Systems of Vector Quasi-equilibrium Problems and Their Applications

Qamrul Hasan Ansari and Jen-Chih Yao

Abstract In this survey chapter, we present systems of various kinds of vector quasi-equilibrium problems and give existence theory for their solutions. Some applications to systems of vector quasi-optimization problems, quasi-saddle point problems for vector-valued functions and Debreu type equilibrium problems, also known as constrained Nash equilibrium problems, for vector-valued functions are presented. The investigations of this chapter are based on our papers: Ansari (J Math Anal Appl 341:1271–1283, 2008); Ansari et al. (J Global Optim 29:45–57, 2004); Ansari and Khan (*Mathematical Analysis and Applications*, edited by S. Nanda and G.P. Rajasekhar, Narosa, New Delhi, 2004, pp. 1–13); and Ansari et al. (J Optim Theory Appl 127:27–44, 2005).

1 Introduction

In the last two decades, vector variational inequalities (VVI) have been investigated [2,32,47,48,55,57,62,65,87,97] and used as tools to solve vector optimization problems (VOP) for differentiable and convex or nonconvex vector-valued functions. A generalized form of VVI for multivalued maps is called a generalized vector variational inequality (GVVI). GVVI has been used to study VOP for nondifferentiable and nonconvex vector-valued functions. The weak (respectively, strong) solution of Stampacchia GVVI provides a sufficient condition (respectively, necessary and sufficient conditions) for a solution of VOP; see, for example, [17, 23, 30, 59, 60] and the references therein.

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In the recent years, vector equilibrium problems (VEP) have been studied in [3, 19, 28, 44, 48–50, 94] and the references therein. It is a unified model of several problems, namely, vector variational inequality problems, vector variational-like inequality problems (also called vector pre-variational inequality problems), vector complementarity problems, vector saddle point problems and vector optimization problems. A comprehensive bibliography on VEP, vector variational inequalities, vector variational-like inequalities and their generalizations can be found in [48]. For further details on generalized vector variational inequality problems, generalized vector variational-like inequality problems and vector equilibrium problems, we refer to [2,17,23,28,37,48,50,56,58,61,62,64,83,94] and the references therein. In [24], we extended a quasi-equilibrium problem, studied in [41, 72], to the case of vector-valued functions, called a *vector quasi-equilibrium problem* (VQEP). We established some existence results for a solution of VQEP with or without a generalized pseudomonotonicity assumption. As a result, we derived the existence results for solutions of vector quasi-optimization problems, vector quasi-saddle point problems, vector quasi-variational inequality problems and vector quasi-variational-like inequality problems [55, 63, 65].

When the involved bifunction in the formulation of VEP (respectively, VQEP) is a multivalued map, then VEP (respectively, VQEP) is called a generalized vector equilibrium problem (GVEP) [respectively, generalized vector quasi-equilibrium problem (GVQEP)]. The GVEP (respectively, GVQEP) includes as special cases generalized implicit vector variational inequality problems, GVVI problems, generalized vector variational-like inequality problems and vector equilibrium problems (respectively, generalized implicit vector quasi-variational inequality problems, generalized vector quasi-variational inequality problems, generalized vector quasi-variational-like inequality problems and vector quasi-equilibrium problems). GVEP and GVQEP has been studied in [7, 8, 11, 13, 18, 21, 33, 46, 57, 74, 83–85, 93] and the references therein.

The system of vector equilibrium problems (SVEP), that is, a family of equilibrium problems for vector-valued bifunctions defined on a product set, is introduced in [14] with applications in vector optimization and the Nash equilibrium problem [80–82] for vector-valued functions. The SVEP contains a system of equilibrium problems, a system of vector variational inequalities, a system of vector variational-like inequalities, a system of optimization problems and the Nash equilibrium problem for vector-valued functions as special cases. In the recent past, systems of scalar (vector) equilibrium problems are used as tools to solve the Nash equilibrium problem for vector-valued functions; see, for example, [14, 15, 22, 39, 98, 99] and the references therein. But, by using SVEP, we cannot establish the existence of a solution of Debreu type equilibrium problem [38], also known as constrained Nash equilibrium problem, for vector-valued functions that extends the classical concept of the Nash equilibrium problem for a noncooperative game. For this purpose, in [5], we introduced a system of vector quasi-equilibrium problems (SVQEP) with or without involving Φ-condensing maps and proved the existence of its solution. Consequently, we established some existence results for a solution of a system of vector quasi-variational-like inequalities. The equivalence between a system of vector quasi-variational-like inequalities and the Debreu type equilibrium problem for vector-valued functions (Debreu VEP) is presented. As an application, we derived some existence results for a solution of the Debreu VEP.

In [15], we introduced a system of generalized vector equilibrium problems (SGVEP) which contains a system of generalized implicit vector variational inequality problems, a system of generalized vector variational inequalities, a system of generalized vector variational-like inequalities and SVEP as special cases. We established some existence results for a solution of SGVEP by using a maximal element theorem for a family of multivalued maps due to Deguire et al. [39]. We also derived some existence results for a solution of a system of generalized implicit vector variational inequality problems, a system of generalized vector variational inequalities, a system of generalized vector variational-like inequalities and SVEP. As an application, we gave some existence results for a solution of the Nash equilibrium problem for differentiable (in some sense) vector-valued functions.

In [10], we introduced a system of generalized vector quasi-equilibrium problems (SGVQEP). It is a very general and unified model of several problems, namely, a system of generalized implicit vector quasi-variational inequality problems, a system of generalized vector quasi-variational inequalities, a system of generalized vector quasi-variational-like inequalities, SVEP, SVQEP and SGVEP. We established some existence results for a solution of SGVQEP with or without involving $\boldsymbol{\Phi}$-condensing maps. As consequences, we proved the existence of solutions of several known problems mentioned above. As applications of our results, we derived the existence results for a solution of Debreu VEP for nondifferentiable (in some sense) functions.

In 1994, Husain and Tarafdar [52] introduced simultaneous variational inequalities and gave some applications to minimization problems. These are further studied by Fu [45] for the vector-valued case with applications to vector complementarity problems. Recently, Lin [67] considered and studied simultaneous vector quasi-equilibrium problems and proved existence results for their solutions. By using these results, Lin derived existence results for a solution of a vector quasi-saddle point problem. In [12], we considered systems of simultaneous generalized vector quasi-equilibrium problems (SSGVQEP) which contain simultaneous generalized vector quasi-equilibrium problems [67], generalized vector quasi-equilibrium problems [46], systems of vector quasi-equilibrium problems [5], systems of generalized vector quasi-variational-like inequalities [10] and simultaneous vector variational inequalities [45] as special cases. By using Kakutani fixed point theorem [54], we established an existence result for solutions of SSGVQEP. We derived several existence results for solutions of above-mentioned problems. These existence results either improve or extend known results in the literature. We also considered systems of vector quasi-saddle point problems (SVQSPP) and systems of quasi-minimax inequalities (SQMI). As applications of our existence results for solutions of SSGVQEP, we proved existence of solutions of SVQSPP and SQMI. We gave another application of our results to establish existence of a solution of Debreu VEP.

Because of the applications to vector optimization, game theory and economics, saddle point problems for vector-valued functions, the theory of (vector) equilibrium problems is emerged as a new direction for the researchers; see the references in this chapter.

In this survey chapter, we present systems of various kinds of vector quasi-equilibrium problems and give existence theory for their solutions and some applications to systems of (quasi-) vector optimization problems, systems of quasi-saddle point problems for vector-valued functions and Debreu VEP. The investigations of this chapter are based on our papers [4, 5, 10, 12].

2 Preliminaries

Throughout the chapter, we use the following notations. Let A be a nonempty subset of a topological vector space $\mathscr{X}$, we denote by int A, $\overline{A}$, co$A$ and $\overline{\text{co}}A$, the interior of A in $\mathscr{X}$, the closure of A in $\mathscr{X}$, the convex hull of A, and the closed convex hull of A, respectively. The family of all subsets of A is denoted by 2^A. If X and Y are topological vector spaces, then $L(X,Y)$ denotes the family of all continuous linear maps from X to Y.

Definition 1 ([26, 27]). Let $\mathscr{X}$ and $\mathscr{Y}$ be topological spaces. A multivalued map $T : \mathscr{X} \to 2^{\mathscr{Y}}$ is called *upper semicontinuous at* $x_0 \in \mathscr{X}$ if for any open set $V \subseteq \mathscr{Y}$ containing $T(x_0)$, there exists an open neighbourhood U of x_0 in $\mathscr{X}$ such that $T(x) \subseteq V$ for all $x \in U$.

T is called *lower semicontinuous at* $x \in \mathscr{X}$ if for any open set $V \subseteq \mathscr{Y}$ such that $V \cap T(x_0) \neq \emptyset$, there exists an open neighbourhood U of x_0 in $\mathscr{X}$ such that $T(x) \cap V \neq \emptyset$ for all $x \in U$.

It is said to be *upper (lower) semicontinuous on* $\mathscr{X}$ if it is upper (lower) semi-continuous at every point $x \in \mathscr{X}$.

Further, T is said to be *continuous* on X if it is upper semicontinuous as well as lower semicontinuous on X.

Lemma 1 ([26]). *A multivalued map $T : \mathscr{X} \to 2^{\mathscr{Y}}$ is* lower semicontinuous at *$x \in \mathscr{X}$ if and only if for any $y \in T(x)$ and for any $x_n \in \mathscr{X}$ such that $x_n \to x$, there exists $y_n \in T(x_n)$ such that $y_n \to y$.*

Definition 2. Let $\mathscr{X}$ and $\mathscr{Y}$ be two topological spaces. A multivalued map $T : \mathscr{X} \to 2^{\mathscr{Y}}$ is said to be:

(i) *Compact* if there exists a compact subset $\mathscr{K} \subseteq \mathscr{Y}$ such that $T(\mathscr{X}) \subseteq \mathscr{K}$
(ii) *Closed* if its graph $Gr(T) = \{(x,y) \mid x \in \mathscr{X},\ y \in T(x)\}$ is closed in $\mathscr{X} \times \mathscr{Y}$

Lemma 2 ([79]). *Let $(E, \|\cdot\|)$ be a normed vector space and $\mathscr{H}$ be a Hausdorff metric on the collection $\mathscr{CB}(E)$ of all nonempty, closed and bounded subsets of E, induced by a metric d in terms of $d(x,y) = \|x-y\|$, which is defined as*

$$\mathscr{H}(U,V) = \max\left\{\sup_{x\in U}\inf_{y\in V}\|x-y\|, \sup_{y\in V}\inf_{x\in U}\|x-y\|\right\},$$

for all $U,V \in \mathscr{CB}(E)$. If U and V are compact sets in E, then for all $x \in U$, there exists $y \in V$ such that

$$\|x-y\| \leq \mathscr{H}(U,V).$$

Definition 3 ([79]). Let (E,d) be a metric space and $\mathscr{H}$ be a Hausdorff metric on $\mathscr{CB}(E)$. A multivalued map $T : E \to \mathscr{CB}(E)$ is said to be *continuous* (in the sense of Nadler) on E if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,y \in E$

$$\mathscr{H}(T(x),T(y)) < \varepsilon \quad \text{whenever} \quad d(x,y) < \delta.$$

Remark 1. The notions of continuity in the sense of Definitions 1 and 3 are equivalent if T is compact valued.

Definition 4 ([101]). Let Ω be a nonempty convex subset of a normed space $(E,\|\cdot\|)$ and Υ be a normed linear space. A nonempty compact-valued multifunction $T : \Omega \to 2^{L(E,\Upsilon)}$ is said to be $\mathscr{H}$*-hemicontinuous* if for any $x,y \in \Omega$, the mapping $\alpha \mapsto \mathscr{H}(T(x+\alpha(y-x),T(x)))$ is continuous at 0^+, where $\mathscr{H}$ is the Hausdorff metric defined on $\mathscr{CB}(E)$.

Definition 5 ([88, 89]). Let $\mathscr{E}$ be a Hausdorff topological vector space and L a lattice with least element, denoted by $\mathbf{0}$. A mapping $\Phi : 2^{\mathscr{E}} \to L$ is called a *measure of noncompactness* provided that the following conditions hold for any $M,N \in 2^{\mathscr{E}}$:

(i) $\Phi(M) = \mathbf{0}$ if and only if M is precompact (i.e., it is relatively compact).
(ii) $\Phi(\overline{\text{conv}}M) = \Phi(M)$, where $\overline{\text{conv}}M$ denotes the closed convex hull of M.
(iii) $\Phi(M \cup N) = \max\{\Phi(M),\Phi(N)\}$.

It follows from (iii) that if $M \subseteq N$, then $\Phi(M) \leq \Phi(N)$.

Definition 6 ([88, 89]). Let $\Phi : 2^{\mathscr{E}} \to L$ be a measure of noncompactness on $\mathscr{E}$ and $D \subseteq \mathscr{E}$. A multivalued map $T : D \to 2^{\mathscr{E}}$ is called Φ*-condensing* provided that if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$ then M is relatively compact.

Remark 2. Note that every multivalued map defined on a compact set is necessarily Φ-condensing. If $\mathscr{E}$ is locally convex, then a compact multivalued map (i.e., $T(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : D \to 2^{\mathscr{E}}$ is Φ-condensing and if $S : D \to 2^{\mathscr{E}}$ satisfies $S(x) \subseteq T(x)$ for all $x \in D$, then S is also Φ-condensing.

The following maximal element theorem for a family of multivalued maps is a main tool to study systems of vector quasi-equilibrium problems and their generalizations.

Theorem 1 ([39, 69]). *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. Let $K = \prod_{i\in I} K_i$. For each $i \in I$, let $S_i,T_i : K \to 2^{K_i}$ be multivalued maps satisfying the following conditions:*

(i) *For each $i \in I$ and for all $x \in K$, $\text{co}S_i(x) \subseteq T_i(x)$, where $\text{co}S_i(x)$ denotes the convex hull of $S_i(x)$.*

(ii) *For each* $i \in I$ *and for all* $x = (x_i)_{i\in I} \in K$, $x_i \notin T_i(x)$, *where* x_i *is the ith component of* x.
(iii) *For each* $i \in I$ *and for all* $y_i \in K_i$, $S_i^{-1}(y_i) = \{x \in K : y_i \in S_i(x)\}$ *is open in* K.
(iv) *There exist a nonempty compact subset* M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exists* $i \in I$ *such that* $S_i(x) \cap N_i \neq \emptyset$.

