1

Multivariate Random Variables

1 Introduction

One-dimensional random variables are introduced when the object of interest is a one-dimensional function of the events (in the probability space $(\Omega, \mathcal{F}, P)$); recall Section 4 of the Introduction. In an analogous manner we now define *multivariate random variables*, or random vectors, as multivariate functions.

Definition 1.1. *An n-dimensional random variable or vector* $\mathbf{X}$ *is a (measurable) function from the probability space* Ω *to* $\mathbb{R}^n$*, that is,*

$$\mathbf{X} : \Omega \to \mathbb{R}^n. \qquad \Box$$

Remark 1.1. We remind the reader that this text does not presuppose any knowledge of measure theory. This is why we do not explicitly mention that functions and sets are supposed to be *measurable*.

Remark 1.2. Sometimes we call $\mathbf{X}$ a *random variable* and sometimes we call it a *random vector*, in which case we consider it a *column vector*:

$$\mathbf{X} = (X_1, X_2, \ldots, X_n)'. \qquad \Box$$

A complete description of the distribution of the random variable is provided by the *joint distribution function*

$$F_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n),$$

for $x_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$.

A more compact way to express this is

$$F_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} \le \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

where the event $\{\mathbf{X} \le \mathbf{x}\}$ is to be interpreted componentwise, that is,

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$$\{\mathbf{X} \leq \mathbf{x}\} = \{X_1 \leq x_1, \ldots, X_n \leq x_n\} = \bigcap_{k=1}^{n} \{X_k \leq x_k\}.$$

In the discrete case we introduce the *joint probability function*

$$p_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} = \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

that is,

$$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

for $x_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$.

It follows that

$$F_{\mathbf{X}}(\mathbf{x}) = \sum_{\mathbf{y} \leq \mathbf{x}} p_{\mathbf{X}}(\mathbf{y}),$$

that is,

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \sum_{y_1 \leq x_1} \cdots \sum_{y_n \leq x_n} p_{X_1,X_2,\ldots,X_n}(y_1, y_2, \ldots, y_n).$$

In the (absolutely) continuous case we define the *joint density (function)*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{d^n F_{\mathbf{X}}(\mathbf{x})}{d\mathbf{x}^n}, \quad \mathbf{x} \in \mathbb{R}^n,$$

that is,

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \frac{\partial^n F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n},$$

where, again, $x_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$.

Remark 1.3. Throughout we assume that all components of a random vector are of the same kind, either all discrete or all continuous. □

It may well happen that in an n-dimensional problem one is only interested in the distribution of $m < n$ of the coordinate variables. We illustrate this situation with an example where $n = 2$.

Example 1.1. Let (X, Y) be a point that is uniformly distributed on the unit disc; that is, the joint distribution of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi}, & \text{for} \quad x^2 + y^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of the x-coordinate. □

Choosing a point in the plane is obviously a two-dimensional task. However, the object of interest is a one-dimensional quantity; the problem is formulated in terms of the joint distribution of X and Y, and we are interested in the distribution of X (the density $f_X(x)$).

Before we solve this problem we shall study the discrete case, which, in some respects, is easier to handle.

Thus, suppose that (X, Y) is a given two-dimensional random variable whose joint probability function is $p_{X,Y}(x, y)$ and that we are interested in finding $p_X(x)$. We have

$$\begin{aligned} p_X(x) &= P(X = x) = P(\bigcup_y \{X = x, Y = y\}) \\ &= \sum_y P(X = x, Y = y) = \sum_y p_{X,Y}(x, y). \end{aligned}$$

A similar computation yields $p_Y(y)$. The distributions thus obtained are called *marginal distributions* (of X and Y, respectively).

The *marginal probability functions* are

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

and

$$p_Y(y) = \sum_x p_{X,Y}(x, y).$$

Analogous formulas hold in higher dimensions. They show that the probability function of a marginal distribution is obtained by summing the joint probability function over the components that are not of interest.

