € 9.25 + 6.50 = 9.00 + 6.00 = 15.00 but then I have to add on the 25 cents and the 50 cents and so it's to much because I've only got € 15.00 to spend. So, I think I'll have a cheaper pizza, for example a margherita which costs € 3.61 (say 3.50). So, I have to start again by doing 1.25 + 3.50 = 3.00 + 1.00 = 4.00 + 25 cents = 4.25 + 50 cents = 4.75 cents. Now I add on the water which is 4.75 + 2.00 = 6. 75 and try again

> adding the dessert doing 6.75 + 6.50 = 6 + 6 = 12.00 + 50 cents = 12. 50 + 75 cents which is about one euro so it's more or less € 13.50.

The less confident children (as in the case reported above) made an estimate that was less precise, whereas the more confident ones attempted a more careful estimate down to the number of cents. Here is another example, P 740:

> I first took away the price of the cover charge, so I did € 15.00 – 1.25 = 14.00. Then I took away another € 0.25 which is half of 0.50: € 15.00 – 0.50 = € 13.50 + 0.25 = € 13.75. Then I decided to take away the price of a dessert which is what I like most, so I do € 13.75 – € 6.50. Here, I can do what I did before and divide 75 cents in two and do 50 and 25. So 13.50 – 6.50 = 7.50 €. But I've got to add on the 25 cents and so € 7.50 + 25 = € 7.75. Now I can have a Coke which is easy because I have to take away 3 euros and that makes € 4.75. With that money I can have a pizza Napoli which costs € 4.64 and still have some money left.

These examples clearly demonstrate that the children exhibited flexibility in their reasoning, by exploring different strategies, and that they often were sensitive to the context and quantities involved, in a way that was meaningful and consistent with a sense-making disposition and closer to the procedures emerging from out-of-school mathematics practice (hypothesis III confirmed). Many other examples of written works also demonstrated that the children did not ignore the relevant, plausible and familiar aspects of reality, nor did they exclude real-world knowledge from their observations and reasoning.

34.5 Conclusion and Open Problems

This paper presented results from a teaching experiment characterized by a sequence of activities based on the use of suitable cultural artifacts, interactive teaching methods, and the introduction of new socio-mathematical norms. An effort was made to create a substantially modified teaching/learning environment that focused on fostering a mindful approach towards realistic mathematical modeling and problem posing. The positive results of the teaching experiment can be attributed to a combination of closely linked factors, in particular the use of suitable cultural artifacts and an adequate balance between problem posing and problem solving activities.

Regarding the use of cultural artifacts, the implementation of this kind of classroom activity requires a radical change on the part of teachers as well (for an analysis sees Bonotto, 2005). These tools differ from those usually mastered by the teacher that are quite always highly structured, rigid, not really suitable to develop alternative processes deriving from circumstantial solicitations, unforeseen interests, particular classroom situations. The teacher has to be ready to *create* and manage open situations, that are continuously *transforming,* that can be mastered after long experimentation and of which he/she cannot foresee the final evolution or result. As a matter of fact, these situations are sensitive to the social interactions that are established, to the students reactions, their ability to ask questions, to find links between school and extra-school knowledge; hence the teacher has to be able to modify on the way the content objectives of the lesson.

In future research, we will take a further look at the role of cultural artifacts not only as mathematizing tools that keep the focus on meaning found in everyday situations and as tools of mediation and integration between in and out-of-school knowledge, but also as possible interface tools between problem posing and problem solving activities.

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Chapter 35
Modeling in Class and the Development of Beliefs about the Usefulness of Mathematics

Katja Maass

Abstract Modeling and applications are regarded as important components of mathematics education within the didactical discussion. Little however is known about the impacts of integrating modeling and application into day-to-day teaching practice. A qualitative study has been carried out which investigated the effects on students during a 15 months intervention. This paper deals with the effects on student's beliefs.

35.1 Introduction

Mathematics is often regarded as important or useful. However, when asked for examples of where they use mathematics, people often can name only elementary situations such as "when shopping". When dealing with higher level mathematics, students often lack the insight into the use of these topics. Such beliefs do not reflect the real relevance of mathematics which plays an important role in a wide range of sciences and for many aspects of daily life. For this reason, an empirical study has been carried out which aimed at exploring students' beliefs and the possibility to change them by integrating modeling in day-to-day teaching practice. The described study deals with the following research questions:

1. What beliefs do students have about mathematics and its usefulness?
2. How can the integration of modelling into day-to-day teaching practice affect students' beliefs about mathematics and its usefulness?
3. What impact do students' beliefs have on their reaction towards modelling tasks?
4. What are modelling competencies?
5. Can students aged 13 be enabled to carry out modelling processes on their own?
6. Which kind of connections exist between mathematical beliefs and modelling competencies of the students?

K. Maass (✉)
University of Education Freiburg, Freiburg, Germany
e-mail: katja.maass@ph-freiburg.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_35,

This paper deals with the questions 1–3, results concerning question 4 can be found in Maass (2006).

35.2 Theoretical Framework

Concerning the discussion of modeling: We refer to the position of Blum, which can be regarded as an important position within Europe: The modelling process begins with a real world problem. By simplifying, structuring and idealizing this problem you get a real model. The mathematizing of the real model leads to a mathematical model. By working within mathematics a mathematical solution can be found. This solution has to be interpreted first and then validated. If the solution or the chosen process do not prove to be appropriate to reality, particular steps or maybe even the entire modelling process need to be worked through again (see e.g. Blum and Niss, 1991).

According to different perspectives of the modelling process, opinions on the relevance of the content differ. For example, de Lange (1989) also accepts context-free problems whilst others emphasize the importance of realistic or authentic problems (see e.g. Galbraith, 1995). In the study reported here, the modelling tasks were supposed to allow students to appreciate the relevance and the usefulness of mathematics to individuals as well as society – one aim, amongst others, which are seen as important reasons for integrating modelling tasks into the curriculum (Blum and Niss, 1991).

Despite the long lasting discussion about modeling and applications, such tasks are rarely integrated in day-to-day mathematics teaching (Blum et al., 2002). Reasons include an inappropriate organizational framework (e.g. lessons of 45 min), the complexity of the modeling tasks which seems to make mathematics education more complicated, and teachers' fears that students cannot solve such tasks (Blum and Niss, 1991).

Concerning the conception of mathematical beliefs: The mathematical beliefs of a student are seen as a filter, influencing all activity and thinking (Pehkonen, 1993). However, this is a complex issue and there is no simple shared understanding of the concept of beliefs (see e.g. Op't Eynde et al., 2002). Furenghetti and Pehkonen (2002) suggest – based on an international review of literature – differences between subjective and objective knowledge, and regard beliefs as subjective knowledge which take into account affective components. Concerning the question of whether beliefs are changeable, they see this in relation to the existence of surface beliefs and deeply rooted beliefs. In this study, we will follow a definition based on Furenghetti and Pehkonen (2002) as well as on Pehkonen and Törner (1996): *Mathematical beliefs consist of relatively long lasting subjective knowledge, as well as of the related attitudes and emotions, about mathematics and mathematics education. Beliefs can be conscious or unconscious.*

Grigutsch (1996) gave an important contribution to characterize students' beliefs within the German discussion. He categorized students' beliefs mainly by four aspects of mathematical belief systems which refer to mathematics as a field of science. Mathematics can be understood as a science which mainly consists of problem solving processes (aspect of process), a science which is relevant for society and life (aspect of application), an exact, formal and logical science (aspect of formalism) or a a fixed body of rules and formulas (aspect of scheme). Similar categories can be found within the international discussion: Ernest (1991) and Dionne (1984) differ between a *traditional perspective*, a *formalist perspective* and a *constructivist* perspective which seem to correspond to the aspect of scheme, of formalism and of process.

Empirical studies point to the fact that student's beliefs about mathematics are mainly scheme-orientated. Students think that mathematics tasks have only one solution and can be solved quickly and teachers have to explain how to do tasks correctly. Additionally, students seldom reflect on the nature of mathematics (Spangler, 1992; Frank, 1988; Kloostermann, 2002).

35.2.1 Transfer of the Theoretical Approach into Practice

Two groups of learners were chosen for the study. They were in parallel classes; and, all were aged 13 at the outset of the study. In order to overcome various difficulties, the "island approach" by Blum and Niss (1991) was chosen for integrating the modelling examples to allow for curriculum expectations although integrating modelling examples into "normal lessons" as well as avoiding resistance by learners and their parents who were used to traditional mathematics classes. At the same time the view was taken that making links to the real world and modelling are not the only components that learners should experience in the complex field of teaching and learning mathematics.

According to the results of several empirical studies, teaching methods were chosen which allowed the students to work independently (e.g. Ikeda and Stephens, 2001) as well as in groups. Group work is considered an appropriate teaching method within this context and within the given school framework. Therefore, phases of group work alternated with phases of classroom discussion and phases of individual work. During the classroom discussions students were engaged to talk to each other whilst the teacher tried to remain as silent as possible.

Six modelling units were developed regarding the theoretical approach and the given framework. So, context-related open modelling problems were selected for the lessons. The majority of context-related problems were authentic in order to make the students realize the relevance of mathematics. They addressed different contexts as well as different mathematical content and they required different amounts of time (1–12 lessons). They were integrated into the mathematical lessons during the data collection period of 15 months. For examples, the teaching units address questions such as:

- How big is the surface area of a Porsche 911? This question has to be answered when a new type of car is designed and the production costs are to be calculated. Duration: 3 lessons. (for details see Maaß, 2006)

- How many people are in a traffic jam of 25 km (Jahnke, 1997)? This question has to be answered when people in the traffic jam are to be provided with water or food. Duration: 3 lessons
- To what extent can the service water in Stuttgart get warmed by solar panels on the roofs? This question is of ecological interest. (Duration: 8 lessons, see Maaß, 2007)

35.2.2 Theoretical Basis and Methods in Data Collection

The goal of this study was to generate hypotheses about the consequences of using modelling in day-to-day teaching. An essential characteristic for the selection of sample survey and evaluation methods was the principle of openness: Since hardly anything was known about the consequences of modelling lessons, hypotheses should not already be brought to the study but rather be developed while dealing with the data and be formulated as results. Data evaluation was done according to the Grounded Theory approach with codes deriving from theory as well as in-vivo-codes (Strauss and Corbin, 1998). A goal of the study was the creation of typologies. Procedures of creating types, comparing cases and contrasting cases play an important role in qualitative research because thus the complex reality is reduced and made concrete (Kelle and Kluge, 1999). One tool to elucidate the results was the construction of ideal types as described by Weber (1904, see Gerhardt, 1990). Ideal types are the result of an idealization and therefore have a theoretical character.

In order to meet the complexity of the research's objectives, a variety of methods in data collection was used: Questionnaires (at the beginning and the end of the study), interviews (one in the middle of the study, one at the end), learner's diaries (filled in continuously). Based on computer-aided data evaluation, typologies were created to explain interrelations between phenomena.

35.3 Results of the Study

The results of the study show that the aspects as reported by Grigutsch can be reconstructed – among others – in the students' minds. Altogether the following beliefs could be reconstructed according to these categories:

- *The usefulness of mathematics (aspect of application):* Mathematics is useful in life; mathematics is useful for your profession; you need mathematics for other subjects at school; mathematics is needed everywhere; mathematics is necessary for general knowledge; you need mathematics only when shopping; mathematics and reality have nothing in common; you only need basic mathematics in life.
- *Aspect of process:* You have to think and try a lot; you have to look for possible ways of solution on your own; you have to solve problems; you need ideas to solve the tasks; you have to reason a lot; different ways of solution are possible.

- *Aspect of scheme*: To solve tasks you have to learn rules; there is only one way of solution; you have to know the algorithm to solve a task; it has to be explained how to solve certain tasks; you have to practise a lot in mathematics; knowledge has to be taught quite quickly in mathematics lessons.
- *Aspect of formalism*: Exact algorithms have to be used in mathematics; a task needs an exact solution; a task needs an unambiguous result; mathematics is a logical science; you have to think logically in mathematics.

Data evaluation, however, showed that these aspects do not sufficiently describe the students beliefs. There seem to be further beliefs which do not focus on mathematics as a science but on mathematics lessons. There were cognitive beliefs about mathematics lessons as well as attitudes concerning mathematics lessons.

35.3.1 Beliefs About the Learning and Teaching of Mathematics – Beliefs of Cognitive Priority

- *Beliefs about the short duration of teaching units within the maths classes:* The students shared the opinion that teaching units should be short and that mathematics classes consist of a lot of short exercises to work on, respectively.
- *Beliefs about a minor importance of words in the exercises:* The students were of the opinion that writing is not substantial for mathematics classes.
- *Beliefs about the necessity of learning:* The students were of the opinion that either one is able to deal with mathematics or not. In consequence they thought that dealing with mathematics must not be labour-consuming.
- *Beliefs about the relevance of mathematics at school:* The students regarded mathematics as an important subject where they are not allowed to fail.

35.3.2 Beliefs About the Learning and Teaching of Mathematics – Attitudes

- *Beliefs about the teaching methods*: The teaching methods were of great importance, e.g. the students liked mathematics lessons because their were allowed to work in groups.
- *Beliefs about the atmosphere within maths classes*: A friendly atmosphere in the lessons was of great importance for these students.
- *Beliefs about understanding*: The understanding of mathematical content had a high impact on the students (the students liked the lessons when they understood the content or disliked them when they didn't understand the content, respectively).

These beliefs about the learning and teaching of mathematics seem to be so significant for some students that beliefs referring to mathematics itself as reported by

Grigutsch (1996) could not be reconstructed. Altogether, the student's beliefs seem to have a heavy impact on the student's reactions towards modelling examples as well as on their development of the beliefs about the usefulness of mathematics. This result – gathered from the evaluation of the data of 35 students – will be illustrated by looking at three cases who can be regarded as prototypes and who show the results very clearly.

Albert: Albert's beliefs about mathematics can be described as process-orientated. He is fairly interested in mathematics and shows a good performance. From the very beginning of the study Albert likes engaging in thinking processes: *"I like the problem solving within the task" (students' diary, 26.06.01)* However, he dislikes modelling tasks, this is shown by his criticism towards *every* context of the modelling tasks dealt with in class.

> I would not do it [a modelling task] again, because I did not like the context [traffic jam]. I would like to do tasks about interesting and useful things, such as calculations about the volume of prism and pyramids [. . .] (students' diary, 26.06.01)

While he criticises the topic "traffic jam" here, he also criticises the topic "Porsche" in addition to "mobile phone" later: *"I did not like the modelling task because the topic "mobile phone" does not fit into mathematics and because it is without solution." (student's diary, 20.12.01)* Further more, he does not seem to regard the results we got as solution for the tasks, maybe because they are not as exact as usually. Other sources show that Albert seems to have a quite narrow view about the usefulness of mathematics.

> What topics to you regard as important for your life?" "Well, things I need later on, such as terms etc, [. . .] because at the point of sale, you have many numbers. But geometry is not so important. You do not have to know how to calculate the surface of a cuboid (1.interview, 24.01.02).

Nevertheless, a change in his view is indicated after 8 months of the study: *"If I had the choice, I would prefer modelling tasks, they are more fun. There you can work in groups and you have to think about things at home." (1. interview, 24.1.02)* This quotation shows that Albert now seems to see a connection between modelling tasks and problem solving and therefore likes modelling. During the following months his interest in modelling tasks is rising; apparently for the same reason:

> I liked the task "sun energy", because we were allowed to do many things alone [. . .] This teaching unit belongs to mathematics because we will need this later on, even more than the rest. Additionally, you have to think logically about things and not only to learn rules and to use them. It is more fun and much more interesting than terms or geometry (student's diary, 25.4.02).

In contrast to the unit "mobile phone" and the other topics, he now regards the context as belonging to mathematics education. Again he seems to see a relation between modelling and problem solving. The same holds for *every* other context after the first half of the study pointing to a change in his beliefs. This change is not only in his attitude towards modelling tasks but also in his beliefs about the usefulness of mathematics:

> Do you think you can use the things you learnt during the modelling units later in your life?" "I think it will do no harm, because knowledge is always liked. Additionally, I learn to work more independently and to critically discover problems in life (student's diary, 2.7.02).

Whereas in the first interview after 8 months Albert showed quite narrow beliefs about the usefulness of mathematics he now emphasizes the use and relevance of general knowledge. He is also referring to becoming a critically aware citizen. Apart from these indications other parts of his texts show that Albert's positive attitudes towards mathematics rise. Whilst at the beginning he only "partly" liked mathematics, he declares mathematics to be "very good" at the end.

Altogether, Albert can be regarded as a prototype for those students whose beliefs mainly consist of process-orientated beliefs. It is indicated that Albert learned to like modelling tasks and by this enhanced his beliefs about the usefulness as well as his positive attitude towards mathematics. The situation however is totally different for Carsten.

Carsten: Carsten's beliefs can be described as mainly scheme-orientated. He shows a good performance in mathematics. In contrast to Albert, Carsten apparently does not like to engage in thinking processes. He criticises for example a young student teacher because she wants the students to solve problems which have not been explained in advance. *"I do not like the mathematics lessons of Mrs. X, because she presents tasks we do not know." (1. questionnaire, 27.4.01)* Additionally, he only sees but little use of mathematics:

> I think is justified [to abandon mathematics from the curriculum] when students are older than 12, because at this time they know all they need to know in life (1. questionnaire 27.4.01).

Carsten wants to be taught mathematical content quite quickly and calls thinking about a topic for any length of time as a "waste". Additionally, he does not like tasks with several ways of arriving at a solution: *"I think there were too many ways to get a solution" (students' diary, 26.6.01).* During the second half of the study Carten's reluctance against modelling tasks rises as is indicated among others by the following quotation: *"I did not like doing a damn modelling task again*" *(student's diary, 31.1.02).*

During the study there seems to be no change of beliefs. His beliefs remain scheme-orientated.

> I liked the teaching unit about mathematical expressions better than modelling tasks because modelling tasks take too much time. I like that tasks about terms only have one way of solution, in contrast to modelling tasks (student's diary, 7.3.02).

In all his remarks he puts a strong emphasis on tasks where formulas have to be manipulated and calls them "*real mathematics*". He seems to separate modelling tasks from real mathematics which in his eyes seem to consist of numbers, calculating and manipulating of formulas. By the end of the study almost everything he says refers to calculating. He cannot see any relation between modelling tasks and mathematics.

> Do you think modelling tasks belong to mathematics lessons?" "No, not really, because I, because you don't have to calculate. And in mathematics lessons you have to calculate, I would say! (2. interview, 11.7.02).

He also still seems to have a narrow view about the usefulness of mathematics. The following remark reinforces this hypothesis:

> Which mathematical topics are relevant for your later life?" "It depends on the profession. If you become an engineer, you need geometry, a salesman only needs plus, minus, times and divided by (2. questionnaire, 2.7.02).

Carsten does not seem to see any relevance of mathematics beyond the use in profession and even there his beliefs seem quite narrow, seeing only elementary calculations as relevant for salesmen.

Altogether, Carsten's reactions are typical for those who have mainly scheme-orientated beliefs. Carsten and Albert show a very different development. Although they both dislike the modelling tasks at the beginning, Albert begins to like modelling tasks as a means to do problem solving and even enhances his beliefs about the usefulness, whereas Carsten continues to dislike them. Up to the end of the study, he cannot see any connection between modelling and mathematics. Presumably, modelling tasks do not fit into his picture of mathematics.

Doro: Doro, a student with low competencies in mathematics, is an example for a further kind of development. Her beliefs mainly referred to the teaching and learning of mathematics and were of affective priority. At the beginning of the study, her attitude towards mathematics seems to be quite negative.

> I do not really like mathematics lessons because we learn things which we will probably never use again, well some things may be useful but it is not so wonderful (1. questionnaire, 27.4.01).

For Doro beliefs or attitudes referring to the lessons and their atmosphere seem to be very important. Her answer to the question "In which situations do you like mathematics?" is as follows: *"Working in groups, Games, working on my own if the tasks are easy, easy topics" (1. questionnaire, 27.4.01).* She only refers to working methods and the simplicity of the tasks, not to any content. The understanding of a topic seems to be very important for her as is shown by the following two quotations where she refers to modelling units:

> Up to now, I have understood everything, because it is interesting! (diary, 26.9.01)
>
> I have understood everything, and if you understand things, then it is fun!(diary, 29.11.01)

For Doro there seems to be a direct relation between a positive attitude and the understanding. Doro also refers quite often to the atmosphere: *"[Mrs. Maaß] was stressed but nevertheless very friendly. That was very good." (diary, 12.12.01)*

She sincerely likes the teaching units about modelling and gives different reasons for this. As was seen above, she seems to understand modelling tasks quite well, whereas she often does not understand traditional mathematics. In addition to this, she apparently sees the use for daily life.

> I liked the teaching unit "mobile phone" very much because it refers to reality. It is better than calculating. The teaching unit was long but this was ok for me, because I can use this later in my life (diary, 20.12.01).

Doro shows a general interest in all the teaching units about modelling. Although she is not used to long tasks within mathematics lesson, this does not have a negative impact on her positive attitude towards modelling tasks. Even her beliefs about the usefulness of mathematics seem to rise. However, the formulation "better than calculating" indicates that she may separate traditional mathematics from modelling.

During the life-time of the study her interest in modelling seems to rise. *"I love reality-orientated tasks." (diary, 25.4.02)* She continues to refer quite often to the atmosphere in class as well as to her interest in group work. So, her beliefs remain mainly attitudes referring to the teaching and learning of mathematics. Additionally, at the end of the study her beliefs about the usefulness of mathematics seem to have changed.

> What did you learn from modelling examples? "A lot! 1. Mathematics is everywhere. 2. Maths lessons can be fun. 3. Everybody needs mathematics. 4. Which mobile contract I have to choose. . .5. Well, many important things" (2. questionnaire,2.7.02).

Altogether, Doro is very enthusiastic about the modelling tasks as is shown by several statements. But despite her eager interest in modelling this does not seem to have changed her attitude towards mathematics in general.

> "Mathematics is something for brainy people [. . .]. Mathematics is a stupid very important subject at school, where you have to pay attention. Otherwise you will fail" [. . .] "In what situations do you like mathematics?" "Reality orientated tasks are very good and they make a change" (2. questionnaire, 2.7.02).

Presumably, she still separates traditional mathematics from modelling at the end of the study. While she likes modelling she dislikes traditional tasks with which she seems to have made negative experiences. This negative attitude which was developed during her 7 years of school life could apparently not be changed within the life time of the study. However, she seems to begin to develop more beliefs about the usefulness of mathematics and she obviously has positive attitudes towards modelling tasks.

Altogether, the development of Doro differs from the development of Albert and Carsten. As in the case of Albert, her beliefs about the usefulness of mathematics seem to be reinforced during the study. In contrast to Albert, however, she is enthusiastic about the tasks from the very beginning although they do not meet her expectations about mathematics. One main reason for this seems to be her better understanding of the modelling tasks in contrast to traditional tasks. For her the modelling tasks seem to be a way of understanding and liking mathematics.

The overall analysis of the data lead to the construction of six ideal types of which three have been illustrated by the case examples in this paper. Altogether, the following ideal types could be construed.

Ideal type A: The students have an application-orientated mathematical belief system and regard the modelling examples positively. The application-orientated beliefs are intensified during the study.

Ideal type B: The students have a process-oriented mathematical belief system and develop a positive attitude towards modelling examples. The application-orientated beliefs increase during the study (see Albert).

Ideal type C: The students have a scheme-oriented mathematical belief system and reject the modelling examples in an emotional way. No application-orientated beliefs are developed until the end of the study (see Carsten).

Ideal type D: The students have a formalism-oriented mathematical belief system and reject the modelling examples in an emotional way. No application-orientated beliefs are developed until the end of the study.

Ideal type E: The students have mainly cognitive shaped beliefs about the learning and teaching of mathematics including beliefs about short-lasting teaching units, minor importance of texts and little necessity of learning. They react in a negative manner towards modelling examples. Only a few application-orientated beliefs are developed.

Ideal type F: The students have mainly affective shaped beliefs about the learning and teaching of mathematics including beliefs on teaching methods, atmosphere and understanding. Furthermore, the students believe that they understand the content quite well. They regard the modelling examples as positive and develop application-orientated beliefs (see Doro).

35.4 Consequences

Regarding the students who reinforced their beliefs about the usefulness of mathematics, it becomes clear that the aims of integrating modelling into mathematics class can be realized. About half of the students deepened their application-orientated beliefs within 15 months. This shows that mathematics lessons can have influence of the beliefs of students. An appropriate view of the usefulness of mathematics can be reached. Perhaps over a longer period of time developing critically aware citizens who view the world from a modelling perspective seems to be possible.

Caution however is needed when thinking about short-term motivation of students. The results of the study point quite clearly to a problematic situation for those students who presumably cannot integrate modelling tasks into their beliefs about mathematics and its education. Their emotional reactions highlight the necessity to start modelling quite early within a school life, preferable in primary school. It seems very important to deal with tasks which clarify that mathematics tasks do not always have only one way of solving them and one exact solution, that estimating is a mathematical proceeding, and that reflection and time may be needed to solve tasks. In order to enable teachers to integrate modelling and application into day-to-day teaching practice modelling has to be integrated into pre-service and in-service teacher training.

The results also support the necessity of further research. A long-term intervention has to be carried out to find out whether the beliefs of all students can be changed if modelling and applications are integrated into day-to-day teaching practice for a longer period. Additionally, research with younger students is needed. Finally, research should also be carried out to investigate how teachers who are themselves beginners in modelling and applications can influence student's beliefs.

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Section 9
How Do Teachers Develop Models of Modeling?

Chapter 36
Insights into Teachers' Unconscious Behaviour in Modeling Contexts

Rita Borromeo Ferri and Werner Blum

Abstract In this paper, we will describe from a cognitive perspective how teachers deal with mathematical modeling problems in the classroom (grades 8–10, 14–16-year-olds) and how their behaviour is obviously dependent on certain features that are widely unknown to them. Our examples are taken from two projects: the COM^2 project where, among other things, the influence of the teachers' mathematical thinking styles on their way of dealing with modeling problems is investigated, and the DISUM project where, among other things, teachers' interventions during their coaching of students' problem solving processes are analysed.

36.1 The Projects COM^2 and DISUM

Within the discussion on mathematical modeling it is still an interesting question how teachers actually deal with modeling problems in the classroom.

A lot of studies make clear how important the role of the teacher during the modeling process is (see e.g. Lesh and Doerr, 2003, Schorr and Lesh, 2003, Blum and Leiß, 2007a). For example, Lesh and Doerr (2003, 126) pointed out that most studies focus on what "the teacher does in a particular situation, but not how the teacher thought about the context, what alternatives she considered, what purposes she had in mind or what elements of the situation she attended to and the meaning of those elements." The latter was the focus of their study and they conclude that the development of teachers' models during modeling-eliciting activities is of great importance. Doing practical work with teachers is a good and effective way of

R. Borromeo Ferri (✉)
University of Hamburg, Hamburg, Germany
e-mail: borromeo@erzwiss.uni-hamburg.de

W. Blum (✉)
University of Kassel
e-mail: blum@mathematik.uni-kassel.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_36,

revealing teachers' thinking. Schorr and Lesh (2003) developed so called "thought-revealing activities" for teachers who participated in a 3-year study. The results of this study also make clear that a big change happened with the teachers on different levels concerning their behaviour while pupils worked on modeling problems.

In this paper, we report two studies where experienced German teachers were observed while dealing with modeling tasks in grades 8–10 (14–16-year-olds). A common feature was that most of the teachers have not reflected on how they deal with modeling problems in the classroom concerning their preferred mathematical thinking style or their preferred types of interventions. Their actions and reactions were often on an unconscious level. One aim of our studies was to make more visible what the teachers actually do. We will now introduce these two studies in more detail (with a stronger emphasis on the first author's study).

The aim of the project COM^2 ("*Co*gnitive-psychological analysis of *modeling* processes in *m*athematics lessons"), which was directed by the first author at the University of Hamburg, is to analyse teachers' and students' actions, interactions and reactions while working on modeling problems in mathematics lessons from a cognitive perspective. For that aim, "theoretical glasses" are needed which make clear how the data were finally analysed and interpreted in this sense. These "glasses" are the theory of *mathematical thinking styles* (Borromeo Ferri, 2004; for roots of this concept in cognitive psychology see Sternberg, 1997). The term *mathematical thinking style* denotes "the way in which an individual prefers to present, to understand and to think through mathematical facts and connections using certain internal imaginations and/or externalized representations. Hence, mathematical thinking style is based on two components: (1) internal imaginations and externalized representations, (2) the holistic respectively the dissecting way of proceeding when solving mathematical problems." (cf. Borromeo Ferri, 2004), 50). Empirically, three mathematical thinking styles of students attending grades 9/10 could be reconstructed:

- *"Visual" (pictorial-holistic) thinking style*: Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations as well as preferences for the understanding of mathematical facts and connections through existing illustrative representations. The internal imaginations are mainly effected by strong associations with experienced situations.
- *"Analytical" (symbolic-dissecting) thinking style:* Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and clearly define their expressed ideas in formalisms.
- *"Integrated" thinking style:* These persons combine visual and analytic ways of thinking to the same extent.

Mathematical thinking styles should not be seen as mathematical abilities but as preferences how mathematical abilities are used. Thinking styles are mostly set on an unconscious level of personality so that an individual does not know about his or her mathematical thinking style. But this has several implications for teaching and learning mathematics (Borromeo Ferri, 2004, 2007). Some pupils described that they do not understand their teachers while being taught math by them. This has nothing to do with the fact that the explanations of the teacher are bad. The hypothesis is that the mathematical thinking style of the teacher does not match with the mathematical thinking style of the learner. Then both the teacher and the learner are not talking in the same "mathematical language". In this paper, the focus will be on teachers only.

The leading questions of the project DISUM ("*D*idaktische *I*nterventionsformen für einen *s*elbständigkeitsorientierten aufgabengesteuerten *U*nterricht in *M*athematik"), directed by the second author together with R. Messner (both University of Kassel) and R. Pekrun (University of München), with research staff D. Leiß, S. Schukajlow and M. Müller, are (see Blum and Leiß, 2007b):

- What cognitive potential do given modeling tasks have, and how is it used by students and teachers – how do they actually deal with such tasks?
- Which effects do different teaching conceptions and different types of teachers' interventions have?

Students and teachers were observed both in laboratory situations (pairs of students working together on modeling tasks, partly with and partly without the support of a teacher) and in the classroom. The theoretical background for our observations of teachers' interventions in particular is the classification according to Leiß and Wiegand (2005): organisational, affective, content-related and strategic interventions. The focus of all observations is the crucial question how the subtle *balance between students' independence and teacher's guidance* is realised in the classroom (according to Maria Montessori's principle: "Help me to do it by myself!")

Both projects also aim at implementing their insights into mathematics teaching and mathematics teacher education. An important instrument for analyses and observations in both projects is our version of the modeling cycle (Fig. 36.1; see Blum/Leiß, 2005, Blum and Leiß, 2007a,b, Borromeo Ferri, 2006). We illustrate it using a modeling task called "Lighthouse."

The real situation is given by the task. The first step is mentally imagining the situation, consisting of the lighthouse, the ship and the surface of the earth in between (a non-trivial step for many students). The resulting "situation model" (Kintsch and Greeno, 1985) has to be simplified, idealised and structured: the earth as a sphere, the ship as a point, and free sight between lighthouse and ship. Mathematisation leads to a mathematical model of the real situation, with $H \approx 30{,}7$ m as the height of the lighthouse, $R \approx 6370$ km as the radius of the earth and S as the unknown distance lighthouse-ship. Mathematical considerations show that there is a right-angled triangle, and the Pythagorean theorem gives

$$S^2 + R^2 = (R + H)^2, \text{ hence } S = \sqrt{2\,RH + H^2} \approx \sqrt{2\,RH} \approx 19{,}81 \text{ km}$$

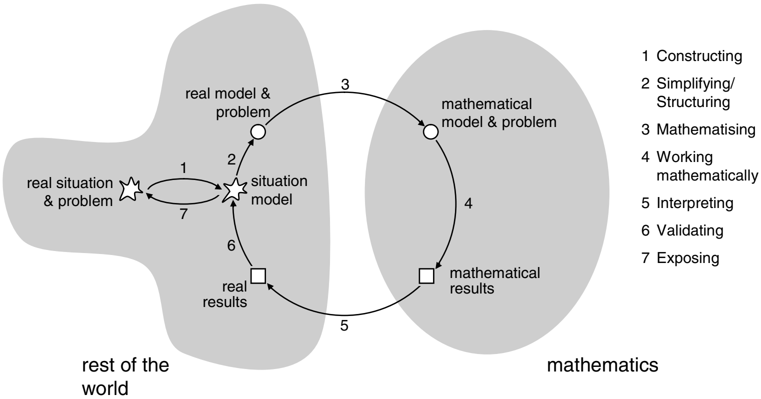

Fig. 36.1 Modeling cycle under a cognitive perspective

Lighthouse
In the bay of Bremen, directly on the coast, a lighthouse called "Roter Sand" was built in 1884, measuring 30.7 m in height. Its beacon was meant to warn ships that they were approaching the coast. How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution.

Interpreting this mathematical result (the fifth step) gives the answer "approximately 20 km" to the initial question. Now this real result has to be validated: Is it reasonable, are the assumptions appropriate (the ship is certainly not a point, etc.)? If need be, the cycle may start once again with new assumptions. The final step is always the exposure of the result.

36.2 Teachers' Mathematical Thinking Styles

The central questions of the COM^2 study were: *How do mathematical thinking styles of teachers influence their way of dealing with mathematical modeling problems in the classroom? Are there differences with respect to the various phases of the modeling cycle* (*real situation, situation model, real model, mathematical model, mathematical results, real results*)? The design of this qualitative study is highly complex, because both teachers and pupils are in the focus. Quantitative research seems to be inappropriate given the focus of the study, the internal cognitive processes of learners and teachers. Three grade 10 classes from different *Gymnasien* (German Grammar Schools) were chosen. The sample is comprised of 65 pupils and 3 teachers (one male, two female). Each individual in a class had to complete a questionnaire on mathematical thinking styles, which has been developed on the basis of the Ph.D. thesis Borromeo Ferri (2004).

Focused interviews were conducted with the teachers to reconstruct in each case his or her mathematical thinking style. Biographical questions were also included and questions were asked about his/her studies of mathematics at university but also about his/her current view of mathematics or about reasons why his/her view of mathematics might have changed in the course of his/her teaching life. In each lesson, pupils worked in groups of five on different modeling problems. The video-camera was directed on one group desk and had a view of the whole class during plenary discussions to record all the interactions of the learners.

Additionally, the teachers were equipped with a minidisc-recorder strapped to their body in order to document the teacher's help or suggestions during modeling as this could possibly influence the students' modeling processes. After videotaping the lessons there was a stimulated recall with each of the teachers where they were shown sequences of their behaviour in the classroom.

The modeling tasks (all taken from the DISUM project) selected for the learners are of central importance, as they delineate the field for the analyses. The tasks were analysed with regard to subject matter aspects and from a cognitive viewpoint. All the data were transcribed and analysed in detail. In accordance with Grounded Theory (Strauss and Corbin, 1990), codes were formed and used in order to break up and reassemble data. . . . On the basis of this data analyses the following theses can be formulated concerning the research questions articulated above:

- A teacher's mathematical thinking style can be reconstructed and manifests itself during individual pupil-teacher conversations, as well as during discussions of solutions, and while imparting knowledge of mathematical facts.
- Teachers who differ in their mathematical thinking styles have preferences for focusing on different parts of the modeling cycle, while discussing the solutions of the problems and while helping students during their modeling processes.
- Teachers were mostly not aware of their behaviour during modeling activities in the classroom, and were astonished about their preferences for certain parts of the modeling process, connected to their mathematical thinking style.

Mrs. R and Mr. P

In the following the results gained from two teachers, Mrs. R. and Mr. P., are presented. On the basis of the interviews, Mrs. R. is reconstructed as a visual thinker, while Mr. P. is reconstructed as an analytic thinker. The following parts of the interviews can only be an illustration to make clear which mathematical thinking style they are attributed to.

Answers from Mrs. R. and Mr. P. to two questions taken from the interview:

Interviewer: Please describe in five terms what mathematics means to you.

Mrs. R.: "Oh (5 s) a good question, okay (3 s), an interesting subject (5 s), logical thinking, ehm (3 s) making connections. Ehm, tasks, yes tasks also belong to it."

Mr. P.: "In five terms describing mathematics, yes, playing with numbers, playing with variables, logical thinking (3 s), building logical connections, yes and there is also a connection to reality. For me, mathematics is the language of physics."

Interviewer: Which view of mathematics do you think you give the pupils while teaching?

Mrs. R.: "That they know that mathematics will be good if they keep the overall view. Often I tell them that I like mathematics. I am not a formalist. When I get a task, the first thing I do is drawing a sketch. For me it is not so important that they do everything formally in a correct way but that they understand that mathematics can help them in their way of thinking."

Mr. P.: "That they mainly learn to recognize structures and, yes, I give them the connection to reality mostly through physics because I am also a physics teacher. But in mathematics I believe that they have to learn to think in structures and that they are able to 'move' within these structures so that they are able to see and to build formulae."

In the following, reactions of the two teachers after students' presentation of solutions of the "Lighthouse task" will be presented to illustrate the theses and to show how the teachers handled this problem.

Reaction of Mr P.: "That was really good. [...] But what I am missing as a maths teacher is that you can use more terms, more abstract terms and that you write down a formula and not only numbers. This way corresponds more to the way of thinking physicians and mathematicians prefer, when you use and transform terms and get a formula afterwards [...]" [Mr. P. developed a formula with the pupils after this statement.]

Reaction of Mrs. R.: "So we have different solutions. But what I recognized and what I missed in our discussion till now is the fact that you are not thinking of what is happening in the reality! When you want to illustrate yourself the lighthouse and the distance to a ship, then think for example of the *Dom* (name of a famous fair in Hamburg). I can see the *Dom* from my balcony. Or ehm, whatever, think of taking off with a plane in the evening and so on. Two kilometres. Is that much? Is that less?"

Mr. P. and Mrs. R during stimulated recall

Interviewer: Do you think that you have an unconscious preference for formalising?

Mr. P.: "I don't think about that, for me that is mathematics, yes, doing mathematics."

Interviewer: You recognized that you are a visual thinker. What about your experiences about "speaking not the same language" with some pupils. Can you give now some reflections about this communication problem?

Mrs. P.: "I had a girl who came from another school in my math class. After a while she came a lot of times after math lesson to me and told me that she is not able to understand me. She did not understand me! So I think, that she means my explanations, my mathematical explanations."

The analysis of Mrs. P's and Mr. R's lessons show that their mathematical thinking styles became evident in the discussion of reality-based tasks in the classroom.

Even more, in relation to modeling, the following interesting connection (of which only the crucial point is mentioned here) can be made:

- Mr P. as an analytical thinker focussed less on interpretation and validation. For him, the subsequent formalisation of task solutions in the form of abstract equations is important. Accordingly, the real situation becomes less important.
- Mrs. R. as a visual thinker interprets and, above all, validates the modeling processes with the learners. This becomes evident in her very vivid, reality-based descriptions she provides for the learners.

36.3 Teachers' Interventions

We concentrate here on the question: *How do teachers intervene, and how do they succeed in realising the balance between students' independence and teachers' guidance?* We report on a "Best Practice Study" with experienced teachers (all participating in a long-term reform project in Germany) in the context of the DISUM project. We observed teachers in 16 grade 9 or 10 classes teaching demanding modeling tasks in their own style. Here is an example of how a teacher handled a mistake in a very productive way. A group of students in a grade 10 Gymnasium class had solved the "Lighthouse" task successfully (distance 20 km). Afterwards, they had refined their model by including the height of the ship (10 m), and the result was that the distance became shorter (16 km), that is the lighthouse can now be seen later from the ship – an obvious nonsense! The reason for that result is that the students used a wrong model, which, in effect, involves $\sqrt{H-h}$ instead of $\sqrt{H}+\sqrt{h}$ in the distance calculation (here H, h are height of lighthouse and ship – the students used only concrete numbers in their calculations, no variables). The teacher let the students finish their work and made his own calculations in parallel, based on this model, and only in the reflection phase after the students' presentation of their wrong solution did he point to this mistake. Here is an excerpt of what the teacher said:

Indeed: if one calculates in the way you did then the following is true: the higher the ship, the later you can see the lighthouse.. ..

I would like you to try to realise the relationship between the height of the ship and the solution of this task by means of a sketch. Is the way this calculation was done really correct? I have calculated exactly in the way Max has presented it here.

Thus, the teacher revealed the inherent cognitive conflict, and then asked the students to deal with this problem and to find the mistake by themselves. In fact, the students discovered their mistake and corrected it independently.

This was an example of an adaptive, independence-preserving intervention. But, these were not often found. Here are two general observations from the study:

- Teachers' interventions were mostly intuitive and not independence-preserving, they were mostly content-related or organisational and rarely strategic. The spectrum of interventions that a specific teacher used was mostly rather narrow, many teachers had their own "intervention style", independent of the individual students' needs.
- Often the teachers' own favourite solutions of the modeling tasks were unconsciously imposed on the students through their interventions, also because of an insufficient knowledge of the "task space" on the teachers' side. Most teachers were very surprised after this was revealed by our study.

In his Ph.D. thesis, Leiß (2007), the author revealed common features of teachers' interventions also in a laboratory environment. For instance, many teachers showed a preference for one-step hints that students were, according to their experiences in maths lessons, immediately able to understand and to use (whereas content-related interventions that aimed at students' own reflections were often not understood and sometimes even refused by students). Most teachers intervened astonishingly often not because of students' difficulties but because their own (mostly formal) demand on the solution was not fulfilled. Also in the laboratory, nearly no strategic interventions could be observed.

There is no space here to report in detail on a recent case study within the DISUM project where two optimised teaching styles were compared, one more teacher-guided ("directive") style and more independence-oriented ("operative-strategic") style. All lessons (ten per class) were videotaped and analysed. There was a pre-test immediately before the teaching unit, a post-test immediately afterwards and a follow-up-test three months later. One of the most important results was that the progress in modeling competency of the students in the "operative-strategic" classes were substantially higher and, in particular, more enduring compared to the "directive" classes (and even more so compared to students working totally alone). The differences in progress resulted mostly from the stronger students, whereas the progress of the weaker students was similar within the two approaches. The best results concerning progress in modeling competency were achieved in the classes were the balance between students' independence and teachers' guidance was, according to our observations and ratings, realised best.

36.4 Implications for Teaching Mathematical Modeling

We have emphasised that our studies also aim at using our insights for improving mathematics teaching and teacher education. Here are some obvious implications resulting from our studies:

- Our version of the modeling cycle is helpful and even indispensable both for teachers (as a basis for their diagnoses and interventions) and for researchers (as a tool for describing actions and cognitive processes in learning environments with modeling tasks). For students, a simpler version seems to be appropriate (we have developed a four-step "Solution Plan" for students, see Blum and Leiß, 2007b).
- It is necessary to make mathematics teachers aware of their own thinking styles and thus to support them in consciously finding an appropriate balance between thinking and acting within mathematics and thinking and acting within the real world. Reflecting on their own thinking styles will also help teachers to better communicate with their students, in particular with those whose thinking styles do not match with theirs.
- It is necessary to make mathematics teachers aware of their own intervention styles and to supply them with a broad spectrum of intervention modes in various teaching situations with modeling tasks, and thus to support them in better realising the afore-mentioned balance between students' independence and teacher's guidance. For that, a lot of reflected experience on the teachers' side is necessary.

We try to implement these aspects in pre-service and in-service teacher education – in particular we try to link our approaches even more closely together. This involves searching for possible connections between teachers' mathematical thinking styles and teachers' intervention styles, both on a descriptive level and for a well-aimed broadening of teachers' repertoire of actions, to support students' successful solving of modelling tasks, and students' developing of modelling competency.

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Chapter 37
Future Teachers' Professional Knowledge on Modeling

Gabriele Kaiser, Björn Schwarz, and Silke Tiedemann

Abstract In this paper, qualitative results are presented for a case study about future math teachers' professional knowledge restricted to the area of modeling competencies. Based on interviews and the results of open questionnaires, the future teachers' competencies were evaluated in the areas of mathematical knowledge (so-called mathematical content knowledge), knowledge of mathematics pedagogy (so-called pedagogical content knowledge in mathematics) and knowledge of educational psychology (so-called general pedagogical knowledge). The study shows that in order to develop a comprehensive understanding of modeling and its pedagogical value, future teachers need appropriate knowledge and competencies in mathematics, mathematics pedagogy and general pedagogy. The study underlines the central role pedagogical content knowledge plays in the development of professional knowledge of teachers.

37.1 Introduction

Teacher education has been criticised for a long time without its effectiveness ever being analysed empirically on a broader base. There are only few empirically based results about the impacts of the various worldwide teacher education systems on future teachers' knowledge and development of competencies (for an overview of the actual situation of empirical research see Blömeke, 2004). Therefore, the

G. Kaiser (✉)
University of Hamburg, Hamburg, Germany
e-mail: gabriele.kaiser@uni-hamburg.de

B. Schwarz (✉)
University of Hamburg, Hamburg, Germany
e-mail: schwarz@erzwiss.uni-hamburg.de

S. Tiedemann (✉)
University of Hamburg, Hamburg, Germany
e-mail: silketiedemann@web.de

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_37, © Springer Science+Business Media, LLC 2010

International Association for the Evaluation of Educational Achievement (IEA) started an international comparative study in 2006, using the education of mathematics teachers of the primary and lower secondary levels as examples (Teacher Education and Development Study: Learning to Teach Mathematics – TEDS-M). In this study the effectiveness of different teacher education systems shall be analysed for the first time. Concerning the future teachers, TEDS-M will record data about their knowledge in three areas: mathematics, mathematics pedagogy, and education and psychology. Beyond that, future teachers will be asked about their beliefs and personality traits. Furthermore, institutional and curricular analyses will be conducted, and the intended and implemented curriculum of the educational institutions will be recorded. The results of this study, involving 20 countries including the USA and Germany, are expected by January 2010.

Due to problems with the conceptualisation of the theoretical foundation on behalf of IEA, a pilot study, the project "Mathematics Teaching in the 21st Century (MT21)" has been conducted in six countries, also including the USA and Germany. As part of this study at the University of Hamburg, several supplementary and more detailed case studies on professional knowledge of future teachers have been carried out. Results that are restricted to the area of modeling competencies will be presented and discussed in this paper. This supplementary study concentrates exclusively on the micro level of MT21, the level of individual competence acquirement, and is limited to the first phase of teacher education within a consecutive structure of teacher education.

37.2 Theoretical Framework of MT21 and TEDS-M and the Objectives of the Supplementary Case Study

The initial ideas of MT21 und TEDS-M consider the central aspects of teachers' professional competencies as defined by Shulman (1986) and developed and differentiated further by Bromme (1994, 1995) and others. For discussions on theoretical professional foundations see Blömeke (2002). TEDS-M and MT21 are based on the conceptions of professional competencies of future mathematics teachers elaborated by Bromme (1992) and Weinert (2001), and they are aimed at collecting data about the requirements for professional tasks of future teachers, such as teaching and making diagnoses. Furthermore, referring to Shulman (1986), the following knowledge areas are distinguished:

(1) Mathematical content knowledge, sub-divided into:
The required cognitive activities of future teachers, based on fundamental ideas of mathematics such as algorithmising or modeling;
Mathematical content areas such as algebra or statistics;
Mathematical levels, i.e. school mathematics of lower secondary level or upper secondary level, school mathematics from a higher standpoint and mathematics at university level.

(2) Pedagogical content knowledge in mathematics sub-divided into:
Mathematical content areas as under point (1);
Teaching-related tasks of mathematics teachers like elementarisation of mathematical concepts or the diagnosis of students errors;
Stimulated cognitive activities of students, including amongst others problem solving or modeling in everyday life situations.
(3) General pedagogical knowledge focusing on teaching and diagnostic questions: Referring to Bromme (1995) mathematical pedagogical content knowledge is understood as the central field where mathematical content knowledge, general conceptions about mathematics, knowledge about curricular conceptions of mathematics teaching, and aspects of teaching experiences as well as knowledge about the students perceptions are interwoven with each other.
(4) Professional competencies include affective and value-oriented aspects apart from cognitive dimensions of knowledge measured via belief components. These aspects will be differentiated according to beliefs about mathematics as a scientific discipline, beliefs about teaching and learning mathematics, beliefs about teaching at school and learning in general, and beliefs about teacher education and professional development.
(5) Personality traits in a professional and non-professional context: As these aspects cannot be considered completely within the framework of the following case study, we will not discuss them further.