Then there exists $\bar{x} \in K$ *such that* $S_i(\bar{x}) = \emptyset$ *for all* $i \in I$.

Remark 3. If for each $i \in I$, K_i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, then condition (iv) of Theorem 1 can be replaced by the following condition:

(iv)′ The multivalued map $S : K \to 2^K$ defined as $S(x) := \prod_{i\in I} S_i(x)$ for all $x \in K$, is Φ-condensing.

(See Corollary 4 in [29]).

Let $\mathscr{Z}$ be a topological vector space and P a closed convex cone in $\mathscr{Z}$ with $\operatorname{int} P \neq \emptyset$. Then, P induces the vector ordering in $\mathscr{Z}$ by setting, $\forall\, x, y \in P$,

$$x \leq_P y \quad \Leftrightarrow \quad y - x \in P;$$

$$x \not\leq_P y \quad \Leftrightarrow \quad y - x \notin P.$$

Since $\operatorname{int} P \neq \emptyset$, we also have the weak ordering in $\mathscr{Z}$ by setting, $\forall\, x, y \in P$,

$$x <_P y \quad \Leftrightarrow \quad y - x \in \operatorname{int} P;$$

$$x \not<_P y \quad \Leftrightarrow \quad y - x \notin \operatorname{int} P.$$

The ordering $\geq_P$, $\not\geq_P$, $>_P$, $\not>_P$ are defined similarly. A cone P is called *pointed* if $P \cap (-P) = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero element of $\mathscr{Z}$.

Definition 7 ([28, 76, 94]). Let $\mathscr{M}$ be a nonempty subset of a topological vector space $\mathscr{E}$, and let $\mathscr{Z}$ be a topological vector space with a proper, closed and convex cone P with apex at the origin and $\operatorname{int} P \neq \emptyset$. A vector-valued function $\phi : \mathscr{M} \to \mathscr{Z}$ is said to be *P-lower semicontinuous* (respectively, *P-upper semicontinuous*) *at* $x_0 \in \mathscr{M}$ if and only if for any neighbourhood V of $\phi(x_0)$ in $\mathscr{Z}$, $\exists$ a neighbourhood U of x_0 in $\mathscr{E}$ such that

$$\phi(x) \in V + P, \quad \forall\, x \in U \cap \mathscr{M}$$

$$(\text{respectively,} \quad \phi(x) \in V - P, \quad \forall\, x \in U \cap \mathscr{M}).$$

Furthermore, ϕ is *P-lower semicontinuous* (respectively, *P-upper semicontinuous*) *on* $\mathscr{M}$ if and only if it is P-lower semicontinuous (respectively, P-upper semicontinuous) at each $x \in \mathscr{M}$.

ϕ is *P-continuous on* $\mathscr{M}$ if and only if it is both P-lower semicontinuous and P-upper semicontinuous on $\mathscr{M}$.

Remark 4. In [28], it is shown that a function $\phi : \mathscr{M} \to \mathscr{Z}$ is P-lower semicontinuous if and only if $\forall\ \alpha \in \mathscr{Z}$, the set

$$L(\alpha) := \{x \in \mathscr{M} : \phi(x) - \alpha \notin \text{int}\, P\}$$

is closed in $\mathscr{M}$.

Similarly, we can show that ϕ is P-upper semicontinuous if and only if $\forall\ \alpha \in \mathscr{Z}$, the set

$$U(\alpha) := \{x \in \mathscr{M} : \phi(x) - \alpha \notin -\text{int}\, P\}$$

is closed in $\mathscr{M}$.

Definition 8 ([28,43,76]).[1] Let $(\mathscr{Z}, P)$ be an ordered topological vector space and $\mathscr{K}$ a nonempty convex subset of a vector space $\mathscr{X}$. A map $\phi : \mathscr{K} \to \mathscr{Z}$ is said to be:

(i) *P-convex* if $\forall\ x,y \in \mathscr{K}$ and $t \in [0,1]$, we have

$$\phi(tx + (1-t)y) \leq_P t\phi(x) + (1-t)\phi(y).$$

(ii) *Properly P-quasiconvex* if $\forall\ x,y \in \mathscr{K}$ and $t \in [0,1]$, we have either

$$\phi(tx + (1-t)y) \leq_P \phi(x)$$

or

$$\phi(tx + (1-t)y) \leq_P \phi(y).$$

(iii) *Properly P-quasiconcave* if $-\phi$ is properly quasiconvex.

(iv) *Natural P-quasiconvex* (or *natural P-quasifunction*) if $\forall\ x,y \in \mathscr{K}$ and $\forall\ t \in [0,1]$,

$$\phi(tx + (1-t)y) \in \text{co}\{\phi(x), \phi(y)\} - P.$$

(v) *P-quasiconvex* (or *P-quasifunction*) if $\forall\ \alpha \in \mathscr{Z}$, the set $\{x \in \mathscr{K} : \phi(x) - \alpha \in -P\}$ is convex.

Remark 5. (a) Every P-convex function is natural P-quasiconvex and every natural P-quasiconvex function is P-quasiconvex, but converse assertions are not true; see, for example, Remark 2.1 in [95].

(b) ϕ is a natural P-quasiconvex function if and only if $\forall\ x,y \in \mathscr{K}$ and $\forall\ t \in [0,1]$, $\exists\ s \in [0,1]$ such that

$$\phi(tx + (1-t)y) \in s\phi(x) + (1-s)\phi(y) - P.$$

(c) If ϕ is a P-quasiconvex function, then the set $\{x \in \mathscr{K} : \phi(x) - \alpha \in -\text{int}\, P\}$ is also convex for all $\alpha \in \mathscr{Z}$.

[1] The terms P-convex, natural P-quasiconvex and P-quasiconvex are used in [28,43,76] instead of P-function, natural P-quasifunction and P-quasifunction which are suggested by Prof. F. Giannessi.

Example 1. Let $\mathscr{K} = [0,1]$, $\mathscr{Z} = \mathbb{R}^2$, $P = \mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0,\ y_2 \geq 0\}$ and define a function $\phi : \mathscr{K} \to \mathscr{Z}$ by $\phi(x) = \left(x^2, 1 - x^2\right)$. Then, the function ϕ is continuous and natural P-quasiconvex, but neither P-convex nor properly P-quasiconvex.

Example 2. Let $\mathscr{K}, \mathscr{Z}, P$ be the same as in Example 1. We define functions $\xi : \mathscr{K} \to \mathscr{Z}$ by

$$\xi(x) = \left(\cos\left(\frac{\pi x}{2}\right), \sin\left(\frac{\pi x}{2}\right)\right)$$

and the function $\tau : \mathscr{K} \to \mathscr{Z}$ by

$$\tau(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Then, the function ξ is continuous and P-quasiconvex, but not natural P-quasiconvex, and the function τ is continuous, but not natural P-quasiconvex and hence, not P-convex.

Throughout the chapter, all topological spaces are assumed to be Hausdorff.

3 System of Vector Quasi-equilibrium Problems

Throughout the chapter, unless otherwise specified, we use the following notations. Let I be any index set (countable or uncountable). For each $i \in I$, let X_i be a Hausdorff topological vector space and K_i be a nonempty convex subset of X_i. We set $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$ and $K^i = \prod_{j\in I,\ j\neq i} K_j$, and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we also write $x = (x^i, x_i)$. For each $i \in I$, let Y_i be a topological vector space and $C_i : K \to 2^{Y_i}$ be a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$. For each $i \in I$, let $P_i = \bigcap_{x\in K} C_i(x)$. For each $i \in I$, we denote by $L(X_i, Y_i)$ the space of all continuous linear operators from X_i into Y_i. We denote by $\langle s_i, x_i\rangle$ the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$. We also assume that $\forall i \in I$, $A_i : K \to 2^{K_i}$ is a multivalued map such that $\forall\, x \in K$, $A_i(x)$ is nonempty and convex, $A_i^{-1}(y_i)$ is open in K $\forall\, y_i \in K_i$ and the set $\mathscr{F}_i := \{x \in K : x_i \in A_i(x)\}$ is closed in K, where x_i is the ith component of x.

We consider the following *system of vector quasi-equilibrium problems* (SVQEP) [5], that is, to find $\bar{x} \in K$ such that for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \ \ : \ \ f_i(\bar{x}, y_i) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

If for each $i \in I$, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+$ $\ \forall\, x \in K$, then SVQEP is known as a *system of quasi-equilibrium problems*; see [9, 98] and the references therein.

If for each $i \in I$ and $\forall x \in K$, $A_i(x) = K_i$ and $C_i(x) = P_i$ a fixed proper closed convex cone with nonempty interior, then SVQEP reduces to a *system of vector equilibrium problems* (SVEP) [14] of finding $\bar{x} \in K$ such that for each $i \in I$,

$$f_i(\bar{x}, y_i) \notin -\operatorname{int} P_i, \quad \forall\, y_i \in K_i.$$

If the index set I is singleton, then SVQEP becomes a vector quasi-equilibrium problem [24] which contains vector quasi-optimization problems, vector quasi-variational inequality problems, vector quasi-variational-like inequality problems and vector quasi-saddle point problems as special cases.

Examples of SVQEP

(1) For each $i \in I$, let $T_i : K \to L(X_i, Y_i)$ and $\eta_i : K_i \times K_i \to X_i$ be two maps. If for each $i \in I$,

$$f_i(x, y_i) = \langle T_i(x), \eta_i(y_i, x_i) \rangle,$$

then SVQEP is equivalent to the following problem of finding $\bar{x} \in K$ such that $\forall\, i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \quad : \quad \langle T_i(\bar{x}), \eta_i(y_i, \bar{x}_i) \rangle \notin -\operatorname{int} C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

It is known as a *system of vector quasi-variational-like inequalities* (SVQVLI). When $\eta_i(y_i, x_i) = y_i - x_i$, then SVQVLI is called a *system of vector quasi-variational inequalities* (SVQVI). If for each $i \in I$, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+ \ \forall\, x \in K$, SVQVI is studied in [9, 98].

If for each $i \in I$, $A_i(x) = K_i \ \forall\, x \in K$, SVQVLI and SVQVI reduce to the following system of vector variational-like inequalities and the system of vector variational inequalities, respectively, studied in [14].

The *system of vector variational-like inequalities* (SVVLI): find $\bar{x} \in K$ such that for each $i \in I$,

$$\langle T_i(\bar{x}), \eta(y_i, \bar{x}_i) \rangle \notin -\operatorname{int} C_i(\bar{x}), \quad \text{for all } \ y_i \in K_i.$$

The *system of vector variational inequalities* (SVVI): find $\bar{x} \in K$ such that for each $i \in I$,

$$\langle T_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C_i(\bar{x}), \quad \text{for all } \ y_i \in K_i.$$

If for each $i \in I$, $Y_i = \mathbb{R}$ and $\operatorname{int} C_i(x) = \mathbb{R}_+$, then SVVI becomes the systems of variational inequalities studied in [20, 35, 86].

In case the index set I is a singleton, SVVI reduces to a vector variational inequality first considered in [47]; see also [48] and the references therein.

(2) For each $i \in I$, let $\varphi_i : K \to Y$ be a given function. The *system of vector quasi-optimization problems* (SVQOP) is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\varphi_i(y) - \varphi_i(\bar{x}) \notin -\operatorname{int} C_i(\bar{x}), \quad \text{for all } \ y \in A(\bar{x}).$$

We can choose $y \in K$ in such a way that $y^i = \bar{x}^i$. Then we have *Debreu VEP* also known as *constrained Nash equilibrium problem* for vector-valued functions which is to find $\bar{x} \in K$ such that for each $i \in I$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\operatorname{int} C_i(\bar{x}), \quad \text{for all } \ y_i \in A_i(\bar{x}).$$

For each $i \in I$ and for all $x \in K$, if $A_i(x) = K_i$, then Debreu VEP reduces to the following *Nash equilibrium problem* for vector-valued functions: Find $\bar{x} \in K$ such that $\forall\, i \in I$,

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, y_i \in K_i.$$

It is clear that every solution of SVQOP is also a solution of Debreu VEP but the converse is not true.

Of course, if for each $i \in I$, φ_i is a scalar-valued function, then Debreu VEP is the same as one introduced and studied by Debreu in [38], see also [80–82]. In this case, a large number of papers have already been appeared in the literature; see [9,98] and the references therein.

Section 3.1 deals with the existence theory of solutions of SVEP and SVQEP with or without involving Φ-condensing maps. Consequently, we get some existence results for a solution of SVQVLI. In Sect. 3.2, we first establish an equivalence between SVQVLI and Debreu VEP and then we derive some existence results for a solution of the Debreu VEP for convex or nonconvex functions.

3.1 Existence Results for Solutions of SVEP and SVQEP

We present the following existence results for solutions of SVEP which are established in [14] by utilizing scalarization technique and by using collectively fixed point theorem for a family of multivalued maps [22].

Theorem 2 ([14]). *Let Y be a topological vector space and $C \subset Y$ be a proper, closed convex cone with apex at the origin* $\mathbf{0}$ *and* $\text{int}\, C \neq \emptyset$. *For each $i \in I$, let K_i be a nonempty compact convex subset of X_i and let $f_i : K \times K_i \to Y$ be a bifunction such that $f_i(x, x_i) = \mathbf{0}$ for all $x = (x^i, x_i) \in K$. Assume that the following conditions are satisfied:*

(i) *For each $i \in I$ and $\forall x \in K$, the function $y_i \mapsto f_i(x, y_i)$ is C-quasiconvex.*
(ii) *For each $i \in I$, f_i is continuous on $K \times K_i$.*

Then the solution set of SVEP is nonempty and compact.

In case K_i is not necessarily compact, we have the following result.

