Chapter 4
Proportionally modular numerical semigroups

Introduction

In [94] the authors introduce the concept of a modular Diophantine inequality. The set of integer solutions of such an inequality is a numerical semigroup. In that manuscript it is shown that the genus of these semigroups can be obtained from the coefficients of the inequality. However, to date we still do not know formulas for the Frobenius number or the multiplicity of the semigroup of solutions of a modular Diophantine inequality.

Later in [92] these inequalities are slightly modified obtaining a wider class of numerical semigroups. The new inequalities are called proportionally modular Diophantine inequalities. In [95] the concept of Bézout sequence is introduced, which became an important tool for the study of this type of numerical semigroup. These sequences are tightly related to Farey sequences (see [40] for the definition and properties of Farey sequences) and to the Stern-Brocot tree (see [38]).

1 Periodic subadditive functions

We introduce the concept of periodic subadditive function. We show that to every such mapping there exists a numerical semigroup. This correspondence also goes in the other direction; for every numerical semigroup and every nonzero element in it, we find a periodic subadditive function associated to them. The contents of this section can be found in [66].

Let $\mathbb{Q}_0^+$ denote the set of nonnegative rational numbers. A subadditive function is a map $f : \mathbb{N} \to \mathbb{Q}_0^+$ such that

(1) $f(0) = 0$,
(2) $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{N}$.

From this definition it is easy to prove our next result, in which we see that every subadditive function has a submonoid of $\mathbb{N}$ associated to it.

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Lemma 5.1. *Let $f : \mathbb{N} \to \mathbb{Q}_0^+$ be a subadditive function. Then*

$$\mathrm{M}(f) = \{x \in \mathbb{N} \mid f(x) \leq x\}$$

is a submonoid of $\mathbb{N}$.

Let m be a positive integer. The map $f : \mathbb{N} \to \mathbb{Q}_0^+$ has *period m* if $f(x+m) = f(x)$ for all $x \in \mathbb{N}$. We denote by $\mathscr{SF}_m$ the set of m-periodic subadditive functions. If $f \in \mathscr{SF}_m$, then we know that $\mathrm{M}(f)$ is a submonoid of $\mathbb{N}$. Clearly, for every $x \in \mathbb{N}$ such that $x \geq \max\{f(0), \ldots, f(m-1)\}$ one has that $x \in \mathrm{M}(f)$, which implies that $\mathbb{N} \setminus \mathrm{M}(f)$ is finite. This proves the following lemma.

Lemma 5.2. *Let m be a positive integer and let $f \in \mathscr{SF}_m$. Then* $\mathrm{M}(f)$ *is a numerical semigroup.*

The use of subadditive functions is inspired in the following result, which is a direct consequence of Lemma 2.6 and Proposition 3.5.

Lemma 5.3. *Let S be a numerical semigroup and let m be a nonzero element of S. Assume that* $\mathrm{Ap}\,(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$ *with* $w(i) \equiv i \bmod m$ *for all* $i \in \{0, \ldots, m-1\}$. *Define* $f : \mathbb{N} \to \mathbb{N}$ *by* $f(x) = w(x \bmod m)$. *Then* $f \in \mathscr{SF}_m$ *and* $\mathrm{M}(f) = S$.

If m is a positive integer and $f \in \mathscr{SF}_m$, then as $0 = f(0) = f(0+m) = f(m)$, we have that $f(m) \leq m$, or equivalently $m \in \mathrm{M}(f)$, as expected.

Lemma 5.4. *Let m be a positive integer and $f \in \mathscr{SF}_m$. Then $m \in \mathrm{M}(f)$.*

Let $\mathscr{S}_m$ be the set of numerical semigroups containing m. As a consequence of the results given so far in this section, we obtain the following result which shows the tight connection between numerical semigroups and periodic subadditive functions.

Theorem 5.5. *Let m be a positive integer. Then*

$$\mathscr{S}_m = \{\mathrm{M}(f) \mid f \in \mathscr{SF}_m\}.$$

We now introduce a family of periodic subadditive functions whose associated semigroups will be the subject of study for the rest of this chapter.

Let a, b and c be positive integers. The map

$$f : \mathbb{N} \to \mathbb{Q}_0^+,\ f(x) = \frac{ax \bmod b}{c}$$

is a subadditive function of period b. Hence

$$\mathrm{S}(a,b,c) = \mathrm{M}(f) = \left\{ x \in \mathbb{N} \;\middle|\; \frac{ax \bmod b}{c} \leq x \right\} = \{x \in \mathbb{N} \mid ax \bmod b \leq cx\}$$

is a numerical semigroup.

A *proportionally modular Diophantine inequality* is an expression of the form $ax \bmod b \leq cx$, with a, b and c positive integers. The integers a, b and c are called the *factor*, *modulus* and *proportion*, respectively. The semigroup $\mathrm{S}(a,b,c)$ is the set of integer solutions of a proportionally modular Diophantine inequality. A numerical semigroup of this form will be called *proportionally modular*.

Example 5.6. $\mathrm{S}(12,32,3) = \{x \in \mathbb{N} \mid 12x \bmod 32 \leq 3x\} = \{0,3,6,\rightarrow\} = \langle 3,7,8\rangle$.

2 The numerical semigroup associated to an interval of rational numbers

We observe in this section that proportionally modular numerical semigroups are precisely the set of numerators of the fractions belonging to a bounded interval. The results of this section also appear in [92].

Given a subset A of $\mathbb{Q}_0^+$, we denote by $\langle A\rangle$, the submonoid of $\mathbb{Q}_0^+$ generated by A, that is,

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid a_1,\ldots,a_n \in A \text{ and } \lambda_1,\ldots,\lambda_n \in \mathbb{N}\}.$$

Clearly $\mathrm{S}(A) = \langle A\rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$ (we use the same letter we are using for proportionally modular numerical semigroups by reasons that will become obvious later). We say that $\mathrm{S}(A)$ is the numerical semigroup *associated* to A.

Given two rational numbers $\lambda < \mu$, we use $[\lambda,\mu]$, $[\lambda,\mu[$, $]\lambda,\mu]$ and $]\lambda,\mu[$ to denote the closed, right-opened, left-opened and opened intervals of rational numbers between λ and μ.

In this section, I denotes any of these intervals with $0 \leq \lambda < \mu$.

Lemma 5.7. *Let $x_1,\ldots,x_k \in I$, then $\frac{1}{k}(x_1+\cdots+x_k) \in I$.*

Proof. As $k(\min\{x_1,\ldots,x_k\}) \leq x_1+\cdots+x_k \leq k(\max\{x_1,\ldots,x_k\})$, we have that $\min\{x_1,\ldots,x_k\} \leq \frac{x_1+\cdots+x_k}{k} \leq \max\{x_1,\ldots,x_k\}$, and thus $\frac{1}{k}(x_1+\cdots+x_k)$ is in I. □

The set $\mathrm{S}(I)$ coincides with the set of numerators of the fractions belonging to I. This fact follows from the next result.

Lemma 5.8. *Let x be a positive rational number. Then $x \in \langle I\rangle$ if and only if there exists a positive integer k such that $\frac{x}{k} \in I$.*

Proof. If $x \in \langle I\rangle$, then by definition $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$ for some $\lambda_1,\ldots,\lambda_k \in \mathbb{N}$ and $x_1,\ldots,x_k \in I$. By Lemma 5.7, $\frac{x}{\lambda_1+\cdots+\lambda_k} \in I$.

If $\frac{x}{k} \in I$, then trivially $k\frac{x}{k} \in \langle I\rangle$. □

We now see that every proportionally modular numerical semigroup can be realized as the numerical semigroup associated to a closed interval whose ends are determined by the factor, modulus and proportion of the semigroup.

Lemma 5.9. *Let a, b and c be positive integers with $c < a$. Then*

$$\mathrm{S}(a,b,c) = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right).$$

Proof. Let $x \in \mathrm{S}(a,b,c) \setminus \{0\}$. Then $ax \bmod b \le cx$. Hence there exists a nonnegative integer k such that $0 \le ax - kb \le cx$. If $k = 0$, then $ax \le cx$, contradicting $c < a$. Thus $k \ne 0$ and $\frac{b}{a} \le \frac{x}{k} \le \frac{b}{a-c}$. By Lemma 5.8, we obtain $x \in \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$.

Now take $x \in \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right) \setminus \{0\}$. By Lemma 5.8 again, there exists a positive integer k such that $\frac{b}{a} \le \frac{x}{k} \le \frac{b}{a-c}$. This implies that $0 \le ax - kb \le cx$, and consequently $ax \bmod b \le cx$. □

Remark 5.10. The condition $c < a$ might seem restrictive. However this is not the case, because if $c \ge a$, then the semigroup $\mathrm{S}(a,b,c)$ is equal to $\mathbb{N}$.

