11

Approximate symbolic models for control

This chapter continues the generalization from exact to approximate similarity, now in the context of symbolic models for control. We discuss approximate feedback composition and refinement, and show how the techniques developed in Chapter 10 can be suitably extended to control systems and switched affine systems. Nonlinear extensions of these results are presented as special topics.

Notation

The following notation is used in this chapter. For any matrix $P \in \mathbb{R}^{n \times n}$, P^T denotes the transposed matrix. Matrix P is said to be symmetric if $P^T = P$, and is said to be positive definite if for every $x \in \mathbb{R}^n$, $x \neq 0$ implies $x^T P x > 0$. We denote by $\mathcal{SP}(n)$ the set of all symmetric and positive definite matrices in $\mathbb{R}^{n \times n}$. The minimum and the maximum eigenvalues of a matrix $P \in \mathbb{R}^{n \times n}$ are denoted by $\lambda_m(P)$ and $\lambda_M(P)$, respectively. For any $x \in \mathbb{R}^n$, $\|x\|$ represents the Euclidean norm of x defined by $\|x\| = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{\frac{1}{2}}$ where x_i is the ith component of the vector x. This norm induces a norm in the space of matrices that can be computed as $\|A\| = \lambda_M^{\frac{1}{2}}(A^T A)$ for any $A \in \mathbb{R}^{n \times m}$. The exponential of any matrix $A \in \mathbb{R}^{n \times n}$ is denoted by e^A and is the analytic function $\sum_{i=0}^{\infty} \frac{1}{i!} A^i$. The ball of radius $r \in \mathbb{R}_0^+$ centered at $x \in \mathbb{R}^n$ is denoted by $\mathcal{B}_r(x)$ and defined as the set of all the points $x' \in \mathbb{R}^n$ satisfying $\|x - x'\| \leq r$. If $Z \subseteq \mathbb{R}^n$ and $\eta \in \mathbb{R}^+$, $[Z]_\eta$ denotes the subset $[Z]_\eta \subseteq Z$ defined by:

$$[Z]_\eta = \left\{ z \in Z \mid z_i = k_i \frac{2}{\sqrt{n}} \eta \text{ for some } k_i \in \mathbb{Z} \text{ and } i = 1, 2, \ldots, n \right\}.$$

Note that we can cover Z by balls of radius η centered at the points in $[Z]_\eta$. This observation is used several times in this chapter.

Given a subset $W \subseteq Z$ we denote by $\imath : W \hookrightarrow Z$ the natural inclusion of W in Z taking $w \in W$ to $\imath(w) = w \in Z$. The identity map on Z is denoted by $1_Z : Z \to Z$ while $\pi_X : X_a \times X_b \times U_a \times U_b \to X_a \times X_b$ denotes the projection

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sending $(x_a, x_b, u_a, u_b) \in X_a \times X_b \times U_a \times U_b$ to $(x_a, x_b) \in X_a \times X_b$. A relation $R \subseteq Z \times W$ is surjective when for every $w \in W$ there is a $z \in Z$ satisfying $(z, w) \in R$.

A metric on a set Z is a function $\mathbf{d} : Z \times Z \to \mathbb{R}_0^+$ satisfying: $\mathbf{d}(z, z') = 0$ iff $z = z'$; $\mathbf{d}(z, z') + \mathbf{d}(z', z'') \geq \mathbf{d}(z, z'')$; $\mathbf{d}(z, z') = \mathbf{d}(z', z)$. A metric $\mathbf{d}$ is said to be norm-induced if $\mathbf{d}(x, y) = \|x - y\|$ for some norm $\|\cdot\|$ and for every $x, y \in Z$. A metric $\mathbf{d} : Z \times Z \to \mathbb{R}_0^+$ on the set Z induces a pseudo-metric on 2^Z, the set of all subsets of Z. Such pseudo-metric, called the Hausdorff pseudo-metric and denoted by $\mathbf{d}_h$, is defined by $\mathbf{d}_h(K, W) = \max\left\{\overrightarrow{\mathbf{d}_h}(K, W), \overrightarrow{\mathbf{d}_h}(W, K)\right\}$, where $\overrightarrow{\mathbf{d}_h}(K, W) = \sup_{k \in K} \inf_{w \in W} \mathbf{d}(k, w)$ is the directed Hausdorff pseudo-metric and $K, W \subseteq Z$. We recall that the Hausdorff pseudo-metric $\mathbf{d}_h$ satisfies all the requirements of a metric except that $W = W'$ implies $\mathbf{d}_h(W, W') = 0$ but $\mathbf{d}_h(W, W') = 0$ does not imply $W = W'$.

A function $f :]a, b[\to \mathbb{R}^n$, $a, b \in \mathbb{R}$, is said to be piecewise continuous if there exists an ordered sequence of real numbers $a = i_1 < i_2 < \ldots < i_k = b$ such that for every $j \in \{1, 2, \ldots, k-1\}$, the restriction of f to the interval $]i_j, i_{j+1}[$ is continuous. A piecewise continuous function $f :]a, b[\to \mathbb{R}^n$ is essentially bounded if there exists a compact set $K \subset \mathbb{R}^n$ such that $f(t) \in K$ for almost all $t \in]a, b[$. When $f :]a, b[\to \mathbb{R}^n$ is an essentially bounded piecewise continuous function, the supremum norm of f, denoted by $\|f\|$, is the supremum of the set $\{r \in \mathbb{R}_0^+ \mid \exists\, t \in]a, b[\quad r = \|f(t)\| \wedge f(t) \in K\}$. The domain of a function $f : Z \to W$ is denoted by $\operatorname{dom} f$.

11.1 Stability of linear control systems

We review a few stability results needed for the study of approximate simulations and bisimulations. The reader is expected to have read Section 8.1.2 where several concepts related to control systems were introduced. Here, we consider affine control systems described by the affine differential equation:

$$\frac{d}{dt}\xi = A\xi + C\chi + D\delta + h \tag{11.1}$$

with $\xi(t) \in \mathbb{R}^n$, $\chi(t) \in \mathbb{R}^m$, $\delta(t) \in \mathbb{R}^l$, $\chi \in \mathcal{C}$, $\delta \in \mathcal{D}$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times l}$, $h \in \mathbb{R}^n$, and $t \in \mathbb{R}_0^+$. We distinguish between two different kinds of inputs: control inputs χ, and disturbance inputs δ. We are thus taking $\mathcal{U} = \mathcal{C} \times \mathcal{D}$ and $\upsilon = (\chi, \delta)$ according to the notion of continuous-time control system introduced in Section 8.1.2. Independently of the nature of the inputs, a solution to (11.1) can always be written in the form:

$$\xi_{x\chi\delta}(\tau) = e^{A\tau}x + \int_0^\tau e^{A(\tau - t)}\left(C\chi(t) + D\delta(t) + h\right) dt. \tag{11.2}$$

Affine control systems are denoted by the septuple $\Sigma = (\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D, h)$ or by the sextuple $\Sigma = (\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$ when $h = 0$. In the later case we speak of a linear control system. Although we are interested in the slightly more general class of affine control systems, it is sufficient to consider the stability properties of linear control systems. In some of the results we will assume the absence of disturbances, *i.e.*, $D = 0$. In such cases we denote Σ by the quadruple $(\mathbb{R}^n, \mathcal{C}, A, C)$.

Definition 11.1 (Input-to-state stability). *A linear control system* $(\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$ *is said to be* input-to-state stable (ISS) *when there exist constants* $\kappa, \lambda, \rho_c, \rho_d \in \mathbb{R}^+$ *such that for any* $x \in \mathbb{R}^n$*, any* $\chi \in \mathcal{C}$*, any* $\delta \in \mathcal{D}$*, and any* $t \in \mathbb{R}^+$*, the following inequality is satisfied:*

$$\|\xi_{x\chi\delta}(t)\| \leq \kappa e^{-\lambda t}\|x\| + \rho_c\|\chi\| + \rho_d\|\delta\|. \tag{11.3}$$

Inequality (11.3) extends inequality (10.3) from linear dynamical systems to linear control systems. The next step is to extended also the concept of Lyapunov function.

Definition 11.2 (ISS Lyapunov function). *Let* $(\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$ *be a linear control system and consider a function* $V : \mathbb{R}^n \to \mathbb{R}$ *satisfying the following three properties:*

1. V *is continuous on* $\mathbb{R}^n$ *and smooth on* $\mathbb{R}^n \backslash \{0\}$*;*
2. $V(x) \geq 0$ *for all* $x \in \mathbb{R}^n$*;*
3. $V(x) = 0$ *implies* $x = 0$*.*

The function V *is an* ISS-Lyapunov function *for* Σ *if there exist constants* $\lambda, \sigma_c, \sigma_d \in \mathbb{R}^+$ *such that for all* $x \in \mathbb{R}^n \backslash \{0\}$*,* $c \in \mathbb{R}^m$*, and* $d \in \mathbb{R}^l$*, the following inequality holds:*

$$\frac{\partial V}{\partial x}(Ax + Cc + Dd) \leq -\lambda V(x) + \sigma_c\|c\| + \sigma_d\|d\|. \tag{11.4}$$

Inequality (11.4) entails the differential inequality:

$$\frac{d}{dt} V \circ \xi \leq -\lambda V \circ \xi + \sigma_c\|\chi\| + \sigma_d\|\delta\|$$

that can be integrated to provide the estimate:

$$\begin{aligned} V \circ \xi(t) &\leq e^{-\lambda t}V(\xi(0)) + \frac{\sigma_c\|\chi\|}{\lambda}(1 - e^{-\lambda t}) + \frac{\sigma_d\|\delta\|}{\lambda}(1 - e^{-\lambda t}) \\ &\leq e^{-\lambda t}V(\xi(0)) + \frac{\sigma_c}{\lambda}\|\chi\| + \frac{\sigma_d}{\lambda}\|\delta\|. \end{aligned} \tag{11.5}$$

Inequality (11.5) can be combined with (10.5) to fully characterize ISS in terms of ISS-Lyapunov functions as stated in the next result.