Theorem 3 ([14]). *Let Y be a topological vector space and $C \subset Y$ be a proper, closed convex cone with apex at the origin* $\mathbf{0}$ *and* $\text{int}\, C \neq \emptyset$. *For each $i \in I$, let K_i be a nonempty convex subset of X_i and let $f_i : K \times K_i \to Y$ be a bifunction such that $f_i(x, x_i) = \mathbf{0}$ for all $x = (x^i, x_i) \in K$. Assume that the following conditions are satisfied:*

(i) *For each $i \in I$ and $\forall x \in K$, the function $y_i \mapsto f_i(x, y_i)$ is C-quasiconvex.*
(ii) *For each $i \in I$, f_i is continuous on each compact convex subset of $K \times K_i$.*

(iii) *For each $i \in I$, there exists a nonempty compact convex subset B_i of K_i, and let $B = \prod_{i \in I} B_i \subset K$ such that for each $x \in K \setminus B$, there exists $\tilde{y}_i \in B_i$ such that*

$$f_i(x, \tilde{y}_i) \in -\mathrm{int}\, C.$$

Then there exists a solution $\bar{x} \in B$ of SVEP.

Remark 6. Let I be a finite index set and for each $i \in I$, let X_i be a reflexive Banach space with norm $||\cdot||_i$ equipped with the weak topology. Consider a Banach space Y equipped with the norm topology. The norm on $X = \prod_{i \in I} X_i$ will be denoted by $||\cdot||$. Then assumption (iii) in Theorem 3 can be replaced by the following condition:

(iii)$'$ There exists $r > 0$ such that for all $x \in K$, $||x|| > r$, there exists $\tilde{y}_i \in K_i$, $||\tilde{y}_i||_i \leq r$ such that

$$f_i(x, \tilde{y}_i) \in -\mathrm{int}\, C.$$

We present the following existence result for solutions of SVQEP without involving Φ-condensing maps. In [5], we proved this result by using maximal element Theorem 1.

Theorem 4 ([5]). *For each $i \in I$, let K_i be a nonempty and convex subset of a Hausdorff topological vector space X_i and $f_i : K \times K_i \to Y_i$ be a bifunction. Assume that the following conditions hold:*

(i) *For each $i \in I$ and $\forall\, x \in K$, $f_i(x, x_i) \notin -\mathrm{int}\, C_i(x)$, where x_i is the ith component of x.*
(ii) *For each $i \in I$ and $\forall\, x \in K$, the vector-valued function $y_i \mapsto f_i(x, y_i)$ is a natural P_i-quasiconvex function.*
(iii) *For each $i \in I$ and $\forall\, y_i \in K_i$, the set $\{x \in K : f_i(x, y_i) \notin -\mathrm{int}\, C_i(x)\}$ is closed in K.*
(iv) *There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\, i \in I$, such that $\forall\, x \in K \setminus N$ $\exists\, i \in I$ and $\exists\, \tilde{y}_i \in B_i$, such that $\tilde{y}_i \in A_i(x)$ and $f_i(x, \tilde{y}_i) \in -\mathrm{int}\, C_i(x)$.*

Then SVQEP has a solution.

Remark 7. (1) The condition (iii) of Theorem 4 is satisfied if the following conditions hold $\forall\, i \in I$:

(a) The multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\mathrm{int}\, C_i(x)\}$ $\forall\, x \in K$, is closed in $K \times K_i$.
(b) For all $y_i \in K_i$, $f_i(\cdot, y_i) : K \to Y_i$ is continuous (in the usual sense) on K.

(2) If for each $i \in I$ and $\forall\, x \in K$, $C_i(x) = C_i$, a (fixed) proper, closed and convex cone in Y_i, then conditions (ii) and (iii) of Theorem 4 can be replaced, respectively, by the following conditions:

(c) For each $i \in I$ and $\forall\, x \in K$, the vector-valued function $y_i \mapsto f_i(x, y_i)$ is a C_i-quasiconvex function.

(d) For each $i \in I$ and $\forall\, y_i \in K_i$ the vector-valued function $x \mapsto f_i(x, y_i)$ is C_i-upper semicontinuous on K.

(3) Theorem 4 extends and generalizes Theorem 6 in [9], Theorem 2.1 in [14] and Corollary 3.1 in [24] in several ways.

(4) If for each $i \in I$, K_i is a nonempty, compact and convex subset of a Hausdorff topological vector space X_i, then the conclusion of Theorem 4 holds without condition (iv).

We mention the following existence result for a solution of SVQEP involving Φ-condensing maps. We proved this result by using maximal element Theorem 1 with Remark 3.

Theorem 5 ([5]). *For each $i \in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, $f_i : K \times K_i \to Y_i$ be a bifunction and let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)\ \ \forall\, x \in K$, be Φ-condensing. Assume that the conditions (i), (ii) and (iii) of Theorem 4 hold. Then SVQEP has a solution.*

In order to derive the existence results for solutions of systems of vector quasi-variational (-like) inequalities, we define a topology on the space $L(\mathscr{E}, \mathscr{Z})$ by the following way:

Let $\mathscr{E}$ and $\mathscr{Z}$ be Hausdorff topological vector spaces. Let σ be the family of bounded subsets of $\mathscr{E}$ whose union is total in $\mathscr{E}$, that is, the linear hull of $\bigcup\{U : U \in \sigma\}$ is dense in $\mathscr{E}$. Let $\mathscr{B}$ be a neighbourhood base of 0 in $\mathscr{Z}$. When U runs through σ, V through $\mathscr{B}$, the family

$$M(U,V) = \{\xi \in L(\mathscr{E}, \mathscr{Z}) : \cup_{x\in U} \langle \xi, x\rangle \subseteq V\}$$

is a neighbourhood base of 0 in $L(\mathscr{E}, \mathscr{Z})$ for a unique translation-invariant topology, called the *topology of uniform convergence* on the sets $U \in \sigma$, or, briefly, the σ-topology (see [42, pp. 79–80] and also [91]).

Lemma 3 ([42]). *Let $\mathscr{E}$ and $\mathscr{Z}$ be Hausdorff topological vector spaces and $L(\mathscr{E}, \mathscr{Z})$ be the topological vector space under the σ-topology. Then, the bilinear mapping $\langle ., . \rangle : L(\mathscr{E}, \mathscr{Z}) \times \mathscr{E} \to \mathscr{Z}$ is continuous on $L(\mathscr{E}, \mathscr{Z}) \times \mathscr{E}$.*

Throughout the chapter, we assume that $L(\mathscr{E}, \mathscr{Z})$ is equipped with σ-topology.

In addition to the assumptions on $C_i : K \to 2^{Y_i}$, in the following corollaries, we further assume that $C_i(x)$ is pointed, $\forall\, i \in I$ and $\forall\, x \in K$. Then the following results can be easily derived, respectively, from Theorems 4 and 5 by setting

$$f_i(x, y_i) = \langle T_i(x), \eta_i(y_i, x_i)\rangle.$$

Corollary 1 ([5]). *For each $i \in I$, let K_i, X_i and W_i be the same as in Theorem 4 and Remark 7, respectively. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be continuous in the second variable such that $\eta_i(x_i, x_i) = 0\ \ \forall\, x_i \in K_i$, and let $T_i : K \to L(X_i, Y_i)$ be* *continuous on K such that the map* $y_i \mapsto \langle T_i(x), \eta_i(y_i, x_i)\rangle$ *is a natural* P_i*-quasiconvex function,* $\forall\, x \in K$*. Assume that there exist a nonempty and compact subset N of K and a nonempty, compact and convex subset* B_i *of* K_i $\forall\, i \in I$*, such that* $\forall\, x \in K \setminus N$ $\exists\, i \in I$ *and* $\exists\, \tilde{y}_i \in B_i$ *such that* $\tilde{y}_i \in A_i(x)$ *and* $\langle T_i(x), \eta_i(\tilde{y}_i, x_i)\rangle \in -\mathrm{int}\, C_i(x)$*. Then SVQVLI has a solution.*

Corollary 2 ([5]). *For each* $i \in I$*, let* K_i*,* X_i*,* A_i*, A and* η_i*,* T_i*,* W_i*,* $L(X_i, Y_i)$ *be the same as in Theorem 5 and Corollary 1, respectively. Then SVQVLI has a solution.*

Remark 8. To the best of our knowledge, there is only one paper [40] appeared in the literature on the scalar quasi-variational-like inequality problems involving Φ-condensing maps. Since the approach in this chapter is different from the one adopted in [40], Corollary 2 is a new result in the literature, not only for the vector case but also for the scalar one.

3.2 Applications of SVQEP

Let $I = \{1, 2, \ldots, n\}$ be a finite index set and for each $i \in I$, let X_i be a normed space and $X = \prod_{i \in I} X_i$. Let Z be a normed space. We recall the following definition.

Definition 9 ([100]). The function $\phi : X \to Z$ is said to be *partial Gâteaux differentiable at* $x = (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \in X$ w. r. t. the jth variable x_j if

$$\left\langle D_{x_j}\phi(x), h_j \right\rangle = \lim_{t \to 0} \frac{\phi(x_1, \ldots, x_{j-1}, x_j + t h_j, x_{j+1}, \ldots, x_n) - \phi(x)}{t} \quad \text{exists,}$$

for all $h_j \in X_j$. $D_{x_j}\phi(x) \in L(X_j, Z)$ is called the *partial Gâteaux derivative of* ϕ *at* $x \in X$ w.r.t. the jth variable x_j.

ϕ is called *partial Gâteaux differentiable on X* if it is partial Gâteaux differentiable at each point of K w.r.t. each variable.

Definition 10 ([96]). Let E be a normed space, Z a normed space with a closed and convex cone P with apex at the origin, M a nonempty subset of E, $\eta : M \times M \to E$ a function. A Gâteaux differentiable function $\phi : M \to Z$ is said to be *P-invex* w.r.t. η if $\forall\, x, y \in M$,

$$\phi(y) - \phi(x) - \langle D_x\phi(x), \eta(y, x)\rangle \in P,$$

where $D_x\phi(x)$ denotes the Gâteaux derivative of ϕ at x.

Definition 11 ([78]). A subset M of a vector space E is said to be *invex w.r.t.* $\eta : M \times M \to E$ if $\forall\, x, y \in M$ and $\forall\, t \in [0, 1]$, $x + t\eta(y, x) \in M$.

Definition 12 ([96]). Let M be an invex set in a normed space E w.r.t. $\eta : M \times M \to E$. A vector-valued function $\phi : M \to Z$ is said to be *P-preinvex* if $\forall\, x, y \in M$ and $\forall\, t \in [0, 1]$,

$$t\phi(y) + (1 - t)\phi(x) - \phi(x + t\eta(y, x)) \in P.$$

Remark 9. It can be easily seen that if M is an invex subset of E w.r.t. $\eta : M \times M \to E$ and $\phi : M \to Z$ is Gâteaux differentiable on M and P-preinvex, then ϕ is P-invex w.r.t. η. But the converse assertion may not be true.

We have the following result which provides a sufficient condition for a solution of Debreu VEP.

Proposition 1 ([5]). *Let I be a finite index set. For each $i \in I$, let X_i and Y_i be normed spaces, K_i a nonempty and convex subset of X_i, $K = \prod_{i \in I} K_i$, $A_i : K \to 2^{K_i}$ nonempty convex-valued multivalued map, $\eta_i : K_i \times K_i \to X_i$, and $\varphi_i : K \to Y_i$ partial Gâteaux differentiable on each open subset of K and P_i-invex w.r.t. η_i in each argument. Then every solution of SVQVLI with $T_i(x) = D_{x_i}\varphi_i(x)$ is also a solution of Debreu VEP.*

Proof. Assume that $\bar{x} \in K$ is a solution of SVQVLI with $T_i(x) = D_{x_i}\varphi_i(x)$. Then for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \quad : \quad \langle D_{x_i}\varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}). \tag{1}$$

Since for each $i \in I$, φ_i is P_i-invex w.r.t. η_i in each argument, we have

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) - \langle D_{x_i}\varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \in P_i \subseteq C_i(\bar{x}). \tag{2}$$

Since $a - b \in P$ and $b \notin -\text{int}\, P \Rightarrow a \notin -\text{int}\, P$, it follows from (1) and (2) that

$$\bar{x}_i \in A_i(\bar{x}) \quad : \quad \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

Hence $\bar{x} \in K$ is a solution of Debreu VEP.

The next result provides the equivalence between SVQVLI and Debreu VEP.

Proposition 2 ([5]). *Let I be a finite index set. For each $i \in I$, let X_i and Y_i be normed spaces, $K_i \subseteq X_i$ nonempty invex w.r.t. $\eta_i : K_i \times K_i \to X_i$, $K = \prod_{i \in I} K_i$, $A_i : K \to 2^{K_i}$ nonempty invex valued multivalued map and $\varphi_i : K \to Y_i$ partial Gâteaux differentiable on each open subset of K and P_i-preinvex in each argument. Then $\bar{x} \in K$ is a solution of SVQVLI with $T_i(x) = D_{x_i}\varphi_i(x)$ if and only if it is a solution of Debreu VEP.*

Proof. Assume that $\bar{x} \in K$ is a solution of SVQVLI. Then by Proposition 1, $\bar{x} \in K$ is a solution of Debreu VEP.

Conversely, let $\bar{x} \in K$ be a solution of Debreu VEP. Then for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \quad : \quad \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}). \tag{3}$$

Since $\bar{x}_i, y_i \in A_i(\bar{x})$ and each $A_i(\bar{x})$ is invex, we have $\bar{x}_i + t\eta_i(y_i, \bar{x}_i) \in A_i(\bar{x})\ \forall\, t \in [0,1]$. Therefore, from (3), we get

$$\varphi_i(\bar{x}^i, \bar{x}_i + t\eta_i(y_i, \bar{x}_i)) - \varphi_i(\bar{x}) \in W_i(\bar{x}) = Y_i \setminus \{-\text{int}\, C_i(\bar{x})\}.$$

Since for each $i \in I$, $W_i(\bar{x})$ is a closed cone, we have

$$\lim_{t \to 0} \frac{\varphi_i(\bar{x}^i, \bar{x}_i + t\eta_i(y_i, \bar{x}_i)) - \varphi_i(\bar{x})}{t} \in W_i(\bar{x}).$$

From the partial Gâteaux differentiability of each φ_i, we get, $\forall\, i \in I$

$$\bar{x}_i \in A_i(\bar{x}) \quad : \quad \langle D_{x_i}\varphi_i(\bar{x}), \eta_i(y_i, \bar{x}_i)\rangle \notin -\mathrm{int}\, C_i(\bar{x}), \quad \forall\, y_i \in A_i(\bar{x}).$$

Hence $\bar{x} \in K$ is a solution of SVQVLI with $T_i(\bar{x}) = D_{x_i}\varphi_i(\bar{x})$ $\forall\, i \in I$.