The *marginal distribution function* is obtained in the usual way. In the two-dimensional case we have, for example,

$$F_{X_1}(x) = \sum_{x' \leq x} p_{X_1}(x') = \sum_{x' \leq x} \sum_y p_{X_1,X_2}(x', y).$$

A corresponding discussion for the continuous case cannot be made immediately, since all probabilities involved equal zero. We therefore make definitions that are analogous to the results in the discrete case. In the two-dimensional case we define the *marginal density functions* as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, DD$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx.$$

The marginal distribution function of X is

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-\infty}^{x} \Big(\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,dy\Big)du.$$

We now return to Example 1.1. Recall that the joint density of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & \text{for} \quad x^2+y^2 \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

which yields

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2}$$

for $-1 < x < 1$ (and $f_X(x) = 0$ for $|x| \ge 1$).

As an extra precaution one might check that $\int_{-1}^{1} \frac{2}{\pi}\sqrt{1-x^2}\,dx = 1$. Similarly (by symmetry), we have

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}\,, \quad -1 < y < 1.$$

Exercise 1.1. Let (X,Y,Z) be a point chosen uniformly within the three-dimensional unit sphere. Determine the marginal distributions of (X,Y) and X. □

We have now seen how a model might well be formulated in a higher dimension than the actual problem of interest. The converse is the problem of discovering to what extent the marginal distributions determine the joint distribution. There exist counterexamples showing that the joint distribution is not necessarily uniquely determined by the marginal ones. Interesting applications are computer tomography and satellite pictures; in both applications one makes two-dimensional pictures and wishes to make conclusions about three-dimensional objects (the brain and the Earth).

We close this section by introducing the concepts of independence and uncorrelatedness.

The components of a random vector $\mathbf{X}$ are *independent* iff, for the joint distribution, we have

$$F_{\mathbf{X}}(\mathbf{x}) = \prod_{k=1}^{n} F_{X_k}(x_k), \quad x_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n,$$

that is, iff the joint distribution function equals the product of the marginal ones. In the discrete case this is equivalent to

$$p_{\mathbf{X}}(\mathbf{x}) = \prod_{k=1}^{n} p_{X_k}(x_k), \quad x_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n.$$

In the continuous case it is equivalent to

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{k=1}^{n} f_{X_k}(x_k), \quad x_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n.$$

The random variables X and Y are *uncorrelated* iff their *covariance* equals zero, that is, iff

$$\operatorname{Cov}(X,Y) = E(X - E\,X)(Y - E\,Y) = 0.$$

If the variances are nondegenerate (and finite), the situation is equivalent to the *correlation coefficient* being equal to zero, that is

$$\rho_{X,Y} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}} = 0$$

(recall that the correlation coefficient ρ is a scale-invariant real number and that $|\rho| \leq 1$).

In particular, independent random variables are uncorrelated. The converse is not necessarily true.

The random variables $X_1, X_2, \ldots, X_n$ are pairwise uncorrelated if every pair is uncorrelated.

Exercise 1.2. Are X and Y independent in Example 1.1? Are they uncorrelated?

Exercise 1.3. Let (X,Y) be a point that is uniformly distributed on a square whose corners are $(\pm 1, \pm 1)$. Determine the distribution(s) of the x- and y-coordinates. Are X and Y independent? Are they uncorrelated? □

2 Functions of Random Variables

Frequently, one is not primarily interested in the random variables themselves, but in functions of them. For example, the sum and the difference of two random variables X and Y are, in fact, functions of the two-dimensional random variable (X,Y).

As an introduction we consider one-dimensional functions of one-dimensional random variables.

Example 2.1. Let $X \in U(0,1)$, and put $Y = X^2$. Then

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}).$$

Differentiation yields

$$f_Y(y) = f_X(\sqrt{y})\,\frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}, \quad 0 < y < 1,$$

(and $f_Y(y) = 0$ otherwise).