Theorem 11.3. *A linear control system* Σ *is input-to-state stable iff* Σ *admits an ISS-Lyapunov function.*

For linear systems, the above theorem can be strengthened by asserting that ISS implies the existence of an ISS-Lyapunov function of the form $V(x) = \sqrt{x^T P x}$ with $P \in \mathcal{SP}(n)$. Moreover, it can be shown that a linear control system $(\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$ is ISS iff the origin is an asymptotically stable equilibrium point for the linear dynamical system $(\mathbb{R}^n, A)$. The ISS assumption is thus very simple to check since Theorem 10.2 asserts that it suffices to determine if all the eigenvalues of the matrix A have negative real part.

Although input-to-state stability is the assumption upon which all the results in this chapter rely, there is a straightforward extension to a wider class of control systems. When a linear control system Σ is not ISS, it may be rendered ISS by suitably designing a linear feedback control law $\chi = K\xi + \chi'$ transforming Σ into the linear control system defined by:

$$\frac{d}{dt}\xi = (A + CK)\xi + C\chi' + D\delta$$

with new control input χ'. ISS is achieved whenever K makes the real part of the eigenvalues of $A + CK$ negative. The results in this chapter remain valid for this larger class of systems even though, for simplicity, we will directly assume input-to-state stability.

11.2 Control and switched systems as systems

11.2.1 Control Systems

In Chapter 10 we introduced the system $S_\tau(\Sigma)$ describing the time-triggered sampled version of a given dynamical system Σ. A simple generalization is available for control systems.

Definition 11.4. *The system $S_\tau = (X_\tau, U_\tau, \underset{\tau}{\longrightarrow})$ associated with a control system $\Sigma = (\mathbb{R}^n, \mathcal{C} \times \mathcal{D}, f)$ and with $\tau \in \mathbb{R}^+$ consists of:*

- $X_\tau = \mathbb{R}^n$;
- $U_\tau = \{\chi \in \mathcal{C} \mid \operatorname{dom} \chi = [0, \tau]\}$;
- $x \overset{\chi}{\underset{\tau}{\longrightarrow}} x'$ *if there exist* $\chi \in U_\tau$, $\delta \in \mathcal{D}$, *and a trajectory* $\xi_{x\chi\delta} : [0, \tau] \to \mathbb{R}^n$ *of* Σ *satisfying* $\xi_{x\chi\delta}(\tau) = x'$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = \imath : X_\tau \hookrightarrow \mathbb{R}^n$.

The output set $Y_\tau = \mathbb{R}^n$ of $S_\tau(\Sigma)$ is naturally equipped with the norm-induced metric $\mathbf{d}(y, y') = \|y - y'\|$. In addition to control systems, we also consider switched systems.

11.2.2 Switched systems

Switched systems are a class of hybrid dynamical systems frequently arising in embedded control applications. We restrict the discussion to the case where the continuous-time dynamics in each finite state is given by an affine dynamical system.

Definition 11.5 (Switched affine system). *A hybrid dynamical system:*

$$\Sigma = (S_a, \{\mathrm{In}_{x_a}\}_{x_a \in X_a}, \{\mathrm{Gu}_{t_a}\}_{t_a \in \underset{a}{\longrightarrow}}, \{\mathrm{Re}_{t_a}\}_{t_a \in \underset{a}{\longrightarrow}}, \{f_{x_a}\}_{x_a \in X_a})$$

is said to be a switched affine system *if the following conditions are satisfied:*

1. $U_a = X_a$;
2. $\underset{a}{\longrightarrow} = \{(x_a, u_a, x'_a) \in X_a \times X_a \times X_a \mid u_a = x'_a\}$;
3. $\mathrm{In}_{x_a} = \mathbb{R}^n$ *for every* $x_a \in X_a$;
4. $\mathrm{Gu}_{(x_a,u_a,x'_a)} = \mathbb{R}^n$ *for every* $(x_a, u_a, x'_a) \in \underset{a}{\longrightarrow}$;
5. $\mathrm{Re}_{(x_a,u_a,x'_a)}(x_b) = x_b$ *for every* $(x_a, u_a, x'_a) \in \underset{a}{\longrightarrow}$ *and* $x_b \in \mathrm{In}_{x_a}$;
6. $f_{x_a}(x_b) = A_{x_a} x_b + h_{x_a}$ *for some matrix* $A_{x_a} \in \mathbb{R}^{n\times n}$, *some vector* $h_{x_a} \in \mathbb{R}^n$, *and all* $x_a \in X_a$, $x_b \in \mathrm{In}_{x_a}$.

In a switched affine system it is possible, at any time and independently of the infinite state, to switch from any finite state to any other finite state without changing the infinte part of the state. This possibility is described by the several requirements in Definition 11.5. The first two requirements ask that for every two finite states $x_a, x'_a \in X_a$ there exists one and only one transition between them: $x_a \overset{x'_a}{\underset{a}{\longrightarrow}} x'_a$. The third and sixth requirements ask that in each finite state $x_a \in X_a$, the switched system behaves like the affine dynamical system $(\mathbb{R}^n, A_{x_a}, h_{x_a})$. The fourth requirement allows for discrete transitions to take place at any time and for any value of the infinite part of the state. Finally, the fifth condition declares that discrete transitions do not alter the infinite part of the state. These restrictions also imply that a switched affine system is completely defined by the finite set of states X_a, and the collection of affine dynamical systems $\{(\mathbb{R}^n, A_{x_a}, h_{x_a})\}_{x_a \in X_a}$. For this reason, we also denote a switched affine system by the triple $\Sigma = (X_a, \mathbb{R}^n, \{A_{x_a}, h_{x_a}\}_{x_a \in X_a})$.

Example 11.6. Switched affine systems provide a useful framework for switching control. Suppose that several affine controllers:

$$c = K_1 x + h_1, c = K_2 x + h_2, \ldots, c = K_p x + h_p,$$

have been designed to control the linear system:

$$\dot{\xi} = A\xi + C\chi, \qquad \xi(t) \in \mathbb{R}^n, \chi(t) \in \mathbb{R}^m, t \in \mathbb{R}^+_0.$$

If these controllers can be used independently of the infinite state $x \in \mathbb{R}^n$, we have a switched affine system Σ described by:

$$(\{1, 2, \ldots, p\}, \mathbb{R}^n, \{A + CK_i, Ch_i\}_{i \in \{1,2,\ldots,p\}}).$$

A software module deciding which controller is executed and when, can now be seen as a supervisory controller acting on the switched affine system Σ. $\triangleleft$

When switched affine systems are viewed as models for switching control, the supervisory controller is typically implemented as a periodic task, with period τ, running on a microprocessor. This implies that discrete transitions only happen at instants that are integer multiples of τ. An appropriate model for this kind of system is $S_\tau(\Sigma)$, capturing only transitions of duration τ.

Definition 11.7. *The system* $S_\tau(\Sigma) = (X_\tau, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau)$ *associated with a switched affine system* $\Sigma = (X_a, \mathbb{R}^n, \{A_{x_a}, h_{x_a}\}_{x_a \in X_a})$ *and with* $\tau \in \mathbb{R}^+$ *consists of:*

- $X_\tau = \mathbb{R}^n$;
- $U_\tau = X_a$;
- $x \overset{u_a}{\underset{\tau}{\longrightarrow}} x'$ *if there exists a solution* $\xi_x : [0, \tau] \to \mathbb{R}^n$ *of the affine dynamical system* $(\mathbb{R}^n, A_{u_a}, h_{u_a})$ *satisfying* $\xi_x(\tau) = x'$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = \imath : X_\tau \hookrightarrow \mathbb{R}^n$.

Note that $S_\tau(\Sigma)$ is both infinite-state as well as metric with a norm-induced metric.

11.3 Approximate feedback composition and controller refinement

The controller refinement process carries over, mutatis mutandis, from the exact to the approximate case. We recall that in Chapter 1 we simplified the representation of the composition $S_a \times_{\mathcal{I}} S_b$ whenever the interconnection relation $\mathcal{I}$ satisfied the condition:

$$(x_a, x_b) \in \pi_X(\mathcal{I}) \implies H_a(x_a) = H_b(x_b).$$

In the current approximate context, we consider the generalized condition:

$$(x_a, x_b) \in \pi_X(\mathcal{I}) \implies \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$$

and make the additional assumption that $\mathbf{d}$ is norm-induced. Note that this assumption entails that $Y_a = Y_b$ are normed vector spaces with the same norm. Under these assumptions, we denote the composition by:

$$S_a \times_{\mathcal{I}}^{\varepsilon} S_b = (X_{ab}, X_{ab0}, U_{ab}, \underset{ab}{\longrightarrow}, Y_{ab}, H_{ab})$$

and simplify its representation to:

- $X_{ab} = \pi_X(\mathcal{I})$;
- $X_{ab0} = X_{ab} \cap (X_{a0} \times X_{b0})$;
- $U_{ab} = U_a \times U_b$;
- $(x_a, x_b) \xrightarrow[ab]{u_a, u_b} (x'_a, x'_b)$ if the following three conditions hold:
  1. $x_a \xrightarrow[a]{u_a} x'_a$ in S_a;
  2. $x_b \xrightarrow[b]{u_b} x'_b$ in S_b;
  3. $(x_a, x_b, u_a, u_b) \in \mathcal{I}$;
- $Y_{ab} = Y_a = Y_b$;
- $H_{ab}(x_a, x_b) = \frac{1}{2}(H_a(x_a) + H_b(x_b))$.