Altogether, due to a lack of space, here we will not give detailed descriptions, especially not concerning the differences between MT21 and TEDS-M. Thus, we refer to the framework of TEDS-M and publications on MT21 (Tatto et al., 2007; Blömeke et al., 2008). Similar conceptualisations of professional knowledge of mathematics teachers are used in other studies as well. We refer to the COACTIV study, which basically deals with questions about the conceptualisation and the measurement of subject-based professional knowledge of mathematics teachers and possible correlations with students' development of achievement (see among others Brunner et al., 2006a and the overview of Baumert and Kunter, 2006). However, further conceptualisations of professional knowledge exist: Thus Ball et al. (2005) distinguish among common knowledge of content, specialised knowledge of content acquired through professional training, and knowledge of students and content – but due to lack of space, we cannot go into detail here.

Within the scope of MT21, we have conducted detailed in-depth case studies in order to analyse future teachers' professional knowledge. Here we will present results concentrated on modeling. The case study focuses on establishing foundations for professional competencies of future teachers during the first phase of teacher education. In MT21 and TEDS-M, modeling often plays an important role, but only as one cognitive activity among others. However, due to its significant didactical and politico-educational relevance we decided to concentrate on this aspect. Within the framework of our case study, modeling will be more differentiated drawing from the didactical debate on modeling (for example see Blum, 1996;

Blum et al., 2007; Maaß, 2004; Kaiser, 2007). We are using the following idealised description of a modeling process:

A real world situation is the process' starting point. Then the situation is idealised, i.e. simplified or structured in order to get a real world model. Then this real world model is mathematised, i.e. translated into mathematics so that it leads to a mathematical model of the original situation. Mathematical considerations during the mathematical modeling process produce mathematical results, which must be reinterpreted into the real situation. The adequacy of the results must be checked, i.e. validated. In the case of an unsatisfactory problem solution, which happens quite frequently in practice, this process must be iterated (Fig. 37.1).

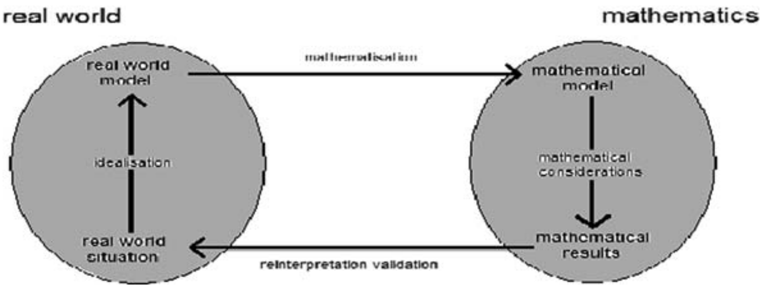

Fig. 37. 1 Description of modeling process (from Kaiser, 1996, p. 68; Blum, 1996, p. 18)

For each phase of the modeling process, the following kinds of competencies are distinguished:

- Sub-competencies for carrying out a single phase of a modeling process like structuring a real world situation, developing a mathematical model, validation of a solution etc.;
- Meta-cognitive modeling competencies like the competence to reflect critically about already executed modeling;
- The competence to argue.

The case study presented in the following is analysed based on the above-described conceptualizations and differentiations. The question being investigated is: What are the mathematical content knowledge, the mathematical pedagogical content knowledge, and the general pedagogical knowledge the future teachers possess?

37.3 Methodical Approach

Within the framework of a number of additional qualitative studies related to MT21, a questionnaire with open items has been developed that concentrates on the areas "modeling and real world context" and "argumentation and proof", admitting more qualitative analyses. This questioning was conducted with 80 future mathematics teachers on a voluntary basis within the scope of pro-seminars and advanced seminars for future teachers at the University of Hamburg. The questionnaire consists of 7 items that are domain-overlapping, designed as so-called "Bridging Items". Each of the items captures several areas of knowledge and related beliefs: 3 items deal with modeling and real world examples, 3 with argumentation and proof, and one is about how to handle heterogeneity when teaching mathematics. Furthermore, demographic information like number of semesters, second subject, attended seminars, and teaching experiences has been collected (for first results see Schwarz, Kaiser and Buchholtz, 2008).

Based on this, 20 future teachers – participating on a voluntary basis – were questioned in more detail by means of problem-centred guided interviews (Witzel, 1985). The guidelines for the interviews contain pre-structured and open questions (ask-back questions). For instance, future teachers were given simple items they worked on during the interview. To obtain information about modeling competencies, the future teachers were given an item that asked them to compare various mobile telephone rates and, by considering specific conditions, to suggest one rate. More specifically, the future teachers were given a table with 3 different mobile telephone rates that varied in their basic charge, charges for calls per minute, at peak times and other times, as well as if connecting to various phone networks. The future teachers were asked to execute a modeling process by referring to the above shown modeling circle and to describe what they were doing. Then they were asked to analyse wrong solutions of students. They were given a student solution in which the charges per minute for calls using different phone networks were added up but not weighted according to the duration of the calls.

The interviews lasted about 45–90 min, and, in order to avoid fear of failure and worries of assessment, future teachers rather than professors or university lecturers conducted the interviews. For the case study presented below, 3 of the most interesting interviews were chosen. The audiotaped interviews were transliterated then evaluated according to qualitative content analysis methods (Mayring, 1997). We referred to methods of structuring content analysis aiming at extracting a specific structure from the material by using criteria that have been defined before. This means that by referring to definitions, to typical passages of the text functioning as so-called anchor examples as well as to encoding rules, an encoding guideline is developed to analyse and to structure the material. Our main content categories were defined in a theory-guided way and oriented towards professional teacher knowledge as described above. The mathematical knowledge is restricted to school mathematics content knowledge and focuses on modeling. For pedagogical content knowledge in mathematics knowledge about the goals of modeling items, about possible ways of lesson planning, and the competence to identify students' misconceptions and problems of understanding were taken into account. For general pedagogical knowledge, motivational aspects and information about how heterogeneity is handled were considered. Then, by means of a developed category system, the interviews and the questionnaires were structured and encoded. A generalization of the results is neither intended nor possible due to the small sample size. However, the results permit insights into possible structures of professional knowledge of future mathematics teachers concerning modeling.

37.4 Case Descriptions

In the following, three future teachers (names are changed) are introduced. They are described according to the above questions and based on the results of the interviews and the questionnaires.

Anita is, at the time of the interview, studying in the 4th semester for becoming a teacher in remedial pedagogy. She has completed 2 practical courses in school and has teaching experience from giving private lessons. She has attended 2 lectures in mathematics, 4 seminars on mathematics pedagogy and general pedagogy especially focused on teaching students with learning handicaps.

Betül is studying to become a teacher at the primary and lower secondary level, and is in her 4th semester. Her second subject is sports. She has already finished one school practicum, is giving private lessons, and helps children doing their homework at an after-school care club. She has already attended 2 compulsory seminars on mathematics and two pro-seminars on mathematics pedagogy.

Christopher is in his 1st semester for becoming a teacher at the upper secondary level of general schools. He has already finished study in physics, his second subject. He has not yet done a school practicum but has extra-school teaching experiences from giving private lessons. Christopher has attended quite a few lectures on mathematics, but only one in mathematics pedagogy.

37.5 Results of the Study

37.5.1 Mathematical Content Knowledge (Restricted to School Knowledge)

These three future teachers show big differences between their knowledge about modeling processes and their competencies to conduct modeling processes by applying the example of mobile telephone rates. Christopher possesses profound knowledge of modeling. The same holds for Betül, who has similar knowledge and related competencies, but she has difficulties with evaluating and validating results, which we concluded because she could not discover students' wrong assumptions. This is in concordance with her description of the modeling circle where she did not consider the phase of validation. Anita has severe problems with several phases of the modeling process. She does not understand the real world situation completely, so makes wrong and over-simplified assumptions. When the mathematics-related phases are finished, she has difficulties reinterpreting the results into the real world perspective. The phase in which the results have to be validated does not exist in her explanations. As she did not point out the validation phase in her description before, it must be assumed that she does not have any knowledge about this phase or that she is not conscious that it is necessary.

As anticipated, the study shows that knowledge about the modeling process and modeling competencies are inseparable from each other; knowledge about the modeling process has an impact upon modeling competencies. Furthermore, modeling difficulties became particularly obvious in the area of the real world, i.e. during the phase when the real world situation has to be interpreted or understood and a real model to be established through simplification and idealisation. This is consistent with results from empirical studies on the execution of modeling processes with students where similar problems occurred (see among others Kaiser and Schwarz, 2006; Houston and Neill, 2003). Taken as a whole, the interdependence of subject matter content knowledge and chosen program of study is obvious: Christopher strives to become a teacher at the *Gymnasium* (school for high attaining students at lower and upper secondary level), Betül to becoming a teacher at primary and lower secondary level schools (for low and intermediate achieving students), while Anita wants to teach at schools for handicapped children. We will repeatedly refer back to the aspect of the dependence of acquired knowledge and developed competencies on future teachers' choice of where they plan to teach after they have finished their studies.

On the level of understanding the characteristics of real world examples the future teachers show significant differences: Anita reduces/simplifies the real world items to word problems; for Christopher, real world context consists predominantly of modeling. He is the one who possesses the highest competence in the area of modeling. This strongly supports the assumption that the comprehension of modeling and the understanding of real world situations are firmly linked to each other. Future teachers having a profound knowledge of modeling processes also have a good understanding of real world situations, so that they do not tend to reduce them to simple word problems.

37.5.2 Pedagogical Content Knowledge in Mathematics

The study points out that future teachers' profound knowledge of modeling processes and modeling problems may not go along with a profound knowledge of the objectives of teaching modeling. This explains why Christopher names only two didactical aims of modeling although he possesses a sound content knowledge of modeling. Betül does not have as sound subject matter content knowledge of modeling as Christopher, but she has very concrete ideas concerning the aims of modeling tasks so that she names several target levels for teaching modeling. Anita, who has the lowest subject matter content knowledge of modeling showing significant gaps of knowledge in this area, does not possess any pedagogical content knowledge concerning teaching modeling items, either. Seen from the perspective of the interrelation between subject matter content knowledge and pedagogical content knowledge (see among others Brunner et al., 2006b) the above described result is understandable: pedagogical content knowledge of modeling objectives depends on content knowledge in modeling. Therefore, a lack of the respective mathematical content knowledge leads to gaps in curricular knowledge as with Anita. General pedagogic knowledge is not firmly bound to mathematical content knowledge, for instance the competence to recognise misconceptions of students, but it constitutes an independent category of knowledge. For this reason Betül is able to acquire pedagogical content knowledge concerning the objectives of teaching modeling although she has lower subject matter content knowledge in modeling.

However, in other areas of pedagogical content knowledge – particularly concerning the curricular possibilities and necessities for teaching modeling – the interdependence between content knowledge of modeling and pedagogical content knowledge are strongly obvious because a profound subject matter content knowledge constitutes a precondition that enables the future teachers to develop pedagogical content knowledge. Thus, in connection with teaching modeling problems, Christopher emphasises the phase of idealisation and simplification of real world situations leading to real world models. In addition to that, he emphasises that enough teaching time should be provided for teaching modeling. Furthermore, he stresses the need to develop the students' meta-knowledge, i.e. knowledge about the necessary phases of a modeling process and about the transferability to other modeling problems. Anita, who has severe problems with the real world phases of modeling, emphasises mathematical activities for teaching modeling. She does not see any necessity to impart meta-knowledge about the modeling process. It seems reasonable that this attitude is strongly influenced by her explicit deficits of mathematical knowledge. Betül performs below Christopher's level concerning modeling content knowledge but is clearly better than Anita, which means that in this area she holds an intermediate position.

In the field of curricular knowledge about modeling problems, only small differences could be observed among the three future teachers. Due to lacking mathematical knowledge, Anita is not able to integrate modeling problems properly into a curriculum, but she develops acceptable ideas about the time that is needed for treating the item about mobile phone rates in a lesson because she has concrete ideas about the knowledge of her future students. The 45 min estimated by Betül and Christopher are too short for tackling such a modeling problem in school, even with well-performing and modeling-experienced students. However, Christopher is the only one who is able to arrange the given modeling task into the curriculum. Once more this shows the interdependence of the competence of understanding a modeling problem and the respective content knowledge.

Each of the three future teachers is able to develop concrete curricular suggestions for treating modeling examples like the example "mobile telephone rates", but in different ways: Anita and Betül would reduce the information given in the work sheet in order to cope with lower performing students, but Betül would work out the problem by herself, while Anita would not. Only Christopher is able to create a concrete lesson plan for tackling the problem at school: he would proceed gradually and not start with computations because the students should first get a feeling for the real world situation. He would not reduce the information in the work sheet in order to clarify the "computational problem" but let the students make assumptions and possibly let them simplify the situation. Beyond that, Christopher emphasises the interdisciplinary attempt of the modeling example and could imagine organising project classes about the topic "mobile telephone". Christopher's ideas contrast strongly with Anita's and Betül's approaches, where both would try to transform the modeling problem into a word problem for which the main focus would be on computation.

In the area of diagnostic competence, the three future teachers show clear differences concerning their acquired subject matter content knowledge on one hand, but on the other hand other relations could be observed. First, in the competence to recognise student misconceptions, only Christopher recognised the misconception in the given solution presented by students where the using of different telephone networks is not weighted, the costs for simultaneous using of three networks are calculated, and variables are treated quite carelessly. Betül and Anita do not see this, but list unimportant aspects like the neglect of installation fees that can be neglected for a good reason: they are the same for all the given rates. Due to his content knowledge of modeling Christopher is able to ascertain the misconceptions of the students which indicates that this component of diagnostic competence strongly depends on components of content knowledge.

Concerning other facets of diagnostic competence, such as the competence to anticipate students' difficulties during the different phases of the modeling process, no differences between the three future teachers could be observed. Each of them only has a vague idea of the potential difficulties which they identify according to their individual ideas of modeling problems, for instance in the area of computation like Anita who reduces the modeling task to a computational problem. Christopher and Betül foresee problems more in the phase of comprehending the real world situation. However, the content knowledge of the modeling cycle does not lead automatically to an understanding of the difficulties students may experience during the various phases of the modeling process. This kind of knowledge either depends on practical experience and only can be acquired in respective practice phases, or depends on general pedagogical knowledge.

37.5.3 General Pedagogical Knowledge

General pedagogical knowledge is, in this case study, examined only concerning knowledge of motivation of students. All three future teachers emphasised the necessity of students being motivated. They all felt that students would be motivated in this case because the topic, "mobile telephone rates", was up-to-date. If students were not motivated, each of the three future teachers would try to find out the reason for the unwillingness or lack of interest to work on the modeling task by communicating with the students and by demonstrating the relevance of the topic. Both Anita and Betül would give up sooner than Christopher if students would not be willing to cooperate.

37.6 Summary and Conclusions

As a whole, the results show the relevance of pedagogical content knowledge for professional teacher knowledge as pointed out by Bromme (1995). Pedagogical content knowledge can be regarded as a conjunction of subject matter content knowledge and general pedagogical knowledge, and is influenced by deficits in one of the other areas of knowledge. How far these dependencies might possibly be generalised we will know later from the results of MT21 and TEDS-M.

Taking into account the opportunities to learn as offered by university seminars and lectures, the future teachers in question have been offered the respective curricular and pedagogical content knowledge within the framework of compulsory seminars or lectures on mathematics pedagogy. In this case these three future teachers attended the same seminar, but their knowledge showed significant differences. Thus, Christopher has acquired a profound knowledge of (school) mathematics concerning modeling and developed respectively sound modeling competencies. In contrast, Anita only has minimal subject matter content knowledge about the modeling circle and therefore has difficulties in the area of pedagogical content knowledge with modeling. Taken together, it also became clear that subject matter content knowledge depends on the program of study chosen by a future teacher: Christopher wants to become teacher for higher-attaining students, Betül wants to teach at primary schools and schools for lower and intermediate attaining students while Anita plans to become teacher at schools for handicapped children. These strong differences are also shown within the framework of a large-scale study, the COACTIV study, that demonstrates clearly evident differences of the acquired competencies between the different kinds of teacher careers (Brunner et al., 2006b). Likewise, another case study of ours confirms this coherency, too (see Kaiser et al., 2007).

The chosen program of study has an impact on the future teachers' attitudes and expectations concerning the expected knowledge of students and therefore also concerning "reasonable tasks". This also influences the subject matter content knowledge that is regarded as necessary: Anita assumes that at schools for handicapped students she will have to deal with low-performing students who need to be only prepared for their everyday life and therefore should not be given too complex problems. The same holds for Betül, who also reduces the modeling problems to computational problems that should not be too complex. Both future teachers think that real world contexts means to work on word problems. In contrast, Christopher imagines higher performing students at the upper secondary level and therefore wants to treat real world contexts comprehensively.

It is remarkable that the area of diagnostic competence of all three future teachers is not highly developed. This might be explained by the fact that they have just started their study in mathematics pedagogy.

As a whole, it needs to be analysed whether the interweaving of mathematical content knowledge, mathematics pedagogical content knowledge and general pedagogical knowledge and their dependence on the chosen teaching career is also found in other areas of school mathematics. Therefore, in this context, it is important to examine the above found interrelations through large-scale studies.

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Chapter 38
Theory Meets Practice: Working Pragmatically Within Different Cultures and Traditions

Fco. Javier García, Katja Maass, and Geoff Wake

Abstract We explore the question of how a research community focused on mathematical modeling might best inform teachers' practices from the viewpoint of an international collaborative project LEMA[1] which seeks to develop a teacher training program for use in different countries with their distinctive cultures and traditions. Following identification of some of the key challenges, we illustrate these from the different perspectives of three of the partners involved. Then, we seek to identify where common ground and indeed differences might be used to enrich such a project. In doing so, we invite other researchers to reflect on their perspectives developed within their own cultural settings and to consider how they might help inform teacher education programs that seek to promote mathematical modeling.

38.1 Introduction

At the beginning of a new century, in many countries, mathematics, as it is taught in schools, seems to remain almost the same as in the past. Students continue to study a mathematics that is only useful inside the classroom, but rarely in the world beyond. Indeed, studies that have explored the use of mathematics in out-of-school situations, such as workplaces (e.g. see Williams and Wake, 2007), illustrate that

F.J. García (✉)
University of Jaén, Jaén, Spain
e-mail: fjgarcia@ujaen.es

K. Maass (✉)
University of Education of Freiburg, Freiburg, Germany
e-mail: katja.maass@ph-freiburg.de

G. Wake (✉)
University of Manchester
e-mail: geoff.wake@manchester.ac.uk

[1]LEMA (Learning and education in and through modelling) was a EU funded Comenius project (Projekt-Number: 128740-CP-1-2006-DE-Comenius-C21, 2006–2009).

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_38,

school/academic mathematics is just one genre of mathematics with a particularly privileged position, which currently focuses on content that is often abstract and without apparent application. While research points to problems of transfer, the inability to use mathematics, and the inability to solve problems in situations such as those met in workplaces or daily life, so far day-to-day teaching practice has not been influenced by such findings.

Perhaps a more likely source of stimulus to bring about change in school mathematics curricula is the OECD PISA international comparative study which attempts to measure how well young people are prepared to meet the challenges of living in the 21st Century. In particular, PISA focuses on the four domains of mathematical, reading and scientific literacies and associated problem solving skills (OECD, 2002). This study identifies modeling as a necessary competency of someone wishing to work with mathematics in a way that allows them to make critical sense of the world about them – while also acknowledging other competencies such as communicating and problem solving that mathematical modeling also promotes. In this climate, therefore, curricula are being developed in nations across Europe and beyond to include modeling or at least some important aspects of modeling. However, our concern is that many teachers know little about modeling. So, it is difficult if not impossible for them to adopt appropriate pedagogies that engage their students with modeling tasks.

This paper presents challenges of a trans-national European Project that seeks to tackle the problem at the teacher level by developing a common course of professional development in mathematical modeling. Here, we focus on how researchers/teacher educators, with their different frames of reference and perspectives, might best inform such a programme for teachers situated in different educational cultures and traditions. This is not considered unproblematic by partners: we recognise the difficulties of informing classroom practice effectively by appropriate research. In general, therefore, we ask "how can theory best inform practice in the context of mathematical modeling within different cultures and traditions?"

38.2 Clarifying Challenges Associated with Operationalizing Theory into Practice

Before presenting, briefly, some personal thoughts and reflections of researchers/teacher educators representing three of the partner nations, we identify a major challenge that is posed by what we term the "teachers' problem" and the "researchers' problem." We illustrate this by reference to a typical scenario, before we consider how these problems articulate with the goal of our project.

(a) The teachers' problem: Teachers face many problems in their everyday classroom practice. For example: How to organize the teaching of a mathematical concept? Or: How to deal with students' mistakes or their attitudes? Consequently, new pedagogical tools often are needed to optimize teaching and learning processes or to overcome specific problems that teachers identify. For instance, consider the teacher who asks, *"every year it seems very difficult to introduce algebra in such a way that students find it useful and interesting: how can I do this so that students can assign some meaning to algebraic expressions and the way they are expected to operate with them*? *What situations can I use to introduce algebra in a meaningful way*?"

Normally, the teacher has a pragmatic need, but not necessarily a strong theoretical basis, for developing the tools that are needed. Teachers therefore conceive of the didactics of mathematics in a *technical* way. That is, it is a scientific domain where they seek specific *tools* to deal with the daily complexity of *their* classroom and the mathematical knowledge and understanding involved in teaching *their* students.

(b) The researchers' problem: On the other hand, research often seems distant from the classroom analysing the processes of teaching and leaning through the perspective of theoretical frameworks. Researchers consider the reality of school in a complex way, taking into account many variables, conditions and constraints, from different domains including:

- mathematical knowledge (for instance, considering the way mathematical knowledge is structured to be taught at school, epistemological models of mathematics, the nature of mathematics in relation to non-mathematical reality);
- "the teacher" and "teaching" (for instance, teachers' beliefs concerning mathematics and the teaching of mathematics, resources and pedagogical tools available for teaching, the impact of technology in the teaching processes);
- "the students" and "learning" (for instance, students' beliefs concerning mathematics and its teaching, general learning models, specific models for the learning of mathematics);
- the "social and cultural environment" (for instance, parents' and authorities' expectations about what mathematics is and how mathematics has to be taught, pedagogical traditions, cultural restrictions).

Given this complexity, researchers' findings are therefore unlikely to produce immediately applicable solutions to ameliorate teachers' concerns.

In terms of the previous example concerning the teaching and learning of algebra, a researcher may reformulate this as, *"If algebra is considered as a language, what intersections are there between daily and algebraic languages that might be used to inform the introduction of, and the process of learning, algebra at school? How can the semantics of algebra be developed and adapted from situations normally found in world problems?"* but may also consider, "*what is algebra? What form should algebra take at school?"* and, *"is it possible to characterise a specific 'algebraic way of thinking'? How could this 'special' thinking be developed? How is it related to a general way of mathematical thinking?".*

As this example shows, research often is not focused on one specific situation but takes a broader view looking for new didactic approaches with general applicability. Consequently, research in mathematics education often is not well positioned or structured as a field of inquiry to attend effectively to teachers' immediate needs. This leads to a recurrent problem: what *research knowledge* can be *transposed* or *transformed* into the educational system in order to directly improve current teaching practices? A challenge we face in developing a teacher training course, then, is to determine how such courses can be informed effectively by appropriate research knowledge. An additional but allied problem occurs when the development is carried out in an international context where the theory-practice divide is exacerbated by the different research cultures and even different philosophical traditions of the partners from the nations involved. Before exploring these issues further we situate them by exploring the perspectives of the partners from three nations.

38.2.1 England

In current research, the English partner works as part of a team that adopts a socio-cultural approach – drawing particularly on ideas of the development of learner identity and using Cultural-Historical Activity Theory (CHAT) (see for example, Engestrom and Cole, 1997) to explore how different pedagogical experiences can offer distinct cultural models of mathematics and what it is like to be or become a mathematics learner. This situates mathematics teaching and learning in a complex social setting with CHAT providing a framework that allows one to identify how a learner's action is mediated by not only "*instruments*" including artefacts and tools, cultural tools, concepts and language genres, but also by the way the *community* operates including its *division of labour* and its associated *norms/expectations/rules.*

In their analysis of classroom activity, the team from England notes that pedagogic practices designed to be more engaging for students by actively encouraging them to work with others and "have a go" can, for some students, allow them to develop or transform cultural models which for example, position "maths as difficult" to one which more positively positions "maths as challenging and I like a challenge" perhaps leading to a more inclusive approach to the subject. The English partner's view is that modeling can, and indeed is likely to, promote discussion-based pedagogies so that students are more inclined to identify mathematics as a field of inquiry with which they can actively and "sociably" engage. Research evidence often points to such pedagogic approaches as being particularly successful in not only stretching the most able, but also engaging those not so able (at least as measured by current assessment processes) in ways that motivate and promote self-efficacy in mathematics (e.g. see Boaler and Greeno, 2000).

With a long history of involvement in curriculum development and design in modeling based curricula, the partners from England have been influenced by the work of Freudenthal and those who have followed in the tradition of Realistic Mathematics Education (RME) who make the useful distinction between horizontal and vertical mathematization. In an attempt to connect mathematics with students' experiences via realisable situations they propose that mathematics can be used to provide models *of* situations (horizontal mathematization) and models *for* (vertical mathematization) the development of mathematics itself. In their design of curriculum materials the followers of Freudenthal attempt to achieve progression in learning by ensuring that learners are guided through situations where they use models in the horizontal sense to allow their development in the vertical sense. These curriculum materials carefully develop a relatively restricted number of models (such as, the empty number line, ratio tables and so on), through a process of guided reinvention, allowing learners access to a range of powerful tools /models both to make sense of realistic situations and develop their mathematical understanding.

It is perhaps not accidental that the English partner refers to, and often adopts and adapts, the RME approach as the cultural and historical system in which his work is situated leads to schools where there is an emphasis on learners developing strong instrumental facility with mathematical procedures rather than a focus on mathematical methods and processes. The RME approach appears to have the potential to allow a blend of both content and process in a situation, as discussion of the national context below will illuminate, where the dominant emphasis in classrooms is on the former.

38.2.2 English National Context

Mathematical modeling has had different status at different times in recent years in the mathematics curriculum at all levels in England: although ostensibly present in the “Using & Applying” strand of the National Curriculum, modeling often is given scant attention in mathematics lessons in schools. The current position is heavily influenced by a pervasive regime of national testing and accountability at institutional level, the results of which are made public and used by parents in their selection of schools for their children. Thus, although key aspects of modeling may be detected in curriculum specifications, its omission from national assessment ensures that it is given little attention as a valid mathematical activity in classrooms on a day-to-day basis. Consequently, recent national interventions designed to affect pedagogic practices in primary and secondary education focus teachers on lesson structures that do not promote mathematical modeling and indeed may be considered to have actually been detrimental in this regard. National strategies for both primary and secondary teachers have strongly promoted lessons that involve whole-class teaching with short bursts of activity – and which place a greater emphasis on oral and mental work than previously. Although such lesson structures and pedagogic practices are not prescribed, they are strongly supported by the teams that inspect schools and their judgements about lessons are important in informing their reports about the effectiveness of schools which are made public.

All is not without hope, however: although, at present, modeling and applications are relatively obscure in both the defined and implemented curriculum, this is likely to change as developments currently underway attempt to reposition the curriculum so that learners have opportunities to develop their mathematical literacy (Steen, 2001) via "functional mathematics".

In implementing a program of professional development focused on mathematical modeling in England, then, the problem is one of convincing teachers that adding a pedagogy based on mathematical modeling to their repertoire is going to be useful to them in their daily practice. This partner therefore has suggested two reasons why mathematical modeling should be a useful pedagogic approach to teaching and learning mathematics: (a) to develop a more inclusive learning community and (b) to allow the development of new mathematics based on models developed *of* situations (in the sense of RME). However, as a previous CHAT analysis of curriculum innovation suggests (Wake et al, 2004), innovations are likely to fail unless each aspect which mediates the action of learners in their community is properly and fully supported. In this case, the current rules and norms associated with assessment and inspection appear to conspire to suggest that the teacher training course is unlikely to succeed unless they are more suitably aligned than at present to support the proposed approach to the curriculum.

38.2.3 Spain

Research of the Spanish team is in the tradition of the Anthropological Theory of Didactics (ATD) as initially developed by Chevallard (1999, 2006). Mathematics is considered both as a human activity (process) and as the product of this activity, and it is modelled in terms of *mathematical praxeologies* (also called mathematical works, from the French *ouvres*). The term *praxeology* is a combination of the Greek words "praxis" (the practical or know-how dimension, including a set of tasks and techniques to solve them) and "logos" (the dimension of knowledge, including the explanation and justification of the activity performed) and it is considered as a *primitive term* to describe any human activity in an integrated way.

At school, mathematical praxeologies are not supposed to emerge suddenly or in a definite form or to "live" as *eternal objects*. They arise as the result of complex ongoing activities (called *process of study*) where new mathematical objects are created and new relations are established. The *process of study*, as a new human action (developed both by teachers and students) can be also described in terms of praxeologies (including didactic problems, didactic techniques and explanations and justifications of the "didactic activity" carried out). From a theoretical point of view, mathematical and didactic praxeologies interplay mutually: the way mathematics is structured determines how it can be studied and, reciprocally, the way a process of study is organised determines the nature of the mathematical knowledge constructed.

Although praxeologies are cultural *artefacts* that describe how humans act and live in society, school tends to neglect the process of construction and *show* only the final product, the mathematical praxeology ("linear functions", "polynomials", "symmetries", ...), as a finalised object, valuable on their own, as a part of the mathematical cultural heritage that school has the obligation to preserve. Using a metaphor, mathematical praxeologies become *monuments* that students *visit* and *honour* at school, but rarely tools for action beyond school.

Research from the ATD considers that a *new epistemology* at school which brings back the process of construction of mathematical praxeologies is needed: "For every praxeology or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way a given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its *raisons d'être* are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve." (Chevallard, 2006).

38.2.4 Spanish National Context

The Spanish curriculum is traditionally divided into general and classical blocks (*domains*) of mathematical content, following mathematical criteria not necessarily appropriate from a didactic point of view. The latest reorganisation of the curriculum extracts problem solving strategies identifying these within a separate block. This can be considered an attempt by policymakers to stress the importance of problem solving in school practices. Modeling is not explicitly included as content to be taught. However, the ultimate purpose of school mathematics is to develop students' mathematical competency, where the ability to use mathematics to solve problems related to daily life and working is included.

In Spain, one of the main challenges is the absence of any tradition of modeling in schools. Previous experiences with teachers show that they agree on the idea of making mathematics more useful but their thinking remains at a level of making in superficial and stereotypical links between mathematics and reality. It is rare for their previous training as mathematicians and as teachers to have included any applications. There is therefore a *dominant culture* defining what mathematics is, where modeling and real applications have no place.

Moreover, modeling is normally not included in materials available for teachers. So, another main challenge is to provide teachers with appropriate tools to build these kinds of tasks. Finally, teaching practices are very traditional, although it seems that more collaborative work is been gradually included in classrooms. Teachers also need didactic tools to manage situations where they are more inclined to guide students in developing mathematics rather than giving a lecture and setting practice exercises.

From the point of view of the ATD, it is assumed that the existing *dominant culture* concerning mathematics can be transformed through an explicit and deep reflection on the *raison d'être* (rational) of the mathematical content taught. It is not only important to seek some real contexts to motivate mathematical content learning. It is also important to encourage teachers to ask themselves why a piece of mathematical knowledge is necessary and useful and what it allows us as citizens to do.

Concerning teaching, the ATD proposes a specific structure of any *process of study* in terms of *didactical moments* (first encounter with a type of tasks, exploration, working technically, justifying and explaining the work done, evaluating the mathematics constructed and the process itself and giving a cultural status to the mathematical knowledge constructed). Previous research results (García, 2005) show that it is an appropriate tool to organize *processes of study* where modeling plays a central role.

38.2.5 Germany

The pivotal question for the German partner is: *Why do students have to learn mathematics?* Reasons include that the facts that mathematics plays a very important role in the development of many scientific disciplines, mathematics is important in a considerable number of professional areas allowing description and forecasting of phenomena for example in nature, utilisation of natural resources as well as design and regulation of industrial and socio-technical systems, and mathematics has an important basis for activity in every day life (Niss, 1994). However, this important role of mathematics as a basis for both science and society is not recognized by many people. While the importance of mathematics is increasing due to the use of technology, it becomes at the same time more and more invisible because it is hidden by technology (the "relevance paradox", Niss, 1994). In consequence, many people claim not to need mathematics in their everyday life and work. So, students ask teachers why they have to learn certain topics; and, quite often their teachers cannot answer this question. Mathematics often remains meaningless to students. Modeling is seen as a means to overcome this problem.

Within mathematics education in Germany, the discussion about modeling has a long tradition which goes back to the 1960s. There are, however, many different positions which can be differentiated according to the aims they see for the integration of modeling (Kaiser and Sriraman, 2006). These aims include the introduction of mathematical concepts (conceptual modeling with e.g. de Lange as representative), the promotion of learning processes (didactical modeling) and the solving of real world problems (realistic modeling), the latter of which goes back to Anglo-Saxon pragmatism (with Burkhardt, Burghes and Pollak as important representatives).

The position of Blum and Niss (1991) gathers its strength from a wide range of aims and arguments: by integrating a modeling and applications approach to school mathematics, students have opportunities to see the relevance of mathematics to society, to learn to apply mathematics in their daily lives, to critically view the mathematics of other people, and to develop problem solving strategies and to communicate with other people about mathematics. Additionally, the learning of mathematics, as well as students' motivation is to be enhanced by the integration of modeling. In order to reach these aims, Blum and Niss focus on problems outside mathematics.

The German partners' position is situated somewhere between "realistic" and "didactical" modeling. Following Blum and Niss a wide range of aims is recognised, including the promotion of learning processes as well as the competency to solve realistic problems. In order to reach the latter, the German partner puts a high emphasis on authenticity of the context. An authentic situation is regarded as an outside mathematical situation embedded in a certain field (e.g. a science or a profession) dealing with phenomena and questions which are relevant within this field and are also regarded as important by experts in this field. In the eyes of the German partner this position meets the necessity to let students experience how mathematics can be applied in authentic situations.

These didactical aspects however are not sufficient to introduce modeling into mathematics lessons. Firstly, whether or not modeling activities support the development of students modeling competencies also depends on the chosen teaching methods. Research has shown that work in groups, discussions among students, and independent work of students best support the development of modeling competencies (see e.g. Galbraith and Clatworthy, 1990). An emphasis on teaching methods is also based in the German discussion about mathematics education and education in general. Secondly, learning is also an affective and emotional matter. For this reason, special emphasis is given to affective aspects as changing students' beliefs (especially about the usefulness of mathematics) and their attitudes towards mathematics.

38.2.6 German National Context

From 2004, there have been new German-wide standards which can be seen as a reaction to the results of PISA. Supporting the demand to integrate realistic modeling tasks in these standards new curricula have been developed in some parts of Germany (e.g. in Baden-Württemberg). These may differ from region to region but they are all based on the new German-wide national standards, within which six main competencies are named which are to be taught within six main areas of content with mathematical modeling being one of these. Additionally, the standards refer to the necessity of showing students the usefulness of mathematics. German teachers therefore have to learn how to explicitly teach modeling. However, most of the teachers are not trained to teach modeling and consequently have a need for training in this. Nevertheless it is still possible for teachers to avoid teaching modeling: regional comparative tests still do not include authentic open modeling tasks, although there have recently been changes towards their inclusion.

The question however remains over how far teachers will be willing to accept and work with these ideas. Research (e.g. Maaß, 2008) shows that teachers often have firmly held beliefs about mathematics which may hinder them from accepting authentic, realistic tasks as mathematics. It is hoped that taking account of these beliefs, proceeding in small steps as well as with continuous reference to the new curriculum may help to overcome this gap between theory and praxis.

38.3 Comparing and Contrasting Perspectives

We return now to consider how we might not only use these different theoretical perspectives and approaches to inform in a very practical way a programme for the development of teachers in mathematical modeling but also use them as an opportunity to enrich the programme whilst ensuring it is as flexible as possible because of the diversity of the teachers that will take part

In brief, the situation can be summarized as follows:

(i) the German partner adopts a theoretical perspective directly focused on modeling while the English and Spanish partners adopt more general theoretical perspectives;
(ii) English and German positions are more focused on the cognitive, affective and psychological aspects whilst the Spanish position is centred on institutional and epistemological aspects;
(iii) English and Spanish teams adopt a social view of the learning processes whilst from the German position this is considered as secondary;
(iv) German and Spanish partners consider modeling within the more general debate of why mathematics should be taught and learnt at school whilst the English debate is moving towards consideration of what it is to be functional with mathematics and how modeling might articulate with this;
(v) the authenticity of the situations to be modelled is considered as a core goal in the design of modeling tasks by the German partner whilst for the Spanish and English partners modeling is considered as a tool for the teaching of mathematics and the authenticity of the tasks, although important, are subordinate to other didactical considerations. We summarise these different perspectives and positions in Fig. 38.1.

In our attempts to enrich the design of the teacher programme thorough the diversity of perspectives, we identify the following opportunities.

The German approach with a particular focus on modeling can provide teachers with useful descriptions of the modeling cycle, both for designing modeling tasks and for its didactical development in the classroom. Moreover, the German partner's concerns about authenticity of tasks can be valuable when identifying and designing tasks. Connecting with the Spanish "institutional" position, the authenticity can be also seen in terms of searching for questions *crucial* for students as social individuals, that is, questions with *social and cultural legitimacy.* In addition, these situations also have to be useful to students in allowing them to develop new mathematical understanding through a process of study as a community. From the Spanish perspective, it is important to work with tasks where more than one solution is possible and where it is not only the problem itself that is important, but also where the evolution of the original problem may give rise to new questions which may in turn give rise to new mathematical knowledge and understanding. This in turn articulates well with the importance that the English partner places on processes of horizontal and vertical mathematization of Realistic Mathematics

	Theoretical framework	Main focus of the didactical system considered	Pedagogical perspective	Nature of modeling tasks	Modeling within the teaching practices	Modeling witihn the educational debate
German partner	Modeling focus	Focused on cognitive and affective aspects	Teaching methods are to support the development of modeling competencies	Focus on authentic contexts and on promoting learning processes	Modeling mainly as a teaching goal	Connections between modeling and why teach and learn mathematics?
English partner	Mathematics education general theory	Focused mainly on cognitive aspects	Educational processes from a social perspective	Authenticity subordinat-ed to other didactical considerat-ions	Modeling mainly as a teaching tool	How might modeling inform pedagogies that make mathematics functional?
Spanish partner	Mathematics education general theory	Focused on institutional and epistemolog-ical aspects	Educational processes from a social perspective	Authenticity subordinat-ed to other didactical considerat-ions	Modeling mainly as a teaching tool	Connections between modeling and why teach and learn mathematics

Fig. 38.1 Summary of different perspectives from partner natures

Education, which may provide an organisational framework to consider not only the modeling of real situations but also the evolution of mathematical knowledge and understanding. Additionally taking into account the English perspective that mathematical activity may considered as a social activity, subject to cultural norms and restrictions, this may provide teachers and researchers with a range of new learning and pedagogical models.

The distinction between modeling as a teaching goal or as a teaching tool appears important. On the one hand, the German research tradition normally considers that modeling should become an explicit teaching object and this is now included in their curriculum. On the other hand, currently in classrooms in England this is not required and modeling if it is to find a place will have to be adopted as a teaching tool. However, from a pragmatic point of view, it is possible to assume that training for teachers may in practice be almost the same for each position. We can consider that modeling may become an explicit teaching object after it has been widely used in a variety of situations. In this case meta-reflection with students is needed in order to analyse the different steps and the process itself necessitating the equipping of teachers with "extra" pedagogic tools so as to be able to cope with this in their classrooms.

38.4 Conclusion

Here we have highlighted a general problem in using research in mathematics education to inform teachers in developing their day-to-day classroom practice. It is clear that these difficulties are compounded when attempting to inform a course of professional development for teachers with an international aspect. This adds theoretical, cultural and philosophical perspectives (on mathematics, mathematics education, teacher training, modeling, theoretical frameworks) that makes the development even more challenging. Here we have tried to make as explicit as possible these different perspectives and following an analysis of where positions both converge and diverge we hope to have identified potential opportunities to enrich our international development. We invite other researchers to reflect on their own perspectives and perhaps consider how we can use these to pragmatically inform such work in teacher education.

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Chapter 39
Secondary Teachers Learn and Refine Their Knowledge During Modeling Activities in a Learning Community Environment

César Cristóbal Escalante

Abstract The focus of this study is to investigate resources, strategies and representations that high school teachers exhibited while working on a set of problems that involves themes related to change and variation. The results of this research indicated that during their problem solving processes, the participants transit from incoherent and limited approaches to systematic and more robust ways to solve the tasks as a result of discussing their ideas within a community that values individual and group's participation.

39.1 Introducton

In past decades the need has emerged for restructuring the mathematical learning environment at different school levels. This restructure intends to provide spaces where students can research and practice mathematics in an effective way. Several research reports have described classroom learning environments as being less focused on the teacher, and more similar to a learning community. In other words, there exists a community where students perform activities as a group, and solve problems and analyze situations collaboratively. This will lead them to develop, share, and assess related ideas, individually and as a group, providing the development and use of criteria to assess these ideas in a collaborative way (Boaler, 2000; Manouchehri, 2003). In this type of learning environments, applications and modeling can be used as a way to facilitate and support students when learning math (NCTM; 2000).

In order to apply these changes, it is necessary that the teachers have a vision of the type of curriculum and instruction that will be used. Several of the teachers that work at a secondary level (and other levels) have learnt mathematics in the traditional way, where problem-solving and modeling are hardly ever considered, and individual work has been promoted. In other words, they have not learnt the

C.C. Escalante (✉)
Cinvestav-IPN, Mexico, Mexico
e-mail: cescrist@uqroo.mx

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_39,

aspects of the contents they have to teach, or the aspects of the ways they have to teach them. Therefore, current and future teachers must be trained in the direction that enhances the development of their knowledge, skills, and values relevant to participate in the proposed reforms (Adler et al, 2005).

In this context and considering the previous mathematical knowledge of these teachers, it is relevant to ask yourself: What type of activities could help teachers increase and refine their mathematical knowledge? How do activities that involve mathematical modeling and application, and collaborative work influence the development of teachers' mathematical knowledge? These questions were used to guide the development of the study.

39.2 Conceptual Framework

The term "learning community" is used to refer to a classroom environment where students are interested in achieving a shared goal (in this case, comprehending mathematics ideas or solving mathematics problems) as they get engaged in collaborative work. Thus, students infer, share, discuss, and explain the arguments that support their assumptions. In this context, students, in a collaborative way, find criteria to explore and assess their own conjectures. The instructor organizes and guides the activities so that students work individually, in small groups, and within the whole group. Furthermore, the teacher must provide activities to help students analyze situations using different resources and representations; stimulate the comparison of their results; promote individual and collective work; and encourage the need of learning, increasing, and redefining new mathematical concepts and topics (NCTM, 2000).

This transformation is not a linear and sporadic process; it depends on the instructor initiative and the ways to structure and organize the learning activities, for example, the instructor should select the tasks and the strategic instruction in terms of the type of knowledge and abilities sought, and the activities that can be developed by the students in this learning situation (Manouchehri, 2003) In this type of environments, students not only learn, explore, and examine concepts, methods, and mathematical processes in classroom activities, but also learn habits that are consistence with mathematical practice. Learning mathematics cannot be separated from this interactive participation that takes place in the classroom throughout the learning period because both of them have influence on each other (Boaler, 2000).

Problem-solving tasks based on real contexts require asking and answering questions arisen from the different perspectives embedded in the situations; therefore, there is not necessarily a unique answer like those required in most math problems. In such approach, there may be several answers, according to the conditions and assumptions made. In order to answer these problems, it is necessary that students represent and explore different approaches to create and redefine previous inferences, changing from their initial incoherent and unorganized inferences to better organized and structured ones. It is useful to observe and trace each student's assumptions to be aware of them and be able to analyze the episodes they go through to understand the situation, as well as the procedures they follow in order to answer the question or questions asked. This will give us information that will allow us to identify certain aspects of the process of the student's cognitive development (Lesh and Doerr, 2003).

A mathematical concept may have different meanings in different contexts and the ways in which it is used may be different, even if the same definition is used. During their academic studies in mathematics, teachers develop some meanings of concepts that should be seen as a part of the broad group of meanings of concepts. When students are solving problems, in a non-mathematical context, which requires the use of their previous knowledge and computational tools or other resources, they need to listen to other students and discuss their ideas openly in order to reflect, improve, and restructure the associated meaning with the concepts involved. This perspective is different from the traditional way of learning. (Biehler, 2005)

39.3 Results and Discussion

The information used was taken from the reports made by the students while working with two of the activities; the first one was assigned in the first part of the course and the other one in the last part of the course. The first activity did not have only one solution and could have been approached from different perspectives by using several mathematical concepts and procedures. The second one required students to use concepts such as successions and recursive procedures. The information was analyzed in order to identify characteristic aspects of the episodes followed by the participants when solving each problem: understanding the problem, devising a plan, carrying out the plan, and looking back throughout the different phases of classroom work, focusing to the mathematical concepts and process, and in the representation used.

Problem A. *The following table provides the number of inhabitants in a town at the beginning of each year for the last 10 years. A planning committee needs to make an estimate of the population that there will be in the following 20 years. Make a procedure for this committee to obtain the result and argue their proposal.*

Year	1996	1997	1998	1999	2000	2001	2002
Population	424	663	800	863	1071	1279	1557
2003	2004	2005	2006	2007	2008	2009	2010
1776	2004	2308					

While working individually, the first approaches to this problem were characterized by the numerical search used to try to identify a pattern in the growth of the population using the data. Therefore, they determined the annual increase of the population for all of years given. Two students determined the arithmetic mean d of the annual increase. They implicitly assumed that the population had a constant

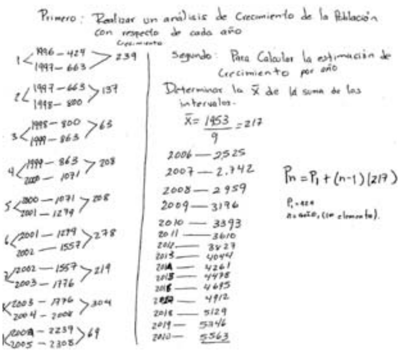


Fig. 39.1 Student E's first approach to problem

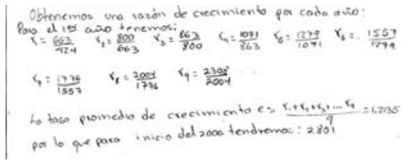
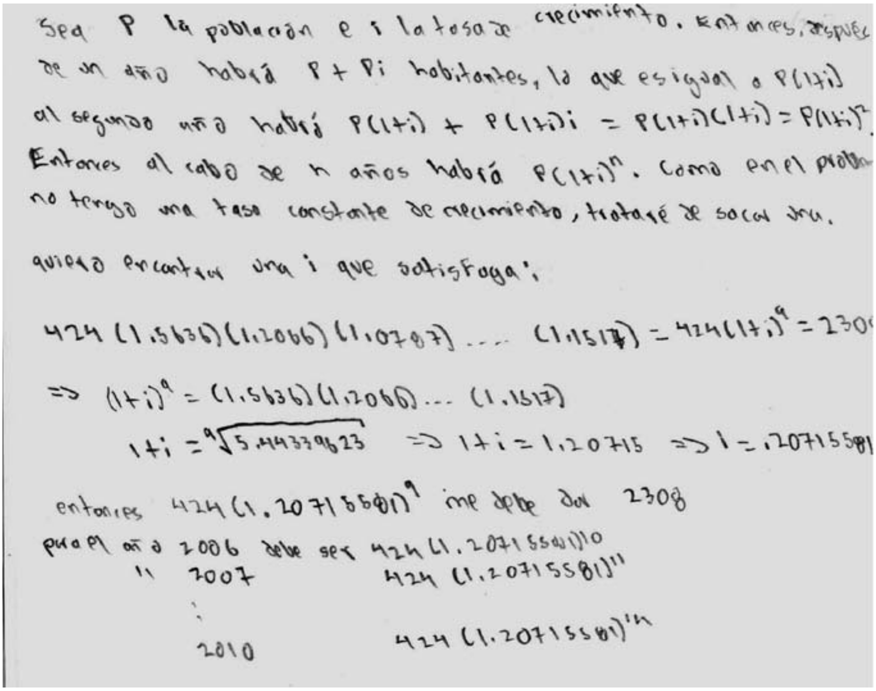


Fig. 39.2 Student C's first approach to problem

annual increase; in other words, that population grows the same amount of inhabitants every year, that amount is the medium annual increase d. They also established that the population in the year n is give by $P_n = P_1 + (n - 1)d$. They used this expression to estimate the population in the following years (see Fig. 39.1).