Remark 10. If for each $i \in I$ and $\forall\, x \in K$, $\eta_i(y_i, x_i) = y_i - x_i$, $A_i(x) = K_i$, $C_i(x) = \mathbb{R}_+$ and $Y_i = \mathbb{R}$, then Proposition 2 reduces to Proposition 4 in [25, p. 269]. Hence Proposition 2 extends Proposition 4 in [25] in several ways.

By using Proposition 1 and Corollary 1, we can easily derive the following existence result for a solution of Debreu VEP.

Theorem 6 ([5]). *Let I be a finite index set. For each $i \in I$, let X_i and Y_i be normed spaces, K_i be a nonempty convex subset of X_i, $K = \prod_{i \in I} K_i$ and W_i be the same as in Remark 7. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ $\forall\, x_i \in K_i$, and $\varphi_i : K \to Y_i$ partial Gâteaux differentiable on K and P_i-invex in each variable such that the function $y_i \mapsto \langle D_{x_i}\varphi_i(x), \eta_i(y_i, x_i)\rangle$ is a natural P_i-quasiconvex function, $\forall\, x \in K$. Assume that there exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\, i \in I$, such that $\forall\, x \in K \setminus N$ $\exists\, i \in I$ and $\exists\, \tilde{y}_i \in B_i$, such that $\tilde{y}_i \in A_i(x)$ and $\langle D_{x_i}\varphi_i(x), \eta_i(\tilde{y}_i, x_i)\rangle \in -\mathrm{int}\, C_i(x)$. Then the Debreu VEP has a solution.*

If the index set I need not be finite and for each $i \in I$, φ_i need not be partial Gâteaux differentiable, then we can also easily derive the following existence results for a solution of Debreu VEP from Theorems 4 and 5 by setting, $\forall\, i \in I$,

$$f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x).$$

Theorem 7 ([5]). *For each $i \in I$, let K_i, K, X_i and W_i be the same as in Theorem 4 and Remark 7, respectively, and let $\varphi_i : K \to Y_i$ be a vector-valued function. Assume that the following conditions hold:*

(i) *For each $i \in I$, φ_i is a natural P_i-quasiconvex function in the ith argument.*
(ii) *For each $i \in I$, φ_i is continuous on K.*
(iii) *There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i $\forall\, i \in I$, such that $\forall\, x \in K \setminus N$ $\exists\, i \in I$ and $\exists\, \tilde{y}_i \in B_i$, such that $\tilde{y}_i \in A_i(x)$ and $\varphi_i(x^i, \tilde{y}_i) - \varphi_i(x) \in -\mathrm{int}\, C_i(x)$.*

Then Debreu VEP has a solution.

Theorem 8 ([5]). *For each $i \in I$, let K_i, K, X_i and W_i be the same as in Theorems 5 and 7, respectively. Assume that the conditions (i) and (ii) of Theorem 5 hold. Then Debreu VEP has a solution.*

Remark 11. (1) If for each $i \in I$ and $\forall\, x \in K$, $C_i(x) = C_i$, a (fixed) proper, closed and convex cone in Y_i, then conditions (i) and (ii) in Theorem 5, and subsequently, in Theorem 6 can be replaced, respectively, by the following conditions:

(i)′ For each $i \in I$ and $\forall\, x \in K$, φ_i is a C_i-quasiconvex function in the ith argument.
(ii)′ For each $i \in I$, φ_i is C_i-upper semicontinuous on K.

(2) Theorem 6 provides the existence of a solution of Debreu VEP involving Φ-condensing map and, consequently, for scalar-valued functions. Therefore, Theorem 6 is a new result in the literature.

4 System of Generalized Vector Quasi-equilibrium Problems

For each $i \in I$, let $F_i : K \times K_i \to 2^{Y_i}$ and $A_i : K \to 2^{K_i}$ be multivalued maps with nonempty values. We consider the following *system of generalized vector quasi-equilibrium problems [10]:*

$$\text{(SGVQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I,\ \bar{x}_i \in A_i(\bar{x}) \quad \text{and} \\ F_i(\bar{x}, y_i) \not\subseteq -\text{int}\, C_i(\bar{x}), \quad \forall y_i \in A_i(\bar{x}). \end{cases}$$

If for each $i \in I$ and $\forall\, x \in K$, $A_i(x) = K_i$, then SGVQEP reduces to the following *system of generalized vector equilibrium problems* (SGVEP) [15]:

$$\text{(SGVEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \\ F_i(\bar{x}, y_i) \not\subseteq -\text{int}\, C_i(\bar{x}), \quad \forall y_i \in K_i. \end{cases}$$

It is introduced and studied in [15] with applications to the Nash equilibrium problem for vector-valued functions.

If I is a singleton set, then SGV(Q)EP reduces to a generalized vector (quasi-) equilibrium problem which contains generalized implicit vector (quasi-) variational inequality problems, generalized vector (quasi-) variational inequality problems, generalized vector (quasi-) variational-like inequality problems and vector (quasi-) equilibrium problems as special cases. For further detail on generalized vector (quasi-) equilibrium problems and their applications, we refer [7, 8, 11, 13, 18, 21, 46, 84, 85, 93, 101] and the references therein.

Examples of SGVQEP

For each $i \in I$, let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, let $T_i : K \to 2^{D_i}$ be a multivalued map with nonempty values. For each $i \in I$, let $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. The problem of *system of generalized implicit vector quasi-variational inequalities* (SGIVQVIP) is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \ \exists \bar{u}_i \in T_i(\bar{x}) \ : \ \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int}\, C_i(\bar{x}).$$

Setting for each $i \in I$,

$$F_i(x,y_i) = \psi_i(T_i(x),x_i,y_i) = \{\psi_i(u_i,x_i,y_i) : u_i \in T_i(x)\}.$$

Then SGVQEP coincides with SGIVQVIP.

For $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ for all $x \in K$ and for each $i \in I$, SGIVQVIP is called the *problem of system of generalized implicit quasi-variational inequalities*. Further, for all $x \in K$ and for each $i \in I$, $A_i(x) = K_i$, it is called the *problem of system of generalized implicit variational inequalities*. Such problem is studied in [22] with application to Nash equilibrium problem [81].

If I is a singleton set, SGIVQVIP reduces to *generalized implicit vector quasi-variational inequality problem*.

The SGIVQVIP contains the following problems as special cases:

(i) For each $i \in I$, let $\theta_i : K \times D_i \to D_i$ and $\eta_i : K_i \times K_i \to X_i$ be bifunctions. If for each $i \in I$,

$$\psi_i(T_i(x),x_i,y_i) = \langle \theta_i(x,T_i(x)), \eta_i(y_i,x_i)\rangle = \{\langle \theta_i(x,u_i), \eta_i(y_i,x_i)\rangle : u_i \in T_i(x)\},$$

then SGIVQVIP reduces to the *problem of system of generalized vector quasi-variational-like inequalities* (SGVQVLIP) (I) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}),\ \ \exists \bar{u}_i \in T_i(\bar{x}) \ \ : \ \ \langle \theta_i(\bar{x},\bar{u}_i), \eta_i(y_i,\bar{x}_i)\rangle \notin -\text{int}\, C_i(\bar{x}).$$

If I is a singleton set, then SGVQVLIP(I) becomes the *generalized vector quasi-variational-like inequality problem*. The strong solution (i.e., $\bar{u}_i$ does not depend on y_i) of SGVQVLIP(I) is studied by Chen et al. [31] and Lee et al. [65], see also the references therein.

If for each $i \in I$, $\theta_i(x,u_i) = u_i$ for all $x \in K$, then SGVQVLIP(I) becomes the following problem denoted by SGVQVLIP(II): Find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}),\ \ \exists \bar{u}_i \in T_i(\bar{x}) \ \ : \ \ \langle \bar{u}_i, \eta_i(y_i,\bar{x}_i)\rangle \notin -\text{int}\, C_i(\bar{x}).$$

For $Y_i = \mathbb{R}$, $C_i(x) = \mathbb{R}_-$ and $A_i(x) = K_i$ for all $x \in K$ and for each $i \in I$, this problem is studied in [22] with application to the Nash equilibrium problem [81].

(ii) If for each $i \in I$,

$$\psi_i(T_i(x),x_i,y_i) = \langle T_i(x), y_i - x_i\rangle = \{\langle u_i, y_i - x_i\rangle : u_i \in T_i(x)\},$$

then SGIVQVIP reduces to the *problem of system of generalized vector quasi-variational inequalities* (SGVQVIP) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}),\ \ \exists \bar{u}_i \in T_i(\bar{x}) \ \ : \ \ \langle \bar{u}_i, y_i - \bar{x}_i\rangle \notin -\text{int}\, C_i(\bar{x}).$$

For each $i \in I$, if F_i is a single-valued map, then SGVQEP reduces to SVQEP (Sect. 3).

4.1 Existence Results for Solutions of SGVQEP

The following results provide the existence of a solution of SGVQEP with or without Φ-condensing maps.

Theorem 9 ([10]). *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and let $F_i : K \times K_i \to 2^{Y_i}$ be a multivalued map with nonempty values. For each $i \in I$, assume that the following conditions hold:*

(i) *For all $x \in K$, $F_i(x, x_i) \not\subseteq -\operatorname{int} C_i(x)$, where x_i is the ith component of x.*
(ii) *For all $x \in K$, the set $\{y_i \in K_i : F_i(x, y_i) \subseteq -\operatorname{int} C_i(x)\}$ is convex.*
(iii) *For all $y_i \in K_i$, the set $\{x \in K : F_i(x, y_i) \not\subseteq -\operatorname{int} C_i(x)\}$ is closed in K.*
(iv) *There exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $F_i(x, \tilde{y}_i) \subseteq -\operatorname{int} C_i(x)$.*

Then the SGVQEP has a solution.

Theorem 10 ([10]). *For each $i \in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, $F_i : K \times K_i \to 2^{Y_i}$ a multivalued map with nonempty values and let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i)–(iii) of Theorem 9 hold. Then the SGVQEP has a solution.*

In order to verify condition (ii) in Theorems 9 and 10, we introduce the following concept.

Definition 13 ([21]). Let W and Z be topological vector spaces and M be a nonempty convex subset of W and let $P : M \to 2^Z$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. For each fixed $x \in M$, a multivalued map $F : M \times M \to 2^Z \setminus \{\emptyset\}$ is called $P(x)$-*quasiconvex-like* if for all $y_1, y_2 \in M$ and $t \in [0, 1]$, we have either

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - P(x),$$

or

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_2) - P(x).$$

To show the class of $P(x)$-quasiconvex-like multivalued is nonempty, we give the following example.

Example 3. Let $M = [0, 1]$, $P(x) = [0, +\infty)$ for all $x \in M$. We define $F : M \times M \to 2^{\mathbb{R}}$ by

$$F(x, y) = [x, y+1] \quad \text{for all } x, y \in M.$$

For all $x, y_1, y_2 \in M$ and $0 \le t \le 1$, we note that

$$\text{if } y_1 \le y_2 \quad \text{then } ty_1 + (1-t)y_2 \le y_2$$

and

$$\text{if } y_1 > y_2 \quad \text{then } ty_1 + (1-t)y_2 \le y_1.$$

Therefore, we have for each $\alpha \in F(x, ty_1 + (1-t)y_2)$,

$$\alpha = \begin{cases} (y_2+1) - [(y_2+1) - \alpha], & y_1 \le y_2, \\ (y_1+1) - [(y_1+1) - \alpha], & y_1 > y_2. \end{cases}$$

Hence, we have either $F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - P(x)$ or $F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_2) - P(x)$. Thus, F is $P(x)$-quasiconvex-like.

Remark 12. (a) If for each $i \in I$, F_i is $C_i(x)$-quasiconvex-like, then the set $\{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int}\, C_i(x)\}$ is convex, for all $x \in K$ (see, e.g., the proof of Theorem 2.1 in [21]).

(b) If for each $i \in I$, X_i is locally convex Hausdorff topological vector space, the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int}\, C_i(x)\}$ for all $x \in K$, is closed on K and for all $y_i \in K_i$, $F_i(.,y_i)$ is upper semicontinuous on K, then condition (iii) of Theorem 9 is satisfied; see, for example, the proof of Theorem 2.1 in [21].

In order to establish existence results for a solution of SGIVQVIP, we modify the definition of $P(x)$-quasiconvex-like multivalued bifunction to a single-valued trifunction.

Definition 14 ([11]). Let W and Z be topological vector spaces, M a nonempty convex subset of W and D a nonempty subset of $L(W,Z)$. Let $T : M \to 2^D \setminus \{\emptyset\}$ and $P : M \to 2^Z$ be multivalued maps such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. For each fixed $x \in M$, a function $\psi : D \times M \times M \to Z$ is called $P(x)$-*quasiconvex-like* if for all $y_1, y_2 \in M$ and $t \in [0,1]$, we have either for all $u \in T(x)$,

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_1) - P(x),$$

or

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_2) - P(x).$$

From Theorems 9 and 10, we derive the following existence result for a solution of SGIVQVIP.

Corollary 3 ([10]). *For each $i \in I$, let K_i be a nonempty convex subset of a locally convex topological vector space X_i and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, $T_i : K \to 2^{D_i}$ be an upper semicontinuous multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. For each $i \in I$, assume that:*

(i) *The multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int}\, C_i(x)\}$ for all $x \in K$, is closed on K.*
(ii) *For all $x \in K$ and $u_i \in T_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\text{int}\, C_i(x)$, where x_i is the ith component of x.*
(iii) *ψ_i is $C_i(x)$-quasiconvex-like.*

(iv) *For all* $y_i \in K_i$, *the map* $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ *is upper semicontinuous on* $D_i \times K_i$.
(v) *There exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *satisfying* $\tilde{y}_i \in A_i(x)$ *and* $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int}\, C_i(x)$ *for all* $u_i \in T_i(x)$.