The apparently arbitrary choice of output map is justified by the following three important properties of approximate composition:

1. $S_a \times_{\mathcal{I}}^{\varepsilon} S_b$ is commutative, *i.e.*, $S_a \times_{\mathcal{I}}^{\varepsilon} S_b \cong_{\mathcal{S}} S_b \times_{\mathcal{I}}^{\varepsilon} S_a$;
2. $S_a \times_{\mathcal{I}}^{\varepsilon} S_b$ generalizes exact composition, *i.e.*, $S_a \times_{\mathcal{I}}^{0} S_b = S_a \times_{\mathcal{I}} S_b$;
3. $S_a \times_{\mathcal{I}}^{\varepsilon} S_b$ satisfies the following version of Proposition 6.3.

Proposition 11.8. *Let S_a and S_b be metric systems with $Y_a = Y_b$ normed vector spaces with the same norm-induced metric, and let $\mathcal{I}$ be an interconnection relation satisfying:*

$$(x_a, x_b) \in \pi_X(\mathcal{I}) \implies \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon.$$

Then, the following holds:

- $S_a \times_{\mathcal{I}}^{\varepsilon} S_b \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_a$*;*
- $S_a \times_{\mathcal{I}}^{\varepsilon} S_b \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_b$*.*

Proof. The proof of this result is the same as the proof of its exact counterpart, Proposition 6.3, except for the computation of the precision. We thus focus on this part and consider only $S_a \times_{\mathcal{I}}^{\varepsilon} S_b \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_a$ since the case $S_a \times_{\mathcal{I}}^{\varepsilon} S_b \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S_b$ can be similarly proved. The desired $\frac{1}{2}\varepsilon$-approximate simulation relation from $S_a \times_{\mathcal{I}}^{\varepsilon} S_b$ to S_a is given by:

$$R_a = \{((x_a, x_b), x'_a) \in X_{ab} \times X_a \mid x_a = x'_a\}.$$

For any $((x_a, x_b), x_a) \in R_a$ it is simple to see that:

$$\begin{aligned}\mathbf{d}(H_{ab}(x_a, x_b), H_a(x_a)) &= \left\| \frac{1}{2} H_a(x_a) + \frac{1}{2} H_b(x_b) - H_a(x_a) \right\| \\ &= \left\| -\frac{1}{2} H_a(x_a) + \frac{1}{2} H_b(x_b) \right\| \\ &= \frac{1}{2} \mathbf{d}(H_a(x_a), H_b(x_b)) \leq \frac{1}{2} \varepsilon\end{aligned}$$

since $(x_a, x_b) \in \pi_X(\mathcal{I})$. □

With the notion of approximate composition at our disposal we venture into approximate feedback composition.

Definition 11.9 (Approximate feedback composition). *A system S_c is said to be ε-*approximately feedback composable *with a system S_a, if there exists an ε-approximate alternating simulation relation R from S_c to S_a. When S_c is ε-approximate feedback composable with S_a, the feedback composition of S_c and S_a, with interconnection relation $\mathcal{F} = R^e$, is given by $S_c \times_{\mathcal{F}}^{\varepsilon} S_a$.*

Proposition 8.7 also admits an approximate version.

Proposition 11.10. *Let S_a, S_b, and S_c be systems with the same output set, assume that S_c is ${}_c\varepsilon_a$-approximately feedback composable with S_a, and let ${}_cR_a$ be the corresponding ${}_c\varepsilon_a$-approximate alternating simulation relation. If there exists a ${}_a\varepsilon_b$-approximate alternating simulation relation ${}_aR_b$ from S_a to S_b then $S_c \times_{{}_cR_a^e}^{{}_c\varepsilon_a} S_a$ is feedback composable with S_b and the corresponding $({}_c\varepsilon_a + {}_a\varepsilon_b)$-approximate alternating simulation relation is given by:*

$${}_{ca}R_b = \{((x_c, x_a), x_b) \in (X_c \times X_a) \times X_b \mid (x_c, x_a) \in {}_cR_a \wedge (x_a, x_b) \in {}_aR_b\}.$$

The proof of this result consists in inserting the word approximate in several locations along the proof of Proposition 8.7 and is therefore omitted. Proposition 11.10 suggests how to refine a controller S_{cont} synthesized to solve a simulation game for an approximate finite-state abstraction S_{abs} of S and a specification S_{spec}. If the simulation game is solved exactly, *i.e.*, with $\varepsilon = 0$, we have:

$$S_{cont} \times_{\mathcal{F}} S_{abs} \preceq_{\mathcal{S}}^{0} S_{spec}.$$

Assuming the abstraction S_{abs} to be related to the original system S by an ε-approximate alternating simulation relation, we can invoke Proposition 11.10 to conclude that $S_{cont} \times_{\mathcal{F}} S_{abs}$ is ε-approximately feedback composable with S. Therefore, using $S'_{cont} = S_{cont} \times_{\mathcal{F}} S_{abs}$ as a controller for S we obtain:

$$S'_{cont} \times_{\mathcal{G}}^{0+\varepsilon} S \preceq_{\mathcal{S}}^{\frac{1}{2}\varepsilon} S'_{cont} = S_{cont} \times_{\mathcal{F}} S_{abs} \preceq_{\mathcal{S}}^{0} S_{spec}$$

which shows the specification approximately simulating the controlled system $S'_{cont} \times_{\mathcal{G}}^{\varepsilon} S$ with precision $\frac{1}{2}\varepsilon$.

11.4 Symbolic models for affine control systems

The abstractions constructed in Chapter 10 for dynamical systems relied on quantizing the set of states and approximating the transitions of $S_\tau(\Sigma)$. A natural generalization to control systems leads to the following construction.

Definition 11.11. *The system* $S_{\tau\eta} = (X_{\tau\eta}, U_{\tau\eta}, \underset{\tau\eta}{\longrightarrow}, Y_{\tau\eta}, H_{\tau\eta})$ *associated with a control system* $\Sigma = (\mathbb{R}^n, \mathcal{C} \times \mathcal{D}, f)$ *and with* $\tau, \eta \in \mathbb{R}^+$ *consists of:*

- $X_{\tau\eta} = [\mathbb{R}^n]_\eta$;
- $U_{\tau\eta} = \{\chi \in \mathcal{C} \mid \operatorname{dom}\chi = [0,\tau]\}$;
- $x \underset{\tau\eta}{\overset{\chi}{\longrightarrow}} x'$ *if there exist* $\chi \in U_{\tau\eta}$, $\delta \in \mathcal{D}$, *and a trajectory* $\xi_{x\chi\delta} : [0,\tau] \to \mathbb{R}^n$ *of* Σ *satisfying* $\|\xi_{x\chi\delta}(\tau) - x'\| \leq \eta$;
- $Y_{\tau\eta} = \mathbb{R}^n$;
- $H_{\tau\eta} = \imath : X_{\tau\eta} \hookrightarrow \mathbb{R}^n$.

The system $S_{\tau\eta}(\Sigma)$ can be regarded as a time and space quantization of a control system Σ. It is constructed by approximating the transitions of $S_\tau(\Sigma)$ so as to enforce departure from and arrival at states in $X_{\tau\eta} = [\mathbb{R}^n]_\eta$. This construction is not guaranteed to result in a system approximately simulated by $S_\tau(\Sigma)$ since the mismatch between outputs of $S_{\tau\eta}(\Sigma)$ and $S_\tau(\Sigma)$ can grow without bounds along any two external behaviors. In Chapter 10 we relied on asymptotic stability to overcome this difficulty in the context of dynamical systems. A similar strategy can be employed for control systems in order to establish the existence of an approximate alternating simulation relation from $S_{\tau\eta}(\Sigma)$ to $S_\tau(\Sigma)$. Moreover, such relation would desirably be surjective since this allows us to relate any state of $S_\tau(\Sigma)$ to a state of $S_{\tau\eta}(\Sigma)$ for which a controller can be designed.

Theorem 11.12. *Let* $\Sigma = (\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D, h)$ *be an affine control system and assume that the linear dynamical system,* $(\mathbb{R}^n, A)$ *admits a Lyapunov function* V *of the form* $V(x) = \sqrt{x^T P x}$ *with* $P \in \mathcal{SP}(n)$. *For any desired precision* $\varepsilon \in \mathbb{R}^+$, *for any desired time quantization* $\tau \in \mathbb{R}^+$, *and for any space quantization* $\eta \in \mathbb{R}^+$ *satisfying:*

$$\eta \leq \min\left\{\gamma^{-1}\underline{\alpha}\varepsilon\left(1 - e^{-\lambda\tau}\right), \overline{\alpha}^{-1}\underline{\alpha}\varepsilon\right\}, \tag{11.6}$$

the relation $R_\varepsilon \subseteq X_{\tau\eta} \times X_\tau$ *defined by:*

$$R_\varepsilon = \{(x_{\tau\eta}, x_\tau) \in X_{\tau\eta} \times X_\tau \mid V(x_\tau - x_{\tau\eta}) \leq \underline{\alpha}\varepsilon\} \tag{11.7}$$

is a surjective ε*-approximate alternating simulation relation from* $S_{\tau\eta}(\Sigma)$ *to* $S_\tau(\Sigma)$.

Proof. The proof consists in showing that R_ε satisfies all the requirements in the definition of approximate alternating simulation relation.

We first note that R_ε is surjective since $\mathbb{R}^n \subseteq \cup_{x\in[\mathbb{R}^n]_\eta}\mathcal{B}_\eta(x)$ implies that for every $x_\tau \in X_\tau = \mathbb{R}^n$ there exists $x_{\tau\eta} \in X_{\tau\eta}$ satisfying $\|x_\tau - x_{\tau\eta}\| \leq \eta$. It follows from the sequence of inequalities (10.9) that $(x_{\tau\eta}, x_\tau) \in R_\varepsilon$.