Example 2.2. Let $X \in U(0,1)$, and put $Y = -\log X$. Then

$$F_Y(y) = P(Y \le y) = P(-\log X \le y) = P(X \ge e^{-y})$$
$$= 1 - F_X(e^{-y}) = 1 - e^{-y}, \quad y > 0,$$

which we recognize as $F_{\text{Exp}(1)}(y)$ (or else we obtain $f_Y(y) = e^{-y}$, for $y > 0$, by differentiation and again that $Y \in \text{Exp}(1)$).

Example 2.3. Let X have an arbitrary continuous distribution, and suppose that g is a differentiable, strictly increasing function (whose inverse g^{-1} thus exists uniquely). Set $Y = g(X)$. Computations like those above yield

$$F_Y(y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X\big(g^{-1}(y)\big)$$

and

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy} g^{-1}(y).$$

If g had been strictly decreasing, we would have obtained

$$f_Y(y) = -f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy} g^{-1}(y).$$

(Note that $f_Y(y) > 0$ since $dg^{-1}(y)/dy < 0$).

To summarize, we have shown that if g is strictly monotone, then

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \cdot \Big|\frac{d}{dy} g^{-1}(y)\Big|. \qquad \square$$

Our next topic is a multivariate analog of this result.

2.1 The Transformation Theorem

Let $\mathbf{X}$ be an n-dimensional, continuous, random variable with density $f_{\mathbf{X}}(\mathbf{x})$, and suppose that $\mathbf{X}$ has its mass concentrated on a set $S \subset \mathbb{R}^n$. Let $g = (g_1, g_2, \ldots, g_n)$ be a bijection from S to some set $T \subset \mathbb{R}^n$, and consider the n-dimensional random variable

$$\mathbf{Y} = g(\mathbf{X}).$$

This means that we consider the n one-dimensional random variables

$$\begin{aligned} Y_1 &= g_1(X_1, X_2, \ldots, X_n), \\ Y_2 &= g_2(X_1, X_2, \ldots, X_n), \\ &\vdots \\ Y_n &= g_n(X_1, X_2, \ldots, X_n). \end{aligned}$$

Finally, assume, say, that g and its inverse are both continuously differentiable (in order for the Jacobian $\mathbf{J} = |d(\mathbf{x})/d(\mathbf{y})|$ to be well defined).

Theorem 2.1. *The density of* $\mathbf{Y}$ *is*

$$f_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} f_{\mathbf{X}}\big(h_1(\mathbf{y}), h_2(\mathbf{y}), \ldots, h_n(\mathbf{y})\big) \cdot \mid \mathbf{J} \mid, & \textit{for} \quad \mathbf{y} \in T, \\ 0, & \textit{otherwise}, \end{cases}$$

where h is the (unique) inverse of g and where

$$\mathbf{J} = \left|\frac{d(\mathbf{x})}{d(\mathbf{y})}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix};$$

that is, $\mathbf{J}$ *is the Jacobian.*

Proof. We first introduce the following piece of notation:

$$h(B) = \{\mathbf{x} : g(\mathbf{x}) \in B\}, \qquad \text{for} \quad B \subset \mathbb{R}^n.$$

Now,

$$P(\mathbf{Y} \in B) = P(\mathbf{X} \in h(B)) = \int_{h(B)} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}.$$

The change of variable $\mathbf{y} = g(\mathbf{x})$ yields

$$P(\mathbf{Y} \in B) = \int_B f_{\mathbf{X}}(h_1(\mathbf{y}), h_2(\mathbf{y}), \ldots, h_n(\mathbf{y})) \cdot \mid \mathbf{J} \mid \, d\mathbf{y}\,,$$

according to the formula for changing variables in multiple integrals. The claim now follows in view of the following result:

Lemma 2.1. *Let* $\mathbf{Z}$ *be an n-dimensional continuous random variable. If, for every $B \subset \mathbb{R}^n$,*

$$P(\mathbf{Z} \in B) = \int_B h(\mathbf{x})\, d\mathbf{x}\,,$$

then h is the density of $\mathbf{Z}$*.* □

Remark 2.1. Note that the Jacobian in Theorem 2.1 reduces to the derivative of the inverse in Example 2.3 when $n = 1$. □

Example 2.4. Let X and Y be independent $N(0, 1)$-distributed random variables. Show that $X+Y$ and $X-Y$ are independent $N(0, 2)$-distributed random variables.