Then the SGIVQVIP has a solution.

Corollary 4 ([10]). *For each* $i \in I$, *let* $K_i, X_i, D_i, \psi_i, T_i$ *and* W_i *be the same as in Corollary 3 and let the multivalued map* $A = \prod_{i\in I} A_i : K \to 2^K$ *defined as* $A(x) = \prod_{i\in I} A_i(x)$ *for all* $x \in K$, *be* Φ*-condensing. Assume that the conditions (i)–(iv) of Corollary 3 hold. Then the SGIVQVIP has a solution.*

We derive the existence results for a solution of SGVQVLIP by using Corollaries 3 and 4.

Corollary 5 ([10]). *For each* $i \in I$, *let* Y_i *be a Hausdorff topological vector space and let* K_i, X_i, D_i, T_i *and* W_i *be the same as in Corollary 3. For each* $i \in I$, *let* $\eta_i : K_i \times K_i \to X_i$ *be affine in the first argument and continuous in the second argument such that* $\eta_i(x_i, x_i) = 0$ *for all* $x_i \in K_i$. *Assume that there exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *satisfying* $\tilde{y}_i \in A_i(x)$ *and* $\langle u_i, \eta_i(\tilde{y}_i, x_i)\rangle \in -\text{int}\, C_i(x)$ *for all* $u_i \in T_i(x)$. *Then the SGVQVLIP has a solution.*

Corollary 6 ([10]). *For each* $i \in I$, *let* $K_i, X_i, Y_i, D_i, \eta_i, T_i$ *and* W_i *be the same as in Corollary 4. For each* $i \in I$, *let* $\eta_i : K_i \times K_i \to X_i$ *be affine in the first argument and continuous in the second argument such that* $\eta_i(x_i, x_i) = 0$ *for all* $x_i \in K_i$. *Let the multivalued map* $A = \prod_{i\in I} A_i : K \to 2^K$ *defined as* $A(x) = \prod_{i\in I} A_i(x)$ *for all* $x \in K$, *be* Φ*-condensing. Then SGVQVIP has a solution.*

The following results provide the existence of a solution of SGVQVIP with or without Φ-condensing maps.

Corollary 7 ([10]). *For each* $i \in I$, *let* K_i, X_i, Y_i, D_i, T_i *and* W_i *be the same as in Corollary 4. Assume that there exist a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B_i *of* K_i *for each* $i \in I$ *such that for each* $x \in K \setminus N$ *there exist* $i \in I$ *and* $\tilde{y}_i \in B_i$ *satisfying* $\tilde{y}_i \in A_i(x)$ *and* $\langle u_i, \tilde{y}_i - x_i\rangle \in -\text{int}\, C_i(x)$ *for all* $u_i \in T_i(x)$. *Then the SGVQVIP has a solution.*

Corollary 8 ([10]). *For each* $i \in I$, *let* K_i, X_i, Y_i, D_i, T_i *and* W_i *be the same as in Corollary 4. Let the multivalued map* $A = \prod_{i\in I} A_i : K \to 2^K$ *defined as* $A(x) = \prod_{i\in I} A_i(x)$ *for all* $x \in K$, *be* Φ*-condensing. Then the SGVQVIP has a solution.*

4.2 Applications

Throughout this section, unless otherwise specified, we assume that the index set I is finite, that is, $I = \{1, \ldots, n\}$. For each $i \in I$, let X_i and Y_i be finite dimensional

Euclidean spaces $\mathbb{R}^{p_i}$ and $\mathbb{R}^{q_i}$, respectively, and K_i be a nonempty convex subset of X_i. Let $K = \prod_{i\in I} K_i$. Let $K = \prod_{i=1}^{n} K_i$. For each $i \in I$, let $C_i : K \to 2^{Y_i}$ be a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$ and $\mathbb{R}_+^{q_i} \subseteq C_i(x)$. Let the multivalued map $A = \prod_{i\in I} A_i :$ $K \to 2^K$ be defined as $A(x) = \prod_{i\in I} A_i(x)$, for all $x \in K$. For each $i \in I$, let $\varphi_i : K \to Y_i$ be a given vector-valued function. We recall the following SVQOP which is to find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\varphi_i(y) - \varphi_i(\bar{x}) \notin -\operatorname{int} C_i(\bar{x}) \quad \forall\, y \in A(\bar{x}),$$

where $\varphi_i(x) = (\varphi_{i_1}(x), \varphi_{i_2}(x), \ldots, \varphi_{i_{q_i}}(x))$ and for each $l \in \mathscr{L} = \{1, \ldots, q_i\}$, $\varphi_{i_l} :$ $K \to \mathbb{R}$ is a function.

As we have seen in Sect. 3 that every solution of SVQOP is also a solution of Debreu VEP, but the converse need not be true.

We recall the following definitions.

Definition 15. A real-valued function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *locally Lipschitz* if for any $z \in \mathbb{R}^p$, there exist a neighbourhood $N(z)$ of z and a positive constant k such that

$$|f(x) - f(y)| \leq k||x - y||, \quad \forall\, x, y \in N(z).$$

The Clarke *generalized directional derivative* [34] of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

The Clarke *generalized gradient* [34] of a locally Lipschitz function f at x is defined as

$$\partial f(x) = \left\{ \xi \in \mathbb{R}^p : f^0(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^p \right\}.$$

If f is convex, then the Clarke generalized gradient coincides with the subdifferential of f in the sense of convex analysis [90].

The generalized invex function was introduced by Craven [36] as a generalization of invex functions [51].

Definition 16. A locally Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *generalized invex at x w.r.t. a given function* $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ if

$$f(y) - f(x) \geq \langle \xi, \eta(y, x) \rangle, \quad \forall\, \xi \in \partial f(x) \text{ and } y \in \mathbb{R}^p.$$

For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be a locally Lipschitz function and let $x \in K$, $x_j \in K_j$. Following Clarke [34], the *generalized directional derivative at x_j in the direction* $d_j \in K_j$ of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ denoted by $\phi_{ij}^0(x; d_j)$ is

$$\phi_{ij}^0(x;d_j) = \lim_{\substack{y_j \to x_j \\ t \downarrow 0}} \sup \frac{1}{t} \Big\{ \phi_i(x_1,\ldots,x_{j-1},y_j+td_j,x_{j+1},\ldots,x_n) - \phi_i(x_1,\ldots,x_{j-1},y_j,x_{j+1},\ldots,x_n) \Big\}.$$

The *partial generalized gradient* [34] of the function $\phi_i(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)$ at x_j is defined as follows:

$$\partial_j \phi_i(x) = \left\{ \xi_j \in X_j : \phi_{ij}^0(x;d_j) \geq \langle \xi_j, d_j \rangle \text{ for all } d_j \in K_j \right\}.$$

Lemma 4 ([34]). *For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be locally Lipschitz. Then for each $i \in I$, the multivalued map $\partial_i \phi_i$ is upper semicontinuous.*

Definition 17. For each $i \in I$, $\phi_i : K \to \mathbb{R}$ is called *generalized invex* at x w.r.t. a given function $\eta_i : K_i \times K_i \to \mathbb{R}^{p_i}$ if

$$\phi_i(y) - \phi_i(x) \geq \langle \xi_i, \eta_i(y_i,x_i) \rangle, \quad \forall\, \xi_i \in \partial_i \phi_i(x) \text{ and } \forall\, y \in K.$$

Proposition 3 ([10]). *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$. Then any solution of SGVQVLIP (II) is a solution of SVQOP with $T_i(x) = \partial_i \varphi_i(x)$ for each $i \in I$ and for all $x \in K$, where $\partial_i \varphi_i(x) = (\partial_i \varphi_{i_1}(x), \partial_i \varphi_{i_2}(x), \ldots, \partial_i \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$.*

Proof. For the sake of simplicity, we denote by $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$ where $u_{i_l} \in \partial_i \varphi_{i_l}(x)$ for all $l \in \mathscr{L}$, and

$$\langle u_i, \eta_i(y_i,x_i) \rangle = \Big(\langle u_{i_1}, \eta_{i_1}(y_i,x_i) \rangle, \ldots, \langle u_{i_{q_i}}, \eta_{i_{q_i}}(y_i,x_i) \rangle \Big) \in \mathbb{R}^{q_i}.$$

Assume that $\bar{x} \in K$ is a solution of the SGVQVLIP (II). Then for each $i \in I$,

$$\forall y_i \in A_i(\bar{x}),\ \exists \bar{u}_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ for all } l \in \mathscr{L} \text{ such that}$$

$$\Big(\langle \bar{u}_{i_1}, \eta_{i_1}(y_i,\bar{x}_i) \rangle, \ldots, \langle \bar{u}_{i_{q_i}}, \eta_{i_{q_i}}(y_i,\bar{x}_i) \rangle \Big) \notin -\text{int}\, C_i(\bar{x}).$$

We can rewrite this as

$$\forall y_i \in A_i(\bar{x}),\ \exists \bar{u}_i \in \partial_i \varphi_i(\bar{x}) \ : \ \langle \bar{u}_i, \eta_i(y_i,\bar{x}_i) \rangle \notin -\text{int}\, C_i(\bar{x}). \tag{4}$$

Since for each $i \in I$ and for all $l \in \mathscr{L}$, φ_{i_l} is generalized invex w.r.t. η_{i_l}, we have

$$\varphi_{i_l}(y) - \varphi_{i_l}(\bar{x}) \geq \langle u_{i_l}, \eta_{i_l}(y_i,\bar{x}_i) \rangle \quad \text{for all } u_{i_l} \in \partial_i \varphi_{i_l}(\bar{x}) \text{ and } y \in A(\bar{x}) = \prod_{i \in I} A_i(\bar{x}),$$

that is, for each $i \in I$

$$\varphi_i(y) - \varphi_i(\bar{x}) \geq \langle u_i, \eta_i(y_i, \bar{x}_i)\rangle \quad \text{for all } u_i \in \partial_i \varphi_i(\bar{x}) \text{ and } y \in A(\bar{x}).$$

Therefore, for each $i \in I$ and for all $u_i \in \partial_i \varphi_i(\bar{x})$, we have

$$\begin{aligned} \varphi_i(y) - \varphi_i(\bar{x}) &\in \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle + \mathbb{R}^{q_i}_+ \\ &\subseteq \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle + \operatorname{int} C_i(\bar{x}). \end{aligned} \tag{5}$$

From (4) and (5), we have $\varphi_i(y) - \varphi_i(\bar{x}) \notin -\operatorname{int} C_i(\bar{x})$. Hence $\bar{x} \in K$ is a solution of the SVQOP.

Rest of the section, unless otherwise specified, $\partial_i \varphi_i(x)$ and $\langle u_i, \eta_i(y_i, x_i)\rangle$ are the same as defined in Proposition 3.

Theorem 11 ([10]). *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exists $r > 0$ such that for all $x \in K$, $||x|| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $||\tilde{y}_i||_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and*

$$\langle u_i, \eta_i(\tilde{y}_i, x_i)\rangle \in -\operatorname{int} C_i(x), \quad \forall\, u_i \in \partial_i \varphi_i(x),$$

where $||\cdot||$ and $||\cdot||_i$ denote the norms on X and X_i, respectively. Then the SVQOP has a solution.

Theorem 12 ([10]). *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then the SVQOP has a solution.*

The following example, provided by one of the referees, shows that if η is affine in the second argument, then it is not necessary that $\eta(x,x) = 0$.

Example 4. Consider the map $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\eta(x,y) = (x + y + 1, 0), \quad \text{for all } x, y \in \mathbb{R}_+ = [0, \infty).$$

Then η is affine in the second argument but $\eta(x,x) \neq 0$ for all $x \in \mathbb{R}_+$.

In the next three corollaries, we set $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$, $\langle u_i, y_i - x_i\rangle = (\langle u_{i_1}, y_i - x_i\rangle, \ldots, \langle u_{i_{q_i}}, y_i - x_i\rangle) \in \mathbb{R}^{q_i}$ and $\partial_i \varphi_i(x) = (\partial_i \varphi_{i_1}(x), \partial_i \varphi_{i_2}(x), \ldots, \partial_i \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$, where $\partial_i \varphi_{i_j}(x)$ $(j = 1, \ldots, q_i)$ is the partial subdifferential in the sense of convex analysis.

Corollary 9 ([10]). *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be convex and lower semicontinuous Assume that there exists $r > 0$ such that for all $x \in K$, $||x|| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $||\tilde{y}_i||_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and*

$$\langle u_i, \tilde{y}_i - x_i\rangle \in -\operatorname{int} C_i(x), \quad \forall\, u_i \in \partial_i \varphi_i(x),$$

where $||\cdot||$ *and* $||\cdot||_i$ *denote the norms on* X *and* X_i*, respectively. Then the SVQOP has a solution.*

Corollary 10 ([10]). *For each* $i \in I$ *and for all* $l \in \mathscr{L}$*, let* $\varphi_{i_l} : K \to \mathbb{R}$ *be convex and lower semicontinuous on* K*. Let the multivalued map* $A = \prod_{i \in I} A_i : K \to 2^K$ *defined as* $A(x) = \prod_{i \in I} A_i(x)$ *for all* $x \in K$*, be* Φ*-condensing. Then the SVQOP has a solution.*

5 System of Generalized Implicit Vector Quasi-equilibrium Problems

As we have seen in the previous sections that systems of vector quasi-equilibrium problems are used as tools to establish the existence of a solution of Debreu VEP, also known as constrained Nash equilibrium problem, both for nondifferentiable and (non)convex vector-valued functions. These are also used to solve mathematical programs with equilibrium constraints [70], fixed point theory for a family of nonexpansive multivalued maps [68] and several related topics. By using different types of maximal element theorems for a family of multivalued maps and different types of fixed point theorems for a multivalued map, several authors studied the existence of solutions of different kinds of systems of vector quasi-equilibrium problems; see, for example, [5,6,9,10,12,39,68,70,71,75,98,99] and the references therein.