The first requirement in Definition 9.6 follows immediately from the definition of $X_{\tau 0}$ and $X_{\tau\eta 0}$, and from the observation that $x_{\tau\eta 0} \in X_{\tau\eta 0} \subset X_{\tau 0}$ implies $(x_{\tau\eta 0}, x_{\tau 0}) \in R_\varepsilon$ for $x_{\tau 0} = x_{\tau\eta 0}$.

The second requirement is a consequence of the definition of R_ε. If $(x_{\tau\eta}, x_\tau) \in R_\varepsilon$, then $V(x_{\tau\eta}-x_\tau) \leq \underline{\alpha}\varepsilon$ which leads, by (10.5), to $\|x_{\tau\eta} - x_\tau\| \leq \varepsilon$.

We now consider the third requirement which requires us to show that $(x_{\tau\eta}, x_\tau) \in R_\varepsilon$ implies:

$$\forall u_{\tau\eta} \in U_{\tau\eta}(x_{\tau\eta}) \quad \exists u_\tau \in U_\tau(x_\tau) \quad \forall x'_\tau \in \mathrm{Post}_{u_\tau}(x_\tau) \quad \exists x'_{\tau\eta} \in \mathrm{Post}_{u_{\tau\eta}}(x_{\tau\eta})$$

with $(x'_{\tau\eta}, x'_\tau) \in R_\varepsilon$. Fix an input $u_{\tau\eta} \in U_{\tau\eta}(x_{\tau\eta})$ and note that it follows from the definition of $U_{\tau\eta}$ that $u_{\tau\eta} \in U_\tau(x_\tau)$. We then choose u_τ to be $u_{\tau\eta}$, *i.e.*, $u_\tau = u_{\tau\eta}$. Let now $x'_\tau \in \mathrm{Post}_{u_\tau}(x_\tau)$. This means that $x'_\tau = \xi_{x_\tau u_\tau \delta}(\tau)$ for some essentially bounded piecewise continuous curve $\delta \in \mathcal{D}$. Consider a state $x'_{\tau\eta} \in \mathrm{Post}_{u_{\tau\eta}}(x_{\tau\eta})$ satisfying $x_{\tau\eta} \xrightarrow[\tau\eta]{u_{\tau\eta},\delta} x'_{\tau\eta}$ in $S_{\tau\eta}(\Sigma)$ and recall that, by definition of $S_{\tau\eta}(\Sigma)$, we have:

$$\|\xi_{x_{\tau\eta} u_{\tau\eta}\delta}(\tau) - x'_{\tau\eta}\| \leq \eta. \tag{11.8}$$

We claim that $(x'_{\tau\eta}, x'_\tau) \in R_\varepsilon$. To prove the claim, consider the sequence of inequalities:

$$\begin{aligned}
V(x'_\tau, x'_{\tau\eta}) &\leq V\left(x'_\tau - \xi_{x_{\tau\eta}u_{\tau\eta}\delta}(\tau)\right) + \gamma\|\xi_{x_{\tau\eta}u_{\tau\eta}\delta}(\tau) - x'_{\tau\eta}\| \\
&\leq V\Bigg(e^{A\tau}x_\tau + \int_0^\tau e^{A(\tau-t)}\left(Cu_{\tau\eta}(t) + D\delta(t) + h\right)dt \\
&\qquad -e^{A\tau}x_{\tau\eta} - \int_0^\tau e^{A(\tau-t)}\left(Cu_{\tau\eta}(t) + D\delta(t) + h\right)dt\Bigg) + \gamma\eta \\
&\leq V\left(e^{A\tau}x_\tau - e^{A\tau}x_{\tau\eta}\right) + \gamma\eta \\
&\leq V\left(\xi_{x_\tau 00}(\tau) - \xi_{x_{\tau\eta}00}(\tau)\right) + \gamma\eta \\
&\leq e^{-\lambda\tau}V\left(\xi_{x_\tau 00}(0) - \xi_{x_{\tau\eta}00}(0)\right) + \gamma\eta \\
&\leq e^{-\lambda\tau}V\left(x_\tau - x_{\tau\eta}\right) + \gamma\eta \\
&\leq e^{-\lambda\tau}\underline{\alpha}\varepsilon + \gamma\eta \\
&\leq \underline{\alpha}\varepsilon
\end{aligned}$$

where the first, second, fifth, seventh, and eight inequalities are a consequence of (10.6), (11.8), (10.16), (11.7), and (11.6), respectively. □

Although we established the existence of a surjective ε-approximate alternating simulation relation from $S_{\tau\eta}(\Sigma)$ to $S_\tau(\Sigma)$, one problem remains unsolved: how do we compute $S_{\tau\eta}(\Sigma)$? We address this problem in two steps. First, we treat the case where disturbance inputs are absent: $D = 0$. By choosing a finite set $\mathcal{C}$ of control inputs curves, it becomes possible to compute $S_{\tau\eta}(\Sigma)$ using numerical methods. The errors introduced by numerical simulation can be explicitly accounted for, as discussed in Chapter 10. In practice, the choice of the set $\mathcal{C}$ is based on domain knowledge about the system and problem being solved. When a solution to a control synthesis problem fails to exist for the abstraction, one can choose a larger set $\mathcal{C}$ and compute a new and more faithful abstraction of the system to be controlled. Ideally, one would like to avoid this iterative process and construct directly a symbolic model that can be used to prove or disprove the existence of a controller. This is possible for the important case where the inputs are kept constant during the intervals $[0, \tau]$, commonly referred to as digital control or sampled-data control. The appropriate system model for this situation is the abstraction $S_{\tau\eta\omega}(\Sigma)$.

Definition 11.13. *The system* $S_{\tau\eta\omega} = (X_{\tau\eta\omega}, U_{\tau\eta\omega}, \underset{\tau\eta\omega}{\longrightarrow}, Y_{\tau\eta\omega}, H_{\tau\eta\omega})$ *associated with a control system* $\Sigma = (\mathbb{R}^n, \mathcal{C} \times \mathcal{D}, f)$ *and with* $\tau, \eta, \omega \in \mathbb{R}^+$ *consists of:*

- $X_{\tau\eta\omega} = [\mathbb{R}^n]_\eta$;
- $U_{\tau\eta\omega} = \{\chi \in \mathcal{C} \mid \chi(t) = \chi(t') \in [\mathbb{R}^m]_\omega \quad \forall t, t' \in [0, \tau] = \operatorname{dom} \chi\}$;
- $x \underset{\tau\eta\omega}{\overset{\chi}{\longrightarrow}} x'$ *if there exist* $\chi \in U_{\tau\eta\omega}$, $\delta \in \mathcal{D}$, *and a trajectory* $\xi_{x\chi\delta} : [0, \tau] \to \mathbb{R}^n$ *of* Σ *satisfying* $\|\xi_{x\chi\delta}(\tau) - x'\| \leq \eta$;
- $Y_{\tau\eta\omega} = \mathbb{R}^n$;
- $H_{\tau\eta\omega} = \imath : X_{\tau\eta\omega} \hookrightarrow \mathbb{R}^n$.

The assumption of piecewise constant inputs is satisfied by most embedded control systems implemented in digital platforms. The frequency of the updates is dictated by the dynamics of the physical system being controlled and by the frequency of the embedded microprocessor executing the control software. Under this assumption we can strengthen Theorem 11.12 from simulation to bisimulation.

Theorem 11.14. *Let* $\Sigma = (\mathbb{R}^n, \mathcal{C}, A, C, h)$ *be an affine control system where* $\mathcal{C}$ *is the set of all constant curves, and assume that the linear control system* $(\mathbb{R}^n, \mathcal{C}, A, C)$ *admits an ISS-Lyapunov function* V *of the form* $V(x) = \sqrt{x^T P x}$ *with* $P \in \mathcal{SP}(n)$. *For any desired precision* $\varepsilon \in \mathbb{R}^+$, *for any desired time quantization* $\tau \in \mathbb{R}^+$, *for any desired input quantization* $\omega \in \mathbb{R}^+$, *and for any space quantization* $\eta \in \mathbb{R}^+$ *satisfying:*

$$\eta \leq \min\left\{\gamma^{-1}\underline{\alpha}\varepsilon\left(1 - e^{-\lambda\tau}\right) - \gamma^{-1}\lambda^{-1}\sigma_c\,\omega, \overline{\alpha}^{-1}\underline{\alpha}\varepsilon\right\}, \tag{11.9}$$

the relation $R_\varepsilon \subseteq X_{\tau\eta} \times X_\tau$ *defined by:*

$$R_\varepsilon = \{(x_{\tau\eta}, x_\tau) \in X_{\tau\eta} \times X_\tau \mid V(x_\tau - x_{\tau\eta}) \leq \underline{\alpha}\varepsilon\} \tag{11.10}$$

is an ε*-approximate bisimulation relation between* $S_\tau(\Sigma)$ *and* $S_{\tau\eta}(\Sigma)$.

Inequality (11.9) describes the tradeoff between precision, time quantization, space quantization, and input quantization. It specializes to (10.7), when inputs are absent, thus making Theorem 10.8 a special case of Theorem 11.14.