The other five students started considering the ratio, instead of the difference, of two consecutive populations, the subsequent population to previous population: $r_k = \frac{P_{k+1}}{P_k}$, which they named ratio of annual growth. Four of them determined the arithmetic mean $\bar{r}$ of the ratios, and assuming, in an implicit way also, that population increases every year based on the average ratio, they arrived at the expression $P_{k+1} = P_k\bar{r} = P_1\bar{r}^k$, which was used to determine the estimate of the population (see Fig. 39.2). One of these five participants considered the geometric mean of the ratio of annual increase, instead of the arithmetic mean, using a recursive process to obtain: $P_{k+1} = P_1 r_1 r_2 ... r_k$ and considered that if the ratios of annual growth were equal then he should arrive at $r^k = r_1 r_2 ... r_k$ and, therefore, $r = \sqrt[k]{r_1 r_2 ... r_k}$, the geometric mean. Like others, he implicitly assumed that the population grew every year constantly, and arrives at an expression similar to the previous one: $P_{k+1} = P_k\bar{r} = P_1\bar{r}^k$, where $\bar{r}$ is the geometric mean (see Fig. 39.3). This student included within his approaches both the concept and ratio of growth. In his first report (Fig. 39.3) he used the name *ratio of growth*, in which ratio is understood as the quotient of two quantities. Therefore, he arrives at the ratio of the subsequent population divided by the previous population $\frac{P_{k+1}}{P_k} = r_k$, but in the following report (after pair work), he used the name factor of growth, since it required using the expression $P(1 + i)^n$ to represent the growth of the population at an annual rate of growth i. Then, he expressed i the rate of growth as $i_k = r_k - 1$. This aspect was


Sea P la población e i la tasa de crecimiento. Entonces, después de un año habrá P + Pi habitantes, lo que es igual a P(1+i) al segundo año habrá P(1+i) + P(1+i)i = P(1+i)(1+i) = P(1+i)^2. Entonces al cabo de n años habrá P(1+i)^n. Como en el problema no tengo una tasa constante de crecimiento, trataré de sacar una.

quiero encontrar una i que satisfaga:

424 (1.5636)(1.2066)(1.0707) .... (1.1517) = 424(1+i)^9 = 230[illegible]

=> (1+i)^9 = (1.5636)(1.2066) ... (1.1517)

1+i = $\sqrt[9]{5.44339623}$ => 1+i = 1.20715 => i = .20715581

entonces 424(1.20715581)^9 me debe dar 2308

para el año 2006 debe ser 424(1.20715581)^{10}

" 2007 424(1.20715581)^{11}

⋮

2010 424(1.20715581)^{14}

Fig. 39.3 Student B's first approach to problem

discussed within the whole group where students agreed that it was a problem of representations and meaning of these concepts.

A different assumption from others was carried out by another participant that used Lagrange polynomials. He implicitly considered that the problem was about determining a function that passed by n given points on the plane. He considered the data as given points on a Cartesian coordinate system: time, population. In his first attempts, he used the last three data and arrived at a second degree polynomial function to determine the population for the following year (see Fig. 39.4).

While working individually, all students assumed, implicitly, that they needed to find a function of time, that it passed by, or got close to, the given values of the population. They used Excel in two different ways: to make calculations using their own procedures, such as calculating averages (they didn't use Excel's procedure); and to make graphics of the given and new data. Apparently, the graphics were only used to observe a regular behavior. None of them used the graphics to describe the behavior o justify any observations. The representations used were numerical, tables and graphics based on those tables (see Fig. 39.5).

The results of pair work indicate predominance in three approaches: the ratio of growth using the arithmetic mean, the ratio of growth using the geometric mean, and the Lagrange polynomials. Excel continues to be used in the same way as in individual work. The procedures were improved in comparison to the

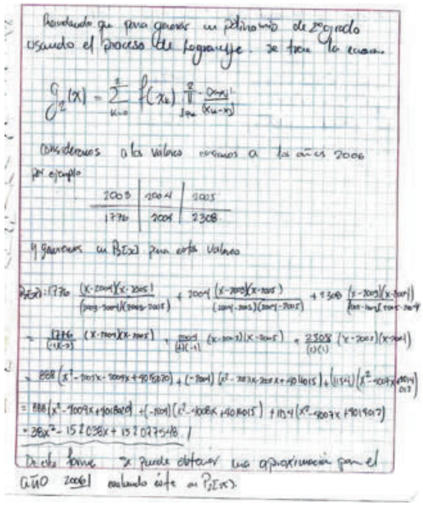
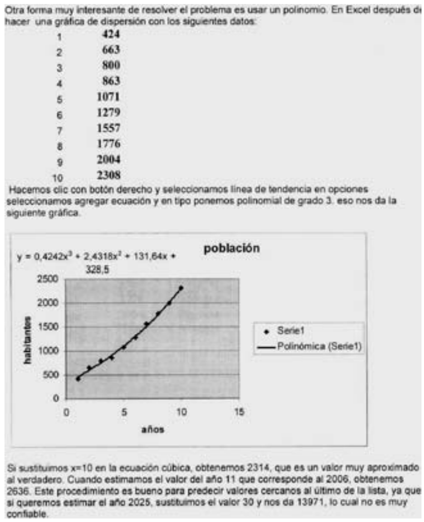


Fig. 39.4 Student H's first approach to problem



Otra forma muy interesante de resolver el problema es usar un polinomio. En Excel después de hacer una gráfica de dispersión con los siguientes datos:

1	424
2	663
3	800
4	863
5	1071
6	1279
7	1557
8	1776
9	2004
10	2308

Hacemos clic con botón derecho y seleccionamos línea de tendencia en opciones seleccionamos agregar ecuación y en tipo ponemos polinomial de grado 3. eso nos da la siguiente gráfica.

Si sustituimos x=10 en la ecuación cúbica, obtenemos 2314, que es un valor muy aproximado al verdadero. Cuando estimamos el valor del año 11 que corresponde al 2006, obtenemos 2636. Este procedimiento es bueno para predecir valores cercanos al último de la lista, ya que si queremos estimar el año 2025, sustituimos el valor 30 y nos da 13971, lo cual no es muy confiable.

Fig. 39.5 Section of Student F's report handed after pair work

first assumptions performed individually (see Fig. 39.6). Also, other approaches emerged, such as using Excel functions to determine the third degree polynomials that best approach the curve (see Fig. 39.5).

In this stage, as participants confronted their ideas regarding the solution, they discussed which should be the solution, but did not develop criteria to take a decision. That is why the reports that students handed in after this activity included one or two assumptions.

The solutions to the problem presented to the group by pairs, were of three types according to the assumption used to identify the behavior of the population: the

Creo que una buena manera de estimar la población para los años siguientes, es obtener una fórmula que esté en función de los años y de la población inicial. Sin embargo, para utilizar este procedimiento, debo suponer que la tasa de crecimiento debe compo rtarse en los años venideros, más o menos de la misma forma en que se ha venido comportando. S upongo que los datos crecen en forma geométrica es decir: $P(1 + i)^n$ donde P es la población, i es la tasa de crecimiento y n es el tiempo en años.
Las razones de crecimiento se calculan en la siguiente tabla:

AÑO	POBLACION	FACTOR DE CRECIMIENTO
1996	424	
1997	663	1,56367925
1998	800	1,2066365
1999	863	1,07875
2000	1071	1,2410197
2001	1279	1,19421102
2002	1557	1,21735731
2003	1776	1,14065511
2004	2004	1,12837838
2005	2308	1,15169661

Con esos factores de crecimiento puedo obtener la media geométrica, es decir multiplicarlos todos y sacar su raíz novena (porque son nueve datos). De esta forma obtenemos:
Tasa media de crecimiento = 1,20715581 -1, o sea, 20.71558% anual. Observemos que si hago n=0 y n=9 en la fórmula mencionada anteriormente, nos dan los datos inicial y final respectivamente, por lo que la función pasa por esos puntos y es aproximado a los valores intermedios.
Sustituyendo la tasa media de crecimiento en la fórmula, nos dan los siguientes resultados:

2006	2786,1156
2007	3363,27562
2008	4059,9977
2009	4901,04979
2010	5916,33071
2011	7141,93297
2012	8621,42585
2013	10407,4043
2014	12563,3585
2015	15165,9311
2016	18307,6418
2017	22100,1761
2018	26678,3559
2019	32204,9323
2020	38876,371
2021	46929,8369
2022	56651,6251
2023	68387,3382
2024	82554,1724
2025	99655,7485

Fig. 39.6 Fragments of the report handed in by Student B after pair work

population grows at a constant annual ratio, with two variants, the ratio of growth is the arithmetic mean of annual growth, and the other with the same assumption but the ratio is equal to the geometric mean of the ratios of annual growth, and the third, the function that describes the population is the eighth degree Lagrange polynomials. At this point, the instructor asked which should be the answer to the problem; in other words, which of all the quantities presented was the solution to the question asked and why. The participants provided arguments to support their proposal, trying, in a certain way, to discredit other pairs' assumptions, but their arguments were valid according to what each had done.

Some pointed out the fact that there were no elements to say if the population was growing at a constant ratio, even though it could be assumed. Furthermore, it was also possible to assume that the population grew at a constant quantity every year or in some other way. After some attempts, other students stated that they needed more information. So, the instructor asked what information they thought they needed. Two students pointed out they way in which the population behaved. The pair that made their assumption using Lagrange polynomials mentioned that it is possible to consider the problem as a determination of the function that passes the closest to the given points, and in this sense, the polynomial that he determined passed by each and everyone of the points given. Other participants agreed. The discussion also included aspects concerning the models and what their solution would be to this particular problem (see Fig. 39.7). In the final reports handed in, 4 of the 7 students agreed that the best estimate was arrived at by using Lagrange polynomials because it provided a function that passed by each and every one of the points. They did not consider what was expressed in the group discussion or did not accept it. The

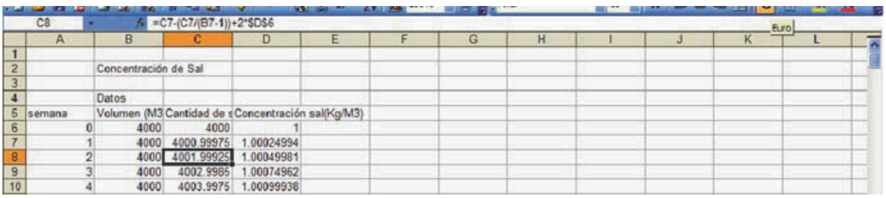
Observamos que la población crece a un ritmo constante y por lo tanto no es muy fiable.
Podemos concluir, que el método de Lagrange es el más aproximado, siempre y cuando se construya un polinomio nuevo para cada estimación de los años posteriores. Si nuestro propósito es encontrar una fórmula que nos prediga la población dentro de 20 años, el método de progresión geométrica es muy aceptable. Sin embargo, debemos tomar en cuenta, que sólo se nos dieron 10 años iniciales y queremos con esos datos estimar para los próximos 20 años, por lo que puede haber muchas variaciones y ningún analista demográfico, estaría de acuerdo en usar nuestras estimaciones, ya que no representaría un comportamiento real del incremento poblacional. Por ejemplo, observemos que de 1996 a 1997 creció mucho la población y de 1998 a 1999 creció muy lento; si ésto pasa para un periodo de 10 años, pueden pasar muchas irregularidades para un periodo del doble de tiempo.

Fig. 39.7 Fragment of the final report handed in by Student B

other three students expressed criteria that could be used to value the information provided by each assumption used (Fig. 39.7).

Problem B. *Initially, a pond has n units of volume of natural water, with a c uniform concentration of salt. Due to the sun, one unit of volume of water evaporates every week. Besides, one unit of volume of the rest is taken out and both replaced with two units natural water so that the concentration of salt does not increase. Describe the concentration of salt in the pond.*

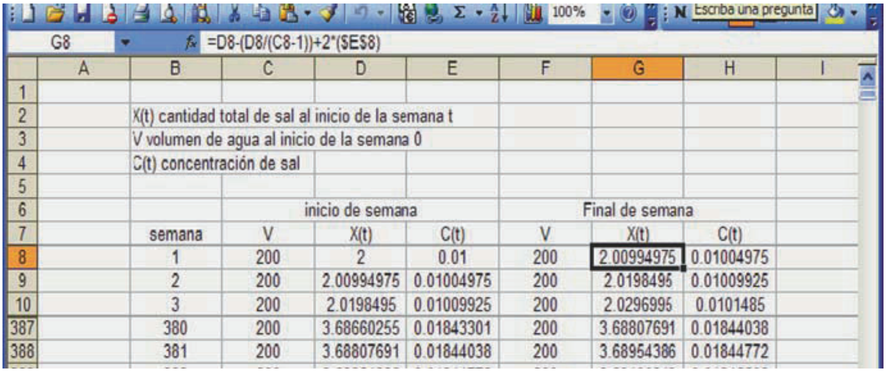
The first approaches to understand the problem, which were made individually, are characterized for being numeric. Students assigned values to the initial volume of water and the initial concentration (or quantity) of salt in the pond, and used procedures in Excel to determine the concentration of salt at the end of every week without giving any extra information (see Fig. 39.8). Their main interest was to determine the quantity of salt as a strategy to get the salt concentration, which determines the quotient of the quantity of salt and volume of the water. On the whole, students identified a recursive process, such as the one carried out by the student G (see Fig. 39.8), which helped them find the amount of salt in terms of what it had been the previous week C7, volume B7, and the initial concentration D6. Furthermore, they were even able to obtain the concentration of salt at the end of the following weeks by repeating the procedure.


C8 =C7-(C7/(B7-1))+2*D6

	A	B	C	D
1				
2		Concentración de Sal		
3				
4		Datos		
5	semana	Volumen (M3	Cantidad de s	Concentración sal(Kg/M3)
6	0	4000	4000	1
7	1	4000	4000.99975	1.00024994
8	2	4000	4001.99925	1.00049981
9	3	4000	4002.9985	1.00074962
10	4	4000	4003.9975	1.00099938

Fig. 39.8 Student G assigned 400 m^3 of V and a concentration of 1 kg/m^3 (Individual report)

At this phase, two students found it difficult to understand and apply the concept of "concentration" when they were representing the process of variation in the concentration of salt in the pond. Pair work helped students improve the representation of the process in Excel by making visible the relevant amounts at the beginning and end of the week, and also, by posing questions such as: What should I obtain? What is the amount of concentration of salt per week? How do I get these sums? We observed that when students work in pairs, they are compelled to explain and clarify the process used to find the answers (see Fig. 39.9).


G8 =D8-(D8/(C8-1))+2*(E8)

	A	B	C	D	E	F	G	H	I
1									
2		X(t) cantidad total de sal al inicio de la semana t							
3		V volumen de agua al inicio de la semana 0							
4		C(t) concentración de sal							
5									
6				inicio de semana			Final de semana		
7		semana	V	X(t)	C(t)	V	X(t)	C(t)	
8		1	200	2	0.01	200	2.00994975	0.01004975	
9		2	200	2.00994975	0.01004975	200	2.0198495	0.01009925	
10		3	200	2.0198495	0.01009925	200	2.0296995	0.0101485	
387		380	200	3.68660255	0.01843301	200	3.68807691	0.01844038	
388		381	200	3.68807691	0.01844038	200	3.68954386	0.01844772	

Fig. 39.9 Modified table carried out by pair of Students C and G

The presentations for the whole group made it possible to assess what had previously been done in pairs, in terms of answering other participants' questions and doubts. This allowed them to identify structural aspects of the procedure and also consider criteria to assess the relation between the initial information and the results obtained.

This activity led students to seek an algebraic representation of the process. The contributions that came up within the group were analyzed in pairs. Each student wrote a new report individually, then they discussed it with their partner, and each pair made another presentation for the whole group. In these presentations, all participants accepted the expression and propositions made by one of the pairs of students (Students G and C). The pair highlighted the recursive nature of the process followed, and used this detail, even though it was used in an incorrect way, to obtain a simple algebraic expression, which guided them to establish that the process in week 2 was a repetition, with the only difference that the initial amount of salt is X_1. The same process was repeated for the following weeks (see Fig. 39.10).

Students used this expression to determine the time in which the concentration would reach a value that is equal to the double of the initial amount of concentration. The previously statement would be represented in the following equation: $2\frac{X_0}{n} = \frac{X_0(n^2-2)^k}{n^{k+1}(n-1)^k}$, using a numeric procedure in Excel.

In the group discussion, nobody doubted about the results obtained, and they did not consider the procedure performed at the first phase, in which a slow increase in the concentration was visible. The instructor took part in the activity to help students identify this difference and guide them to reflect about what they had done, the model concepts, and the representations of a phenomena or situation. The students worked on the problem again in pairs. Some of them were able to obtain the symbolic representation of the process and then informed the group of this result. During the discussion, it became evident that different approaches, if they are correct, should lead to the same result.

$$X_1 = X_0 - \frac{X_0}{n-1} + 2c = (\frac{n-2}{n-1})X_0 + 2\frac{X_0}{n} = X_0 \frac{(n^2-2)}{n(n-1)}$$

$$si\ consideramos\ que\ al\ inicio\ de\ la\ semana\ 2\ hay\ X_1\ de\ sal:$$

$$X_2 = X_1 \frac{(n^2-2)}{n(n-1)} = X_0 \frac{(n^2-2)^2}{n^2(n-1)^2}\ ,\ldots y\ tambien\quad X_k = X_0 \frac{(n^2-2)^k}{n^k(n-1)^k}$$

$$la\ concentración\ en\ la\ semana\ k\ es: C(k) = X_0 \frac{(n^2-2)^k}{n^{k+1}(n-1)^k}$$

Fig. 39.10 Wrong recursive representation, made by student G, which is initially accepted by the class

39.4 Reflection

While students were searching the answers to the activities, they developed attitudes and concepts they had not shown previously; for instance, they established criteria to value the answers given to the questions asked, and searched for more general answers instead of particular ones. Students also valued some procedures more than others. During pair work and group work, students who worked with numerical approach accepted other types of approaches, especially if “advanced” mathematical concepts and theories were used (for example, calculus). Even though the numerical approach in Excel (not using paper and pencil) could provide them the necessary information to answer the questions asked in the problem, and another one that required describing the situation. Possibly due to the students knowledge of the tool or it may also be because of beliefs developed during their previous academic training (Schoenfeld, 1985). The discussion concerning the situation that had multiple answers allowed students to pay attention to mathematical concepts, such as precision, exactitude, approximation, function, and others, of which students proved to have a limited understanding associate to algorithmic activities.

While working (individual – pair – group – pair – individual), students developed an attitude of collaboration and commitment to carry out the different tasks that the activities required. However, they were always in a classroom environment in which they were required to complete all the activities the way the teacher asked for. This became evident because several of the students asked the teacher to assess them individually or in pairs, and were surprised or upset when the teacher answered their questions with another question related to the one they had previously asked, or suggested they asked another pair. The participation and keenness of the teacher to transform the learning situation in the classroom into a learning community is very important since he must consider the heterogeneity in students' education and their experiences. It is essential that the teacher emphasizes how the discussion of students' ideas is a fundamental process to be able to get more knowledge and refine the previous. It's also important that the teacher considers the different possible answers to a problem in which students will make use of different forms of representation and different tools as useful elements to help and guide them throughout their problem-solving activities, and at the same time, make students develop the habit of asking questions and seeking their answers. This is an essential aspect in the transformation of a group of students into a learning community.

It is important to reflect on relevant problem solving behaviours that the participants exhibited during the development of the sessions. The initial approaches to the problem shown by the participants, in general, were incoherent and seemed to aim at representing the problem algebraically. However, examining those first attempts led them to recognize that it was crucial to visualize and comprehend relevant data and particular relationships associated with the task. In this context, the participants recognized that there may be various ways to approach the problem and even those with certain limitations are important to discuss meaning and mathematical properties embedded in the solution process. Thus, the participants conceptualized learning as a continuous process in which their ideas and approaches to tasks are refined as a result of examining openly not only what they think of the problem but also discussing and criticizing the ideas and approaches of other participants. In this context, the structure and development of the sessions seemed to be fundamental ingredients to generate a learning community that values and accepts that learning takes place within an inquiry environment that reflect principles that are consistent with the practice and development of the discipline. In particular, the participants recognized that initial incoherent attempts to solve the problems can be transformed into robust approaches when the learning environment values and promotes the active participation of the learners.

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Chapter 40
An Investigation of Teachers' Shared Interpretations of Their Roles in Supporting and Enhancing Group Functioning

Betsy Berry

Abstract This study investigates the work done by two teams of elementary teachers as they designed teacher level classroom tools to help them support and enhance their students working in groups solving model eliciting activities (Lesh et al., Handbook of research design in mathematics and science education, pp. 591–645, 2000). The goal of this study was to investigate two different teams of teachers' shared interpretation of the teacher's role in supporting and enhancing students' functioning well in groups. These interpretations are described as they are expressed in the form of two tools designed for optimizing those roles, a group observation tool, and a teacher self-coaching tool. The results of the study include: (1) the tested and refined tools that the teams produced which expressed their ways of thinking, and (2) the shared interpretations of the teams and the researcher of the themes and trends of thinking across the development of those tools.

40.1 Introduction

The following three assumptions shaped the rationale for this research study.

- Good teaching is more about seeing and interpreting than it is about doing.
- Many teachers find it challenging and problematic to facilitate group problem-solving in their classrooms.
- An important aspect of teacher learning is collaboration and participation in *joint work*.

The Making Mathematics More Meaningful (M4) project in Crawfordsville, Indiana was the professional development initiative that provided both the problem that was explored in this study and the site and participants for the investigation of that problem. In the first year and a half of the three-year project, K-8 classroom

B. Berry (✉)
Indiana University-Purdue University, Fort Wayne, IN, USA

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_40,

teachers participated in extensive professional development during summer workshops and in on-going follow-up grade level meetings during the school year. At each of those sessions teachers were asked to express their “successes and surprises” and “challenges and concerns.” One of the areas of concern identified in nearly all sessions was the functioning of student groups during classroom activities and group problem solving. This concern was also often voiced by teachers participating in the Case Studies for Kids Project, a research project at Purdue University. It was this concern that formed the context for the investigation that will be described in this paper. Two additional features of the M4 professional development project, reflection and collaboration, provided the format for the design research methodology (Brown, 1992; Collins, 1992; Lesh, 2003; and Lesh and Kelly, 2002) that was employed in this exploration.

40.1.1 Goals and Scope of the Study

The goal of the study was to investigate two teams of teachers’ shared interpretations of their roles in supporting and enhancing their fourth and fifth grade students functioning well in groups as they participated in model eliciting activities. These interpretations were described as they were expressed in the form of two tools designed for optimizing those roles, a group observation tool, and a teacher self-coaching tool. The product of the study was the team’s shared interpretations expressed in the form of those tools that they designed. To get this shared interpretation, the teachers were put in a design research study in which they (the teams of teachers) participated in an iterative series of three situations where they expressed, tested, and revised those interpretations.

The study consisted of two teams of classroom teachers participating in a model eliciting activity for teachers that was designed by the researcher to address a problem raised during grade level follow-up sessions of the M4 project. These teams of teachers addressed the *problem* of supporting and enhancing students functioning well in groups by designing tools for their (the teachers’) use during their teaching while participating in three cycles of designing, testing, and revising of observation and coaching tools. Each cycle included three events: (a) a planning and design session focusing on the development or revision of the tools; (b) implementation sessions where the teachers tested the tools in their classrooms with their students; and (c) a session for reflection and evaluation of the tools. The tools that the teachers designed focused on the observation of students functioning in groups as well as a teacher self-coaching tool. The object of this study was the collective interpretation of the teacher teams as expressed in the design and modifications of those tools.

40.1.2 Models and Modeling

In mathematics education, the phrase *models and modeling* has evolved from research in science and mathematics learning where groups of students are asked to solve open ended problems that require students to mathematize their thinking about some real world phenomena (Lesh and Doerr, 2003); Lehrer and Schauble, 2003). As the students design, test, and revise their models that meet the criteria of a particular problem situation, they are also developing models to construct, describe, or explain mathematically significant systems that they are exploring. Similarly, this perspective has been expanded and applied to the development of teachers' models of teaching as they develop conceptual systems of interpretation that include the complexity of their world, their knowledge of mathematics content, of pedagogy, of the development of students' thinking, and of all the other aspects of teaching that impacts and influences those models (Doerr and Lesh, 2003). That is the perspective that is taken in this study. Teachers participated in a teacher level *model eliciting activity* to address a problem in their real world of teaching as their students participated in three model eliciting activities in their fourth and fifth grade classrooms. Table 40.1 provides a summary of the three student level activities used in this study. These activities were developed for third through fifth grade students by Lyn English and her associates for related research projects at the Queensland University of Technology in Brisbane, Australia and were adapted for this research study. They were selected for their open-endedness and for their appropriateness for the grade levels that the participating teachers were facilitating.

Table 40.1 Description of Model-Eliciting Activities Used in Study

Title	Description of problem statement	Mathematical idea(s)
Snack chip Consumer guide	Develop a consumer guide to help people determine which type of snack chip is the best to buy	Estimation, data analysis
Beans, beans, beautiful beans	Determine which of the light conditions is suited to growing beans to produce the greatest crop and predicting the weight of beans produced under certain light conditions	Representations of data, making predictions
Paper airplane contest	Recommend to the judges of the contest how to select a winner using data from preceding years' contests	Manipulating qualitative and quantitative data, weighting variables,

40.2 Research Design

The framework for this study can be thought of as a *multi-tier design experiment*. (Lesh and Kelly, 2000) with three tiers: (1) researcher/facilitator; (2) teacher team/researchers; (3) student group problem-solvers. The students participated in the model eliciting activities where they were expressing, testing and revising their solutions to the problems in groups in the classroom. The teacher teams designed the observation and self-coaching tools they used, tested them in their classrooms with their students and revised those tools during each cycle. The researcher/facilitator designed the teacher-level model eliciting activity and worked closely in the design, reflection, and revision process with the teachers but was not involved in the classroom testing of the tools. This paper describes the researcher/facilitator and teacher/researcher levels of those tiers only.

In much the same way that a city planner might give specifications for a new bridge across an urban river, such as location, and traffic needs, budget, etc, but the bridge designers and engineers would be responsible for the actual bridge that meets those specifications, in this study the "specs" were expressed in the following guiding questions provided by the researcher. However, the resulting tools were the design of the teacher "engineers." The city planner might well be an involved party in the design of the bridge, just as this researcher was an involved participant in the design of the classroom tools, however, the teachers were the "experts" in this study and were doing the testing and analyzing of their created products, just as the engineers are the experts in the bridge design.

***Guiding Questions*:**

1. What are the characteristics of a well-functioning group?
2. What are the characteristics of a useful observation tool and teacher self-coaching tool?
3. How well, to what extent, do these tools serve our purpose of supporting and enhancing our students functioning well in groups?
4. To what extent could someone else use these tools effectively?

Two teams of teachers participated in this study. One team consisted of three fourth grade teachers; the other team included two fifth grade teachers. These teams participated in three design iterations in order to develop a refined and generalizable set of observation tools and teacher self-coaching tools that supported the functioning of student groups as they solved model eliciting activities. None of these teachers had used model-eliciting activities in their classrooms prior to this study. All of these teachers, however, had been involved in professional development activities during the preceding two years that was facilitated by the researcher as part of an Indiana Department of Education Mathematics Professional Development Partnership Grant awarded to their school corporation. All said that they use groups in working on projects in their classrooms but not necessarily on open-ended mathematics problems.

The following diagram illustrates a sample cycle of this research project: The areas of interest and the events when data were collected for this study are entitled "Session 1" and "Session 2." These represent the planning and reflecting and redesigning meetings that were held before and after every classroom implementation. The teacher-student implementation sessions are included here to signify the testing of the teacher-designed tools in their classrooms. The teacher-participants made notes and comments on their designed tools and brought those back to the reflection sessions as part of the evaluation and re-design process (Fig. 40.1).

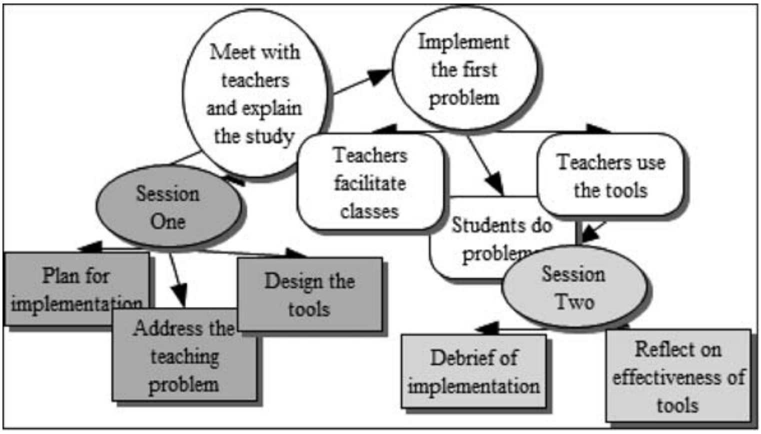

Fig. 40.1 A sample cycle of the design study

40.2.1 Tool Design

Throughout the course of the three cycles of design, test, and revise, several themes appeared in the development of the observation tool by one or both of the design teams. They include:

- Compatible groups and working together
- Staying on task
- Understanding the task
- Strategies
- Outcome – final product

These themes were revealed in different degrees as each cycle of observation tools was designed and used and revised. The teams worked independently for three cycles and then designed a fourth and final observation tool together. The final product is included in the appendix.

The design process began as the teams were asked to brainstorm about the characteristics of a well-functioning group of elementary students and things that they as teachers might look for or listen for as their students worked. The results of the two teams were:

We see in these notes that some common concerns and topics were raised as the teams discussed their past experiences and their potential ideals for their students when they work in groups. Both teams indicated a need or concern for their groups

What makes a well-functioning group in 4th grade?

1. Must have a leader – one who can take charge and give out responsibilities – set an example
2. Ability group – so the groups would be even. . .heterogeneous groups
3. Compatible group members – different strengths as a group member
4. Make sure they are well informed – guidelines are set
5. Materials available
6. Environment set – such as places in room for groups
7. Discipline problems need to be considered

Things to look for

- Group members participating – all of them
- Being able to come to a common agreement – being able to back the differences
- Do they all have a job or purpose
- All working to the best of their ability with specific task
- All working to achieve a final product
- Using different types of organizational skills

What makes a well-functioning group in 5th grade?

- Heterogeneous: because not looking for a grade, need some leadership, reading ability, etc. to be successful
- Homogeneous: disadvantage some groups wouldn't succeed, time period would need to be different for different groups
- Separation of behavior problems

  Steps to create heterogeneous groups (3 members each)
- Select top functioning students (overall)
- Separate any behavior problem students
- Finish 2nd person in group with other lowest kids
- Round out each 3–4 member group with middle kids

Things to look for in working groups

- Being on task – working to solution
- Effective communication – one talking at a time
- Trying different ways/discussion
- Logical methods being applied
- Framework/organization – order of process.

to be heterogeneous or "even" in ability. Both groups discussed a need for leadership in the group and that discipline problems or behavior had to be considered in making up the groups. Other common topics included organization skills and working toward a solution or final product or being on task. Three of the themes emerged as important for the fourth grade team on their first observation tool: compatible groups, understanding the task, and staying on task. The themes that were revealed as important in the first fifth grade observation tool included: staying on task, communication, strategies, and final product.

The first coaching tool was designed in a similar way. The groups were asked to consider what the teacher does and might say and what choices and decisions he or she might make during group problem-solving to support and enhance the students. They were asked to select two or three topic areas and then design potential questions or comments as suggestions to themselves. Three common themes or categories appeared in some form on the tools that the two design teams developed: understanding/organization, examples/strategies, and goals/final product/presentation. In addition the two teams developed some individual themes that they included on these tools. These individual themes are: degree of intervention, discussion – degree, and logical reasoning.

This process of design, test, and revise continued for three cycles of implementations by these two teacher-teams with their students as they implemented three different MEA's. Each cycle produced a new set of tools. Following their debrief sessions after their last testing cycle, the two teams joined as one group and designed an observation tool and a teacher coaching tool that they believed they could share with other colleagues who had not been a part of this design process. As part of this discussion we revisited the Guiding Questions that we had used throughout our work to direct our design of the tools. Each of the teams now shared their sequence of tools that they had designed over the three cycles with the other group, and through discussion and negotiation the two teams designed together a final set of tools. Space does not allow for the inclusion of the evolution of the tools in this paper, but the final products are included as Appendix A.

40.3 Discussions and Implications

The design of the final set of tools culminated the work of the two teams in their three cycles of research for this design study. These tools are not meant to represent a "perfect" set of tools. These teams of teachers or any other teams of teachers could take this final set of tools and continue to test and revise it. What this set of tools does provide is a representation of these teams of teachers shared interpretations of the most important, most useful, or most essential categories and ideas as they developed over the life of this study. However, the fourth and final pair of tools designed through the collaborative effort of the two teams working together could be presented to any group of teachers as a generic starting point for them to use in their classrooms as they facilitate MEAs or other open-ended, problem-based, investigative activities in mathematics or other content areas. Several of the tools designed that were specific to the three MEAs implemented in this study could be offered as potentially good instruments for other teachers to use and adapt when they implement the same MEAs with their students.

As indicated in the *Guiding Questions* that framed the design of the final tools by the teacher teams, a criterion for testing and revising the tools entailed considering that they were shareable and generalizable. The shareability component of design research (Lesh and Kelly, 2000) refers to the notion that another person or group of people could use these tools for the purpose of enhancing and supporting students' group functioning while solving MEAs. The generalizability principle (Lesh and Kelly, 2000) implies that others could reuse the tools for other purposes and in other settings by adapting or slightly modifying the tools to fit the respective needs of the new situation. The shareability of the tools is embedded in their design. Two teams of two and three teachers made contributions to the design, testing and revision of the tools. In addition to sharing these tools between and among the two design teams, the tools were shared within the school corporation as part of the M4 fourth and fifth grade level follow-up meetings and they are useable throughout the models and modeling community to teachers, researchers and other educators involved with the implementation of MEAs. The tools are generalizable in the sense that they are useable in any of a myriad of group problem solving or other collaborative learning environments. Although the focus of this study was on mathematics classrooms and on the implementation of MEAs, the tools designed are adaptable to other content areas, other age groups, and other learning activities and other countries.

These two teams of teachers set out to investigate a concern or problem raised by them or their colleagues within the domain of the M4 project. The concern was how to facilitate an environment to enhance and support students to function well in groups when solving open-ended mathematics problems. The strategy to address the issue was to design a set of tools for teachers and to test and revise them over several implementations of MEAs. The teams addressed this problem through the format of a design study (Designed Based Research Collective, 2003) and in the context of a models and modeling perspective (Doerr and Lesh, 2003). The study conducted here supports suggestions existing within that perspective that the tools designed by the teacher teams gave frame and substance to their thinking – that the artifacts *represented* their interpretations (Doerr and Lesh, 2003). While the nature, quality, and quantity of teacher learning was not within the scope of this research project, the themes of interest and the interpretations shared by the teachers and embodied in the tools they created, give us a small window of evidence into their evolving conceptual systems of teaching and learning with respect to students' functioning in groups. There was no pre-test or post-test to analyze or measure these teachers' learning within the context of the work that they did together, but the process itself provides a trail of documentation of those evolving systems, and it was this process and evidence that was provided in the results of the study.

Appendix

Observation guide final (Combined groups)							
	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Compatible groups (work well together)							
Active participation by each team member							
Understanding the task							
Create a plan to solve the task							
Explanation supporting the why and how							
Outcome & presentation of final product							

Teacher coaching tool final (Combined groups)	
Category one: Questions	Category two: Group dynamics
• Lead questions with statements • Ask open-ended questions • Have questions ready – ask same to each group	• Advise teachers to group in different ways • (Easiest to divide)
Category three: Monitoring	Category four: Suggested questions
• Continually move around • Listen and check on groups • Stop every once in awhile to check on progress	• What is the big picture? • Why are you having difficulty? • Reason it out • Show me the data
Category five: Final product	Category six
• Give time to share	

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Chapter 41
Mathematical Modeling: Implications for Teaching

Maria Salett Biembengut and Nelson Hein

Abstract In this paper we present the main implications of Modeling in the teaching of Mathematics where empirical data was obtained from the use of mathematical modeling for teachers through Courses of Continuing Education. The objectives of the research were to verify the possibilities and difficulties in establishing modeling as a teaching methodology. The experiment was conducted in four Courses given to 105 teachers. The main difficulty in terms of teachers' education was their lack of experience with tasks of this nature. It is rather rare for teachers' Mathematics training programs to include any orientation regarding Modeling, whether in the use of the process or its formal teaching. In spite of the difficulties, research has shown that the adoption of mathematical models in teaching can lead to better achievements for teachers and students, becoming one of the chief agents for change.

41.1 Introduction

Movements in international educational concerning mathematical modeling in teaching have also influenced Brazil (practically at the same time, starting in the 1970s), with Brazilian educators collaborating to represent the country in that part of the international academic community involved in Mathematics Education. Similarly, Mathematical Modeling, whether as a discipline in Engineering Courses and Economics, among others, or as a proposal for integrating mathematics with reality for primary and secondary students has become one of the methodological processes used in school education.

M.S. Biembengut (✉)
University Regional of Blumenau (FURB), Blumenau, Brasil
e-mail: salett@furb.br

N. Hein (✉)
University Regional of Blumenau (FURB), Blumenau, Brasil
e-mail: hein@furb.br

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_41,

The amount of research and reports of classroom experiences presented at Mathematics Education events (regional, statewide and national) and at the National Modeling Congress (held on a biennial basis since 1999) has increased significantly along with the number of teachers who have become interested in the subject through extension and post-graduate courses and publications. Furthermore, various undergraduate courses for mathematics teachers have sought to include Modeling in their course curricula, either as a separate subject or as part of the overall discipline known as Teaching Methodology in Mathematics. However, despite the growing interest in modeling we see that many math teachers still do not use mathematical modeling as a classroom teaching practice. Except for some isolated cases, mathematics learning has fallen short of what had been hoped for, according to data recently revealed by the National Institute for Evaluation and Research, the governmental evaluation agency responsible for Education (Biembengut, 2004).

Mathematics not only greatly contributes to exercising the intellect but is also the language of science. Mathematical modeling follows the rules of scientific investigation, in other words, the four conditions that should be satisfied in the scientific proposition (Bassanezi, 2002; Maturana and Varela, 2001). Over last decade we have made research into the development of a method that we refer to as (mathematical) Modeling Education, which uses the main principles of modeling in the teaching process and in the learning of Mathematics at every school level.

With the primary purpose of identifying and understanding both the possibilities and difficulties faced by teachers when trying to make mathematical modeling a classroom teaching practice, we held four Continuing Education Courses for teachers over the course of two semesters in 2006. A total of 105 grade school teachers participated (5th to 8th grades). The course was made up of two parts: the first was for teachers to learn how to make a mathematical model; the second was how to adapt the modeling process to the teaching of mathematics by following the method. Analysis of data obtained from this research was based on work developed by the teachers, in the observation of activities they engaged in during the process and in interviews conducted at the end of the course. In this article we present the main implications of Modeling Education in the teaching of mathematics, based on this experimental activity. Therefore, we will first present a brief explanation of Modeling Education as a method for teaching and researching.

41.2 Modeling Education as a Method for Teaching and Researching

Mathematical modeling, like teaching methodology and learning, starts with a theme/subject and then develops questions about it. These questions are then answered by using mathematical tools and existing research on the theme. It is highly pleasing to investigate a subject that can help students acquire a significant amount of mathematical knowledge or to develop knowledge on the theme being studied.

Many professors defend the idea that each student can choose a theme/subject of any area of his/her interest, do research on it, formulate questions, and, with the teacher's guidance, design a mathematical model. In this way, students become co-responsible for their learning and the teacher becomes an advisor. Learning becomes richer, considering that the student does not just learn mathematics inserted in the context of another area of knowledge but also has his critical and creative senses stirred. In formal teaching, there are factors such as curriculum, class schedules, number of students per class and availability for the teacher to follow up on the students' work. Because of these variables we had to adapt the process of mathematical modeling to teaching methodology and learning, establishing a method that we call Modeling Education (Biembengut, 1997; Biembengut, 2004).

Modeling Education is a *teaching* and *research method* that uses the essence of modeling. Research has shown that the application of the method is expected to provide the student with the following: integration of mathematics with other areas of knowledge; interest in the application of mathematics; improvement in grasping mathematical concepts; incentive for creativity in the formulation and resolution of problems; ability to use machines (graphic calculators and computers); capacity to act in a group; orientation in doing research; and capacity for reporting research (Biembengut and Hein, 2007). Modeling Education is guided by the teaching of the program content, starting with applied mathematical models and then moving into several areas of knowledge, at the same time guiding students in the direction of research. It can be implemented at any educational level, from elementary school teaching to the university.

41.2.1 Modeling Education as a Teaching Method

Modeling Education as a method of teaching mathematics aims at providing the student with a better apprehension of mathematical concepts, training him/her to read, to interpret, to formulate and to solve specific situation-problems, as well as to awaken his critical and creative senses. To apply it in teaching, the teacher chooses a theme/subject of any area of knowledge that can be of interest to students (according to the content of the program) and designs a mathematical model, adapting it to teaching. Alternatively, the teacher can choose an existing mathematical model and adapt it to the development of the program content. That model will then serve as a guide. This involves the teacher in a series of activities/stages (Biembengut and Hein, 2007):

1st. Exhibition of the theme – The teacher begins the class by giving a brief explanation of the theme/subject to the students, urging them to raise questions about the chosen theme.

2nd. Delimitation of the problem – The teacher selects one or more questions which allow him to develop the program content. If it is possible and/or convenient, he can propose to the students that they do research on the subject, either a bibliographical study or an interview with some expert in the subject area.

3rd. Formulation of the problem – The teacher starts to formulate the problem by showing hypotheses, calculating or organizing the data in such a way that it asks for mathematical content for its resolution.

4th. Development of the program content – Here the program content is presented (concept, definition, properties etc.), linking it with the subject that generated the process.

5th. Presentation of similar examples – Immediately after these steps, similar examples are presented, enlarging the options for application, and thus avoiding any chance that the content becomes limited to the theme or presented subject. Stimulus and guidance in the use of technological devices like calculators and/or computers that are part of classroom practice is also important.

6th. Formulation of a mathematical model and solution of the problem departing from the model – The teacher proposes to the students that they come back to the problem that generated the process and solve it.

7th. Interpretation of the solution and validation of the model – Ending at this stage, the student evaluates the result – validation. This allows the student to develop a better understanding or comprehension of the obtained results.

41.2.2 Modeling Education as a Research Method

The central objective of this work is to create conditions for students to learn how to do research and design applied mathematical models from some other area of knowledge. This work is accomplished parallel to the development of the program content. We have suggested that students group themselves according to their interests and personal preferences and that the school period be divided into at least five stages, so that the proposals can be accomplished and the teacher can properly guide the class. The stages are:

1st. Theme choice – Groups are formed with a maximum of four students, and each group chooses a theme/subject according to individual interests. The groups, with the teacher's guidance, should be responsible for the choice and direction of their own work. When the theme/subject is chosen, the teacher proposes that the groups collect data using special bibliographies and/or by interviewing experts in the area.

2nd. Becoming familiar with the theme to be modeled – In this second stage students should already be familiar with the theme and have much data at their disposal. The teacher proposes that a series of questions be raised and that they synthesize the research. That synthesis allows the teacher to learn about the theme and to select, as a suggestion, about three questions to be addressed by each group. These questions should include mathematical notions that are part of the program.

3rd. Delimitation of the problem and formulation – After having delimited the problem or selected the questions to be answered, the teacher starts to formulate the problem from the questions that ask for more elementary mathematical concepts for their solution. When the group has a good background for the theme that they are working on, an interview with a specialist/expert can greatly contribute to the work.

4th. Design of a mathematical model, solution and validation – Once the problem has been formulated, the group tries to construct a model that not only allows for a solution of the specific questions at stake but that can also be applied to other solutions or that can be used to predict certain results.

5th. Organization of the written work and oral report – It is essential that the work be published. Therefore, in this stage, the groups should present the developed work in writing and orally, in a seminar, to the other students or to those for whom it may be of interest.

Modeling Education can be used either as a *teaching method* or as a *research method*. When using it as a teaching method, it is suggested that the teacher guides the students' research by using a specialized bibliography and/or by interviewing a specialist/expert about the theme of the principal model(s). When using Modeling Education as a research method, development of the program content can be given in the traditional form. Using just one of the approaches or both – teaching and research – the premise is to promote mathematical knowledge and the ability to apply it in other areas of knowledge; in other words, the objective is to supply students with elements which allow them to develop their potential, providing them with the capacity to think critically and independently.

41.3 Important Occurrences in Modeling Education

Starting from this premise, we have been applying, directly or indirectly, Modeling Education for less than two decades with teachers sympathetic to the method at the elementary, secondary and university levels. We have also been developing projects with teachers through continuing education courses and disciplines taught in specialization courses in mathematical education. Despite a number of continuing education courses that we have held and the interest shown by the majority of teachers who participated, we do not have information available about the possible adoption of modeling in their classroom practices nor any data concerning how modeling was understood and/or adopted by teachers. This information will allow us to know where they are at the moment and what roots or influences lay in the past as well as the roads that need to be followed for the future implications in designing Modeling Education courses.

With this in mind, we conducted four continuing education courses for four groups, training a total of 105 teachers who agreed to participate and contribute to the research over the course of two semesters in 2006. Each course included 90 h of instruction. Each course followed the orientation described above as the method for teaching and research. As a *teaching method* we used five mathematical models and remade each one (following the routine described in Modeling Education as a teaching method above) in order for the teachers to be able to become aware of the art of mathematical modeling, to clarify mathematics content, and to verify the possibility of adapting the example to their classroom practices. As a *research method* (following the routine described in Modeling Education as a research method above), teachers formed groups of four people, with each group agreeing on and selecting a theme that interested them and developing a mathematical modeling project that could also be used in their classroom teaching practices. Based on observations made during course activities, from the work done by participants and from interviews conducted at the conclusion of the course, some essential results were obtained that allow us to re-structure the teacher training process. The following are the main advantages and difficulties presented by the teachers who participated in the course, which could have implications for their classroom teaching practices.

41.3.1 Main Advantages

a) In relation to the mathematical models used to orient the process:

- Teachers gained a better understanding of the developed content due to the approach to the target area and its application;
- Teachers felt safer implementing classroom activities, greatly facilitating the way classes are conducted and also facilitating inter-relationships with other disciplines, by coming back to the orientating model, solving it and evaluating it.

b) In relation to the modeling work:

- Teachers were led to act/do and not just to receive prepared concepts/materials without understanding the meaning of what they had been studying; to do research, which is not a routine activity in the classroom, in spite of being part of the curriculum; to imprint knowledge, creativity and critical sense, mainly in the formulation and validation of the model; to interact with and to learn about the work of the other groups in the seminar; and to apply the norms of the scientific methodology when composing a written report of the work.
- Teachers were allowed to be more attentive to students' difficulties, became aware of their work in a gradual way, especially when guiding the students, and were able to change their criteria and evaluation instruments.