Note also that the inequality $ax \bmod b \le cx$ has the same set of integer solutions as $(a \bmod b)x \bmod b \le cx$. Hence we can, in our study of Diophantine proportionally modular inequalities, assume that $0 < c < a < b$.

Example 5.11. $\mathrm{S}(44,32,3) = S(12,32,3) = \mathrm{S}\left(\left[\frac{32}{12}, \frac{32}{9}\right]\right) = \mathrm{S}\left(\left[\frac{8}{3}, \frac{32}{9}\right]\right) = \mathbb{N} \cap (\{0\} \cup \left[\frac{8}{3}, \frac{32}{9}\right] \cup \left[\frac{16}{3}, \frac{64}{9}\right] \cup \left[8, \frac{32}{3}\right] \cup \cdots) = \{0,3,6,\rightarrow\}$.

Numerical semigroups associated to closed intervals are always proportionally modular. Its factor, modulus and proportion are determined by the ends of the interval. This result is a sort of converse to Lemma 5.9.

Lemma 5.12. *Let a_1, a_2, b_1 and b_2 be positive integers with $\frac{b_1}{a_1} < \frac{b_2}{a_2}$. Then*

$$\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}(a_1b_2, b_1b_2, a_1b_2 - a_2b_1).$$

Proof. Note that $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}\left(\left[\frac{b_1b_2}{a_1b_2}, \frac{b_1b_2}{b_1a_2}\right]\right)$. The proof now follows by Lemma 5.9. □

With this we can show that the numerical semigroup associated to a bounded interval is proportionally modular.

Lemma 5.13. $\mathrm{S}(I)$ *is a proportionally modular numerical semigroup.*

Proof. As $\mathrm{S}(I) = \langle I \rangle \cap \mathbb{N}$, we have that $\mathrm{S}(I)$ is a submonoid of $\mathbb{N}$. Take α and β in I with $\alpha < \beta$. Then $\mathrm{S}([\alpha,\beta]) \subseteq \mathrm{S}(I)$ because $[\alpha,\beta] \subseteq I$. By Lemma 5.12 and Theorem 5.5, we know that $\mathrm{S}([\alpha,\beta])$ is a numerical semigroup, and thus has finite complement in $\mathbb{N}$. This forces $\mathrm{S}(I)$ to have finite complement in $\mathbb{N}$, which proves that it is a numerical semigroup.

Let $\{n_1,\ldots,n_p\}$ be the minimal generating system of $\mathrm{S}(I)$. By Lemma 5.8, there exist positive integers $d_1,\ldots,d_p$ such that $\frac{n_i}{d_i} \in I$ for all $i \in \{1,\ldots,p\}$. After rearranging the set $\{n_1,\ldots,n_p\}$, assume that

$$\frac{n_1}{d_1} < \cdots < \frac{n_p}{d_p}.$$

Then $\mathrm{S}\left(\left[\frac{n_1}{d_1}, \frac{n_p}{d_p}\right]\right) \subseteq \mathrm{S}(I)$, and by Lemma 5.8 again, $\{n_1, \ldots, n_p\} \subseteq \mathrm{S}\left(\left[\frac{n_1}{d_1}, \frac{n_p}{d_p}\right]\right)$. Thus $\mathrm{S}\left(\left[\frac{n_1}{d_1}, \frac{n_p}{d_p}\right]\right) = \mathrm{S}(I)$. In view of Lemma 5.12, $\mathrm{S}(I)$ is proportionally modular. □

With all these results we obtain the following characterization for proportionally modular numerical semigroups, which states that the set of solutions of a proportionally modular Diophantine inequality coincides with the set of numerators of all the fractions in a bounded interval.

Theorem 5.14. *Let S be a numerical semigroup. The following conditions are equivalent.*

1) S is proportionally modular.
2) There exist rational numbers α and β, with $0 < \alpha < \beta$, such that $S = \mathrm{S}([\alpha, \beta])$.
3) There exists a bounded interval of positive rational numbers such that $S = \mathrm{S}(I)$.

3 Bézout sequences

In this section we introduce the concept of Bézout sequence. As we have mentioned at the beginning of this chapter, this is one of the main tools used for the study of the set of integer solutions of a proportionally modular Diophantine inequality. This sequences and their relation with proportionally modular numerical semigroups are the main topic of [95].

A sequence of fractions $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a *Bézout sequence* if $a_1, \ldots, a_p$, $b_1, \ldots, b_p$ are positive integers such that $a_{i+1}b_i - a_ib_{i+1} = 1$ for all $i \in \{1, \ldots, p-1\}$. We say that p is the *length* of the sequence, and that $\frac{a_1}{b_1}$ and $\frac{a_p}{b_p}$ are its *ends*.

Bézout sequences are tightly connected with proportionally modular numerical semigroups. The first motivation to introduce this concept is the following property.

Proposition 5.15. *Let a_1, b_1, a_2 and b_2 be positive integers such that $a_1b_2 - a_2b_1 = 1$. Then $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \langle b_1, b_2 \rangle$.*

Proof. Let $x \in \langle b_1, b_2 \rangle \setminus \{0\}$. Then $x = \lambda b_1 + \mu b_2$ for some $\lambda, \mu \in \mathbb{N}$, not both equal to zero. As

$$\frac{b_1}{a_1} \leq \frac{\lambda b_1 + \mu b_2}{\lambda a_1 + \mu a_2} = \frac{x}{\lambda a_1 + \mu a_2} \leq \frac{b_2}{a_2},$$

in view of Lemma 5.8, $x \in \mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right)$.

From Lemma 5.12, by using that $a_1b_2 - a_2b_1 = 1$, we know that

$$\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}(a_1b_2, b_1b_2, 1).$$

If $x \in \mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right)$, then $a_1 b_2 x \bmod b_1 b_2 \leq x$, and thus $b_2(a_1 x \bmod b_1) \leq x$. Since

$$x = \frac{x - (a_1 x \bmod b_1) b_2}{b_1} b_1 + (a_1 x \bmod b_1) b_2,$$

for proving that $x \in \langle b_1, b_2 \rangle$, it suffices to show that $\frac{x-(a_1 x \bmod b_1)b_2}{b_1} \in \mathbb{Z}$ (we already know that it is nonnegative). Or equivalently, that $(a_1 x \bmod b_1) b_2$ and x are congruent modulo b_1. Note that $(a_1 x \bmod b_1) b_2 = a_1 b_2 x \bmod b_1 b_2 = (1 + b_1 a_2) x \bmod b_1 b_2 = x + b_1 a_2 x + k b_1 b_2 = x + b_1(a_2 x + k b_2)$ for some integer k. □

Remark 5.16. Assume now that $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a Bézout sequence. From Lemma 5.8 a positive integer belongs to $\mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right)$ if and only if there exists a positive integer k such that $\frac{x}{k} \in \left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]$. Note that $\frac{x}{k} \in \left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]$ if and only if $\frac{x}{k} \in \left[\frac{a_i}{b_i}, \frac{a_{i+1}}{b_{i+1}}\right]$ for some $i \in \{1, \ldots, p-1\}$. This is equivalent to $x \in \mathrm{S}\left(\left[\frac{a_i}{b_i}, \frac{a_{i+1}}{b_{i+1}}\right]\right)$ in view of Lemma 5.8 again. Proposition 5.15 states then that $x \in \mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right)$ if and only if $x \in \langle a_i, a_{i+1} \rangle$ for some $i \in \{1, \ldots, p-1\}$. That is,

$$\mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right) = \langle a_1, a_2 \rangle \cup \langle a_2, a_3 \rangle \cup \cdots \cup \langle a_{p-1}, a_p \rangle.$$

This also proves the following.

Corollary 5.17. *Let $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ be a Bézout sequence. Then*

$$\mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_p}{b_p}\right]\right) = \langle a_1, a_2, \ldots, a_p \rangle.$$

Example 5.18. Let us find the integer solutions to $50x \bmod 131 \leq 3x$. We know that the set of solutions to this inequality is $\mathrm{S}\left(\left[\frac{131}{50}, \frac{131}{47}\right]\right)$. As

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{11}{4} < \frac{25}{9} < \frac{39}{14} < \frac{131}{47}$$

is a Bézout sequence, we have that $\mathrm{S}\left(\left[\frac{131}{50}, \frac{131}{47}\right]\right) = \langle 131, 76, 21, 8, 11, 25, 39 \rangle = \langle 8, 11, 21, 25, 39 \rangle$.

In this example we have given the Bézout sequence connecting the ends of the interval defining the semigroup of solutions to the Diophantine inequality. We will soon learn how to construct such a sequence once we know the ends of an interval.