Proof. We only present the main steps since the proof mirrors the proof of Theorem 11.12. The first important step is to show that the third requirement in Definition 9.2 holds. For this, we consider a pair $(x_\tau, x_{\tau\eta\omega}) \in R_\varepsilon$, we assume that $x_\tau \xrightarrow[\tau]{u_\tau} x'_\tau$, and we seek to show that $(x'_\tau, x'_{\tau\eta\omega}) \in R_\varepsilon$ where $x'_{\tau\eta\omega}$ satisfies $x_{\tau\eta\omega} \xrightarrow[\tau\eta\omega]{u_{\tau\eta\omega}} x'_{\tau\eta\omega}$ for an input $u_{\tau\eta\omega} \in U_{\tau\eta\omega}(x_{\tau\eta\omega})$ close to u_τ in the sense:

$$\|u_{\tau\eta\omega} - u_\tau\| \leq \omega.$$

Note that such input always exists since $\mathbb{R}^m \subseteq \cup_{u \in [\mathbb{R}^m]_\omega} \mathcal{B}_\omega(u)$. The membership $(x'_\tau, x'_{\tau\eta\omega}) \in R_\varepsilon$ follows from the following sequence of inequalities where we use $x'' = x_\tau - x_{\tau\eta\omega}$, $u'' = u_\tau - u_{\tau\eta\omega}$, and the inequality (11.5):

$$\begin{aligned}
V(x'_\tau - x'_{\tau\eta\omega}) &\leq V(x'_\tau - \xi_{x_{\tau\eta\omega} u_{\tau\eta\omega}}(\tau)) + \gamma \|\xi_{x_{\tau\eta\omega} u_{\tau\eta\omega}}(\tau) - x'_{\tau\eta\omega}\| && (11.11)\\
&\leq V\left(e^{A\tau} x_\tau + \int_0^\tau e^{A(\tau - t)}(Cu_\tau + h)dt\right. && (11.12)\\
&\qquad \left. - e^{A\tau} x_{\tau\eta\omega} - \int_0^\tau e^{A(\tau - t)}(Cu_{\tau\eta\omega} + h)dt\right) + \gamma\eta && (11.13)\\
&\leq V\left(e^{A\tau} x'' + \int_0^\tau e^{A(\tau - t)} Cu'' dt\right) + \gamma\eta && (11.14)\\
&\leq V \circ \xi_{x'' u''}(\tau) + \gamma\eta && (11.15)\\
&\leq e^{-\lambda\tau} V \circ \xi_{x'' u''}(0) + \frac{\sigma_c}{\lambda}\|u''\| + \gamma\eta && (11.16)\\
&\leq e^{-\lambda\tau} V(x_\tau - x_{\tau\eta\omega}) + \frac{\sigma_c}{\lambda}\|u''\| + \gamma\eta && (11.17)\\
&\leq e^{-\lambda\tau} \underline{\alpha}\varepsilon + \frac{\sigma_c}{\lambda}\omega + \gamma\eta && (11.18)\\
&\leq \underline{\alpha}\varepsilon. && (11.19)
\end{aligned}$$

The reverse direction is similarly shown. If $(x_\tau, x_{\tau\eta\omega}) \in R_\varepsilon$ and $x_{\tau\eta\omega} \xrightarrow[\tau\eta\omega]{u_{\tau\eta\omega}} x'_{\tau\eta\omega}$, then we claim that $(x'_\tau, x'_{\tau\eta\omega}) \in R_\varepsilon$ where x'_τ is given by $x_\tau \xrightarrow[\tau]{u_{\tau\eta\omega}} x'_\tau$. The claim follows directly from inequalities (11.11) through (11.19) by using $u_\tau = u_{\tau\eta\omega}$. □

Example 11.15. To illustrate Theorem 11.12 we consider the linear control system defined by:

$$A = \begin{bmatrix} -1 & 1 \\ -8 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the origin is not an asymptotically stable equilibrium point for $(\mathbb{R}^n, A)$, we first design the feedback control law:

$$u = Kx = 7x_1 - 6x_2 + u'$$

rendering the origin an asymptotically stable equilibrium point for the linear dynamical system $(\mathbb{R}^n, A + CK)$ where:

$$A + CK = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Using the function $V(x) = \sqrt{x^T P x}$ with:

$$P = \begin{bmatrix} 1 & \frac{1}{16} \\ \frac{1}{16} & 1 \end{bmatrix}$$

as a Lyapunov function we obtain:

$$\gamma = \frac{17}{4\sqrt{15}}, \quad \lambda = \frac{16 - \sqrt{2}}{17}, \quad \underline{\alpha} = \frac{15}{16}, \quad \overline{\alpha} = \frac{17}{16}.$$

For a sampling time $\tau = 0.25$ and a precision $\varepsilon = 0.1$ we conclude from (11.6) that η needs to be smaller than 0.017. We choose $\eta = \frac{\sqrt{2}}{100} \approx 0.014$, restrict Σ to the set $[-1, 1] \times [-1, 1]$, and define $\mathcal{C}$ as the finite set consisting of constant input curves assuming values on $\{-0.5, -0.25, 0, 0.25, 0.5\}$. Although $S_{\tau\eta}(\Sigma)$ is only guaranteed to be approximately simulated by $S_\tau(\Sigma)$, several control problems that are difficult to solve directly on $S_\tau(\Sigma)$ become fairly straightforward computations on $S_{\tau\eta}(\Sigma)$. Consider the safety game for system $S_\tau(\Sigma)$ and specification set $[-0.35, -0.15] \times [-0.15, 0.15]$. It is quite difficult to solve this problem on $S_\tau(\Sigma)$, but it is immediate to solve it on $S_{\tau\eta}(\Sigma)$ due to its finite-state nature. According to the discussion in Section 11.3, if we synthesize a controller S_{cont} solving the safety game for system $S_{\tau\eta}(\Sigma)$ and specification set W, the controller $S_{cont} \times_{\mathcal{F}} S_{\tau\eta}(\Sigma)$ solves the safety game for system $S_\tau(\Sigma)$ and specification set $W^{\frac{1}{2}\varepsilon}$. Therefore, we define W as $W = [[-0.3, -0.1] \times [-0.1, 0.1]]_\eta$ and use the operator F_W defined in Chapter 6 to solve the safety game. The solution of this game is shown in Figure 11.1 where transitions with the same source and destination are not displayed to keep the figure legible.

Example 11.16. The synthesis of trajectories satisfying desired specifications can also be easily done on $S_{\tau\eta}(\Sigma)$. Assume that we are interested in designing a periodic trajectory passing through $(0.2, 0)$ and $(-0.2, 0)$. Since $S_{\tau\eta}(\Sigma)$ is finite-state, this problem reduces to a simple search on a graph. A possible solution is shown in Figure 11.2 where, in addition to the transitions of $S_{\tau\eta}(\Sigma)$, we also show several trajectories of the closed-loop system. More elaborate control problems can be solved on $S_{\tau\eta}(\Sigma)$ with similar ease by resorting to the synthesis algorithms in Chapter 6. $\triangleleft$

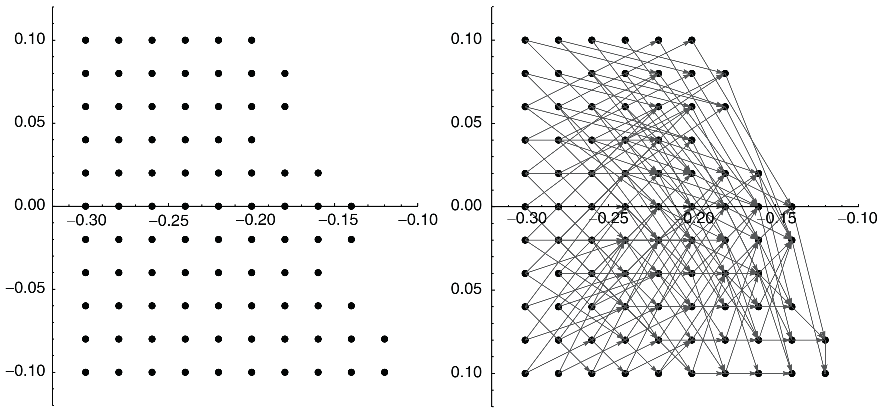

Fig. 11.1. Solution to the safety game for system $S_{\tau\eta}(\Sigma)$ and specification set $W = [[-0.3, -0.1] \times [-0.1, 0.1]]_{\eta}$. The left figure shows the states in W from which it is possible to control the system to remain within W. The right figure shows the corresponding transitions for which the source and destination are not the same state.

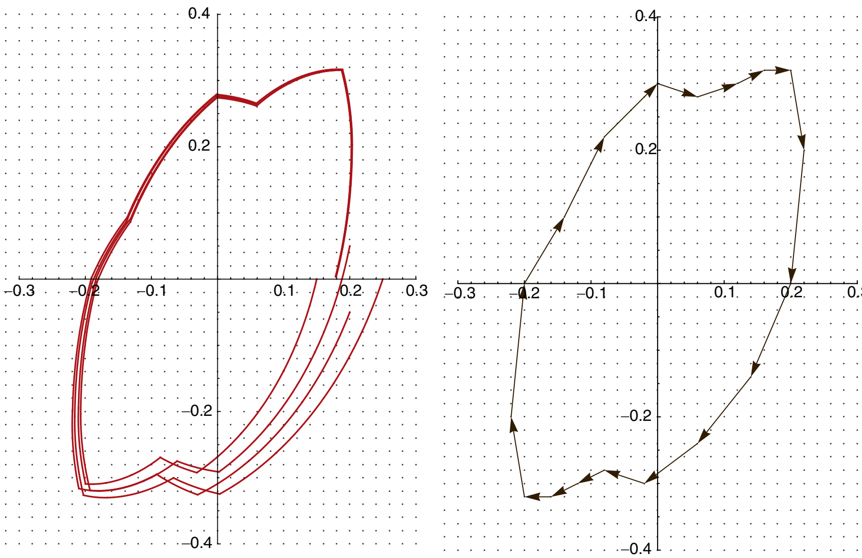

Fig. 11.2. Periodic trajectory passing through the points $(-0.2, 0)$ and $(0.2, 0)$. The left figure shows the transitions of $S_{\tau\eta}(\Sigma)$. The right figure shows several trajectories of the controlled system for different initial conditions.