41.3.2 Main Difficulties

The main difficulty centers on teachers' training/education and their lack of experience with tasks of this nature when they were students themselves. It is rather rare for teachers' mathematics training programs to include any orientation on modeling, whether in the use of the process or its formal teaching. The inclusion of this content has been more frequent in Brazil in the last decade in extension courses or in disciplines taught in Masters' Degree Programs in Mathematics Education. We identified two types of difficulties presented by the teachers who participated in the course: one during the process of designing a mathematical modeling project and the other gleaned from the interview given at the end of the course.

a) Difficulties experienced by teachers in the stage of Modeling Education as a research method:

   - *Interpretation of the context*. In traditional teaching, especially in mathematics, learners are rarely presented with situations or problems that ask for a reading and an interpretation of their context and then asked to proceed to the formulation and explanation of that context. Without that living experience, a student or professional will lose this capacity. Any attempt to rescue the individual is not an easy task (Kovacs, 1997; George, 1973). Traditional teaching doesn't make the student qualified to read the context, or to read, in a wide sense, the words. Abilities are rarely developed to enable the reading of music, the interpretation of a piece of art or poetry, or the understanding of a historical context, a political situation, or a statistical result, among others. This is one of the most important flaws of current education. In these terms, whenever these teachers were faced with a text or context, they experienced serious difficulties in reading, understanding, and interpreting, that is, in making a reading.
   - *Availability for research*. The themes chosen demand research, research which has to be done outside the course. Since the majority of teachers did not have any time available for out-of-class orientation activities, however, this was not always possible. When discussing work in modeling, it is worth pointing out that the time that students have available for collecting data and studying has an important effect on developing quality work.
   - *Choices of the theme/subject*. The choice of a theme was not an easy task. Each group of teachers chose a subject of interest, but they did not always achieve expected results. For instance, some groups chose themes whose data didn't bring any additional knowledge to what concerns mathematics, and others chose themes whose data was not easy to obtain. In this case, in order to avoid problems in the middle of the process, we had to encourage participants to look for other topics.
   - *Work in groups*. A theme chosen by the group didn't assure that all, in fact, were interested in it. This required a deeper involvement from each participant in order to accomplish what was proposed. This lack of involvement, along

with the lack of commitment by some teachers, compromised the work of the group where the essence of this work is cooperation and teamwork in the construction of knowledge.

b) Difficulties presented by the teachers in interviews regarding the implementation of Modeling Education as a teaching method and research method.

- *Improvement*. Since Modeling and Modeling Education have been hailed as teaching methods for less than three decades and considering the geographical dimensions of our country, improvement/continuing education courses and specialization courses in this subject have not yet reached all educators in the area. Limited to 90 h, this course was not enough for complete training in the method. It was just enough to "signal" the subject, provoking a certain motivation on the part of the teachers.
- *Bibliography*. There are few studies in Portuguese about the use of modeling and Modeling Education in teaching that have been published or that are available or easily accessible to teachers. Despite the fact that most areas of knowledge apply mathematical models (Physics, Chemistry, Biology, Economics, etc.), teachers state that they do not have sufficient knowledge in these areas to be able to interpret these mathematical models in the scarce amount of time available to them.
- *Guidance*. A single course or text about this subject does not provide sufficient background for the teacher to immediately put the method of Modeling Education into practice. Confidence and ability are acquired with time. Professional orientation in the subject – explaining difficulties and aiding in the planning and organization of the activities – would provide a safer ground for the teacher. One way could be supervision by virtual means; in this case, we would require researchers to plan activities that include this kind of guidance.
- *Planning*. To plan means to establish the strategies that should be used in explaining and referring problems to the learning process, to the structure and form adopted, and to the practices applied towards better evaluation of the process and results. For the majority of participating teachers, however, planning is included in the content lists and in the forms of evaluation that they will adopt over the course of the school year. Later they will adopt a textbook to follow. In implanting mathematical modeling in teaching, the teacher will need to employ careful planning in deciding when to teach a particular topic of the program and to present similar examples, all integrated into the students' work. For example, in order to be able to guide students in their work, the teacher will have to know something of the theme/subject. The larger the number of students' groups and the larger the number of themes, the longer the time the teacher will need to study the projects. This demands that the teacher have enough time available to do the task, which the teacher does not have.

41.4 Final Considerations

We understand that the objective of teaching, at all levels, should be to provide students with opportunities to acquire knowledge and to develop attitudes and abilities that help them to fully interact with society. It is with this objective that we have been defending Modeling Education as a teaching and research method. Modeling Education, however, is not a panacea to overcome all problems of school practice in the teaching of mathematics. Research points out that it can represent progress in the teaching of mathematics in the classroom because when it is used Mathematics stops being a mere transmission of resolution techniques and becomes a tool or structure in another area of knowledge (Blum et al., 2007). This demands a deeper commitment to studies, to research and to interpretation of context, for teachers as well as for the students. In other words, it means much more work!

As researchers, we hope to implement effective teaching at all levels of education because there is an imperative need for continuous improvement, especially for teachers who work in training and education programs. It is not enough to teach mathematics content, but it is also necessary to make a commitment to the training of these future teachers. We are not at work in an area simply to explore it. If basic education is precarious, it is because teacher education courses are also precarious.

The research described here, in spite of the resulting difficulties, shows that the adoption of mathematical models in teaching, whether in presentation form or in the creation process, when appropriately adapted to the realities of the school, is a means of providing teachers and students with better chances of success, becoming one of the chief agents for change. With this in mind, the preliminary challenges for researchers stem from recognizing the problems that emerge from our precarious professional training and by confronting the changes inherent to such a situation. There are no mysteries in the basic elements involved in the process for us to reach that condition; all the necessary technologies, tools and elements of changes exist. The real challenge is to decide to be committed to a new way of action. This new perspective needs to be embraced by each of us (Brown, 1991).

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Chapter 42
A Professional Development Course with an Introduction of Models and Modeling in Science

Genaro Zavala, Hugo Alarcon, and Julio Benegas

Abstract This contribution proposes a short professional development course we designed for high school and introductory courses at the college level of physics. The design is based on the results of Physics Education Research by using a contructivistic, active-learning approach so that teachers, besides being instructed content knowledge, they are also trained in pedagogical knowledge by living the experience. Models and modeling in science are introduced in one of the activities of the course in which teachers reflect, based upon their own experience, their students' alternative conceptions and difficulties resulting in a model-eliciting activity in which teachers offer their own model of teaching practice. Results of content knowledge and results on teachers' models will be presented.

42.1 Introduction

The learning of basic science in the high school and college level is generally a priority in developed societies. As in mathematics, the large advances, in the last two decades, in the comprehension of the important issues of learning physics have lead to the development of curricular materials scientifically designed to improve physics learning. In the process, the research of students difficulties, the design and implementation of the educational materials, the design of evaluation tools, and the implementation of training courses for teachers; a fast growing field has emerged

G. Zavala (✉)
Tecnologico de Monterrey, Monterrey, México
e-mail: genaro.zavala@itesm.mx

H. Alarcon (✉)
Tecnologico de Monterrey, Monterrey, México
e-mail: halarcon@itesm.mx

J. Benegas (✉)
Universidad de San Luis, San Luis, Argentina
e-mail: jbenegas@unsl.edu.ar

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_42, © Springer Science+Business Media, LLC 2010

known as Physics Education Research (PER). While new educational strategies continue to appear and different studies show that those strategies can significantly improve learning of the physical principles (Hake, 1998; Redish and Steinberg, 1999), it is evident that, in the classrooms of the majority of educational systems, there has been little change, if any. Different factors contribute to this lack of impact on school classrooms, and in this contribution we propose to work with a fundamental condition for change: teacher preparation for new methodological active learning approaches. Regarding this issue, there have been reports about comprehensive initiatives for re-preparing physics teachers with the objective to change local school systems (Stein, 2001) or multinational programs for teacher education (Pinto, 2005). Although these programs are very appropriate way to prepare teachers and change science teaching, they involve the availability of large resources and the need of getting teachers together long period of time; conditions that in many places are difficult to meet, therefore, short teacher development courses are more feasible.

Model-Eliciting Activities (MEA), in models and modeling research, are open-ended problems with multiple answers which objective is not the solution but the model students (or teachers) use to solve it (Lesh and Doerr, 2003). It has been applied to mathematics and it has been proposed for science. MEAs have also been used for teacher development courses (Schorr and Lesh, 2003) in mathematics but we have not seen literature reporting the use of MEAs in those types of courses in physics.

In this report we present the objectives and characteristics of a short professional development course for in-service physics teachers. The general approach underlying this course is to consider science teacher education in the general context of constructivistic learning, leaving the foundations for an action-research oriented teaching practice. That is, constructivism is seen not only as a theoretical framework to understand how pupils learn science, but also how teachers, building on their own experiences as teachers, actively construct their new knowledge about science teaching. Consequently the main objective of this professional development course was neither to stress rigorous physics treatment nor to stress general pedagogic approaches, but rather to facilitate the development of critical physics teachers who, by using thought.-revealing activities and reflecting on their own teaching and profiting from the curricular advances provided by PER, are able to implement in their courses an iterative virtuous cycle of planning and executing instruction complemented with the formative evaluation necessary to provide the positive feedback for the following course implementation.

A detailed description of the course is reported elsewhere (Zavala et al., 2007). In this contribution we present a general description of the course explaining the main features including the explanation of an iterative process that can be repeated depending on the time constraints and a small description of the pedagogical strategy chosen, one of the most recognized educational strategy in the PER literature. The following section is related to one of the activities on the iterative process, the reflection of the main students' alternative conceptions and difficulties which results in a model-eliciting activity where teachers offer their own model based upon their conceptions of learning and teaching. Next a section of general results of content knowledge is presented followed by a final summary of the contribution.

42.2 General Description of the Course

A teacher development course should address the issues of content and pedagogical knowledge. Authors with long experience in offering special courses for pre and in-service physics teachers postulate that instruction in science should not be separated from instruction in methodology. Discipline courses do not address the critical issue of student learning difficulties and generally leave the participants with the problem of how to adapt the material covered in the development course to their own classrooms. General courses on teaching practices and pedagogy lack discipline specific information about alternative conceptions and the most appropriate educational strategies to deal with them. It is clear, therefore, that separation of content and pedagogical knowledge decreases the value of both for teachers (McDermott and Dewatter, 2000).

Mestre (2001) has summarized the implications of the research in Physics Education for the education of science and physics teachers. He points to the following central issues:

- Physics content and pedagogy should be integrated.
- Construction and sense-making of physics knowledge should be encouraged.
- Teaching of content should be a central focus.
- Ample opportunities should be available for learning "the process of doing science".
- Qualitative reasoning based on physics concepts should be encouraged.
- Ample opportunities should be provided for students to apply their knowledge flexibly across multiple contexts.
- Metacognitive strategies should be explicitly taught to students.
- Formative assessment should be used frequently to monitor students' understanding and to help tailor instruction to meet students' needs.

Taking into account these guidelines, the course was designed to provide in-service teachers with a first hand knowledge of a teaching methodology that favors a deep conceptual learning of basic principles through the active participation of the student in constructing their own knowledge. The course therefore followed the recommendation of the National Science Education Standards (National Research Council, 1996) in the sense that professional development training of teachers "requires building understanding and ability for lifelong learning and should provide teachers the opportunity to learn and use the skills of research to generate new knowledge about science and the teaching and learning of science". For that reason the following basic features were included in the present course understanding that the knowledge content of the course was mechanics:

- The teaching strategy presented (Tutorials for Introductory Physics, McDermott et al., 1998, hereafter Tutorials) is an outcome of the research in the teaching and learning of physics in the last 30 years.
- Participants reflect on their own experiences as teachers in order to point out the more common student's difficulties on the subject of each activity eliciting their own model of learning and confront it with that of other participants.
- Participants reflect on their own (traditional) teaching experiences and compare them with the learning model propose by Tutorials eliciting their own model of teaching and confront it with that of other participants.
- Participants read and discuss of scientific literature reports relevant to the learning of the subject matter of each Tutorial activity and compare with what they previously reflected.
- The research-based test introduced (Force Concept Inventory, Hesteness et al., 1992) in the course in order to evaluate the results of instruction serves as a way of reproducing as close as possible the atmosphere of control instruction advocated for a regular physics course.

The idea behind this course structure is that using the results, techniques and language of educational research should help the in-service teachers to develop a critical view of their teaching, as a necessary first step to improve their instruction (McDermott, 2001). It is furthermore seem that reflection and discussion about student's learning difficulties and teaching practices should help in-service teachers to shift their attention from themselves as teachers to their students as learners (Van Zee and Roberts, 2001). During the course activities, reflection of their teaching practices and of the list of their own student's difficulties constitute a vivid evidence of what a teacher really think that is important not only to themselves but also to the instructor/researcher of the course. This sort of "prior knowledge" should be taken into account as the starting point of any teacher development course if the active learning of science teaching, based on the general principles of constructivism, is a central course premise (Fig. 42.1).

The course started with the administration of the FCI test before any Tutorial instruction (Pre-test), and finished with the administration of the FCI after all instruction (Post-test) followed by a whole group discussion of the proposed active learning methodology, teacher practices and models of learning as well as the value of teaching under a controlled atmosphere of formative evaluation. In this teacher development course a central issue is the use of formative assessment to improve instruction. We strongly advocate throughout evaluation of course procedures and results in a cyclic process of planning, executing and evaluating instruction in order to improve teaching and learning. A key point in this process is to select an appropriate evaluation instrument to measure the results of instruction, i.e., the objectives and procedures used in assessing student learning must be consistent with the goals set for instruction. Since the teaching strategy methodology chosen for the description of the present teacher development course, Tutorials in Introductory Physics, addresses the issue of conceptual learning of basic Newtonian mechanics in contexts closely related to physical reality, it seemed very appropriate to introduce the

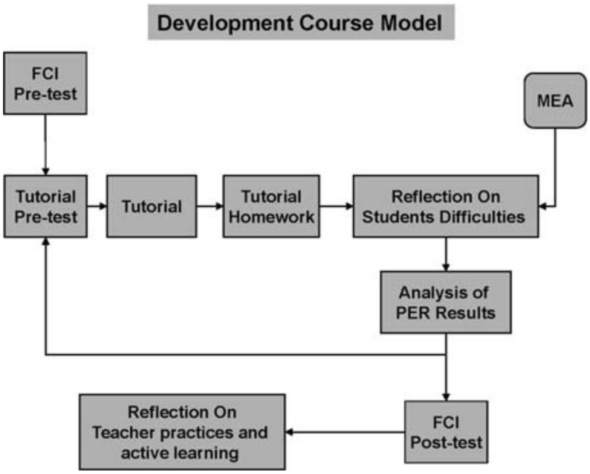


Fig. 42.1 General scheme of the course activities

multiple choice test Force Concept Inventory (FCI, Hesteness et al., 1992) as an example of a method to evaluate participant learning. FCI is one of the few scientifically developed multiple choice tests based on the extensive research on student understanding of the main concepts of elementary physics. Because FCI includes as distractors a rather complete taxonomy of alternative conceptions and learning difficulties, it has been shown to be very helpful in determining the degree and extend of these difficulties in a particular student population. Systematic results in the field of PER have shown that this kind of tests, administered before (pre-test) and after instruction (post-test) provides an objective measurement of the impact of a teaching strategy in the conceptual learning of the subject matter. It also allows for important comparison of different student populations (Hake, 1998). These characteristics made the FCI an excellent choice for a practical example of how to implement a virtuous cycle of planning, implementation and evaluation of instruction.

The activities between Tutorial Pre-test and Analysis of PER results are iterated as many times as different Tutorials are worked through in the course. While the aims and characteristics of the Tutorial pedagogical constituents (pre test, Tutorial and Homework) are discussed elsewhere (McDermott et al., 1994), it is very important that teachers go through them in the same way as regular students are supposed to do. This process not only assures teacher knowledge of the subject matter (physics) and pedagogical activities, but it also gives them the possibility of anticipating student's learning difficulties, preparing questions to guide that learning through an inquiring process. Among the several active learning methodologies that have been developed in the last decade as a practical result of PER Tutorials in Introductory Physics (McDermott et al., 1998) shows three distinct advantages: (a) it can easily be adapted to almost any kind of course structure or activities, since it covers the usual curriculum of basic physics, (b) it is very low demanding in classroom time, material, and human resources, and (c) Literature reports (Redish and Steinberg, 1999) objectively indicate that Tutorials is one of the most effective teaching strategies for introductory physics.

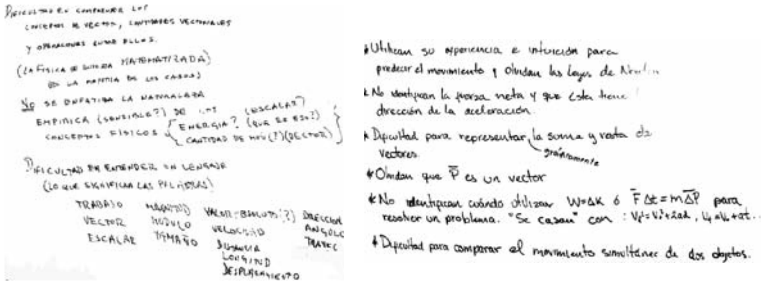
The Reflection on Students Difficulties section was included not only to discuss, in small groups, what the teachers think the students' difficulties with the subject matter are, but also, to elicit the teachers' model about their own scheme of what the important issues are regarding to learning and teaching of physics. It also helps participants to understand the kind of questions and problems that can be used to guide student's learning and to evaluate students' conceptual comprehension. Participants' contributions on the subject were compared and discussed against the results of educational research in the Analysis of PER Results section that ended each Tutorial cycle. This later section was included in order to make the interested in-service teacher aware of the kind of information that can be obtained from PER literature. It seems clear that putting together that information with the knowledge and experience that the in-service teachers have about their own students and their own model is a necessary and fundamental step to improve instruction.

42.3 Model-Eliciting Activity During the Course

A very important part of the development course is the reflection of the main students' alternative conceptions and difficulties that, based upon experience, teachers have observed. Since teachers work in groups and discuss what the main alternative conceptions their students have, the activity helps teachers to reflect something they might not have done before. It is also important because after the group discussion, a general discussion where each group presents and confronts their results and an instructor's presentation of literature reports to contrast with teachers results are organized. In addition, the activity plays another very important role; it serves as a model-eliciting activity. To learn more about the nature of teachers' developing knowledge, it is productive to focus on tasks in which the products that are generated reveal significant information about the ways of thinking like those produced by MEA.

Schorr and Lesh (2003) mentioned the steps one must involve to have a thought-revealing activities for teachers: (i) to find a situation that elicit a model, (ii) to produce a conceptual tool which makes evident of the model and (iii) to be able to share and reuse the activity. Reflection of the main students' alternative conceptions and difficulties is an activity which can elicit two types of models. A model that reveals the teachers' interpretation of students' difficulties and a model of teachers' conceptual systems related to their own teaching practice. This reflection activity comes after working with a Tutorial in which they were confronted with physical concepts through scientific reasoning. The result of that experience will have a different effect for each teacher and on each of them the reflection activity will reveal a different conceptual system related to their own idea of what the important issues in the learning and teaching of the subject matter are. In the activity, teachers work in groups discussing their ideas and they document them by writing a list on the whiteboard leaving evidence of their models. Finally, all participants take part on a general discussion sharing their ideas with the rest of the course participants.

An example of results one can obtain from this activity is presented in Fig. 42.2. Teachers were discussing in groups trying to build a list, based upon their own experience, what the main students' difficulties during a Tutorial related to changes in energy and momentum could be. Figure 42.2a, b present the result written on a whiteboard. The writing is in Spanish; however, language is not important to understand the difference between the teachers' models in this case, since the difference is so evident.



(a) (b)

Fig. 42.2 Two teams reflecting on students difficulties on energy and momentum

In Fig. 42.2a the teachers' model is related to the conceptual understanding of the principles, they emphasize the empiric nature of physics, they think language, i.e. the meaning of words is important and they even criticize the mathematization of physics. On the other hand, Fig. 42.2b shows a group of teachers who emphasize the use of the laws instead of using experience, they talk about procedures the students do not learn, and they worry about the students' confusion to use one equation instead of another to solve a problem. The general discussion in which each group presents their results might help each teacher to modify their model to make it more complex and sophisticate (Schorr and Lesh, 2003).

42.4 Results of Implementations of the Course

The course has been offered twice, during the summer academic recesses of 2004 and 2005 (hereafter referred as 2004 and 2005 courses) in the Tecnologico de Monterrey, a private university system in Mexico. In the 2004 course, three Tutorials where implemented: 'Representations of Motion', 'Newton's Second and Third Laws', and 'Changes in Energy and Momentum'. In the 2005 course we added the Tutorial 'Forces', as described below.

The participants of both courses were high school teachers and college professors who had a wide distribution of ages and teaching experience. Formal education ranged from B.Sc. degrees in science and engineering to M.Sc. and Ph.D. degrees in physics. The courses were offered at the Monterrey Campus of the Tecnologico de Monterrey. They were advertised nationally, so several participants came from other regions and institutions from the same university system, although a substantial fraction worked in the Monterrey area.

Participants' conceptual knowledge of force and motion was measured using the multiple-choice test Force Concept Inventory. The FCI is a 30-item, multiple-choice test built upon the results of educational research on the main difficulties and alternative conceptions about force and motion held by students at the high school and university levels both at the beginning and at the end of the short course.

The results can be summarized in two main points:

42.4.1 The State of Content Knowledge

FCI pre-test data from the participants in the 2004 and 2005 courses show that their initial state of disciplinary knowledge regarding force and motion is not adequate for an effective instruction of basic physics. About 50% of both populations have a performance below the 85%, a value considered by Hesteness et al. (1992) as a threshold for the mastery of Newtonian thinking. These authors also suggest that this figure could even be a reachable goal for high school students. It is clear that a basic condition for such ambitious objective is that teachers' conceptual knowledge be at least at that level. FCI pre-test data also show that the main conceptual shortcomings of the present samples of physics instructors are:

- believing that a (net) force in the direction of motion is necessary to keep an object moving at constant velocity (a manifestation of the force proportional to velocity alternative conception),
- failing to identify the type of motion that results when the applied force in not in the direction of motion and,
- incorrectly applying Newton's 3rd Law to practical situations involving motion at constant or varied velocity.

Finally, we note that participants in both courses (especially the 2005 course) also show weakness in one-dimensional kinematics (e. g., when comparing motion at constant velocity with accelerated motion along two rectilinear tracks). Since understanding of kinematics is required for Newton's Laws (and more complex physics concepts), this point should be taken into consideration when designing development courses for in-service and pre-service physics teachers.

42.4.2 The Effects of Tutorials in Modifying Content Knowledge

We have used the pre/post FCI data to draw some conclusion about the effect of just a few Tutorials on the conceptual knowledge of course participants about force and motion. The first conclusion is that after the development course the participants have a modest but clear improvement in their conceptual knowledge. An analysis using paired samples t-test showed that the pre/post distributions are different with a reasonable degree of confidence. Results show also that this gain is present in most FCI dimensions, but particularly so in those closely related to the subject matter of the Tutorials used in the respective development course. Pre-test results show the low performance of participants in kinematics. In post-test participants perform better although not excellent (particularly in 2005). However, in both courses, 2004 and 2005, only one tutorial (representation of motion) in kinematics was used. There are three other kinematics tutorials which can be used to further increase understanding (i. e. velocity, acceleration in one dimension and motion in two dimensions). Also we note that after the first course we realized that participants have some difficulties in identifying forces. For that reason in the second course we introduced the Tutorial "Forces", which addresses explicitly the issues of identification and notation of forces. Among other benefits of this change, we noted that the notation provided a unification of language that improved participants' communication. We think that this extra Tutorial was an important factor in the higher overall gain attained in the 2005 course. This and the fact that only one kinematics tutorial was used as discussed above confirm the important effect that just one Tutorial activity can have on student's knowledge (Abbot et al., 2000).

42.5 Summary

In this report we presented the objectives, characteristics and results of two implementations of a short professional development course for in-service physics teachers. In this course, constructivism is seen not only as a theoretical framework to understand how pupils learn science, but also how teachers, building on their own experiences as teachers, actively construct their new knowledge about science teaching by using thought-revealing activities and reflecting on their own teaching and profiting from the curricular advances provided by PER. Although the activity of reflection of the main students' alternative conceptions and difficulties does not follow closely the rules for a MEA (Lesh et al., 2000), it does elicit teacher's models regarding their own practice. This result could be used by the instructor/researcher to guide the course/research and by the teacher to understand how a MEA is used and confront the different models of teachers. This latter is an activity that will transform their model from a general and simple model to a more sophisticate and deeper model. The last part of this contribution showed that the content knowledge can be improved with this course, something that in conjunction with having lived the pedagogical knowledge constitutes one of the main objectives of the course.

Acknowledgments Genaro Zavala and Hugo Alarcon acknowledge financial support from the Tecnologico de Monterrey Research Chair in Physics Education Research under grant CAT-140.

Julio C. Benegas aknowledges the financial support of the Tecnologico de Monterrey for his participation in the course here describe.Part of this research has been carried out under the Project P-320102 financed by the Universidad Nacional de San Luis, Argentina.

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Chapter 43
Modeling as Isomorphism: The Case of Teacher Education

Sergei Abramovich

Abstract This chapter describes modeling activities designed for technology-enhanced mathematics teacher education courses aimed at the development of grade-appropriate mathematical concepts. Conceptualized in terms of isomorphism, these activities deal with various problem-solving topics in arithmetic, algebra, geometry, and discrete mathematics. The chapter is based on work done with prospective K-12 teachers and it includes their reflections on the use of technology in modeling.

43.1 Introduction

The material of this chapter shares pedagogical ideas that emphasize the importance of teaching mathematics to prospective K-12 teachers through modeling with new technologies in a variety of problem-solving contexts. The notion of modeling is discussed in terms of isomorphism. This abstract concept, applied "when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure" (Hofstadter, 1999, p. 49), has rather concrete meanings and can serve as a conceptual underpinning of mathematical modeling. Indeed, the latter can be described as a process of exploring properties of objects (contextual situations) that belong to one system by mapping them onto another system that is structurally identical (isomorphic) to it. Through this process, one creates a model (either physical or symbolic), explores its properties, develops methods of investigation, comes up with meaningful results, and then *interprets* these results in the language of the original system.

Typically, a model that is used to describe a situation is simpler than the situation itself. Sometimes, however, a model can be at least as complex; yet, the model enables the development of methods of investigation that are not available within

S. Abramovich (✉)
State University of New York at Potsdam, Potsdam, NY, USA
e-mail: abramovs@potsdam.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_43,

the system being described. Often, the best standards-based problem-solving situations from the K-12 curricula can be associated with elementary-but-powerful models that enable one to develop the appreciation of how mathematics can be approached, "at least initially, . . . from an experientially-based direction rather than from an abstract/deductive one" (Conference Board of the Mathematical Sciences, 2001, p. 96). Modeling pedagogy also has the potential to develop an *expertise* in bridging seemingly disconnected structures, whereas success of that pedagogy depends on one's ability to interpret (possibly hidden) isomorphic relationships between the model and the object (or situation) it describes. More specifically, the chapter analyses various modeling activities that were designed for technology-enhanced mathematics teacher education courses taught by the author and used both as learning and practice-oriented tools. These activities cover problem-solving topics in arithmetic, algebra, geometry, and discrete mathematics. The chapter reflects work done with the teachers within the courses aimed at the development of grade-appropriate mathematical concepts through modeling and applications. Technology is understood broadly in terms of cultural tools (Cobb, 1995) to include learning environments utilizing a spreadsheet, dynamic geometry program, as well as concrete materials. The chapter shows a variety of ways to use these tools to demonstrate isomorphic relationships between objects under study and their models. My claim is that it is the use of these tools that makes mathematics learning possible in a way that has not been available in their absence (Noss and Hoyles, 1996; Johnson and Lesh, 2003).

Another aspect of modeling highlighted in the chapter concerns two ways in which isomorphic relationships can be extended. One way is to change the structure of the object (situation), develop a new model, and then refine the corresponding methods of exploring the model. Another way is to change the structure of the model first, then to change the object (situation), and finally to develop new results, and then see how to modify the object (situation) in order to make new results applicable to it. In other words, the first approach (Fig. 43.1) describes the case of a model being dependent on object but not vice versa; the second approach (Fig. 43.2) describes the case when contextual inquiry results from the meaningful change of parameters of a model, a process requiring a higher level of mathematical thinking.

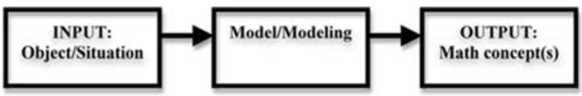

Fig. 43.1 Model does not change unless input changes

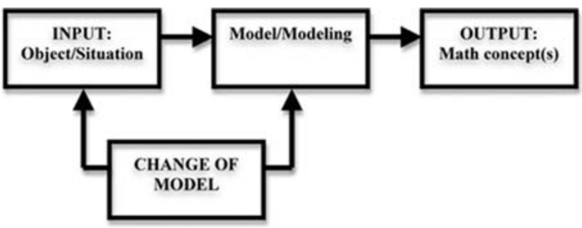

Fig. 43.2 Change of model within a model affects input

43.2 Modeling with Counters

The use of concrete materials (or their electronic analogs) at the elementary level is commonly viewed as an important precursor to the formal learning of mathematical concepts. One can say that this hands-on approach to mathematics *is* modeling. Indeed, by acting on manipulatives in a teacher-guided learning environment, one can create a physical model that is structurally isomorphic to an intended symbolic representation of a mathematical situation. Once the model is constructed, it can be interpreted symbolically in the form of numerical (or, more generally, algebraic) relationships. The illustrations of this section show how modeling with counters followed by the appropriate interpretation of the model can lead to the development of rather sophisticated mathematical concepts.

43.2.1 Illustration 1

Partition of integers into addends. Consider the following problem: Find all ways to represent a positive integer n as a sum of counting numbers. This mathematical situation can be modeled by using two-color counters as suggested in the following task adapted from New York State mathematics core curriculum (New York State Education Department, 1998): *Using four two-color counters, create different combinations of red and yellow and stop when you believe you found all of them.*

This task was given to a second grade pupil of average mathematical abilities. The pupil's response is shown in Fig. 43.3 (red/yellow=shaded/non-shaded). It represents a physical model constructed through trial and error that is short of one combination only. However, the true intellectual accomplishment of the pupil is manifested by his reflective analysis of the model; particularly, by the use of the term "brothers." In this, the pupil really attempted to *interpret* the model. From this interpretation, a generalized conceptual system (English, 2003) can emerge. The development of such a system is the main purpose of using concrete materials as modeling tools. In fact, a hidden meaning of the task is to create conditions for generalization by moving from a physical model to its symbolic interpretation. By interpreting each combination of counters numerically, one can develop the understanding of what changes and what remains invariant when the number of counters involved becomes a variable quantity.

For example, the second combination in Fig. 43.3 can be described in terms of the action on counters as follows: two yellow counters when added to two red counters give the total of four counters. In a decontextualized form, this verbal description can be represented numerically as $2 + 2 = 4$. But the fifth combination in Fig. 43.3, being associated by the pupil with the second combination, has the same numerical meaning! By pairing "brothers," one puts all red-yellow combinations of four counters in eight groups each of which has a unique numerical interpretation. One can see that the physical model (almost completely) constructed by the second grader is structurally isomorphic to eight representations of four as a sum of counting

Fig. 43.3 A model constructed by a 2nd grader

numbers with regard to the order of addends ($4 = 1 + 1 + 1 + 1$, $4 = 1 + 1 + 2$, $4 = 1 + 2 + 1$, $4 = 2 + 1 + 1$, $4 = 1 + 3$, $4 = 3 + 1$, $4 = 2 + 2$, $4 = 4$).

Moving towards generalization, the teachers can prove (using recursive reasoning) that the number of combinations of red and yellow doubles as one moves from k to $k + 1$ counters. This requires a reorganization of the model (without qualitatively changing it) and then a formal interpretation of this reorganization to make a transition from four counters to n counters. As a result, the output (Fig. 43.1) yields 2^{n-1} pairs of "brothers." In other words, one can conclude there exists 2^{n-1} ways to represent n as a sum of counting numbers with regard to order.

43.2.2 Illustration 2

Fibonacci numbers. As an extension of the above situation (the change of context), one can be asked to *create different size samples of two-sided (red and yellow) counters in which no two red counters appear in a sequel*. In doing so, one can discover that there are two ways to create a one-counter sample, three ways to create a two-counter sample, and five ways to create a three-counter sample. Furthermore, by interpreting the model so constructed, one can put all possible combinations that make the four-counter sample into two distinct groups. In the first group, the first counter is yellow – the total number of possible arrangements of that type equals the total number of elements in the three-counter sample. In the second group, the first counter is red – the total number of possible arrangements of that type equals to the total number of elements in the two-counter sample. Through this activity, prospective teachers can be introduced to Fibonacci numbers in which every number beginning from the third equals the sum of the previous two. In other words, there exists an isomorphism between Fibonacci numbers and the counter-based physical model through which these numbers can be discovered. Once again, it is through the interpretation of a hands-on activity that a new conceptual system, a recursion, can be developed.

43.2.3 Illustration 3

Combinatorics. Another way to extend modeling with counters is to change the model first and then find a matching context (Fig. 43.2). This can be done in a number of ways also. For example, one can asked to *find all possible arrangements of two red and three yellow (one-sided) counters*. This activity represents a change of model (over the original one demonstrated in Fig. 43.3), as the number of counters in each color is specified and cannot be changed through manipulation. In finding all ways of creating such combinations, one develops a new context associated with finding all three-combination of five counters, or, more generally, all m-combination of n objects. In other words, yellow/red color indicates that the object is chosen/ignored. One can show that in this case, there exists an isomorphism between the five-counter (physical) model and a five-letter word YYYNN – a symbolic model in which the letters Y and N mean, respectively, that the corresponding object is included/not included in the combination. Because there exist 10 ways to permute letters in the word YYYNN, the five counters can be arranged in 10 different ways. Consequently, the number of three-combination of five objects is 10. This context can be extended further to allow for the introduction of combinations with repetitions through the variation of the two-letter symbolic model. Finally, from both models generalized conceptual systems can be developed.

It should be noted that in a transition from one context to another, the need for a new symbolic model should be motivated by a didactic pair that includes a physical model representing new context and a counterexample demonstrating the deficiency of the old (symbolic) model. In fact, a physical model should be provided first followed by a counterexample. To clarify this remark, the following classroom episode is worth mentioning. One student argued against the author's counterexample designed to support the change of symbolic model (in a transition from one kind of combination to another), using a seemingly flawless argument for the preservation of the old model. What allowed the author to demonstrate a flaw in the student's argument was the absence of isomorphism between a symbolic model that the latter was advocating for and a physical model that correctly described the new context. In that way, whereas a counterexample can be recommended as a tool in justifying the need for new symbolic model in transition from one context to another, this tool alone may not suffice without demonstrating an isomorphism between the physical and the symbolic models describing a new context.

43.3 Modeling with Spreadsheets

This section describes modeling activities used by the author with prospective teachers of secondary mathematics. The *Principles and Standards for School*

Mathematics (National Council of Teachers of Mathematics, 2000, p. 335) includes an interesting example of a problem-solving activity recommended for grades 9 – 12 that can be interpreted in terms of isomorphism. It shows how the $n \times n$ multiplication table and a checkerboard of the same size can be mapped onto each other so that the former (numeric) structure can be utilized in counting all possible rectangles on the latter (geometric) structure. In other words, the multiplication table serves as a mathematical model for a counting problem in plane geometry. The following illustration shows how extending this counting problem/situation (i.e., changing input) brings about a new model from which one can develop new concepts.

43.3.1 Illustration 4

Counting rectangles with special properties. On a square size checkerboard find the total number of rectangles at least one side of which is an odd (even) number. This extension represents the change in situation that requires the change of model. The existing model – the $n \times n$ multiplication table – should be modified to reflect the absence of rectangles with both sides having even (odd) dimensions. First, consider the former case on a square checkerboard of even size $n = 2l$. All rectangles with two even dimensions can be put into l^2 groups so that rectangle of the size $(2k) \times (2m)$ is associated with an odd product $(n - 2k + 1)(n - 2m + 1)$ in the $n \times n$ multiplication table; $k, m = 1, 2, \ldots, l$. For example, when $n = 10$, $k = 1$, $m = 2$, one has a rectangle of the size 2×4 associated with the cell containing the product 7×9 (in that order). In other words, all rectangles of the size $(2k) \times (2m)$ can be mapped onto the cell $(n - 2k + 1)(n - 2m + 1)$ in the $n \times n$ multiplication table. Therefore, the total number of rectangles with two even dimensions on an even size square checkerboard equals to the sum of all odd numbers in the multiplication table of that size.

In finding this sum, one can use a spreadsheet. Figure 43.4 displays all odd products in the 10×10 table suggesting that the sum of the products equals to $(1 + 3 +\ldots+ 9) + 3(1 + 3 +\ldots+ 9)+\ldots+ 9(1 + 3 +\ldots+ 9) = (1 + 3 +\ldots+ 9)^2 = 625$. In

X	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	1		3		5		7		9	
2										
3	3		9		15		21		27	
4										
5	5		15		25		35		45	
6										
7	7		21		35		49		63	
8										
9	9		27		45		63		81	
10										

Fig. 43.4 2×4 and 4×6 rectangles related to products 9×7 and 7×5 respectively

general, the sum of odd numbers in the $(2l) \times (2l)$ multiplication table $S_{\text{odd}}(2l) = (1 + 3 + 5 + \ldots + 2l - 1)^2 = (l^2)^2 = l^4$.

Finally, by substituting $n = 2l$ in formula (1), one can find $S_{\text{even}}(2l)$ – the sum of all even numbers in the $(2l) \times (2l)$ multiplication table – as follows:

$$S_{\text{even}}(2l) = (1 + 2 + 3 + \ldots + 2l)^2 - l^4 = l^2(l + 1)(2l + 1).$$

This sum equals to the number of rectangles with at least one odd dimension on the $2l \times 2l$ checkerboard; *and*, in a similar way, one can establish an isomorphism between rectangles with at least one even dimension on a square checkerboard of odd size and the cells of a corresponding multiplication table. Note that one can also count rectangles with at least one odd dimension on an odd size checkerboard as well as rectangles with at least one even dimension on an even size checkerboard (Abramovich, 2007). This would require somewhat more complicated counting techniques. However, the idea of modeling as isomorphism and spreadsheet applications remain unchanged for these more complex considerations.

43.3.2 Illustration 5

Modeling with the Geometer's Sketchpad. Modeling activities using The Geometer's Sketchpad (*GSP*) can be carried out by teachers in a geometric context of edge-to-edge tessellations with regular polygons. For example, the triple of regular polygons with 3, 10 and 15 sides enables such a tessellation since $60° + 144° + 156° = 360°$. One can check to see that $1/3 + 1/10 + 1/15 = 1/2$. The last equality, showing an isomorphism between angular and fractional relations, is not a coincidence but rather a special case of the general relationship $1/n_1 + 1/n_2 + 1/n_3 = 1/2$ that enables an edge-to-edge tessellation using regular polygons with n_1, n_2, and n_3 sides. In other words, a triple of polygons that allows for an edge-to-edge tessellation can be mapped onto the triple of fraction circles that make up a half-circle.

One can partition one-half into three distinct fractions by using *GSP*-based fraction circles (Abramovich and Brouwer, 2004). The reason for establishing an isomorphism between regular polygons and fraction circles is because in the latter world one can use a system in partitioning a unit fraction into three like fractions, whereas the development of such a system in the world of regular polygons is more complex.

By comparing basic elements of the two physical models one can discover that an internal angle of a regular polygon and the central angle of the corresponding fraction circle are supplementary angles. This observation can be easily verified through the identity $\frac{360°}{n} + \frac{n-2}{n}180° = 180°$. This identity can be seen as a symbolic model drawn from a physical one. Its interpretation brings about the above statement regarding supplementary angles which can, in turn, can be generalized to the form

$$\frac{2\alpha}{n} + \frac{n-2}{n}\alpha = \alpha \tag{1}$$

where α is an arbitrary angle. The next illustration will show where this generalization can lead.

43.3.3 Illustration 6

Modeling a partial difference equation. The change in the model calls for its appropriate interpretation and the development of new contextual inquiries. For example, Equation (1) can be associated with what one may refer to as a double angle triangle in which two angles (measured α and 2α) are in a two-to-one ratio. This implies $0 < \alpha < 60°$. Consider a situation when each of the angles is divided into n congruent angles by the line segments l_k and l_m (k, m = 1, 2,. . ., n) dropped from the corresponding vertex to the opposite side of the triangle. One can show that the angles $f(k,m)$ formed at the intersection of the line segments l_k and l_m and the sides that include angle 2α, satisfy what can be referred to as a partial difference equation (Heins, 1941)

$$f(k+1,m) = f(k,m) + 2,\ f(k,1) = \frac{(k+1)\alpha}{n},\ f(1,m) = \frac{2\alpha}{n} \tag{2}$$

Thus, n^2 angles (all being multiples of α/n) can be identified and a question about their total sum can be posed. This question, representing a new context, was motivated by the change of model within a model (Fig. 43.2-type modeling). In turn, Equation (2) can be modeled numerically within a spreadsheet (Fig. 43.5) in which each of the integers is complemented by a hidden factor α/n. The sequence 2, 12, 39, 92, 180, . . . representing the sums of the integers for n = 1, 2, 3, 4, 5, . . . can be shown to be represented by the third degree polynomial $S(n) = an^3 + bn^2 + cn + d$. Using a spreadsheet function MINVERSE (that returns the inverse matrix for a given matrix) yields $a = 1.5$, $b = -0.5$, $c = 1$, $d = 0$. One can also be asked to

Fig. 43.5 Spreadsheet developed from a double angle triangle

k\m	*1*	*2*	*3*	*4*	*5*
1	2	3	4	5	6
2	2	5	6	7	8
3	2	7	8	9	10
4	2	9	10	11	12
5	2	11	12	13	14

find the smallest n for which the sum of the angles modeled within the spreadsheet is greater than 200α. For that, one has to model the inequality $S(n)/n > 200$ within a spreadsheet to get $n > 12$. Other questions about the double angle triangle can be formulated and resolved through spreadsheet modeling. Therefore, just like in the case of a checkerboard, the notion of isomorphism enables one to use a spreadsheet as an effective tool in modeling geometric problems.

43.4 Teachers' Voices from a Modeling Classroom

Many teachers believe that modeling activities involving concrete materials (manipulatives) are only associated with young children, however age level does not limit such modeling of mathematical content. Indeed, throughout the K-12 curriculum, the use of manipulatives as modeling tools facilitates one's conceptual understanding and the development of problem-solving skills. This approach allows one to develop a physical model and then internalize it through appropriate mathematical interpretation. As one teacher put it: "*I think using manipulatives helps the understanding of formal mathematics because it brings you back to the base of the problem and allows you to visually see what is going on in the problem. You can take the simple problem with the manipulatives and apply it to more difficult problems*." This remark suggests that the teacher has developed appreciation of using concrete materials in modeling. Moreover, it indicates that the teacher considers a physical model as a springboard into an extended context that may include generalization.

By using tools of technology in mathematical modeling, teachers can discover isomorphic relationships among different concepts. Through such experiential approaches to mathematics one can construct an "empirical situation in which [familiar] objects are differently related to one another" (Dewey, 1929, p. 86). One example of such an empirical situation is the connection between edge-to-edge tessellations with regular polygons and the representation of one-half fraction circle as the sum of three fraction circle pieces. As another teacher noted: "*I first used the fraction circles to experiment. As I tried more examples, I started to think about the process I went through. It is a great way to represent sums of unit fractions geometrically*."

Another example of this kind is the connection (recognized, in fact, by one of the teachers) between angles of a polygon and the corresponding fraction circle. This connection can be interpreted in more general terms by making an invariant angular element of the model (the straight angle) a variable parameter. This, in turn, can lead to a new model describing complex angular relationships developed within a net of intersecting segments stemming from two vertices of the triangle of a special type (Illustration 6). Reflecting on the activities, a teacher noted: "*In mathematics, it is important to experiment and formulate ideas about a solution for the problem. Once you have experimented, it is important to extend this knowledge into theoretical knowledge. When you create a theory, this theory can be applied to many different* *types of problems*." This note indicates the teacher's understanding of the value of modeling in developing theoretical knowledge from an experiential situation.

Furthermore, through experimentation made possible by the use of technology, one can see new directions in which mathematical explorations can proceed, discover connections among concepts, and experience mathematics as a set of interconnected ideas that make sense. In the words of yet another teacher: "*Throughout this class, I have also seen that there are a lot of connections in math. Learning about one topic or idea can carry over to another area. These connections help students to gain a more concrete understanding of mathematics*." The use of the word "concrete" by the teacher indicates the emergence of meaning through constructing isomorphic relationships that make mathematical concepts less abstract. The author believes that promoting sense-making modeling pedagogy based on the notion of isomorphism fosters teachers' modeling abilities and, ultimately, can positively affect how mathematics is taught in schools.

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Chapter 44
Mathematical Modeling and the Teachers' Tensions

Andréia Maria Pereira de Oliveira and Jonei Cerqueira Barbosa

44.1 Introduction

Debates about modeling in Mathematics Education have increased in the last decades (Blum et. al., 2002). One way to characterize modeling is as a learning environment in which students are invited to use mathematics to solve problems which refer to real situations (Barbosa, 2003, 2006; Skovsmose, 2000). Blum et al. (2002) have pointed out that programs of teacher education that include mathematical modeling are rare; and, Niss, Blum and Galbraith (2007) emphasize that it is important that teachers themselves have authentic experiences that involve modeling.

Barbosa (2001) and Holmquist and Lingefjärd (2003) provide evidence concerning the role of modelling in expanding the teachers' mathematical knowledge, in developing their abilities to solve applied mathematical problems, and in challenging their belief systems. But experiences "doing modeling" is not enough to support the teachers in developing these activities in their teaching practices. Empirical studies by Doerr (2007), Kaiser and Maaβ (2007), and Julie and Mudaly (2007) emphasize the influence of teachers' beliefs and understandings about the role of modeling beyond school classrooms. So, designing modelling tasks in classrooms requires more than just "doing modeling." More studies are needed that explore pedagogical dimensions of modeling in schools, and in teacher education. For example, Blomhoj and Kjeldsen (2006) have followed this direction by exploring some important teachers' dilemmas in conducting modeling in their classrooms. Their findings describe three dilemmas: the understanding of the phases in the process of modeling; the goal of the modeling activity, motivation or mathematics teaching; and, finally, how to develop autonomy in students in during projects work.

A.M. Pereira de Oliveira (✉)
State University of Feira de Santana, Feira de Santana, Brazil

J.C. Barbosa (✉)
State University of Feira de Santana, Feira de Santana, Brazil
e-mail: joneicb@uol.com.br

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_44,

44.2 Discursive Approach to Tension

A key concept in our research is tension. Our purpose is to investigate tensions that occur when teachers try to engage students in mathematical modeling activities. We use the notion of tension as a theoretical category related to concerns, uncertainties and dilemmas. Following a discursive socio-cultural approach (Edwards, 2004; Lerman, 2001; Wertsch, 1993), tension does not represent the manifestation of some interior psychological instance. Instead, it is constituted in social environment through discourse.

In this study, we explore the preceding idea using the case of a teacher who manifests some uncertainties about how to act in the mathematics class at certain moments during modeling activities. Theses uncertainties are related to the legitimacy that the teacher attributes to his/her actions in a social context, which is expressed through discourses such as: What do I do now? Am I on the right track? If I do that or do not do that, is it right?

In this paper, we focus on the following question: *What tensions does the teacher experience when achieving activities of mathematical modeling in his/her teaching practice?*

44.3 Context and Methodology

The research context was the Boli's class. He was developing the first modeling experience at his school in Conceição do Jacuípe City, in the Brazilian Northeast. He has been a teacher for 22 years and his lessons were organized by mainly using a textbook.

At the time of data collection, Boli was finishing a training program for non-certified teachers at the State University of Feira de Santana. The first author was a professor for Boli in two semesters devoted to mathematical modeling. He and his colleagues engaged in modeling activities by themselves. Then, after that, the pedagogical dimension was addressed through the reading of papers and the analysis of classroom cases. However, the main task was to develop modeling at a real secondary classroom as a kind of project work. This was the context where data were collected.

The nature of the research is qualitative (Denzin and Lincoln, 2000) because focus is on the understanding the tensions that the teacher has experienced when carrying out mathematical modeling. The Boli's modeling-based lessons were filmed and transcribed, and semi-structured interviews were carried out after each lesson. As part of the tasks for the training course, Boli had to write a narrative about the lessons, and it was incorporated into the data *corpus*.