As another consequence of Proposition 5.15, we obtain that every numerical semigroup with embedding dimension two is proportionally modular.

Corollary 5.19. *Every numerical semigroup of embedding dimension two is proportionally modular.*

Proof. Let S be a numerical semigroup of embedding dimension two. There exist two relatively prime integers a and b greater than one such that $S = \langle a,b\rangle$. By Bézout's identity, there exist positive integers u and v such that $bu - av = 1$. Proposition 5.15 ensures that $S = \langle a,b\rangle = \mathrm{S}\left(\left[\frac{a}{u},\frac{b}{v}\right]\right)$. Theorem 5.14 tells us that S is proportionally modular. □

Next we will show that given two positive rational numbers, there exists a Bézout sequence whose ends are these numbers. First, we see that the numerators and denominators of the fractions belonging to an interval whose ends are rational numbers admit special expressions in terms of the numerators and denominators of these ends.

Lemma 5.20. *Let a_1,a_2,b_1,b_2,x and y be positive integers such that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$. Then $\frac{a_1}{b_1} < \frac{x}{y} < \frac{a_2}{b_2}$ if and only if $\frac{x}{y} = \frac{\lambda a_1+\mu a_2}{\lambda b_1+\mu b_2}$ for some λ and μ positive integers.*

Proof. Necessity. If $\frac{a_1}{b_1} < \frac{x}{y} < \frac{a_2}{b_2}$, then it is not difficult to show that (x,y) belongs to the positive cone spanned by (a_1,b_1) and (a_2,b_2) (that is, to the set of pairs of the form $r(a_1,b_1)+s(a_2,b_2)$ with r and s positive rational numbers). Hence there exist positive rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ such that $(x,y) = \frac{p_1}{q_1}(a_1,b_1)+\frac{p_2}{q_2}(a_2,b_2)$. Thus $q_1q_2x = p_1q_2a_1 + p_2q_1a_2$ and $q_1q_2y = p_1q_2b_1 + p_2q_1b_2$, and consequently $\frac{x}{y} = \frac{q_1q_2x}{q_1q_2y} = \frac{p_1q_2a_1+p_2q_1a_2}{p_1q_2b_1+p_2q_1b_2}$.

Sufficiency. Follows from the fact that for any positive integers a,b,c and d, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ (this has already been used in Proposition 5.15). □

The next result gives the basic step for constructing a Bézout sequence whose ends are two given rational numbers.

Lemma 5.21. *Let a_1, a_2, b_1 and b_2 be positive integers such that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ and* $\gcd\{a_1,b_1\} = 1$. *Then there exist $x,y \in \mathbb{N}\setminus\{0\}$ such that $\frac{a_1}{b_1} < \frac{x}{y} < \frac{a_2}{b_2}$ and $b_1x - a_1y = 1$.*

Proof. Observe that $b_1x - a_1y = 1$ if and only if $x = \frac{1+a_1y}{b_1}$. As $\gcd\{a_1,b_1\} = 1$, the equation $a_1y \equiv -1 \bmod b_1$ has infinitely many positive solutions. Hence $\frac{x}{y} = \frac{1+a_1y}{b_1y} = \frac{a_1}{b_1} + \frac{1}{b_1y}$ fulfills the desired inequalities for y a large-enough solution to the equation $a_1y \equiv -1 \bmod b_1$. □

Among all possible values arising from the preceding lemma, we fix one that will enable us to apply induction for proving Theorem 5.23. As we will see next, this choice will allow us to effectively construct a Bézout sequence with known ends.

Lemma 5.22. *Let a_1, a_2, b_1 and b_2 be positive integers such that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$,* $\gcd\{a_1, b_1\} = \gcd\{a_2,b_2\} = 1$ *and $a_2b_1 - a_1b_2 = d > 1$. Then there exists $t \in \mathbb{N}$, $1 \le t < d$ such that* $\gcd\{ta_1+a_2, tb_1+b_2\} = d$.

Proof. In view of Lemma 5.21, there exist $x, y \in \mathbb{N}$ such that $\frac{a_1}{b_1} < \frac{x}{y} < \frac{a_2}{b_2}$ with $b_1 x - a_1 y = 1$. Now, from Lemma 5.20, we have that $\frac{x}{y} = \frac{\lambda a_1 + \mu a_2}{\lambda b_1 + \mu b_2}$ for some $\lambda, \mu \in \mathbb{N} \setminus \{0\}$. As $b_1 x - a_1 y = 1$, we know that $\gcd\{x, y\} = 1$ and thus $x = \frac{\lambda a_1 + \mu a_2}{\gcd\{\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2\}}$ and $y = \frac{\lambda b_1 + \mu b_2}{\gcd\{\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2\}}$. By substituting these values in $b_1 x - a_1 y = 1$ we deduce that $\gcd\{\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2\} = \mu(a_2 b_1 - a_1 b_2) = \mu d$. Hence $\mu \mid \lambda a_1 + \mu a_2$ and $\mu \mid \lambda b_1 + \mu b_2$, and consequently $\mu \mid \lambda a_1$ and $\mu \mid \lambda b_1$. By using now that $\gcd\{a_1, b_1\} = 1$, we deduce that $\mu \mid \lambda$. Let $\alpha = \frac{\lambda}{\mu} \in \mathbb{N} \setminus \{0\}$. We have then that $d = \gcd\{\alpha a_1 + a_2, \alpha b_1 + b_2\}$.

Note that if $d = \gcd\{a, b\}$, then $d \mid (a - kd, b - \bar{k}d)$ for all $k, \bar{k} \in \mathbb{N}$. By applying this fact, we deduce that if $t = \alpha \bmod d$, then $d \mid \gcd\{ta_1 + a_2, tb_1 + b_2\}$. Besides, $b_1 \frac{ta_1 + a_2}{d} - a_1 \frac{tb_1 + b_2}{d} = \frac{b_1 a_2 - a_1 b_2}{d} = \frac{d}{d} = 1$. Hence $\gcd\{\frac{ta_1 + a_2}{d}, \frac{tb_1 + b_2}{d}\} = 1$ and thus $\gcd\{ta_1 + a_2, tb_1 + b_2\} = d$.

Since $t = \alpha \bmod d$, obviously $t < d$; also $t \neq 0$, because $\gcd\{a_2, b_2\} = 1 \neq d$. □

Now we are ready to show that for every two positive rational numbers, we can construct a Bézout sequence connecting them.

Theorem 5.23. *Let a_1, a_2, b_1 and b_2 be positive integers such that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$, $\gcd\{a_1, b_1\} = \gcd\{a_2, b_2\} = 1$ and $a_2 b_1 - a_1 b_2 = d$. Then there exists a Bézout sequence of length less than or equal to $d + 1$ with ends $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$.*

Proof. We proceed by induction on d. For $d = 1$ the result is trivial. Now assume that the statement holds for all the integers k with $1 \leq k < d$. By Lemma 5.22, we know that there exists a positive integer t, $1 \leq t < d$ such that $\gcd\{ta_1 + a_2, tb_1 + b_2\} = d$. Let $x_1 = \frac{ta_1 + a_2}{d}$ and $y_1 = \frac{tb_1 + b_2}{d}$. Since $\frac{x_1}{y_1} = \frac{ta_1 + a_2}{tb_1 + b_2}$, Lemma 5.20 asserts that $\frac{a_1}{b_1} < \frac{x_1}{y_1} < \frac{a_2}{b_2}$. Moreover, $b_1 x_1 - a_1 y_1 = b_1 \frac{ta_1 + a_2}{d} - a_1 \frac{tb_1 + b_2}{d} = \frac{b_1 a_2 - a_1 b_2}{d} = \frac{d}{d} = 1$ and $a_2 y_1 - b_2 x_1 = a_2 \frac{tb_1 + b_2}{d} - b_2 \frac{ta_1 + a_2}{d} = \frac{t(a_2 b_1 - a_1 b_2)}{d} = \frac{td}{d} = t < d$. By applying the induction hypothesis to $\frac{x_1}{y_1} < \frac{a_2}{b_2}$, we deduce that there exists a Bézout sequence $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \cdots < \frac{x_s}{y_s} < \frac{a_2}{b_2}$ with $s \leq t$. Hence, $\frac{a_1}{b_1} < \frac{x_1}{y_1} < \frac{x_2}{y_2} < \cdots < \frac{x_s}{y_s} < \frac{a_2}{b_2}$ is a Bézout sequence of length less than or equal to $t + 2 \leq d + 1$. □

Remark 5.24. The proof of Theorem 5.23 gives an algorithmic procedure to compute a Bézout sequence with known ends $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$. Thus we have a procedure to compute a system of generators of $\mathrm{S}\left(\left[\frac{a_1}{b_1}, \frac{a_2}{b_2}\right]\right)$. We must first compute the least positive integer t such that $\gcd\{ta_1 + a_2, tb_1 + b_2\} = d$, and then repeat the procedure with $(\frac{ta_1 + a_2}{d})/(\frac{tb_1 + b_2}{d}) < \frac{a_2}{b_2}$.