For each $i \in I$, let $W_i : K \to 2^{Y_i}$ be a multivalued map defined as $W_i(x) = Y_i \setminus (-\mathrm{int}\, C_i(x))$ for all $x \in K$ such that its graph is closed. For each $i \in I$, let $F_i : K_i \to 2^{Y_i}$ be a multivalued map with nonempty values, $A_i : K \to 2^{K_i}$ be a multivalued map with nonempty convex values such that $A(x) = \prod_{i \in I} A_i(x)$, and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function. We consider the following *Systems of Generalized Implicit Vector Quasi-Equilibrium Problems* (SGIVQEP) [4]:

Problem 1. Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\forall \bar{u}_i \in F_i(\bar{x}) \quad : \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\mathrm{int}\, C_i(\bar{x}), \quad \forall y_i \in A_i(\bar{x}).$$

Problem 2. Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\exists \bar{u}_i \in F_i(\bar{x}) \quad : \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\mathrm{int}\, C_i(\bar{x}), \quad \forall y_i \in A_i(\bar{x}).$$

Problem 3. Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\forall y_i \in A_i(\bar{x}),\ \exists \bar{u}_i \in F_i(\bar{x})\ (\bar{u}_i \text{ depends on } y_i) \quad : \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\mathrm{int}\, C_i(\bar{x}).$$

Problem 4. Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\forall y \in A(\bar{x}) \text{ and } \forall v_i \in F_i(y) \quad : \quad \psi_i(v_i, y_i, \bar{x}_i) \notin \mathrm{int}\, C_i(\bar{x}),$$

where y_i is the ith component of y.

Problem 5. Find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\forall y \in A(\bar{x}), \ \exists v_i \in F_i(y) \ (v_i \text{ depends on } y) \ : \ \psi_i(v_i, y_i, \bar{x}_i) \notin \operatorname{int} C_i(\bar{x}),$$

where y_i is the ith component of y.

Remark 13. Problem 1 $\Rightarrow$ Problem 2 $\Rightarrow$ Problem 3 and Problem 4 $\Rightarrow$ Problem 5.

The solutions of Problems 1, 2 and 3 are called general solution, strong solution and weak solution, respectively. In view of Remark 13, every general solution is a strong solution and every strong solution is a weak solution. But the converse assertions may not be true.

When $A_i(x) = K_i$ for all $x \in K$ and for each $i \in I$, Problems 1–5 are called *systems of generalized implicit vector equilibrium problems* (SGIVEP) considered and studied in [1]. In this case, the existence results for solutions of these problems are investigated by introducing different kinds of generalized pseudomonotonicities. In this case, Nash equilibrium problem for vector-valued functions can be solved by using Problems 1–5 but not Debreu VEP.

As we have seen in Sect. 4 that Problem 3 provides a sufficient condition (which is in general not necessary) for a solution of a SVQOP that includes Debreu VEP for nondifferentiable and nonconvex functions. But, in this case, Problem 2 provides necessary and sufficient conditions for a solution of a SVQOP.

If for each $i \in I$, $A_i(x) = K_i$ for all $x \in K$, Problem 3 is called a *system of generalized implicit vector equilibrium problems* and it is introduced and studied in [15]. It is also used to give the existence of a solution of the Nash equilibrium problem for nondifferentiable and nonconvex functions. Further, if $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ and $A_i(x) = K_i$ for all $x \in K$, Problem 3 was studied by in [22]. As an application of our results, we established some existence results for solutions of systems of optimization problems and the Nash equilibrium problem.

When I is a singleton set, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_+$ for all $x \in K$, the existence of a solution of Problem 2 is studied in [46].

When I is a singleton set, $A_i(x) = K_i$ for all $x \in K$ and $\psi_i(u_i, x_i, y_i) = \langle u_i, \eta_i(y_i, x_i) \rangle$ (respectively, $\psi_i(u_i, x_i, y_i) = \langle u_i, y_i - x_i \rangle$), then Problem 2 provides necessary and sufficient conditions for solutions of vector optimization problems for nondifferentiable and nonconvex functions (respectively, for nondifferentiable, but convex functions). See, for example, [2, 17] and the references therein. In this case, Problem 1 is considered and studied in [2, 30, 62].

When I is a singleton set, Problems 2 and 3 are studied by Kum and Lee [58, 64]. They proved the existence of solutions of these problems under some kind of pseudomonotonicity assumptions.

In Sect. 5.1, we give some relationships among Problems 1–5 by using different kinds of generalized pseudomonotonicities. Section 5.2 is devoted to the existence results for a solution of Problem 1 under lower semi-continuity of the family of multivalued maps involved in the formulation of the problem. The existence of a solution of Problem 1 and so Problems 2 and 3 without any coercivity condition but for Φ-condensing maps is also established. In Sect. 5.3, we establish the existence of a strong solution of our SGVQEP by using $\mathscr{H}$-hemicontinuity assumption in the setting of real Banach spaces. We also present an existence result for a weak solution under generalized pseudomonotonicity and u-hemicontinuity assumptions. Basically, besides establishing existence results for solutions of Problems 1–3 without any coercivity condition but for Φ-condensing maps, we extend the results of [1] for SGIVEP to SGIVQEP. Our results provide the existence of solutions of Problems 1–5 under some kind of pseudomonotonicity assumption and under lower semicontinuity assumption which is one of main motivations of this section.

5.1 Relationships Among Problems 1–5

Throughout this section, for each $i \in I$, we assume that X_i and Y_i are locally convex Hausdorff topological vector spaces and K_i is a nonempty convex subset of X_i, and C_i is the same as defined in the previous section. We set $K = \prod_{i \in I} K_i$, $X = \prod_{i \in I} X_i$, and $Y = \prod_{i \in I} Y_i$.

We recall different kinds of generalized pseudomonotonicities introduced in [1].

Definition 18 ([1]). Let $\{\psi_i\}_{i \in I}$ be a family of mappings $\psi_i : D_i \times K_i \times K_i \to Y_i$. A family $\{F_i\}_{i \in I}$ of multivalued maps $F_i : K \to 2^{K_i}$ with nonempty values is called:

(i) *Generalized strongly pseudomonotone w.r.t.* $\{\psi_i\}_{i \in I}$ if for all $x, y \in K$ and for each $i \in I$,

$$\forall u_i \in F_i(x) \;:\; \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x) \Rightarrow \forall v_i \in F_i(y) \;:\; \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x).$$

(ii) *Generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i \in I}$ if for all $x, y \in K$ and for each $i \in I$,

$$\exists u_i \in F_i(x) \;:\; \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x) \Rightarrow \forall v_i \in F_i(y) \;:\; \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x).$$

(iii) *Generalized weakly pseudomonotone w.r.t.* $\{\psi_i\}_{i \in I}$ if for all $x, y \in K$ and for each $i \in I$,

$$\exists u_i \in F_i(x) \;:\; \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x) \Rightarrow \exists v_i \in F_i(y) \;:\; \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x).$$

(iv) *Generalized pseudomonotone*$^+$ *w.r.t.* $\{\psi_i\}_{i \in I}$ if for all $x, y \in K$ and for each $i \in I$,

$$\forall u_i \in F_i(x) \;:\; \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x) \Rightarrow \exists v_i \in F_i(y) \;:\; \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x).$$

(v) *u-Hemicontinuous w.r.t.* $\{\psi_i\}_{i \in I}$ if for all $x, y \in K$ and $\alpha \in [0,1]$ and for each $i \in I$, the multivalued map

$$\alpha \mapsto \psi_i(F_i(x + \alpha(y - x)), x_i, y_i)$$

is upper semicontinuous at 0^+, where

$$\psi_i(F_i(x+\alpha(y-x)),x_i,y_i) = \{\psi_i(w_i,x_i,y_i) : w_i \in F_i(x+\alpha(y-x))\}.$$

Remark 14. Definition (i) ⇒ Definition (ii) ⇒ Definition (iii); Definition (iv) ⇒ Definition (iii); Definition (i) ⇒ Definition (iv); that is, Definition (i) ⇒ Definition (iv) ⇒ Definition (iii).

In the next three lemmas, we discuss the relationships among Problems 1–5.

Lemma 5 ([4]).

(a) Problem 3 ⇒ Problem 4 if $\{F_i\}_{i\in I}$ is generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.
(b) Problem 3 ⇒ Problem 5 if $\{F_i\}_{i\in I}$ is generalized weakly pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.
(c) Problem 1 ⇒ Problem 5 if $\{F_i\}_{i\in I}$ is generalized pseudomonotone$^+$ w.r.t. $\{\psi_i\}_{i\in I}$.
(d) Problem 1 ⇒ Problem 4 if $\{F_i\}_{i\in I}$ is generalized strongly pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.
(e) Problem 2 ⇒ Problem 4 if $\{F_i\}_{i\in I}$ is generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.

Lemma 6 ([4]). *For each $i \in I$, assume that the following conditions hold:*

(i) For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i,x_i,x_i) \in \mathscr{C}_i = \bigcap_{x\in K} C_i(x)$.
(ii) For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i,x_i,\cdot)$ is $\mathscr{C}_i$-convex, that is, for all $s_i \in L(X_i,Y_i)$, $x,y \in K$ and $\alpha \in [0,1]$,

$$\psi_i(s_i,x_i,\alpha x_i+(1-\alpha)y_i) \in \alpha\psi_i(s_i,x_i,x_i)+(1-\alpha)\psi_i(s_i,x_i,y_i)-\mathscr{C}_i.$$

(iii) For all $s_i \in L(X_i,Y_i)$, $x,y,z \in K$ and $\alpha \in [0,1]$,

$$\psi_i(s_i,x_i+\alpha(y_i-x_i),z_i) = (1-\alpha)\psi_i(s_i,x_i,z_i).$$

(iv) $\{F_i\}_{i\in I}$ is u-hemicontinuous w.r.t. $\{\psi_i\}_{i\in I}$.

Then Problem 5 ⇒ Problem 3 as well as Problem 4 ⇒ Problem 3.

Proposition 4 ([4]). *Under the conditions of Lemmas 5(a) and 6, Problems 3, 4 and 5 are equivalent.*

Lemma 7. *For each $i \in I$, let $(X_i,\|\cdot\|)$ and Y_i be real Banach spaces and K_i be a nonempty convex subset of X_i. Let $K = \prod_{i\in I} K_i$. For each $i \in I$, assume that the following conditions hold:*

(i) For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i,x_i,x_i) \in \mathscr{C}_i = \bigcap_{x\in K} C_i(x)$.
(ii) For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i,x_i,\cdot)$ is $\mathscr{C}_i$-convex, that is, for all $s_i \in L(X_i,Y_i)$, $x,y \in K$ and $t \in [0,1]$,

$$\psi_i(s_i,x_i,tx_i+(1-t)y_i) \in t\psi_i(s_i,x_i,x_i)+(1-t)\psi_i(s_i,x_i,y_i)-\mathscr{C}_i.$$

(iii) For all $s_i \in L(X_i,Y_i)$, $x,y,z \in K$ and $t \in [0,1]$,

$$\psi_i(s_i,x_i+t(y_i-x_i),z_i) = (1-t)\psi_i(s_i,x_i,z_i).$$

(iv) ψ_i *is continuous in the first argument.*
(v) F_i *is* $\mathscr{H}$*-hemicontinuous and for all* $x \in K$, $F_i(x)$ *is a nonempty compact set in* Y_i.
(vi) The family $\{F_i\}_{i\in I}$ *is generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.

Then Problems 2 and 4 are equivalent.

5.2 Existence Results Under Lower Semicontinuity

For each $i \in I$, we assume that the graph of the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int}\, C_i(x)\}$ for all $x \in K$, is closed. For each $i \in I$, we also assume that $A_i : K \to 2^{K_i}$ is a multivalued map such that for all $x \in K$, $A_i(x)$ is nonempty and convex, $A_i^{-1}(y_i)$ is open in K for all $y_i \in K_i$ and the set $\mathscr{F}_i := \{x \in K : x_i \in A_i(x)\}$ is closed in K, where x_i is the ith component of x.

We extend and generalize Definition 14 for a family of trifunctions.

Definition 19 ([1]). For each $i \in I$, let $F_i : K \to 2^{D_i}$ be a multivalued map with nonempty values. A family $\{\psi_i\}_{i\in I}$ of functions $\psi_i : D_i \times K_i \times K_i \to Y_i$ is called *$C_i(x)$-quasiconvex-like w.r.t.* $\{F_i\}_{i\in I}$ if for all $x \in K$, $y_i', y_i'' \in K_i$ and $t \in [0,1]$, we either have $\forall u_i \in F_i(x)$,

$$\psi_i(u_i, x_i, ty_i' + (1-t)y_i'') \in \psi_i(u_i, x_i, y_i') - \text{int}\, C_i(x),$$

or

$$\psi_i(u_i, x_i, ty_i' + (1-t)y_i'') \in \psi_i(u_i, x_i, y_i'') - \text{int}\, C_i(x).$$

Definition 20 ([1]). For each $i \in I$, let $F_i : K \to 2^{D_i}$ be multivalued map with nonempty values. A family $\{\psi_i\}_{i\in I}$ of functions $\psi_i : D_i \times K_i \times K_i \to Y_i$ is called *simultaneously $C_i(x)$-quasiconvex-like w.r.t.* $\{F_i\}_{i\in I}$ if for all $x \in K$, $y_i', y_i'' \in K_i$ and $t \in [0,1]$, we either have $\forall u_i', u_i'' \in F_i(x)$,

$$\psi_i(tu_i' + (1-t)u_i'', x_i, ty_i' + (t)y_i'') \in \psi_i(u_i', x_i, y_i') - \text{int}\, C_i(x),$$

or

$$\psi_i(tu_i' + (1-t)u_i'', x_i, ty_i' + (1-t)y_i'') \in \psi_i(u_i'', x_i, y_i'') - \text{int}\, C_i(x).$$

We present an existence result for a solution of Problem 1 under lower semicontinuity of the family of multivalued maps involved in the formulation of the problem.

Theorem 13 ([4]). *For each* $i \in I$, *let* K_i *be a nonempty convex subset of a Hausdorff topological vector space* X_i. *Let* $K = \prod_{i\in I} K_i$. *For each* $i \in I$, *let* $F_i : K \to 2^{K_i}$ *be a lower semicontinuous multivalued map with nonempty convex values and* $\psi_i : D_i \times K_i \times K_i \to Y_i$ *be a function such that the following conditions are satisfied:*

(i) For all $x \in K$, *the family* $\{\psi_i\}_{i\in I}$ *of functions* ψ_i *is simultaneously* $C_i(x)$*-quasiconvex-like w.r.t.* $\{F_i\}_{i\in I}$.