The data analysis was inspired by the *Grounded Theory* (Charmaz, 2006) approach to research, which is intended to generate theoretical understandings based on the empirical evidences collected. All material was edited and coded as we tried to grasp a theoretical understanding for the research focus and to integrate it in literature.

44.4 The Tensions in Boli's Modeling-Based Lessons

In this section, we describe the tensions experienced by Boli when developing modelling-based lessons. Boli organized the modeling environment according to what Barbosa (2003) calls *Case 2*. In other words, the teacher presents a problem and the students collect data and investigate the relevant situations. The problem presented to the students was the following one: "How could the minimum wage sustain a family in Conceição do Jacuípe City?" This question was part of a project called "Basic food for all". The project was carried out during eight lessons, 2 h each one, from August 2006 to October 2006.

We describe three tensions that we have identified in Boli' s classes. They are: the tension of *"deciding what to do next",* the tension of *"students" involvement'* and the tension of *"students" domination of the mathematical content'*.

44.4.1 The Tension of "Deciding What to do Next"

In the first class, when Boli started the modelling task by inviting students to think aloud about minimum wages, he was not sure how to continue the activity and asked the teacher (the first author in this chapter): "*Should I ask for items of the basic food list now or write them on the blackboard?"* Then, when continuing the lesson, Boli again asked the teacher if it would be appropriate to set up students in groups: *"Do I set up the groups now?"* . . . Boli clarified his reason:

> I thought that I had to ask questions for students before. In other words, I think there is an order for questions. I planned all, but it's confused in the classroom. I ask or not the question? [. . .] Students talking a lot make me nervous [. . .] I think I had to follow a plan in order to make students to understand the task, and so on (From interview).

Boli was mindful of prior plans and the fact that, when students developed models, development often did not progress along pre-planned paths. So, he was uncertain about whether to ask leading questions.

In the second class, Boli debated with students about the list of products that would be needed to compose a basic food package. They wanted to define the amount for each product. But, different groups had generated different amounts; and, Boli did not expect this to happen. So, he was uncertain how to manage these divergences. *"What do I do? The amounts of basic food packages are different. . . . New things always happen and I had to seek help with you; we could examine what was occurring and so to plan the next class. [. . .] Every time I got worried about the project, I asked you for guidelines" (From interview).*

Particularly, in the fourth interview, Boli reported similar dilemmas as well as concerns related to previous planning: *"I get upset when unplanned things appear, but it is ok. I read on modeling, and it is ok. My worry is the following: what should I do next? Sometimes, I asked you* (first author). *I try to make the best at this moment" (From interview).*

Boli' s dilemma was that he had a planned guide to support his lesson, but unexpected things happened because modeling involves students expressing, and testing, and revising their own ways of thinking – which often did not develop as Boli had expected. We name this the tension of deciding *what to do next*.

44.4.2 The Tension of "Students" Involvement'

Another of the concerns expressed by Boli was to make sure that students were fully engaged in the modelling activities – since they sometimes seemed apathetic. For example, in the second class, when he was presenting a task, Boli had to stop his explanation and ask for attention. Getting silence, Boli continued to say some words about the task, but again a stop was necessary: "*Pay attention, folks! Have you got the prices which I asked you? So pay attention here. I want some answers, ok?*" However, the students continued indifferent despite Boli's questions. So, he kept asking for attention: "*Hey, pay attention here, come on!*"

Boli reported he was annoyed with his class because he did not get their attention: *"I got worried with the noisy-ness in the classroom. They need to be more attentive [. . .] I got boring at the initial point, because there was a group speaking aloud. I think the project is good for all, so they should have more interest" (From interview).* Boli expected more attention from students in classroom because he believed that the task would be good for them. The non attention was unexpected.

Before the last class, Boli started the class speaking about the end of the project. The students were a bit noisy at this moment, which made Boli ask for attention: *"Hey people, let's listen to me? Please pay attention!"* Continuing the class to introduce the notion of function, he observed students were not attentive: *"[. . .] I make an expression relating to these. . . pay attention!" "[. . .] Then, it just looks at. . . Please group, pay attention"*. In the last interview, he mentioned that the students were not concentrated on the activity and said: *"What annoyed me was that some students did not have much interest. [. . .] I was annoyed with this group. They must take part. It is a task that I am interested in, and it should interest them too" (From interview).*

When Boli analysed the students' involvement and compared it with his plan, he thought the students should be more engaged. This episode brings us to consider the *students' involvement* as a kind of tension.

44.4.3 The Tension of "Students" Domination of the Mathematical Content'

Boli' s concerns and dilemmas also included some about the mathematical content. Many times Boli had to interrupt the lesson to address mathematical misconceptions expressed by students. An example occurred in the fifth class, when Boli and a student group were developing calculations involving percentages concerning family expenditures. He found that some students were not able to make the calculations, and this missing skills prevented progress on the task. So, Boli asked: *"Have you ever studied ratio and percentage in 6th grade?"* On his narrative, he wrote what he observed. *"I was surprised that they did not remember percentage and rules of proportion"*. This situation made him to spend unexpected time dealing with students' difficulties which occurred "out of the blue".

Another episode occurred in the sixth class, when Boli tried to understand the calculations of a family' expenditure done by students. Errors occurred. So, he announced: *"These calculations are wrong, because they were done fast... let's check them!"(From observation)*. This was an unexpected situation because Boli thought students would not have problems with calculations.

> "As it was a mathematics class, you must have to learn calculations. They need some support in that. I had this concern" ... "Consider the case of the percentage. If I didn't explain how to calculate percentage, the task wouldn't be interesting. They were doing a modeling task and learning mathematics at the same time" (From interview).

Boli' s concern about the inadequate skills appeared when he had to address the students' difficulties for calculations. This case makes us to name this tension *students' domination of the mathematical content*. Missing skills were not expected since students had studied these topics in previous school years.

44.5 Discussion and Conclusion

In this chapter, we present three tensions expressed by Boli during the first experience with mathematical modelling in his classroom, which had been specified in the following terms:

- The tension of *deciding what to do* in the modeling environment, referring to a dilemma on which to do at any given instant. The teacher is undecided about different directions for lesson;
- The tension of the *students' involvement*, referring to a concern about making sure of the students' involvement. The teacher expects that students be involved in the task, but they seems indifferent or disperse;
- The tension of the *students' domination of mathematical content*, referring to a concern on what students know about mathematical ideas and algorithms. The teacher expects that students have some mathematical knowledge, but they seem to have many difficulties.

The data suggest the discursive dimension for the Boli's tensions, which involved three unexpected events: students thinking often evolves along unplanned directions for the task, students often develop divergent ways of thinking about the task, and un-mastered skills often prevent student progress. Thus, experience with modeling demanded new actions which were not familiar to Boli. So, tensions occurred when two discourses diverged: one referring to the planned "itinerary" and the other referring to what actually happens in the classroom.

Doerr (2007) describes such tensions as involving the pedagogical knowledge needed by teachers when he/she teaches mathematics through modeling. Results describe din this paper suggest that such pedagogical knowledge should become an important part of teacher education programs.

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Chapter 45
A Case Study of Two Teachers: Teacher Questions and Student Explanations

Lisa B. Warner, Roberta Y. Schorr, Cecilia C. Arias, and Lina Sanchez

Abstract We focus on the ways in which two middle school teachers interacted with their students at various points during a long-term University-based professional development project. Our specific focus in this report is on the types of changes that occurred in the ways in which the teachers provided opportunities for their students to defend and justify solutions and communicate directly with their peers.

45.1 Introduction and Theoretical Framework

This research is one component of a multi-year longitudinal project[1] in which researchers from Rutgers University partnered with middle school teachers in the Newark Public Schools as part of a district-wide systemic initiative in mathematics. In the current study, we focus on the ways in which two of the participating teachers interacted with their students in order to help them develop powerful mathematical ideas, take ownership of these ideas and communicate their thinking directly with each other. This paper extends prior research (e.g., Schorr & Lesh, 2003; Schorr

L.B. Warner (✉)
Rutgers University, Newark, NJ, USA
e-mail: lwarnerB@gmail.com

R.Y. Schorr (✉)
Rutgers University, Newark, NJ, USA
e-mail: schorr@rci.rutgers.edu

C.C. Arias (✉)
Rutgers University, Newark, NJ, USA
e-mail: carias@pegasus.rutgers.edu

L. Sanchez (✉)
Rutgers University, Newark, NJ, USA
e-mail: linas@rci.rutgers.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_45, © Springer Science+Business Media, LLC 2010

et al., 2007; Warner et al., 2006) in which the closely connected and interdependent relationship between teacher and student actions/reactions is documented. The goal of this report is to document the nature and type of changes that occur in two teachers' classroom practice, noting in particular, changes in their behaviors and, to the extent possible, the corresponding changes that take place in the mathematical problem solving behaviors of their students.

Warner (2008) states that specific student behaviors are associated with the growth of mathematical ideas. Her findings suggest that as understanding grows, there is a general shift in the types of behaviors students exhibit. For example, there are major shifts in the ways in which students communicate with each other and their teacher. While students may talk with their peers early on in the solution process (when working in small groups), the talk tends to be highly focused on attempts to make sense of initial conceptualizations and representations. As understanding grows, students tend be more focused on justifying and defending their own ideas to each other and consequently, may ask different types of questions to each other – questions that are more focused on linking and understanding different strategies, representations, and solution paths.

This report addresses the following critical question associated with these shifts in student behaviors: what types of teacher actions may contribute to these changes in student behaviors? More specifically, we address changes in the ways in which teachers question their students and in the ways in which students defend and justify their solutions – especially to each other.

This work is grounded in a models and modeling perspective (see Schorr & Koellner-Clark, 2003, for a more complete description). Briefly stated, we note that all teachers (indeed, all people) have internal "models" or explanatory systems for making sense of particular situations. These models are relatively stable and allow teachers to recognize and contextualize situations and make decisions relating to them. Schorr and Koellner-Clark note that teachers' models for teaching and learning are robust, and therefore do not change simply because they see or hear about a new tactic or strategy. Indeed, in many instances, "teachers may express new beliefs about children, teaching, or learning, [but] unless fundamental changes in their models occur, their practice will remain relatively unchanged. . ." (pp. 197–198).

The research in this report is designed to understand and document changes over time in teachers' models for helping students to explain and justify their answers and make sense of problem situations. We are particularly interested in understanding the interdependent relationship between changes in teacher knowledge and understanding and changes in student knowledge and understanding. Our research, as well as the research of others (e.g., Lesh et al., 2007), indicates that as teachers develop new models relating to the teaching and learning process, they notice new things about their students and their own teaching practices. This, in turn, causes the teachers to continue to revise their approaches to teaching in an iterative manner.

For the purposes of this study, we focus on several types of teacher behaviors that have been documented in the literature as contributing to increased opportunities for student understanding (Davis et al., 1997; Groves & Doig, 2005; Schorr & Lesh, 2003; Stein et al., 2000). These include (but are not limited to): encouraging students to share ideas, provide explanations (often using public displays of explanations), and, questioning students to elicit their ideas and then listening closely to the ideas for the purpose of building instruction based upon the thinking that is shared.

Boaler and Brodie (2004) highlight the importance of the *different* questions teachers ask in shaping the nature and flow of classroom discussions and the cognitive opportunities offered to students in the process. Others (e.g., Martino & Maher, 1999; Mason, 1998) recommend that teachers ask questions that promote student interaction to help extend their ideas and justify their solutions. To this last point, Mason notes that teacher questions or comments can be used as springboards upon which students base their actions. Depending upon the circumstance, students may simply try to guess what the teacher is thinking and offer an idea consistent with what they think the teacher wants to hear. However, when teachers ask questions in a manner that genuinely supports students as they learn to extend and justify conclusions, students often share their *own* ideas.

45.2 Methodology

The research reported on in this study takes place in Newark, the largest school district in the state of New Jersey. The two teachers that are the subject of this paper were chosen because of the extended length of time they participated in the project. More specifically, during the first two years of the present project, Ms. E and Mr. C participated in both workshop and classroom-based professional development experiences with University researchers and graduate students (as will be described below). Ms. E continued to participate in all aspects of the project for another two years. Mr. C, however, did not participate during the third year (he taught science that year). He resumed involvement during the fourth year of the project.

Two key components of the professional development (PD) involved weekly or bi-weekly meetings at Rutgers University as well as on-site support to the teachers as they implemented project activities in their classrooms (see Schorr et al., 2007, for a more complete description). During all aspects of the PD, the teachers had the opportunity to consider, amongst other things, mathematical ideas, classroom implementation strategies, student knowledge development, and classroom cultures in which proof, justification, sense making, and high cognitive demand were the norm.

During most of the PD sessions, teachers explored math problems that had three main characteristics: First, they had the potential for eliciting high cognitive demand in students (see Stein et al., 2000). Second, they were designed to elicit a number of important mathematical ideas (including discrete mathematics). Third, the mathematical topics were associated with the Newark Public School curriculum.

Data: For the purpose of this research, Ms. E and Mr. C were videotaped during selected sessions when they implemented problems that were the same or similar to those that they explored during the PD sessions. There were at least 18 such sessions videotaped for each teacher, involving four different mathematical tasks.

Data for analysis came from two primary sources: videotapes and descriptive field notes. Videotapes and descriptive field notes (written by the researchers) were compiled. At least two video cameras captured different views of the teacher and students. In addition, teachers were asked to provide written reflections immediately after each session. Student artifacts were also collected.

Analysis: Researchers identified key episodes based upon teacher actions that contributed to or served as an obstacle to students communicating ideas directly to each other and attempting to make sense of each other's ideas. We traced these moments back in time in an effort to document specific behaviors or actions on the part of the students (these included asking clarifying questions, attempting to link representations, setting up hypothetical situations, etc. For additional examples of some of the types of student behaviors/actions included in the coding process, see Warner, 2008 and Warner et al., 2002). We also coded for the same types of actions/behaviors on the part of the teachers. These were then summarized and transcribed. We then searched for relationships between the teacher behaviors and student behaviors. They were checked and verified by at least two researchers.

45.3 Results

In the next sections, we will describe some of the changes in each teacher's actions over time and how this appears to have impacted their students and vice versa.

To begin, we note that during all problem-solving sessions, both teachers allowed their students to work on the problem tasks in small groups over extended periods of time. Both also expressed a desire to allow students to build their own ideas and share them with others. They felt that asking students to explain their thinking was very important, and that questioning was a particularly useful way to elicit student thinking.

Consistent with prior research (e.g., Schorr & Koellner-Clark, 2003; Warner et al., 2006), we note that early on, Ms. E and Mr. C tended to ask questions that appeared to be based upon their own thinking rather than that of their students. They knew that they needed to ask questions, but were not quite sure how. In an effort to make sense of the situation, they often asked questions at inopportune times, or at the expense of the individual ideas of the students. In fact, without realizing it, they often interrupted their students in the middle of a thought or tried, through questioning, to "channel" student thoughts in a particular direction. As time elapsed, each of the teachers became more aware of these actions, and in turn, shifted their ways of questioning students, and having students share ideas with each other. These changes will be described below.

Ms. E: In personal conversations and in her field notes, Ms. E discussed how she questioned students and encouraged students to talk with each other. Early on in the project, she noted, "There are days when I revert back to my old ways-giving more answers than questions. Each time I do, I notice the difference in the students' behavior and response. They are always less responsive when I offer too much information. They appear more confused when I give them an algorithm. They produce very little when the tasks are low-level and repetitive" (Ms. E, personal communication, January 19, 2005).

Ms. E also decided that it was important to let students have extended periods of time to work on a problem, often as long as they wanted. One consequence was that some students went off in what turned out to be unproductive directions. Upon reflection, Ms. E noted that she was not sure when to step in and when to let the students explore. When she did step in, she confessed to having a "repertoire" of questions that she asked again and again. In fact, she noted at the PD sessions with researchers that she had four generic questions written on an index card and asked the same questions to each group. Her comments and our field notes suggest that student thinking had little, if any, influence on what she asked next; rather, she used her pre-recorded questions. We infer from this that her model for teaching had not changed; rather she had "appended" a series of new strategies to her older teaching model (see Schorr and Koellner-Clark 2003 for a more complete description of this aspect of teacher change).

After repeated reviews of her video-taped classroom activity with peers and researchers, she noted that the students were in fact talking more, but so was she, and she was not really paying attention to their ideas, but rather channeling them to fit her questions and ways of interpreting and solving the problem. As time went on, she stopped looking at the index cards and began listening more closely to her students' explanations.

We will use an episode from Ms. E's seventh grade classroom, involving combinatorics, to illustrate this. In this case, the students were investigating a problem involving a mathematical structure of the form: n(n – 1)/2. The problem essentially involved finding the fewest number of calls that would be needed in order for 15 people to call each other exactly once. (Ms. E explored this exact problem with her peers at the PD sessions).

After working in small groups for a class period, Ms. E. asked her students to share their ideas at the overhead projector in the front of the room. A young man, Juan, justified his general statement of n(n – 1)/2 using an area model. Two girls, Sattima and Dominique, built on his idea by building another form of the generalization: (n × n – n)/2. As they presented their new general statement to the class, and their classmates asked them questions about the representation, Sattima and Dominique made sense of their representation and linked representations to each other as they justified their solution. The episode began immediately after the students explained their use of a five by five array (see Fig. 45.1) as an example of why they multiplied the number of people by itself, subtracted the number of people then divided by two. Notice that the teacher only asked two questions in the beginning of the following excerpt, encouraging the students to justify their generalization. In response, the students set up their own hypothetical situations based on the existing problem, asked each other for justifications and linked representations to each other.

[Teacher] (*Students applauding their peers' presentation*) But more than clapping, actually, I wanted you to find out if it works. [Shahidah] It does work. [De'Quan] We know it works. [Teacher] Why? [Dominique] Look, this right here,

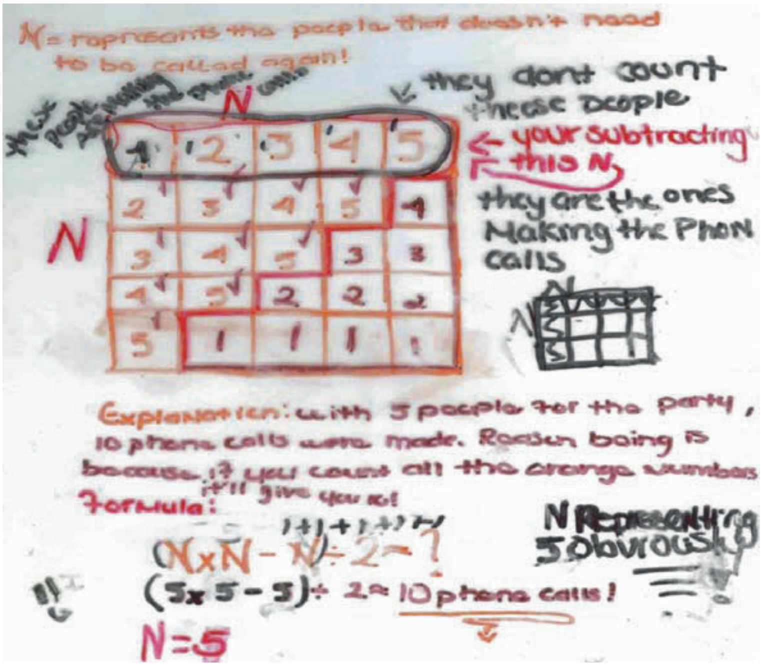

Fig. 45.1 An area model

we subtract it by… [Sattima] But we just did a different formula. [Dominique] And we did, we did N × N take away N divided two. So like, if you got 15…*(Student starts to set up a hypothetical situation but gets interrupted by a question from the audience.)* [Aquan] Hold up…why are you dividing by two? [Sattima & Dominique] Because there's two sides. [De'Quan] Isn't that Juan's formula? *(Comparing this formula to that of Juan's that was presented earlier.)* [Aquan] All right, if you don't do that, if you don't do that…it's like, it's like, you're still gonna do what the problem did? [Dominique] Huh? [Aquan] If the problem just gave five people and you don't do no chart are you still gonna have…you still be dividing by two? [Dominique] I don't get it. [Sattima] Oh, I hear what you're saying. There's no chart. And there's two sides. Yeah…you will still divide it by two…it's two…[Dominique] It's N × N. [Sattima] It's two…umm…It's two different things you're dealing with. [Valentina] Why are you subtracting N? [Aquan] Because. [Dominique] You subtract these people out (*pointing at the people on the top row*) [Sattima] Say there are four people (*setting up a hypothetical situation*). Well, say there's two people…it be one phone call getting made…You know how its unnecessary to make two phone calls…so it will be one…

The students continued by questioning the use of the same variable for both the length and width and asking why they used multiplication.

In the above excerpt, the students demanded explanations and justifications from each other and the teacher was only involved in a "moderating" role. The students took on the role of asking for, in fact demanding, explanations and justifications. In fact, they even began setting up hypothetical situations based on the existing problem. The teacher, in turn, spent very little time asking any questions, thereby leaving opportunities for the students to ask the questions themselves. Further, rather than going off on blind explorations, the teacher encouraged the students to continue to explore and justify the solution, even after they had developed a generalized solution.

Mr. C: In Mr. C's class, all student discourse was directed at him. The students did not question each other nor did they express interest in each other's ideas. Mr. C was the one who decided when a justification was fully developed and did not appear to notice whether or not the other students understood the explanation that was offered. Over time, as a result of reflection, during the PD sessions, while viewing his own classroom videos and the videos of others, Mr. C began to realize the impact of his actions. He told researchers that he wanted to "pull back" a bit from inserting his own questions, and wait and see if the students had questions for each other. In a reflection he wrote: "When questioning a group in front of the class I will hold off with my questions until I feel the rest of the students have exhausted their questions and comments. Even then I just try to ask a question or make a comment that will lead to more questioning" (Mr. C, personal communication, April 7, 2005).

We will use an episode from Mr. C's eighth grade classroom, in which the students were exploring a mathematical task that involved rational numbers, to illustrate this. The problem itself was one that was used at a PD session. The problem involved fractions and decimals. As part of the problem, the teachers had to consider some of the mathematical ideas involved in sharing two chocolate bars equally among three people. A central question emerged: was it true that .9 repeating was equal to or less than one? Mr. C argued vehemently that .9 repeating was not equal to one. He was eventually convinced otherwise, and felt that his students would benefit from solving the same problem as well.

During the very next week, his students worked on the same problem. When asked to share two chocolate bars equally among three team members and five chocolate bars equally among nine team members, some of the students in the class argued that they were able to represent the amount that each team member received using fractions but not decimals, since they encountered repeating but non-terminating decimal numbers. Rather than provide an explanation for the students, Mr. C encouraged the students to grapple with this idea and make sense of the repeating decimals in connection to the problem context. He highlighted several students' thinking by encouraging them to share ideas at the chalkboard and overhead projector.

One student, Jamie, in considering .9 repeating, made a connection between the current problem and an exponential function that they had explored several weeks earlier: [Jamie] When it is in decimals, it keeps on repeating. It will get really close to the next number but it never gets there. Just like. . .remember when we were talking about the graph. . .remember how we said it gets really, really close. . . it will be that close but it never gets there. . .like that. [Anna] Fractions and decimals are all parts of a whole number. They are equivalent to each other. It doesn't matter which way you choose to represent it. It is still the same.

Mr. C then asked another student, Miguel, to share his thinking at the chalkboard (see Fig. 45.2 below). Miguel presented his thinking as follows: [Miguel] You have three over three equals one, right? But one over three equals point three repetent [sic], right? So, point three, point three, point three repetent, add them and you get point nine repetent and. . .in the real world you want one, not point nine repetent. So, how can you get from point nine repetent to one in the real world? [Jamie] (*interrupting Miguel by responding to him*) It's because. . . [Miguel] without one third. . . [Jamie] Yo [sic]. . . [Juan] Round it up. . . [Jamie] Yo. . . It's not because you round it (*directing this to Juan*), it is because a fraction is really a division problem (*then directing his response to Miguel*). Exactly, that is why it is equal to one. Every fraction is a division problem. [Miguel] So where is that point whatever (*directing his question to Jamie*)? [Jamie] Only in math it is possible. You can argue it. [Miguel] Mathematics is saying that one equals point nine repetent (*writing 1 = .9. . . on the chalkboard*) but in the real world you can't have that. [Mr. C] So, is that true or not? Is one equal to point nine repeating (*with a neutral tone of voice*)?

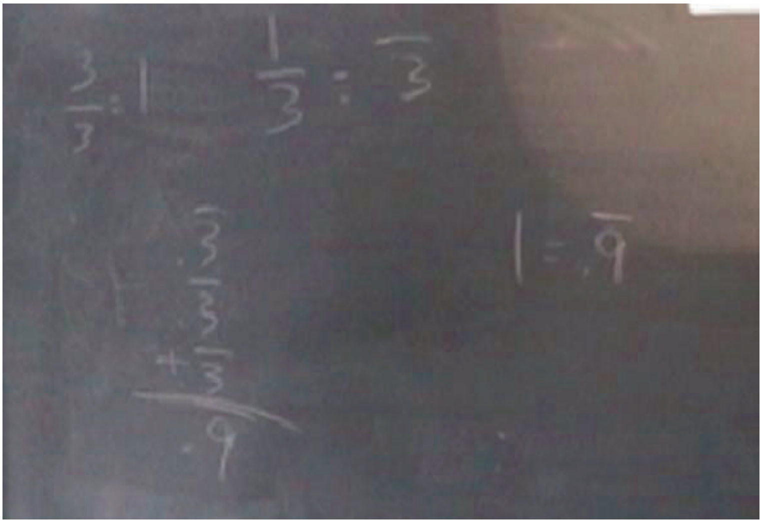

Fig. 45.2 Does 1 = .9. . .?

Whereas before, Mr. C would have simply allowed some student talk about this idea, and then promptly tell them the "right" answer, this time he encouraged the students to explore the validity of their own arguments. As a result, another student (Jasmine) came to the chalkboard and used a ruler to draw a line that was three inches in length and then split it up into one-inch segments. Some of the students in the audience argued that the inch-long piece would be one-third the length of the chocolate bar but not the decimal equivalent of one-third. Others argued that the length of the piece was equal to both one-third and point three repeating.

In this example, we see evidence that Mr. C was genuinely interested in encouraging students to question each other and validate their own ideas. The students responded to this by sharing ideas and connecting their ideas to previously explored contexts.

Ms. E and Mr. C: As can be seen from the excerpts above, both of the teachers encouraged their students to explain, question, justify, and build on their own or others' thinking. This resulted in a corresponding increase in their students' willingness to actually talk about their ideas with their peers. As both teachers repeatedly asked students to justify their ideas, their students began to spontaneously justify their ideas and ask their peers for justification, even when the teacher was not present. We see this as some evidence of more fundamental shifts in their models for teaching, especially as it relates to student questioning.

45.4 Discussion and Conclusions

The two teachers in our study displayed several types of behaviors as they helped students build mathematical ideas. Initially their goal was just to encourage the students to explain their mathematical thinking-whether or not the explanations were complete or understood by others. One way in which they did this was to ask questions to the students. At first, they would often ask the students questions based on their own preconceived notions about what a "correct" solution should be. This behavior appeared to reflect little, if any, deep change in previously established models for teaching and learning. They either asked general questions as in the case of Ms. E, or directive questions, as in the case of Mr. C As time went on, both of the teachers progressed in different ways: Ms. E went from using generic questions to highly specific questions that built on students' ideas. Mr. C went from using directive questions to guiding questions that capitalized on students' emergent ideas.

As the teachers encouraged students to discuss their ideas and justify their thinking to each other, their students began to spontaneously talk to each other about their ideas and provide justifications, even without teacher provocation. Over the course of the project, both of the teachers felt that their behaviors were helping their students build mathematical ideas. The teachers noted that the students began to build on each other's ideas in unique ways. This appeared to help students develop, as Kaput (1999) says, a sense of ownership of the mathematical ideas.

The teachers also reported that there were still occasions when they were not quite sure how to apply these practices more generally. This occurred especially when they were working on a problem involving a mathematical idea that they were not as familiar with or had not considered in depth. In those cases, they often intervened more often, or in Ms. E's case, had doubts regarding whether and how to intervene. This provides evidence that her older model for teaching was still, to some extent, highly operational, especially when dealing with situations that didn't appear to map into anything that she had directly experienced before.

An analysis of this type, while limited in scope, contributes to the body of knowledge that reveals how teachers make shifts in their models for teaching and learning. It also underscores the great difficulty that teachers may have in making fundamental and deep shifts on a more routine basis and over a wide variety of new and unique situations.

45.5 Notes

[1] The material contained herein is based upon work supported by the U.S. National Science Foundation (NSF) under grant numbers 0138806 (The Newark Public Schools Systemic Initiative in Mathematics) and ESI-0333753 (MetroMath: The Center for Mathematics in America's Cities). Any opinions, findings and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF, Rutgers University or the Newark Public Schools.

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Chapter 46
Pre-service Teachers' Perceptions of Model Eliciting Activities

Kelli Thomas and Juliet Hart

Abstract This paper explores the role of model eliciting activities (MEAs) in the preparation of United States pre-service teachers who will work with elementary-aged students. Specifically, this paper focuses on how teachers can be assisted in facilitating and interpreting experiences that support students in identifying and using mathematical structures, conceptual systems, and modeling perspectives. Preliminary results from a qualitative study investigating teachers' perceptions of teaching mathematics from a modeling perspective are included. Data from 16 participating pre-service teachers collected as they worked through a MEA, included audiotapes of workgroups, work samples, and a focus group interview. Common themes among the teachers regarding their perceptions were generated using grounded theory methodology. Implications for mathematics teacher education are described.

46.1 Introduction

Many pre-service and practicing teachers of primary students in the United States struggle with their own dispositions toward mathematics and what it means to help children learn mathematics. Often, teachers in the United States employ a narrow vision of mathematics that focuses on developing concepts and skills in isolation from related ideas, personal experiences, or real world situations (Ma, 1999). This superficiality can be due, in part, to mathematics instruction that seeks to cover large quantities of fragmented ideas and bits of information that are unconnected to other knowledge and conceptual systems. Evidence of shallow understanding by

K. Thomas (✉)
The University of Kansas, Lawrence, KS, USA
e-mail: kthomas@ku.edu

J. Hart (✉)
Arizona State University, Phoenix, AZ, USA
e-mail: juliet.hart@asu.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*, DOI 10.1007/978-1-4419-0561-1_46,

students exists when they do not, or cannot, use knowledge to make clear distinctions, build arguments, solve problems, or develop more complex understandings of other related phenomena.

With specific reference to teacher education, research on pre-service teachers' attitudes towards teaching mathematics shows that elementary teachers' lack of knowledge and understanding of mathematics results in negative attitudes toward the subject (Hill et al., 2005). The combination of insufficient experiences with mathematics and related anxiety about teaching it leads to concerns about elementary teachers' potential effectiveness in teaching mathematics to children, particularly those from traditionally marginalized populations such as students with learning disabilities (Bursal and Panzokas, 2006).

46.2 Theoretical Perspective

In a world where our use of mathematical thinking and understanding is increasingly complex and multifaceted, it becomes critical for students to learn to hypothesize, describe, explain, construct, analyze relationships, and communicate their developing understanding of mathematical concepts and conceptual systems. This is particularly true as professionals, in fields that use mathematics, report that the nature of their work and problem solving have changed dramatically during the last twenty years (Lesh and Zawojewski, 2007). It is not enough for users of mathematics to have memorized procedures and basic skills that they subsequently apply to problem situations. Preparing students for their futures beyond school will require that they approach mathematics in ways that supports deciphering and resolving complex problems that elicit mathematical thinking and novel concept development. A mathematical models and modeling perspective can be used to involve students in solving complex problems that represent real-world situations. We view models and the modeling process as an approach for mathematically making sense of complex systems (English, 2006; English and Watters, 2005; Lesh and Doerr, 2003). Models are conceptual systems that students develop to represent, interpret, explain, and mathematically describe a situation (Lesh and Doerr, 2003). Teachers play an important role in facilitating and interpreting the types of experiences that support students in identifying and using mathematical structures, conceptual systems, and ultimately models and modeling perspectives. How these intellectual dispositions can best be fostered for teachers, and by extension for their students, continues to be the source of much academic conversation across disciplines (English, 2003; English, 2006; Lesh, 2007; Lesh and Doerr, 2003; Schorr and Koellner-Clark), 2003.

46.3 Purpose of the Study

Investigations into how a models and modeling perspective can shape mathematics teaching, learning, and problem solving in elementary schools are imperative for providing a foundation for educational reforms. In this paper we explore ideas about developing pre-service teachers' knowledge and understanding of model eliciting activities appropriate for elementary students. In our view, model eliciting activities (MEAs) are realistic, complex problems that engage students in mathematical thinking beyond traditional school mathematics where the solution involves the creation of conceptual tools or models that can be used to communicate, make sense of, and resolve realistic situations (Lesh and Zawojewski, 2007). We also share pre-service teachers' perceptions related to teaching mathematics from a models and modeling perspective. The following questions framed a qualitative study of pre-service elementary mathematics teachers after completing a model eliciting activity (MEA).

- What are pre-services teachers' orientations toward a models and modeling perspective for helping elementary students learn mathematics?
- How can pre-service teacher knowledge and understanding of the use of models and modeling be developed?
- What are pre-service teachers' perceptions of using a models and modeling perspective with students including students with special needs?

46.4 Research Methods

46.4.1 Data Collection

As participant observer researchers, we utilized a qualitative approach to investigate issues related to a model and modeling perceptive with pre-service elementary mathematics teachers. One of us is a mathematics educator and was teaching the pre-service teachers enrolled in the elementary mathematics methods course, and one of us is a special education educator who also teaches pre-service teachers in the teacher education program. The pre-service teacher participants were enrolled in an elementary mathematics course during the spring semester of their senior year. Sixteen pre-service teachers were presented with a MEA (the Big Foot problem) and asked to work through the activity as students. While the pre-service teachers worked in groups of four on the Big Foot problem, we gathered participant observer field notes to capture the pre-service teachers' orientations toward working through a MEA. The conversation of one group of four was audio-taped to provide a more finely delineated record of the pre-service teachers' work. Additionally, we gathered pre-service teacher work samples from all four groups which provided evidence of the groups' progression through the activity. Following the work on the problem, we conducted an hour-long focus group, which was audio taped, facilitating a discussion about issues related to a models and modeling perspective in the elementary classroom. The following semi-structured questions/prompts were used to focus the discussion.

- Generally, share your perceptions of MEAs and a models and modeling perspective to mathematics learning.

- Describe the approach your group took to the problem and the processes you used.
- What are the benefits of MEAs for elementary school students?
- What are challenges of using MEAs with elementary school students?
- Share your thoughts about issues of planning, designing, and assessment related to using MEAs with students.
- What do you perceive to be benefits and challenges for using MEAs for students with special needs?

46.4.2 Data Analysis

We analyzed the data using a grounded theory approach (Bogdan and Biklen, 2007; Straus and Corbin, 1998) and the constant comparative method, facilitated by a coding process (Strauss and Corbin, 1998). The constant comparative method is a qualitative, dialectical process that involves the researcher simultaneously coding and analyzing the data in order to develop concepts. By continually comparing specific incidents in the data, the investigator is able to refine these concepts, identify their properties, explore their interconnectedness or relationships to one another, and integrate them into a coherent theory. As data have been analyzed and themes emerged, additional data will be collected to expand and confirm themes in future iterations of this study with additional participant teachers. The overall objective of this data analytic strategy is to begin exploring and describing pre-service teachers' orientations and perceptions related to using MEAs within a models and modeling perspective. We also use the data analysis process to begin to understand how knowledge and understanding of models and a modeling perspective can be developed for teachers.

46.5 Results

Preliminary results based on this study can be characterized by four major themes or areas of commonality. We present each theme that emerged with quotes from the participants to illustrate the findings.

1. Ambiguity of modeling eliciting activities
2. Benefits of MEAs
3. Challenges and constraints of MEAs
4. Beliefs and Expectations about MEAs and Disability.

46.5.1 Ambiguity of Modeling Eliciting Activities

Our data suggested that pre-service teachers struggle themselves with the ambiguity of modeling eliciting activities. Their reactions to working through a MEA were somewhat mixed. Some suggested the activity was "fun" while others said it was "frustrating" in that they did not know the precise process or procedure that should be followed. Comments from the focus group such as, "Well we just kept, we just kept messing up and then we just kept coming up with things and saying oh, wait that just can't be right it doesn't logically make sense," characterize the source of the frustration. One student reasoned, "I think we're used to having a formula where you just plug in the numbers and not having to come up with one on our own." The frustration that several of the pre-service teachers acknowledged as they worked on the Bigfoot activity may be attributable to their own experiences as elementary-aged learners of mathematics. While many found the MEA frustrating, others believed this type of experience was preferable to more traditional mathematics instruction. The rationale for this orientation toward the MEA is evident in the following statements from the focus group. "I liked this better, because, I am the kind of math learner that never can understand why or when to use a particular formula. So, it seemed to make sense I could choice which one to use this time and seemed like that would be a good idea and I could see why it would be a good idea." The audio-taped workgroup discussion, observer field notes, and the corresponding work samples confirmed that the general orientation toward the MEA – whether one of frustration or one of preference – was consistent with each groups' level of persistence with the activity and level of openness to the cyclical on-going process of working through the activity to generate a model. The pre-service teachers who expressed frustration were much more focused on their own lack of comfort with the ambiguity of the problem than those students who did not become frustrated.

46.5.2 Benefits of MEAs

The pre-service teachers identified benefits, as well as challenges or constraints to using a models and modeling perspective with elementary-aged students which seemed to flow from the two distinct perspectives toward the MEA. Pre-service teachers who worked in groups that indicated they would prefer this approach to learning mathematics identified benefits for students such as the following:

- There are lots of different answers, there's not just one answer.
- I say they (students) see themselves as a creator. Like they see that they can create equations.
- It makes it more engaging and interesting to them.
- They get to integrate all of their knowledge into the problem like how to use it and what's appropriate to use in a real situation rather than them being told what to use.
- It's like Bloom, it requires a higher level of thinking instead of like we had said a worksheet, it causes them to think.

46.5.3 Challenges and Constraints of MEAs

Pre-service teachers who expressed frustration with the MEA identified challenges or constraints consistent with their own discomfort and often went further to suggest that those challenges could be overcome by the teacher taking a more active role in providing structure and guidance. For example one pre-service teacher commented, “Depending on the age of the kids, I think just brainstorming before you started would work well. I mean I know we went into it blind and that was the whole purpose. But if you had younger kids who some of these processes were still very new to them, maybe brainstorming some different ways to solve the problem and just maybe doing a little bit of a review before you started. It might give them just a little bit of a jump start on the problem.” Another contextualized a constraint in terms of her own beliefs about herself as a learner of mathematics stating, “I would definitely say grouping students. Just making sure like because you know some kids are not that great at math like when I was in elementary school I know I would have struggled with this because I am not a math person. So just making sure you have an equal amount of kids that aren’t all at like a higher level in one group and all at a lower level in another group so that they can help each other.”

Conversely, when discussing potential challenges or constraints, the students who expressed positive orientations toward a models and modeling perspective spoke in terms of why a particular challenge should not overshadow the benefits to students making statements such as: “I really like what Katie and I did with our science investigation where they would have to come up with a plan for conducting this investigation. And then to get their plan approved and to explain it to us and get it approved before they could go ahead and do the investigation.”

46.5.4 Beliefs and Expectations about MEAs and Disability

The pre-service teachers’ perceptions of using MEA for students with special needs were fairly consistent. Generally, their perceptions aligned with the notion that in order for students with special needs to be successful, the teacher would have to provide directive assistance and structure that students could mimic in working through an MEA. These sentiments seemed to suggest reduced expectations for students with disabilities to perform well on MEAs, and a lack of understanding regarding the potential for MEAs to promote higher order thinking and mathematical discourse in a population often left out of activities that might actually engage them more effectively than traditional approaches. These orientations were presented through comments such as the following:

- They need more, they may need more direction. You know, as to like read the problem now come up with a way, you know you probably need to break it down more. Okay, first do this process, think about this and take these things into consideration.

- Another thing is like with structure. Some kids require a lot of structure in their activities and this doesn't really have, it's kind of go off and do your own thing. If you could somehow manipulate that and get more structure in there for them then I think it would work better.

One notable contradiction to this view was expressed by a pre-service teacher with ADHD. She shared her own experience with the MEA by saying, "I know that I was with it [snapping fingers] from the very beginning to the end and I had no problems paying attention at all. So I know that it is great for ADHD. But then, at the same time like, she [another student] mentioned earlier dyslexia or you know like autism, or downs syndrome. I don't know how you could apply that and would it be the same thing as like a learning disability or you know like anything else we studied." It is interesting that rather than take the stance that students with special learning needs would have to have more instructional structure or that MEAs would be frustrating for them, she approached the notion with a sense of intellectual curiosity and acknowledgement that she would need to know more.

These results suggest that it is important to acknowledge and understand the interactions between one's experiences/orientations and views of teaching and learning mathematics through a models and modeling perspective because those interactions likely will shape the choices teachers make instructionally. Pre-service teachers' orientations (dispositions, expectations, perceptions, and beliefs) toward a models and modeling perspective are effected by and influence their views of learning and teaching mathematics.

46.6 Conclusion

The preliminary results of this study have implications for teacher education and developing pre-service teachers' knowledge, and understanding of a models and modeling perspective toward learning mathematics. This study particularly highlights the connection between students' experiences as learners of mathematics, and their subsequent orientations toward MEAs. Increasing attention has been drawn to the need for developing students' abilities to engage collaboratively in mathematically rich experiences. Opening the dialogue among professionals of varying practical and theoretical backgrounds is essential for building fundamental understandings of how models and modeling can be effectively utilized to promote the learning of mathematics in primary classrooms.

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Section 10
How Do New Technologies Influence Modeling in School?

Chapter 47
Modeling Practices with *The Geometer's Sketchpad*

Nathalie Sinclair and Nicholas Jackiw

Abstract In this paper, we seek to establish insight into varieties of modeling that occur, recurrently, in students', teachers', and curriculum developers' experiences with *The Geometer's Sketchpad*. In this process, we contrast the conventional and practical sense of mathematical modeling – modeling of situations and phenomena to generate and predict plausible outcomes – to at least two other available forms or types of modeling practice that we find especially relevant to mathematics education.

47.1 Introduction

In the discourse and research surrounding students' use of technology-based simulation and construction environments, the term "mathematical modeling" is often invoked to both name and justify the richness of particular paradigms of technology interaction. *Modeling* becomes an activity, a goal, and a mantra. In this paper, we seek to establish insight into varieties of modeling that occur, recurrently, in students', teachers', and curriculum developers' experiences with *The Geometer's Sketchpad*. In this process, we contrast the conventional sense of mathematical modeling – modeling of situations and phenomena to generate and predict plausible outcomes – to at least two other available forms or types of modeling practice, that we find especially relevant to mathematics education in their foundational didactic intent.

N. Sinclair (✉)
Simon Fraser University, Burnaby, Canada
e-mail: nathsinc@sfu.ca

N. Jackiw (✉)
KCP Technologies, Emeryville, CA, USA
e-mail: njackiw@keypress.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_47, © Springer Science+Business Media, LLC 2010

47.2 Of Microworlds and Models

The word "model" and the act of "modeling" have a wide variety of meanings in mathematics education, yet all of them seem circumscribed by educators' common interest in providing students with greater opportunities for mathematical agency and helping them bridge connections between the pure world of mathematics (with fixed solutions and "perfect" forms) and the more messy, ambiguous, or subjective world of experience. The "model" thus conceived has strong family resemblances, from an educational technology perspective, to the "microworld" – the latter idea being the foundation of contemporary educational discourse concerning the relation of educational technology to students' modeling activity – and, indirectly, to central considerations of student agency and power.

Let's explore some of the strong affinities between mathematical modeling and Papert's notion of the microworld. Since its coinage thirty years ago in Logo research, the term "microworld" – and the corresponding (micro)world-view – have dominated research in the constructive use of digital technology in mathematics education. Microworlds have been described as "concrete embodiments of mathematical structures" that are extensible (so that the tools and objects of the environment can be built to create new ones), transparent (so that its inner workings are visible) and rich in representations (Edwards, 1995). Microworlds, with their focus on student construction through programming, fit well with the constructionist view of learning (Papert, 1991).

In these early days, using mathematical microworlds involved learners in the act of programming – and this was seen as fundamental to the activity of working with and within them. However, over time, microworlds were developed in which users did not interact directly with "code" but, instead, with microworld-specific tools which were more accessible on one level (in their domain-specificity) and yet more opaque on another (one could not necessarily see how the tools, themselves, were programmed). As Hoyles et al. (2002) point out, these microworlds were first seen as poor substitutes for the constructionist agenda. However, the new Logo microworlds they describe, such as *Matchsticks*, came to be appreciated for two crucial ideas: (a) providing tools that focused on mathematical essentials; while (b) leaving open doors for further student manipulation and exploration. Thus the locus of student mathematical agency shifted intelligent creation of Logo source code to the skillful direct manipulation of a broader, and in specific contexts, more appropriate, set of mathematical tools.

Compared with microworlds, models have a longer history in mathematics and have served broader conceptual and rhetorical positions in education. Models have been described as "a class of representational forms that include an ensemble of particular properties," as ways of mathematicizing the world, and as "mobile and extensible" representational forms that have a real-world referent (see Lesh and Doerr, 2003). Unlike models then, microworlds need not have any real-world referent – though of course they often do. (Famously, the Logo turtle is an instance of the system itself offering the promise of a real-world referent – the embodied turtle – on which to scaffold a more abstract information architecture – the intrinsic planar geometry of "turtle graphics.") Further, models are not limited to computer-based environments. However, representational shifts are common aspects of both models and microworlds, as are the possiblities of extension toward generality or applicability. Both also involve important shifts – both in their making and using – between the "concrete" and the "abstract." We intend the terms *concrete* and *abstract* to be evocative or at best provocative, but by no means definitive. Concrete situations and ideas are transparent, accessible, tangible, and manipulable; abstract situations and ideas are comparatively formal or theoretical, likely self-referential or referencing only other abstractions, and are often initially opaque.

In this paper, we remain rooted in the technology-based tradition of microworlds, with their dual emphasis on programming and direct manipulation, but we adopt the use of the word model because of the essentially referential nature of the *Sketchpad* examples we will be describing. In contrast to Logo, interactive geometry software environments such as *Sketchpad* and *Cabri* are characterized primarily by their ability to offer direct, continuous manipulation of mathematical objects and relationships (Jackiw and Finzer, 1993). Geometric constraints – expressed visually – impose a deterministic rule-base that transforms inputs (the set of independent points or parameters in the construction) into outputs (the constructed dependents of these inputs). Dynamic dragging explores the parameter space of the inputs while simultaneously determining – and displaying – the "behavior" of the output(s).

As we will discuss later, the primary inputs of *Sketchpad* – the point, the circle, the line – are themselves models of mathematical ideas, which can then be used to build more elaborate models both of other mathematical ideas (triangles, for example) and of non-mathematical phenomena. In contrast with Logo then, the programming language and the result of the program (what the user sees on the screen) collapse into one and the same thing: "The distinction between programmer and user disappears; the two coalesce into one – the student" (Laborde, 1998). No intermediary algebraic or numerical language exists in which one must first translate the referent in order to represent it.

47.3 Mathematical Modeling

The most conspicuous didactic application of "mathematical modeling" is the pretechnological word problem. A story context establishes some "real-world" referent; the student's job is to identify an appropriate formalism, exercise some mathematical apparatus, and translate the result back into the terms of the story. But the word problem is insincere: both the student and the teacher understands that the real-world referents are entirely pretextual, and exist only to be stripped away from the real work, which is "solving." The word problem usually also fails to genuinely involve modeling in that (on the front end) the situation is usually already instrumented mathematically – no real translation to mathematics is necessary – and (on the back end) no solution to the real-world context is acceptable or even deemed relevant unless it is the one to the preconceived mathematical "problem." Of course, not all word-problems need be so shallow in their engagement of genuine modeling;

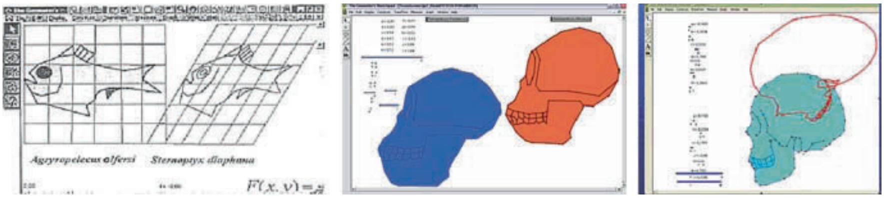


Fig. 47.1 Evolution by linear transformation

and, not all technology-situated activities claimed as "student modeling" offer more depth than the typical disappointing word problem – these are conventions.