Example 5.25 ([95]). We start with the fractions $13/3 < 6/1$. Here $d = 5$ and so there exists $t \in \{1, \ldots, 4\}$ such that $\gcd\{13t + 6, 3t + 1\} = 5$. The choice $t = 3$ fulfills the desired condition, whence we can place $\frac{3 \times 13 + 6}{3 \times 3 + 1} = 9/2$ between $13/3$ and $6/1$. Now we proceed with $9/2 < 6/1$, and obtain $d = 3$. In this setting $\gcd\{1 \times 9 + 6, 1 \times 2 + 1\} = 3$. Thus we put $\frac{9+6}{2+1} = \frac{5}{1}$ between $9/6$ and $6/1$. Finally for $5/1 < 6/1$, it holds that $d = 1$ and consequently the process stops. A Bézout sequence for the given ends is

$$\frac{13}{3} < \frac{9}{2} < \frac{5}{1} < \frac{6}{1}.$$

Observe that Bézout sequences connecting two ends are not unique, since if $\frac{a}{b} < \frac{c}{d}$ is a Bézout sequence, then so is $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

4 Minimal generators of a proportionally modular numerical semigroup

We have seen the connection between systems of generators of a proportionally modular numerical semigroup and Bézout sequences. In this section we will try to sharpen this connection in order to obtain the minimal system of generators of a proportionally modular numerical semigroup. We follow the steps given in [95].

A Bézout sequence $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is *proper* if $a_{i+h}b_i - a_ib_{i+h} \geq 2$ for all $h \geq 2$ such that $i, i+h \in \{1,\ldots,p\}$. Every Bézout sequence can be refined to a proper Bézout sequence, by just removing those terms strictly between $\frac{a_i}{b_i}$ and $\frac{a_{i+h}}{b_{i+h}}$ whenever $a_{i+h}b_i - a_ib_{i+h} = 1$.

Example 5.26. The Bézout sequence $\frac{5}{3} < \frac{12}{7} < \frac{7}{4} < \frac{9}{5}$ is not proper, and $\frac{5}{3} < \frac{7}{4} < \frac{9}{5}$ is proper.

Lemma 5.27. *Let $\frac{a}{u} < \frac{b}{v} < \frac{c}{w}$ be a Bézout sequence. Then $b = \frac{a+c}{d}$ with $d = cu - aw$.*

Proof. The proof follows easily by taking into account that $bu - av = cv - bw = 1$. □

The next result shows that the maximum of the set of numerators of a proper Bézout sequence is always reached at one of its ends.

Lemma 5.28. *Let $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ be a proper Bézout sequence. Then*

$$\max\{a_1, a_2, \ldots, a_p\} = \max\{a_1, a_p\}.$$

Proof. We proceed by induction on p. For $p = 2$, the statement is trivially true. We assume as induction hypothesis that $\max\{a_2,\ldots,a_p\} = \max\{a_2, a_p\}$. We next show that $\max\{a_1,\ldots,a_p\} = \max\{a_1, a_p\}$. If $\max\{a_2, a_p\} = a_p$, then the result follows trivially. Let us assume then that $\max\{a_2, a_p\} = a_2$. If we apply Lemma 5.27 to the Bézout sequence $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{a_3}{b_3}$, then we obtain that $a_2 = \frac{a_1+a_3}{a_3b_1 - a_1b_3}$, and as this Bézout sequence is proper, $a_3b_1 - a_1b_3 \geq 2$. Hence $a_2 \leq \frac{a_1+a_3}{2} \leq \frac{2\max\{a_1,a_3\}}{2}$. We distinguish two cases depending on the value of $\max\{a_1, a_3\}$.

- If $\max\{a_1, a_3\} = a_3$, then we deduce that $a_2 \leq a_3$. Since $\max\{a_2,\ldots,a_p\} = a_2$, this implies that $a_2 = a_3$. By using that $\frac{a_2}{b_2} < \frac{a_3}{b_3}$ is a Bézout sequence and $a_2 = a_3$, we obtain that $a_2(b_2 - b_3) = 1$, whence $a_2 = 1$. Since $a_1 \geq 1$, we conclude that $\max\{a_1,\ldots,a_p\} = a_1$.

- If $\max\{a_1,a_3\} = a_1$, then $a_2 \le a_1$, and the proof follows easily. □

As a consequence of this result, we have that the numerators of the fractions of a proper Bézout sequence are arranged in a special way.

Proposition 5.29. *Let $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ be a proper Bézout sequence. Then $a_1,\ldots,a_p$ is a convex sequence, that is, there exists $h \in \{1,\ldots,p\}$ such that*

$$a_1 \ge a_2 \ge \cdots \ge a_h \le a_{h+1} \le \cdots \le a_p.$$

Two fractions $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ are said to be *adjacent* if

$$\frac{a_2}{b_2+1} < \frac{a_1}{b_1}, \text{ and } b_1 = 1 \text{ or } \frac{a_2}{b_2} < \frac{a_1}{b_1-1}.$$

As we will see later, this is the second condition required to obtain Bézout sequences whose numerators represent minimal systems of generators.

First we show that 1 cannot be the numerator of a fraction in a Bézout sequence of length two with adjacent ends.

Lemma 5.30. *If $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ is a Bézout sequence whose ends are adjacent, then $1 \notin \{a_1,a_2\}$.*

Proof. Assume that $a_1 = 1$. Then $1 = a_2b_1 - a_1b_2 = a_2b_1 - b_2$. Since $\frac{a_2}{b_2+1} < \frac{1}{b_1}$, we have that $a_2b_1 < b_2+1$, in contradiction with $a_2b_1 = b_2+1$.

Suppose now that $a_2 = 1$. Observe that in this setting $b_1 \neq 1$, since otherwise $\frac{a_1}{1} < \frac{1}{b_2}$ and thus $a_1b_2 < 1$. Hence $\frac{1}{b_2} < \frac{a_1}{b_1-1}$ and therefore $b_1 - 1 < a_1b_2$. But this is impossible because $1 = a_2b_1 - a_1b_2 = b_1 - a_1b_2$. □

Proposition 5.31. *If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence whose ends are adjacent, then $\{a_1,\ldots,a_p\}$ is the minimal system of generators of the numerical semigroup $S = \langle a_1,\ldots,a_p\rangle$.*

Proof. We use induction on p. For $p = 2$, we know by Lemma 5.30 that a_1 and a_2 are integers greater than or equal to 2 with $\gcd\{a_1,a_2\} = 1$. Thus the statement is true for $p = 2$.

From Lemma 5.28, we know that $\max\{a_1,\ldots,a_p\} = \max\{a_1,a_p\}$. We distinguish two cases, depending on the value of $\max\{a_1,a_p\}$.

- Assume that $\max\{a_1,\ldots,a_p\} = a_1$. Obviously $\frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence. We prove that its ends are adjacent. Clearly $\frac{a_p}{b_p+1} < \frac{a_2}{b_2}$. Note also that $b_1 \neq 1$, since otherwise the inequality $\frac{a_1}{1} < \frac{a_2}{b_2}$ would imply that $a_2 > a_1$, contradicting that $a_1 = \max\{a_1,\ldots,a_p\}$. Since $a_1b_2 < a_2b_1$ and $a_2 \le a_1$, we have that $a_1b_2 - a_1 < a_2b_1 - a_2$. Hence, if $b_2 \neq 1$, we have that $\frac{a_p}{b_p} < \frac{a_1}{b_1-1} < \frac{a_2}{b_2-1}$. This proves that $\frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence with adjacent ends. By using the induction hypothesis, we have that $\{a_2,\ldots,a_p\}$ minimally generates $\langle a_2,\ldots,a_p\rangle$. Since $a_1 = \max\{a_1,\ldots,a_p\}$, in order to prove that $\{a_1,\ldots,a_p\}$

is a minimal system of generators of $\langle a_1,\dots,a_p\rangle$, it suffices to show that $a_1 \notin \langle a_2,\dots,a_p\rangle$. In view of Corollary 5.17 we know that $\langle a_2,\dots,a_p\rangle = \mathrm{S}\left(\left[\frac{a_2}{b_2},\frac{a_p}{b_p}\right]\right)$. Hence, if $a_1 \in \langle a_2,\dots,a_p\rangle$, then by Lemma 5.8 there exists a positive integer y such that $\frac{a_2}{b_2} \le \frac{a_1}{y} \le \frac{a_p}{b_p}$. This leads to $\frac{a_1}{b_1} < \frac{a_1}{y} \le \frac{a_p}{b_p}$ and consequently $\frac{a_1}{b_1-1} \le \frac{a_1}{y} \le \frac{a_p}{b_p}$, contradicting that $\frac{a_1}{b_1}$ and $\frac{a_p}{b_p}$ are adjacent.
- Assume now that $\max\{a_1,\dots,a_p\} = a_p$. The proof follows by arguing as in the preceding case, but now using that in this setting $\frac{a_1}{b_1} < \cdots < \frac{a_{p-1}}{b_{p-1}}$ is a proper Bézout sequence with adjacent ends. □

We see next that the converse to this result also holds: every proportionally modular numerical semigroup is minimally generated by the numerators of a proper Bézout sequence with adjacent ends. The key to this result is the following lemma.