We put $U = X + Y$ and $V = X - Y$. Inversion yields $X = (U + V)/2$ and $Y = (U - V)/2$, which implies that

$$\mathbf{J} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

By Theorem 2.1 and independence, we now obtain

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\big(\frac{u+v}{2}, \frac{u-v}{2}\big) \cdot \mid \mathbf{J} \mid \\ &= f_X\big(\frac{u+v}{2}\big) \cdot f_Y\big(\frac{u-v}{2}\big) \cdot \mid \mathbf{J} \mid \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{u+v}{2})^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{u-v}{2})^2} \cdot \frac{1}{2} \\ &= \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{1}{2}\frac{v^2}{2}}, \end{aligned}$$

for $-\infty < u,\, v < \infty$. □

Remark 2.2. That $X+Y$ and $X-Y$ are $N(0,2)$-distributed might be known from before; or it can easily be verified via the convolution formula. The important point here is that with the aid of Theorem 2.1 we may, in addition, prove independence.

Remark 2.3. We shall return to this example in Chapter 5 and provide a solution that exploits special properties of the multivariate normal distribution; see Examples 5.7.1 and 5.8.1. □

Example 2.5. Let X and Y be independent Exp(1)-distributed random variables. Show that $X/(X+Y)$ and $X+Y$ are independent, and find their distributions.

We put $U = X/(X+Y)$ and $V = X+Y$. Inversion yields $X = U \cdot V$, $Y = V - UV$, and

$$\mathbf{J} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v.$$

Theorem 2.1 and independence yield

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, v-uv) \cdot \mid \mathbf{J} \mid = f_X(uv) \cdot f_Y(v(1-u)) \cdot \mid \mathbf{J} \mid \\ &= e^{-uv} \cdot e^{-v(1-u)} \cdot v = ve^{-v} \end{aligned}$$

for $0 < u < 1$ and $v > 0$, and $f_{U,V}(u,v) = 0$ otherwise, that is,

$$f_{U,V}(u,v) = \begin{cases} 1 \cdot ve^{-v}, & \text{for} \quad 0 < u < 1,\ v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that $U \in U(0,1)$, that $V \in \Gamma(2,1)$, and that U and V are independent. □

As a further application of Theorem 2.1 we prove the convolution formula (in the continuous case); recall formula (7.2) of the Introduction. We are thus given the continuous, independent random variables X and Y, and we seek the distribution of $X+Y$.

A first observation is that we start with *two* variables but seek the distribution of just *one* new one. The trick is to put $U = X + Y$ and to introduce an auxiliary variable V, which may be arbitrarily (that is, suitably) defined. With the aid of Theorem 2.1, we then obtain $f_{U,V}(u, v)$ and, finally, $f_U(u)$ by integrating over v.

Toward that end, set $U = X + Y$ and $V = X$. Inversion yields $X = V$, $Y = U - V$, and

$$\mathbf{J} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1,$$

from which we obtain

$$f_{U,V}(u, v) = f_{X,Y}(v, u - v) \cdot | \mathbf{J} | = f_X(v) \cdot f_Y(u - v) \cdot 1$$

and, finally,

$$f_U(u) = \int_{-\infty}^{\infty} f_X(v) f_Y(u - v)\, dv,$$

which is the desired formula.

Exercise 2.1. Derive the density for the difference, product, and ratio, respectively, of two independent, continuous random variables. □

2.2 Many-to-One

A natural question is the following: What if g is not injective? Let us again begin with the case $n = 1$.