A *Sketchpad* example of a perhaps "more authentic" tour through the mathematical modeling trajectory is illustrated in Fig. 47.1. It involves a school science project initiated by two Croatian secondary school students with a keen interest in paleontology. The activity emerged because the students were surprised to notice that a specific fish fossil resembled a sheared version of a fossil specimen of a distinct and considerably later species (Fig. 47.1, *left*), and because they realized their current algebra schoolwork gave them a formal vocabulary – matrix representations of linear transformations – for describing shears and dilations. So, the students launched a *Sketchpad* project to "predict" the morphological evolution of entire hominid skulls based on extrapolating linear mappings between component fragments, such as a jawbone or nose. The presence of two jawbones – of different hominids – in the fossil record forms the basis of a linear transformation which we can then apply to components of one fossil to predict the form of corresponding components of the other. Though in rare instances reasonable (Fig. 47.1, *center*), the ultimate utter wrongness of the hypothesis of a consistent and linear evolution of all conjoined biological "components" gives rise to anatomically hilarious predictions (Fig. 47.1, *right*) which the students explore at great length in the *Sketchpad* and *Powerpoint* files documenting their work.

Viewed from our working paradigm of "mathematical modeling," however, this student project appears almost canonical. It begins in a non-mathematical domain (fossil study), and addresses a question that arises there authentically (some form of: "what can we say about Missing Link(s)?"). A mathematical structure is identified that has relevant representational potential; and, a phenomenon in the domain (two jawbones) is used to instantiate that structure (by defining a linear matrix). The mathematics is then exercised predictively (applying the matrix to coordinates modeling other, non-jawbone components), and the results (coordinate data) are translated back into the domain (as a hypothetical skull), where their plausibility is assessed in the terms of the domain (apparently, with laughter).

Since mathematical modeling that takes place in *Sketchpad* uses not only the software's mathematically-representational (geometric) "language" but also a set of consistent, general-purpose tools for manipulating representational structures expressed in that language, models built in the environment easily take on a mathematical life of their own – beyond the original intentions or motives of the modeler.

Fig. 47.2 Kristen's model of a ferris wheel in *Sketchpad*

This was perhaps the case for Kristen's amusement park (Fig. 47.2, *left*), which features a compound Ferris wheel that can be set into motion through an action button. She constructed these objects using points, segments, circles; the button encapsulates a series of dynamic transformations she's specified for certain of those geometric shapes. Once this model exists, though, common *Sketchpad* usage tropes – like investigating "the trace" (or locus) of objects under motion – open up new dimensions of investigation, such as her exploration of the path of the chairs of her Ferris wheel, which generate mathematical curves resembling "epicycles" of planetary movement (Fig. 47.2, *right*).

As suggested in Fig. 47.3, both examples of mathematical modeling described above involved taking a concrete domain of application (evolution and the Ferris wheel) and translating the elements of that domain into the more abstract language of mathematics, thereby producing a model. In its traditional and most widely-construed trajectory, the mathematical modeling process involves using analytic techniques to translate a messy, real-life phenomenon into a precise, causal and controllable structure that produces deterministic effects or outcomes. These outcomes are calculated, and then translated back into the complex environment of the real world. The goal is to arrive at solutions and predictions about the original phenomenon by way of a detour through a technical simulation that we've "programmed" in the language of, and according to the control structures and data representations, mathematics. However, by remaining in (or close to) the language of the referent, a *Sketchpad* model can give rise almost immediately to further mathematical generativity, without having to first return to some external, source domain.

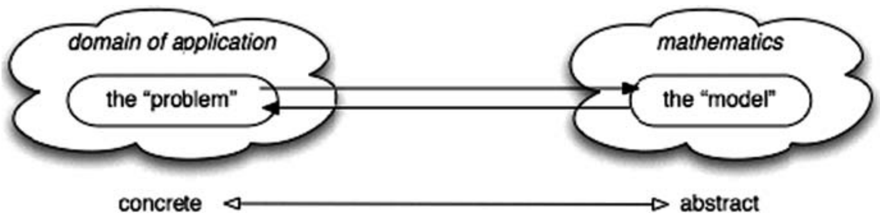


Fig. 47.3 Mathematical modeling

Figure 47.3 locates the model as an abstraction constituted within the domain of mathematics, and in this view it seems almost tautological to say that the “model” is thus “more abstract” – more metaphorical, intangible, inaccessible – than the concrete phenomenon being modeled. But in fact this is not definitional: *Sketchpad*-based models can occupy a variety of positions within the concrete/abstract continuum. In the next section, we explore a practice in which models intercalate and mediate between concrete and abstract ideas.

47.3.1 Virtual Manipulatives

The ability to use geometric primitives such as circles, lines and points to create not only polygons but also a much wider and more diverse set of objects leads naturally to a broader conception of modeling in *Sketchpad*, and to the possibility of expanding to other areas of mathematics such as algebra and number concepts, and even to other forms of modeling. One relatively common use of *Sketchpad* is to create virtual manipulatives that model other concrete manipulatives used in mathematics and school mathematics. Take for example a *Sketchpad*-based Chipboard model (see Fig. 47.4) developed for *The Connected Mathematics Project* (CMP) middle school mathematics curriculum. The referent here is a concrete manipulative used to model the addition and subtraction of integers.

Students place the chips in one section of a workspace and move them to another in order to represent addition or subtraction. Positive integers are represented by black chips and negative ones by red chips (White in Fig.47.4). For example, to model 3 + –4, the student would move three black and four red chips to the workspace and, by eliminating “zeroes” (pairs of red and black chips that cancel each other out), be able to count the one red chip left over. In the *Sketchpad*-based model, students drag chips over from the left side of the screen, across a dividing line, to the right side, and the result of their actions (the sum or difference) is calculated for them.

The *Sketchpad*-based model seems to differ only slightly from the referent model, but these differences introduce sharp pedagogic differentials. The virtual manipulative gives continuous feedback on the interaction between the visually represented chips and the more abstract notion of integer summation (in the sum displayed at

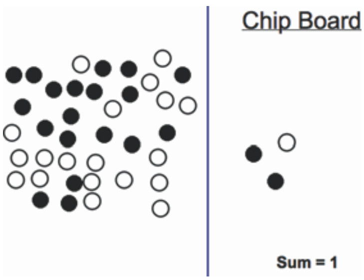


Fig. 47.4 The Sketchpad-based chipboard model

lower-right). The presence of this automatic calculation means the student no longer actually computes the sum or difference and thus – from an activity design point of view – it no longer makes sense to ask students for the result of 3 + –4. *Sketchpad* calculates it for them, but only *if they model it correctly*. This means that the model becomes, instead of a tool used to perform an operation, an object of reflection. This changes the power relations in the classroom: the teacher no longer controls the solution and the students now control the model.

We can't help notice the multiple levels of play in modeling language involved in this example. The student is manipulating (in *Sketchpad*) a model of a Chipboard, which is (in CMP) a model of addition and subtraction; the *Sketchpad* version of this model is built out of *Sketchpad*-circles and *Sketchpad*-segments, which themselves are models of geometric abstractions. The previous descriptive paragraphs model student interactions with this model (in our paper), just as both the *Sketchpad*-Chipboard-authors and CMP-Chipboard-authors modeled them in designing their respective manipulatives. As students move along various strands of this web of models and modeling relationships, the focus and relevant parts of modeling activity changes – for instance, the relationship of *Sketchpad*-circle to mathematics-circle is relatively moot at the point the *Sketchpad*-circle is used as a proxy for a chip (which itself is a proxy for a value). Is it surprising that students are able to navigate these multiple layers of abstraction with such little difficulty?

The Chipboard virtual manipulative makes little use of *Sketchpad*'s dynamic capabilities or of the resulting levels of encapulstion or objectification that the transition from process (occuring over time and imbued with motion) to object variously described as encapsulation, objectification, or reification. Objectified, encapsulated, or reified processes such as functions or loci can be re-infused with time and motion in *Sketchpad*'s dynamic visualization, and be represented in process-oriented terms. At the same time, each process can also be objectified within the environment and then treated as a higher-level process. An example of the dual process-object interaction is evident in the *Sketchpad*-based Rubberband model (Fig. 47.5) that was developed to model the use of rubber bands to create dilated imagines of given shapes (as done, for example, in CMP's *Stretching and Shrinking*).

In the *Sketchpad*-based model, a given pre-image is "stretched" to produce a dilated image: the student traces the contour of the pre-image while the "tip" of the elastic traces the contour of the image. However, two levels of objectification

Fig. 47.5 The *Sketchpad*-based rubberband model

become possible. The first level involves constructing the image as a locus. Then, students can both drag the position of the anchor point to investigate the effect on the locus (Fig. 47.5, *center*), and can also change the number of rubber bands in order to create different scaled images (Fig. 47.5, *right*) of the pre-image (including fractional and negative values). The *Sketchpad* model not only supports the objectification of the process into a locus, which can then undergo further manipulation but also offers a more precise rendition of the original image (as compared with the unsteady tracing with the rubber bands, which often end up flying overhead). The latter affordance also motivates the creation of other "perfect" models that would never operate as well in real life, and it is perhaps this aspect of things that leads to the types of "virtual manipulatives" one sees frequently built in *Sketchpad* for other than "educational purposes" – the Peaucellier linkages and clockwork gearings and other mathematical mechanisms and automata. Making models in environments such as *Sketchpad* – as in algebra – does away with the inconveniences of friction, gravity, and snapped rubber bands, while – very unlike algebra – still offering palpable representations of bodies and forms responding to forces through motion.

Thus, both the Chipboard and the Rubberband are models that mediate between a concrete manipulative and a more abstract mathematical concept. This time, as shown in Fig. 47.6, the manipulative partially inhabits the domain of mathematics, having been conceived by its designers as being evocative of mathematical relationships or structures, and the model remains within mathematics too, in fact, taking on more mathematical power as a consequence of its precision and transformability.

Comparing the migration of the "model" leftward – toward the concrete – from Fig. 47.3 to Fig. 47.6, one can't help wonder whether Fig. 47.6 can ever be completely inverted, with the "model" occupying space at far left, in the purview of the comparatively concrete. Though less common than the other types of modeling we've discussed, we believe such a modeling practice exists, and its home is the field of mathematics itself.

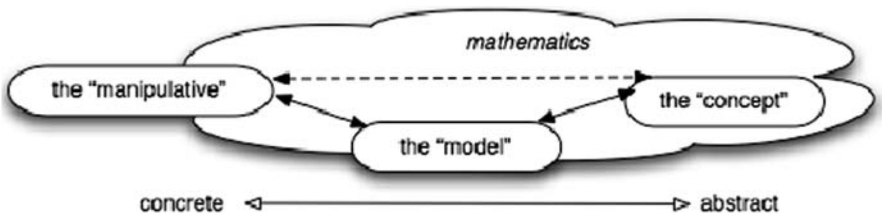


Fig. 47.6 Virtual manipulatives

47.3.2 Modeling of Mathematics

In pure mathematics, modeling signifies a different process than in applied mathematics, and with that different process, a different goal. Here the intent is to model mathematics itself – to embed in one mathematical structure or language a flexible representation of some other mathematical structure or language – often with a goal that is directly pedagogical. Thus process reasons from the abstract (the challenging mathematical concept or construct) to the concrete (the model) rather than the other way around. In a famous example, Poincaré's disk model of hyperbolic geometry (Fig. 47.7) introduces, instantiates, and explains how a geometry might appear and behave consistently, while violating Euclid's own fifth postulate.

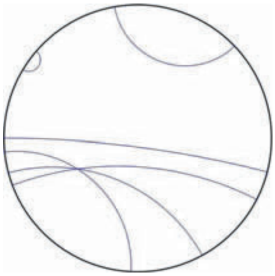


Fig. 47.7 Lines on the Poincaré disk

Poincaré's disk is a model that has only a theoretical referent – the axioms of hyperbolic geometry – and renders that referent accessible through the (still abstract, but comparatively more "concrete" and accessible) representational structure of the Euclidean plane. Moreover, because the model is described in the *language* of geometry – that is, in the language of the referent – any insights or relationships discerned by interacting directly with the model remain close to the referent, and plausibly relevant to it, without much retranslation back from the model to the referent. Thus when "modeling mathematics" (as opposed to when doing "mathematical modeling"), the source and the target of the idea to be modeled are mathematical structures, but the target domain involves more perspiciousally accesible, and often visual, representations that help broach intellectual access to more theoretical and abstract ideas. Figure 47.8 models this class of models.

A simpler – and more school-oriented – example of "modeling of mathematics"-style modeling can be found in the number line used throughout many elementary school curricula. As a geometric model of the real numbers, number lines make accessible – metaphorically tangible and demonstrable – many properties of the

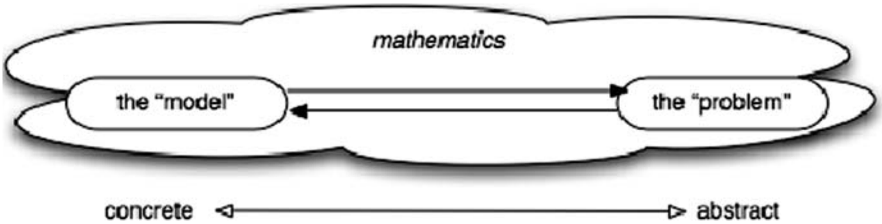
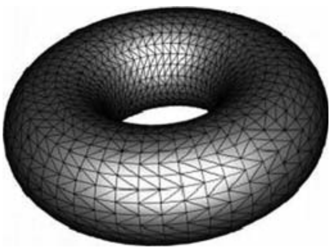


Fig. 47.8 Modeling of mathematics

reals such as order, density and scale. These are all mathematical concepts collocated in the geometric model and its numeric referent – the "common domain" effect keeps the distance between corresponding halves of the model's metaphors short.

A torus model (see Hawkins & Sinclair, 2007) provides a more detailed instance of this process within the *Sketchpad* environment. Again we find in it many levels of modeling going on. In learning to physically and then mentally manipulate the mathematical idea of a doubly-connected 2-manifold, students often move from a physical model (a torus) – or more commonly, the idea of some physical referent (perhaps: an instructor's verbal invocation of a torus, or a doughnut) – to a two-dimensional drawing – which is to say, model – of it (Fig. 47.9, *left*), to a two-dimensional mathematical notation (Fig. 47.9, *right*) that while still depictive, is both more formal and more flexible. The physical torus has the advantage of being manipulable in one's hands, but it has basic drawbacks. The object can never be seen all at once: if you wanted to draw a line on the torus, you'd only be able to see part of it from any one angle. The torus also imposes an extrinsic view of topological surfaces: one sees the object (the *surface* of the torus) from above instead of being experiencing it "from within." The transitions from 3-D physical model to 2-D physical model to 2-D notational model focuses us first on the properties of the surface and then on an intrinsic perspective of that surface, drawing us gradually to the heart of the topological idea (Meyerhoff, 1992).

The Hawkins & Sinclair *Sketchpad* model adopts the same representational notation as Fig. 47.9, *right*, but deploys it within a dynamic geometry environment. Thus, the *Sketchpad* artifact is much more of a "model of mathematics" (though in this case a "model of mathematical notation") than it is a "virtual manipulative" – its model of the torus is an intrisic model of that surface topological connectivity (a "flat torus"), rather than any sort of representation of a doughnut-shaped object floating in space. The dynamic flat torus sketch contains an empty rectangle representing the fundamental region of the torus, along with a set of custom tools in which users construct geometric objects such as points, circles, and rays, all of which reflect the fundamental region's bounded domain and the intrinsic surface connectivity between its opposite edges. (Thus the objects produced by these tools are quite different from the objects produced by "regular" *Sketchpad* tools, which



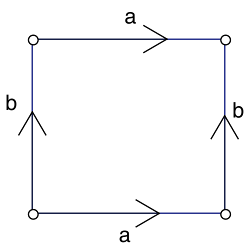


Fig. 47.9 Models of the torus

naturally live in the unbounded Euclidean plan.) The *Sketchpad* model is clearly a "topology microworld," where objects can be both constructed (using the visual "programming language" of the topology toolkit) and directly manipulated. The implementation paradigm of Sketchpad-based microworlds like this is that of the "drawing world" (Jackiw, 1997) where, from the user's perspective, *Sketchpad's* traditional tools are replaced by surrogate, authored tools. Thus the experience and exploration of the mathematics of the microworld is based directly on the experience of manipulating these surrogate tools.

To take a closer look at how this works in the context of this topological "model of mathematics," for example, consider the *Flat Torus Ray* tool. This tool will create an object such as the one shown in Fig. 47.10. As you click and drag the tool within the fundamental region, you extend a ray – or rather, a segment with an arrowhead on one end – away from its anchored base. Continue extending the ray and discover that it never leaves the fundamental region (see Fig. 47.10), but instead wraps around it like the striping on a candy cane, returning from the edge opposite the one it departed, continuing in the same direction. At any point, the ray can be extended by dragging its arrowed end with the mouse. Once placed on the screen, the two points defining the ray can be dragged to different locations in the fundamental region. While an example ray can be similarly depicted in a static mathematical diagram, in *Sketchpad*, unlike in Fig. 47.10, one doesn't just encounter the object as an accomplished fact, but rather is present at (and – importantly – responsible for) its very genesis, as the ray extends dynamically first across, and then around and around, the model.

Just as *Sketchpad* can model the Euclidean triangle, the custom tools provided in its topology models offer, in effect, ways of modeling rays and circles on the given topological surface. Doing so can help build surface orientation and intuition, and may expand one's example space of such objects (rays of different slopes, circles of different sizes). In addition, the *Sketchpad*-based topology model includes its' own modeling environment for representing topological results, problems and theorems.

For example, consider the idea of topological *closure*. The question here – what kind of lines will be closed (will return to their own starting point) on a topological surface? – for, say, torus geometry can be immensely difficult to visualize given an actual torus. However, the notational structure of *Sketchpad* flat torus model provides some more accessible visible evidence and confirmation. Beginning with a ray, one can drag its endpoint so as to change the slope of the ray (as measured

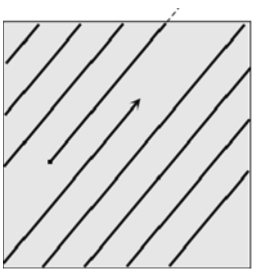


Fig. 47.10 A ray on the flat torus

intrinsically in the fundamental region). One quickly finds that some slopes close more immediately than others. For example, starting with a slope of 1 (Fig. 47.11, *left*), the ray closes upon itself right away. However, dragging B to produce a slope of 2 increases the length of the path. Figure 47.11, *right* shows a slope of 3, which does not seem to produce a closed path at first, until one extends the ray to meet A. Do all rays produce closed paths?

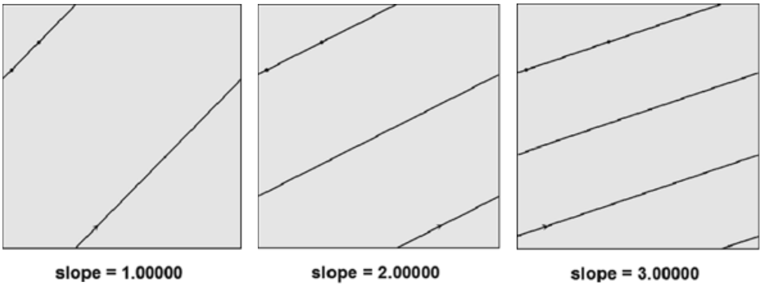


Fig. 47.11 A series of rays with various slopes on the torus

Rays of certain slopes will require more extension then others, and the act of extending them gives a visceral feeling for their pathlength. However, a slope with an irrational number value would mean extending the ray forever, without ever closing. This can be hard to see in the *Sketchpad* model given the finite resolution of the pixel screen. The line will begin to fill in the entire region until it "looks" closed. To show that the line with irrational slope is dense in the plane, however, one could simulate a common topology proof strategy by picking a point not on the line and an arbitrarily small (epsilon) neighborhood, and extending the line until it enters that neighborhood. In the *Sketchpad* model, a different approach might involve zooming in on an area where the ray seems to overlap itself, as in Fig. 47.12, *left* (this can be achieved by creating an *Expand* button that controls the scale of the fundamental region). Figure 47.12, *right*, shows that repeated zooming will always produce a space between the suspected overlapping lines. Like the *Sketchpad* line, the "zoom in" effect of dilation is physically restricted to the dimensions and resolution of the screen, but tangibly infinite through continued dragging.

As evidenced in this *Sketchpad*-based torus model, the interplay between process and object again provides the possibility for enactive interaction with mathematical ideas, while also supporting increasing levels of abstracted, de-temporalized entities. Figure 47.8 locates this type of modeling as a "mathematically concrete" roadmap and orientation to the "mathematically abstract," with both model and referent constituted within the mathematical domain. *Sketchpad*-based models for mathematics exploit this structure, and offer several forms of accessibility (the visual, enactive interpretations of "pure" structures) that do not ultimately or irreversibly compromise the idealized mathematical structures.

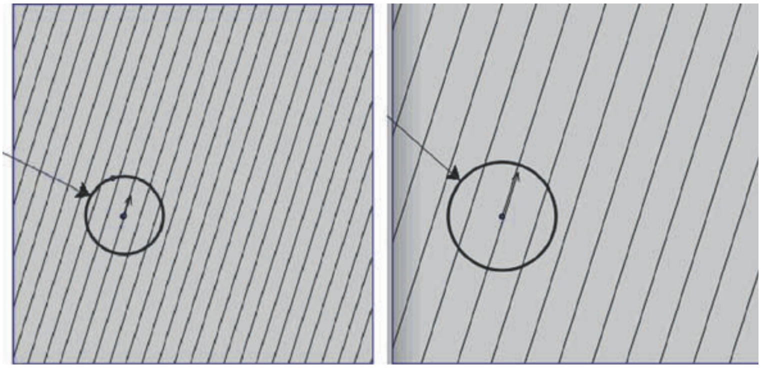


Fig. 47.12 *left*: A closed path?; *right*: Zooming in

47.4 Reflections

In our attempt to unpack the overloaded term "modeling" in the contexts of student practice and, particularly, in the technology milieu of the microworld, we have examined *Sketchpad* examples that chart three different categories of modeling. These categories vary in how their relation of the model to the modeled situates itself on a hypothetical axis that distinguishes access between (relatively) concrete and (relatively) abstract mathematical ideas. In the first, mathematical modeling, the model encodes abstract mathematical representations that seek to interpret a concrete, real-world phenomenon. In the second, the modeling of virtual manipulatives, the model mediates between a concrete, physical model for mathematical learning and the associated abstract mathematical concept. Finally in the third, modeling of mathematics, the model instantiates a concrete representation of theoretical mathematical concepts. While the first and the second categories have long histories outside technology contexts, to which the addition of *Sketchpad* introduces unique characteristics and paradigms, the second category appears technology-specific, and introduces a new genre to modeling practice.

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Chapter 48
A Principal Components Model of Simcalc Mathworlds

Theodore Chao, Susan B. Empson, and Nicole Shechtman

Abstract This study examines changes in student knowledge through a pretest/posttest assessment using data from an experimental research project on the effects of dynamic software on student outcomes in 7th-grade classrooms. Through a principal components analysis, we found that students who used the software had test results that clustered concepts differently than students who did not, grouping together test questions involving various representations of rate and proportionality. Students who did not use the software had test results that grouped test questions together based on surface features and proximity. We present a model showing how increased access to dynamic software leads to an increased ability to fuse concepts, where students add to their mental models and link various representations of the same mathematical concepts together.

48.1 Introduction

In 1994, the Scaling Up project was created to serve a need for innovative curriculum that would prepare students for higher-level mathematics as well as scale easily (Roschelle et al., 2007). This program would increase calculus readiness, equity to learning opportunities, and build upon a scaleable structure. The data used in this report comes from the most recent phase of the Scaling Up research project. The Scaling Up project utilizes multiple means of educational access in order to

T. Chao (✉)
The University of Texas at Austin, Austin, TX, USA
e-mail: teddychao@mail.utexas.edu

S.B. Empson (✉)
The University of Texas at Austin, Austin, TX, USA
e-mail: empson@mail.utexas.edu

N. Shechtman (✉)
SRI International, Menlo Park, CA, USA
e-mail: nicole.shechtman@sri.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_48,

create various ways in which students can take part in higher levels of mathematical thinking at an earlier age, using reform-oriented educational techniques and highly researched technology.

The main driving force behind the Scaling Up project lies in Jim Kaput's (1994) idea of "democratizing access to the mathematics of change and variation." Kaput, whose philosophies are deeply intertwined with the Scaling Up project's history, wrote often about creating new ways of using technology to fuel understanding of calculus concepts (Kaput, 1994). At the heart of the Scaling Up project lies a software component called SimCalc MathWorlds, a form of dynamic software that helps students engage in constructing their own knowledge (Roschelle et al., in press). In the Scaling Up project, students use SimCalc MathWorlds to create their own dynamic representations of rate and proportion, playing out one of Kaput's key ideas that dynamic representation through technology amasses large gains in conceptual understanding (Kaput, 1992).

Currently, the Scaling Up project involves teachers from a large, southern state, each one part of a delayed treatment model in which every teacher will eventually get a chance to implement the Scaling Up curriculum. The data in this report is from year 2 of 4, shortly after the first large treatment and control groups finished teaching the unit for the first time. The treatment group received professional development and support for teaching the Scaling Up curriculum as well as the Charles A. Dana Center's TEXTEAMS supplemental material, while the control group received professional development that consisted only of TEXTEAMS material. This structure was to account for a possible Hawthorne affect, since the TEXTEAMS material are not considered a stand-along curriculum like the Scaling Up unit.

The MathWorlds replacement unit creates multiple opportunities for students to engage in learning concepts of rate and proportionality, democratizing access for all learners (Roschelle et al., in press). Specifically designed for students to interact highly with MathWorlds, thereby creating personal dynamic representations of rate and change, the replacement unit utilizes the principle that dynamic representation through technology leads to large gains in conceptual understanding (Kaput, 1992). The core of the 3-week long replacement unit involved high interactions with the MathWorlds software, in which students learn to analyze, augment, build, and interact with various rate graphs. Students used a workbook to guide them through activities involving connecting graphs with real-world situations and creating piece-wise linear graphs based upon varying rates of speed.

Our study explores what gains in student understanding look like in regards to how students quantify mathematical understanding, and how student results cluster this understanding. While a novice versus expert framework, used extensively in Micki Chi's work in science understanding, might emerge (Chi, Feltovich, & Glaser, 1981) we question the validity of using a 3-week unit to create "experts". Rather, our model explores how students' increased interaction with dynamic software helps create mental models that fuse together multiple representations.

Our study explores what student learning looks like within a framework of democratized access to technology; showing how opportunities to learn cluster in ways that align with treatment type.

48.2 Method

The dataset involves 1621 students from 117 7th-grade teachers in 4 different educational regions of the state. Educational development centers of these regions performed the initial solicitation to bring these teachers into the study. Teachers were told to administer the pretests and posttests before and after they started the unit. Both tests are identical and measure student gain made during the 3-week treatment, with each 30-item test covering concepts involving rate and proportionality.

Principal Components Analysis is a statistical analysis technique that clumps test questions together into component groups based on student response (Meyers et al., 2006). Using this analysis on the pre/posttest results, we found significant components in which students using the SimCalc MathWorlds unit clustered conceptual understanding differently than students who had not been involved with the replacement unit.

In order to perform the analysis, the student dataset was split into four parts: treatment group pretest scores, treatment group posttest scores, control group pretest scores, and control group posttest scores. Individual principal components analyses with a varimax rotation were run on each group, creating components describing student understanding for that particular group. To analyze the results, the treatment group components were analyzed for changes in student learning that occurred during the MathWorlds unit, and those components juxtaposed with the components of the control group.

In order to determine the number of principal components to accept, three criteria are normally followed: (1) All components with eigenvalues greater than one, (2) All components where the cumulative variance is at least greater than 50%, and (3) The number of components above the "elbow" on the scree plot of components and eigenvalues. In this analysis, only the primary components was analyzed because it represented a significant sample, with the treatment pretest component representing 21.51% of the variance, the treatment posttest component representing 22.49% of the variance, the control pretest component representing 21.84% of the variance, and the control posttest component representing 21.8% of the variance.

48.3 Results

The results suggest that students exposed to the experimental unit were able to increase their understanding of rate and proportionality in ways that fused multiple test questions together into a singular component, showing better understanding of how various representations of rate and proportionality relate to each other. The students saw that questions involving slope, linear functions, graphs, and tables were conceptually equivalent and not separate mathematical topics.

In contrast, students who did not participate in the unit had results that clustered test questions together based only upon the similarity of surface features of the questions or their physical proximity to each other on the test, as seen in Table 48.1. For instance, these control students grouped together items that dealt with

Table 48.1 The primary component for the control group, students who were not involved with the MathWorlds unit, consisted almost entirely of questions involving piece-wise graphs. After the unit, the posttest shows the primary component still involves piece-wise graphs. The control students still see all piece-wise graph problems as the same

Question	Pretest concept	Question	Posttest concept
8a	Slope/Speed	13c	Piece-wise graphs
12b	Piece-wise graphs	14a	Piece-wise graphs
13a	Piece-wise graphs	14b	Piece-wise graphs
14a	Piece-wise graphs	14c	Piece-wise graphs
14b	Piece-wise graphs	14d	Piece-wise graphs
14c	Piece-wise graphs	15	Piece-wise graphs
1d	Piece-wise graphs		

piece-wise linear graphs, before and after the treatment, even if the underlying concepts behind the questions differed.

For their primary component group, the control students grouped together items that dealt with Piece-Wise Linear Graphs, before and after the treatment, even if the underlying concepts behind the questions differed. This is very different than the primary component group for the students who used SimCalc MathWorlds, in Table 48.2. They stopped grouping together Piece-Wise Linear Graphs questions, fusing together questions with such varied surface features as Slope and Speed, Linear Functions, Graphs, Tables, and Piece-Wise Linear Graphs together into one conceptual strand. These students seemed to understand that these separate representations of a linear function were related to each other on a deeper level. The Appendix lists the test questions comprising each component. For the treatment group, the pretest component consists completely of questions that involve Piece-Wise Linear Graphs. But the posttest component consists of test questions involving a variety of concepts,

Table 48.2 The treatment group, students who participated in the MathWorlds unit, showed major differences in the primary component. Again, the pretest shows the primary component consisting entirely of Piece-Wise Linear Graph questions. But after the unit, the posttest shows students viewing questions involving Slope/Speed, Linear Functions, Tables, Graphs, and Piece-Wise Linear Graphs as conceptually equivalent. Based on this complement, students have fused multiple representations of rate and proportion problems into a singular conceptual strand

Question	Pretest concept	Question	Posttest concept
12b	Piece-wise graphs	8a	Slope/Speed
13	Piece-wise graphs	8b	Slope/Speed
14a	Piece-wise graphs	9b	Linear functions
14b	Piece-wise graphs	10a	Tables
14c	Piece-wise graphs	10b	Linear functions
14d	Piece-wise graphs	10c	Graphs
15	Piece-wise graphs	11	Graphs
		15	Piece-wise graphs

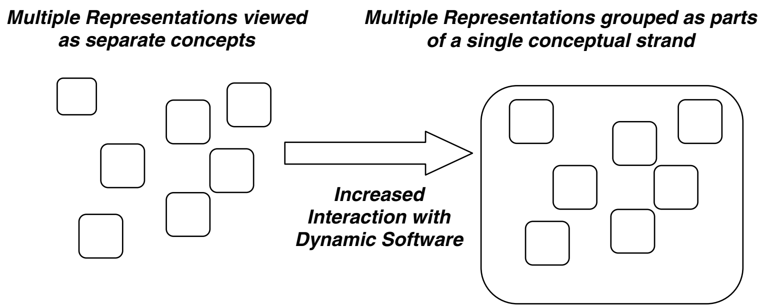

Fig. 48.1 A mental model of student understanding of multiple concepts

showing that after interacting with the MathWorlds unit, students were able to conceptually group together questions involving multiple representations of rate and proportionality. In contrast, the control group's component for both the pretest and posttest involved many of the same Piece-Wise Liner Graph questions.

The Scaling Up research project creates notable differences in the way students cluster concepts together, helping link conceptual knowledge beyond surface features. In just 3-weeks, students working with SimCalc MathWorlds showed a statistically significant difference in their understanding of rate and proportionality compared to the students who did not.

48.4 Discussion and Conclusion

Our study supports the results of Bodemer et als., work showing how interactive media suppers integration of multiple representations (2005) as well as Brenner et als., work on the effectiveness of multiple representations towards algebraic understanding (1997).

These results are valid only because this study assumes that gains were made from pretest to posttest in the treatment group, otherwise the results document deficits in student understanding. Luckily, previous researchers have found high gains in the treatment students (Roschelle et al., 2007). Additionally, the results show how similar the groups appear within this sort of analysis; if all components were taken into consideration, rather than just the primary component, the differences between groups starts to disappear. Also, the size of the principal component in the treatment group grew, while the control group's principal component stayed the same size.

Finally, one last threat to the validity of these results is the differences found in the pretest components for the treatment and control groups. In theory, the pretest components for both groups should be identical because the treatment has yet to start. Yet, before the treatment even started, the two groups already started to show differences. Perhaps subtle changes had already started to happen in the treatment teacher classrooms even before they started the Scaling Up unit, simply due to practices or beliefs they picked up in the summer training workshops.

What is sure is that the Scaling Up project is proving to be quite successful, even after only 2 years of data. The results shown here only add to the numerous data that can be gleaned from this project, showing that Jim Kaput's vision of democratizing access to the mathematics of change and variation learning can create a powerful and scalable revolution in learning.

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Chapter 49
Modeling Random Binomial Rabbit Hops

Sibel Kazak

Abstract Six fourth-grade students engaged in data modeling to make sense of the pattern of variability in binomial distributions. The students modeled five random rabbit-hops by tossing a fair coin to determine the most likely locations of the rabbits along a number line ranging from –5 to +5. To make sense of their empirical distributions, the students generated inscriptions of "paths" showing the possible ways to get to each final location and used these both to prove why certain locations were impossible and why central locations were more likely than extreme ones. Using the NetLogo Model (Wilensky, 1998) also helped some students to notice a particular distribution shape (i.e., a symmetric mound shape) in large number of trials. One student, for example, used the results of the NetLogo simulation to explain the distribution shape and to quantify the probability distribution in terms of the "number of ways" the events could occur.

49.1 Introduction

The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000) has introduced data and probability strand as one of the content standards students should learn from Prekindergarten through Grade 12. Given the critical importance of the knowledge of data and probability to make informed decisions in business, health, and everyday life, solid understandings of these two related topics are essential in our daily lives. However, data and probability are treated as separate topics in the curriculum prior to high school. The conceptual link between the two topics is not explored until the discussion of statistical inference (in advanced levels). Recognizing this as a problem, several statistics educators have recently advocated for developing the connections between data and probability in the earlier grades (e.g., Franklin et al., 2007; NCTM, 2000; Shaughnessy, 2003).

S. Kazak (✉)
University of Massachusetts Amherst, Amherst, MA, USA
e-mail: skazak@srri.umass.edu

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_49,

To begin to address the superficial division between data and probability topics in curricula, this study aimed to examine the role of the notion of probability distribution through data modeling as a way to link these topics together in early grades (Kazak, 2006). In this study, I drew heavily on two lines of research – on students' reasoning about distributions and on students' probabilistic reasoning. Recent classroom-based studies (e.g., Cobb, 1999; Lehrer and Schauble, 2000) have focused on the notion of distribution as a big idea in statistics and have investigated students' reasoning about distributions in the context of statistical reasoning and data modeling. Researchers fostered students' reasoning about data as aggregates through understanding of the qualitative aspects of distribution in relation to the center, spread or variability, and shape. Further, Shaughnessy (2003) suggests that students should be introduced probability through data. This approach basically assumes that data should motivate probability questions. He also points out the close link between the notion of sample space in probability and variability in data.

In the seminal work on children's development of probability concepts, Piaget and Inhelder (1975) had children experiment with random mixtures of objects and anticipated that children could build upon their intuition of random mixture to reason about the fortuitous distributions. They argued that children's conceptions of probability developed in relation to the formation of ratio and proportional reasoning, and combinatoric operations. Their work was extended through by recent studies (e.g., Horvath and Lehrer, 1998; Pratt, 2000; Vahey, 1997) focusing on the notion of distribution in probability contexts. This body of research documented students' reasoning about sample spaces and data obtained from common randomizing devices, such as coins, spinners, and dice, and rolling real dice.

Another well-known study by Kahneman and Tversky (1973) documented that people used persistent erroneous conceptions, such as the representativeness heuristic, in judging the likelihood of uncertain events in the absence of well-developed schemas for probabilistic thinking. For example, according to the representativeness heuristic, people evaluate the probability of an uncertain event based on the degree to which it represents some essential features of its parent population or the randomness of the process that generates it.

For investigating students' probabilistic reasoning, I used a different approach by building on understanding of variability towards the notions of probability and expected number through data modeling. More specifically, I focused on the idea of probability distribution and students' reasoning about the way the distribution was shaped. In this paper, I examine how students made sense of variability in a binomial distribution when modeling five random rabbit-hops by tossing a fair coin to determine the most likely locations of the rabbits along a number line ranging from –5 to +5.

49.2 Study

The small-group design study was conducted with six fourth-grade (9-year-old) students. The primary goal of the study was to examine the development of young students' reasoning about distribution and probability through a sequence of tasks in which students engaged in data modeling in chance situations with physical experiments and simulations. Based on the students' initial knowledge about probability and probabilistic reasoning on interviews conducted before the study, I assigned students to two groups with similar background: Emily, Alicia, and Alex formed Group 1 and Caleb, Josh, and Maya were in Group 2.

The study took place over eleven sessions each lasting 1–1.5 h. The instructional mode was a combination of individual and group work followed by a group discussion. The data sources included student-produced artifacts and videotapes of the individual interviews and the group interactions. The qualitative analysis of data focused on the ways the students arrived at their responses and their understandings of data and probability. The focus of this paper is on students' investigation of a data-modeling task during three 1–1.5-h-long sessions (episodes six through nine).

Task and Software. Students were introduced to the binomial distribution through modeling random rabbit-hops in the following problem: *Suppose there are a number of rabbits on a land where each rabbit can choose to hop only right or left. For each hop, they are as just likely to hop right as left. We want to know where a rabbit is likely to be after five hops* (Adapted from Wilensky, 1997).

This problem built on the idea of equiprobable outcomes, which was established in two contexts, one that involved dropping marbles in the split-box[1] and the other flipping a fair coin (Kazak, 2006). Following the initial discussion of the rabbit problem, students flipped a coin to decide whether a rabbit would hop right or left and marked the final location of the rabbit after five hops along a number line ranging from –5 to 5. The results of ten trials were recorded on graph paper and then discussed as a group. Subsequently, students used the *NetLogo* modeling-and-simulation environment, Binomial Rabbits (Wilensky, 1998), to conduct a large number of trials to investigate the binomial distribution of rabbits. Using the *NetLogo* model (see the modified version in Fig. 49.1), students simulated five random hops (with P(hop right) = 0.50) with various numbers of rabbits (i.e., 10, 100, 500, 1000, and 10000).

49.3 Initial Predictions

Even before flipping a coin to determine the random rabbit hops, students had some strong intuitions about the random hops. Students in Group 1 drew two possible paths, such as RLRRR and LLLLL (R stands for "right" and L stands for "left"), labeling the start and end points (see Fig. 49.2). They said that any of these could happen. In Group 2, Josh developed a representation of the possible end points after

[1] The apparatus [adapted from Piaget and Inhelder (1975)] was an inclined box with a centered funnel-like opening on the upper part. Marbles are dropped through the opening, striking the top of the partition and going either right or left into the two slots. In this task, students were expected to observe that a marble was equally likely to go right or left, and thus the resulting distribution of many marbles would be expected to be uniform.

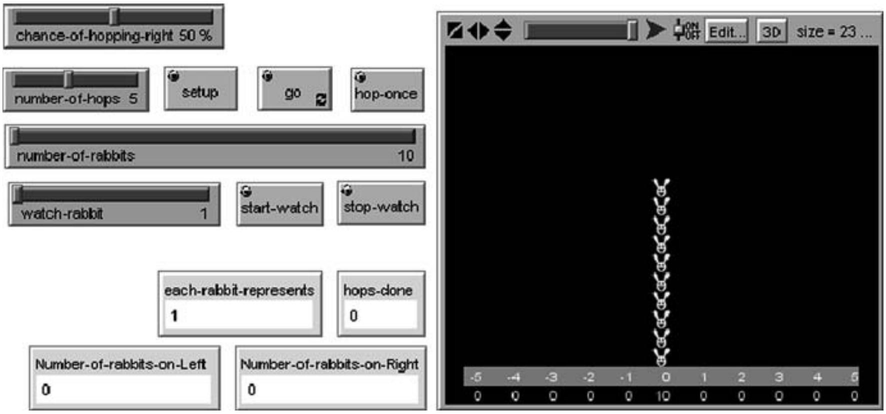

Fig. 49.1 The *NetLogo* interface: Hopping rabbits task

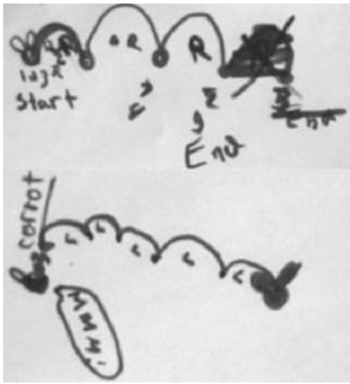

Fig. 49.2 Students in Group 1 initially generated a few possible paths for five random rabbit-hops

five hops. He first generated the representation shown on the top of Fig. 49.3 in which the big circle in the center was the beginning point and the five smaller dots on each side (labeled T for "tails" or H for "heads" after the introduction of coin flipping) were other possible ending points. On this representation, he began to demonstrate how a rabbit could end up on these locations by moving a rabbit object

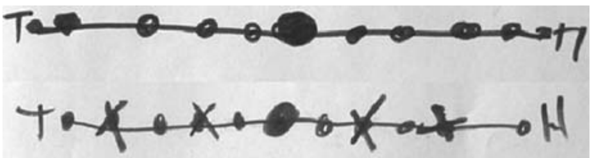

Fig. 49.3 Students in Group 2 began to show possible and impossible outcomes for five random rabbit-hops in their initial predictions

one hop at a time. After Caleb's trial of a random five-hop sequence on the representation, Josh and Caleb noted that some of the marked places on the right and left sides of the starting point were impossible to get after five hops. They crossed those locations out on the representation (bottom of Fig. 49.3). None of the students, at this point, saw that it was impossible for a rabbit to end up at the beginning point after five hops.

Prior to the coin simulation, students' predictions about the distribution of rabbits after five hops indicated a limited view of possible results. They expected an equal number of rabbits on each side of 0 along the number line based on the equally likely hops to right or left determined by a fair coin (in both groups students assigned heads to right and tails to left), rather than equal on both sides with a symmetry and more towards the middle. After developing an intuition about possible final locations, Josh and Caleb made their predictions more specific while others just guessed where the rabbits would most likely be after five hops. For example, Josh believed that –1 and +1 would be the most likely outcomes because there was a "50-50 chance" of getting heads or tails and thus the rabbits would hop back and forth mostly. Caleb paid attention to the symmetry around 0 when he predicted that –3 and +3 had both the same chance because they were just the opposites in terms of the paths, such as HHHHT and TTTTH.

49.4 Inscriptions and Simulations

As they began modeling the problem by flipping a coin, students in each group developed two different ways to record the results. Students in Group 1 used "paths" or arches (Fig. 49.4) and those in Group 2 drew connected dots for each trial (see Fig. 49.5) to keep a history of each outcome they got from flipping the coin. I call these artifacts, *inscriptions*, drawing upon Latour's (1990) description of tools for modeling aspects of the world, communicating and persuading one's ideas to the public, and transforming them into new forms of ideas. In this study, inscriptions became useful tools for students to represent each random hop, to keep track of them, and to show the final location after five hops. Also, students used these drawings to support their claims about impossible and most likely outcomes. For example, Alicia cited the density of paths in the middle (around 0) in Fig. 49.4 to

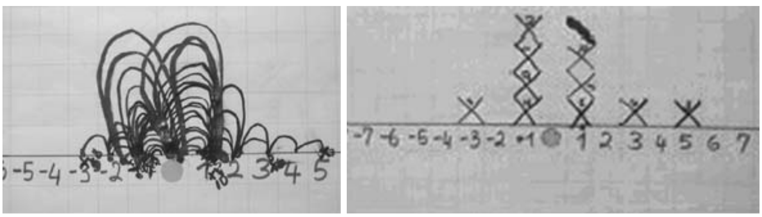

Fig. 49.4 Inscriptions of results of random rabbit-hops made by Group 1

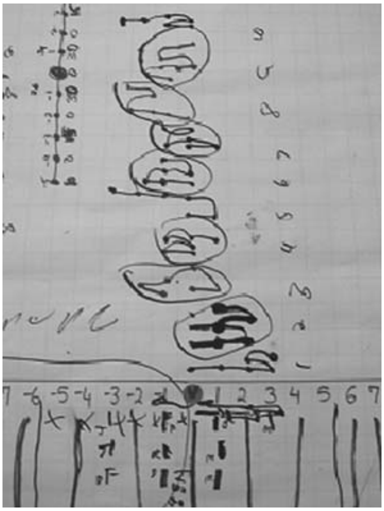

Fig. 49.5 Inscriptions of results of random rabbit-hops made by Group 2

support her claim that the most likely locations were in the middle, i.e., "more lines in the middle. . .less lines on the sides."

When students were asked to explain why middle locations were more likely than extreme ones, they tended to identify the central locations (–1 and +1) as "easier-to-get" and the extreme ones (–5 and +5) as "harder-to-get" by using paths. Some examples of student responses are as follows:

Josh: 50-50 chance that the coin will land on heads or tails. So, most likely it is going to go something, I mean you can see a lot of zigzags [He circles all the paths for 1 s in their representation shown in Fig. 49.5]

Emily: Because it is easy to get [showing path: HHHTT].

Alicia: It's harder to get there because they usually go like [showing path: RRLLR] and less like [showing path: RRRRR].

These arguments that students offered as to why values in the center were more likely seem consistent with the representativeness heuristic (Kahneman and Tversky, 1972). Paths like TTTHH were considered more likely to happen than those like HHHHH or TTTTT because the former had a near equal number of heads and tails, and thus are more representative of the equal probability of heads and tails from a fair coin. This reasoning persisted throughout the experiment in all but Josh's case. Josh went on to develop an analysis of the "number of ways" the events could occur and used this to explain the distribution shape after using the *NetLogo* simulation.

The *NetLogo* environment provided students with additional insights into the way the distribution of rabbits was shaped as they could run the simulation repeatedly using anywhere from 10 to 10000 rabbits. Students began to develop an understanding of the quantification of likelihood of outcomes through expected numbers based on the empirical distributions they got from the *NetLogo* simulations. They often gave predictions of the expected numbers of the various final locations. Before running 100 repetitions, for example, Josh predicted "Sixty for ones, around thirty for threes and around ten for fives" anticipating that the corresponding outcomes on each side would be symmetric (i.e., "even" or "close-to-even" on –1 and +1). Josh's initial expectations for 100 rabbits were proportional to his expectations for 10 rabbits. However, he often tended to refine his predictions after running a simulation to adjust his expected numbers according to data. For example, his first approximate expectations for 500 rabbits were "two-hundred for ones, [about] one-hundred-fifty for threes, and [about] fifty for fives." After running the simulation and examining the results in which he got 311 rabbits for ones, 162 for threes and 27 for fives, he responded that "I was more right with threes. I was really, really off with the ones. Actually I wasn't. And I was very incredibly amazingly off with fives." Accordingly, he changed his expected numbers to 290 for ones, 184 for threes, and 26 for fives. These expected numbers were indeed closer to the theoretical probabilities, i.e., P(1) = P(–1) = 0.31, P(3) = P(–3) = 0.16, and P(5) = P(–5) = 0.03 in comparison with his predictions indicating P("ones") = 0.58, P("threes") = 0.37, and P("fives") = 0.05.