(ii) For all $x \in K$ and for all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\text{int}\, C_i(x)$.
(iii) For each fixed y_i, the map $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ is continuous on $D_i \times K_i$.
(iv) There exist a nonempty compact subset M of K and a nonempty compact convex subset N_i of K_i for each $i \in I$ such that for all $x \in K \setminus M$, there exist $i \in I$ and $\tilde{y}_i \in N_i$ such that $\tilde{y}_i \in A_i(x)$ and $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int}\, C_i(x)$ for all $u_i \in F_i(x)$.

Then Problem 1 has a solution.

The following result provides the existence of a solution of Problem 1 without any coercivity condition but for Φ-condensing maps.

Theorem 14 ([4]). *For each $i \in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i and let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i)–(iii) of Theorem 13 hold. Then Problem 1 has a solution.*

5.3 Existence Results Under Pseudomonotonicity

In this section, we present some existence results for a solution of the SGIVQEP under generalized pseudomonotonicity assumption.

Theorem 15 ([4]). *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. Let $K = \prod_{i\in I} K_i$. For each $i \in I$, let $F_i : K \to 2^{K_i}$ be a multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function such that the following conditions are satisfied:*

(i) The family $\{F_i\}_{i\in I}$ of multivalued maps F_i is generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.
(ii) For all $x \in K$, the family $\{\psi_i\}_{i\in I}$ of functions ψ_i is $C_i(x)$-quasiconvex-like w.r.t. $\{F_i\}_{i\in I}$.
(iii) For all $x \in K$ and for all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\text{int}\, C_i(x)$.
(iv) For each fixed $(v_i, y_i) \in D_i \times K_i$, the map $x_i \mapsto \psi_i(v_i, y_i, x_i)$ is continuous on K_i.
(v) There exist a nonempty compact subset M of K and a nonempty compact convex subset N_i of K_i for each $i \in I$ such that for all $x \in K \setminus M$, there exist $i \in I$ and $\tilde{y}_i \in N_i$ such that $\tilde{y}_i \in A_i(x)$ and $\psi_i(u_i, x_i, \tilde{y}_i) \in -\text{int}\, C_i(x)$ for all $u_i \in F_i(x)$.

Then Problem 4 has a solution.

Theorem 16 ([4]). *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. For each $i \in I$, let $F_i : K \to 2^{K_i}$ be a multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function such that the following conditions are satisfied:*

(i) The family $\{F_i\}_{i\in I}$ of multivalued maps F_i is u-hemicontinuous and generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.

(ii) The family $\{\psi_i\}_{i\in I}$ *of functions* ψ_i *is* $\mathscr{C}_i$*-convex in the third argument.*
(iii) For all $s_i \in L(X_i,Y_i)$, $x,y,z \in X$ *and* $\alpha \in [0,1]$,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i).$$

(iv) For all $x \in K$ *and for all* $u_i \in F_i(x)$, $\psi_i(u_i,x_i,x_i) \in \mathscr{C}_i$.
(v) For each fixed $(v_i,y_i) \in D_i \times K_i$, *the map* $x_i \mapsto \psi_i(v_i,y_i,x_i)$ *is continuous on* K_i.
(vi) There exist a nonempty compact subset M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exist* $i \in I$ *and* $\tilde{y}_i \in N_i$ *such that* $\tilde{y}_i \in A_i(x)$ *and* $\psi_i(u_i,x_i,\tilde{y}_i) \in -\mathrm{int}\, C_i(x)$ *for all* $u_i \in F_i(x)$.

Then Problem 3 has a solution.

The following result provides the existence of a strong solution of Problem 2.

Theorem 17 ([4]). *For each* $i \in I$, *let* K_i *be a nonempty convex subset of a real Banach space* X_i *and* Y_i *be a real Banach space. For each* $i \in I$, *let* $F_i : K \to 2^{K_i}$ *be a multivalued map with nonempty compact values and* $\psi_i : D_i \times K_i \times K_i \to Y_i$ *be a function such that the following conditions are satisfied:*

(i) The family $\{F_i\}_{i\in I}$ *of multivalued maps* F_i *is* $\mathscr{H}$*-hemicontinuous and generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.
(ii) The family $\{\psi_i\}_{i\in I}$ *of functions* ψ_i *is* $\mathscr{C}_i$*-convex in the third argument.*
(iii) For all $s_i \in L(X_i,Y_i)$, $x,y,z \in X$ *and* $\alpha \in [0,1]$,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i).$$

(iv) For all $x \in K$ *and for all* $u_i \in F_i(x)$, $\psi_i(u_i,x_i,x_i) \in \mathscr{C}_i$.
(v) For each fixed $(v_i,y_i) \in D_i \times K_i$, *the map* $x_i \mapsto \psi_i(v_i,y_i,x_i)$ *is continuous on* K_i.
(vi) There exist a nonempty compact subset M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exist* $i \in I$ *and* $\tilde{y}_i \in N_i$ *such that* $\tilde{y}_i \in A_i(x)$ *and* $\psi_i(u_i,x_i,\tilde{y}_i) \in -\mathrm{int}\, C_i(x)$ *for all* $u_i \in F_i(x)$.

Then Problem 2 has a solution.

6 System of Simultaneously Generalized Vector Quasi-equilibrium Problems

Throughout the section, unless otherwise specified, I is any index set (finite or infinite). For each $i \in I$, let X_i and Y_i be two nonempty convex subsets of locally convex topological vector spaces E_i and F_i, respectively, and Z_i be a real topological vector space. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $C_i : X \to 2^{Z_i}$ be a multivalued map such that for all $x \in X$, $C_i(x)$ is a closed convex cone with apex at the origin $\mathbf{0}$. For each $i \in I$, let $P_i = \bigcap_{x\in X} C_i(x)$ such that P_i defines a vector ordering on Z_i. For each $i \in I$, let $S_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be multivalued maps with nonempty values, and $f_i : X \times Y \times X_i \to Z_i$ and $g_i : X \times Y \times Y_i \to Z_i$ be trifunctions.

We consider the following problems of *system of simultaneous generalized vector quasi-equilibrium problems* (SSGVQEP) [12]:
SSGVQEP(I): Find $(\bar{x},\bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x},\bar{y},x_i) \in C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$g_i(\bar{x},\bar{y},y_i) \in C_i(\bar{x}), \quad \forall\, y_i \in T_i(\bar{x}).$$

SSGVQEP(II): Find $(\bar{x},\bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x},\bar{y},x_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$g_i(\bar{x},\bar{y},y_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, y_i \in T_i(\bar{x}).$$

SSGVQEP(III): Find $(\bar{x},\bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x},\bar{y},x_i) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$g_i(\bar{x},\bar{y},y_i) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall\, y_i \in T_i(\bar{x}),$$

in this case we assume that int C_i is nonempty for each $i \in I$.

Remark 15. For each $i \in I$ and $\forall\, x \in X$, let $C_i(x)$ be a pointed cone and $P_i = \bigcap_{x\in X} C_i(x)$, then P_i is also pointed. Indeed,

$$\begin{aligned} P_i \cap (-P_i) &= \Big(\cap_{x\in X} C_i(x)\Big) \bigcap \Big(\cap_{x\in X} (-C_i(x))\Big) \\ &= \bigcap_{x\in X} \Big(C_i(x) \cap (-C_i(x))\Big) = \{\mathbf{0}\}. \end{aligned}$$

Therefore, for each $i \in I$, P_i is pointed.

Remark 16. If for each $i \in I$ and $\forall\, x \in X$, $C_i(x)$ is also pointed, then every solution of SSGVQEP(I) is a solution of SSGVQEP(II) and every solution of SSGVQEP(II) is a solution of SSGVQEP(III). But the reverse implication does not hold.

Indeed, let $(\bar{x},\bar{y}) \in X \times Y$ be a solution of SSGVQEP(I), then for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x},\bar{y},x_i) \in C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$g_i(\bar{x},\bar{y},y_i) \in C_i(\bar{x}), \quad \forall\, y_i \in T_i(\bar{x}).$$

Since for each $i \in I$ and $\forall\, x \in X$, $C_i(x)$ is a pointed cone, we have $C_i(x) \cap (-C_i(x)) = \{\mathbf{0}\}$ and therefore

$$C_i(x) \cap \Big(-C_i(x) \setminus \{\mathbf{0}\}\Big) = \emptyset.$$

Hence

$$f_i(\bar{x},\bar{y},x_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$g_i(\bar{x},\bar{y},y_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, y_i \in T_i(\bar{x}).$$

The second statement follows from the fact that $-\mathrm{int}\, C_i(x) \subseteq -C_i(x) \setminus \{\mathbf{0}\}$, $\forall\, x \in X$ and for each $i \in I$.

For each $i \in I$, we denote by $L(E_i, Z_i)$ the space of all continuous linear operators from E_i into Z_i and let Y_i be a nonempty subset of $L(E_i, Z_i)$. For each $i \in I$, let $g_i \equiv 0$, then SSGVQEP(I) reduces to the following *problem of system of generalized implicit vector quasi-variational inequalities*:
SGIVQVIP(I): Find $(\bar{x},\bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ satisfying

$$f_i(\bar{x},\bar{y},x_i) \in C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x}).$$

Analogous, we can define SGIVQVIP(II) and SGIVQVIP(III) (problems of system of generalized implicit vector quasi-variational inequalities) corresponding to SSGVQEP(II) and SSGVQEP(III), respectively.

The SGIVQVIP contains the problem of system of generalized vector quasi-variational-like inequalities (SGVQVLIP) as a special case. Recently, the weak formulation of SGVQVLIP is studied in [10]. We used SGVQVLIP as a tool to prove the existence of a solution of Debreu type equilibrium problem for nondifferentiable and nonconvex vector-valued functions.

When for each $i \in I$, $X_i = Y_i$, $S_i \equiv T_i$ and $f_i \equiv g_i$, then SSGVQEP is called a *system of vector quasi-equilibrium problems*. In this case, SSGVQEP(III) is considered and studied in [5] for $f_i(x,y,y_i) = h_i(x,y_i)$ with further applications to systems of generalized vector quasi-variational-like inequalities and Debreu type equilibrium problems for vector-valued functions.

When I is a singleton set and $g_i \equiv 0$, then SSGVQEP(I) is considered and studied in [46].

When I is a singleton set, $X = Y$, $S_i \equiv T_i$, $S_i(x) = X$, $f_i(x,y,x_i) = \varphi(x,y)$, $g_i(x,y,y_i) = \phi(x,y)$, then SSGVQEP(III) reduces to the problem of *simultaneous vector variational inequalities* which is considered and studied by Fu [45] for a fixed cone C. If $C = \mathbb{R}_+$, then the problem of simultaneous vector variational inequalities becomes the problem of simultaneous variational inequalities, which is introduced and studied by Husain and Tarafdar [52] with applications to optimization problems.

By making suitable choices of f_i and g_i, we can derive several systems of quasi-variational inequalities and systems of (quasi-) equilibrium problems studied in the literature; see, for example, [5, 9, 10, 14–16] and the references therein.

6.1 Existence Results for Solutions of SSGVQEP

In this section, we present an existence result for a solution of SSGVQEP and derive existence results for solutions of SGIVQVIP(I), simultaneous generalized vector quasi-equilibrium problem and a system of generalized vector quasi-variational-like inequalities.

Theorem 18 ([12]). *For each $i \in I$, let E_i, F_i and Z_i be real locally convex topological vector spaces and F_i be also quasi-complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and $Y_i \subseteq F_i$ a nonempty convex set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values and $T_i : X \to 2^{Y_i}$ a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *$C_i : X \to 2^{Z_i}$ is a closed multivalued map such that $\forall\, x \in X$, $C_i(x)$ is a closed convex cone with apex at the origin, and $P_i = \bigcap_{x\in X} C_i(x)$.*
(ii) *P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.*
(iii) *$f_i : X \times Y \times X_i \to Z_i$ is a continuous function such that:*

(a) *$\forall\, x \in X$ and $y \in Y$, $f_i(x,y,x_i) \geq_{P_i} \mathbf{0}$.*
(b) *$\forall\, (x,y) \in X \times Y$, the map $u_i \mapsto f_i(x,y,u_i)$ is properly quasi-convex.*

(iv) *$g_i : X \times Y \times Y_i \to Z_i$ is a continuous function such that:*

(a) *$\forall\, x \in X$ and $y \in Y$, $g_i(x,y,y_i) \geq_{P_i} \mathbf{0}$.*
(b) *$\forall\, (x,y) \in X \times Y$, the map $v_i \mapsto g_i(x,y,v_i)$ is properly quasi-convex.*

Then there exists a solution $(\bar{x},\bar{y}) \in X \times Y$ of SSGVQEP(I).

If for each $i \in I$, $g_i \equiv 0$, then we have the following result.

Corollary 11. *For each $i \in I$, let E_i, F_i and Z_i be real locally convex topological vector spaces and F_i be also quasi-complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and $Y_i \subseteq F_i$ a nonempty convex set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values and $T_i : X \to 2^{Y_i}$ a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *$C_i : X \to 2^{Z_i}$ is a closed multivalued map such that $\forall\, x \in X$, $C_i(x)$ is a closed convex cone with apex at the origin, and $P_i = \bigcap_{x\in X} C_i(x)$.*
(ii) *P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.*
(iii) *$f_i : X \times Y \times X_i \to Z_i$ is a continuous function such that:*

(a) *$\forall\, x \in X$ and $y \in Y$, $f_i(x,y,x_i) \geq_{P_i} \mathbf{0}$.*
(b) *$\forall\, (x,y) \in X \times Y$, the map $u_i \mapsto f_i(x,y,u_i)$ is properly quasi-convex.*

Then there exists a solution $(\bar{x},\bar{y}) \in X \times Y$ of SGIVQVIP(I).

Remark 17. Corollary 11 is an extension of Theorem 1 in [46] to the system of quasi-equilibrium problems with a moving cone.

When I is a singleton set, then we have the following result.