Lemma 5.32. *Let S be a proportionally modular numerical semigroup other than $\mathbb{N}$. Then there exist two minimal generators n_1 and n_p of S and positive integers b_1 and b_p such that $S = \mathrm{S}\left(\left[\frac{n_1}{b_1},\frac{n_p}{b_p}\right]\right)$. Moreover, $\frac{n_1}{b_1}$ and $\frac{n_p}{b_p}$ are adjacent.*

Proof. Let α and β be two positive rational numbers such that $\alpha < \beta$ and $S = \mathrm{S}([\alpha,\beta])$ (Theorem 5.14). By Lemma 5.8, we know that if n is a minimal generator of S then there exists a positive integer x such that $\alpha \le \frac{n}{x} \le \beta$. Note that $\gcd\{n,x\} = 1$, since if $\gcd\{n,x\} = d \ne 1$, then $\alpha \le \frac{n/d}{x/d} \le \beta$, which would mean that $\frac{n}{d}$ is in S, contradicting that n is a minimal generator of S. Let $a(n) = \max\left\{x \in \mathbb{N}\setminus\{0\} \mid \alpha \le \frac{n}{x}\right\}$. We are assuming that $S \ne \mathbb{N}$, thus if n_i and n_j are two distinct minimal generators of S, then $\frac{n_i}{a(n_i)} \ne \frac{n_j}{a(n_j)}$, because $\gcd\{n_i,a(n_i)\} = \gcd\{n_j,a(n_j)\} = 1$, and $\frac{n_i}{a(n_i)} = \frac{n_j}{a(n_j)}$ would imply that $n_i = n_j$. Hence there exists an arrangement of the minimal generators $n_1,\dots,n_p$ of S such that $\alpha \le \frac{n_1}{a(n_1)} < \frac{n_2}{a(n_2)} < \cdots < \frac{n_p}{a(n_p)} \le \beta$. For all $i \in \{1,\dots,p-1\}$, let $b(n_i) = \min\left\{x \in \mathbb{N}\setminus\{0\} \mid \frac{n_i}{x} \le \frac{n_p}{a(n_p)}\right\}$. Then there exists a permutation σ on the set $\{1,\dots,p-1\}$ such that

$$\alpha \le \frac{n_{\sigma(1)}}{b(n_{\sigma(1)})} < \frac{n_{\sigma(2)}}{b(n_{\sigma(2)})} < \cdots < \frac{n_{\sigma(p-1)}}{b(n_{\sigma(p-1)})} < \frac{n_p}{a(n_p)} \le \beta.$$

Note that $\alpha \le \frac{n_{\sigma(1)}}{a(n_{\sigma(1)})} \le \frac{n_{\sigma(1)}}{b(n_{\sigma(1)})}$ since $b(n_{\sigma(1)}) \le a(n_{\sigma(1)})$, and that $\frac{n_p}{a(n_p)+1} < \alpha$ due to the maximality of $a(n_p)$. Hence $\frac{n_p}{a(n_p)+1} < \frac{n_{\sigma(1)}}{b(n_{\sigma(1)})}$. Besides, it is clear from the definition of $b(n_{\sigma(1)})$ that if $b(n_{\sigma(1)}) \ne 1$, then $\frac{n_p}{a(n_p)} < \frac{n_{\sigma(1)}}{b(n_{\sigma(1)})-1}$.

In order to conclude the proof, it suffices to show that S is the numerical semigroup $T = \mathrm{S}\left(\left[\frac{n_{\sigma(1)}}{b(n_{\sigma(1)})},\frac{n_p}{a(n_p)}\right]\right)$. Since $\left[\frac{n_{\sigma(1)}}{b(n_{\sigma(1)})},\frac{n_p}{a(n_p)}\right] \subseteq [\alpha,\beta]$, we have that $T \subseteq S$. As $\frac{n_{\sigma(1)}}{b(n_{\sigma(1)})} < \frac{n_{\sigma(2)}}{b(n_{\sigma(2)})} < \cdots < \frac{n_{\sigma(p-1)}}{b(n_{\sigma(p-1)})} < \frac{n_p}{a(n_p)}$, by Lemma 5.8 we deduce that $\{n_1,\dots,n_p\} \subseteq T$. Thus $S = T$. □

Proposition 5.33. *Let S be a proportionally modular numerical semigroup with $\mathrm{e}(S) = p \geq 2$. Then there exist an arrangement $n_1, \ldots, n_p$ of the set of minimal generators of S and positive integers $b_1, \ldots, b_p$ such that $\frac{n_1}{b_1} < \frac{n_2}{b_2} < \cdots < \frac{n_p}{b_p}$ is a proper Bézout sequence with adjacent ends.*

Proof. By Lemma 5.32, we know that there exists n_1 and n_p minimal generators of S and positive integers b_1 and b_p such that $S = \mathrm{S}\left(\left[\frac{n_1}{b_1}, \frac{n_p}{b_p}\right]\right)$ and the limits of this interval are adjacent.

As we pointed out in the proof of Lemma 5.32, since n_1 and n_p are minimal generators of S and in view of Lemma 5.8, it must hold that $\gcd\{n_1, b_1\} = \gcd\{n_p, b_p\} = 1$.

If we apply Theorem 5.23 to $\frac{n_1}{b_1} < \frac{n_p}{b_p}$ and refine the resulting Bézout sequence, then we obtain a proper Bézout sequence $\frac{n_1}{b_1} < \frac{x_1}{y_1} < \cdots < \frac{x_l}{y_l} < \frac{n_p}{b_p}$ whose ends are adjacent. From Proposition 5.31, we conclude that $\{n_1, x_1, \ldots, x_l, n_p\}$ is the minimal system of generators of S. □

We end this section by giving an arithmetic characterization of the minimal systems of generators of a proportionally modular numerical semigroup (and thus a characterization of these semigroups). The following easy modular computations will be useful to establish this description. Given positive integers a and b with $\gcd\{a, b\} = 1$, by Bézout's identity, there exist integers u and v such that $au + bu = 1$. We denote by $a^{-1} \bmod b$ the integer $u \bmod b$.

Lemma 5.34. *Let n_1 and n_2 be two integers greater than or equal to two such that $\gcd\{n_1, n_2\} = 1$. Then $n_2(n_2^{-1} \bmod n_1) - n_1((-n_1)^{-1} \bmod n_2) = 1$.*

Proof. Since $n_2(n_2^{-1} \bmod n_1) \equiv 1 \bmod n_1$ and $n_2^{-1} \bmod n_1 < n_1$, we have that $\frac{n_2(n_2^{-1} \bmod n_1) - 1}{n_1}$ is an integer less than n_2. Besides,

$$n_2(n_2^{-1} \bmod n_1) - n_1 \frac{n_2(n_2^{-1} \bmod n_1) - 1}{n_1} = 1,$$

which implies that $n_1 \frac{n_2(n_2^{-1} \bmod n_1) - 1}{n_1} \equiv -1(\bmod n_2)$. Hence $\frac{n_2(n_2^{-1} \bmod n_1) - 1}{n_1}$ equals $(-n_1)^{-1} \bmod n_2$. Thus $n_2(n_2^{-1} \bmod n_1) - n_1((-n_1)^{-1} \bmod n_2) = 1$. □

The above-mentioned characterization is stated as follows.

Theorem 5.35. *A numerical semigroup S is proportionally modular if and only if there is an arrangement $n_1, \ldots, n_p$ of its minimal generators such that the following conditions hold:*

1) $\gcd\{n_i, n_{i+1}\} = 1$ *for all* $i \in \{1, \ldots, p-1\}$,
2) $n_{i-1} + n_{i+1} \equiv 0 \bmod n_i$ *for all* $i \in \{2, \ldots, p-1\}$.