While conducting several computer simulations, the students continued to express their ideas about why the central locations were more likely than the extreme ones in terms of the paths. For example, Alicia began with saying, "usually the bunnies are in the middle but I don't know" and then explained, "because it usually goes like [showing path: HHHTT]." Similarly, the other students often used different paths in their explanations throughout the task (i.e., Emily: It looks like more because there is less likely chance to go [showing path: HHHHH] or [showing path: TTTTT] than it is [showing path: HHHTT]). However, only Josh linked the possible different paths for each location to the number of ways the events could occur.

When explaining why the rabbits were more likely to end up on –1 and +1 based on the *NetLogo* simulation results, Josh stated "because there is only two ways to get to threes and there is plenty of ways to land on ones." To justify his reasoning, he began to list all the possible ways for each location (see Fig. 49.6) using the strategy

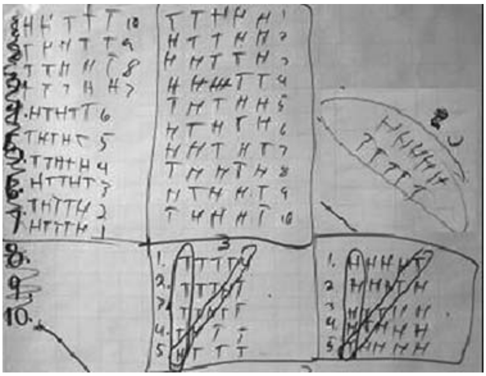

Fig. 49.6 Josh's list of all possible ways to get to each final location (i.e., 1 and –1, 3 and –3, 5 and –5) after five hops

he developed in an earlier task where the students arranged five cardboard bears in a row using five blue and five red bears in as many different ways as they could.

After listing all the possible paths for each location in Group 1, Alicia seemed to continue to reason with the "easier-to-get" results. For instance, she said that "it was a better chance to get one because it is easier to get one. Because you don't usually get like HHHHH, you usually get like LLRRR." When I asked her to explain what she meant by "easier," she directly referred to the coin flipping: "if you are flipping a coin, and say you landed on. You don't usually land on HHHHH or TTTTT. You usually land on like TTTHH." It was not clear in her language whether she referred to a specific path (i.e., TTTHH) or any combinations of three heads and two tails.

After generating the list of all possible ways to get to each distance (e.g., ± 1), Josh expressed the corresponding probabilities as part-to-whole ratios ($P(\pm 1) = 20/32$, $P(\pm 3) = 10/32$, $P(\pm 5) = 2/32$). If he separated the number of ways for the locations on the negative and positive sides of the number line, Josh could have worked out the quantification in a binomial distribution using the sample space and the combinatoric operations.

49.5 Conclusions

The aim of this paper was to investigate how fourth graders developed an understanding of the pattern of variability in a binomial distribution when modeling random rabbit hops through flipping a coin. I followed the students' reasoning at different stages of the task. First, the students speculated about where the rabbits were most likely to be after five hops spontaneously by considering the possible outcomes using the paths when the problem was given. They developed some intuitions about the possible outcomes of the event by drawing different paths for five random hops and determining the possible/impossible outcomes. It could be argued that trying to make sense of the random event in a spatial space, such as on a land or a piece of paper, might prompt students to consider the possible outcomes initially in this task.

Second, students were introduced to the idea of flipping a coin to determine the random rabbit hops. Some students who initially identified the possible/impossible outcomes used the equiprobability of the outcomes, i.e., "50-50 chance of getting heads or tails," to make predictions about the most likely locations of the rabbits on the number line from –5 to 5. These students also noticed the symmetry around 0 (the starting point) based on the sequences of tails and heads, such as HHHHT and TTTTH leading to the opposite paths.

Flipping a coin to determine the random rabbit-hops and using the inscriptions of paths helped students to prove why certain locations were impossible and why central locations were more likely than extremes. Consistent with previous research (e.g., Kahneman and Tversky, 1972; Konold et al., 1993), these students began to use intuitive theories when judging the likelihoods of sequences of five coin-tosses. When they identified "easy-to-get" and "hard-to-get" locations based on paths, such as HHHTT and HHHHH respectively, students tended to reason by representativeness because it seemed to support why the distribution was shaped the way it was.

Finally, the students explored the binomial distribution of rabbits after five hops using the *NetLogo* model. The *NetLogo* simulation environment enabled students to run simulations repeatedly for large number of rabbits and to examine the resultant distributions. The students, who expected about equal number of rabbits on each side of 0 based on the equally likely hops to right and left earlier, began to focus on the symmetry of the specific final locations. Fostering student reasoning about the expected number of rabbits for each final location supported the development of quantification of the chance of landing on those after five hops. Students were also encouraged to develop the idea of number of ways the events could occur as a way to justify why there was more chance that the rabbits would end up on the middle locations than on the extreme ones. After generating all possible ways to land on each location, some students (e.g., Caleb, Emily, and Josh) began to reason that the rabbits were more likely to land on –1 and +1 because they had more possible ways to happen. Alicia seemed to still believe some paths were more likely to happen than others based on representativeness. Consistent with Piaget and Inhelder (1975), the development of quantification of theoretical probability[2] required students to construct the combinatoric operations that involved generating all possible ways the events could occur and to establish the relationships of the individual cases with the whole distribution.

In conclusion, the use of "paths" as inscriptions of random rabbit-hops in this data-modeling task was an emergent approach not only in identifying the possible and impossible outcomes in a random binomial situation but also in reasoning about the number of ways the events could occur. Moreover, inscriptions of paths enabled Josh to build a connection between the pattern of variability in the empirical distributions and the quantification of probability distribution in terms of the number of ways. One concern raised by this study was that students could justify the most likely and the least likely locations based on the representativeness heuristic and it could be difficult for some students to change that belief even when the underlying sample space for five random hops was available to them.

Acknowledgment Writing this paper was enabled through the support by the grant ESI-0454754 from the National Science Foundation (*ModelChance Project*).

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[2]Theoretical probability refers to the ratio of "the number of ways the event could occur" to "the number of all possible ways in the sample space."

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Chapter 50
Investigating Mathematical Search Behavior Using Network Analysis

Thomas Hills

Abstract In math solution networks, search behavior represents a novel way to study mathematical thinking and learning. This paper demonstrates how to pose problems with interesting mathematical solution networks – and also how to address questions about problem solving processes used within these networks. The paper also introduces a tool based on equation edit distance, and shows how this can be used to structure and think about solutions in a mathematical space.

50.1 Introduction

Some of the central tenets of constructivist learning follow from the idea that learning is a process of moving outwards from a central knowledge core. In *Adaptation and Intelligence* (1980), Piaget described the process of learning as very much similar to a process of biological adaptation, where new conceptual models come from progressive differentiations of pre-existing models. The closer the pre-existing model is to the objective, the more likely it is that the objective will be reached. In this way, learning, like evolution, acts via the modulation of existing material. Moreover, "the smaller the gap between the new and familiar becomes" the more likely it is that "novelty, instead of constituting an annoyance avoided by the subject, becomes a problem and invites searching" (Piaget, 1954, pg 354). The basic pattern is one of a general local-to-global cognitive search process (Hills and Stroup, 2004; Hills, 2006) and variations and extensions of this theory of learning speak to ideas such as the zone of proximal development (Vygotsky, 1978), scaffolding (Wood et al., 1976; Pea, 2004), metacognition (Flavell, 1979; Schoenfeld, 1992), situated learning and the principles of transfer (Anderson et al., 1996; Bransford and Schwartz, 1999), and means-end analyses in general problem-solving (Newell and Simon, 1972). In the present work, the focus is on characterizing the "search"

T. Hills (✉)
University of Basel, Basel, Switzerland
e-mail: thomhills@gmail.com

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_50,

in a specific domain – mathematics – where the goal here is to introduce a methodology for characterizing trajectories of mathematical search in a known solution space. This includes both a novel task for exploring mathematical thinking (the math search task) and a cognitive hypothesis about the similarity between mathematical expressions (equation edit distance).

One way that learning trajectories have been characterized is to ask individuals at various stages in their understanding how they can explain a particular phenomenon (e.g., Smith et al., 1993). With this methodology, diSessa (1993) characterized what he called phenomenological primitives for physics understanding and points along the way towards deeper understandings. For most problems we can present, the complete space of possible solutions is not known. However, if we have a complete solution space, we can move further towards characterizing these waypoints on the path to understanding. This would be equivalent to knowing all possible trajectories for understanding a given process, and further, using this space, we could begin to explore how different learning environments, involving different social, methodological, and theoretical approaches facilitate different paths to understanding and how these understandings are then more or less robust and transferable in different circumstances. Furthermore, we can construct hypotheses about how solutions are related to one another in cognition and how cognitive processes borrow from the old to construct the new. Different hypotheses will generate different cognitive landscapes, with different solutions being nearer one another. The landscapes can then be compared against observed data to determine which cognitive hypotheses are better representations of the cognitive landscape associated with a particular problem environment.

To investigate the cognitive landscape associated with basic mathematics, I created a math search task, involving search for solutions in a high-dimensional mathematical space (inspired by conversations with Walter Stroup). For example, provided with the problem "*=6*" and a list of five numbers "*1 2 3 4 5*", using each number no more than once, how many solutions can you find by combining two or more of the five numbers in basic a mathematical expression. "*2 × 3*", "*5 + 1*", "*4/2 + 5 – 1*", and "*3 × 2*" are all valid and unique solutions. In this problem, there are a total of 678 solutions. By allowing subjects to leave one number set – never to return – for another number set, we turn the task into a *cognitive foraging problem*, where patches represent a specific number set and individuals can travel between patches much like honeybees travel between flowers.

Cognitive foraging problems are a broad class of problems involving finding multiple solutions to a given problem (i.e., resources within patches) and travel between problems (i.e., travel time between patches). Similar tasks are the SCRABBLE task (Hills et al., 2007) and the verbal fluency task (Lezak, 1995). Cognitive foraging problems offer a unique way to study cognition because they expose multiple levels of cognitive processing. Bottom-up processes can be observed in terms of how solutions are dependent on the display of the problem. For example, in the SCRABBLE task, children are more likely to be influenced by the ordering of the SCRABBLE letters than are adults (Schneiderman et al., 1978). Cognitive foraging problems offer similar affordances for studying working memory and top down processes.

Tackling mathematics problems can involve similar processes of cognitive navigation, where movement in the solution space is potentially a process of moving outward from the familiar to the novel. Using the mathematical search task presented below, we can follow how individuals and groups navigate a mathematical solution space and begin to investigate underlying processes in new and interesting ways. To help facilitate this analysis, I also present a method for calculating the similarity between mathematical equations, which I present here as a cognitive hypothesis about how problem solving uses components of previous mathematical expressions to build new expressions. The remainder of this article focuses on a particular application of this methodology and focuses on a very basic question: are there general patterns, based on equation size and similarity with prior equations, for trajectories in a mathematical solution space?

50.2 Methods

Participants: 77 English speaking undergraduate students at Indiana University participated in the experiment. All participants were recruited on a volunteer basis with no financial reward for participation.

The Math Search Task: Participants were seated in front of a computer and asked to follow written instructions that appeared on the screen. Instructions guided participants through a series of tasks. The task that is relevant for the present work is the math search task. In the math search task participants saw the screen shown in Fig. 50.1 and were asked to submit valid equations that would provide correct solutions. Order of precedence rules applied and divisions were calculated before multiplications. Instructions were as follows.

> In this task, you will be presented with a number and you are to find as many mathematical solutions as you can. For example, you will see =8 and you could enter 4 × 2 or 9 – 1 or 4 × 2 × 1. Order of operations is important – division and multiplication take place before addition and subtraction. Also, the ordering of numbers is important. You can enter 4 × 2 and 2 × 4 and get two solutions for =8. Each number can only be used once in each solution and numbers you have already used for this solution will turn to grey. To enter a solution, move the mouse over the number and click on it. Then move the mouse over the

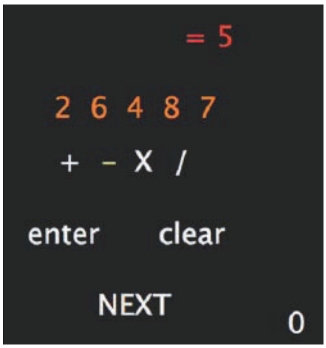

Fig. 50.1 A computer screen where participants entered solutions to math problems

> operation and click on it. When you are ready to enter the solution, click on "enter". To clear the solution, click on "clear". You should try to find as many solutions as you can for each set of numbers. When you can't find anymore, you can press "NEXT" to go onto the next number set. When you press "NEXT" you will have to wait for a short period (90 s) before the next number set is shown. To finish the session, you will need to find 60 solutions.

Participants were trained by asking them to submit 20 solutions using as many number sets as they needed. Then, later, participants were asked to submit 60 solutions, and the data I will look at here is from these 60 solutions.

Equation Edit Distance: Because the number of unique solutions can be computed for a given problem, if we have a hypothesis for how individuals move between one solution and the next, then we can visualize the solution space and speak quantitatively about how math learners move around in a math solution space. The method I develop here is analogous to string edit distance (Sankoff and Kruskal, 1983), in which the distance between two strings is defined as the minimum number of insertions, deletions, or substitutions that are needed to move from one sequence to another. For example, with string edit distance, the distance between BOAT and BLOT is two, one insertion (the L) and one deletion (the A), or two substitutions (replace the O with L and the A with O).

Equation edit distance calculates the distance between two equations as the minimum number of insertions, deletions, or substitutions of nested mathematical units that are required to move from one equation to another. *Mathematical units* are the minimal divisions within a mathematical expression that, when permuted, still generates the same solution. For example, in the equation $4 + 5 - 2$ there are 3 mathematical units: $+4$, $+5$, -2. Any permutation of these units still has the same solution $= 7$. In the following equation, $4 + 3 \times 1$, there are two additive units, $+4$ and $+ 3 \times 1$, but there are also two multiplicative units within the second additive unit, $+3$ and $+1$. Permutations within a multiplicative unit or among the additive units all generate the same solution (e.g., $1 \times 3 + 4 = 7$).

Finding distances between two equations involves finding the number of shared mathematical units. This is done by searching for shared units at each operational level, beginning with the operation level of highest precedence. Set one equation as the match and the other as the target. After a mathematical unit with the highest precedence operation is identified in the match equation, it is compared with all units in the target equation. If it is present in both, a shared unit is tallied, and both are simplified. If it is not present, then no shared unit is tallied, and the unit is simplified in the match equation. Then the next unit of highest precedence is found in the match equation and searched for in the other equation. After all units at a given operational level are simplified in the match, then units at the same operational level are simplified in the target.

The highest possible distance between two equations is the number of mathematical units in the longest equation. To calculate the similarity, I take the total number of shared units and divide by the total number of units in the longest equation. Any permutation of another equation therefore has a similarity of 1 (e.g., $1 \times 3 + 4$ and $4 + 3 \times 1$), whereas two completely novel expressions have a similarity of 0 (e.g., $1 \times 3 + 4$ and $5 + 1$). As with string edit distance, there are a number of ways to tune the subtleties of distance by allocating different costs to different kinds of alterations between match and target. In the analyses I perform here, I let 3×1 and 1×3 have a similarity of 1, but it is perfectly reasonable to assume there is some similarity cost associated with the transposition. It is not the focus of the work described here to determine these more subtle cognitive differences, but they are an interesting focus for future work.

The Network: The total number of solutions for each problem were calculated by creating a program that ran through all possible equations that could be generated from the number set and evaluating these for their equivalency to the problem. For the problem $2\ 1\ 6\ 3\ 9 = 3$, there are 490 solutions. Between each pair of solutions, the equation edit distance generates a similarity measurement. For the above problem with 490 solutions, this creates a 490×490 matrix with the similarity between equations i and j at position i,j in the matrix. Letting each equation be a node in a network, with edge weights between nodes i,j set to the similarity between equations i and j, we can construct a visual description of the solution space (Fig. 50.2).

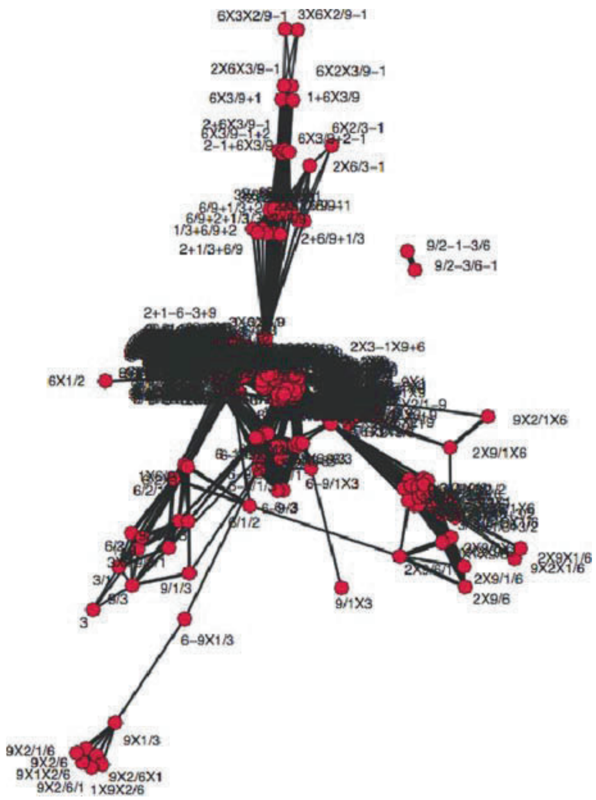

Fig. 50.2 The solution similarity network computed using equation edit distance on the problem, $9\ 6\ 1\ 3\ 2 = 3$. Only edges with a similarity value greater than 0.35 are shown. Isolated nodes are not shown. Thicker edges represent greater similarity. Solutions with high numbers of similar permutations are clustered in the center

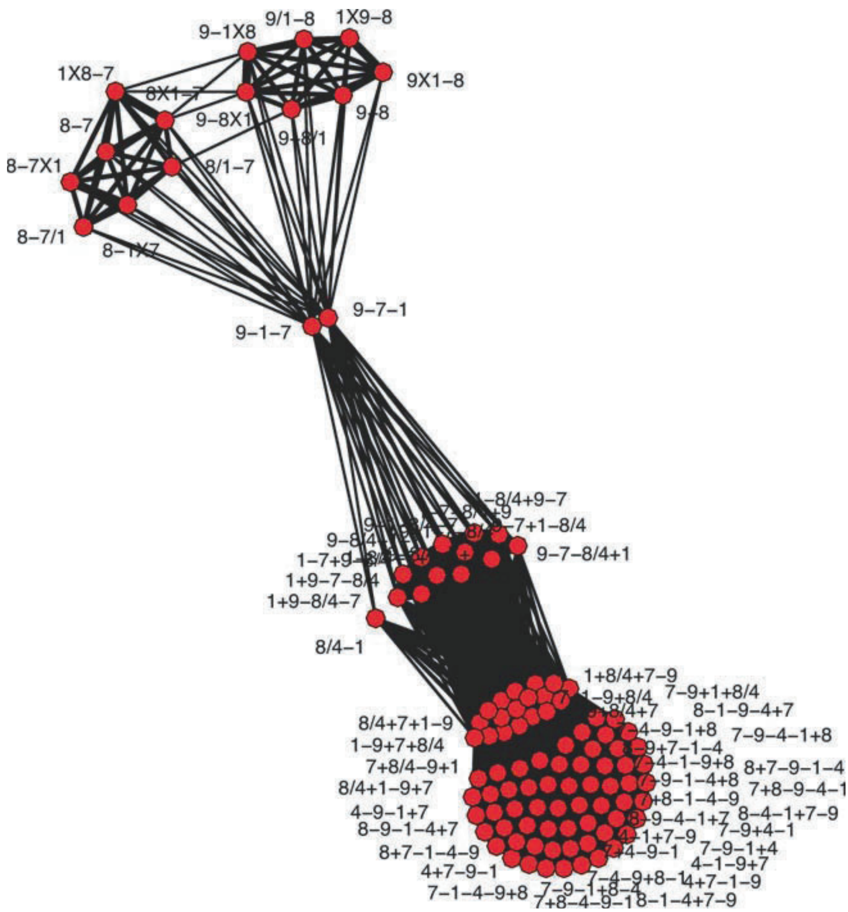

Fig. 50.3 The solution similarity network computed using equation edit distance on the problem, *8 7 9 1 4 = 1*. Only edges with a similarity value greater than 0.3 are shown. Thicker edges represent greater similarity. Isolated nodes are not shown. The ball of solutions in the *lower right* corner is largely made up of permutations of the mathematical units in the expression *7 − 1 + 8 − 9 − 4*

Network structures are quite variable for different problems and can produce a wide variety of configurations based on the similarity properties of the underlying space. Figure 50.2 represents a structure with a fairly complicated layout with more sparsely connected equations on the periphery and more densely connected nodes towards the center. Figure 50.3 presents the solution similarity network for a smaller solution space, for the problem *8 7 9 1 4 = 1*, with approximately 110 solutions.

50.3 Results

Many different kinds of questions can be explored using the math search task and the solution similarity network. In the present paper I want to focus on a very specific set of analyses to demonstrate how the math search task and solution networks might be used to address a specific problem. The problem is the following: is there a general pattern, based on size and similarity, for trajectories in a mathematical solution space? A simple null hypothesis is that solutions generated for a specific problem are randomly chosen from the space of all possible solutions. Equation edit distance allows us to develop a cognitively plausible alternative hypothesis, in which we propose that solutions are generated based on their similarity to prior solutions. This similarity hypothesis is generally consistent with ideas of constructivism, and more specifically with Piaget's statements about "the gap between the new and the familiar" made in the introduction.

An alternative hypothesis to similarity, one based simply on the concept of cognitive load, is that participants in the math search task will generate short solutions first and longer solutions later. Cognitive load refers to the natural constraints on working memory during problem solving and has been a useful tool for thinking about instructional design (Miller, 1956; Sweller, 1994). The basic consequence is that it is easier to generate a solution like *9 – 8* than it is to generate a solution like *7 − 1 + 8 − 9 − 4*. But we can add a further element of rigor to this question by asking if participants in the task are providing the shortest possible solutions. In other words, if there are 20 solutions using one operation (e.g., 3×2 and *7 – 1*), do subjects submit all 20 solutions with one operation before moving on to solutions with two operations (e.g., $3 \times 2 \times 1$)?

Figure 50.4 shows how the number of operations in a solution is related to the entry number (the order in which a solution was submitted) for the problem *9 6 1 3 2 = 3*, with approximately 490 solutions. There is one entry with zero operations, *3*, and approximately 10 entries with 1 operation. Several things are evident from the figure. First, solution size *does* start out small and grow in size. Second, by around entry 20, participants have reached a plateau of approximately three operations. Finally, subjects appear to everywhere be submitting entries longer than the shortest possible entry length. This suggests that our initial interpretation of cognitive load theory as a simple rule for generating the shortest possible solution is incorrect. If we simply assume that participants compute the easiest set of solutions (based on

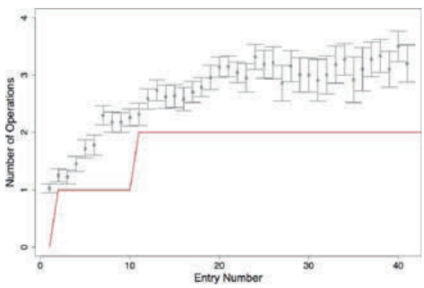

Fig. 50.4 The number of operations in a solution is plotted versus the entry number (the order of solution entry) for the problem *9 6 1 3 2 = 3*. The observed number of operations are shown with unfilled circles. Error bars are standard error of the mean. The solid line shows the entry size that would have been submitted if participants always chose the shortest possible equation size

length) for a given problem, we make incorrect inferences about the solutions they are likely to submit.

If similarity, as defined by equation edit distance, is playing a role in the types of solutions that are being generated, then it should take longer to generate novel solutions than it does to generate solutions of similarity one. Figure 50.5 shows the difference in the amount of time it takes to submit answers that are strict permutations of the previous answer (equation edit distance similarity of one) versus answers that are not strict permutations, for a variety of solution sizes. The results suggest that strict permutations take less time to generate than novel solutions. Moreover, it takes no more time to generate a permutation of the last equation if it had three operations than it does to find a novel solution with one operation.

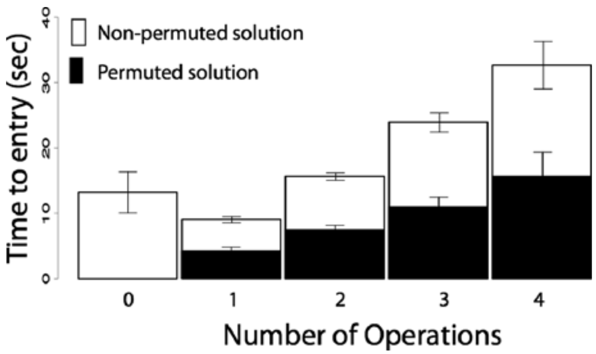

Fig. 50.5 The amount of time taken to submit an equation using a particular number of operations separated by whether or not the equation submitted is a strict permutation of the last equation (similarity equal to one) or not. Error bars are standard error of the man. This figure uses all equation entries taken over all problem spaces

Because longer equations also share high similarity with more equations than shorter equations (compare the clusters of high density with those of low density in Figs. 50.2 and 50.3), it may be that participants are using equation similarity to leapfrog through the solution space towards longer equations. While shorter equations act as points of entry into the solution space, longer equations are the attractors in this space. When subjects find these equation clusters they tend to stay in them. If this were the case, then longer equations entries would, on average, be followed by equation entries of higher similarity than shorter equation entries. But that result alone is not sufficient to determine that subjects are actually using this similarity to guide there working memory processes. A random submission following a longer equation will have a higher similarity to the prior equation on average than a random submission following a shorter equation. Therefore, we must extend this question by asking if equations that follow longer entries are of higher similarity to the prior entry than would be expected if random submissions followed every entry. In other words, are submissions more similar to longer equations than would be expected by chance? The answer is that subsequent equation entries are more similar to longer equations than would be expected by chance (Fig. 50.6). This turns out to be true

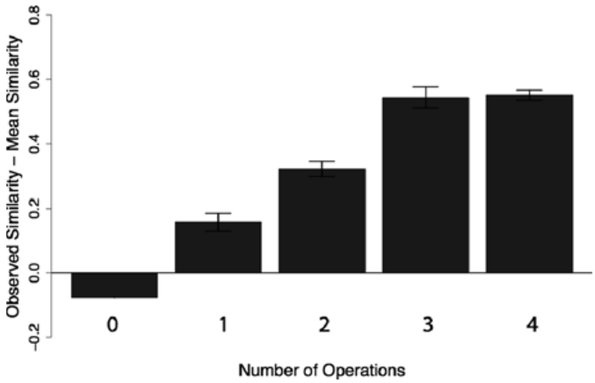

Fig. 50.6 The difference between the observed similarity between subsequent equation entries and the mean similarity of all equations to the prior equation, separated by the number of operators in the prior equation. This is for the problem $9\ 6\ 1\ 3\ 2 = 3$. Error bars are standard error the mean. The results show that longer equations are followed by more similar equations than would be expected by chance

for all equation sizes except sizes of length one (with zero operators). Zero operator equations tend to be followed by equations that share no similarity with that expression – they don't include the number that was in the prior equation. However, longer equations are all more similar than expected and as they get longer, they become increasingly similar to the previous equation. This result supports the hypothesis that participants are using similarity to leapfrog through the space and that they are doing so with increasing frequency as they move to longer equations.

50.4 Discussion

Search in math solution networks represents a novel way to study mathematical thinking and learning. Here, I have demonstrated how to pose problems with mathematical solution networks and also how one might address questions with these networks in a specific context. I also introduce the tool of equation edit distance and show how this can be used to structure and think about solutions in a mathematical space. The basic problem I posed here was what contributions do size and similarity play in the trajectories that our undergraduate students take in these mathematical solution spaces. The preliminary results that I provide suggest that size does play a role, but less so than one would think – students are everywhere submitting longer equations than they could be submitting if they were really looking for the simplest expressions. On the other hand, similarity to the last expression, based on equation edit distance, offers predictive power with respect to both time of entry and what the next expression will look like. With equations using two or more numbers, subsequent equations tend to borrow mathematical units from prior equations and they do more so than we would expect based on random solution generation. While I only show results for a few specific networks, the quantitative results generalize to other networks in the task. Overall, the results are consistent with ideas presented in prior work (Hills and Stroup, 2004; Hills, 2006) and follows along nicely with the inferences one may take from a constructivist view of working memory and problem solving. But again, the main goal here is to offer a new way of thinking about and studying mathematical processing, based on solution trajectories in math networks.

I believe there are a wide variety of uses for this tool in educational contexts, which range from evaluation purposes to research on the kinds of environments that influence trajectories in a mathematical space. We may want to know how a particular problem influences the trajectories that students are likely to take, and how the duration of time they spend in a particular network can influence the kinds of solutions they are likely to come up with. By analyzing the kinds of operations that students submit, we can gain an understanding of their familiarity with the utility of particular operators. For example, Fig. 50.7 shows how the operations that students submit changed over the course of entries for the $9\ 6\ 1\ 3\ 2 = 3$ problem. Information presented in this way allows us to see that subtractions are more common than additions (solid thin lines), even though addition and subtraction operations are equally frequent over all possible equations (dotted thin lines).

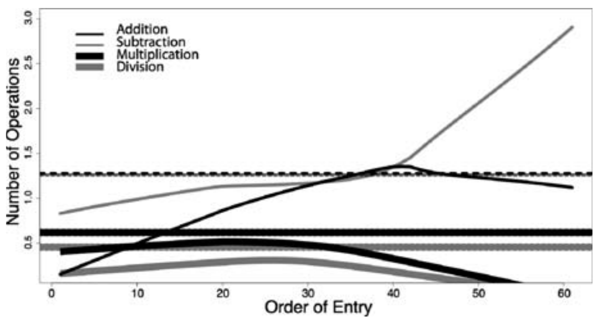

Fig. 50.7 The number of operations by order of entry separated by type of operation. Thick lines represent the average number of operations submitted at a particular point in the order of entry. Dotted straight lines represent the average number of occurrences of an operation over all possible equation entries

The math search task can also be posed easily to groups of networked students simultaneously, which can offer new avenues of study for the effect of social interaction in mathematical thinking and learning. This work can then be easily juxtaposed with individual solution trajectories. Similarly, one can use the math network task and equation edit distance to study how novices differ from experts in mathematical problem solving. Further applications may include different kinds of formative assessment or methods for mapping out how groups of students approach and meet curricular goals.

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Chapter 51
Mathematical Modeling and Virtual Environments

Stephen R. Campbell

Abstract With the advent of new tools for constructing increasingly popular web-based virtual environments[1] such as Second Life <http://www.secondlife.com>, educators face fundamental new questions. What kinds of teaching and learning experiences are possible in web-based mult-user virtual environments (VEs) that are not possible in physical environments (PEs)? How do teaching and learning experiences in VEs relate to teaching and learning experiences in PEs? With the special regard to mathematics education herein, what kinds of mathematics and mathematical applications can be modeled, simulated, and rendered more practically and intuitively in VEs than in PEs? What theories and methodologies might best be brought to bear in researching the teaching and learning of mathematical modeling and applications in virtual environments? This chapter offers background, overviews, inroads into formulating these and other related questions.

51.1 Introduction

The most essential component of mathematics is structure. In this view, mathematics can be seen as the structural components that are latent in all actual objects, relations and processes, and ideally, in all *possible* objects, relations, and processes. These considerations point to the importance of intuiting such structure. In using mathematics to construct virtual worlds, they also point to the importance of mathematical

S.R. Campbell (✉)
Simon Fraser University , British Columbia, Canada
e-mail: sencael@sfu.ca

[1]Virtual *environment* will be used here in lieu of Virtual *reality*, as the latter "... typically focuses on immersive 3D experiences (e.g., using head-mounted displays) and 3D input devices (e.g., data gloves)... [virtual environments, however] neither require nor preclude immersion." (Chittaro and Ranon, 2007). Both will be included here in reference to virtual *worlds*. Moreover, *Multi-User Virtual Environment* is being used in lieu of *Massively Multiplayer Online Game* (MMOG), as MUVEs such as Second Life, while they do not preclude gaming, enable much more than games – they are large-scale interactive social virtual environments with uses ranging from entertainment and collaboration to education and commerce.

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_51,

modeling and applications in these domains. Recently, virtual environments have emerged, most notably "Second Life" <http://www.secondlife.com>. Second Life (SL) is a massively multi-user on-line immersive social interaction virtual environment where users custom-design their own "avatars" or virtual bodies. Avatars can walk, run, dance, fly, and teleport from location to location. Most importantly, users can socially and collaboratively interact in real time through their avatars with the avatars of others. Although many virtual environments have educational potential, SL is of primary interest here.

My main task for this chapter is to consider various initiatives, possibilities, and implications of Multi-User Virtual Environments (MUVEs) for mathematics education, with special emphasis on mathematical modeling and applications. I begin with brief background treatments of the computational technologies and commercial developments underlying the construction and implementation of MUVEs. I then present some educational initiatives using these technologies and developments for constructing and implementing MUVEs. My thesis here is that MUVEs provide affordances that are uniquely and particularly well-suited for teaching and learning mathematical modeling and applications. I then discuss new opportunities and methodologies for educational research in these areas. Philosophical issues abound, some of which I consider in my concluding remarks.

51.2 Background

Multi-User Virtual Environments (MUVEs), and virtual worlds more generally, are applications of mathematical modeling *extraordinaire*. Mathematics is involved in the technologies that render these worlds, and in the mathematics used for modeling these worlds in the first place. There are no physical phenomena, biological processes, or social, political, geographical, or astronomical contextualizations that, in principle, fall beyond the scope and purview of implementation in virtual worlds. From the microscopic to the macroscopic, one may argue whether the physical world is in essence a mathematical world, but there can be little doubt that virtual worlds are quintessentially Pythagorean Worlds. When it comes to the construction and rendering of virtual environments, mathematically speaking, all things really do accord in number.

MUVEs are typically designed and developed using 3D web-based graphic languages such as *Virtual Reality Modeling Language* (VRML), or *eXtensible 3D* (X3D) graphics language, both of which are open source,[2] or proprietary languages such as *Java3D* (a 3D extension of Java) and *Shockwave 3D* (Chittaro and Ranon, 2007). There is also a variety of file interchange formats that enable transfer of 3D graphic content image data and associated physical attributes and propensities between proprietary software applications ranging from computer

[2]X3D is a new International Standards Commission (ISO) standard proposed as a successor to VRML. Development of specifications for X3D and VRML fall under the purview of the Web3D Consortium <http://www.web3d.org>.

aided design (e.g., *STL*) and geographical information systems (*GDAL*) to gaming (e.g, *COLLADA*). It will be both helpful and appropriate then, to first spend a brief amount of time on the kinds of mathematics that are implemented in these languages and file formats, and to understand ways in which they can be used in tandem with computers and the internet to construct virtual environments that, in turn, may be used for teaching and learning mathematical modeling and applications. The roots of virtual environments can be traced to the emergence of raster and vector graphics:

- *Raster Graphics*. A raster data set is a digital representation that typically renders everyday images, but also any actual or possible multidimensional data set. Raster data sets are characterized by three main parameters: the *size* of the data set (e.g., discrete dimensions $h \times w$ for images and $h \times w \times d$ for volumes); the data *resolution*, which defines the density of discrete elemental regions (referred to as picture elements or *pixels* for 2D images, and volume elements or *voxels* for 3D volumes); and the *dynamic range* of the content data, which defines the amount of information that can be represented per region (defined as the range of values that can be represented with a given number of digital bits). Raster graphics provide image representations constrained by size, resolution, and dynamic range, and as such, they do not scale well (e.g., resulting in familiar blocky pixilation effects when such images are expanded), and storage requirements can increase exponentially with increased resolution[3].
- *Vector Graphics*. Vector graphics help overcome the scaling and storage limitations of raster graphics. Whereas the mathematical primitives in raster images are numerical values falling within a given dynamic range, vector graphics are typified by the use of mathematical primitives such as lines, polygons, ellipses, and more advanced Bézier curves and surfaces for modeling and applications (e.g., for problems in mechanics using finite element analysis). Parameters defining these higher order primitives are "vectorized" and thus can be flexibly transformed to construct even higher order static and dynamic objects and eventually, entire virtual worlds. Vector graphics output streams are typically "rasterized" for digital display purposes.

51.3 Major Application Developments

Raster and vector graphics constitute major applications of mathematical modeling at a very basic level. They can be viewed in part as the mathematization of art, or as enabling new mathematical art forms. Precursors of raster graphic representations can be found in pixilated art from ancient Australian aboriginal times to the works of Monet. Similarly, precursors of vector graphics are evident from prehistoric times in the cave paintings of Lascaux to the more modern works of Escher. Just as art represents and expresses, so too, through the medium of computers and the use of

[3]Data interpolation and compression techniques can mitigate these effects.

computer graphics, is mathematics used to represent and express, and now in some ways previously inconceivable. Consider, for instance:

- *Computer-Aided Drafting/Design and Manufacturing (CAD/CAM)*. One of the earliest, most established and pervasive use of computers for commercial and military applications involve computer-aided drafting (design more generally) and manufacturing. *AutoCAD* is a well-known and widely used software package in this regard. A standard file interchange format for 3D CAD/CAM objects is STL, which represents 3D objects as a set of oriented triangles, using a vector normal and the coordinates for each of the three vertices for each triangle, all in the positive octant of a 3D Cartesian co-ordinate system. The CAD/CAM industry makes extensive use of vector graphics, and has been a major motivator and contributor to the evolution of vector graphics technology.
- *Cartography and Geographical Information Systems (GIS)*. Cartography, or mapmaking, enables the visualization of spatial data sets, and it is another area revolutionized through the advent of computers. Computer mapping, using global positioning systems (GPS), has come to fruition through geographical information systems (GIS), which enable limitless varieties of spatial data to be dynamically referenced to models of the Earth itself. ESRI's *ArcGIS* is a well-known and widely used software package in this regard. Such data sets, be they raster or vector, can be transferred from one software application to another through open source libraries (viz. GDAL/OGR). Like CAD/CAM, GIS provides extraordinary new avenues for mathematical modeling and applications, as well as being another major motivator and contributor to the evolution of raster and vector graphics technology.
- *Simulations and Gaming*. Simulations work hand in hand with modeling (Campbell, 2004), by simulating various aspects of reality subject to assumptions and limitations that have been encapsulated within a model. Computer simulations have become commonplace, ranging from simulating climate change to the trajectories of spacecrafts relative to planetary motion. Computer simulations, drawing on raster and vector advances in CAD/CAM and GIS, have also been extensively adopted and advanced by the gaming industry, a growing and vibrant industry that has captured a significant and dramatically increasing share of the entertainment market.
- *Virtual Worlds*. The gaming industry is at the leading edge in developing virtual worlds, and in driving technological advances in support of these worlds. *World of Warcraft*, for instance, is a massively multiplayer online role-playing game (MMORG) with close to 10 million players worldwide. Advances in sound, graphics, and physics cards, and increases in processor speeds are largely driven by the gaming industry, which has also developed what have become open programming (X3D/VRML)[4] and file interchange standards for interactive 3D applications (COLLADA).

[4]For an overview of VRML and X3D, see Chittaro and Ranon (2007).

Virtual environments such as Second Life are based on these technologies, with gaming subsumed to more general social and educational aspects of interaction and collaboration. What, then, of the educational implications?

51.4 Educational Considerations

A number of virtual worlds have been developed for educational purposes (see Dieterlie and Clarke, in press; Chittaro and Ranon, 2007), both with and without added virtual reality hardware (i.e., vehicle simulators, Cave Automatic Virtual Environments or CAVEs, head mounted displays, data gloves, haptic feedback units, and so on). My focus here is restricted mainly to multi-user web-accessible virtual environments that do not require additional hardware[5]. Even with this more restricted focus, no attempt can be made here for a comprehensive review of implementations. Rather, some contexts and examples are presented to offer a few insights into the possibilities and limitations of educational applications:

- *Contexts*. Virtual environments apply to a wide variety of educational contexts. These contexts range from traditional formal classroom schooling, to distance education, e-learning, to vocational and medical training, to informal contexts such as museums and science worlds, to athletics and special needs education. A wondrous oddity of web-based virtual environments is that they can apply to almost any educational context while being almost always accessible from almost any location in the world. The educational possibilities for virtually any subject or activity can be supported to one extent or another. As technology improves and becomes more accessible, these possibilities will only expand.
- *Possibilities*. With traditional schooling, individuals are enrolled in a specific program with a specific teacher and syllabus and subsequently their meetings are confined to a specific place with a specific group of learners at specific times. Imagine places where learners can visit, explore, and constructively engage with learning materials at will, and to the extent their time, interests, and abilities will allow. Imagine virtual learning spaces where kindred spirits with common interests could visit whenever they liked, from wherever they happen to be. The wondrous oddity of virtual environments could have profound affects on our traditional notions of mathematical enculturation (cf., Bishop, 1988). Imagine a special place, a collaborative learning space within a massive web-based MUVE like Second Life that is populated in various innovative analytic, visual, and other intuitive ways with mathematical structures, concepts, procedures, tools, and other resources. The possibilities seem endless. I

[5]It is important to note that resolutions for virtual environments such as Second Life can be greatly enhanced in resolution with dedicated graphics boards and physics accelerators, and that latency times are dramatically reduced with higher internet access speeds. Increased resolution and decreased latency make for more seamless and enjoyable virtual experiences.

discuss a few possibilities for mathematical modeling and applications further below.

- *Initiatives*. In mathematics education, Kaufmann (2006) has been developing Virtual and Augmented Reality (VR/AR) technologies for teaching and learning geometry for a number of years, and with encouraging results. Elliot (2005, AquaMOOSE 3D: A constructionist approach to math learning motivated by artistic expression, unpublished doctoral dissertation) has developed a standalone virtual environment called aquaMOOSE that enables learners to engage, construct, explore, and investigate parametric equations. As important and interesting as these educational initiatives are, the exposure and access they have to learners, and the collaborative opportunities for learning, is limited in comparison with the over 9 million subscriptions to Second Life.
- *Limitations*. MUVEs are not without their own limitations. They are, after all, virtual worlds, and unlike our natural immersion in real physical worlds, experiencing them does require computer and communications technology. Although MUVEs simulate a 3D immersive experience, one is vicariously immersed in this experience through one's avatar via a computer monitor.

51.5 Second Life

Second Life (SL) and MUVEs like it are considered, in the words of IBM's CEO, "the next phase of the internet's evolution (Kirkpatrick, 2007). Creations in SL range from the simplistic to the sublime and richly complex. SL is a burgeoning virtual environment that has major campuses that have been constructed by corporations such as IBM, Dell, and many others. Sweden recently constructed "embassy" in SL, and other countries have been following suit. The architectural structures in SL are impressive, as one visit to IBM's rapidly developing campus will attest. Another popular destination is Svarga, a wonderland with Lord-of-the-Rings architectural overtones, a virtual island replete with mathematically simulated streams and waterfalls. Particularly interesting in Svarga is a simulated ecosystem. At any place and at anytime, pointing at any object and right clicking on one's mouse illustrates how the mathematical structure underlying objects created in SL can be made visible for inspection and in some cases for alteration as one sees fit.

A visit to the National Oceanographic and Atmospheric Agency (NOAA) will lead one to real time weather simulation over the continental United States, or to simulations of a tsunami and global warming. Grab a rope hanging off of a weather balloon to get a closer look at a typhoon simulation. The NOAA and other sites in SL exemplify the possibilities for actually experiencing mathematical models and simulations. Let your avatar take a ride caught up in a tornado in a Martian landscape in the National Aeronautics and Space Administration/Jet Propulsion Laboratories (NASA/JPL) SL site. These examples are incipient and in early phases of development. The pedagogical possibilities for immersive experiences of mathematical models and simulations in virtual environments are seemingly endless. The

International Society for Technology in Education (ISTE) has taken notice and is amongst the leaders in exploring the educational possibilities of SL.

51.6 Questions of Interest to Mathematics Education

With a sense of MUVEs, the technologies on which they are based, and with some sense of their potential for radical and innovative changes in social interaction, it is time to pose some questions regarding their potential for teaching and learning mathematics in general, and mathematical modeling and applications in particular:

- *Question 1*. What kinds of mathematical structures can be most readily visualized in MUVEs like SL? Geometrical structure certainly constitutes the most obvious answer to this question, but what of arithmetic and algebraic functions and relations, and more advanced topics such as topology and dynamic systems theory? How might the calculus and higher math be rendered more intuitive in virtual environments?
- *Question 2*. Relating to Question 1, what kinds of physical phenomena can be most readily modeled with mathematics in virtual environments? Consider the possibility of allowing avatars to flow within the dynamic structures of vector fields, such as a simulated Martian tornado, or even flowing along in a stream, be it embodied as a fish swimming in a river, or as an electron in an electromagnetic field. Virtual environments offer unprecedented means for such experiences, and thus for teaching mathematical modeling and applications.
- *Question 3*. To what extent can learners, through their avatars, become engaged in constructing novel and everyday kinds of objects using a provided palette of mathematical tools, and to what extent will this help learners to become more conscious and appreciative of the mathematical structures latent in actual real world objects, events, and relations?
- *Question 4*. How important will it be to teach and learn mathematical methods involved in implementing and embodying content, including mathematical modeling and applications, on computers and in machinery and artifacts produced by machines in general? Raster and vector graphics, Bézier functions, and finite element methods discussed above serve well as cases in point in this regard, not to mention the "coolness" and "relevance" motivational factors.
- *Question 5*. What are the limitations and/or enhancements of social learning interactions in virtual environments between learners and automated characters such as "math girl" (Jungic, 2007) for things like mathematical problem-solving and knowledge transfer?
- *Question 6*. To what extent can learning mathematics in virtual environments alleviate anxieties that learners might otherwise experience in actual classroom environments? Social interaction in SL is generally known to be less inhibitory.
- *Question 7*. How practical might it be to buy or rent virtual property to construct a virtual mathematics environment, such as a classroom, museum, or even an entire institute dedicated to the learning of mathematics, that learners can access

on-line from wherever they have internet access with sufficient bandwidth and computer display capabilities?

- *Question 8*. What novel forms of research in mathematics education might be possible from observing learner interactions with MUVEs or from within such environments, and what are the ethical implications for conducting such research? Novel approaches are possible, and work is already underway (e.g., Campbell and ENL Group, 2007); Chittaro and Ranon, 2006).

It is not possible to attempt answers to all these questions here. Nor should this list be considered to be comprehensive or complete by any means. Having said that, some cursory considerations in the sections that follow shall be offered for Questions 2 and 8 respectively.

51.7 Mathematical Modeling and Applications Within MUVES

Computers and the internet are being widely used for mathematical modeling and applications, and there are many senses in which mathematical modeling and applications can already be considered to be virtual in nature. Consider the National Oceanic and Atmospheric Administration's (NOAA's) real time weather simulation in SL. Consider remote sensing in general <http://www.artpo.ssc.nasa.gov/m2m>, which can range from the measurement and modeling of various aspects of Earth dynamics from crust motion and rotation changes to surface elevation and wind velocity. Consider further the detailed planetary model accessible through Google Earth. Clearly, remote sensing can be integrated with Earth models within SL, and it is only a matter of time that virtual models such as these take on a "reality" extending beyond what is possible through direct experience of the physical world alone.

A sea change in mathematics education may very well be at hand, one that shifts priorities away from understanding *how* to do the math toward a better understanding of *what* the math is doing. A significant challenge for mathematics educators will be to become knowledgeable of various assumptions and limitations of mathematical models such as these (Campbell, 2001). Moreover, understanding mathematical aspects of modeling and applications such as sampling densities and frequencies, spatial and temporal scaling factors, measurement estimations and associated errors, coordinate systems and transformations, and so on, should increase in importance accordingly – at least to the extent that mathematics educators are concerned with educating citizens to become more literate, adept, and critically-minded regarding mathematical modeling and applications.