Example 2.6. A simple one-dimensional example is $y = x^2$. If X is a continuous, one-dimensional, random variable and $Y = X^2$, then

$$f_Y(y) = f_X(\sqrt{y})\, \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\, \frac{1}{2\sqrt{y}}\, .$$

Note that the function is 2-to-1 and that we obtain *two terms.* □

Now consider the general case. Suppose that the set $S \subset \mathbb{R}^n$ can be partitioned into m disjoint subsets $S_1, S_2, \ldots, S_m$ in $\mathbb{R}^n$, such that $g : S_k \to T$ is 1 to 1 and satisfies the assumptions of Theorem 2.1 for each k. Then

$$P(\mathbf{Y} \in T) = P(\mathbf{X} \in S) = P(\mathbf{X} \in \bigcup_{k=1}^{m} S_k) = \sum_{k=1}^{m} P(\mathbf{X} \in S_k)\, , \tag{2.1}$$

which, by Theorem 2.1 applied m times, yields

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_{k=1}^{m} f_{\mathbf{X}}(h_{1k}(\mathbf{y}), h_{2k}(\mathbf{y}), \ldots, h_{nk}(\mathbf{y})) \cdot | \mathbf{J}_k |\, , \tag{2.2}$$

where, for $k = 1, 2, \ldots, m$, $(h_{1k}, h_{2k}, \ldots, h_{nk})$ is the inverse corresponding to the mapping from S_k to T and $\mathbf{J}_k$ is the Jacobian.

A reconsideration of Example 2.6 in light of this formula shows that the result there corresponds to the partition $S = (\mathbb{R} =) S_1 \cup S_2 \cup \{0\}$, where $S_1 = (0, \infty)$ and $S_2 = (-\infty, 0)$ and also that the first term in the right-hand side there corresponds to S_1 and the second one to S_2. The fact that the value at a single point may be arbitrarily chosen takes care of $f_Y(0)$.

Example 2.7. Steven is a beginner at darts, which means that the points where his darts hit the board can be assumed to be uniformly spread over the board. Find the distribution of the distance from one hitting point to the center of the board.

We assume, without restriction, that the radius of the board is 1 foot (this is only a matter of scaling). Let (X, Y) be the hitting point. We know from Example 1.1 that

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi}, & \text{for} \quad x^2 + y^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We wish to determine the distribution of $U = \sqrt{X^2 + Y^2}$, that is, the distribution of the distance from the hitting point to the origin. To this end we introduce the auxiliary random variable $V = \arctan(Y/X)$ and note that the range of the arctan function is $(-\pi/2, \pi/2)$. This means that we have a 2-to-1 mapping, since the points (X, Y) and $(-X, -Y)$ correspond to the same (U, V). By symmetry and since the Jacobian equals u, we obtain

$$f_{U,V}(u, v) = \begin{cases} 2 \cdot \frac{1}{\pi} \cdot u, & \text{for} \quad 0 < u < 1,\ -\frac{\pi}{2} < v < \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $f_U(u) = 2u$ for $0 < u < 1$ (and 0 otherwise), that $V \in U(-\pi/2, \pi/2)$, and that U and V are independent. □

3 Problems

1. Show that if $X \in C(0, 1)$, then so is $1/X$.
2. Let $X \in C(m, a)$. Determine the distribution of $1/X$.
3. Show that if $T \in t(n)$, then $T^2 \in F(1, n)$.
4. Show that if $F \in F(m, n)$, then $1/F \in F(n, m)$.
5. Show that if $X \in C(0, 1)$, then $X^2 \in F(1, 1)$.
6. Show that $\beta(1, 1) = U(0, 1)$.
7. Show that if $F \in F(m, n)$, then $1/(1 + \frac{m}{n}F) \in \beta(n/2, m/2)$.
8. Show that if X and Y are independent $N(0, 1)$-distributed random variables, then $X/Y \in C(0, 1)$.
9. Show that if $X \in N(0, 1)$ and $Y \in \chi^2(n)$ are independent random variables, then $X/\sqrt{Y/n} \in t(n)$.