When disturbance inputs are present, it is still possible to construct a finite-state system which is ε-approximate anternatingly bisimilar to $S_\tau(\Sigma)$ by a careful analysis of reachable sets. From equality (11.2) we know that all the possible contributions of the control and disturbance inputs to ξ are captured by the reachable sets:

$$\mathcal{R}_{\tau\mathcal{C}} = \left\{ x \in X_\tau : x = \int_0^\tau e^{A(\tau-t)} C\chi(t)dt,\ \chi \in \mathcal{C} \right\},$$
$$\mathcal{R}_{\tau\mathcal{D}} = \left\{ x \in X_\tau : x = \int_0^\tau e^{A(\tau-t)} D\delta(t)dt,\ \delta \in \mathcal{D} \right\}.$$

Through these sets we can indirectly quantize the inputs leading to the system $S_{\tau\eta\eta}(\Sigma)$.

Definition 11.17. *The system $S_{\tau\eta\eta} = (X_{\tau\eta\eta}, U_{\tau\eta\eta}, \underset{\tau\eta\eta}{\longrightarrow}, Y_{\tau\eta\eta}, H_{\tau\eta\eta})$ associated with an affine control system $\Sigma = (\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$, with $\tau, \eta \in \mathbb{R}^+$, and with a set $\mathcal{D}_\eta \subseteq [\mathbb{R}^n]_\eta$ satisfying $\mathbf{d}_h(\mathcal{D}_\eta, \mathcal{R}_{\tau\mathcal{D}}) \leq \eta$, consists of:*

- $X_{\tau\eta\eta} = [\mathbb{R}^n]_\eta$;
- $U_{\tau\eta\eta}$ *is any subset of* $[\mathbb{R}^m]_\eta$ *satisfying* $\mathbf{d}_h(U_{\tau\eta\eta}, \mathcal{R}_{\tau\mathcal{C}}) \leq \eta$;
- $x_{\tau\eta\eta} \overset{\chi}{\underset{\tau\eta\eta}{\longrightarrow}} x'_{\tau\eta\eta}$ *if there exist* $\chi \in U_{\tau\eta\eta}$, $\delta \in \mathcal{D}_\eta$, *and a trajectory* $\xi_{x_{\tau\eta\eta}00} : [0,\tau] \to \mathbb{R}^n$ *of* Σ *satisfying:*
$$\|\xi_{x_{\tau\eta\eta}00}(\tau) + \chi + \delta - x'_{\tau\eta\eta}\| \leq \eta; \tag{11.20}$$
- $Y_{\tau\eta\eta} = \mathbb{R}^n$;
- $H_{\tau\eta\eta} = \imath : X_{\tau\eta\eta} \hookrightarrow \mathbb{R}^n$.

The construction of $S_{\tau\eta\eta}(\Sigma)$ requires the knowledge of the reachable sets $\mathcal{R}_{\tau\mathcal{C}}$ and $\mathcal{R}_{\tau\mathcal{D}}$. However, the computation of these sets does not need to be exact. Using the method described in Section 7.6, we can compute approximations $\widehat{\mathcal{R}}_{\tau\mathcal{C}}$ and $\widehat{\mathcal{R}}_{\tau\mathcal{D}}$ to the sets $\mathcal{R}_{\tau\mathcal{C}}$ and $\mathcal{R}_{\tau\mathcal{D}}$ with approximating errors $e_\mathcal{C}$ and $e_\mathcal{D}$, *i.e.*:

$$\mathbf{d}_h(\widehat{\mathcal{R}}_{\tau\mathcal{C}}, \mathcal{R}_{\tau\mathcal{C}}) \leq e_\mathcal{C} \qquad \mathbf{d}_h(\widehat{\mathcal{R}}_{\tau\mathcal{D}}, \mathcal{R}_{\tau\mathcal{D}}) \leq e_\mathcal{D}.$$

Hence, we can redefine $X_{\tau\eta\eta}$ as $[\mathbb{R}^n]_{\eta - e_\mathcal{C} - e_\mathcal{D}}$ and declare the existence of a transition $x_{\tau\eta\eta} \overset{u}{\underset{\tau\eta\eta}{\longrightarrow}} x'_{\tau\eta\eta}$ when $\|\xi_{x_{\tau\eta\eta}00}(\tau) + \widehat{\chi} + \widehat{\delta} - x'_{\tau\eta\eta}\| \leq \eta - e_\mathcal{C} - e_\mathcal{D}$ for some $\widehat{\chi} \in \widehat{\mathcal{R}}_{\tau\mathcal{C}}$ and $\widehat{\delta} \in \widehat{\mathcal{R}}_{\tau\mathcal{D}}$. With this new state set and transition relation we have:

$$\begin{aligned} \|\xi_{x_{\tau\eta\eta}00}(\tau) + \chi + \delta - x'_{\tau\eta\eta}\| &= \|\xi_{x_{\tau\eta\eta}00}(\tau) + \widehat{\chi} + \widehat{\delta} - x'_{\tau\eta\eta} + \chi - \widehat{\chi} + \delta - \widehat{\delta}\| \\ &\leq \|\xi_{x_{\tau\eta\eta}00}(\tau) + \widehat{\chi} + \widehat{\delta} - x'_{\tau\eta\eta}\| \\ &\quad + \|\chi - \widehat{\chi}\| + \|\delta - \widehat{\delta}\| \\ &\leq \eta - e_\mathcal{C} - e_\mathcal{D} + e_\mathcal{C} + e_\mathcal{D} \leq \eta \end{aligned}$$

thus maintaining the validity of the next result intact.

Theorem 11.18. *Let $\Sigma = (\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D, h)$ be an affine control system and assume that the linear control system $(\mathbb{R}^n, \mathcal{C}, \mathcal{D}, A, C, D)$ admits an ISS-Lyapunov function V of the form $V(x) = \sqrt{x^T P x}$ with $P \in \mathcal{SP}(n)$. For any desired precision $\varepsilon \in \mathbb{R}^+$, for any desired time quantization τ, and for any space quantization $\eta \in \mathbb{R}^+$ satisfying:*

$$\eta \leq \min\left\{\frac{1}{3}\gamma^{-1}\underline{\alpha}\varepsilon(1-e^{-\lambda\tau}), \overline{\alpha}^{-1}\underline{\alpha}\varepsilon\right\}, \tag{11.21}$$

the relation $R_\varepsilon \subseteq X_\tau \times X_{\tau\eta\eta}$ defined by:

$$R_\varepsilon = \{(x_\tau, x_{\tau\eta\eta}) \in X_\tau \times X_{\tau\eta\eta} \mid V(x_\tau - x_{\tau\eta\eta}) \leq \underline{\alpha}\varepsilon\} \tag{11.22}$$

is an ε-approximate alternating bisimulation relation between $S_\tau(\Sigma)$ and $S_{\tau\eta\eta}(\Sigma)$.

Proof. We first show that R_ε is an ε-approximate alternating simulation from $S_\tau(\Sigma)$ to $S_{\tau\eta\eta}(\Sigma)$. The first two requirments in Definition 9.6 are proved as in Theorem 11.12.

Regarding the third requirement, let $(x_\tau, x_{\tau\eta\eta}) \in R_\eta$ and recall that we need to show that:

$$\forall u_\tau \in U_\tau(x_\tau)\ \exists u_{\tau\eta\eta} \in U_{\tau\eta\eta}(x_{\tau\eta\eta})\ \forall x'_{\tau\eta\eta} \in \mathrm{Post}_{u_{\tau\eta\eta}}(x_{\tau\eta\eta})\ \exists x'_\tau \in \mathrm{Post}_{u_\tau}(x_\tau)$$

satisfying $(x'_\tau, x'_{\tau\eta\eta}) \in R_\varepsilon$. Choose any $u_\tau \in U_\tau(x_\tau)$ and let $u_{\tau\eta\eta}$ be any input in $U_{\tau\eta\eta}(x_{\tau\eta\eta})$ satisfying:

$$\left\|\int_0^\tau e^{A(\tau-t)}Cu_\tau(t)dt - u_{\tau\eta\eta}\right\| \leq \eta. \tag{11.23}$$

Note that such input exists by definition of $U_{\tau\eta\eta}$. Let now $x'_{\tau\eta\eta}$ be any state in $\mathrm{Post}_{u_{\tau\eta\eta}}(x_{\tau\eta\eta})$. This means that:

$$\|\xi_{x_{\tau\eta\eta}00}(\tau) + u_{\tau\eta\eta} + \delta_{\tau\eta\eta} - x'_{\tau\eta\eta}\| \leq \eta \tag{11.24}$$

for some $\delta_{\tau\eta\eta} \in \mathcal{D}_\eta$. By definition of $\mathcal{D}_\eta$, there exists $\delta_\tau \in \mathcal{D}$ such that:

$$\left\|\int_0^\tau e^{A(\tau-t)}D\delta_\tau(t)dt - \delta_{\tau\eta\eta}\right\| \leq \eta. \tag{11.25}$$