51.8 Educational Research with and Within MUVEs

Yellowlees and Cook (2006) have developed a virtual schizophrenia clinic (*qua* research laboratory) at UC Davis in SL. Their objective is to help others experience the world of the schizophrenic on a first hand basis in a non-threatening manner. Their research experience is instructive in a number of ways. First, they implemented this virtual clinic in SL for a tenth of the cost of a previous project that required specialized hardware and software. Consequently, the SL version had much greater "educational reach," viz., accessible to anyone with a computer with sufficient memory and an internet connection with sufficient bandwidth. Secondly, they drew their participant base from within SL itself, and administered their clinic in such a way as to monitor and collect psychometric data from those individuals that was automatically emailed for downloading into a spreadsheet for analysis. Yellowlees and Cook (ibid) note that "this type of assessment process could be further developed to allow online students to answer examination questions within [this] three dimensional environment" (p. 537). Thirdly, they note, citing Caruso, that "[i]ntegration of virtual reality, simulation and modeling into educational curricula has been identified as a goal by a recent large survey of information technology use by students" (p. 534). Subscribers to SL are known to be comprised of a predominantly younger demographic. It seems reasonable to infer that younger students are quite comfortable with learning in virtual worlds.

As learners interact with virtual environments via computer monitors, and eventually, one may readily imagine, through physiological means, such as eye-movement, voice, and various bio-potential fields such as electroencephalography, observing, measuring, recording, analyzing, and interpreting these interactions will become of increasing interest and importance to educational researchers. Logging and recording learners' journeys, encounters, interactions, and in short, their virtual experiences will enable research into those learning experiences in unprecedented ways. Some questions that come to mind concern how it is that learners will orient and navigate through virtual environments? How will their sense of identity evolve? What implications for collaborative learning and instructional design? Virtual environments suggest that as mathematics educators, we should re-conceive our means of and priorities in teaching and learning mathematics, especially mathematical modeling and applications, and no less, as educational researchers, our ways of conducting research into these phenomena as well.

51.9 Some Philosophical Reflections

Why should (mathematics) educators be concerned with virtual environments? After all, they are not real. Such an admittedly strawman view is but another manifestation of Cartesian dualism, which those familiar with my previous work are aware I whole-heartedly reject (e.g., Campbell, 2001, 2003; Campbell and Dawson, 1995). Having lived experiences through virtual environments is somewhat akin to lucid dream states (cf., Laberge, 1985). The key existential factor relating physical to virtual presence is exactly that: *presence*; a sense of *being*, coupled with a sense of *becoming* in the here and now. There is no essential difference in the sense of the "now." What differs is the nature of the "here." There are material constraints placed upon our presence in the physical world that do not necessarily obtain in SL. In virtual worlds, the reality of experience remains while the reality that is experienced, and the kinds of possibilities that can be realized, may differ.

Environments such as SL are characterized as *virtual* worlds in order to distinguish them from the world(s) in which we are embodied and experience actual *physical* presence. It does *not* follow however, that virtual worlds are not real or that they are not embodied. Indeed, in an important sense virtual environments in fact are quite real and embodied, but in different ways. First, those who are engaged in virtual worlds, such as SL for instance, are having *experiences* – real lived experiences. Secondly, they are embodied, albeit virtually, through their avatars. Again, how will such differences enable and affect ways in which we learn and experience ourselves, interact and collaborate with others, and conduct research? Clearly, virtual worlds give due pause for reconsidering fundamental ontological, epistemological, and ethical premises about what is real, what can be known, and how we should act. For my part here, I will simply offer a succinct characterization of virtual worlds as mathematical actualizations of collective imagination.

51.10 Concluding Remarks

With the advent of new tools for constructing increasingly popular web-based virtual environments such as Second Life, educators, and educational researchers, face fundamental new questions. What kinds of teaching and learning experiences are possible in virtual environments (VEs) that are not possible in physical environments (PEs)? How do teaching and learning experiences in VEs relate to teaching and learning experiences in PEs? With regard to mathematics education, what kinds of mathematics and mathematical applications can be modeled, simulated, and rendered more practically and intuitively in VEs than in PEs? What theories and methods might best be brought to bear in researching the teaching and learning of mathematical modeling and applications in virtual environments?

This chapter does not provide comprehensive answers to these and other related questions posed herein. Rather, it begins to map out the terrain and offer some inroads into addressing and answering them. Moreover, addressing and seeking answers to these questions may be particularly well suited for building true partnerships across the areas of education, mathematics, physics, computer science, geography and others, which in turn may lead to a synthesis of research expertise that extends well beyond existing research themes in these disciplines. It is also time for mathematics education to deconstruct and reconstruct itself. It is becoming far less important to teach learners how to do math. Calculators and computers do a very good job of that. It is becoming imperative to help learners become more conscious of what math is, i.e., the nature of its objects and procedures, and perhaps most importantly, what the math is doing with regard to modeling and applications. Virtual environments provide both motives and means for making such changes.

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Section 11
What is The History of Modeling in Schools?

Chapter 52
On the Use of Realistic Fermi Problems in Introducing Mathematical Modelling in Upper Secondary Mathematics

Jonas Bergman Ärlebäck and Christer Bergsten

Abstract This paper reports a study investigating potential uses of Fermi problems to introduce mathematical modelling to Swedish upper secondary school students. The work of three groups of students' is analysed using an analytic tool referred to as the "modelling activity diagram", adapted and developed by the author. It was observed that the processes involved in a mathematical modelling cycle were richly represented in groups' solution. The students also frequently used their personal extra-mathematical knowledge in the solving process in three different ways: a creative, a verifying and a social way.

52.1 Introduction

From the curriculum documents governing mathematics education in Swedish upper secondary school, one can identify an increased emphasis on mathematical modelling. However, as Lingefjärd (2006) states, *"it seems that the more mathematical modeling is pointed out as an important competence to obtain for each student in the Swedish school system, the vaguer the label becomes"* (p. 96). It is natural to ask why this is the case and what one can do about it. Research in this area is lacking in Sweden, but challenges and barriers to overcome are likely to be similar to those reported by Burkhardt (2006). Some support for this assumption is found in research reports on the dominant role of the use of traditional textbooks in Swedish mathematics classrooms, greatly influencing class organisation as well as content (Skolverket, 2003).

J.B. Ärlebäck (✉)
Linköpings Universitet, Linköping, Sweden
e-mail: jonas.bergman.arleback@liu.se

C. Bergsten (✉)
Linköpings Universitet, Linköping, Sweden
e-mail: christer.bergsten@liu.se

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_52,

The study reported here is part of a bigger research project aiming both to get an overall picture of the past and present state and status of mathematical modelling in the Swedish upper secondary school, and to design, implement and evaluate small lesson sequences on mathematical modelling, in line with the Swedish national curriculum.

The aim of the pilot study which this paper reports is to investigate the potential of using Fermi problems to introduce mathematical modelling at the Swedish upper secondary level. More specifically the research question can be stated as follows. *What mathematical modelling sub-activities do realistic Fermi problems activate?*

To answer this question, we first briefly discuss Fermi problems and mathematical modelling, before looking at the previous research done in connection with this area. Then, we describe the methodology, the definition of *realistic* Fermi Problems, and the concept of mathematical modelling sub-activities, before the result, discussion and conclusion are presented.

52.2 Fermi Problems

The term *Fermi problem* originates from the 1938 Nobel Prize winner in physics Enrico Fermi (1901–1954). He posed the now classical question *How many piano tuners are there in Chicago? Then,* by using a few reasonable assumptions and estimates, he gave an astoundingly accurate and reasonable answer. Fermi was of the opinion that a good physicist (as well as any thinking person) could estimate any quantity accurate up to a factor 10 just "using one's head" – that is just by reasoning and by using intelligent estimates. These types of problems are called *Fermi problems*. To our knowledge Fermi himself did not define the characteristics of such problems and different authors in the literature emphasise different things.

Ross and Ross (1986) write that *"[t]he essence of a Fermi problem is that a well-informed person can solve it (approximately) by a series of estimates."* (p. 175) and that *"[t]he distinguishing characteristic of a Fermi problem is a total reliance on information that is stored away in the head of the problem solver… Solving Fermi problems presents an artificial challenge"* (p. 181). With a moderately free interpretation of the meaning of "well-informed person" most authors agree with the first of these quotes, but concerning the latter there is diversty. Others, such as Peter-Koop (2004) and Sriraman and Lesh (2006), are of the opinion that the concept of Fermi problems is better and more useful if one allows the problems not to be purely intellectual in nature, but situated in a the real world and everyday context.

Other characteristics ascribed to Fermi problems by some authors are their accessibility, or self-differentiating nature – meaning that the problem can be worked on and solved in different grades as well as at different levels of complexity (Kittel and Marxer, 2005). Also, as expressed by Sowder (1992), there should not exist an exact answer: *"[s]uch problems must be answered with an estimate, since the exact answer is not available"* (p. 372).

Some authors define the characteristics sequentially and more implicitly by describing the steps that are needed, or the understandings or insights that need to be achieved, to successfully come up with an answer. For example, Dirks and Edge (1983) list four *"things typically required"* when solving Fermi problems, namely *"sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge"* (p. 602).

52.3 Mathematical Modelling

There are many ways that one can work with, look at, or describe mathematical modelling in mathematics education (Blum et al., 2007; Kaiser and Sriraman, 2006). From our perspective, mathematical modelling is the entire (iterative and/or cyclic) problem solving process here illustrated in Fig. 52.1. When analysing this complex problem solving process in more detail, one can do so using the notion of *competences* (Maaβ, 2006) to divide the modelling cycle into sub-processes or sub-activities. For example, Borromeo Ferri describes the modelling process in terms of 6 *phases* (real situation, mental representation of the situation, real model, mathematical model mathematical result, real results) and *transitions* (understanding the task, simplifying/structuring the task, mathematizing, working mathematically, interpreting and validating).

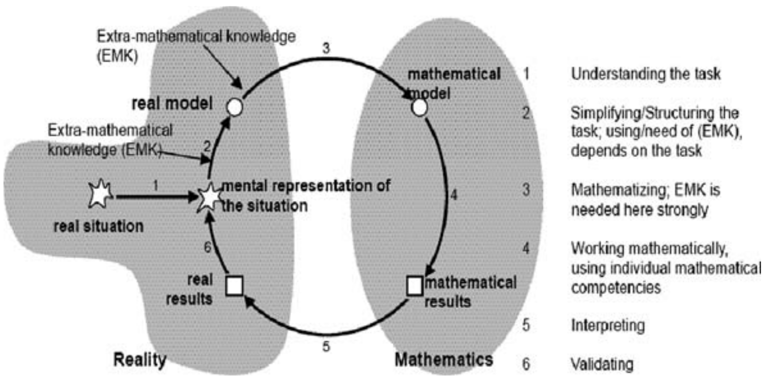

Fig. 52.1 The modelling cycle (Borromeo Ferri, 2006a, p. 92)

52.4 Previous Research

In mathematics education, Fermi problems are in principle mentioned in two different contexts at the lower educational level. First and foremost, they are mentioned in connection with (measure) estimates (or sometimes called numerosity problems); and secondly, they are mentioned in connection with modelling. Recently, Fermi problems also been suggested and used in fostering students' critical thinking (Sriraman and Lesh, 2006; Sriraman et al., 2009). The focus in this paper is on the use of Fermi problems in connection with modelling. However, for a review of the research in estimates see Sowder (1992) and the more recent Hogan and Brezinski (2003).

Although there exist extensive literature about problem solving, only a handful of papers explicitly use Fermi problems. Furthermore, most texts are theoretical in the sense that they discuss "what one can do with Fermi problem" (see the references in the section about Fermi problems above). However, Peter-Koop (2003, 2004) used Fermi problems in third and fourth grade to, among other things, investigate students' problem solving strategies. She concluded that (a) Fermi problems were solved in a sensible and meaningful way by the students, (b) the students developed new mathematical knowledge, and (c) solution processes *"revealed multi-cyclic modelling processes in contrast to a single modelling cycle as suggested in the literature"* (2004, p. 461). However, the impact and use of these results on a broader scale are unclear.

Beerli (2003) reports Swiss teaching materials for grade 7–9 which emphasize Fermi problems in a thematic way throughout; and, because of the way problem solving is connected to reality, opportunities are provided to use mathematization and modelling as means to develop mathematical knowledge and skills. Beerli also suggests that Fermi problems could effectively be used in assessment (pp. 90–91).

52.5 Methodology

As a part of our research, we designed several small units[1] on mathematical modelling in line with the national curriculum for the Swedish upper secondary school. A natural question to ask is how to introduce the subject in a gentle, efficient and interesting way. Reading a paper entitled *Modeling conceptions revisited* by Sriraman and Lesh (2006), where they say in a paragraph about estimation and Fermi problems that *"estimation activities can be used as a way to initiate mathematical modeling"* (p. 248), caught our interest. However, we could not find any reported research on this specific matter, so we decided to design this small pilot study.

Based on the earlier overview of the meaning and use of Fermi problems, we describe our use of the concept by giving it the following definition. Realistic Fermi problems are characterized by:

- their *accessibility*, that is they can be approached by every student and solved on different levels of complexity, they do not necessarily demand any specific pre-mathematical knowledge;
- their clear real-world connection, that is being *realistic*;

[1]By a modelling unit we mean a sequence of lessons with a planned focus on mathematical modelling.

- the *specifying and structuring of the relevant information and relationships* needed to tackle the problem;
- the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities;
- (in connection with the last two points above) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities.

These characteristics were used for guidance when constructing the problems used in the study; and from now on, whenever the term Fermi problems is used in this paper, we refer to problems with these characteristics.

Comparing the phases and transitions in the modelling process described according to Borromeo Ferri (2006a) and the character of a Fermi problem as presented above, one can see that there are obvious similarities. One might therefore try to describe and analyse the process of solving Fermi problems using this framework as it is. However, when estimation involves different sorts of quantitieswhen solving Fermi problems, we felt that this activity also really needed to be incorporated into the framework to give a more nuanced picture of the problem solving process.

52.6 Developing an Analytical Tool

For the present study, to get a schematic picture of the Fermi problem solving process, we developed and used an adapted version of Schoenfeld's "graphs of problem solving" (Schoenfeld, 1985). Hence, starting from the view of modelling presented above, and including the central estimation feature of realistic Fermi problem, we identified the following six *modelling sub-activities*:

Reading: this involves the reading of the task and getting an initial understanding of the task

Making model: simplifying and structuring the task and mathematizing

Estimating: making estimates of quantitative nature

Calculating: doing maths, for example performing calculations and rewriting equations

Validating: interpreting, verifying and validating results, calculations and the model itself

Writing: summarizing the findings and results in a report, writing up the solution

Here, the activity of *reading* is similar to Borromeo Ferri's "understanding the task", *making model* incorporates both "simplifying/structuring the task" and "mathematizing", *calculating* is the same as "working mathematically", and *validating* is both "interpreting" and "validating". The reason for these fusions is that it is often hard to separate "simplifying/structuring the task" from "mathematizing" and vice versa, and that to some extent "interpreting" and "validating" are intertwined. Now, a *modelling activity diagram* is an analytical tool using these categories to picture the problem solving process. An example of how such a diagram can look like is shown in Fig. 52.2 below.

52.7 Method

Using the characteristics of realistic Fermi problems, two such tasks were constructed for this pilot study. This paper reports students' work on one of these problems (the problem the students started with), *The Empire State Building Problem*:

There is an information desk on the street level in the Empire State Building. The two most frequently asked questions to the staff are:

- *How long does the tourist elevator take to the top floor observatory?*
- *If one instead decides to walk the stairs, how long does this take?*

Your task is to write short answers to these questions, including the assumptions on which you base your reasoning, to give to the staff at the information desk.

An a priori analysis of the problems was made identifying what the students *must* estimate and model to be able to solve the problem. Then, we also identified possible extensions and more elaborated features to incorporate in the situation. Information that students reasonably must use in the case of the Empire State Building problem are the height of the Empire State Building, the speed of the elevator and the speed when walking the stairs. As for more elaborate extensions to include in the model, we have the elevator queuing time and the capacity of the elevator. The time for getting in and out of the elevator might also be considered. In the problem on walking the stairs on the other hand one could start thinking of how to model the endurance and one's fitness.

Seven students, six male and one female, enrolled in a university preparatory year taking the upper secondary courses in mathematics, volunteered and divided themselves in three groups, A, B and C. After a short introduction dealing with ethical issues of the study and urging the students to do their best and to think aloud, the groups were placed in different rooms equipped with videotape recorders and were set to work. The two problems were distributed one at a time, and the groups worked on each problem for as long as they wanted.

The work of all three groups on the problems captured on the videotapes was transcribed using a modified and simplified version of the TalkBank conversational analysis codes[2] as a guide for the transcription. The students' written short answers were also collected.

[2] www.talkbank.org

The transcriptions were coded by the categories of the six modelling sub-activities described above. The coding was validated by looking at the video-recordings, as well as the written short answers from the three groups. This process was repeated to refine the coding and test the reliability of the process. The final result of this analysis was graphed in a *modelling activity diagram* for each group (see Fig. 52.2).

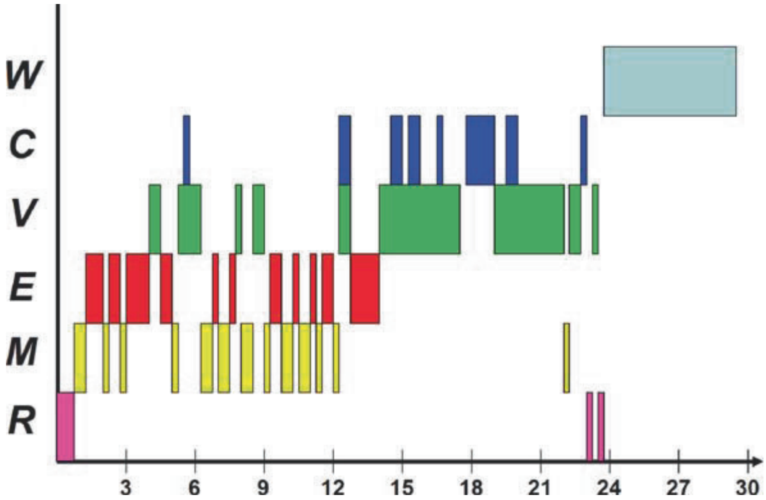

Fig. 52.2 The modelling activity diagram for the Empire State Building problem (group A)

52.8 Results

When given the problems, the students' first reactions involved surprise and frustration of not knowing the data needed to solve the problem. – "It's just to estimate everything!", "It's just to make something up! "That's a bad question since we don't get to know how high it ((the Empire State Building)) is!". – However, after this initial shock, the groups were very active and spent approximately 25–30 min engaged in solving each part of the problem.

In the first part of the *Empire State Building Problem*, none of the groups elaborated their solution to incorporate the suggested (or other) extensions of the problem from the a priori analysis presented above. They simply calculated that time equals the height divided by the average speed. However, in dealing with the second part of the problem, they developed quite different ways to approach how to model the endurance of the stair climbers. Group A used the same model as in the problem with the elevator (using an estimated average speed), and group C a variant of the same approach estimating the average time needed to walk the stairs of one floor and some additional time for resting before walking the next one. Group B developed a more advanced model where the time taken for a given floor depended on how high up in the building the floor is situated. Some of the key estimations done by the groups, and their answers to the problem, can be found in Table 52.1 below. It could be noticed that the mathematical demands were kept at a very elementary level throughout the process.

Table 52.1 Some of the groups' estimated quantities and their answers to the problems

Group	Estimated height (m)	Estimated elevator speed (m/s)	Answer, time for using the elevator	Answer, time for using the stairs
A	300	5	60 s	40 min
B	175	3,6	48,6 s	16 min 15 s
C	350	2	2 min 55 s	1 h 55 min

Figure 52.2 shows the result of analysing the data and constructing the three *modelling activity diagrams* for the solving processes of the groups. Fig. 52.2 shows only the *modelling activity diagram* for group A. But, this diagram is representative for all three groups. On the vertical axis the sub-activities *Reading, Making model, Estimating, Validating, Calculating* and *Writing* are displayed. Time in minutes is displayed on the horizontal axis.

This diagram displays four major phases; first a short initial *reading* phase followed by an 11 min phase consisting of the dialectic interplay between *making a model* and *estimating* (with some minor elements of *validating* and *calculating*). Then there is a third phase of *validating* with a parallel element of *calculating* (approximately 10 min), and finally a 7 min phase of *writing*.

A typical segment of the group work, which was categorized as *making model* deals with negotiating and agreeing on how to structure the problem and which assumptions or idealizations to make. This is briefly illustrated in the following excerpt which takes place around 3 min into the discussion of group A. The group just estimated the height of the Empire State Building to be 200 m and now turned to properties of the elevator:

Bertil: But in the case of a tourist elevator (.) shall we then assume that it just goes from the ground to the absolute top ((*floor*))?
Anders: Yes.
Claes: Without stopping.
Anders: Yes.

Estimation segments are often initiated with a direct question as for example "Yes, but how high is the poor building?", "How fast can an elevator run?" or "How many floors are there?". The discussions that follow such a question are all aimed to produce an estimate of some quantity, a number:

Bertil: Shall we estimate a speed by which an elevator can travel upwards? A typical elevator in a building of a regular height of say 200 m (.) going all the way up (.) in meters per seconds.

Claes: Yes, say that (.) a typical elevator might travel by the speed of 3–4, 3 m per seconds
Bertil: Shall we say 5? It's a quite fast elevator
Claes: Mm, yes, let's settle for 5 ((*m/s*)).
Bertil: 5
Anders: Hm ((*nodding his head*))

Questions also often start up segments of *validation* ("Shall we really say 100 floors?"), as do statements of doubt ("It feels like that's to fast (.) for that building"). In these segments previous assumptions, estimates, calculations and results are looked at critically and are either made manifest or rejected in favour of better versions, as in the following example where the calculated elevator time evokes questioning of the previous estimated height of the *Empire State Building* (of 200 m):

Claes: It feels like that's too fast (.) for that building.
Anders: Hm (.) it does.
Claes: I think it is (.) and now I'm stretching it (.) I think it is 300 ((*m*)) (.) I think it is 280 m, that I was off with a hundred ((*m*)) before.
Bertil: We can say that it ((*the Empire State Building*)) is 200 m, that's fine.
Claes: Yes ((*10 s of quiet mumbling, Claes is working on his calculator*)) 60 s, 1 min to go up there. Is that reasonable?
Anders: Yes.
Bertil: If we assume, how high
Claes: Yes, "cause that"s almost probable
Bertil: Yes, 300 m (.) 300 m
Claes: 1 min then. One minute can be really long.
Bertil: Yes, especially in an elevator.

Calculating is an activity normally preformed by one of the group members in the background of some other activity, so here the video recording is crucial for the coding. Occasionally the whole group focus on the actual calculation, but regardless of how it is obtained and by whom, the result of a calculation is important for how the solving process evolves.

The sub-activities *reading* and *writing* are rather self-explaining, but it is notable that when the group came to writing down their answers (in the form of a letter) they just reproduced and retold what they said before without any reflections or critical scrutiny.

From the constructed modelling activity diagrams one can observe that the students engage in all of the predefined different sub-activities, that they do spend a considerable amount of time in each sub-activity, and that they go back and forth between the different types of activities numerous times. In other words, the processes involved in the mathematical modelling cycle pictured in Fig. 52.1 are richly represented in the groups' problem solving processes.

It could also be noted in the data that the students frequently used their personal extra-mathematical knowledge in the solving process. It seems that they did this in at least three different ways: in a *creative* way to construct a model or to make an estimate, in the process of *validating* a result or an estimate, and finally in a *social* way as a narrative anecdote. In the following excerpt Claes shares a personal experience from an amusement park in a creative way in hope to easier get an estimate of the elevator speed. However, in this specific case his reasoning makes him question the estimate of the height of the Empire State Building (200 m) the group agreed on 2 min earlier. Hence he is also using this piece of extra-mathematical knowledge to (involuntary) initiate a validating process:

Claes: Hm (.) Did anyone ride the FREE FALL[3]
Anders: ((*in unison with Bertil*)) Nope.
Claes: I was thinking that since (.) hm, it is 90 m ((*high*)), how long does it take to get up there? (.) I think it takes 15, 20–25 s (.) and that's 90 m.
Bertil: Yes.
Claes: It ((*the Empire State Building*)) must be higher than 200 m.

52.9 Discussion

The modelling activity diagram is another way to describe students' modelling processes as has been done in many empirical studies (Borromeo Ferri, 2006a). Borromeo Ferri (2006b, 2007) pictured what she called *"individual modelling routes"* of her students by drawing arrows in the modelling cycle shown in Fig. 52.1. In this context, the modelling activity diagram can be used to visualize a group or an individual student's modelling sub-activities in a more linear way along a timeline. Thus, it provides a simple dynamical picture of the activities involved. From the results presented above, the modelling activity diagram shows that Fermi problem might serve well as a means to introduce mathematical modelling at this school level. All modelling sub-activities are richly represented and contributed in a dialectic progression towards a solution to the task.

In the data material one can note that the group dynamics are essential for the evolution of and activation of the different sub-activities during the problem solving process. It is the discussions and interactions in the groups, when different beliefs and opinions are confronted, that drives and forms the modelling process.

Borromeo Ferri (2006a, 2006b, 2007) also noted that students' use of extra-mathematical knowledge in the modelling process when deriving the mathematical model and validating results, and in the latter case differentiates between *"intuitive"* and *"knowledge-based validation"* (2006a, p. 93). She notes that students mostly only make what she calls *"inner-mathematical validation"* and that validating for

[3]FREE FALL is an attraction in the amusement park Gröna Lund in Stockholm, Sweden.

students means *"calculating"*. However, the data in this study provides numerous examples of validation of the model, the estimates as well as the calculations.

One of the key reasons for using realistic Fermi problems was to urge the groups into discussions about the problem setting and how to solve it. These expectations were confirmed, but the study also suggests that to some extent such problems take the focus away from the mathematics, which we believe students experience hard to discuss, and makes the problem available in an indirect way through the discussions about how to structure the problem and what (and how) to estimate. In this respect the realistic feature of the problem is central. Although the mathematics was kept at a very elementary level, one could have tried to deepen it by explicitly asking for, say, an equation relating height and time spent in the elevator or in the stairs.

For the validity of the study using adult students in a special university preparatory course may be questioned but the mathematical pre-knowledge were the same as for adolescent students in school taking the same courses, and the increased extra-mathematical knowledge that came into play when working on the problems studied were not of a kind that could not have been experienced also by the latter group. That the participants chose the groups themselves we consider to be an advantage for this pilot study, facilitating openness in the discussions. An alternative method of using a grounded theory approach could also have been applied to the data. However, the results of a study can only be interpreted within the research framework chosen, which in this case was linked to the mathematical modelling perspective where the pre-defined categories used make it easier to relate to and locate the results in previous research.

52.10 Conclusions and Future Research

Returning to the research question, one can conclude that all the modelling sub-activities proposed by the framework (*reading, making model, estimating, calculating, validating* and *writing*) are richly and dynamically represented when the students get engaged in solving realistic Fermi problems. Thus this pilot study shows that small group work on realistic Fermi problems may provide a good and potentially fruitful opportunity to introduce mathematical modelling at upper secondary school level.

This research may be continued in a number of ways. First, the tool *modelling activity diagram* as an instrument of analysis has a potential to be developed further in different directions, depending on what it will be used for. One idea is to incorporate the group dynamics into the diagram by indicating in a sub-activity segment how much each group member contributes to the discussion. One could also try to modify the framework to be more general; *reading* in this study is just reading, but in a more general setting *reading* could stand for the gathering of any external information. In some situations it may be needed to try to split the sub-activity *making model* into the two sub-activities *structuring* and *mathematizing*.

Second, in a setting of teaching mathematical modelling, a joint follow-up session with the three groups in close connection to the problem solving session could

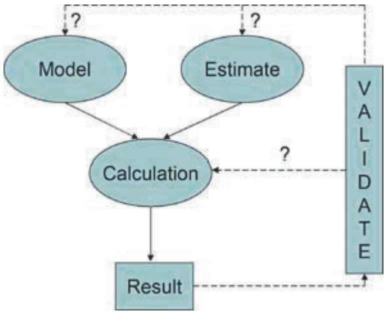

Fig. 52.3 Possible validation routes

serve well as a meta-cognitive activity, discussing the problem and their way of solving it, highlighting the processes of mathematical modelling and possibly end up with some version of modelling cycle. In future research the outcome of such an intervention might be fruitful to investigate.

Third, the data also pose interesting questions about how the students validate their results, models and estimates. – Why do they choose to do this the way they do? A (calculated) result depends on the model developed, the estimates done and the performed calculation. So, in validating a result it is desirable that all these three "influences" are looked upon critically (Fig. 52.3). Since all of these three types of validating are present in the data it would be interesting to investigate this phenomenon in more detail.

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Chapter 53
The Dutch Maths Curriculum: 25 Years of Modelling

Pauline Vos

Abstract In the Netherlands, mathematics education is intertwined with applications as a result of the inspirational work by Hans Freudenthal and his colleagues, who developed a treatise known as Realistic Mathematics Education (RME). In this paper I will present a retrospective on 25 years of curriculum revisions in the Netherlands, exemplified by two nation-wide projects that established new routines into modern Dutch mathematics education. The first project established a new mathematics curriculum for the lesser gifted students in grades 7–10. In this curriculum modelling serves as a vehicle for students' construction of mathematical knowledge. The second project established an annual modelling competition for teams of students at senior secondary level in the social sciences streams.

53.1 Introduction

The ICMI-14 Study on Modelling and Applications in Mathematics Education has resulted in an impressive volume with many reports on recent developments and findings in the field of applications and modelling in mathematics education (Blum et al., 2007). Despite its huge amount of encyclopaedic information, it does not highlight the developments in one country that deviates from many others with respect to modelling and applications in mathematics education: the Netherlands.

For readers interested in the projects carried out in different countries, the ICMI-14 Study has one chapter with country reports from the US, Australia, Ontario, and South Africa. Additionally, the chapter on lower secondary education contains short descriptions of how the implementation of modelling and applications into the mathematics curriculum is hindered or encouraged in selected countries. Here, the Netherlands appears with a report on how modelling and applications are mandated within the mathematics curriculum, but its successful implementation is

P. Vos (✉)
University of Amsterdam, Amsterdam, Netherlands
e-mail: f.p.vos@uva.nl

R. Lesh et al. (eds.), *Modeling Students' Mathematical Modeling Competencies*,
DOI 10.1007/978-1-4419-0561-1_53,

hindered by inadequate teacher training and textbooks with over-scaffolded, simplified exercises. The other chapters on applications at different educational levels do not contain examples from the Dutch modelling practice. Two other Dutch contributions in the volume are normative; they describe how modelling should have a place in the design of mathematics curricula (Gravemeijer, 2007) and in its assessment (Vos, 2007). But, these contributions illustrate their point with confined empirical data and not with data from everyday Dutch mathematics classes. Somehow, the rich Dutch practice in modelling and applications did not find its way into the ICMI-14 Study, and I will suggest a reason at the end of this paper.

To supplement the ICMI-14 study, I will use this ICTMA-13 opportunity to give a retrospective of the Dutch mathematics curriculum, which embeds modelling and applications at all levels of learning. Because of limited space, I will focus on two nation-wide projects, which have not yet attracted (inter-)national research attention. The first project targeted students at the lower secondary level, especially the lesser gifted ones. The second project targeted pre-university students, especially in the social sciences streams. Both projects serve as show-pieces of how modelling is not only reserved for the mathematically gifted students (e.g. future engineers).

53.2 Backgrounds

In the Netherlands mathematics education has been inspired by Hans Freudenthal and his colleagues, who developed a treatise known as Realistic Mathematics Education (RME). Freudenthal (1973) characterized mathematics as being an integral part of real-life, perceiving it as an activity and not as a set of rules. As such, mathematics is a creative and organizing activity, in which unknown regularities, relations and structures are to be discovered. Successive RME-authorities emphasized the conceptual learning of abstract mathematics through using contexts. De Lange (1987) used *conceptual mathematization* as a tool to assist students in their process to develop mathematical concepts starting from concrete situations. Gravemeijer (2007) used *emergent modelling* as a domain specific instructional strategy, through which students are assisted to construct conceptual knowledge in a long term process of successive steps of abstraction. For both authors, well-designed exercises embedded within contexts are the basis of learning mathematics. The contexts are constructed, yet sufficiently imaginable to students.

In 1971 the Dutch Ministry of Education strengthened national mathematics education by establishing an institute for the development of mathematics education (later: Freudenthal Institute). In that same year a curriculum reform at primary level (Wiskobas) was started. For most mathematicians and mathematics teachers, the project at primary school level went largely unnoticed, but the ensuing project was overwhelming: in the school year 1981/82 we experienced the start of a curriculum reform at senior secondary level with the introduction of *Mathematics A* as a subject preparing for the social sciences (Hewet/Hawex project). This new school subject was set in parallel to the classical, abstract mathematics, which targeted future engineers, and which was subsequently named *Mathematics B*. Thus, Mathematics B had an emphasis on abstract functions, derivatives, vectors, and so forth, while Mathematics A started from mathematical models in the social and economical sciences and cut abstract mathematics to a minimum in order to enable future students in psychology or sociology (predominantly girls) to continue to learn mathematics (mainly discrete mathematics and statistics) throughout their secondary schooling. With national exams[1] adjusted to the new content approach, all schools had to comply to the new curriculum and had to offer both Mathematics A and B.

In 1993 the next curriculum reform established a new common core curriculum at junior secondary level, described in more detail in this paper. The history of Dutch mathematics curriculum reforms did not stop there, with ensuing reforms at senior technical vocational level (Twin), calculus within Advanced Mathematics (Profi) and others, but space limits a full overview.

53.3 Theoretical Framework

This paper will describe two curriculum interventions that connected Dutch daily-life mathematics education to the use of contexts. I will analyze the projects by identifying the benefiting target groups and by using the curriculum distinction between *modelling-as-vehicle* and *modelling-as-content*, as introduced by Julie and Mudaly (2007). Modelling-as-vehicle is an approach to the teaching of abstract mathematical concepts embedded in contexts, whether it be at the introduction of a concept or in the aftermath as an application of concepts. The other approach is to have a contextual problem as a starting point without the prescription that the learning of certain mathematical concepts is the outcome of the model-building process. Julie and Mudaly stress that modelling-as-vehicle and modelling-as-content are an idealization and to be perceived as extremities of a continuum.

Another framework to describe curriculum innovations with respect to modelling was found in Burkhardt (2006). He states that the large-scale implementation of modelling into mathematics curricula faces four *barriers*: systemic inertia (habits, beliefs, teaching traditions), the real world (the messiness of data), limited professional development and the role of research (too much focused on insight). Besides, Burkhardt points at *levers* that may tackle each barrier, such as curriculum descriptions, illustrative examples, well-engineered material to support assessment, professional development, students' increased motivation, and a research approach that is more design-oriented. I will use this framework in the description of the two curriculum projects to identify observed barriers and levers.

[1] In the Netherlands, entry into tertiary education depends on achievement in the national exit exams (grades 10-11-12, depending on the track). For the university social sciences the requirement is Mathematics A (or B) on the grade 12 diploma; for engineering it is Mathematics B.

53.4 The Mathematics Curriculum for Junior Secondary and for the Lower Tracks

In 1993 a new mathematics core curriculum was legislated for junior secondary level (grades 7/8) and for the final years of the lower ability track (grades 9/10 of the *vmbo*[2]). This curriculum was developed not as a watering down of curricula at a higher level, but it intended to give all future citizens basic, mathematical abilities. The first designs were created within the *Wiskunde 12–16 project* by a mixed team of teachers, trainers, and researchers. The designers stressed that the usefulness of mathematics should not have its justification from future adult life alone, but already should be experienced during the learning process (Kok et al., 1992).

To the dismay of many mathematicians, contents like *sets* and *congruency* were abandoned. Alphabetic variables (x, a) were postponed through the use of *word formula* (a formula with variables stated in words, see the Appendix). The new curriculum emphasized data modelling and interpreting (through tables, graphs, etc), visual 3-d geometry, approximation, the use of ICT and other topics considered relevant to daily life in the 21st century (Kok et al., 1992). Contexts were not to be used at the end of each topic as in applications. Instead, mathematical concepts were introduced to the students through a range of examples of mathematical models. Also, open problems were introduced. A German colleague commented on a common Dutch textbook for the lowest track at grade 7 level: *"There are real open modelling tasks in them, you would normally not find this in German books. Very interesting."* (Katja Maass, 19-03-07, personal communication).

National assessment was adjusted to the new approach. Generally, problems describe an appealing daily life situation (often with authentic photographs) followed by questions. The examples in Appendix 1 were developed for 14-year old students by the W12-16 curriculum designers (cited in Kuiper et al., 1997). One problem is on the prediction of the growth of students, derived from their parents' length. The other problem is on the reconstruction of the position of a photographer. Both have an identifiable context, although the context "Length" is clearly constructed: the "school doctor" is used as an analogy for a researcher collecting demographic data and the mathematical model is a word formula that transforms the mean of the parents' length, which indeed is a genetic approximation of children's future length. In the two exercises, the real-world contexts are already modelled into a formula and a scale drawing. The main task for the students is to work with the mathematical model (reworking, simplifying, interpreting). Several integrated mathematics topics can be discerned (substituting values into a formula with three variables, fractions, order of operations, conversion of cm to m, translating a 3-d shape into a 2-d representation, reasoning on visibility). There are a number of similarities between the problems in the context-based Dutch mathematics curriculum for junior secondary schools on the one hand, and the mathematics problems

[2]The lowest track in secondary education, Voorbereidend Middelbaar Beroeps Onderwijs (= vocational education). It serves approximately 60% of all Dutch students.

from the PISA-tests on the other hand (Dekker et al., 2006): the presence of a title indicating the theme, the amount of clarifying text, the photographs, and the provided mathematical model. In the problems the mathematical procedures are largely prescribed by the availability of the model. Thus, the problems do not display a *modelling-as-content* approach. Extrapolating towards the curriculum, the emphasis is on *modelling-as-vehicle*, with a strong emphasis on the use of contexts throughout the learning process, whether it be at the introduction, the consolidation or the assessment of mathematical concepts. However, some RME-based researchers have now questioned the indiscriminate use of contexts in the curriculum (Dekker et al., 2006; Wijers et al., 2004).

The experimental curriculum materials were made widely available, together with publications on curricular backgrounds (Kok et al., 1992). The strategy to explain the intentions of the curriculum to teachers and textbook authors was “being concrete”: offering many exemplary problems. Additionally, anecdotal reports of experimental lessons were published in the two teacher journals.

Moreover, with national exams being binding, the new format of problems was a very strong lever for the curriculum implementation, directing teachers and textbook authors to prepare students on the use of contexts in mathematics. Three years before the legislation, a large exercise on professional development was undertaken to introduce teachers to the new content and its approach. In autumn 1990 in all provincial capitals, two-day workshops were organized by the Dutch Association of Mathematics Teachers, with not one teacher having to travel more than 50 km. These introductory workshops were attended by approximately 10% of all Dutch mathematics teachers (Verhage and Wijers, 1991). In the ensuing years, the regional conferences were repeated.

To frame the project in light of the four barriers as identified by Burkhardt (2006), the barrier of systemic inertia was contested by systemic intervention (exams), the barrier of the real world was contested by constructing (or adapting) real world problems and adding real life elements, and the barrier of lack of professional development was contested by organizing teacher meetings. The fourth barrier identified by Burkhardt, is the role of research and development being too remote from practice, only identifying problems and barely offering solutions. This barrier could not be identified, as the new curriculum was purely developed through an artisan engineering approach, without theory development nor strong research base. Although a university professor did chair the project team, I could not trace one PhD project and not one publication in a scientific journal dedicated to the design and development of this context-based curriculum. Only one independent researcher evaluated the implementation of the curriculum (Vos, 2002). Thus, academic indolence did not hinder the project, but the project did not enhance research either.

As for the question on which target group did benefit from the context-based mathematics curriculum, I will present data on the mathematics performance of the lesser gifted students in the Netherlands at the age of 14 years, even if this is assessed in an abstract manner. I will use TIMSS data, because this test is unbiased towards mathematical applications and modelling. Table 53.1 displays the TIMSS percentile

Table 53.1 TIMSS 2003 mathematics scores for grade 8 students, by percentiles and selected countries*

Country	P5	P25	P50	P75	P95	Slope
Australia	368	450	506	561	634	2.78
England	373	445	497	552	627	2.66
Flandres	398	495	545	588	643	2.52
Netherlands	417	488	540	587	644	2.39
Sweden	378	452	501	548	614	2.46
USA	369	450	505	560	635	2.78

*Percentile scores from Mullis et al. (2004), p. 410.

scores of grade 8 students in selected countries, with P5 being the least gifted 5% of the students. At P50 we see the median score. For each country the scores increase per column as subsequent percentiles contain better performing students. The five percentile scores of a country yield a *slope* (the gradient of the regression line, as calculated by Excel), which is an indicator of the discrepancy between the lesser and the more gifted students.

The table shows that the Netherlands has a high scoring population, and the score at P5 is highest and the slope is lowest. Although the evidence is indicative, this suggests that the least gifted students benefit from the Dutch contexts-based curriculum, because within this curriculum they can use common sense strategies and they are not hindered by abstract symbols, to which they cannot connect meaning. Other authors have also stated that a curriculum, which develops mathematical literacy, is particularly appropriate for the construction of mathematical knowledge by the lesser gifted students (Dekker et al., 2006), but further research is needed here.

53.5 The Math Alympiad

Since 1989 an annual modelling competition is organized, offering open-ended modelling problems, which are often derived from an authentic real-life problem. It has been named *Math Alympiad*, with the “A” being associated to the subject of *Mathematics A* at senior secondary schools. This subject prepares for the social and economical sciences with an emphasis on discrete mathematics and statistics (as described in the Backgrounds). The competition was initiated because of growing discomfort on the limitations of written tests, which did not enable students to engage in a full modelling cycle. As a result, Jan de Lange came up with the idea of a competition on modelling, in which teams of three or four students (grades 11–12) dedicate a full day to merely one modelling task (De Haan and Wijers, 2000).

The annual Math Alympiad has two rounds: the first round is the preliminary, which is administered at the schools. Here the modelling task has to be completed within a day (a Friday in November). The second round is the final, in which the twelve best teams from the first round compete at a central location. The final task is harder than the preliminary task, and it is to be completed within two days (a Friday and Saturday in April).

Each Math Alympiad round only contains one problem. Over the years, problems have been taken from politics, sociology, archeology, sports, the life sciences, and so forth, with one area clearly being avoided: the area of engineering. All Alympiad problems are available from http://www.fi.uu.nl/alympiade/en/. Often the problems have an authentic origin; for example an article in Nature induced the committee to design a task on biodiversity (De Haan and Wijers, 2000). The typical Math Alympiad problem is complex and asks for mathematics as an organizing activity (Freudenthal, 1973) with students acting as policy consultants and mathematics being instrumental in their advisory task. Students are asked to justify their assumptions and to weigh options. There is never only one correct answer, nor is there a prescribed procedure available. The Math Alympiad clearly aims at *modelling-as-content*.

The Math Alympiad started small with 14 schools with only one team per school. The success was immediately evident and every year more schools joined in. See Table 53.2. Presently, more than 150 schools (approx. 20% of all Dutch schools) with around ten teams per school participate in the preliminaries. It means that more than 4000 students engage in a day of modelling. Each school can send in the work of two teams as candidates for the finals. The exercise to select the best teams then is immense, with students' work often being hard to compare. Internationally the Alympiad is growing as well, with schools from Denmark, the Antilles and Nordrhein-Westfalen (Germany) having joined ranks. Recently, Japanese authorities have shown interest (Goris, 2006).

The Math Alympiad targets at students who take Mathematics A as an exam subject. Therefore, the target group does not include the students who take mathematics at its most abstract level. Some participants even show a dislike of mathematics, and Goris (2005) notes how students react skeptically to the idea of doing mathematics for a whole day. However, at the end of the day there is overwhelming enthusiasm. De Haan (1997) observes that the best motivators to new participants are the students who competed in the previous year. Goris (2005) describes the bonding process within the teams. After an intense working day and even despite the foul Dutch weather, all teams come to deliver their assignment as a team, instead of sending one representative to hand in the work. Thus, the experience of a large modelling task and the group work yields satisfaction and motivation, which serves as a lever to students' initial hesitation.

To overcome the barrier of system inertia, several levers can be observed. The competitive element serves as a lever, because schools are keen to see their names mentioned in national publications. Also a new legislation on exams in 1998 prescribed schools to include an additional format named "practical assignment", which consists of a coherent, complex task. Many schools have since imposed participation in the preliminaries of the Math Alympiad as part of the Math A exam. Thus, legislation led to an increase in the number of participating schools.

The barrier of the real world (the messiness of data) is overcome in the Math Alympiad by constructing the problems. Not one problem is fully original, but all

Table 53.2 Frequencies of schools (teams) participating in the Math Alympiad

School year	Netherlands	Denmark	Antilles*	Nordrhein-Westfalen
1989/90	14 (14)			
1990/91	48 (117)			
1991/92	49 (127)			
1992/93	53 (135)			
1993/94	87 (261)			
1994/95	101 (347)			
1995/96	113 (660)	2 (8)		
1996/97	106 (835)	5 (9)		
1997/98	112 (>1000)	6 (19)	2 (16)	
1998/99	113 (>1000)	45 (80)	2 (17)	
1999/00	101 (>1000)	30 (82)	4 (29)	
2000/01	111 (>1000)	26 (65)	4 (26)	
2001/02	127 (>1000)	23 (59)	4 (28)	2 (10)
2002/03	141 (>1000)	20 (50)	4 (25)	10 (100)
2003/04	146 (>1000)	10 (30)	4 (23)	10 (100)
2004/05	148 (>1000)	1 (8)	2 (22)	10 (100)
2005/06	161 (>1000)	1 (10)	2 (13)	10 (100)

*Data from Germany and from the Antilles (Aruba, Curaçao, St Maarten) are inaccurate.

problems remained close enough to reality, with many real-life aspects added to give the tasks sufficient fidelity to reality. The barrier of insufficient professional development is overcome by the annual organization of teacher network meetings on the Alympiad. Here, issues are discussed on how to prepare students for the tasks and on the assessment of students' work. Another motivator for teachers is students' enthusiasm, which is remarkable considering the fact that Dutch students in the Math Alympiad are on average not highly talented for abstract mathematics as they did not elect Math B as exam subject (with its large calculus component). Finally, the fourth barrier identified by Burkhardt, is the role of research and development. Again, this barrier could not be identified at the Math Alympiad, as it was purely developed through an engineering approach, starting small and growing organically without theory development nor strong research base. Despite a large number of reports in teacher journals and two anniversary volumes on the Alympiad (De Haan and Wijers, 2000; Wijers and Hoogland, 1995), I could not trace one PhD project or scientific journal publication dedicated to research on the Alympiad. Thus, academic indolence did not hinder the project, but the project did not enhance research either.

53.6 Conclusions

From 1981 onwards, several curriculum projects established new routines into Dutch mathematics education. In this paper I have reviewed two projects. The 1993 curriculum brought *modelling-as-a-vehicle* with contexts being used throughout the development of concepts. There are indications that in particular the lesser gifted students benefit from this approach. The second innovation, the Math Alympiad is a team competition with *modelling-as-content* and especially pre-university students in the social and economical sciences benefit from it.

A number of levers were identified, overcoming barriers of the curriculum implementation (Burkhardt, 2006). A strong lever is the national exam format, which enforces textbook authors, schools and teachers to adapt to certain approaches. Another lever is the dissemination of exemplary exercises and the organization of teacher meetings. Moreover, students' enthusiasm from working on modelling tasks serves as a lever. The barrier of the real world (messiness of data) was overcome by constructing problems, although the original meaningfulness remained intact. Finally, the barrier of "the nature of research" was not observed in the two cases. However, the lack of a research-based approach might have been the reason why the two curriculum innovations did not make it into the ICMI-14 Study.

Appendix

Examples of typical mathematics classroom problems for 14-year-old Dutch students, translated from the national option test, administered as a national option within TIMSS-95 (Kuiper et al., 1997).

Length

To calculate any girl's future height as a grown-up, the school doctor uses this formula:

$$\text{Length daughter (in cm)} = \frac{\text{length father (cm)} + \text{length mother (cm)} - 12}{2} + 3$$

Use this formula in the questions below.

A. The father of Danielle is 1,82 m tall, her mother is 1,68 m. How tall will Danielle grow according to the formula?
B. Is it possible that – according to the doctor's formula – a daughter will grow taller than her father? Explain your answer. If you want you can use an example.

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