Corollary 12. *Let E, F and Z be real locally convex topological vector spaces and F be also quasi-complete. Let $X \subseteq E$ be a nonempty compact convex set and $Y \subseteq F$ a nonempty convex set. Let $S : X \to 2^X$ be a continuous multivalued map with nonempty closed convex values and $T : X \to 2^Y$ a continuous multivalued map with nonempty compact convex values. Assume that the following conditions are satisfied:*

(i) *$C : X \to 2^Z$ is a closed multivalued map such that $\forall\, x \in X$, $C(x)$ is a closed convex cone with apex at the origin, and $P = \bigcap_{x\in X} C(x)$.*
(ii) *P^* has a weak* compact convex base B^* and Z is ordered by P.*
(iii) *$f : X \times Y \times X \to Z$ is a continuous function such that:*

  (a) *$\forall\, x \in X$ and $y \in Y$, $f(x,y,x) \geq_P \mathbf{0}$.*
  (b) *$\forall\, (x,y) \in X \times Y$, the map $u \mapsto f(x,y,u)$ is properly quasi-convex.*

(iv) *$g : X \times Y \times Y \to Z$ is a continuous function such that:*

  (a) *$\forall\, x \in X$ and $y \in Y$, $g(x,y,y) \geq_P \mathbf{0}$.*
  (b) *$\forall\, (x,y) \in X \times Y$, the map $v \mapsto g(x,y,v)$ is properly quasi-convex.*

Then there exists a solution $(\bar{x},\bar{y}) \in X \times Y$ of the simultaneous generalized vector quasi-equilibrium problem *(SGVQEP): find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,*

$$f(\bar{x},\bar{y},x) \in C(\bar{x}), \quad \forall\, x \in S(\bar{x})$$

and

$$g(\bar{x},\bar{y},y) \in C(\bar{x}), \quad \forall\, y \in T(\bar{x}).$$

In addition to the assumptions on $C_i : K \to 2^{Z_i}$, in the following corollary, we further assume that $C_i(x)$ is pointed, for each $i \in I$ and for all $x \in K$. Then the following result can be easily derived from Corollary 3.1 by setting

$$f_i(x,y,u_i) = \langle \theta_i(x,y), \eta_i(u_i,x_i) \rangle.$$

Corollary 13. *For each $i \in I$, let E_i and Z_i be real locally convex topological vector spaces and let $L(E_i,Z_i)$ be quasi-complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and $Y_i \subseteq L(E_i,Z_i)$ a nonempty convex set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values and $T_i : X \to 2^{Y_i}$ a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *$C_i : X \to 2^{Z_i}$ is a closed multivalued map such that $\forall\, x \in X$, $C_i(x)$ is a nonempty closed convex pointed cone, and $P_i = \bigcap_{x\in X} C_i(x)$.*
(ii) *P_i^* has a weak* compact convex base B_i and Z_i is ordered by P_i.*

(iii) $\theta_i : X \times Y \to Y_i$ *and* $\eta_i : X_i \times X_i \to X_i$ *are continuous bifunctions such that for each* $i \in I$*:*

(a) $\forall\, x_i \in X_i,\ \eta_i(x_i, x_i)\rangle \geq_{P_i} 0.$
(b) $\forall\, (x,y) \in X \times Y$*, the map* $u_i \mapsto \langle \theta_i(x,y), \eta_i(u_i, x_i)\rangle$ *is properly quasi-convex.*

Then there exists a solution $(\bar{x},\bar{y}) \in X \times Y$ *of the problem of system of generalized vector quasi-variational-like inequalities (SGVQVLIP)(I): find* $(\bar{x},\bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$*,* $\bar{x}_i \in S_i(\bar{x})$*,* $\bar{y}_i \in T_i(\bar{x})$ *and*

$$\langle \theta_i(\bar{x},\bar{y}), \eta_i(x_i,\bar{x}_i)\rangle \in C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x}).$$

Remark 18. It is worth to mention that the weak formulation of SGVQVLIP(III) is considered and studied in [10]. Corollary 13 provides the existence of a solution of a more general problem than SGVQVLIP(III).

6.2 Systems of Vector Quasi-saddle Point Problems

In this section, we define systems of quasi-saddle point problems and systems of quasi-minimax inequalities. As application of the results of previous section, we derive existence results for solutions of these problems.

Let X, Y, X_i, Y_i, Z_i and C_i be the same as defined in the formulations of SSGVQEP. Let $\ell_i : X_i \times Y_i \to Z_i$ be a bifunction. We consider the following *systems of quasi-saddle point problems*.

SVQSPP(I): Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i,\bar{y}_i) - \ell_i(\bar{x}_i,\bar{y}_i) \in C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$\ell_i(\bar{x}_i,\bar{y}_i) - \ell_i(\bar{x}_i,y_i) \in C_i(\bar{x}), \quad \forall\, y_i \in T_i(\bar{x}).$$

SVQSPP(II): Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i,\bar{y}_i) - \ell_i(\bar{x}_i,\bar{y}_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$\ell_i(\bar{x}_i,\bar{y}_i) - \ell_i(\bar{x}_i,y_i) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, y_i \in T_i(\bar{x}).$$

SVQSPP(III): Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i, \bar{y}_i) - \ell_i(\bar{x}_i, \bar{y}_i) \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, x_i \in S_i(\bar{x})$$

and

$$\ell_i(\bar{x}_i, \bar{y}_i) - \ell_i(\bar{x}_i, y_i) \notin -\text{int}\, C_i(\bar{x}), \quad \forall\, y_i \in T_i(\bar{x}).$$

Remark 19. If for each $i \in I$ and $\forall\, x \in X$, $C_i(x)$ is a convex pointed cone, then every solution of SVQSPP(I) is a solution of SVQSPP(II) and every solution of SVQSPP(II) is a solution of SVQSPP(III). But the converse implication is not true.

If I is a singleton set and $Z = \mathbb{R}$ then SVQSPP(I), SVQSPP(II) and SVQSPP(III) are called a *quasi-saddle point problem* (QSPP). Of course, if I is a singleton set, $S_i(x) = X_i$ and $T_i(x) = Y_i$, $\forall\, x \in X$ and $Z_i = \mathbb{R}$, then above-mentioned SVQSPPs reduce to the classical saddle point problem. A study of saddle point for set-valued maps can be found in [77].

For each $i \in I$, let ℓ_i be a real-valued bifunction. We also consider the following *problem of system of quasi-minimax inequalities* (SQMIP): find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ and

$$\min_{u_i \in S_i(\bar{x}_i)} \max_{v_i \in T_i(\bar{x}_i)} \ell_i(u_i, v_i) = \ell_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x}_i)} \min_{u_i \in S_i(\bar{x}_i)} \ell_i(u_i, v_i).$$

When I is a singleton set, SQMIP is called *quasi-minimax inequality problem* (QMIP). A study of a minimax type inequality for vector-valued functions can be found in [60, 66].

As application of Theorem 18, we derive the following existence result for a solution of SGVQSPP(I).

Theorem 19. *For each $i \in I$, let E_i, F_i and Z_i be real locally convex topological vector spaces and also F_i be quasi-complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and $Y_i \subseteq F_i$ a nonempty convex set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values and $T_i : X \to 2^{Y_i}$ a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *$C_i : X \to 2^{Z_i}$ is a closed multivalued map such that $\forall\, x \in X$, $C_i(x)$ is a closed convex cone with apex at the origin, and $P_i = \bigcap_{x\in X} C_i(x)$.*
(ii) *P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.*
(iii) *$\ell_i : X_i \times Y_i \to Z_i$ is a continuous function such that:*
    (a) *For each fixed $y_i \in Y_i$, $x_i \mapsto \ell_i(x_i, y_i)$ is properly quasi-convex.*
    (b) *For each fixed $x_i \in X_i$, $y_i \mapsto \ell_i(x_i, y_i)$ is properly quasi-concave.*

Then the SVQSPP(I) has a solution.

If I is a singleton set and $Z = \mathbb{R}$, then we have following existence result for a solution of the quasi-saddle point problem.

Corollary 14. *Let E and F be real locally convex topological vector spaces and also F be quasi-complete. Let $X \subseteq E$ be a nonempty compact convex set and $Y \subseteq F$*

a nonempty convex set. Let $S : X \to 2^X$ be a continuous multivalued map with nonempty closed convex values and $T : X \to 2^Y$ a continuous multivalued map with nonempty compact convex values. Assume that $\ell : X \times Y \to Z$ is a continuous function such that:

(a) *For each fixed $y \in Y$, $x \mapsto \ell(x,y)$ is quasi-convex.*
(b) *For each fixed $x \in X$, $y \mapsto \ell(x,y)$ is quasi-concave.*

Then the QSPP has a solution.

As a consequence of Theorem 19, we have the following existence result for a solution of the system of quasi-minimax inequalities.

Theorem 20. *Let E_i, F_i, X_i, Y_i, X, Y, S_i and T_i be the same as in Theorem 3.1. For each $i \in I$, assume that $\ell_i : X_i \times Y_i \to \mathbb{R}$ is a continuous function satisfying the following conditions:*

(i) *For each fixed $y_i \in Y_i$, $x_i \mapsto \ell_i(x_i,y_i)$ is quasi-convex.*
(ii) *For each fixed $x_i \in X_i$, $y_i \mapsto \ell_i(x_i,y_i)$ is quasi-concave.*

Then the SQMIP has a solution.

If for each $i \in I$, X_i and Y_i are nonempty compact convex sets, and $S_i(x) = X_i$ and $T_i(x) = Y_i$, $\forall\, x \in X$, then from Theorem 20 we derive the following corollary which can be seen as an extension of Sion's minimax theorem [92] for a family of continuous bifunctions.

Corollary 15. *For each $i \in I$, let X_i and Y_i be nonempty compact convex subsets of E_i and F_i, respectively. For each $i \in I$, assume that $\ell_i : X_i \times Y_i \to \mathbb{R}$ is a continuous function satisfying the following conditions:*

(i) *For each fixed $y_i \in Y_i$, $x_i \mapsto \ell_i(x_i,y_i)$ is quasi-convex.*
(ii) *For each fixed $x_i \in X_i$, $y_i \mapsto \ell_i(x_i,y_i)$ is quasi-concave.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$,

$$\min_{u_i\in X_i} \max_{v_i\in Y_i} \ell_i(u_i,v_i) = \ell_i(\bar{x}_i,\bar{y}_i) = \max_{v_i\in Y_i} \min_{u_i\in X_i} \ell_i(u_i,v_i).$$

If I is a singleton, then Theorem 20 reduces to the Corollary 3.2 in [73].

Corollary 16. *Let E, F, X, Y, S and T be the same as in Corollary 14. Assume that $\ell : X \times Y \to \mathbb{R}$ is a continuous function satisfying the following conditions:*

(i) *For each fixed $y \in Y$, $x \mapsto \ell(x,y)$ is quasi-convex.*
(ii) *for each fixed $x \in X$, $y \mapsto \ell(x,y)$ is quasi-concave.*

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{y})$ and

$$\min_{u\in S(\bar{x})} \max_{v\in T(\bar{x})} \ell(u,v) = \ell(\bar{x},\bar{y}) = \max_{v\in T(\bar{x})} \min_{u\in S(\bar{x})} \ell(u,v).$$

6.3 Debreu Type Equilibrium Problem

In this section, we give another application of Corollary 11 to prove the existence of a solution of the Debreu VEP.

Let X, X_i, Z_i and C_i be the same as defined in the formulations of SSGVQEP. For each $i \in I$, let $\varphi_i : X \to Z_i$ be a vector-valued function and let $X^i = \prod_{j\in I,\ j\neq i} X_j$ and we write $X = X^i \times X_i$. For $x \in X$, x^i denotes the projection of x onto X^i and hence we write $x = (x^i, x_i)$. We consider the following Debreu VEP:
Debreu VEP(I): Find $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \in C_i(\bar{x}), \quad \forall\, y_i \in S_i(\bar{x}).$$

Debreu VEP(II): Find $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -C_i(\bar{x}) \setminus \{\mathbf{0}\}, \quad \forall\, y_i \in S_i(\bar{x}).$$

Debreu VEP(III): Find $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$ and

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\mathrm{int}\, C_i(\bar{x}), \quad \forall\, y_i \in S_i(\bar{x}),$$

in this case we assume that $\mathrm{int}\, C_i$ is nonempty for each $i \in I$.

Of course, if for each $i \in I$, φ_i is a scalar-valued function, then Debreu VEPs are the same as the one introduced and studied by Debreu in [38], see also [80–82]. In this case, a large number of papers have already been appeared in the literature; see, for example, [9,98] and the references therein. In [5], we introduced and studied Debreu VEP(III) and established several existence results for its solution with or without involving Φ-condensing maps. It is the first paper in the literature in which the Debreu type equilibrium problem for vector-valued functions is considered.

As in the case of SSGVQEPs, if for each $i \in I$ and $\forall\, x \in X$, $C_i(x)$ is also pointed, then every solution of Debreu VEP(I) is a solution of Debreu VEP(II) and every solution of Debreu VEP(II) is a solution of Debreu VEP(III). But the reverse implication does not hold.

Let $\mathscr{Z}^*$ be the dual of a locally convex topological vector space $\mathscr{Z}$, $P^* \subseteq \mathscr{Z}^*$ the polar cone of P, that is, $P^* = \{z^* \in \mathscr{Z}^* : \langle z^*, z\rangle \geq 0,\ \ \forall z \in P\}$. We assume that P^* has a weak* compact convex base B^*. This means that $B^* \subseteq P^*$ is a weak* compact convex set such that $0 \notin B^*$ and $P^* = \bigcup_{\lambda\geq 0} \lambda B^*$; see, for example, [53].

Theorem 21. *For each $i \in I$, let E_i and Z_i be real locally convex topological vector spaces and E_i be also quasi-complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values. For each $i \in I$, assume that the following conditions are satisfied:*

(i) *$C_i : X \to 2^{Z_i}$ is a closed multivalued map such that $\forall\, x \in X$, $C_i(x)$ is a closed convex cone with apex at the origin, and $P_i = \bigcap_{x\in X} C_i(x)$.*
(ii) *P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.*
(iii) *$\varphi_i : X \to Z_i$ is continuous and properly quasi-convex in each argument.*

Then there exists a solution $\bar{x} \in X$ of the Debreu VEP(I).

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