We then choose x'_τ to be the element of $\mathrm{Post}_{u_\tau}(x_\tau)$ given by $x'_\tau = \xi_{x_\tau u_\tau \delta_\tau}(\tau)$ and we claim that $(x'_\tau, x'_{\tau\eta\eta}) \in R_\varepsilon$. The claim is a direct consequence of the following chain of inequalities where we used (10.6), (11.5), (11.23), (11.25), (11.22), (11.24), (11.21), and $x'' = x_\tau - x_{\tau\eta\eta}$:

$$\begin{aligned}
V(x'_\tau - x'_{\tau\eta\eta}) &\leq V\left(x'_\tau - e^{A\tau}x_{\tau\eta\eta} - \int_0^\tau e^{A(\tau-t)}\left(C\chi_\tau(t) + D\delta_\tau(t) + h\right)dt\right) \\
&\quad +\gamma\left\|x'_{\tau\eta\eta} - e^{A\tau}x_{\tau\eta\eta} - \int_0^\tau e^{A(\tau-t)}\left(C\chi_\tau(t) + D\delta_\tau(t) + h\right)dt\right\| \\
&\leq V\left(e^{A\tau}x_\tau - e^{A\tau}x_{\tau\eta\eta}\right) \\
&\quad +\gamma\left\|e^{A\tau}x_{\tau\eta\eta} + \int_0^\tau e^{A(\tau-t)}\left(C\chi_\tau(t) + D\delta_\tau(t) + h\right)dt - x'_{\tau\eta\eta}\right\| \\
&\leq V\left(\xi_{x''00}(\tau)\right) \\
&\quad +\gamma\left\|e^{A\tau}x_{\tau\eta\eta} + \int_0^\tau e^{A(\tau-t)}hdt + u_{\tau\eta\eta} + \delta_{\tau\eta\eta} - x'_{\tau\eta\eta}\right\| \\
&\quad +\gamma\left\|\int_0^\tau e^{A(\tau-t)}C\chi_\tau(t)dt - u_{\tau\eta\eta}\right\| \\
&\quad +\gamma\left\|\int_0^\tau e^{A(\tau-t)}D\delta_\tau(t)dt - \delta_{\tau\eta\eta}\right\| \\
&\leq e^{-\lambda\tau}V\left(\xi_{x''00}(0)\right) + \gamma\left\|\xi_{x_{\tau\eta\eta}00} + u_{\tau\eta\eta} + \delta_{\tau\eta\eta} - x'_{\tau\eta\eta}\right\| \\
&\quad +\gamma\eta + \gamma\eta \\
&\leq e^{-\lambda\tau}\underline{\alpha}\varepsilon + 3\gamma\eta \leq \underline{\alpha}\varepsilon.
\end{aligned}$$

The proof that R_ε^{-1} is an ε-approximate alternating simulation from $S_{\tau\eta\eta}(\Sigma)$ to $S_\tau(\Sigma)$ is similar and thus omitted. □

The previous result can also be used in the context of verification when $C = 0$. In this case, we regard the affine control system (11.1) as a closed-loop system affected by an adversarial input δ and the verification objective is to prove that a certain property holds, independently of δ.

11.5 Symbolic models for switched affine systems

The abstraction techniques developed for dynamical and control systems remarkably generalize to switched affine systems. At this point, the reader should be able to foresee how such generalization unfolds. The first step consists in quantizing the states and approximating the transitions of a switched affine system.

Definition 11.19. *The system* $S_{\tau\eta}(\Sigma) = (X_{\tau\eta}, U_{\tau\eta}, \underset{\tau\eta}{\longrightarrow}, Y_{\tau\eta}, H_{\tau\eta})$ *associated with a switched affine system* $\Sigma = (X_a, \mathbb{R}^n, \{A_{x_a}, h_{x_a}\}_{x_a \in X_a})$ *and with* $\tau, \eta \in \mathbb{R}^+$ *consists of:*

- $X_{\tau\eta} = [\mathbb{R}^n]_\eta$;
- $U_{\tau\eta} = X_a$;
- $x \xrightarrow[\tau\eta]{u_a} x'$ *if there exists a solution* $\xi_x : [0,\tau] \to \mathbb{R}^n$ *of the affine dynamical system* $(\mathbb{R}^n, A_{u_a}, h_{u_a})$ *satisfying* $\|\xi_x(\tau) - x'\| \leq \eta$;
- $Y_{\tau\eta} = \mathbb{R}^n$;
- $H_{\tau\eta} = \imath : X_{\tau\eta} \hookrightarrow \mathbb{R}^n$.

A close analysis of the proof of Theorem 11.14 reveals that its conclusion does not depend on the particular form of the differential equation $\dot{\xi} = A\xi + C\chi + h$ but only on the inequality $\frac{\partial V}{\partial x}(Ax + Cc) \leq -\lambda V(x) + \sigma_c\|c\|$. For many affine switched systems it is possible to find a single Lyapunov function V satisfying the inequalities:

$$\frac{\partial V}{\partial x} A_{x_a} x \leq -\lambda V(x) \qquad \forall x_a \in X_a.$$

When this is the case we say that V is a *common Lyapunov function* for Σ. The arguments in the proof of Theorem 11.14 apply directly to this case and provide the following corollary.

Corollary 11.20. *Let* $\Sigma = (X_a, \mathbb{R}^n, \{A_{x_a}, h_{x_a}\}_{x_a \in X_a})$ *be a switched affine system admitting a common Lyapunov function* V *of the form* $V(x) = \sqrt{x^T P x}$ *with* $P \in \mathcal{SP}(n)$. *For any desired precision* $\varepsilon \in \mathbb{R}^+$, *for any desired time quantization* $\tau \in \mathbb{R}^+$, *and for any space quantization* $\eta \in \mathbb{R}^+$ *satisfying:*

$$\eta \leq \min\left\{\gamma^{-1}\underline{\alpha}\varepsilon\left(1 - e^{-\lambda\tau}\right), \overline{\alpha}^{-1}\underline{\alpha}\varepsilon\right\}, \tag{11.26}$$

the relation $R_\varepsilon \subseteq X_\tau \times X_{\tau\eta}$ *defined by:*

$$R_\varepsilon = \{(x_\tau, x_{\tau\eta}) \in X_\tau \times X_{\tau\eta} \mid V(x_\tau - x_{\tau\eta}) \leq \underline{\alpha}\varepsilon\} \tag{11.27}$$

is an ε*-approximate bisimulation relation between* $S_\tau(\Sigma)$ *and* $S_{\tau\eta}(\Sigma)$.

This result can be used in two different ways. When the inputs X_a are regarded as adversarial, $S_{\tau\eta}(\Sigma)$ can be used to verify properties that hold independently of the disturbance input. When X_a is regarded as a set of control inputs, then $S_{\tau\eta}(\Sigma)$ can be used for control design.

Corollary 11.20 can also be extended to the case when there exists a Lyapunov function V_{x_a} for every linear dynamical system $(\mathbb{R}^n, A_{x_a})$. It is well known that existence of such Lyapunov functions does not imply the existence of a common Lyapunov function for Σ. In this case, a more elaborate construction is required, building upon the concept of dwell time used to study the stability properties of switched systems.

Example 11.21. We revisit the boost DC-DC converter of Chapter 1, represented in Figure 1.7. This is a switched affine system with two modes of operation corresponding to the two positions of the switch. The dynamics in mode 1 is described by (1.16) and (1.17) while the dynamics in mode 2 is described by (1.18) and (1.19). The values of the components, given in the per unit system, are:

$$C = 70, L = 3, R_C = 0.005, R_L = 0.05, v_s = 1, R_0 = 1.$$

Before proceeding with our analysis, we make the linear change of coordinates defined by :

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

to better condition the problem numerically.

The purpose of the boost DC-DC converter is to regulate the voltage across the load resistor R_0. This objective can be reformulated as the regulation of the current flowing through the inductor, which is one of the infinite state variables. In order to synthesize a controller, we regard this problem as an instance of a safety game where the specification set W contains the desired values for the current. Although safety games are difficult to solve on $S_\tau(\Sigma)$, we can use Corollary 11.20 to construct the finite-state abstraction $S_{\tau\eta}(\Sigma)$ and solve the safety game on $S_{\tau\eta}(\Sigma)$. One possible common Lyapunov function is $V(z) = \sqrt{z^T P z}$ with:

$$P = \begin{bmatrix} 1.0224 & 0.0084 \\ 0.0084 & 1.0031 \end{bmatrix}$$

and satisfying:

$$\frac{\partial V}{\partial z} A_1 z \le -0.0139V \qquad \frac{\partial V}{\partial z} A_2 z \le -0.0138V.$$

Therefore, $\lambda = \min\{0.0138, 0.00139\} = 0.0138$. A bound for γ can be computed using the expression in the proof of Proposition 10.5: $\gamma = 1.0256$. We select a sampling time $\tau = 0.2$ and a precision of $\varepsilon = 3$. Although this precision is not useful for practical purposes, it will keep the symbolic model $S_{\tau\eta}(\Sigma)$ small so that it can be easily visualized. From inequality (11.26) we obtain the bound $\eta \le 0.00807$ and set $\eta = \frac{\sqrt{2}}{200} \approx 0.0071$. With these parameters we construct $S_{\tau\eta}(\Sigma)$ and consider the safety game with specification set $W = [1.2, 1.6] \times [5.6, 5.8]$ that is easily solved by iterating the operator F_W studied in Chapter 6. In Figure 11.3, the reader can find the points in W where mode 1 should be used and the points in W where mode 2 should be used. The fixed-point of F_W is displayed in Figure 11.4 and the closed-loop system $S_c \times_{\mathcal{F}} S_{\tau\eta}(\Sigma)$ is represented in the book's cover. $\lhd$

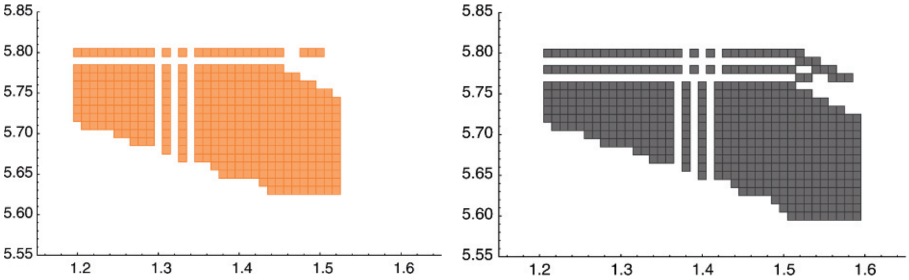

Fig. 11.3. Solution of the safety game for system $S_{\tau\eta}(\Sigma)$ and specification set $W = [1.2, 1.6] \times [5.6, 5.8]$. The points in W where mode 1 should be used are shown in the left figure and the points in W where mode 2 should be used are shown in the right figure.