Proof. Necessity. By Proposition 5.33, we know that (possibly after a rearrangement of $n_1,\ldots,n_p$) there exist positive integers $b_1,\ldots,b_p$ such that $\frac{n_1}{b_1} < \cdots < \frac{n_p}{b_p}$ is a Bézout sequence. Hence $\gcd\{n_i,n_{i+1}\} = 1$ for all $i \in \{1,\ldots,p-1\}$. In view of Lemma 5.27, we obtain that $n_i = \frac{n_{i-1}+n_{i+1}}{n_{i+1}b_{i-1}-n_{i-1}b_{i+1}}$ for all $i \in \{2,\ldots,p-1\}$ and consequently $n_{i-1}+n_{i-1} \equiv 0 \bmod n_i$ for all $i \in \{2,\ldots,p-1\}$.

Sufficiency. From Lemma 5.34 and Condition 2), it is not hard to see that

$$\frac{n_1}{n_2^{-1} \bmod n_1} < \frac{n_2}{n_3^{-1} \bmod n_2} < \cdots < \frac{n_{p-1}}{n_p^{-1} \bmod n_{p-1}} < \frac{n_p}{(-n_{p-1})^{-1} \bmod n_p}$$

is a Bézout sequence. By Corollary 5.17 and Theorem 5.14, we conclude that S is a proportionally modular numerical semigroup. □

Example 5.36. This theorem gives a criterium to check whether or not a numerical semigroup is proportionally modular. We illustrate it with some examples.

(1) The semigroup $\langle 6,8,11,13\rangle$ is not proportionally modular, since $\gcd\{6,8\} \neq 1$.
(2) We already know that the semigroup $\langle 8,11,21,25,39\rangle$ is proportionally modular. Let us check it again by using the last theorem. In the arrangement of the generators described in Theorem 5.35, 8 and 11 lie together (in view of Proposition 5.29, this arrangement yields a convex sequence). It does not really matter if we start with 8,11 or 11,8, since if an arrangement fits the conditions of Theorem 5.35 so does its symmetry. The next generator we must place is 21. As $21+11 = 32 \equiv 0 \bmod 8$ and $21+8 \not\equiv 0 \bmod 11$, thus 21 goes at the left of 8. Proceeding in this way with 25 and 39, we conclude that the generators arranged as 21,8,11,25,39 fulfill the conditions of Theorem 5.35.
(3) Let us see that $\langle 5,7,11\rangle$ is not proportionally modular. The generators 5 and 7 must be neighbors in the sequence. Hence we start with 5,7. If we want to place 11, then we must check if $11+7$ is a multiple of 5 or $5+11$ is a multiple of 7. None of these two conditions hold, and thus there is no possible arrangement of 5,7,11 that meets the requirements of Theorem 5.35.

5 Modular numerical semigroups

Given a, b and c positive integers, we leave open the problem of finding formulas to compute, in terms of a, b and c, the Frobenius number, genus and multiplicity of $\mathrm{S}(a,b,c)$. In this section we present the results of [94], which show that a formula for the genus of $\mathrm{S}(a,b,1)$ can be given in terms of a and b.

A *modular Diophantine inequality* is an expression of the form $ax \bmod b \leq x$, with a and b positive integers. A numerical semigroup is *modular* if it is the set of solutions of a modular Diophantine inequality.

Remark 5.37. 1) Every numerical semigroup of embedding dimension two is modular (see the proof of Proposition 5.15 and Corollary 5.19).

2) There are proportionally modular numerical semigroups that are not modular (for instance $\langle 3,8,10\rangle$ as shown in [92, Example 26]; this is proposed as an exercise at the end of this chapter).

Easy computations are enough to prove the following two results. We write them down because we will reference them in the future.

Lemma 5.38. *Let a and b be two integers such that $0 \le a < b$ and let $x \in \mathbb{N}$. Then*

$$a(b-x) \bmod b = \begin{cases} 0, & \text{if } ax \bmod b = 0, \\ b-(ax \bmod b), & \text{if } ax \bmod b \neq 0, \end{cases}$$

Lemma 5.39. *Let a and b be integers such that $0 \le a < b$. Then $ax \bmod b > x$ implies that $a(b-x) \bmod b < b-x$.*

As a consequence of this we obtain the following property, which shows that the modulus of a modular numerical semigroup behaves like the Frobenius number in a symmetric numerical semigroup.

Proposition 5.40. *Let S be a modular numerical semigroup with modulus b. If $x \in \mathbb{N} \setminus S$, then $b-x \in S$.*

As every integer greater than b belongs to $\mathrm{S}(a,b,1)$, in order to compute the genus of $\mathrm{S}(a,b,1)$ we can focus on the interval $[0,b-1]$. Next we see when for x in this interval, both x and $b-x$ belong to $\mathrm{S}(a,b,1)$.

Lemma 5.41. *Let $S = \mathrm{S}(a,b,1)$ for some integers $0 \le a < b$, and let x be an integer such that $0 \le x \le b-1$. Then $x \in S$ and $b-x \in S$ if and only if $ax \bmod b \in \{0,x\}$.*

Proof. Necessity. Assume that $ax \bmod b \neq 0$. As $x \in S$, we have that $ax \bmod b \le x$. If $ax \bmod b < x$, then by Lemma 5.38, we have that $a(b-x) \bmod b = b-(ax \bmod b) > b-x$, and consequently $b-x \notin S$, which contradicts the hypothesis. We conclude that $ax \bmod b = x$.

Sufficiency. If $ax \bmod b = 0$, then clearly $x \in S$. Moreover, by Lemma 5.38, we have that $a(b-x) \bmod b = 0$ and thus $b-x$ is an element of S.

If $ax \bmod b = x \neq 0$, then again $x \in S$, and Lemma 5.38 states that $a(b-x) \bmod b = b-(ax \bmod b) = b-x$, which implies that $b-x \in S$. □

We consider both possibilities separately. Easy modular calculations characterize them.

Lemma 5.42. *Let a and b be positive integers, and let x be an integer such that $0 \le x \le b-1$. Then $ax \bmod b = 0$ if and only if x is a multiple of $\frac{b}{\gcd\{a,b\}}$.*

Lemma 5.43. *Let a and b be positive integers, and let x be an integer such that $0 \le x \le b-1$. Then $ax \bmod b = x$ if and only if x is a multiple of $\frac{b}{\gcd\{a-1,b\}}$.*

With this we can control the set of integers x in $[0,b-1]$ such that $x \in \mathrm{S}(a,b,1)$ and $b-x \in \mathrm{S}(a,b,1)$.

Lemma 5.44. *Let $S = \mathrm{S}(a,b,1)$ for some integers a and b such that $0 < a < b$. Let $d = \gcd\{b,a\}$ and $d' = \gcd\{b,a-1\}$, and let x be an integer such that $0 \le x \le b-1$. Then $x \in S$ and $b - x \in S$ if and only if*

$$x \in X = \left\{0, \frac{b}{d'}, 2\frac{b}{d'}, \ldots (d'-1)\frac{b}{d'}, \frac{b}{d}, 2\frac{b}{d}, \ldots, (d-1)\frac{b}{d}\right\}.$$

Moreover, the cardinality of X is $d' + d - 1$.

Proof. By Lemma 5.38 we know that $x \in S$ and $b - x \in S$ if and only if $ax \bmod b \in \{0,x\}$. By using now Lemmas 5.42 and 5.43, we know that this is equivalent to $x \in X$.

Note that $\gcd\{a-1,a\} = 1$ and thus $\gcd\{d',d\} = 1$. If $sb/d' = tb/d$ for some $s,t \in \mathbb{N}$, then $sd = td'$ and since $\gcd\{d',d\} = 1$, we deduce that there exists $k \in \mathbb{N}$ such that $sd = td' = kd'd$. Hence $s = kd'$ and $t = kd$. Therefore the cardinality of X is $d' + d - 1$. □

The number of gaps of $\mathrm{S}(a,b,1)$ can now be easily computed as we show in the following theorem.

Theorem 5.45. *Let $S = \mathrm{S}(a,b,1)$ for some integers a and b with $0 \le a < b$. Then*

$$\mathrm{g}(S) = \frac{b + 1 - \gcd\{a,b\} - \gcd\{a-1,b\}}{2}.$$

Proof. Let d, d' and X be as in Lemma 5.44. By using Proposition 5.40 and Lemma 5.44, we deduce that for the set $Y = \{0,\ldots,b-1\} \setminus X$, the cardinality of $(Y \cap S)$ equals that of $(Y \setminus S)$. Hence the cardinality of Y is $2\mathrm{g}(S)$. From Lemma 5.44, we deduce that $2\mathrm{g}(S) = b - (d + d' - 1)$. □

Open Problem 5.46. How are the minimal generators of a modular numerical semigroup characterized? More precisely, which additional condition(s) must be imposed in Theorem 5.35 to obtain a characterization of modular numerical semigroups in terms of their minimal generators?