10. Show that if $X \in \chi^2(m)$ and $Y \in \chi^2(n)$ are independent random variables, then $(X/m)/(Y/n) \in F(m,n)$.
11. Show that if X and Y are independent $\text{Exp}(a)$-distributed random variables, then $X/Y \in F(2,2)$.
12. Let X and Y be independent random variables such that $X \in U(0,1)$ and $Y \in U(0,\alpha)$. Find the density function of $Z = X + Y$.
    Remark. Note that there are two cases: $\alpha \geq 1$ and $\alpha < 1$.
13. Let X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 1, & \text{for} \quad 0 \leq x \leq 2,\ \max(0, x-1) \leq y \leq \min(1, x), \\ 0, & \text{otherwise.} \end{cases}$$

    Determine the marginal density functions and the joint and marginal distribution functions.
14. Suppose that $X \in \text{Exp}(1)$, let Y be the integer part and Z the fractional part, that is, let

$$Y = [X] \quad \text{and} \quad Z = X - [X].$$

    Show that Y and Z are independent and find their distributions.
15. Ottar jogs regularly. One day he started his run at 5:31 p.m. and returned at 5:46 p.m. The following day he started at 5:31 p.m. and returned at 5:47 p.m. His watch shows only hours and minutes (not seconds). What is the probability that the run the first day lasted longer than the run the second day?
16. A certain chemistry problem involves the numerical study of a lognormal random variable X. Suppose that the software package used requires the input of $E\,Y$ and $\text{Var}\,Y$ into the computer (where Y is normal and such that $X = e^Y$), but that one knows only the values of $E\,X$ and $\text{Var}\,X$. Find expressions for the former mean and variance in terms of the latter.
17. Let X and Y be independent $\text{Exp}(a)$-distributed random variables. Find the density function of the random variable $Z = X/(1+Y)$.
18. Let $X \in \text{Exp}(1)$ and $Y \in U(0,1)$ be independent random variables. Determine the distribution (density) of $X + Y$.
19. The random vector $\mathbf{X} = (X_1, X_2, X_3)'$ has density function

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{2}{2e-5} \cdot x_1^2 \cdot x_2 \cdot e^{x_1 \cdot x_2 \cdot x_3}, & \text{for} \quad 0 < x_1, x_2, x_3 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

    Determine the distribution of $X_1 \cdot X_2 \cdot X_3$.
20. The random variables X_1 and X_2 are independent and equidistributed with density function

$$f(x) = \begin{cases} 4x^3, & \text{for} \quad 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

    Set $Y_1 = X_1\sqrt{X_2}$ and $Y_2 = X_2\sqrt{X_1}$.

(a) Determine the joint density function of Y_1 and Y_2.
(b) Are Y_1 and Y_2 independent?

21. Let $(X, Y)'$ have density

$$f(x, y) = \begin{cases} \frac{x}{(1+x)^2 \cdot (1+xy)^2}, & \text{for} \quad x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that X and $X \cdot Y$ are independent, equidistriduted random variables and determine their distribution.

22. Let X and Y have joint density

$$f(x, y) = \begin{cases} cx(1-y), & \text{when} \quad 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $Y - X$.

23. Suppose that $(X, Y)'$ has a density function given by

$$f(x, y) = \begin{cases} e^{-x^2 y}, & \text{for} \quad x \geq 1,\, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X^2 Y$.

24. Let X and Y have the following joint density function:

$$f(x, y) = \begin{cases} \lambda^2 e^{-\lambda y}, & \text{for} \quad 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Show that Y and $X/(Y - X)$ are independent, and find their distributions.

25. Let X and Y have joint density

$$f(x, y) = \begin{cases} cx, & \text{when} \quad 0 < x^2 < y < \sqrt{x} < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of XY.

26. Suppose that X and Y are random variables with a joint density

$$f(x, y) = \begin{cases} \frac{1}{y} e^{-x/y} e^{-y}, & \text{when} \quad 0 < x, y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Show that X/Y and Y are independent standard exponential random variables and exploit this fact in order to compute $E\,X$ and $\operatorname{Var} X$.

27. Let X and Y have joint density

$$f(x, y) = \begin{cases} cx, & \text{when} \quad 0 < x^3 < y < \sqrt{x} < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of XY.