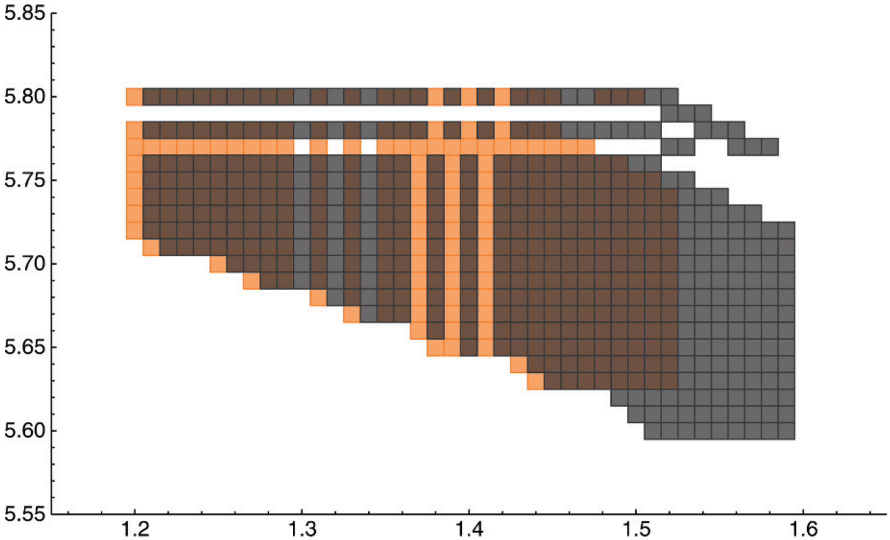

Fig. 11.4. Solution of the safety game for system $S_{\tau\eta}(\Sigma)$ and specification set $W = [1.2, 1.6] \times [5.6, 5.8]$. The fixed-point of the operator F_W is represented as the superposition of the images in Figure 11.3.

11.6 Advanced topics

In this section we show how Theorem 11.12 and Theorem 11.14 can be generalized to nonlinear control systems. The exposition will be swift and relies on advanced control theoretical concepts.

We make extensive use of comparison functions of class $\mathcal{K}$ and $\mathcal{KL}$ to simplify the arguments. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$

if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r,s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed r, the map $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.

In a nonlinear context we need to replace the asymptotic stability assumption with the stronger assumption of incremental stability.

Definition 11.22 (Incremental global asymptotic stability). *A control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ is* incrementally globally asymptotically stable (δ–GAS) *if it is forward complete and there exists a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon \in \mathcal{U}$, the following inequality is satisfied:*

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon}(t)\| \leq \beta(\|x - x'\|, t). \tag{11.28}$$

We also need the stronger notion of incremental input-to-state stability.

Definition 11.23 (Incremental global input-to-state stability). *A control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ is* incrementally globally input-to-state stable (δ–ISS) *if it is forward complete and there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function ρ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon, \upsilon' \in \mathcal{U}$, the following inequality is satisfied:*

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \beta(\|x - x'\|, t) + \rho(\|\upsilon - \upsilon'\|). \tag{11.29}$$

It is clear that δ–ISS implies δ–GAS since (11.28) can be obtained from (11.29) by setting $\upsilon = \upsilon'$. Both δ–GAS and δ–ISS can be characterized by dissipation inequalities.

Definition 11.24 (δ–GAS Lyapunov function). *A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a* δ–GAS Lyapunov function *for a control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$, if there exist $\lambda \in \mathbb{R}^+$ and $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that for any $x, x' \in \mathbb{R}^n$ and any $u \in \mathbb{R}^m$ we have:*

$$\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|)$$
$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq -\lambda V(x, x').$$

Function V is called a δ–ISS Lyapunov function for Σ, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and σ such that for any $x, x' \in \mathbb{R}^n$ and any $u, u' \in \mathbb{R}^m$ we have:

$$\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|)$$
$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\lambda V(x, x') + \sigma(\|u - u'\|).$$

The following result completely characterizes δ–GAS and δ–ISS in terms of existence of Lyapunov functions.

Theorem 11.25. *For any control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ the following holds:*

1. *if the elements of $\mathcal{U}$ assume values on compact set $K \subseteq \mathbb{R}^m$, then Σ is δ–GAS if and only if it admits a δ–GAS Lyapunov function;*
2. *if the elements of $\mathcal{U}$ assume values on closed and convex set $K \subseteq \mathbb{R}^m$ containing the origin, and if $f(0,0) = 0$, then Σ is δ–ISS if it admits a δ–ISS Lyapunov function. Moreover if the elements of $\mathcal{U}$ assume values on compact set $K \subseteq \mathbb{R}^m$, existence of a δ–ISS Lyapunov function is equivalent to δ–ISS.*

Theorems 11.12 and 11.14 can now be generalized to the nonlinear context.

Theorem 11.26. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ be a control system admitting a δ-GAS Lyapunov function V satisfying:*

$$V(x, x') - V(x, x'') \leq \gamma(\|x' - x''\|)$$

for some class $\mathcal{K}_\infty$ function γ and for every $x, x', x'' \in \mathbb{R}^n$. For any desired precision $\varepsilon \in \mathbb{R}^+$, for any desired time quantization $\tau \in \mathbb{R}^+$, and for any space quantization $\eta \in \mathbb{R}^+$ satisfying:

$$\eta \leq \min\left\{\gamma^{-1}\left((1 - e^{\lambda\tau})\overline{\alpha}(\varepsilon)\right), \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)\right\}, \tag{11.30}$$

the relation $R_\varepsilon \subseteq X_{\tau\eta} \times X_\tau$ defined by:

$$R_\varepsilon = \{(x_{\tau\eta}, x_\tau) \in X_{\tau\eta} \times X_\tau \mid V(x_\tau, x_{\tau\eta}) \leq \underline{\alpha}(\varepsilon)\} \tag{11.31}$$

is a surjective ε-approximate simulation relation from $S_{\tau\eta}(\Sigma)$ to $S_\tau(\Sigma)$. Moreover, if V is a δ-ISS Lyapunov function and $\mathcal{U}$ contains only constant curves, then for any desired precision $\varepsilon \in \mathbb{R}^+$, for any desired time quantization $\tau \in \mathbb{R}^+$, for any desired input quantization $\omega \in \mathbb{R}^+$, and for any space quantization $\eta \in \mathbb{R}^+$ satisfying:

$$\eta \leq \min\left\{\gamma^{-1}\left(\underline{\alpha}(\varepsilon)(1 - e^{-\lambda\tau}) - \lambda^{-1}\sigma\omega\right), \overline{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)\right\}, \tag{11.32}$$

the relation (11.31) is an ε-approximate bisimulation relation between $S_{\tau\eta\omega}(\Sigma)$ and $S_\tau(\Sigma)$.

Proof. The proof parallels the proof of Theorems 11.12 and 11.14. The only modification is the replacement of the sequence of inequalities used to prove the third condition in Definitions 9.5 and 9.6. We only provide the details for the inequalities (11.11) through (11.19) since the same argument applies to the remaining ones.

$$\begin{aligned}
V(x'_\tau, x'_{\tau\eta\omega}) &= V(x'_\tau, \xi_{x_{\tau\eta\omega}u_{\tau\eta\omega}}(\tau)) + \gamma(\|\xi_{x_{\tau\eta\omega}u_{\tau\eta\omega}}(\tau) - x'_{\tau\eta\omega}\|) \\
&\leq V(\xi_{x_\tau u_\tau}(\tau), \xi_{x_{\tau\eta\omega}u_{\tau\eta\omega}}(\tau)) + \gamma(\eta) \\
&\leq e^{-\lambda\tau} V(x_\tau, x_{\tau\eta\omega}) + \frac{\sigma}{\lambda}\omega + \gamma(\eta) \\
&\leq e^{-\lambda\tau}\underline{\alpha}(\varepsilon) + \frac{\sigma}{\lambda}\omega + \gamma(\eta) \leq \underline{\alpha}(\varepsilon).
\end{aligned}$$

□

11.7 Notes

The results in this chapter are quite recent and based on [PGT08, GPT09, PT09]. Earlier work relating stability properties of control systems to the existence of approximate simulation relations appeared in [Tab06, Tab08a]. In [GPT09], the reader can find a nonlinear version of Corollary 11.20 that does not require a common Lyapunov function. Instead, it relies on the concept of dwell time from the switched systems literature. The generalization of Theorem 11.18 to nonlinear systems is reported in [PT09].

The boost DC-DC example is taken from [GPM04] and was also used in [GPT09]. In this reference, the interested readers can find a more detailed treatment of Example 11.21.

The discussion of δ-ISS properties in Section 11.6 follows [Ang02] where the proof of Theorem 11.25 can be found.

Although we only used the notions of approximate simulation and bisimulation to construct finite-state abstractions, they can also be used to construct infinite-state abstractions to simplify controller design problems [GP09].

Controller synthesis based on finite-state approximate models had already been discussed in [RO98, MRO02] although the notion of approximation used in these references corresponds that what we defined as a simulation relation.

As mentioned in Section 10.5, the abstraction techniques discussed in Part IV have not yet been extended to hybrid systems. The main difficulty consists in inferring, from the entrance of a single trajectory in a guard set, the entrance of the surrounding trajectories in the same guard set . The exception of switched systems, discussed in Section 11.5, is easy to explain since for this class of hybrid systems the guards coincide with the invariant sets. A very recent and promising research direction that may lead to the desired extension is a direct study of the stability properties of hybrid systems and its corresponding Lyapunov functions [CTG07, CGT08].