6 Opened modular numerical semigroups

In this section we characterize those proportionally numerical semigroups that are irreducible. The idea is extracted from [97].

Recall that a numerical semigroup of the form $\{0, m, \rightarrow\}$ with m a positive integer is called a half-line. We say that a numerical semigroup S is an *opened modular* numerical semigroup if it is either a half-line or $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ for some integers a and b with $2 \le a < b$.

Note that the half-line $\{0, m, \rightarrow\} = \mathrm{S}([m, 2m])$ and thus it is a proportionally modular numerical semigroup in view of Theorem 5.14. The semigroups of the form $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ are also proportionally modular by Theorem 5.14.

We are going to see that every irreducible proportionally modular numerical semigroup is of this form. The idea is to compute the genus of these semigroups by using what we already know for modular numerical semigroups. As the Frobenius number for opened modular numerical semigroups is easy to compute, we can then search which of these semigroups have the least possible number of gaps in order to get the irreducibles.

The next result shows that opened modular numerical semigroups play the same role in the set of proportional numerical semigroups as irreducible numerical semigroups do for numerical semigroups in general.

Proposition 5.47. *Every proportionally modular numerical semigroup is the intersection of finitely many opened modular numerical semigroups.*

Proof. Let S be a proportionally modular numerical semigroup. If $S = \mathbb{N}$, then clearly S is a half-line and thus opened modular. So assume that $S \neq \mathbb{N}$. By Theorem 5.14, there exist rational numbers α and β with $1 < \alpha < \beta$ such that $S = \mathrm{S}([\alpha, \beta])$. Let $h \in \mathrm{G}(S)$. If $h \geq \alpha$, in view of Lemma 5.8, there exists $n_h \in \mathbb{N}$ such that $n_h \geq 2$ and $\frac{h}{n_h} < \alpha < \beta < \frac{h}{n_h - 1}$. Define $S_h = \mathrm{S}\left(\left]\frac{h}{n_h}, \frac{h}{n_h-1}\right[\right)$, which contains S. If $h < \alpha$, set $S_h = \{0, h+1, \rightarrow\}$. Observe that in this setting $\mathrm{m}(S) > h$ (use Lemma 5.8), and consequently S_h contains S. Hence $S \subseteq \bigcap_{h \in \mathrm{G}(S)} S_h$. If $x \notin S$, then $x \in \mathrm{G}(S)$ and by Lemma 5.8 (or simply by the definition in the half-line case) $x \notin S_x$. This proves that $\bigcap_{h \in \mathrm{G}(S)} S_h \subseteq S$, and thus both semigroups coincide. □

In this section, a and b represent two integers such that $2 \leq a < b$, and d and d' will denote $\gcd\{a, b\}$ and $\gcd\{a-1, b\}$, respectively.

Lemma 5.48.

$$\{b+1, \rightarrow\} \subseteq \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right).$$

Proof. Let n be a positive integer. As $a(b+n) - (a-1)(b+n) = b+n > b$, there exists a positive integer k such that $(a-1)(b+n) < kb < a(b+n)$. This implies that $\frac{b}{a} < \frac{b+n}{k} < \frac{b}{a-1}$. Lemma 5.8 ensures that $b+n \in \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. □

Lemma 5.49. *Let x be a nonnegative integer. Then*

$$x \in \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right) \quad \textit{and} \quad x \notin \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$$

if and only if

$$x \in \left\{\lambda \frac{b}{d} \,\middle|\, \lambda \in \{1, \ldots, d\}\right\} \cup \left\{\lambda \frac{b}{d'} \,\middle|\, \lambda \in \{1, \ldots, d'\}\right\}.$$

Proof. Let $T = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-1}\right]\right)$ and let $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. By Lemma 5.8, if $x \in T \setminus S$, then there exists a positive integer k such that either $\frac{x}{k} = \frac{b}{a}$ or $\frac{x}{k} = \frac{b}{a-1}$. This implies that either x is a multiple of $\frac{b}{d}$ or $\frac{b}{d'}$. As by Lemma 5.48, $\{b+1, \rightarrow\} \subseteq \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$, this forces $x \in \{\lambda \frac{b}{d} \mid \lambda \in \{1, \ldots, d\}\} \cup \{\lambda \frac{b}{d'} \mid \lambda \in \{1, \ldots, d'\}\}$.

For the other implication, take $x \in \{\lambda \frac{b}{d} \mid \lambda \in \{1,\ldots,d\}\} \cup \{\lambda \frac{b}{d'} \mid \lambda \in \{1,\ldots,d'\}\}$. Then either $x = \lambda \frac{b}{d}$ or $x = \lambda \frac{b}{d'}$. In both cases $x \in T$ by Lemma 5.8. Assume that $\lambda \frac{b}{d} \in S$. Then again by Lemma 5.8, there exists a positive integer k such that

$$\frac{b}{a} < \frac{\lambda \frac{b}{d}}{k} < \frac{b}{a-1}.$$

And this implies that $(a-1)\lambda < dk < a\lambda$. As a is a multiple of d, both dk and $a\lambda$ are multiples of d. Since $dk < a\lambda$, we have that $dk \leq a\lambda - d$. Hence $(a-1)\lambda < a\lambda - d$, which leads to $d < \lambda$, in contradiction with the choice of λ. This proves that $\lambda \frac{b}{d} \notin S$. In a similar way it is easy to show that $\lambda \frac{b}{d'}$ is not in S. □

We have achieved enough information to compute the Frobenius number and genus, with the help of Theorem 5.45, of an opened proportionally modular numerical semigroup that is not a half-line.

Theorem 5.50. *Let a and b be two integers with $2 \leq a < b$. Let $d = \gcd\{a,b\}$ and $d' = \gcd\{a-1,b\}$. Then*

$$\mathrm{F}\left(\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)\right) = b \text{ and } \mathrm{g}\left(\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)\right) = \frac{1}{2}(b-1+d+d').$$

Proof. By Lemma 5.48 and Proposition 5.49, $\mathrm{F}\left(\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)\right) = b$. As $\gcd\{d,d'\} = 1$, $\lambda \frac{b}{d} \neq \lambda' \frac{b}{d'}$ for any $\lambda \in \{1,\ldots,d-1\}$ and $\lambda' \in \{1,\ldots,d'-1\}$. By Proposition 5.49 this implies that

$$\mathrm{g}\left(\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)\right) = \mathrm{g}\left(\mathrm{S}\left(\left[\frac{b}{a},\frac{b}{a-1}\right]\right)\right) + d + d' - 1.$$

We obtain the desired formula by using Theorem 5.45. □

Open Problem 5.51. Even though we know formulas for the Frobenius number and genus of an opened modular numerical semigroup, a formula for the multiplicity in terms of a and b is still unknown.

From the formula given in Theorem 5.50 and the characterization of irreducible numerical semigroups established in Corollary 4.5, we get the following consequence.

Corollary 5.52. *Let a and b be integers such that $2 \leq a < b$.*

1) $\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)$ *is symmetric if and only if* $\gcd\{a,b\} = \gcd\{a-1,b\} = 1$.
2) $\mathrm{S}\left(\left]\frac{b}{a},\frac{b}{a-1}\right[\right)$ *is pseudo-symmetric if and only if* $\{\gcd\{a,b\},\gcd\{a-1,b\}\} = \{1,2\}$.

Example 5.53. $\mathrm{S}\left(\left]\frac{7}{3},\frac{7}{3-1}\right[\right) = \langle 3,5\rangle$ is an example of the first statement. And $\mathrm{S}\left(\left]\frac{8}{7},\frac{8}{7-1}\right[\right) = \langle 3,5,7\rangle$ illustrates the second assertion of the last corollary.

The next result characterizes irreducible half-lines.

Lemma 5.54. *Let S be an irreducible numerical semigroup. Then S is a half-line if and only if $S \in \{\mathbb{N}, \langle 2,3\rangle, \langle 3,4,5\rangle\}$.*

Proof. If S is a half-line, there exists a positive integer m such that $S = \{0, m, \rightarrow\}$. Hence $S = \langle m, m+1, \ldots, 2m-1\rangle$ and $\mathrm{e}(S) = \mathrm{m}(S)$. As S is irreducible, by Remark 4.21 and Lemma 4.15, either S has embedding dimension two or is of the form $\langle 3, x+3, 2x+3\rangle$. Since S is a half-line, S must be either $\langle 2,3\rangle$ or $\langle 3,4,5\rangle$. □

If S is not a half-line, then $\mathrm{m}(S) < \mathrm{F}(S)$. This, with the help of Lemma 5.8, translates to the following conditions in a proportionally modular numerical semigroup.