28. Let X and Y have joint density

$$f(x,y) = \begin{cases} cx, & \text{when} \quad 0 < x^2 < y < \sqrt{x} < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of X^2/Y.

29. Suppose that $(X,Y)'$ has density

$$f(x,y) = \begin{cases} \frac{2}{(1+x+y)^3}, & \text{for} \quad x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of
(a) $X+Y$,
(b) $X-Y$.

30. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & \text{when} \quad 0 < x, y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $2X+3Y$.

31. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} xe^{-x-xy}, & \text{when} \quad x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X(1+Y)$.

32. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} c\frac{x}{(1+y)^2}, & \text{when} \quad 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X/(1+Y)^2$.

33. Suppose that X, Y, and Z are random variables with a joint density

$$f(x,y,z) = \begin{cases} \frac{6}{(1+x+y+z)^4}, & \text{when} \quad x, y, z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X+Y+Z$.

34. Suppose that X, Y, and Z are random variables with a joint density

$$f(x,y,z) = \begin{cases} ce^{-(x+y)^2}, & \text{for} \quad -\infty < x < \infty,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X+Y$.

35. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} \frac{c}{(1+x-y)^2}, & \text{when} \quad 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X - Y$.

36. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} c \cdot \cos x, & \text{when} \quad 0 < y < x < \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of Y/X.

37. Suppose that X and Y are independent $\text{Pa}(1,1)$-distributed random variables. Determine the distributions of XY and X/Y.

38. Suppose that X and Y are random variables with a joint density

$$f(x,y) = \begin{cases} c \cdot \log y, & \text{when} \quad 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution (density) of $Z = -\log(Y/X)$.

39. Let $X_1 \in \Gamma(a_1, b)$ and $X_2 \in \Gamma(a_2, b)$ be independent random variables. Show that X_1/X_2 and $X_1 + X_2$ are independent random variables, and determine their distributions.

40. Let $X \in \Gamma(r, 1)$ and $Y \in \Gamma(s, 1)$ be independent random variables.
(a) Show that $X/(X+Y)$ and $X+Y$ are independent.
(b) Show that $X/(X+Y) \in \beta(r, s)$.
(c) Use (a) and (b) and the relation

$$X = (X+Y) \cdot \frac{X}{X+Y}$$

in order to compute the mean and the variance of the beta distribution.

41. Let X_1, X_2, and X_3 be independent random variables, and suppose that $X_i \in \Gamma(r_i, 1)$, $i = 1, 2, 3$. Set

$$Y_1 = \frac{X_1}{X_1 + X_2},$$

$$Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3},$$

$$Y_3 = X_1 + X_2 + X_3.$$

Determine the joint distribution of Y_1, Y_2, and Y_3. Conclusions?

42. Let X and Y be independent $N(0,1)$-distributed random variables.
(a) What is the distribution of $X^2 + Y^2$?
(b) Are $X^2 + Y^2$ and X/Y independent?
(c) Determine the distribution of X/Y.

43. Let X and Y be independent random variables. Determine the distribution of $(X - Y)/(X + Y)$ if
(a) $X, Y \in \operatorname{Exp}(1)$,
(b) $X, Y \in N(0, 1)$ (see also Problem 5.10.9(c)).
44. A random vector in $\mathbb{R}^2$ is chosen as follows: Its length, Z, and its angle, Θ, with the positive x-axis, are independent random variables, Z has density

$$f(z) = ze^{-z^2/2}, \quad z > 0,$$

and $\Theta \in U(0, 2\pi)$. Let Q denote the point of the vector. Determine the joint distribution of the Cartesian coordinates of Q.
45. Show that the following procedure generates $N(0, 1)$-distributed random numbers: Pick two independent $U(0, 1)$-distributed numbers U_1 and U_2 and set $X = \sqrt{-2 \log U_1} \cdot \cos(2\pi U_2)$ and $Y = \sqrt{-2 \log U_1} \cdot \sin(2\pi U_2)$. Show that X and Y are independent $N(0, 1)$-distributed random variables.