Lemma 5.55. *Let α and β be rational numbers such that $1 < \alpha < \beta$ and let $S = \mathrm{S}([\alpha, \beta])$. If S is not a half-line, then*

$$\frac{\mathrm{F}(S)}{\mathrm{F}(S) - 1} < \alpha < \beta < \mathrm{F}(S).$$

Proof. As we have mentioned above, $\mathrm{m}(S) < \mathrm{F}(S)$. By Lemma 5.8, there exists a positive integer k such that $\alpha \le \frac{\mathrm{m}(S)}{k} \le \beta$ ($k < \mathrm{m}(S)$ because $\alpha > 1$). This leads to $\alpha \le \frac{\mathrm{m}(S)}{k} < \frac{\mathrm{F}(S)}{k} \le \frac{\mathrm{F}(S)}{1}$. As $\mathrm{F}(S) \notin S$, Lemma 5.8 forces $\mathrm{F}(S)$ to be greater than β. Besides, $\beta \ge \frac{\mathrm{m}(S)}{k} \ge \frac{\mathrm{m}(S)}{\mathrm{m}(S)-1} > \frac{\mathrm{F}(S)}{\mathrm{F}(S)-1}$. By using again that $\mathrm{F}(S) \notin S$ and Lemma 5.8, $\frac{\mathrm{F}(S)}{\mathrm{F}(S)-1} < \alpha$. □

We can now prove that every irreducible proportionally modular numerical semigroup is opened modular.

Lemma 5.56. *Let S be an irreducible proportionally modular numerical semigroup that is not a half-line. Then there exists an integer k such that $2 \le k < \mathrm{F}(S)$ and $S = \mathrm{S}\left(\left]\frac{\mathrm{F}(S)}{k}, \frac{\mathrm{F}(S)}{k-1}\right[\right)$.*

Proof. By Theorem 5.14, there exist rational numbers α and β such that $1 < \alpha < \beta$ and $S = \mathrm{S}([\alpha, \beta])$. From Lemmas 5.8 and 5.55 we deduce that there exists an integer k with $2 \le k < \mathrm{F}(S)$ such that $\frac{\mathrm{F}(S)}{k} < \alpha < \beta < \frac{\mathrm{F}(S)}{k-1}$. Let $T = \mathrm{S}\left(\left]\frac{\mathrm{F}(S)}{k}, \frac{\mathrm{F}(S)}{k-1}\right[\right)$. Theorem 5.50 ensures that $\mathrm{F}(T) = \mathrm{F}(S)$. The inequalities $\frac{\mathrm{F}(S)}{k} < \alpha < \beta < \frac{\mathrm{F}(S)}{k-1}$ imply that $S \subseteq T$. The irreducibility of S forces by Theorem 4.2 that S must be equal to T, since both have the same Frobenius number. □

With all this information, by using Corollary 4.5 it is not hard to prove the following characterization of irreducible modular numerical semigroups.

Theorem 5.57. *Let S be a proportionally modular numerical semigroup.*

1) *S is symmetric if and only if $S = \mathbb{N}$, $S = \langle 2,3\rangle$ or $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ for some integers a and b with $2 \le a < b$ and $\gcd\{a,b\} = \gcd\{a-1,b\} = 1$.*
2) *S is pseudo-symmetric if and only if $S = \langle 3,4,5\rangle$ or $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ for some integers a and b with $2 \le a < b$ and $\gcd\{a,b\} = \gcd\{a-1,b\} = \{1,2\}$.*

Exercises

Exercise 5.1. Let a, b and c be positive integers with $\gcd\{a,b\} = 1$. Prove that $S = \langle a, a+b, a+2b, \ldots, a+cb\rangle$ is a proportionally modular numerical semigroup.

Exercise 5.2. Let S be a proportionally modular numerical semigroup with minimal system of generators $\{n_1 < n_2 < \cdots < n_e\}$ and $e \geq 3$. Prove that $\langle n_1, \ldots, n_{e-1}\rangle$ is also a proportionally modular numerical semigroup.

Exercise 5.3 ([25]). Given integers a, b and c such that $0 < c < a < b$, prove that

$$\mathrm{S}(a,b,c) = \mathrm{S}(b+c-a,b,c).$$

Exercise 5.4 ([95]). Prove that a numerical semigroup S is proportionally modular if and only if there is an arrangement $n_1, \ldots, n_e$ of its minimal generators such that the following conditions hold:

1) $\langle n_i, n_{i+1}\rangle$ is a numerical semigroup for all $i \in \{1, \ldots, e-1\}$,
2) $\langle n_{i-1}, n_i, n_{i+1}\rangle = \langle n_{i-1}, n_i\rangle \cup \langle n_i, n_{i+1}\rangle$ for all $i \in \{2, \ldots, e-1\}$.

(*Hint:* Use Theorem 5.35.) Observe that this result sharpens the contents of Remark 5.16.

Exercise 5.5. Let $S = \langle 7, 8, 9, 10, 12\rangle$. Prove that S is not proportionally modular. However $S = \langle 12, 7\rangle \cup \langle 7, 8\rangle \cup \langle 8, 9\rangle \cup \langle 9, 10\rangle$.

Exercise 5.6. Find two proportionally modular numerical semigroups whose intersection is not proportionally modular.

Exercise 5.7. Give an example of a proportionally modular numerical semigroup $S \neq \mathbb{N}$ such that $S \cup \{\mathrm{F}(S)\}$ is not proportionally modular.

Exercise 5.8 ([25]). For integers a, b and c with $0 < c < a < b$, prove that

$$\mathrm{F}(\mathrm{S}(a,b,c)) = b - \left\lfloor \frac{\zeta b}{a} \right\rfloor - 1,$$

where $\zeta = \min\left\{ k \in \{1, \ldots, a-1\} \mid kb \bmod a + \left\lfloor \frac{kb}{a} \right\rfloor c > (c-1)b + a - c \right\}$.

Exercise 5.9 ([94]). Let $ax \bmod b \leq x$ be a modular Diophantine inequality (with as usual $0 < a < b$). We define its *weight* as $\mathrm{w}(a,b) = b - \gcd\{a,b\} - \gcd\{a-1,b\}$.

a) Prove that if two modular Diophantine inequalities have the same set of integers solutions, then they have the same weight.
b) Find an example showing that the converse of a) does not hold in general.
c) Prove that $\mathrm{w}(a,b)$ is an odd integer greater than or equal to $\mathrm{F}(\mathrm{S}(a,b,1))$.
d) Show that $\mathrm{S}(a,b,1)$ is symmetric if and only if $\mathrm{w}(a,b) = \mathrm{F}(\mathrm{S}(a,b,1))$.
e) Show that $\mathrm{S}(a,b,1)$ is pseudo-symmetric if and only if $\mathrm{w}(a,b) = \mathrm{F(S(a,b,1))+1}$.

Exercise 5.10 ([94]). Let a and b be integers with $0 < a < b$. Prove that $b \geq \mathrm{F}(\mathrm{S}(a,b,1)) + \mathrm{m}(\mathrm{S}(a,b,1))$ and that the equality holds if and only if

$$\mathrm{m}(\mathrm{S}(a,b,1)) \neq \min\left\{\frac{b}{\gcd\{a,b\}}, \frac{b}{\gcd\{a-1,b\}}\right\}.$$

Exercise 5.11 ([94]). Given integers a and b with $0 < a < b$, show that

$$b \leq 12\mathrm{g}(\mathrm{S}(a,b,1)) - 6.$$

Exercise 5.12. Prove that $S = \langle 3, 8, 10\rangle$ is a proportionally modular numerical semigroup that is not modular.

Exercise 5.13 ([94]). Let a and b be positive integers. Prove that

a) $\mathrm{m}(\mathrm{S}(a,ab,1)) = b$,
b) $\mathrm{F}(\mathrm{S}(a,ab,1)) = \left\lceil \frac{(a-1)(b-1)}{b} \right\rceil b - 1$.

Exercise 5.14 ([94]). Let a and b be integers such that $0 < a < b$ and $b \bmod a \neq 0$. Show that

a) $\mathrm{F}(\mathrm{S}(a,b,1)) = b - \left\lceil \frac{b}{a} \right\rceil$ if and only if $(a-1)(a-(b \bmod a)) < b$,
b) if $(a-1)(a-(b \bmod a)) < b$, then $\mathrm{m}(\mathrm{S}(a,b,1)) = \left\lceil \frac{b}{a} \right\rceil$.